BANACH ALGEBRAIC PROPERTIES OF THE FOURIER AND FOURIER-STIELTJES ALGEBRAS OVER LOCALLY COMPACT GROUPOIDS

by

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ABSTRACT

The Fourier and Fourier-Stieltjes algebras are two Banach algebras that can be associated to a locally compact group and were introduced in the paper *L'algèbre de Fourier d'un groupe localement compact* by Eymard. These algebras formed connections between other widely studied algebraic objects defined for groups- the group C^*-algebra is the continuous linear predual to the Fourier-Stieltjes algebra, and the group von Neumann algebra is the continuous linear dual to the Fourier algebra. Since their introduction, many group properties as well as Banach algebraic properties have been linked to these algebras, allowing one to study these as opposed to the groups or properties themselves.

Separately, groups have been generalized to an object known as a groupoid. Groupoids have received much attention as of recent due to their wide array of applications, from the study of discrete graphs, to dynamical systems, to topology. The Fourier and Fourier-Stieltjes algebras are built out of the harmonic analysis of groups, so it is natural to ask whether there are sufficient tools to make analogous definitions over groupoids. While the answer is yes, there is no unique way to extend these definitions to the groupoid setting in a way that will reduce to the usual notion when the groupoid is a group.

We study the *continuous* Fourier and Fourier-Stieltjes algebras, particularly their structure and norms. Our main motivating question was to explicitly determine these algebras for specific families of groupoids. We also study restriction style results, asking what objects we obtain when we restrict the domain of functions in Fourier or Fourier-Stieltjes algebras to subgroups of the base groupoid. We establish connections between these restricted algebras and the Fourier/Fourier-Stieltjes algebras of the subgroups. In line with this study, we examine the behavior of the restriction maps to

particular classes of groupoids, including transitive groupoids, group bundles, and HLS groupoids.

In the transitive case, we show that the restriction map on the Fourier-Stieltjes algebra is surjective, as well as for group bundles with a particular topology. We also identify the Gelfand spectrum of these group bundles and find an explicit description for their Fourier and Fourier-Stieltjes algebras. In the HLS case, we explicitly construct pre-images for the restriction map on the Fourier algebra, and demonstrate that it is not always surjective on the Fourier-Stieltjes algebra. We further apply these restriction results to obtain a number of Banach algebraic corollaries based on the idea of hereditary properties; those which are preserved under continuous homomorphism with dense range. We also present a preliminary probe into Fourier coefficient spaces for groupoids.

Chapter 1

INTRODUCTION

Every locally compact group G has associated to it two commutative Banach algebras: the Fourier-Stieltjes algebra $B(G)$ and the Fourier algebra $A(G)$. This area of research got its start with Eymard's paper "L'algèbre de Fourier d'un groupe localement compact", [31], which translates to "The Fourier algebra of a locally compact group". The definitions for $B(G)$ and $A(G)$ are found in Chapters 2 and 3 respectively. If $\pi : G \to \mathcal{U}(\mathcal{H})$ is a continuous unitary representation of G, for $\xi, \eta \in \mathcal{H}$ the map $\pi_{\xi,\eta} : G \to \mathbb{C}$ given by $\pi_{\xi,\eta}(x) = \langle \pi(x)\xi, \eta \rangle$ is called a coefficient function. The collection of all coefficient functions for equivalence classes of unitary representations is the Fourier-Stieltjes algebra $B(G)$. The space $B(G)$ forms a commutative algebra under the operations of pointwise scalar multiplication, and pointwise addition and multiplication of functions. If $\varphi \in B(G)$, define the norm of φ to be

$$\|\varphi\| = \inf_{\varphi = \pi_{\xi,\eta}} \|\xi\| \, \|\eta\|$$

the infimum taken over all representations of $\varphi = \pi_{\xi,\eta}$ where π is a continuous unitary representation of G. Under this norm, $B(G)$ is a complete Banach algebra. Alternatively, (and originally) $B(G)$ can be realized as $C^*(G)^*$, the continuous linear dual to the full group C^*-algebra (the C^*-algebra is defined in Section A.3).

All locally compact groups possess a Haar measure, a regular Borel measure μ such that $\mu(xS) = \mu(S)$ for all measurable sets $S \subseteq G$ and every $x \in G$ (this last property is called the left invariance of the measure). This measure is unique up to multiplication by a positive constant. The construction of this measure can be seen in [32, Theorem 2.10]. Let μ be a Haar measure of the locally compact group G. The

Fourier algebra $A(G)$ is a sub-algebra of $B(G)$ obtained by restricting attention to only the coefficient functions of the left regular representation $L : G \to \mathcal{U}(L^2(G, \mu))$ where $(L(x)f)(z) = f(x^{-1}z)$. The Fourier algebra is contained in $C_0(G)$, the collection of complex valued functions which vanish at infinity, and is also the predual of the group von Neumann algebra $VN(G)$.

The Fourier algebra $A(G)$ itself suffices to form a Banach algebra, without the need for closure, as shown in [31]. This is not true for a general representation π. In his PhD thesis [8], Arsac initiated the study of Fourier coefficient spaces. For a continuous unitary representation π, the Fourier coefficient space associated to π is $A_\pi(G) := \overline{\text{span}}\,\{\pi_{\xi,\eta} : \xi, \eta \in \mathcal{H}_\pi\}$, the closure taken with respect to the $B(G)$ norm. Each A_π is a Banach space (as a closed subspace of a Banach space). Since their introduction, the Fourier coefficient spaces have seen a wide array of uses. In [2] the authors make use of these spaces to classify projections (functions f such that $f * f = f$ under the convoloution product) in $L^1(G)$ for unimodular groups, and in [86] the authors study the Gelfand spectrum of the algebra generated by these spaces.

The Fourier and Fourier-Stieltjes algebras have been widely studied for their connections between other group algebraic constructs (in addition to their connections to $VN(G)$ and $C^*(G)$). One connection in particular is the fact that G is amenable as a group if and only if $A(G)$ has a bounded approximate identity. We discuss this and other connections in Section 2.2.3. These algebras as well as their coefficient spaces also enjoy a plethora of restriction and hereditary results. The algebra $A(G)|_H$, which is the elements of $A(G)$ whose domain has been restricted to a closed subgroup H can be identified with $A(H)$ (as shown by Herz [40]). This result extends to the Fourier coefficient spaces, that $A_\pi(G)|_H = A_{\pi|_H}(G)$. The same restriction of $B(G)$ results in a closed subspace of $B(H)$, though it will not necessarily be the entire space. These results are discussed in more detail in Section 2.3.5.

Groupoids were first introduced in [15] during a discussion of semigroups as another generalization of the notion of a group. We introduce the formal definition of a groupoid in Section 2.1. Informally, a groupoid $\mathcal{G}$ will be a similar algebraic object to

a group. A groupoid will only require a partial product rather than a full one, and will not require an identity element. In lieu of an identity, groupoids have their unit space $\mathcal{G}^{(0)}$ whose elements act like an identity. The unit space consists of elements of the form xx^{-1}, and when the products are defined we have $axx^{-1} = a = yy^{-1}a$. Multiplication of an element by its inverse is always defined, and we use this to define the range and source maps r and s. For an element x, $r(x) = xx^{-1}$ and $s(x) = x^{-1}x$. The range and source maps mentioned earlier can be used to form the fibers of a groupoid: for a unit space element u set $\mathcal{G}^u := r^{-1}(\{u\})$, $\mathcal{G}_u := s^{-1}(\{u\})$, and $\mathcal{G}_v^u := \mathcal{G}_v \cap \mathcal{G}^u$. The set $\mathcal{G}_u^u$ will always be a group, called the isotropy subgroup at u. The quintessential example of a groupoid (which is not itself a group) is the group bundle, the disjoint union of groups.

Groupoids have received their fair share of attention, as they arise in a variety of contexts. Groupoids can be formed from graphs, which connect to Cuntz-Krieger algebras and dynamical systems as in [52], and the fundamental groupoid of a topological space is a topological invariant, shown in [17]. If the range map of a groupoid is a local homeomorphism, the groupoid is called étale. When $\mathcal{G}$ is an étale groupoid we can further associate a group to this known as the *topological full group*, formed from particular open sets of the groupoid, which, as seen in [60] can be used as a tool in the study of topological dynamical systems. Recently, groupoids have been receiving attention serving as models for operator algebras (see [74, 62, 64, 57, 51, 28]). With this, it is natural to ask whether a given C^*-algebra can be realized as the full or reduced C^*-algebra for some locally compact groupoid. This question was answered in the negative in [18].

A combination of the recent interest in groupoids and the deep functorial results of the Fourier/Fourier-Stieltjes algebra motivates our study. It is immediate to ask whether or not there are analogous versions of these algebras in the groupoid case. One requirement we impose is that any definition reduces to the the existing definitions of $B(G)$ and $A(G)$ when the groupoid is a group. There are at least two approaches to defining $B(\mathcal{G})$ and $A(\mathcal{G})$. The two approaches follow a similar path, both mimicking

the group case:

1. Begin with a locally compact groupoid $\mathcal{G}$;
2. Define a replacement for continuous unitary representations;
3. Define the analog of coefficient functions for the operators in (2);
4. Consider the collection of all such functions in (3) for $B(\mathcal{G})$, and restrict to some version of a left regular representation for $A(\mathcal{G})$.

The first approach, due to [76] replaces the Hilbert spaces of unitary representations with *measurable* Hilbert fields. The second, which our research follows from, is due to [70] which uses *continuous* Hilbert fields. Both of these objects are ways of considering collections of Hilbert spaces over an arbitrary topological space. The former approach requires a particular norm map to be measurable while the latter insists on continuity. The analogue of a continuous unitary representation will be called a $\mathcal{G}$-Hilbert bundle, and the associated function on $\mathcal{G}$ a coefficient of the $\mathcal{G}$-Hilbert bundle. We detail the specifics of this process and construction in Section 2.4. Choosing the approach we have allows us to develop similar functorial results for $B(\mathcal{G})$ and $A(\mathcal{G})$ as presented in the group case. Proceeding with measurable Hilbert bundles and measurable functions invokes the Haagerup tensor product to show a number of duality results involving the groupoid Fourier Algebra and the operator algebras. These results were announced in [75].

The thesis is organized as follows. Chapter 2 contains notations and background of relevant fields to the thesis. As mentioned, we formally define groupoids and cover many of their initial properties in Section 2.1. We also introduce the notions of a topological and locally compact groupoid here. These groupoids are the algebraic objects we have based our work on. We present relevant definitions from Banach algebra theory, Hilbert space theory, and include a discussion of group amenability which highlights the importance of the algebras $B(G)$ and $A(G)$ in Section 2.2 and its subsections.

Section 2.3 houses the constructions and properties of $B(G)$ and $A(G)$. We begin in Section 2.3.1 where we cover group representation theory. Following this, we present the original definition of $B(G)$ in Section 2.3.2 and prove it is equivalent to the collection of all coefficient functions. The next subsection, Section 2.3.3, mirrors the previous, detailing $A(G)$. In the last two subsections we discuss Arsac's coefficient spaces and the functorial properties of $B(G)$ and $A(G)$. Section 2.4 is where we formally construct the groupoid algebras $B(\mathcal{G})$ and $A(\mathcal{G})$. The final section in this chapter, Section 2.5, houses a discussion of the differences between r-discrete and étale groupoids, which are often mistakenly used interchangeably. We also briefly comment on different approaches to the algebras $B(\mathcal{G})$ and $A(\mathcal{G})$.

Chapter 3 is arguably our main chapter. It begins with Section 3.1, where we establish several restriction style results of general locally compact groupoids. We begin by detailing several technicalities that will aid us in subsequent sections. Firstly, we show that if both $\mathcal{G}_u^v$ and $\mathcal{G}_u^w$ are non-empty, then these sets are homeomorphic in the induced topology (see Proposition 3.1.2). This further implies that if $\mathcal{G}_u^v$ is non-empty, the isotropy subgroups $\mathcal{G}_u^u$ and $\mathcal{G}_v^v$ are isomorphic. In this subsection we also show in Lemma 3.1.3 that if λ^u (a measure we impose on the fiber $\mathcal{G}^u$) restricted to $\mathcal{G}_u^u$ is non-zero, then it is the Haar measure of the group.

Our main results in this chapter mimic Hertz' result: we show in Theorem 3.1.10 that the restriction of functions in $B(\mathcal{G})$ to $\mathcal{G}_u^u$ is a continuous algebra homomorphism $B(\mathcal{G}) \to B(\mathcal{G}_u^u)$ for every choice of $u \in \mathcal{G}^{(0)}$. When $\mathcal{G}$ is étale, Theorem 3.1.12 shows the surjectivity of this map when restricted to $A(\mathcal{G})$. If $\mathcal{G}$ is not étale, the restriction map is still a continuous Banach space homomorphism (seen in Proposition 3.1.17). From here, we focused on particular classes of groupoid in an effort to strengthen these results.

A transitive groupoid will be any groupoid $\mathcal{G}$ such that $\mathcal{G}_v^u$ is nonempty for every choice of $u, v \in \mathcal{G}^{(0)}$. We study the restriction map in the case when $\mathcal{G}$ is a transitive groupoid in Section 3.2.1. When $\mathcal{G}$ is transitive and the range map r possesses a continuous section, the restriction map on $B(\mathcal{G})$ is surjective (Theorem 3.2.3).

Furthermore, since $\mathcal{G}_v^u$ is non-empty for every pair u, v, the isotropy subgroups of a transitive groupoid are mutually isomorphic. We leverage this fact in Proposition 3.2.5 to classify $\mathcal{G}$-Hilbert bundles of a transitive groupoid with discrete unit space in terms of the representations of the isotropy subgroup. This then implies Corollary 3.2.5.1, that every $\mathcal{G}$-Hilbert bundle of a transitive groupoid with discrete unit space is unitarily equivalent to a $\mathcal{G}$-Hilbert bundle with a constant Hilbert field.

Our project initiated with the study of group bundles, the disjoint union of groups, due to the simplicity in their construction. One would be foolish to assume that the simplicity of the construction translates into the simplicity of the theory. We focus on locally compact group bundles whose topology induces the discrete topology on the unit space in Section 3.2.2. In this setting, we demonstrate in Proposition 3.2.7 that the restriction map on $B(\mathcal{G})$ is surjective.

We close this chapter with Section 3.2.3 and Section 3.2.4 on HLS groupoids, which are a type of group bundle constructed from a discrete group G with an approximating sequence $\{K_n\}_{n\in\mathbb{N}}$. An approximating sequence of a discrete group is a sequence of nested, finite index, normal subgroups with trivial total intersection. The HLS groupoids were first introduced by [42] and utilized further by [90] to disprove a standing conjecture concerning groupoid C^*-algebras. In the group setting, [43] showed that a group G is amenable if and only if $C^*(G) = C^*_{red}(G)$, and it was questioned whether or not this result held in the groupoid case. For groupoids, there are at least two notions of amenability (topological and measure wise, defined in [5]). The same source shows the implication that topological amenability implies measure wise, and either notion implies the equality of the full and reduced C^* algebras.

In [90], it is demonstrated that the HLS groupoid associated to the approximated group $(G, \{K_n\})$ is topologically amenable if and only if the group itself is amenable. The paper further makes use of this to construct an HLS groupoid which is not amenable, yet its full and reduced C^* algebras coincide, completing the counterexample.

HLS groupoids are always étale, thus by Theorem 3.1.12 the restriction map on

$A(\mathcal{G})$ will be surjective when $\mathcal{G}$ is an HLS groupoid. The proof is non-constructive in general. When $\mathcal{G}$ is the HLS groupoid associated to the approximated group $(G, \{K_n\}_{n\in\mathbb{N}})$ we provide an explicit pre-image for elements of $A(G)$ under the restriction map in Theorem 3.2.13. In Corollary 3.2.16.1 we show that this implies the restriction map is a quotient map for HLS groupoids. We further demonstrate in Proposition 3.2.17 that the representation formed on G by the linear actions of a $\mathcal{G}$-Hilbert bundle is always weakly contained in the collection of representations of the other groups in the bundle. We use this idea in Corollary 3.2.18.1 to give a groupoid $\mathcal{G}$ such that the restriction map on $B(\mathcal{G})$ is not necessarily surjective. Finally, we demonstrate in Proposition 3.2.20 that $A(\mathcal{G}) = \overline{B(\mathcal{G}) \cap C_c(\mathcal{G})}^{\|\cdot\|_{B(\mathcal{G})}}$.

We apply the results given in Chapter 3 in Chapter 4 to obtain a number of interesting corollaries. Banach algebras possess a variety of hereditary properties, by which we mean a property preserved under continuous homomorphisms with dense range. Given such a homomorphism $\theta : A \to B$, if B fails to have a hereditary property so does A. (Weak) Banach algebra amenability, existence of a bounded approximate identity, and contractibility are all hereditary properties, thus give negative results in $A(\mathcal{G})$ in Corollaries 4.1.6.1, 4.1.14.1, and 4.1.16.1 respectively. Paterson [70] questions whether $A(\mathcal{G})$ has a bounded approximate identity, and these corollaries provide a partial answer in conditions when existence can fail.

Weak amenability by itself has received its fair share of attention. In [46], Johnson demonstrates that $A(G)$ will not be weakly amenable for any compact simple Lie group (particularly for $G = SO(3)$).

Following this, [72] extended this construction to any non-abelian, compact, connected Lie group and [36] shows that you can drop the word "Lie". Johnson's work inspired [80] to examine a connected topic, operator amenability. With this notion, $A(G)$ is operator amenable if and only if G is amenable as a group. The looser notion of operator weak amenability for general $A(G)$ was settled independently by [85] and [82] while the question of amenability of $A(G)$ was answered in [34].

We continue generating applications of Theorems 3.1.10 and 3.1.12 in Proposition 4.2.2 and Proposition 4.2.3 by showing $B(\mathcal{G}) \simeq \ell^\infty \oplus_{t\in T} B(G_t)$ and $A(\mathcal{G}) \simeq c_0 \oplus_{t\in T} A(G_t)$ when $\mathcal{G} := \sqcup_{t\in T} G_t$ is a group bundle with T discrete.

Eymard [31] additionally showed that when $\mathcal{G}$ is a locally compact group, the Gelfand spectrum of $A(G)$, $\sigma(A(\mathcal{G}))$ (the collection of non-zero multiplicative linear functionals on $A(\mathcal{G})$) can be homeomorphically identified with G via point evaluation functionals. [70] questions whether the same is true for locally compact groupoids. In Proposition 4.3.4, we establish this is true when $\mathcal{G} := \sqcup_{t\in T} G_t$ is a group bundle with T discrete. We further question whether this is true in the general case, and outline a proof method to show this.

Harmonic analysis on groups predates the introduction of $B(G)$ and $A(G)$ and is a rich and deep field in and of itself. A continuous unitary representation $\pi : G \to \mathcal{U}(\mathcal{H})$ of a group G is called irreducible if the only subspaces of $\mathcal{H}$ that satisfy $\pi(x)(M) \leq M$ for every $x \in G$ are the 0 subspace and the entire space $\mathcal{H}$. Schur's lemma provides conditions for irreducibility, and his orthogonality relations pave the way for the Peter-Weyl theorem. The Peter-Weyl theorem provides a decomposition of the left regular representation over a compact group in terms of the group's irreducible representations. This in turn gives a decomposition of $L^2(G)$. We examine results attributed to [4] which attempted to extend these harmonic analytic results to continuous $\mathcal{G}$-Hilbert bundles over locally compact groups $\mathcal{G}$ in Section 5.1. We acknowledge where we have arrived at different conclusions, and show valid versions when $\mathcal{G}$ is a compact group bundle.

We do this to study Fourier coefficient spaces of compact groupoids in Section 5.2, which are defined analogously to Arsac's in the group case. We provide preliminary results of an ongoing project to study these spaces. Firstly, we show in Proposition 5.2.3 that every element of $B(\mathcal{G})$ when $\mathcal{G}$ is the union of a finite number of groups has a $\mathcal{G}$-Hilbert bundle and continuous sections which achieves the norm in $B(\mathcal{G})$. We also give a concrete description of $A_{\mathfrak{H}}$ in Proposition 5.2.6 under the same assumptions.

Our motivation in this study was to develop a formula for $\|u\|_{A(\mathcal{G})}$ such that if $\mathcal{G}$

is a locally compact group, $\|u\|_{A(G)} = \sum_{\pi \in \widehat{G}} d_\pi \|\pi(f)\|_1$ where $\|\pi(f)\|_1$ is the trace class norm of the operator $\pi(f)$, defined in Proposition 2.3.6. The formula itself appears in which appears in [21]. Proposition 4.2.3, when coupled with the existing norm formula provides the first generalization to the groupoid case when $\mathcal{G}$ is a union of a finite number of compact groups as seen in Proposition 5.2.7.

The propositions that can be used to derive this norm formula in the compact group case heavily depend on duality results not present in our body of work (e.g., the duality between $C^*(G)$ with $B(G)$, and $VN(G)$ with $A(G)$). Many of the arguments that can be used to circumvent these tools depend on the harmonic analysis of compact groups. We close this dissertation by presenting partial results for generalizations to the compact group setting (in Proposition 5.2.8) and pose questions that would generalize the proofs needed to the groupoid setting.

Chapter 2

NOTATIONS AND BACKGROUND

In this chapter, we will define groupoids and provide the necessary background and extensions to representation theory needed to define and analyze analogous objects to the Fourier and Fourier-Stieltjes algebras. We will begin with Section 2.1 by covering elementary properties and definitions of groupoids as presented in [84] as well as [69], [73], and [63]. We will also lift the notion of a topological group to the groupoid setting. Following this, throughout Section 2.2 and its subsections we will detail the theory of Banach algebras. Section 2.2.1 provides the definition of a Banach algebra as well as key examples we will use throughout the thesis. We will also cover linear maps between Banach spaces. Following this is Section 2.2.2 where we will cover Hilbert spaces. Our later results concern properties of Banach algebras preserved under continuous homomorphisms with dense range. One of the most important of these properties is amenability, which we discuss in Section 2.2.3. In Sections 2.3 and 2.4, we will introduce the Fourier and Fourier-Stieltjes algebras in the group and groupoid cases respectively.

2.1 Groupoids, Topological Groupoids, and Haar Systems

The following definitions, along with further development and explanation can be found in [69], and [84] with the latter also providing examples.

Definition 2.1.1. A *groupoid* is a set $\mathcal{G}$ together with a subset $\mathcal{G}^{(2)} \subseteq \mathcal{G} \times \mathcal{G}$, a product map $\mathcal{G}^{(2)} \to \mathcal{G}$, $(a, b) \mapsto ab$ and an involutive inverse map $\mathcal{G} \to \mathcal{G}$, $a \mapsto a^{-1}$ such that:

1. If $(a, b), (b, c) \in \mathcal{G}^{(2)}$, then $(ab, c), (a, bc) \in G^{(2)}$ and $(ab)c = a(bc)$;
2. $(b, b^{-1}) \in \mathcal{G}^{(2)}$ for all $b \in \mathcal{G}$, and if $(a, b) \in \mathcal{G}^{(2)}$, then $a^{-1}(ab) = b$, and $(ab)b^{-1} = a$.

The key generalizations that differentiate a group from a groupoid are the lack of an identity element, and the lack of a full product (elements are only composable if the pair is in $\mathcal{G}^{(2)}$). From this definition we may immediately deduce the following proposition concerning cancellation in groupoids.

Proposition 2.1.2. *[84, Lemma 2.1.3] Let $\mathcal{G}$ be a groupoid, and suppose $(a,c),(b,c) \in \mathcal{G}^{(2)}$ such that $ac = bc$. Then $a = b$. Similarly, if $(c,a),(c,b) \in \mathcal{G}^{(2)}$ such that $ca = cb$ then $a = b$.*

Proof. By Definition 2.1.1, (1) and (2), we have $a = acc^{-1} = bcc^{-1} = b$. The proof is identical for the second conclusion. □

Groupoids come equipped with *range* and *source* maps, denoted r and s respectively with

$$r(a) := aa^{-1} \quad \text{and} \quad s(a) := a^{-1}a.$$

These maps can be (equivalently) used to define another incredibly important subset of any groupoid: the *unit space*. The unit space, $\mathcal{G}^{(0)} := r(\mathcal{G}) = \{aa^{-1} : a \in \mathcal{G}\}$. As the inversion map is involutive, the unit space is equivalently $s(\mathcal{G})$. Another equivalent definition is given in [84], which we will detail in Proposition 2.1.4 for completeness and to illustrate techniques of proof with groupoids. To aid in its proof, and recall more properties for general groupoids, we present the following.

Proposition 2.1.3. *[84, Lemma 2.1.4] Let $\mathcal{G}$ be a groupoid. Then the following are true:*

1. *$(a,b) \in \mathcal{G}^{(2)}$ if and only if $s(a) = r(b)$;*
2. *$r(ab) = r(a)$ and $s(ab) = s(b)$ for all $(a,b) \in \mathcal{G}^{(2)}$;*
3. *$(ab)^{-1} = b^{-1}a^{-1}$ for every $(a,b) \in \mathcal{G}^{(2)}$;*
4. *$r(u) = s(u) = u$ for all $u \in \mathcal{G}^{(0)}$.*

Proposition 2.1.4. *[84, Corollary 2.1.6] If $\mathcal{G}$ is a groupoid, then*

$$\mathcal{G}^{(0)} = \left\{a \in \mathcal{G} : (a,a) \in \mathcal{G}^{(2)} \text{ and } a^2 = a\right\}.$$

Proof. Let $x \in \mathcal{G}^{(0)}$. Then $x = aa^{-1}$ for some $a \in \mathcal{G}$. By the second groupoid axiom, we can see that $r(c)c = c = cs(c)$ for each $c \in \mathcal{G}$. Combining this with Proposition 2.1.3 (4) we have that $x = r(x)x = x^2$. Next assume that $x \in \mathcal{G}$ such that $(x,x) \in \mathcal{G}^{(2)}$ and $x^2 = x$. Then we have $x^2 = x = xs(x)$ and left cancellation gives $x = s(x) \in \mathcal{G}^{(0)}$. □

We will need three more crucial subsets of a groupoid $\mathcal{G}$: the fibers of the unit space under the range and source maps. Notationally, for $u \in \mathcal{G}^{(0)}$, we set $\mathcal{G}^u := r^{-1}(\{u\})$, $\mathcal{G}_u := s^{-1}(\{u\})$, and $\mathcal{G}_u^v := \mathcal{G}_u \cap \mathcal{G}^v$. These fibers do not typically form algebraic objects in their own right, however, the set $\mathcal{G}_u^u$ always forms a group, called the *isotropy subgroup at u*.

Proposition 2.1.5. *Let $\mathcal{G}$ be a groupoid, and $u \in \mathcal{G}^{(0)}$. Then $\mathcal{G}_u^u$ is a group under the product inherited from the partial product of $\mathcal{G}$.*

Proof. For any $a, b \in \mathcal{G}_u^u$ we have $r(a) = r(b) = s(a) = s(b)$. By Proposition 2.1.3, this implies that the partial product inherited from $\mathcal{G}$ is in fact a full product. With respect to this product, $r(ab) = r(a) = u = s(b) = s(ab)$, so $\mathcal{G}_u^u$ is closed with respect to this product. As $u \in \mathcal{G}^{(0)}$, again by Proposition 2.1.3 we have $r(u) = s(u) = u$, so $u \in \mathcal{G}_u^u$. We claim that u is the identity element. As $u = xx^{-1}$ for some $x \in \mathcal{G}$, $au = a = ua$ for any $a \in \mathcal{G}_u^u$ by the definition of a groupoid. For any $a \in \mathcal{G}_u^u$, we have both $r(a^{-1}) = s(a) = u$ and $s(a^{-1}) = r(a) = u$, so G_u^u is closed under the inversion map. With this, $aa^{-1} = r(a) = u$ and $a^{-1}a = s(a) = u$, so the groupoid inverses form proper group inverses in $\mathcal{G}_u^u$. Finally, the partial product in $\mathcal{G}$ is associative when the triple product is defined, so $\mathcal{G}_u^u$ is associative in its full product. □

As we've seen, groupoids possess many comforting properties that groups do, like cancellation and the socks-shoes property for inverting products, though their proofs are longer and more involved with the lack of an identity element. In the classes

of groupoids we work with, the composability or noncomposability of elements will be immediately obvious, and so the formality of justifying why a particular pair has a defined product will be omitted. Before moving forward, we present a list of examples of groupoids before moving on to view some of the consequences of equipping groupoids with topologies.

Example 2.1.6. Any group is a groupoid where the unit space is the singleton identity of the group, and $\mathcal{G}^{(2)}$ is the entire product $\mathcal{G} \times \mathcal{G}$.

Example 2.1.7 (Matrix Groupoids)**.** Fix any $n \in \mathbb{N}$. Let $[n] = \{1, \ldots, n\}$ and define $R_n := [n] \times [n]$. Further define $(i,j)(j,k) = (i,k)$ and $(i,j)^{-1} = (j,i)$. This leads to $R_n^{(0)} = \{(i,i) : 1 \leq i \leq n\}$, which can be identified with $[n]$.

The above example generalizes to any set X, with $R_X := X \times X$, and further generalizes in the below example.

Example 2.1.8 (Equivalence Relations)**.** Let R be any equivalence relation on a set X. Equip R with the partial product $(x,y)(y,z) = (x,z)$ and inversion $(x,y)^{-1} = (y,x)$ (the same conventions as the matrix example). Then R becomes a groupoid, with $R^{(0)} = \{(x,x) : x \in X\}$ which can be identified with X in the natural way.

Example 2.1.9 (Transformation Groupoids)**.** Let X be a set, and let Γ be a group acting on X by bijections. Define $\mathcal{G} := \Gamma \times X$. The groupoid product is defined for elements of the form $(g, h \cdot x)(h, x)$ and is defined to be (gh, x). The inverse is given by $(g,x)^{-1} = (g^{-1}, g \cdot x)$. Combining these, we see that $r((g,x)) = (e_\Gamma, g \cdot x)$. Similarly, $s((g,x)) = (e_\Gamma, x)$ which can be identified with $x \in X$. These give the unit space of the transformation groupoid can be identified with X, as any action by a group element is a bijection.

Example 2.1.10 (Group Bundles)**.** The disjoint union of groups, known as a *group bundle*, also forms a groupoid. Analysis on this class of groupoid is a major focal point of this thesis, so we will devote Sections 3.2.2, 4.3, and Section 3.2.3 exclusively to these.

We next explore the idea of a topological groupoid. In the group case, one starts with a locally compact group, which is a group that has been equipped with a locally compact topology that makes the multiplication and inversion maps continuous. Locally compact groups will always possess a *Haar measure*; a left (or right) translation invariant regular Borel measure. This is discussed in detail in Section A.2. Mimicking this development in the groupoid case will form the basis for groupoid abstract harmonic analysis just as was the case for a locally compact group.

Definition 2.1.11. A *topological groupoid* is a groupoid $\mathcal{G}$ endowed with a topology under which the map $\gamma \mapsto \gamma^{-1}$ is continuous from $\mathcal{G} \to \mathcal{G}$, and the map $(g, h) \to gh$ is continuous with respect to the relative topology on $\mathcal{G}^{(2)}$ as a subset of $\mathcal{G} \times \mathcal{G}$. A *locally compact* (respectively *Hausdorff*) groupoid is a topological groupoid whose topology is locally compact (respectively Hausdorff).

An immediate consequence of Definition 2.1.11 is the continuity of the range and source maps, as r can be expressed as the composition of the continuous maps which send $x \mapsto (x, x^{-1}) \mapsto xx^{-1}$. If we take $\mathcal{G}$ to be Hausdorff, then it is easy to verify that $\mathcal{G}^{(0)}$ is also Hausdorff. Singleton sets are closed in Hausdorff spaces (Theorem 1.19 of [1]). This will imply that the fibers $\mathcal{G}^u$ and $\mathcal{G}_u$ are closed, as the preimages of closed sets under continuous functions. The Haar measure on locally compact groups allows for a theory of integration to be developed, which is discussed in more detail in Section 2.2.1 and Section A.2. The generalization to groupoids is the notion of a *Haar system* as defined below.

Definition 2.1.12. [69, Definition 2.2.2] A *left Haar system* for a locally compact groupoid $\mathcal{G}$ is a family $\{\lambda^u\}_{u \in \mathcal{G}^{(0)}}$, where each λ^u is a positive, regular Borel measure on the locally compact Hausdorff space $\mathcal{G}^u$, such that the following three conditions are satisfied:

1. The support of each λ^u is the whole of $\mathcal{G}^u$;
2. For any $g \in C_c(\mathcal{G})$, the function $g^0 : \mathcal{G}^{(0)} \to \mathbb{C}$, defined $g^0(u) = \int_{\mathcal{G}^u} g d\lambda^u$ belongs to $C_c(\mathcal{G}^{(0)})$;

3. For any $x \in \mathcal{G}$ and $f \in C_c(\mathcal{G})$,

$$\int_{\mathcal{G}^{s(x)}} f(xz) d\lambda^{s(x)}(z) = \int_{\mathcal{G}^{r(x)}} f(z) d\lambda^{r(x)}(z).$$

For $f, g \in C_c(\mathcal{G})$ we may define a groupoid convolution product

$$(f * g)(x) = \int_{\mathcal{G}^{r(x)}} f(y) g(y^{-1}x) d\lambda^{r(x)}(y). \tag{2.1}$$

This definition is enough to show that each λ^u is a Radon measure, and some sources (such as [91]) take this as their definition. We show this in the proof of Lemma 3.1.3.

Local compactness is not enough to guarantee the existence of a Haar system; see [83] for an example of a compact group bundle which does not possess a Haar system. It is known that if a groupoid $\mathcal{G}$ possesses a Haar system, then its range map will be open, i.e., $r : \mathcal{G} \to \mathcal{G}^{(0)}$ maps open sets to open sets ([91, Proposition 1.23]). However the converse fails: taking $\mathcal{G} := X \times X$ to be the full equivalence relation groupoid on X, $\mathcal{G}$ has a Haar system only when X has a Radon measure. Via [74, Lemma 1.3], a group bundle will possess a Haar system exactly when the range map r is open.

In further contrast to the group case, the Haar system of a groupoid need not be unique. Considering the equivalence relation groupoid in the preceding paragraph, any full Radon measure on X leads to a Haar system on $X \times X$. As such, if X is compact, taking any probability measure on X gives a distinct Haar system.

While groupoids are not guaranteed Haar systems, an important class referred to as the étale groupoids have at least one very easily described Haar system.

Definition 2.1.13. A locally compact, Hausdorff groupoid will be called *étale* if the range (or equivalently source) map is a local homeomorphism[1] $\mathcal{G} \to \mathcal{G}$.

All étale groupoids have Haar systems consisting of counting measures (see Section 2.5). The collection of étale groupoids has received a large amount of attention (see [44, 49, 24, 50, 66, 11, 30, 61, 39, 20, 7]).

[1] A local homeomorphism is a function f between topological spaces such that every point has an open neighborhood U such that $f(U)$ is open and $f|_U$ is a homeomorphism.

Example 2.1.14. Consider the groupoids defined in Examples 2.1.7-2.1.10. It is relatively quick to deduce the following facts.

1. Group bundles (and by extension groups viewed as groupoids) with Haar systems will be étale precisely when the individual groups are discrete in the relative topology.

2. Transformation groupoids $\Gamma \times X$ will be étale if and only if the group Γ is discrete (see [19, Example 2.9]).

Unless otherwise specified, the groupoids seen in this thesis can be assumed to be both locally compact and Hausdorff if not explicitely stated.

2.2 Banach Algebra Theory

This section is designed to house results and definitions from the theory of Banach Algebras that we will use implicitly in our work. The definitions and proofs of Section 2.2.1 and Section 2.2.2 are sourced from [12] as well as a variety of Banach algebra textbooks, such as [68, 77, 88]. We use the details of Section 2.2.3 from [59].

2.2.1 Banach Algebras and Examples

We begin with the idea of a *normed space*, which is a vector space V along with a function referred to as a norm $\|\cdot\| : V \to \mathbb{R}_+$ (the set of non-negative real numbers) such that for every $x, y \in V$, $\lambda \in \mathbb{C}$, $\|x+y\| \leq \|x\| + \|y\|$, $\|\lambda x\| = |\lambda| \, \|x\|$, and $\|x\| = 0 \iff x = 0$. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in V is *Cauchy* if for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that $\|x_n - x_m\| < \varepsilon$ for $n, m \geq N$. In a normed space (or just a metric space), any convergent sequence will be Cauchy, though the converse is not always true. A space in which every Cauchy sequence converges is called *complete*. A complete normed space is called a *Banach Space*.

Definition 2.2.1. A *Banach Algebra* is a complete normed vector space over $\mathbb{R}$ or $\mathbb{C}$ equipped with an associative product which distributes over vector addition which further obeys the norm inequality $\|xy\| \leq \|x\| \, \|y\|$ for every pair of elements x, y.

For any Banach space in this thesis we will take our scalar field to be $\mathbb{C}$. Prototypical examples of Banach algebras include the set of square matrices over $\mathbb{R}$ or $\mathbb{C}$, and the vector spaces $\mathbb{R}^n$ and $\mathbb{C}^n$ equipped with componentwise multiplication. These Banach algebras are all finite-dimensional. One can build a collection of infinite dimensional Banach spaces from any measurable space $(X, \mathcal{F}, \mu)$. For details on defining integration over these spaces, see Section A.2. The elements of these L^p spaces will be equivalence classes of complex-valued functions over locally compact groups where $f \sim g$ if and only if $\mu(\{x : f(x) \neq g(x)\}) = 0$ where μ is the Haar measure for the group. For readability, we will simply say $f \in L^p(G)$ instead of $[f] \in L^p(G)$.

Example 2.2.2 (L^p Spaces)**.** Let $(X, \mathcal{F}, \mu)$ be a measurable space and let $1 \leq p < \infty$. Define

$$L^p(X, \mu) := \left\{ f : X \to \mathbb{C} \;\middle|\; \int_G |f|^p \, d\mu < \infty \right\}.$$

Equip $L^p(X, \mu)$ with the norm $\|f\|_p = \sqrt[p]{\int_G |f|^p \, d\mu}$ and pointwise operations. Then $L^p(G, \mu)$ is a Banach space. The linearity of integration shows that $L^p(G, \mu)$ is a vector space. That $\|\cdot\|_p$ forms a norm is Minkowski's inequality [79, pg. 396]. The completeness of the space with respect to norm is known as the Riesz-Fischer Theorem [79, pg. 398].

Example 2.2.3 (The Group Algebra)**.** Let G be a locally compact group with Haar measure μ. If we define the convolution product

$$(f * g)(x) = \int_G f(y) g(y^{-1}x) d\mu(y),$$

then $L^1(G, \mu)$ is a Banach algebra. To see this, if we choose $f, g \in L^1(G, \mu)$ we have

$$\begin{aligned}
\|f * g\|_1 = \int_G |(f * g)(x)| \, d\mu(x) &= \int_G \left(\left| \int_G f(y) g(y^{-1}x) d\mu(y) \right| \right) d\mu(x) \\
&\leq \int_G \int_G \left| f(y) g(y^{-1}x) \right| d\mu(y) d\mu(x) \\
&= \left(\int_G |f(y)| \, d\mu(y) \right) \left(\int_G \left| g(y^{-1}x) d\mu(x) \right| \right)^2 \\
&= \|f\|_1 \|g\|_1 ,
\end{aligned}$$

and so $L^1(G,\mu)$ is closed under the convolution product. It can be quickly checked that the convolution product distributes over vector addition, and it is a straightforward calculation to check that convolution is also associative. The same calculation that shows $f * g$ has finite L^1 norm also shows the norm inequality required for Banach algebras.

$L^1(G,\mu)$ also possesses an involution $f \mapsto f^*$ with $f^*(x) = \Delta(x^{-1})\overline{f(x^{-1})}$ where Δ is the modular function of the group (see Section A.2). The case when $p = \infty$ gives the space $L^\infty(G)$, which forms a C^*-algebra under the norm $\|f\|_\infty = \sup\{|f(x)| : x \in G\}$ and pointwise operations on G. For some details of C^*-algebras, see Appendix A.3.

Let X and Y be Banach spaces over the same scalar field. A *linear operator* will refer to a linear map $T : X \to Y$. We say that a linear operator is *bounded* if there exists a constant C such that $\|T(x)\|_Y \leq C\|x\|_X$ for every $x \in X$. T is an *isometry* if $\|T(x)\|_Y = \|x\|_X$. The collection of all bounded linear operators $X \to Y$ is denoted $B(X,Y)$ with the second space omitted if $X = Y$. A linear operator $T : X \to Y$ is bounded if and only if it is continuous by [12, Theorem 2.2].

For normed spaces X, Y, $B(X,Y)$ also forms a normed space under the operator norm and vector addition given by pointwise addition on X. For $T \in B(X,Y)$ set

$$\|T\| := \sup\{\|T(x)\| : \|x\| \leq 1\}.$$

Equivalently, $\|T\| = \inf\{C > 0 : \|T(x)\| \leq C\|x\| \quad \forall x \in X\}$. That these define a norm is due to [12, Theorem 2.4] which further states that if Y is complete, then $B(X,Y)$ is complete. The special case of $B(X,Y)$ when $Y = \mathbb{C}$ is referred to as the *continuous linear dual of* X, denoted $X^* := B(X,\mathbb{C})$. The continuous dual space X^* is always a Banach space as $\mathbb{C}$ is complete.

Example 2.2.4. Let G be a locally compact group with Haar measure μ. Suppose that $1 \leq p < \infty$ and q such that $\frac{1}{p} + \frac{1}{q} = 1$ with $q = \infty$ if $p = 1$. Then $L^p(G)^*$ is isometrically

[2] Via the Tonelli-Fubini theorem.

isomorphic to $L^q(G)$. Hölders inequality guarantees that for every function $g \in L^q(G)$ there is a continuous linear functional T_g on $L^p(G)$ given by $T_g(f) = \int_G f(x)g(x)d\mu(x)$. The injectivity of the map $g \mapsto T_g$ and the fact that every linear functional on $L^p(G)$ is of the form T_g for some $g \in L^q(G)$ can be seen in the proof of Theorem 1.4.1 of [87].

Recall that a Banach algebra is a vector space equipped with two operations: vector addition and multiplication. By the definition of a vector space, the former operation will always have an identity. The latter may not. For further details on *approximate identities*, see Section 4.1.1, though we will define the object now.

Definition 2.2.5. [29, B29] Let A be a normed algebra. An *approximate identity* of A is a net of elements $\{\psi_\lambda\}_{\lambda \in I}$ such that $\|\psi_\lambda * x - x\| \to_\lambda 0$ and $\|x * \psi_\lambda - x\| \to_\lambda 0$ for every $x \in A$. If $\sup\{\|\psi_\lambda\| : \lambda \in I\} < \infty$ then the net is a *bounded approximate identity*.

2.2.2 Hilbert Spaces

Hilbert spaces are special cases of Banach spaces that introduce additional structure which adds notions of geometry to the vector space. Hilbert spaces are critical to abstract harmonic analysis and so we wish to mention some of their important properties. As we take our vector spaces over the field $\mathbb{C}$, an *inner product space* will be a vector space V along with a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ such that $\langle \cdot, \cdot \rangle$ is linear in its first coordinate, $\langle u, v \rangle = \overline{\langle v, u \rangle}$ for every u, v (which implies conjugate linearity in the second coordinate), and $\langle u, u \rangle \geq 0$, with equality if and only if $u = 0$. Every inner product space is also a normed space, the norm given by $\|u\| = \sqrt{\langle u, u \rangle}$. Inner product spaces also obey the Cauchy-Schwarz inequality, which states

$$|\langle u, v \rangle| \leq \|u\| \, \|v\| .$$

Inner products may also be decomposed into a linear combination of those with matching entries via the polarization identity:

$$\langle u, v \rangle = \frac{1}{4} \left(\langle u + v, u + v \rangle - \langle u - v, u - v \rangle + i \langle u + iv, u + iv \rangle - i \langle u - iv, u - iv \rangle \right)$$

Two vectors in an inner product space will be called *orthogonal* if $\langle u, v\rangle = 0$. For a subset of an inner product space $S \subseteq V$, the *orthogonal compliment of* S, $S^{\perp}$ is defined to be the collection of vectors orthogonal to every vector of S. When S is a subspace of V, then $S^{\perp}$ is also a subspace.

Definition 2.2.6. A *Hilbert space* is an inner product space which is complete with respect to the norm induced by the inner product.

Example 2.2.7. The space $\mathbb{C}^n$ is a Hilbert space under the dot product. If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ then

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i \overline{y_i}.$$

Example 2.2.8. For a locally compact group G, with Haar measure μ, $L^2(G, \mu)$ is a Hilbert space under the inner product

$$\langle f, g\rangle = \int_G f(x)\overline{g(x)} d\mu(x).$$

Given the additional structure the inner product provides, Hilbert spaces have a beautiful characterization of their continuous dual spaces in the Riesz Representation Theorem.

Proposition 2.2.9. *[25, Theorem 3.4] Let $\mathcal{H}$ be a Hilbert space and $\varphi \in \mathcal{H}^*$. Then there is a unique vector $v_\varphi \in \mathcal{H}$ such that $\varphi(u) = \langle u, v_\varphi\rangle$. Moreover, $\|\varphi\|_{\mathcal{H}^*} = \|v_\varphi\|$.*

The first corollary of the Riesz Representation Theorem to mention is the fact that $\mathcal{H}$ can be isometrically identified with $\mathcal{H}^*$ via the antilinear map $v \mapsto f_v = \langle \cdot, v\rangle$. The second consequence[3] is the existence of the adjoint map. Given a $T \in B(\mathcal{H}_1, \mathcal{H}_2)$, there is a unique map $T^* \in B(\mathcal{H}_2, \mathcal{H}_1)$ such that $\langle Tu, v\rangle_{\mathcal{H}_2} = \langle u, T^*v\rangle_{\mathcal{H}_1}$ for every $u \in \mathcal{H}_1$ and $v \in \mathcal{H}_2$. In the finite-dimensional case, T^* can be seen as the map whose matrix form is the conjugate transpose of a matrix representation for T.

[3] This idea can be generalized to Banach Spaces. If $T \in B(X, Y)$ then there is a unique $T^* \in B(Y^*, X^*)$ such that $(T^*\varphi)(x) = \varphi(Tx)$ for $\varphi \in Y^*$. For details see page 31 in [12].

If $\mathcal{H}$ is a Hilbert space, we may equip $B(\mathcal{H})$ with several topologies, the topology induced by the operator norm being just one of them. Other notable examples are the weak operator and the strong operator topologies, whose definitions are best described in terms of converging nets. We take these as our definitions, though they are consequences of another proposition in [25].

Definition 2.2.10. [25, Proposition 1.3] Let $\mathcal{H}$ be a Hilbert space. Then:

1. A net $\{\varphi_\lambda\}_{\lambda\in I}$ in $B(\mathcal{H})$ converges to φ in the weak operator topology if $\langle\varphi_\lambda u, v\rangle$ converges to $\langle\varphi u, v\rangle$ for every pair of $u, v \in \mathcal{H}$.

2. A net $\{\varphi_\lambda\}_{\lambda\in I}$ in $B(\mathcal{H})$ converges to φ in the strong operator topology if $\|\varphi_\lambda u - \varphi u\|$ converges to 0 for every $u \in \mathcal{H}$.

Proposition 2.2.11. *If $\mathcal{H}$ is a Hilbert space and $\{\varphi_\lambda\}_{\lambda\in I}$ is a net in $B(\mathcal{H})$ converging to φ in the operator norm, then it also converges in the strong operator topology. Moreover, if the net $\{\varphi_\lambda\}_{\lambda\in I}$ converges to φ in the strong operator topology, then it converges in the weak operator topology.*

2.2.3 Amenability

The subject of amenability began in the context of locally compact groups and was motivated by the Banach-Tarski paradox. This result stated that, assuming the axiom of choice, one could disassemble a ball into a finite number of pieces, and through only rigid body transformations reassemble those pieces into two identical copies of the original ball. The key points of the proof of said paradox are the existence of non-measurable sets, and some innate property of the free group $\mathbb{F}_2$ on two generators that allows for a paradoxical decomposition. It was this property that was studied and generalized into amenability. To discuss the amenability of groups, we first define a mean on $L^\infty(G)$ as in Definition 1.1.1 of [59]. Let G be a locally compact group with its Haar measure. For $\varphi \in L^\infty(G)$, set $\varphi_x(g) = \varphi(x^{-1}g)$. A *mean* is a functional $m \in L^\infty(G)^*$ such that $m(\mathbf{1}) = \|m\| = 1$ where $\mathbf{1}$ is the constant function with value 1. A mean m is is *left invariant* if $m(\varphi_x) = m(\varphi)$ for every $\varphi \in L^\infty(G)$ and every $x \in G$.

Definition 2.2.12. [59, Definition 1.1.4] A locally compact group G is *amenable* if there is a left invariant mean on $L^\infty(G)$.

It is precisely the fact that $\mathbb{F}_2$ is *not* amenable that allows the Banach-Tarksi paradox to occur. Example 1.1.5 of [59] explains that any compact group is amenable, as well as any abelian group. Moreover any countable discrete group which contains $\mathbb{F}_2$ is not amenable. Since its formalization, amenability of groups has received a substantial amount of attention and numerous equivalent conditions have been shown. Some, but not all equivalent conditions are given below and can be found in their appropriate references. The book "Amenable Locally Compact Groups" [71] presents these and others.

Proposition 2.2.13. *Let G be a locally compact group. Then the following are equivalent:*

1. *G is amenable (as in Definition 2.2.12).*
2. *All irreducible representations of G are weakly contained in the left regular representation[4] [43].*
3. *The full and reduced group C^*- algebras[5] coincide [43].*
4. *The group's Fourier Algebra[6] has a bounded approximate identity [54].*
5. *$L^1(G)$ is amenable as a Banach algebra [45]*

The idea of linking amenability of groups to the structure of their group algebra (Proposition 2.2.13, (5)) is what prompted the definition and study of the amenability of Banach algebras. This will be discussed further in Section 4.1.2.

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[4] *We discuss unitary representations in Section 2.3 and weak containment in Section A.3.*

[5] *See Appendix A.3 for a description of these objects.*

[6] *This is defined and studied in Section 2.3.*

2.3 Fourier and Fourier-Stieltjes Algebras for Groups

The titular algebras of this section are built from the representation theory of a locally compact group. To organize the content, we will divide this section as follows. Section 2.3.1 contains the background on unitary representations necessary to define the Fourier-Stieltjes and Fourier algebras. The following two sections Section 2.3.2 and Section 2.3.3 define these algebras respectively and prove that they are Banach algebras. Once we have defined the Fourier algebra, it is natural to ask about similar constructions for other unitary representations. These spaces are referred to as Fourier coefficient spaces, and are discussed in Section 2.3.4. Finally, much analysis on group algebras is done via restriction results, i.e., results centered around taking functions in the group algebras and restricting their domain to subsets of the group. A brief description of existing restriction results which motivates our study is contained in Section 2.3.5

2.3.1 Representation Theory of Groups

By a *continuous unitary representation*, we mean a homomorphism $\pi : G \to \mathcal{U}(\mathcal{H})$ from G to the group of unitary operators on some non-zero Hilbert space such that the map $x \mapsto \langle \pi(x)\xi, \eta \rangle$ is continuous for every pair $\xi, \eta, \in \mathcal{H}$. Some sources, such as [32] initially define the continuity of a representation with the strong operator topology instead, i.e. the map which sends x to $\pi(x)\xi$ is continuous for every $\xi \in \mathcal{H}$. This is not a concern for our purposes, as the weak and strong operator topologies coincide on $\mathcal{U}(\mathcal{H})$. The map $x \mapsto \langle \pi(x)\xi, \eta \rangle$ from $\mathcal{G} \to \mathbb{C}$ will be denoted $\pi_{\xi,\eta}$ and referred to as the *coefficient function of π with respect to ξ and η*. We reserve this notation for coefficient functions of unitary representations and introduce different notation when defining analogous functions for groupoids. Let $\mathbb{C}^*$ be the group of non-zero complex numbers under multiplication.

Example 2.3.1 (The Trivial Representation)**.** Let G be any locally compact group. Define $\iota : G \to \mathbb{C}^*$ by $\iota(x) = 1$ for all $x \in G$.

Example 2.3.2 (The Left Regular Representation)**.** Let G be any locally compact group. Define

$$L : G \to \mathcal{U}(L^2(G))^{7}$$

$$x \mapsto L_x$$

with $(L_x f)(z) = f(x^{-1}z)$.

We can also define the right regular representation $R : G \to \mathcal{U}(L^2(G))$ by $x \mapsto R_x$ with $(R_x f)(z) = \sqrt{\Delta(x)} f(zx)$ where Δ is the modular function of the group. It is easily verified that L and R are both continuous representations. Given any two representations π, σ we can define $\pi \oplus \sigma$ and $\pi \otimes \sigma$ as follows:

$$\pi \oplus \sigma : G \to \mathcal{U}\left(\mathcal{H}_\pi \oplus \mathcal{H}_\sigma\right)$$

$$(\pi \oplus \sigma)(x)(\xi \oplus \eta) = \pi(x)\xi \oplus \sigma(x)\eta$$

and

$$\pi \otimes \sigma : G \to \mathcal{U}\left(\mathcal{H}_\pi \otimes \mathcal{H}_\sigma\right)$$

$$(\pi \otimes \sigma)(x)(\xi \otimes \eta) = \pi(x)\xi \otimes \sigma(x)\eta.$$

In some sense, we can "see" both π and σ in the direct sum $\pi \oplus \sigma$. If there is a closed subspace $M \leq \mathcal{H}_\pi$ such that $\pi(x)M \leq M$ for every $x \in G$, M is said to be an *invariant* subspace. Restricting each $\pi(x)$ to M gives another representation, the *sub-representation* $\pi^M : G \to \mathcal{U}(M)$. In this language, both π and σ are sub-representations of $\pi \oplus \sigma$. Every representation has at least two invariant subspaces, the zero subspace and the entire Hilbert space. If these are the only two invariant subspaces, the representation is called *irreducible*, and *reducible* otherwise. Define the *dimension of a representation* to be the dimension of the Hilbert space it acts on. If G is abelian, every irreducible representation will be one-dimensional ([32, Corollary

[7] When G is a locally compact group we implicitly assume that $L^p(G)$ uses the Haar measure.

3.6]), and if G is compact each irreducible representation is finite-dimensional ([32, Theorem 5.2]).

Two representations π and σ of a group G are said to be *unitarily equivalent*, denoted $\pi \simeq \sigma$, if there exists a unitary operator $T : \mathcal{H}_\pi \to \mathcal{H}_\sigma$ such that $T\pi(x) = \sigma(x)T$ for every $x \in G$. Visually, two representations are unitarily equivalent if there is some unitary T such that the following diagram commutes for every x.

$$\begin{array}{ccc} \mathcal{H}_\pi & \xrightarrow{\pi(x)} & \mathcal{H}_\pi \\ {\scriptstyle T}\downarrow & & \downarrow{\scriptstyle T} \\ \mathcal{H}_\sigma & \xrightarrow[\sigma(x)]{} & \mathcal{H}_\sigma \end{array}$$

Example 2.3.3. For a locally compact group G with Haar measure μ, the left and right regular representations are unitarily equivalent. To see this, define $U : L^2(G) \to L^2(G)$, $Uf = \frac{\check{f}}{\sqrt{\Delta}}$ where $\check{f}(x) = f(x^{-1})$. Then:

$$\|Uf\|_2^2 = \int_G \left| \frac{f(x^{-1})}{\sqrt{\Delta(x)}} \right|^2 d\mu(x) = \int_G \left| \frac{f(x)}{\sqrt{\Delta(x^{-1})}} \right|^2 d\mu(x^{-1}) = \int_G |f(x)|^2 \, d\mu(x) = \|f\|_2^2$$

so U is an isometry. Furthermore,

$$UUf(x) = U\left(\frac{\check{f}}{\sqrt{\Delta}}\right)(x) = \frac{f\left((x^{-1})^{-1}\right)}{\sqrt{\Delta(x^{-1})}\sqrt{\Delta(x)}} = f(x)$$

so U is invertible. In particular, it is surjective so that U is a unitary map. Finally, note that

$$(L_x U f)(z) = Uf(x^{-1}z) = \frac{f(z^{-1}x)}{\sqrt{\Delta(zx^{-1})}}$$

and

$$(UR_x f)(z) = \frac{R_x f\,(z^{-1})}{\sqrt{\Delta(z)}} = \frac{f\,(z^{-1}x)}{\sqrt{\Delta(z)}}\sqrt{\Delta(x)},$$

which are equal since Δ is a homomorphism, so U intertwines L and R.

A bounded operator that satisfies the commuting condition above, but isn't necessarily unitary, is called an *intertwining operator*. The space of all intertwining operators from π to σ is denoted $C(\pi, \sigma)$ with the second argument omitted if $\pi = \sigma$.

Lemma 2.3.4. *Let G be a locally compact group, and $\pi : G \to \mathcal{U}(\mathcal{H})$ a continuous unitary representation. Let $\{\xi_\alpha\}_{\alpha\in I}$ be a net converging to ξ in $\mathcal{H}$, and let $\{x_\alpha\}_{\alpha\in I}$ be a net converging to x in G. Then $\{\pi(x_\alpha)\xi_\alpha\}_{\alpha\in I}$ converges to $\pi(x)\xi$ in $\mathcal{H}$.*

Proof. Note that

$$\|\pi(x_\alpha)\xi_\alpha - \pi(x)\xi\| = \|\pi(x_\alpha)(\xi_\alpha - \xi) + \pi(x_\alpha)\xi - \pi(x)\xi\| \leq \|\xi_\alpha - \xi\| + \|\pi(x_\alpha)\xi - \pi(x)\xi\|.$$

As $\{\xi_\alpha\}_{\alpha\in I}$ is a convergent net, there is an index α_1 such that $\|\xi_\alpha - \xi\| < \frac{\varepsilon}{2}$ for all $\alpha \geq \alpha_1$. Since π is continuous in the strong operator topology, there is an α_2 such that $\|\pi(x_\alpha)\xi - \pi(x)\xi\| < \frac{\varepsilon}{2}$ for $\alpha \geq \alpha_2$. Choosing $\alpha_0 > \max\{\alpha_1, \alpha_2\}$ we have $\|\pi(x_\alpha)\xi_\alpha - \pi(x)\xi\| < \varepsilon$ for all $\alpha \geq \alpha_0$. □

2.3.2 The Fourier-Stieltjes Algebra of a Group

The Fourier-Stieltjes algebra is a Banach algebra built from the coefficient functions of (equivalence classes of) unitary representations. This, and the similarly named Fourier algebra were introduced in a groundbreaking paper [31] by Dr. Pierre Eymard. Since their introduction, a great deal of theory has been developed surrounding these objects, linking them to other imporant algebraic constructions arising from locally compact groups (such as the group algebra $L^1(G)$, the group C^* algebra, etc). The next portion of this section will be spent defining this algebra and the Fourier algebra, as well as proving that they are in fact Banach algebras as claimed. To rip the metaphorical bandaid off, let us present the Fourier-Stieltjes algebra of a locally compact group. Let $\widetilde{G}$ denote the set of all (classes of) unitary representations of G.

Definition 2.3.5 (Fourier-Stieltjes Algebra)**.** Let G be a locally compact group. The Fourier-Stieltjes algebra is defined to be

$$B(G) := \left\{\pi_{\xi,\eta} : \pi \in \widetilde{G}, \xi, \eta \in \mathcal{H}_\pi\right\}$$

under pointwise addition, scalar multiplication, and pointwise multiplication. The norm is given by

$$\|\varphi\|_{B(G)} := \inf_{\varphi=\pi_{\xi,\eta}} \|\xi\| \cdot \|\eta\|,$$

where the infimum is taken over all such coefficient function representations of $\varphi = \pi_{\xi,\eta}$.

We take a winding path towards showing $B(G)$ as defined is indeed a Banach algebra, partly due to the way the object was initially introduced. Eymard [31, Chapter 2] defines $B(G)$ in terms of functions of positive type, and provides a different norm on the space. We mention both of these now, as they will prove invaluable in our proof. Via [32], Section 3.3, a function $\varphi \in L^\infty(G)$ is *of positive type* if it defines a positive functional on $L^1(G)$, that is, if for every $f \in L^1(G)$,

$$\int_G (f^* * f)(x)\varphi(x)dx \geq 0. \tag{2.2}$$

We denote by $P(G)$ the collection of functions of positive type over a locally compact group G. Using the definition of the convolution product and performing a change of variables we may express the inequality Equation (2.2) in the equivalent form

$$\int_G \int_G \overline{f(y)} f(x)\varphi(y^{-1}x)dydx \geq 0. \tag{2.3}$$

This association gives that (linear combinations of) functions of positive type exist in bijective correspondence with elements of $C^*(G)^*$, the continuous linear dual to the group C^*-algebra. This fact naturally gives rise to the original norm used by Eymard: for $u \in B(G)$,

$$\|u\|_B = \sup_{f \in L^1(G), \|f\|_{C^*(G)}=1} \left| \int_G f(x)u(x)dx \right|. \tag{2.4}$$

To avoid confusion in this section, we denote the norm candidate for our $B(G)$ by $\|\cdot\|_{B(G)}$ and the original norm by $\|\cdot\|_B$. We define $B(G)$ using the $\|\cdot\|_{B(G)}$ norm as it mimics the form of the norm used when defining the same algebra over a groupoid. To make use of these connections, we prove the following proposition linking functions of positive type to coefficient functions.

Proposition 2.3.6. *Let G be a locally compact group. Then a function $\varphi : G \to \mathbb{C}$ is a linear combination of functions of positive type if and only if $\varphi = \pi_{\xi,\eta}$ for some unitary representation π and $\xi, \eta \in \mathcal{H}_\pi$.*

Proof. It suffices to show that φ is a function of positive type if and only if there is a representation π with $\varphi = \pi_{\xi,\xi}$ for $\xi \in \mathcal{H}_\pi$, as this implies our proposition via the polarization identity. First assume that $\varphi = \pi_{\xi,\xi}$ for some unitary representation. For $f \in L^1(G)$ define $\pi(f) : \mathcal{H}_\pi \to \mathcal{H}_\pi$ by $\pi(f) := \int_G f(x)\pi(x)dx$, understood weakly via $\langle \pi(f)\xi', \eta' \rangle = \int_G f(x) \langle \pi(x)\xi', \eta' \rangle \, dx$. Direct computation shows

$$\begin{aligned}
\int_G \int_G \overline{f(y)} f(x) \varphi(y^{-1}x) dy dx &= \int_G \int_G \overline{f(y)} f(x) \left\langle \pi(y^{-1}x)\xi, \xi \right\rangle dy dx \\
&= \int_G \int_G \overline{f(y)} f(x) \langle \pi(x)\xi, \pi(y)\xi \rangle \, dy dx \\
&= \int_G \overline{f(y)} \langle \pi(f)\xi, \pi(y)\xi \rangle \, dy \\
&= \overline{\langle \pi(f)\xi, \pi(f)\xi \rangle} \\
&\geq 0.
\end{aligned}$$

Next, let φ be a function of positive type. Define on $L^1(G)$

$$\langle f, g \rangle_\varphi := \int_G \int_G f(x) \overline{g(y)} \varphi(y^{-1}x) dy dx.$$

We note that this defines a positive, semi-definite, hermitian form on $L^1(G)$. Further define $N := \left\{ f : \langle f, f \rangle_\varphi = 0 \right\}$. As $\left| \langle f, g \rangle_\varphi \right| \leq \left| \langle f, f \rangle_\varphi \right| \left| \langle g, g \rangle_\varphi \right|$, we have that $f \in N$ if and only if $\langle f, g \rangle_\varphi = 0$ for every $g \in L^1(G)$. It is easy to see that N is closed, and it follows quickly from the previous sentence. Using this fact, $\langle \cdot, \cdot \rangle_\varphi$ induces an inner product on $L^1(G)/N$. Define $\mathcal{H}_\varphi := \overline{L^1(G)/N}$, the completion taken with respect to the norm induced by $\langle \cdot, \cdot \rangle_\varphi$. Denote $\tilde{f}$ to be the image of f in $L^1(G)/N$ under the quotient map.

Another direct computation shows that

$$\begin{aligned}
\langle \lambda_x f, \lambda_x g \rangle_\varphi &= \int_G \int_G f(x^{-1}y) \overline{g(x^{-1}z)} \varphi(y^{-1}z) dy dz \\
&= \int_G \int_G f(y) \overline{g(x^{-1}z)} \varphi(y^{-1}x^{-1}z) dy dz \\
&= \int_G \int_G f(y) \overline{g(z)} \varphi(y^{-1}z) dy dz \\
&= \langle f, g \rangle_\varphi
\end{aligned}$$

so $\lambda_x(N) \subseteq N$. We can then define $\pi_\varphi(x)(\tilde{f}) := \widetilde{\lambda_x f}$, which is a continuous unitary representation $G \to \mathcal{U}(\mathcal{H}_\varphi)$. What remains to show is that there exists a vector ξ such that $\varphi(x) = \langle \pi_\varphi(x)\xi, \xi \rangle_\varphi$. To construct this, let $\{\psi_u\}_{u \in I}$ be a bounded approximate identity in $L^1(G)$ (so that $\{\psi_u^*\}_{u \in I}$ is also a bounded approximate identity). Then $\left\langle \tilde{f}, \tilde{\psi}_u \right\rangle_\varphi = \int_G (\psi_u^* * f)(x)\varphi(x)dx \to_u \int_G f(x)\varphi(x)dx$. It follows that $\lim_u \left\langle v, \tilde{\psi}_u \right\rangle_\varphi$ exists for every $v \in \mathcal{H}_\varphi$ so $\tilde{\psi}_u$ weakly converges to ξ for some $\xi \in \mathcal{H}_\varphi$. We claim that $\langle \pi_\varphi(x)\xi, \xi \rangle = \varphi(x)$. This is true if and only if the two sides of the equation define the same linear functional on $L^1(G)$, i.e. for every $f \in L^1(G)$

$$\begin{aligned}
&\int_G \langle \pi_\varphi(x)\xi, \xi \rangle_\varphi f(x)dx = \int_G \varphi(x)f(x)dx = \lim_u \int_G \varphi(x)f(x)\psi_u(x)dx \\
&\iff \langle \pi_\varphi(f)\xi, \xi \rangle_\varphi = \lim_u \left\langle \tilde{f}, \tilde{\psi}_u \right\rangle_\varphi \\
&\iff \lim_u \left\langle \pi_\varphi(f)\xi, \tilde{\psi}_u \right\rangle_\varphi = \lim_u \left\langle \tilde{f}, \tilde{\psi}_u \right\rangle_\varphi \\
&\iff \pi_\varphi(f)\xi = \tilde{f}.
\end{aligned}$$

The final equality is true if and only if $\langle \pi(f)\xi, \tilde{g} \rangle_\varphi = \left\langle \tilde{f}, \tilde{g} \right\rangle$ for every $\tilde{g} \in \mathcal{H}_\varphi$. This is immediate from the following lemma.

Lemma 2.3.7. *Let $\mathcal{H}$ be a Hilbert space. Then $u = v$ if and only if $\langle u, w \rangle = \langle v, w \rangle$ for every $w \in \mathcal{H}$.*

A final omitted direct calculation verifies $\langle \pi(f)\xi, \tilde{g} \rangle_\varphi = \left\langle \tilde{f}, \tilde{g} \right\rangle$, completing the proof. □

Corollary 2.3.7.1. *If φ is a continuous function of positive type of a locally compact group G, then $\|\varphi\|_\infty = \varphi(e_G)$.*

Proof. We have $\varphi(x) = \langle \pi(x)\xi, \xi \rangle$, and so $|\varphi(x)| \leq \|\xi\|^2 = \varphi(e_G)$. □

As stated previously, the norm $\|\cdot\|_B$ arises naturally as an operator space norm when considering $B(G)$ as $C^*(G)^*$. It is less obvious why $\|\cdot\|_{B(G)}$ defines a norm, a fact we clarify in the next proposition, simultaneously demonstrating that $\|\cdot\|_B$ and $\|\cdot\|_{B(G)}$ are equivalent.

Proposition 2.3.8. *[31, Lemma 2.14] Let G be a locally compact group. For any $\varphi \in B(G)$ such that $\varphi = \pi_{\xi,\eta}$, we have $\|\varphi\|_B \leq \|\xi\| \cdot \|\eta\|$. Conversely, if $u \in B(G)$ there exists a representation π and vectors $\xi, \eta \in \mathcal{H}_\pi$ such that $\varphi = \pi_{\xi,\eta}$ and $\|\varphi\|_B = \|\xi\| \cdot \|\eta\|$.*

Corollary 2.3.8.1. *For a locally compact group G, $\|\cdot\|_{B(G)}$ is a norm on $B(G)$ and it is equivalent to $\|\cdot\|_B$.*

We will refer to the representation and pair of vectors which achieve the norm in $B(G)$ given by Proposition 2.3.8 as the *Eymard representation* and *Eymard vectors*. With this, we have all the tools necessary to accomplish the goal of this subsection: show that $B(G)$ is a Banach algebra.

Proposition 2.3.9. *Let G be a locally compact group, and $B(G)$ as defined in Definition 2.3.5. Then $B(G)$ is a Banach algebra.*

Proof. That $B(G)$ is both a vector space and an algebra is immediate from the existence of the tensor and direct product of representations since $\pi_{\xi,\eta} + \sigma_{\xi',\eta'} = (\pi \oplus \sigma)_{\xi \oplus \xi', \eta \oplus \eta'}$ and $\pi_{\xi,\eta}\sigma_{\xi',\eta'} = (\pi \otimes \sigma)_{\xi \otimes \xi', \eta \otimes \eta'}$. It becomes a normed space by pairing it with the $\|\cdot\|_{B(G)}$. $B(G)$ is complete with respect to this norm as it is complete with respect to the $\|\cdot\|_B$ norm due to the identification that $B(G) = C^*(G)^*$. Finally, let $\varphi_1, \varphi_2 \in B(G)$, and let π, π' be Eymard representations for φ_1 and φ_2 with ξ, η, ξ', η' Eymard vectors for φ_1 and φ_2 respectively. Then we have

$$\|\varphi_1\varphi_2\|_{B(G)} \leq \|\xi \otimes \xi'\| \cdot \|\eta \otimes \eta'\| = \|\xi\| \cdot \|\eta\| \cdot \|\xi'\| \cdot \|\eta'\| = \|\varphi_1\|_{B(G)} \cdot \|\varphi_2\|_{B(G)}.$$

□

Alternatively we could have also used the equivalence of the two norms on $B(G)$ to show the last inequality.

2.3.3 The Fourier Algebra of a Group

The Fourier algebra is also a collection of certain coefficient functions, in this case restricting to coefficient functions of the left regular representation. As was the

case with the Fourier-Stieltjes algebra, this was not the original definition of the Fourier algebra. We devote some time to showing the equivalence of the definitions, and will end with small remarks that we will need in subsequent chapters. We do this in two steps: first, providing the original definition of the Fourier algebra as well as proving it is indeed a Banach algebra, then proving the theorem which shows their equivalence. The results themselves can be sourced originally [31], though the statements and proofs appearing here are due to Kaniuth and Lau in their book [48]. We continue the assumption that G is a locally compact group.

Definition 2.3.10. Let G be a locally compact group. The Fourier Algebra is defined to be

$$A(G) := \left\{\lambda_{f,g} : \lambda \text{ is the left regular representation of } G, f, g \in L^2(G)\right\}$$

under pointwise addition, scalar multiplication, and pointwise multiplication. The norm is the Fourier-Stieltjes norm, $\|\cdot\|_{B(G)}$.

We will shortly, in Proposition 2.3.13, provide the original definition of $A(G)$ and in the process prove that it is a Banach algebra. Proposition 2.3.16 shows that the definition we proceed with and the original definition are equivalent. Prior to this, we will need the following two results.

Proposition 2.3.11. *[48, Lemma 2.3.1] Let $f, g \in L^2(G)$. Then $f * \tilde{g} \in A(G)$ and $\|f * \tilde{g}\|_{B(G)} \leq \|f\|_2 \cdot \|g\|_2$ with $\tilde{g}(x) = \overline{g(x^{-1})}$*

Proof. Establishing the former statement gives the latter as a direct consequence of the fact that $\|\pi_{\xi,\eta}\|_{B(G)} \leq \|\xi\| \cdot \|\eta\|$. By direct calculation:

$$(f * \tilde{g})(x) = \int_G f(y)\overline{g(y^{-1}x)}dy = \langle \lambda_{x^{-1}} f, g\rangle = \lambda_{f,g}(x^{-1}).$$

□

As later proofs will benefit from the notation, define $\check{f}(x) = f(x^{-1})$.

Proposition 2.3.12. *[48, Proposition 2.3.2] Let G be a locally compact group with Haar measure μ, K a compact subset of G and U an open subset of G such that $K \subseteq U$. Then there exists a function u on G which is a finite linear combination of functions in $P(G) \cap C_c(G)$ and satisfies*

$$0 \leq u \leq 1, \;\; u|_K = 1 \text{ and } u|_{G \setminus U} = 0.$$

Proof. As K is compact, we may find a compact symmetric (meaning $S^{-1} = S$) neighborhood V containing e_G such that $KV^2 \subseteq U$. Let $\mathbb{1}_S$ denote the characteristic function of the set S. Define

$$u(x) := \mu(V)^{-1} \left(\mathbb{1}_{KV} * \mathbb{1}_V\right)(x) = \mu(V)^{-1} \cdot \mu\left(xV \cap KV\right)$$

for $x \in G$. We immediately have that $0 \leq u(x) \leq 1$. If $x \in K$ then $\mu(xV \cap KV) = \mu(xV) = \mu(V)$, so $u(x) = 1$ for each $x \in K$. If $x \notin KV^2$ we have $xV \cap KV = \varnothing$ and so $u(x) = 0$. In particular, $\operatorname{supp}(u) \subseteq KV^2$, a compact set, and $u(x) = 0$ when $x \notin U$. Finally, the polarization identity

$$4(f * g^*) = (f+g)*(f+g)^* - (f-g)*(f-g)^* + i(f+ig)*(f+ig)^* - i(f-ig)*(f-ig)^*$$

for $f, g \in L^1(G)$ shows that u is a finite linear combination of functions of the form $h * h^*$ where $h \in L^\infty(G)$ and h has compact support, leading us to the conclusion that $h * h^* \in P(G) \cap C_c(G)$. □

Continuing from the next proposition, we will develop Eymard's Fourier algebra as originally presented. It will be denoted $\mathcal{A}(G)$. Following this, we will show that $\mathcal{A}(G)$ and our $A(G)$ coincide.

Proposition 2.3.13. *[48, Proposition 2.3.3] For $1 \leq j \leq 10$, define a subset M_j of $B(G)$ as follows:*

$M_1 = \{f * \tilde{g} : f, g \in C_c(G)\}$;

$M_2 = \left\{h * \tilde{h} : h \in C_c(G)\right\}$;

$M_3 = \{f * \tilde{g} : f, g \in L^\infty(G) \text{ with compact support}\}$;

$M_4 = \left\{h * \tilde{h} : h \in L^\infty(G) \text{ with compact support}\right\}$;
$M_5 = B(G) \cap C_c(G)$;
$M_6 = P(G) \cap C_c(G)$;
$M_7 = \left\{u \in P(G) : \Delta^{-1/2} u \in L^1(G)\right\}$;
$M_8 = P(G) \cap L^2(G)$;
$M_9 = \left\{h * \tilde{h} : h \in L^2(G)\right\}$;
$M_{10} = \{f * \tilde{g} : f, g \in L^2(G)\}$.
Let E_j denote the linear span of M_j. Then

$$E_1 = E_2 \subseteq E_3 = E_4 \subseteq E_5 = E_6 \subseteq E_7 \subseteq E_8 \subseteq E_9 = E_{10} \subseteq A(G),$$

and all of these subspaces have the same closure with respect to $\|\cdot\|_{B(G)}$ denoted $\mathcal{A}(G)$. Moreover, $\mathcal{A}(G)$ is an ideal in $B(G)$.

Proof. The equalities $E_1 = E_2$, $E_3 = E_4$ and $E_9 = E_{10}$ all follow from the polarization identity, while the containments $E_2 \subseteq E_3$, $E_4 \subseteq E_5$ and $E_6 \subseteq E_7$ are clear.

Next, we show the containment $E_5 \subseteq E_6$. Choose $v \in B(G) \cap C_c(G)$ and $v = v_1 - v_2 + i(v_3 - v_4)$, where $v_k \in P(G)$. By Proposition 2.3.12 there exists a function u of the form $u = \sum_{j=1}^n c_j u_j$ where $c_j \in \mathbb{C}$ and $u_j \in P(G) \cap C_c(G)$ with $u = 1$ on $\text{supp}(v)$. Then

$$v = uv = \sum_{j=1}^{n} c_j u_j [v_1 - v_2 + i(v_3 - v_4)],$$

so $v \in E_6$. The reverse containment is evident from the characterisation of elements of $B(G)$ via functions of positive type. The containment $E_7 \subseteq E_8$ is a direct consequence of [38, Proposition 12], which states that any continuous function of positive type of the form of the elements of E_7 is square integrable. The final containment, $E_8 \subseteq E_9$ follows from the fact that every $u \in P(G) \cap L^2(G)$ can be written in the form $u = f * \tilde{f}$ with $f \in L^2(G)$, shown in [29, Theorem 13.8.6].

Finally, we show that each space has the same closure. It suffices to show that E_1 is dense in E_{10}. Thus, let $f, g \in L^2(G)$ and let $\varepsilon > 0$ be given. Then there are

functions $h, k \in C_c(G)$ with $\|f-h\|_2 < \varepsilon$ and $\|g-k\|_2 < \varepsilon$. By Proposition 2.3.11,

$$\begin{aligned}\left\|f*\tilde{g}-h*\tilde{k}\right\|_{B(G)} &= \left\|(f-h)*\tilde{g}+h*(\tilde{g}-\tilde{k})\right\|_{B(G)} \\ &\leq \|(f-h)*\tilde{g}\|_{B(G)} + \left\|h*(\tilde{g}-\tilde{k})\right\|_{B(G)} \\ &\leq \|f-h\|_2 \cdot \|g\|_2 + \|h\|_2 \|g-k\|_2 \\ &\leq \varepsilon(\|g\|_2 + \|f\|_2 + \varepsilon).\end{aligned}$$

We conclude $M_{10} \subseteq \overline{M_1}^{\|\cdot\|_{B(G)}}$ which shows $E_{10} \subseteq \overline{E_1}^{\|\cdot\|_{B(G)}}$. The fact that $\mathcal{A}$ is an ideal follows from the fact that E_5 is an ideal. □

With this result, we have provided Eymard's original definition of the Fourier algebra and we can rephrase our goal of equivalent definitions to show that $\mathcal{A}(G) = E_{10}$, without the need for closure. The justification of this will be proved in Proposition 2.3.16. Its proof requires a few more definitions, and a dip into Von Neumann theory. Along the way we will also see that $A(G)^* = VN(G)$, in Proposition 2.3.15.

Definition 2.3.14. [31, Definition 3.9] Denote by $VN(G)$ the von Neumann algebra generated by the operators $\lambda(f)$ for $f \in C_c(G)$ (recall the definition of this operator in the proof of Proposition 2.3.6).

An alternative characterization of $VN(G)$ is given in a remark immediately following the definition, when Eymard cites [29, Chapter 1, Section 5]. $VN(G)$ can be realized as the set of bounded linear operators on $L^2(G)$ which commute with the right convolution operators for fixed f. If A is a Banach space, for $v \in A$ and $\varphi \in A^*$ we denote the duality via $\langle v, \varphi \rangle = \varphi(v)$.

Proposition 2.3.15. *[48, Theorem 2.3.9] Let G be a locally compact group. For any $\varphi \in \mathcal{A}(G)^*$ there exists a unique operator $T_\varphi \in VN(G)$ such that*

$$\langle T_\varphi f, g\rangle_{L^2(G)} = \left\langle \varphi, \overline{g} * \check{f} \right\rangle = \left\langle \varphi, (f*\tilde{g})\check{} \right\rangle$$

for all $f, g \in L^2(G)$. The mapping $\varphi \mapsto T_\varphi$ from $\mathcal{A}(G)^* \to VN(G)$ is a surjective linear isometry and has the following additional properties.

(i) If $u = \sum_{j=1}^{\infty}(g_j * \check{f}_j)$, $f_j, g_j \in L^2(G)$, *with* $\sum_{j=1}^{\infty} \|f_j\|_2 \cdot \|g_j\|_2 < \infty$, *then*

$$\langle \varphi, u \rangle = \sum_{j=1}^{\infty} \langle T_\varphi(f_j), g_j \rangle_2 .$$

(ii) $\varphi \to T_\varphi$ *is a homeomorphism for the* w^**-topology on* $\mathcal{A}(G)^*$ *and the ultraweak topology on* $VN(G)$.

Proof. Given any $\varphi \in \mathcal{A}(G)^*$, for any $f, g \in L^2(G)$,

$$\left|\left\langle \varphi, \overline{g} * \check{f} \right\rangle\right| \leq \|\varphi\| \cdot \left\|\overline{g} * \check{f}\right\|_{B(G)} \leq \|\varphi\| \cdot \|f\|_2 \, \|g\|_2 .$$

Fixing f, the map $g \mapsto \left\langle \varphi, \overline{g} * \check{f} \right\rangle$ defines a conjugate linear functional on $L^2(G)$. Hence, there is a unique $f_\varphi \in L^2(G)$ such that $\left\langle \varphi, \overline{g} * \check{f} \right\rangle = \langle f_\varphi, g \rangle_{L^2(G)}$ for every $g \in L^2(G)$. Define $T_\varphi : L^2(G) \to L^2(G)$ by $T_\varphi(f) = f_\varphi$. Then T_φ is linear, and $\|T_\varphi(f)\|_2 \leq \|\varphi\| \cdot \|f\|_2$ for every $f \in L^2(G)$, so $\|T_\varphi\| \leq \|\varphi\|$. For any $f, g \in L^2(G)$ and $h \in L^1(G)$, we have

$$\begin{aligned}
\langle T_\varphi(\lambda(h)f)f), g \rangle_{L^2(G)} &= \langle T_\varphi(f * h), g \rangle_{L^2(G)} = \left\langle \varphi, \overline{g} * \check{(f * h)} \right\rangle \\
&= \left\langle \varphi, \overline{g} * \check{h} * \check{f} \right\rangle = \left\langle \varphi, \overline{g * h^*} * \check{f} \right\rangle \\
&= \langle T_\varphi(f), g * h^* \rangle_{L^2(G)} = \langle T_\varphi(f), \lambda(h)^* g \rangle_{L^2(G)} \\
&= \langle \lambda(h) T_\varphi(f), g \rangle_{L^2(G)} .
\end{aligned}$$

So, T_φ commutes with the right regular representation operators and this demonstrates $T_\varphi \in VN(G)$. If u is as stated in (i), the series $\sum_{j=1}^{\infty}(g_j * \check{f}_j)$ is absolutely convergent and so

$$\langle \varphi, u \rangle = \sum_{j=1}^{\infty} \left\langle \varphi, g_j * \check{f}_j \right\rangle = \sum_{j=1}^{\infty} \left\langle T_\varphi(f_j), \overline{g}_j \right\rangle_{L^2(G)}$$

and $|\langle \varphi, u \rangle| \leq \|T_\varphi\| \sum_{j=1}^{\infty} \|f_j\|_2 \cdot \|g_j\|_2$. Since this holds for all representations of $u \in A(G)$, $|\langle \varphi, u \rangle| \leq \|T_\varphi\| \cdot \|u\|_{B(G)}$. It then follows that $\varphi \to T_\varphi$ is an isometry, and the above equation yields statement (ii). What remains to show is that every $T \in VN(G)$ is of the form T_φ for some $\varphi \in \mathcal{A}(G)^*$. Let $T \in VN(G)$ and note that given $f, g \in C_c(G)$,

$T(u) := T(f*\tilde{g}) = T(f)*\tilde{g}$ is continuous. So, for $u \in E_1$ we can set $\langle \varphi_T, u\rangle = T(\check{u})(e_G)$. This definition does not depend on the representation of u.

Next, we show that φ_T is a bounded linear functional on E_1. To do this, we cite Kaplansky's density theorem to produce a net $\{h_\alpha\}_\alpha$ in $C_c(G)$ such that $\|\lambda(h_\alpha)\| \leq \|T\|$ for all α and $\|Tg - \lambda(h_\alpha)g\| \to 0$ for each $g \in L^2(G)$. For $u = \sum_{j=1}^n (f_j * g_j^*)$, with $f_j, g_j \in C_c(G)$, we have

$$\langle \varphi_T, u\rangle = \lim_\alpha \left\langle \varphi_{\lambda(h_\alpha)} f_j, g_j \right\rangle = \lim_\alpha \left\langle \varphi_{\lambda(h_\alpha)}, u \right\rangle$$

and so

$$|\langle \varphi_T, u\rangle| = \lim_\alpha \left|\left\langle \varphi_{\lambda(h_\alpha)}, u \right\rangle\right| \leq \|u\|_{B(G)} \sup_\alpha \|\lambda(h_\alpha)\| \leq \|u\|_{B(G)} \cdot \|T\|.$$

Given that E_1 is dense in $\mathcal{A}(G)$, φ_T extends uniquely to a bounded linear funcitonal on $\mathcal{A}(G)$, also denoted φ_T, of norm at most $\|T\|$. By the definition of φ_T,

$$\begin{aligned}
\langle T_{\varphi_T}(f), g\rangle_{L^2(G)} &= \langle f_{\varphi_T}, g\rangle_{L^2(G)} = \left\langle \varphi_T, \overline{g} * \check{f} \right\rangle \\
&= \left\langle \varphi_T, \left(f * \tilde{g}\right)\check{}\right\rangle = T(f * \tilde{g})(e_G) \\
&= (T(f) * \tilde{g})(e_G) = \langle T(f), g\rangle_{L^2(G)}
\end{aligned}$$

and hence $T_{\varphi_T} = T$. □

With this (quite lengthy) proof, we have shown that $\mathcal{A}(G)^* = VN(G)$ and classified the ultraweakly continuous linear functionals on $VN(G)$. This identification is used in the next proposition to show the equivalence $\mathcal{A}(G) = A(G)$.

Proposition 2.3.16. *[48, Theorem 2.4.3] Let G be a locally compact group. Then $\mathcal{A}(G)$ is precisely the set of all functions $f * \tilde{g}$ where $f, g \in L^2(G)$.*

The remainder of this section will be spent showcasing a few miscellaneous results about both $B(G)$ and $A(G)$ which have motivated our study and further demonstrate the importance of the algebras.

Proposition 2.3.17. *[31, Example 3.6] If G is a compact group, then $B(G) = A(G)$.*

Proof. This is an immediate corollary of the Peter Weyl theorem (which we will discuss in Chapter 5.1) as the left regular representation decomposes in terms of the group's irreducible representations. □

Proposition 2.3.18. *[48, Corollary 2.6.5] Let G be a locally compact group, and H a closed subgroup of G. Let $I(H) = \{u \in A(G) : u|_H = 0\}$. Then the restriction map $u \mapsto u|_H$ from $A(G) \to A(H)$ induces an isometric isomorphism $u + I(H) \mapsto u|_H$ from the quotient algebra $A(G)/I(H)$ onto $A(H)$.*

This last proof is omitted, but is a nice consequence of properties of how $A(H)$ interacts with $A(G)$.

2.3.4 Fourier Coefficient Spaces

As previously shown, the Fourier algebra $A(G)$ is the vector space formed by the coefficient functions of the left regular representation. We have shown that there is no need to consider the closure nor the span of these functions. This is not true for a general representation π, though these spaces are still important. Their study was initiated by Arsac in [8], though the results have been translated and presented in [48].

Definition 2.3.19. For a representation π of a locally compact group G, define

$$A_\pi(G) := \overline{\text{span}}\,\{\pi_{\xi,\eta} : \xi, \eta \in \mathcal{H}_\pi\}$$

the closure taken with respect to the $B(G)$ norm.

Then A_π is a Banach space for each π. Defining $VN_\pi(G)$ to be $\{\pi(x) : x \in G\}''$, one can show that A_π^* is isometrically isomorphic to VN_π (for a proof, see [48, Lemma 2.8.2]. The space A_π has an incredibly useful characterization:

Proposition 2.3.20. *[48, Theorem 2.8.4] Let π be a representation of a locally compact group G. Then A_π is exactly the collection of functions which can be expressed in the form*

$$u = \sum_{n=1}^{\infty} \pi_{\xi_n,\eta_n}$$

for $(\xi_n), (\eta_n) \in \mathcal{H}_\pi$, $\sum_{n=1}^{\infty} \|\xi_n\| \, \|\eta_n\| < \infty$. *Moreover, given* $u \in A_\pi$, *there are sequences* $(\xi_n), (\eta_n) \in \mathcal{H}_\pi$ *such that* $u = \sum_{n=1}^{\infty} \pi_{\xi_n, \eta_n}$ *and* $\|u\|_{A(G)} = \sum_{n=1}^{\infty} \|\xi_n\| \, \|\eta_n\|$.

From here we pivot to discussing properties of the assignment $\pi \mapsto A_\pi$. For representations π, σ of a locally compact group G, we say that π and σ are *disjoint* if neither is equivalent to a sub-representation of the other. For unitary representations π and σ, $A_\pi \cap A_\sigma = \{0\}$ if and only if π, σ are disjoint. The spaces A_π respect the direct sum of representations in several nice ways, seen in the following propositions.

Proposition 2.3.21. *[48, Proposition 2.8.8] Let* $\{\pi_i\}_{i \in I}$ *be a family of unitary representations of* G, *and* π *their direct sum. Then* A_π *is exactly the functions of* $B(G)$ *which can be written as* $u = \sum_{i \in I} u_i$ *where* $u_i \in A_{\pi_i}$ *and* $\sum_{i \in I} \|u_i\| < \infty$.

Proposition 2.3.22. *[48, Proposition 2.8.11] Let* $(\pi_i)_{i \in I}$ *be a family of pairwise disjoint unitary representations of* G, *and* π *their direct sum. Then there exists a unique decomposition* $u = \sum_{i \in I} u_i$ *with* $u_i \in A_{\pi_i}$ *and* $\|u\| = \sum_{i \in I} \|u_i\|$.

This can also be strengthened in the case when G is compact to pairwise non-unitarily equivalent representations. In Chapter 5.2 we provide a definition of Fourier spaces for groupoids.

2.3.5 Functorial Properties of $A(G)$

Globally in analysis, restriction style results can be an incredibly powerful tool. By this, we (broadly) mean results concerning the restriction of the domain for the class of functions being studied. These, and the inverse extension problems allow us to move between possibly problematic domains to ones in which we have more control. The first major result of this kind in abstract harmonic analysis was due to Herz [40] where he shows that for any closed subgroup H of a locally compact group G, $A(G)|_H = A(H)$, though this was known for abelian groups prior to Herz. This is the most general version of this result one can hope for, as for a Hausdorff group G, a subgroup H needs to be open or closed to guarantee local compactness ([65, Corollary 29.3]), and in a topological group open subgroups are also closed ([32, Proposition 2.1]). Conversely,

every function in $A(H)$ can be extended to a function in $A(G)$ with the same norm ([48, Theorem 2.6.4]).

Given these results, it is natural to ask whether there are similar results for $B(G)$. In full generality, the answer is no, as [26, pg. 92] provides an example where $B(G)|_H \subsetneq B(H)$. As a consolation, the range of this restriction map is closed ([16, Proposition 2.3]). Remark 2.4 of the same source shows that this proof method can be used to extend Herz' result to the coefficient spaces A_π, so $A_\pi(G)|_H = A_{\pi|_H}(G)$. This lets us reimagine Herz' result to show

$$A(G)|_H = A_{\lambda_G}(G)|_H = A_{\lambda_G|_H}(G) = A(H).$$

Ghandehari has studied the restriction map applied to those functions in $B(G)$ which vanish at infinity, and shows the closedness of $B_0(G) = B(G) \cap C_0(G)$ restricted to a closed subgroup H in [37, Proposition 4.1].

The question as to when $B(G)|_H = B(H)$ for locally compact G and closed subgroup H has been answered in the affirmative in the following cases:

1. G is compact, discrete, or locally compact abelian ([48, Section 7.3]).
2. H is the connected component containing the identity ([48, Theorem 7.2.4]).
3. H is closed and either compact or contained in the center of G ([48, Corollary 7.1.6]).
4. G is nilpotent and H is an infinite cyclic subgroup ([47, Theorem 3.2]).
5. Every subgroup of SIN-groups ([26]).

2.4 Fourier and Fourier-Stieltjes Algebras for Groupoids

First recall the locally compact group case, where representations are defined as a linear action on a Hilbert space associated to each group element (homomorphisms $G \to \mathcal{U}(\mathcal{H})$ for some Hilbert space. These representations give rise to (coefficient) functions. The collection of every coefficient function for every representation forms

the Fourier-Stieltjes algebra, and the collection of coefficient functions corresponding to the left regular representation forms the Fourier algebra.

In the locally compact groupoid case, defining a suitable replacement for each component of the algebras becomes a bit trickier. We present the initial definitions and development as in [70], which further cites inspiration from [76] and language from [29], specifically Definition 10.1.2. Roughly speaking, while a group requires unitary representations over Hilbert spaces, groupoids will require $\mathcal{G}$-Hilbert bundles. To that end, let us first present the definition of a continuous Hilbert field over a topological space. If T is a topological space, for each $t \in T$ let E_t be a complex Banach space. We may interpret each element of $\Pi_{t\in T}E_t$ as a function $x : T \to \sqcup E_t$ such that $x(t) \in E_t$ for every $t \in T$. Such functions are referred to as *vector fields* in [29].

Definition 2.4.1. [29, Definition 10.1.2] A *continuous field of Banach spaces* $\mathscr{E}$ over a topological space T is a family $(E_t)_{t\in T}$ of Banach spaces, with a set $\Gamma \subseteq \Pi_{t\in T}E_t$ such that:

1. Γ is a complex linear subspace of $\Pi_{t\in T}E_t$;
2. For every $t \in T$, the set $\{x(t) : x \in \Gamma\}$ is dense in E_t;
3. For every $x \in \Gamma$, the function $t \to \|x(t)\|_{E_t}$ is continuous;
4. Let $x \in \Pi_{t\in T}E_t$ be a vector field; if for every $t \in T$ and every $\varepsilon > 0$, there exists $x' \in \Gamma$ such that $\|x(\tau) - x'(\tau)\| < \varepsilon$ for all τ in some neighborhood of t, then $x \in \Gamma$.

In the case where each Banach space is also a Hilbert space, the definition above is referred to as a *continuous Hilbert field*, though this thesis may omit the continuous adjective as we assume each of our Hilbert fields are continuous. Paterson uses slightly differing terminology than above when referring to the elements of Γ; while Dixmier refers to these elements as *continuous vector fields* Paterson refers to them as *continuous sections*. We shall use the latter terminology. Item 4 of Definition 2.4.1 states that

any vector field which is locally close to an element of Γ must be a continuous vector field itself. This means that a section x is continuous if for every $t_0 \in T$ and for every $\varepsilon > 0$, there is a neighborhood V of t_0 and a continuous section x' such that $\tau \in V$ implies that $\|x(\tau) - x'(\tau)\| < \varepsilon$.

Example 2.4.2. If T is discrete, then $\Gamma = \Pi_{t\in T} E_t$, and the continuous field coincides with the direct product of the Banach spaces.

Example 2.4.3. [70, pg 7] Suppose $E_t = E$ for every $t \in T$. As elements of $\prod_{t\in T} E$ may be identified with functions from $T \to E$, $C(T, E)$ is clearly a collection of vector fields. In this case, the set of continuous sections $\Gamma = C(T, E)$ and this continuous Hilbert field is called the *constant field*.

Given a continuous Hilbert field, we may form a groupoid from isomorphisms between its fibers.

Example 2.4.4. Let $\mathfrak{H} := (\{\mathfrak{H}_x\}_{x\in X}, \Gamma)$ be a continuous Hilbert field over a topological space X with continuous sections Γ. Define $\mathcal{G}$ to be the collection of all isomorphisms $S : \mathfrak{H}_x \to \mathfrak{H}_y$ where $x, y \in X$. The partial product is defined to be function composition, and inversion the inversion of functions. The unit space $\mathcal{G}^{(0)}$ is the collection of identity maps $\{I_{\mathfrak{H}_x}\}_{x\in X}$. For an isomorphism $S : \mathfrak{H}_x \to \mathfrak{H}_y$, $r(S) = I_{\mathfrak{H}_y}$ and $s(S) = I_{\mathfrak{H}_x}$.

For the duration of this thesis, if $(\{\mathfrak{H}_x\}_{x\in X}, \Gamma)$ is a continuous Hilbert field then Γ may be assumed to be the set of continuous sections. Often it is difficult to exactly describe the collection of continuous sections Γ. Dixmier provides an alternative characterization in the following proposition.

Proposition 2.4.5. *[29, Proposition 10.2.3] Let $\mathfrak{H} := (\{\mathfrak{H}_t\}_{t\in T}, \Gamma)$ be a continuous field of Banach spaces over T. Suppose that there exists a set $\Xi \subseteq \prod_{t\in T} \mathfrak{H}_t$ satisfying the first three axioms of Definition 2.4.1. Then there is a unique subset $\Gamma \subseteq \prod_{t\in T} \mathfrak{H}_t$ which satisfies all four axioms and contains Ξ.*

The sections of Ξ will be referred to as the *fundamental family of sections*, or the *fundamental sections*.

Example 2.4.6. Let $\mathcal{G}$ be a locally compact groupoid with Haar system $\{\lambda^u\}_{u\in\mathcal{G}^{(0)}}$. Define the continuous Hilbert field $L^2(\mathcal{G}) := \{L^2(\mathcal{G}^u, \lambda^u)\}_{u\in\mathcal{G}^{(0)}}$. The functions $f \in C_c(\mathcal{G})$ can be considered continuous sections, in the sense that $u \mapsto f^u = f|_{\mathcal{G}^u}$. We will justify why $C_c(\mathcal{G})$, interpreted as continuous sections, forms a fundamental family of sections.

Lemma 2.4.7. *For a locally compact groupoid $\mathcal{G}$ with Haar system $\{\lambda^u\}_{u\in\mathcal{G}^{(0)}}$, $C_c(\mathcal{G})$ is a fundamental family of sections as defined in Proposition 2.4.5 thus uniquely determines a set of continuous sections of $L^2(\mathcal{G})$.*

Proof. $C_c(\mathcal{G})$ is clearly a complex linear space. If we fix $u \in \mathcal{G}^{(0)}$, we claim that the set $\{f^u : f \in C_c(\mathcal{G})\} \subseteq L^2(\mathcal{G}^u, \lambda^u)$ is dense in $L^2(\mathcal{G}^u, \lambda^u)$. By Remark A.1.8, any $f \in C_c(\mathcal{G}^u)$ may be extended to an $\tilde{f} \in C_c(\mathcal{G})$, so the collection of $\{f^u : f \in C_c(\mathcal{G})\}$ contains $C_c(\mathcal{G}^u)$, demonstrating the claim. By (2) of Definition 2.1.12, for every $g \in C_c(\mathcal{G})$, the function $g^0(u) = \int_{\mathcal{G}^u} g(x)d\lambda^u(x)$ belongs to $C_c(\mathcal{G}^{(0)})$. For $f \in C_c(\mathcal{G})$, setting $g := |f|^2$ gives the continuity of the map $u \mapsto \sqrt{\|f|_{\mathcal{G}^u}\|_{L^2(\mathcal{G}^u)}}$ establishing the claim. □

Remark 2.4.8. [70, pg. 10] provides an alternative method of determining the continuous sections of the left regular field. A function $F : \mathcal{G} \to \mathbb{C}$ is a continuous section of the left regular field if and only if $u \mapsto \|F^u\|$ is continuous and $u \mapsto \langle F^u, g^u\rangle$ is continuous for all $g \in C_c(\mathcal{G})$.

Example 2.4.9. Let $\mathfrak{H} := (\{\mathfrak{H}_x\}_{x\in X}, \Gamma)$ and $\mathfrak{K} := (\{\mathfrak{K}_x\}_{x\in X}, \Gamma')$ be two continuous Hilbert bundles over a topological space X. Define the *direct sum of Hilbert fields* $\mathfrak{H} \oplus \mathfrak{K}$ to be the continuous Hilbert field with fibers $\{\mathfrak{H}_x \oplus \mathfrak{K}_x\}_{x\in X}$ and fundamental family of sections $\Gamma \oplus \Gamma'$ where an element $(f \oplus g)(x) = f(x) \oplus g(x)$. Similarly, define the *tensor product of Hilbert fields* $\mathfrak{H} \otimes \mathfrak{K}$ to be the continuous Hilbert field with fibers $\{\mathfrak{H}_x \otimes \mathfrak{K}_x\}_{x\in X}$ and fundamental family of sections $\Gamma \otimes \Gamma'$ where an element $(f \otimes g)(x) = f(x) \otimes g(x)$.

Proposition 2.4.10. *[29, Proposition 10.1.9 (ii)] Let $\mathfrak{H} := (\{\mathfrak{H}_t\}_{t\in T}, \Gamma)$ be a continuous field of Banach spaces over T. If $y \in \Gamma$ and $g : T \to \mathbb{C}$ is continuous on T then $gy \in \Gamma$.*

Proposition 2.4.11. *[29, Proposition 10.1.10] Let $\mathfrak{H} := (\{\mathfrak{H}_t\}_{t\in T}, \Gamma)$ be a continuous field of Banach spaces over T. For every $t_0 \in T$ and every $\xi \in \mathfrak{H}_{t_0}$ there exists an $x \in \Gamma$ such that $x(t_0) = \xi$.*

Proof. By axiom 2 of Definition 2.4.1, given $\xi \in \mathfrak{H}_{t_0}$ there is a sequence $\{\xi_n\}_{n\in\mathbb{N}} \subseteq \mathfrak{H}_{t_0}$ such that $\sum_{n\in\mathbb{N}} \xi_n = \xi$ and for every n there exists an $x_n \in \Gamma$ with $x_n(t_0) = \xi_n$. For each $n \in \mathbb{N}$ define the function f_n on T which is 1 when $\|x_n(t)\| \leq \|\xi_n\|$ and $\|\xi_n\| / \|x_n(t)\|$ otherwise. Then each f_n is continuous and so by Proposition 2.4.10, $y_n = f_n x_n \in \Gamma$. This implies that $y_n(t_0) = \xi_n$ and $\|y_n(t)\|_{\mathfrak{H}_t} \leq \|\xi_n\|$ for every $t \in T$, so $\sum_{n\in\mathbb{N}} y_n(t)$ converges to an element $x(t) \in \mathfrak{H}_t$ for every t. We then have $x(t_0) = \sum_{n\in\mathbb{N}} \xi_n = \xi$, and for each $\varepsilon > 0$ there is an $m \in \mathbb{N}$ with $|x(t) - \sum_{n=1}^{m} y_n(t)| < \varepsilon$ on T, so $x \in \Gamma$. □

We can also define a notion of equivalence for continuous Banach fields.

Definition 2.4.12. [14, Definition 2.8] Let $\mathfrak{H} := (\{\mathfrak{H}_t\}_{t\in T}, \Gamma)$ and $\mathfrak{G} := (\{\mathfrak{G}_t\}_{t\in T}, \Gamma')$ be two continuous fields of Banach spaces over T. A collection of bounded linear operators $\{T_t : \mathfrak{H}_t \to \mathfrak{G}_t\}_{t\in T}$ will be called a *morphism* of $\mathfrak{H}$ and $\mathfrak{G}$ if the induced map $T : \mathfrak{H} \to \mathfrak{G}$ satisfies $\{T \circ \xi : \xi \in \Gamma\} \subseteq \Gamma'$ where $(T \circ \xi)(t) = T_t(\xi(t))$.

Moving forward, let $\mathfrak{H} := (\{\mathfrak{H}_t\}_{t\in T}, \Gamma)$ be a continuous Hilbert field over the space T. Denote the norm of each Hilbert space fiber $\mathfrak{H}_u$ as $\|\cdot\|_u$ or $\|\cdot\|_{\mathfrak{H}_u}$ for $u \in T$. The subindices will be omitted when the space the argument belongs to is clear. We define the *section norm*

$$\|\xi\|_\Delta = \sup_{u\in T} \|\xi(u)\|_u .$$

When it is clear the argument is a continuous section we will drop the Δ from the notation. We further define $\Delta_b(\mathfrak{H})$ and $\Delta_0(\mathfrak{H})$ to be the subspaces of Γ of bounded sections, and those which vanish at infinity respectively under the section norm. Define $\Delta_c(\mathfrak{H})$ to be the subset of $\Delta_0(\mathfrak{H})$ of continuous sections with compact support. [29, Proposition 10.1.9] shows that all three of Δ_b, Δ_c, and Δ_0 are $C_0(T)$-modules under pointwise multiplication as the module action.

Lemma 2.4.13. *[70, pg 6] The set $\Delta_b(\mathfrak{H})$ is a Banach space under norm with pointwise operations.*

Proof. That $\Delta_b(\mathfrak{H})$ forms a vector space is due to the inequalities

$$\|\xi+\eta\| = \sup_{u\in T}\|\xi(u)+\eta(u)\|_u \leq \sup_{u\in T}\|\xi(u)\|_u + \sup_{u\in T}\|\eta(u)\|_u < \infty.$$

Suppose that $\{\xi_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\Delta_b(\mathfrak{H})$. Then, for a given $\varepsilon > 0$, there is an N such that $n, m \geq N$ implies that $\|\xi_n - \xi_m\| < \varepsilon$. By definition of the section norm, $\|\xi_n(t) - \xi_m(t)\|_t < \varepsilon$ for every $t \in T$. As the sequence $(\xi_n(t))_{n\in\mathbb{N}}$ belongs to $\mathfrak{H}_t$, it converges given that Hilbert spaces are complete. Define a vector field $\xi(t) := \lim_{n\to\infty}\xi_n(t)$. Then, given the inequality $\|\xi_n(t) - \xi_m(t)\| < \varepsilon$ for sufficiently large n, m and a fixed t, letting $m \to \infty$ and then taking the supremum over t establishes that $\|\xi_n - \xi\| < \varepsilon$. So $\{\xi_n\}_{n\in\mathbb{N}}$ converges to ξ in section norm. By axiom 4 of Definition 2.4.1 this also implies that ξ is a continuous section. To see that ξ is also bounded, let $\varepsilon = 1$. Then we may find an N such that $\|\xi_n - \xi\| < 1$ for $n \geq N$. Then

$$\|\xi\| \leq \|\xi - \xi_N\| + \|\xi_N\| < 1 + \|\xi_N\| < \infty$$

so $\xi \in \Delta_b(\mathfrak{H})$, and $\Delta_b(\mathfrak{H})$ is complete. □

These continuous Hilbert fields and their continuous sections will be our replacement for vectors and a single Hilbert space. With this extension, we may now define the analogue of representations. Let $\mathcal{G}$ be a locally compact groupoid, and $\mathfrak{H}$ a continuous Hilbert bundle over $\mathcal{G}^{(0)}$.

Definition 2.4.14. [70, Section 3] Let $\mathcal{G}$ be a topological groupoid. A continuous Hilbert field $\mathfrak{H} := (\{\mathfrak{H}_u\}_{u\in\mathcal{G}^{(0)}}, \Gamma)$ is called a *$\mathcal{G}$-Hilbert bundle* over $\mathcal{G}^{(0)}$ if for each $x \in \mathcal{G}$, there is a given unitary map $\pi(x) : \mathfrak{H}_{s(x)} \to \mathfrak{H}_{r(x)}$ such that

1. For each $\boldsymbol{\xi}, \boldsymbol{\eta} \in \Delta_b(\mathfrak{H})$ the map $\mathcal{G} \to \mathbb{C}$ defined as $x \mapsto \langle \pi(x)\boldsymbol{\xi}(s(x)), \boldsymbol{\eta}(r(x))\rangle_{\mathfrak{H}_{r(x)}}$ is continuous;

2. The map $x \mapsto \pi(x)$ is a groupoid homomorphism from $\mathcal{G}$ into the isomorphism groupoid of $\mathfrak{H}$ (as defined in 2.4.4), namely, for $(x, y) \in \mathcal{G}^{(2)}$ $\pi(xy) = \pi(x)\pi(y)$.

Remark 2.4.15. It suffices to check Definition 2.4.14 (1) for $\boldsymbol{\xi}, \boldsymbol{\eta} \in \Xi$, the fundamental sections.

Definition 2.4.16. Let $\mathfrak{H}$ and $\mathfrak{H}'$ be two $\mathcal{G}$-Hilbert bundles with linear actions $\{\mathcal{T}(x)\}_{x\in\mathcal{G}}$ and $\{\varrho(x)\}_{x\in\mathcal{G}}$ respectively. A morphism of $\mathfrak{H}, \mathfrak{H}'$ given by $\{T_u : \mathfrak{H}_u \to \mathfrak{H}'_u\}_{u\in\mathcal{G}^{(0)}}$ is called *intertwining* if $\varrho(x)T_{s(x)} = T_{r(x)}\mathcal{T}(x)$ for every $x \in \mathcal{G}$. Two $\mathcal{G}$-Hilbert bundles are *unitarily equivalent*, denoted $\mathfrak{H} \simeq \mathfrak{H}'$ if there is a collection of unitary intertwining operators [8].

Visually, two $\mathcal{G}$-Hilbert bundles are unitarily equivalent the following diagram commutes for every x.

$$\begin{array}{ccc} \mathfrak{H}_{s(x)} & \xrightarrow{\mathcal{T}(x)} & \mathfrak{H}_{r(x)} \\ \Big\downarrow{\scriptstyle \mathcal{U}_{s(x)}} & & \Big\downarrow{\scriptstyle \mathcal{U}_{r(x)}} \\ \mathfrak{H}'_{s(x)} & \xrightarrow[\varrho(x)]{} & \mathfrak{H}'_{r(x)} \end{array}$$

For brevity, when the set of continuous sections of a Hilbert field is understood, we will suppress Γ in the notation and simply say "$\mathfrak{H}$ is a $\mathcal{G}$-Hilbert bundle". As in the case of representations, the direct sums and tensor products of $\mathcal{G}$-Hilbert bundles are themselves $\mathcal{G}$-Hilbert bundles via direct and tensor products of the underlying fibers and maps ([70, pg 7]). Let $\mathfrak{H}$ and $\mathfrak{K}$ be two $\mathcal{G}$-Hilbert bundles with linear actions $\{\mathcal{T}(x)\}$ and $\{\varrho(x)\}$ respectively. Let $\mathfrak{H} \oplus \mathfrak{K}$ be the $\mathcal{G}$-Hilbert bundle with Hilbert field $\{\mathfrak{H}_u \oplus \mathfrak{K}_u\}_{u\in\mathcal{G}^{(0)}}$ (defined in Example 2.4.9) and action $(\mathcal{T} \oplus \varrho)(x) : \mathfrak{H}_{s(x)} \oplus \mathfrak{K}_{s(x)} \to \mathfrak{H}_{r(x)} \oplus \mathfrak{K}_{r(x)}$ given by $(\mathcal{T}(x) \oplus \varrho(x))(u \oplus v) = \mathcal{T}(x)u \oplus \varrho(x)v$. Similarly, let $\mathfrak{H} \otimes \mathfrak{K}$ be the $\mathcal{G}$-Hilbert bundle with Hilbert field $\{\mathfrak{H}_u \otimes \mathfrak{K}_u\}_{u\in\mathcal{G}^{(0)}}$ and action $(\mathcal{T} \otimes \varrho)(x) : \mathfrak{H}_{s(x)} \otimes \mathfrak{K}_{s(x)} \to \mathfrak{H}_{r(x)} \otimes \mathfrak{K}_{r(x)}$ given by $(\mathcal{T}(x) \otimes \varrho(x))(u \otimes v) = \mathcal{T}(x)u \otimes \varrho(x)v$.

Definition 2.4.17. The map $x \mapsto \langle \mathcal{T}(x)\boldsymbol{\xi}(s(x)), \boldsymbol{\eta}(r(x))\rangle_{\mathfrak{H}_{r(x)}}$ will be referred to as the *coefficient of* $\mathfrak{H}$ *associated with* $\boldsymbol{\xi}$ *and* $\boldsymbol{\eta}$ and written $\mathfrak{H}_{\boldsymbol{\xi},\boldsymbol{\eta}}$.

[8] We have defined unitary equivalence when the groupoids are the same. This definition will be true generally so long as the unitary operators also carry the continuous sections of one $\mathcal{G}$-Hilbert bundle to the other.

To differentiate between coefficient functions of group representations, and coefficients of $\mathcal{G}$-Hilbert bundles, we will reserve lower case greek letters and unbolded symbols for representations, and accented text with bold symbols to refer to $\mathcal{G}$-Hilbert bundles and their sections.

Example 2.4.18. (The Trivial Bundle) Take, for every $u \in \mathcal{G}^{(0)}$, $\mathfrak{H}_u = \mathcal{H}$ for some fixed Hilbert space. Then $\Gamma = C(\mathcal{G}^{(0)}, \mathcal{H})$ and defining $\mathfrak{T}(x) = I_{\mathcal{H}}$ turns $\mathfrak{H}$ into a $\mathcal{G}$-Hilbert bundle. For $\boldsymbol{\xi}, \boldsymbol{\eta} \in \Delta_b(\mathfrak{H})$ the coefficient $\mathfrak{H}_{\boldsymbol{\xi},\boldsymbol{\eta}}(x) = \langle \boldsymbol{\xi}(s(x)), \boldsymbol{\eta}(r(x)) \rangle_{\mathcal{H}}$.

Example 2.4.19. (The Left Regular Bundle) Let $\mathcal{G}$ be a locally compact groupoid with Haar system $\{\lambda^u\}_{u \in \mathcal{G}^{(0)}}$. The left regular bundle is formed from the continuous Hilbert field $L^2(\mathcal{G}) := \{L^2(\mathcal{G}^u, \lambda^u)\}_{u \in \mathcal{G}^{(0)}}$ as presented in Example 2.4.6. With the continuous Hilbert field $L^2(\mathcal{G})$ in hand, define $L_x : L^2(\mathcal{G}^{s(x)}) \to L^2(\mathcal{G}^{r(x)})$ by $(L_x f)(z) = f(x^{-1}z)$. This will turn $L^2(\mathcal{G})$ into a $\mathcal{G}$-Hilbert bundle, denoted Λ. To justify the codomain, suppose $f \in L^2(\mathcal{G}^{s(x)})$. Then $(L_x f)(z) \neq 0$ if and only if $x^{-1}z \in \operatorname{supp}(f)$, or $z \in x \operatorname{supp}(f) \subseteq \mathcal{G}^{r(x)}$. Moreover, by left invariance of the Haar system,

$$\|L_x f\|_{L^2(\mathcal{G}^{r(x)})} = \int_{\mathcal{G}^{r(x)}} \left| f(x^{-1}z) \right| d\lambda^{r(x)} = \int_{\mathcal{G}^{s(x)}} \left| f(xx^{-1}z) \right| d\lambda^{s(x)} = \|f\|_{L^2(\mathcal{G}^{s(x)})}$$

showing both $L_x f$ is an $L^2(\mathcal{G}^{r(x)})$ function and L_x is an isometry. If we fix $g \in L^2(\mathcal{G}^{r(x)})$ and define $f(z) = g(xz)$, we have $L_x f = g$.

Finally, note that for $f, g \in C_c(\mathcal{G})$

$$\Lambda_{f,g}(x) = \int_{\mathcal{G}^{r(x)}} f^{s(x)}(x^{-1}z)\overline{g^{r(x)}(z)} d\lambda^{r(x)}(z).$$

By Equation (2.1) $\Lambda_{f,g}$ can be seen as the convolution between $\overline{f}$ and $\check{g}$, where $\check{g}(z) = g(z^{-1})$. [70, Proposition 9] shows that left convolution defines a continuous function on $\mathcal{G}$, thus our coefficients are continuous.

Later, we will intimately work with the elements of $C_c(\mathcal{G})$ as both sections and coefficients of Λ. To help differentiate which norm we consider, we use $\|f\|_{\Delta}$ to denote the section norm. With the notion of $\mathcal{G}$-Hilbert bundles in hand, we are prepared to

extend the definition of functions of positive type and the Fourier-Stieltjes algebra to groupoids.

Page 6 of [70] defines a continuous function φ over a groupoid to be *positive definite*, or *of positive type* if

$$\int_{\mathcal{G}^u}\int_{\mathcal{G}^u} \varphi(y^{-1}x)f(x)\overline{f(y)}d\lambda^u(x)d\lambda^u(y) \geq 0$$

for every $f \in C_c(\mathcal{G})$ and $u \in \mathcal{G}^{(0)}$. The set of all functions of positive type is denoted $P(\mathcal{G})$. We then have an analog to Proposition 2.3.6.

Proposition 2.4.20. *[70, Theorem 1] Let $\mathcal{G}$ be a locally compact groupoid and $\varphi \in C(\mathcal{G})$. Then φ is of positive type if and only if $\varphi = \mathfrak{H}_{\boldsymbol{\xi},\boldsymbol{\xi}}$ for some $\mathcal{G}$-Hilbert bundle $\mathfrak{H}$ and $\boldsymbol{\xi} \in \Delta_b(\mathfrak{H})$.*

Definition 2.4.21 (The Fourier-Stieltjes Algebra of $\mathcal{G}$)**.** The *Fourier-Stieltjes algbera* of a locally compact groupoid $\mathcal{G}$ is defined as

$$B(\mathcal{G}) := \{\mathfrak{H}_{\boldsymbol{\xi},\boldsymbol{\eta}} : \mathfrak{H} \text{ is a } \mathcal{G}\text{-Hilbert Bundle, } \boldsymbol{\xi},\boldsymbol{\eta} \in \Delta_b(\mathfrak{H})\}$$

under pointwise addition, scalar multiplication, and pointwise multiplication. Equip $B(\mathcal{G})$ with the norm

$$\|\varphi\|_{B(\mathcal{G})} := \inf_{\varphi=\mathfrak{H}_{\boldsymbol{\xi},\boldsymbol{\eta}}} \|\boldsymbol{\xi}\| \cdot \|\boldsymbol{\eta}\|$$

where the infimum is taken over all coefficient representations of $\varphi = \mathfrak{H}_{\boldsymbol{\xi},\boldsymbol{\eta}}$.

Further equip $B(\mathcal{G})$ with involution $\mathfrak{H}^*_{\boldsymbol{\xi},\boldsymbol{\eta}} = \mathfrak{H}_{\boldsymbol{\eta},\boldsymbol{\xi}}$. That $B(\mathcal{G})$ is both a vector space and an algebra is due to the direct sum and tensor products of $\mathcal{G}$-Hilbert bundles, as $\mathfrak{H}_{\boldsymbol{\xi},\boldsymbol{\eta}} + \mathfrak{H}'_{\boldsymbol{\xi}',\boldsymbol{\eta}'} = (\mathfrak{H} \oplus \mathfrak{H}')_{\boldsymbol{\xi}\oplus\boldsymbol{\xi}',\boldsymbol{\eta}\oplus\boldsymbol{\eta}'}$ and $\mathfrak{H}_{\boldsymbol{\xi},\boldsymbol{\eta}}\mathfrak{H}'_{\boldsymbol{\xi}',\boldsymbol{\eta}'} = (\mathfrak{H} \otimes \mathfrak{H}')_{\boldsymbol{\xi}\otimes\boldsymbol{\xi}',\boldsymbol{\eta}\otimes\boldsymbol{\eta}'}$. To justify the required norm inequality, we use the tensor product expression above to see

$$\left\|\mathfrak{H}_{\boldsymbol{\xi},\boldsymbol{\eta}}\mathfrak{H}'_{\boldsymbol{\xi}',\boldsymbol{\eta}'}\right\|_{B(\mathcal{G})} \leq \|\boldsymbol{\xi}\otimes\boldsymbol{\xi}'\|\,\|\boldsymbol{\eta}\otimes\boldsymbol{\eta}'\| \leq \|\boldsymbol{\xi}\|\,\|\boldsymbol{\xi}'\|\,\|\boldsymbol{\eta}\|\,\|\boldsymbol{\eta}'\| = \|\boldsymbol{\xi}\|\,\|\boldsymbol{\eta}\|\,\|\boldsymbol{\xi}'\|\,\|\boldsymbol{\eta}'\|\,.$$

Taking the necessary infimum provides the result.

The last thing to justify is that $B(\mathcal{G})$ as proposed is complete with respect to $\|\cdot\|_{B(\mathcal{G})}$. To show this, we relate functions in $B(\mathcal{G})$ to functions of positive type on the

groupoid $\mathcal{G} \times I_2$ where I_2 is the groupoid with base set $\{1,2\} \times \{1,2\}$, partial product $(i,j)(j,k) = (i,k)$ and inversion $(i,j)^{-1} = (j,i)$.

A function $f : \mathcal{G} \times I_2 \to \mathbb{C}$ can naturally be identified with a matrix valued function

$$\begin{bmatrix} f((x,1,1)) & f((x,1,2)) \\ f((x,2,1)) & f((x,2,2)) \end{bmatrix}$$

where each of the four entries are functions on $\mathcal{G}$. This identification is what allows us to state the following proposition, which is used to show that $\|\cdot\|_{B(\mathcal{G})}$ is complete.

Proposition 2.4.22. *[70, Proposition 5] A function $\varphi \in B(\mathcal{G})$ if and only if there is a function $f \in P(\mathcal{G} \times I_2)$ which can be identified with the matrix valued function*

$$\begin{bmatrix} \rho(x) & \varphi(x) \\ \varphi^*(x) & \tau(x) \end{bmatrix},$$

(i.e., $f((x,1,1)) = \rho(x)$, $f((x,1,2)) = \varphi(x)$, etc.) where $\tau, \rho \in P(\mathcal{G})$.

This proposition leads to a corollary as in [76, Corollary 1.1] where $\|\varphi\|_{B(\mathcal{G})}$ is equal to $\inf \|f\|_{C(\mathcal{G})}$ the infimum being taken over all such $f \in P(\mathcal{G} \times I_2)$ such that $\varphi(x) = f((x,1,2))$.

Proposition 2.4.23. *Let $\mathcal{G}$ be a locally compact groupoid. Then $B(\mathcal{G})$ as defined is a unital commutative Banach algebra.*

Proof. Earlier discussions show that $B(\mathcal{G})$ forms an algebra under its pointwise operations and that it obeys the norm inequality $\|\varphi_1\varphi_2\|_{B(\mathcal{G})} \leq \|\varphi_1\|_{B(\mathcal{G})} \|\varphi_2\|_{B(\mathcal{G})}$. That $B(\mathcal{G})$ is complete is seen as in the proof of [76, Proposition 1.4] and uses Proposition 2.4.22, with $L^\infty(\mathcal{G})$ replaced with $C(\mathcal{G})$ and continuous Hilbert fields used in place of measurable fields.

Let $\mathcal{I}$ denote the trivial $\mathcal{G}$-Hilbert bundle of Example 2.4.18 with Hilbert field $(\{\mathcal{H}\}_{u \in \mathcal{G}^{(0)}}, C(\mathcal{G}^{(0)}, \mathcal{H}))$. Let $f : \mathcal{G}^{(0)} \to \mathcal{H}$ be the continuous and bounded section $f(u) = \alpha$ for some fixed unit vector α. By Example 2.4.18, we have that $\mathcal{I}_{f,f}(x) = \langle I_{\mathcal{H}} f(s(x)), f(r(x)) \rangle = \langle \alpha, \alpha \rangle = 1$ independently of the choice of α. Then $\mathcal{I}_{f,f}$ is clearly the multiplicative identity of $B(\mathcal{G})$. □

When $\mathcal{G}$ is a locally compact group, $B(\mathcal{G})$ coincides with the known Fourier-Stieltjes algebra. The Fourier algebra will be defined using coefficients of the left regular bundle, so to discuss the algebra our group needs a Haar system. Recall that the left regular bundle Λ can be defined as the Hilbert field $L^2(\mathcal{G}) = \{L^2(\mathcal{G}^u, \lambda^u)\}_{u\in\mathcal{G}^{(0)}}$ with the set of fundamental continuous sections given by $C_c(\mathcal{G})$, the section corresponding to $f \in C_c(\mathcal{G})$ being the map $u \mapsto f^u = f|_{\mathcal{G}^u}$.

Definition 2.4.24. Let $A_{cf}(\mathcal{G})$ be the set of coefficients $\Lambda_{f,g}$ with $f, g \in C_c(\mathcal{G})$. From this, let $A_c(\mathcal{G})$ be the algebra generated by $A_{cf}(\mathcal{G})$ under pointwise operations. The *Fourier Algebra* of a locally compact groupoid is given by $A(\mathcal{G}) := \overline{A_c(\mathcal{G})}^{\|\cdot\|_{B(\mathcal{G})}}$.

By definition, $A(\mathcal{G})$ is a closed sub-algebra of $B(\mathcal{G})$ thus $A(\mathcal{G})$ is itself a Banach algebra. As with the Fourier-Stieltjes algebra, if $\mathcal{G}$ is a group then $A(\mathcal{G})$ coincides with the definition in the group case.

Remark 2.4.25. By definition, we must have that $A_{cf}(\mathcal{G}) \subseteq C_c(\mathcal{G})$. The set $C_c(\mathcal{G})$ forms an algebra under convolution ([91, Proposition 1.34]) and we can express a coefficient $\Lambda_{f,g}$ as $\overline{g} * f^*$ with $f^*(z) = \overline{f(z^{-1})}$. The product of continuous and compactly supported functions remains so, thus $A_c(\mathcal{G}) \subseteq C_c(\mathcal{G})$.

2.5 Notes and Comments

This section is, globally, to house remarks pertaining to topics already discussed that would detract from the goal of the section the topics were originally mentioned in.

2.5.1 Étale and r-discrete groupoids

Historically, there is a larger class of groupoids that predates the étale groupoids: the r-discrete groupoids. We discuss these groupoids here to shine a light on the differences between r-discrete and étale as the terminology is often used interchangably. The proofs here are similar to those in [91] and [73].

Definition 2.5.1. Let $\mathcal{G}$ be a locally compact Hausdorff groupoid. $\mathcal{G}$ is called *r-discrete* if $\mathcal{G}^{(0)}$ is open in $\mathcal{G}$.

The above definition very naturally leads to the following lemma.

Proposition 2.5.2. *[91, Lemma 1.25] If $\mathcal{G}$ is r-discrete, then for each $u \in \mathcal{G}^{(0)}$ the sets $\mathcal{G}^u$ and $\mathcal{G}_u$ are discrete spaces.*

Lemma 2.5.3. *Suppose $\mathcal{G}$ is any topological groupoid such that r is a local homeomorphism. Then $\mathcal{G}$ is r-discrete.*

For an r-discrete groupoid, a number of useful conditions are all equivalent, including the one above in Lemma 2.5.3. We will define the étale groupoids using any of those equivalent conditions.

Proposition 2.5.4. *[73, Proposition I.2.8] Let $\mathcal{G}$ be a locally compact, r-discrete groupoid. Then the following are equivalent:*

1. *Counting measures form a Haar system for $\mathcal{G}$.*
2. *r is an open map.*
3. *r is a local homeomorphism.*
4. *$\mathcal{G}$ has a topological basis consisting of open bisections (sets B such that $r|_B$ and $s|_B$ are homeomorphisms).*

Any locally compact groupoid satisfying the conditions of Proposition 2.5.4 is called *étale*. Most commonly, authors will define an étale groupoid to be one in which the range map is a local homeomorphism. Combining this definition with Lemma 2.5.3, every étale groupoid is r-discrete, but not every r-discrete groupoid is necessarily étale.

As clarified by [91], historically there was not much of a distinction between the étale groupoids and r-discrete ones as C^*-algebraists tended to assume the existence of a Haar system for their own purposes. Conversely, with the widespread usage of the word "étale", it is not always specified whether results pertaining to étale groupoids

also apply to the r-discrete family. We are primarily concerned with the étale case (as opposed to the more general r-discrete).

2.5.2 Alternative approaches to $B(\mathcal{G})$ and $A(\mathcal{G})$

In [76] Renault studies *measurable* versions of the algebras $B(\mathcal{G})$ and $A(\mathcal{G})$ in the context of *measurable* Hilbert fields. The definitions are identical except for the two continuity conditions replaced with measurability (e.g. the coefficients need only be measurable, not necessarily continuous).

Paterson suggests three different definitions for the Fourier algebra in [70], each of which mimic one presentation of the algebra in the group case. All three definitions reduce to the existing Fourier algebra in the case when the groupoid is a proper group, however at the time of publication Paterson was unaware whether the three definitions coincide for general groupoids. We will acknowledge when definitions coincide for the groupoids studied in this thesis. The first of the alternate definitions is to define $A(\mathcal{G})$ to be the closure of coefficients coming from the set of sections which vanish at infinity. The second, due to [67, pg 186] suggests defining $A(\mathcal{G})$ to be the closure of $C_c(\mathcal{G}) \cap B(\mathcal{G})$ in the $B(\mathcal{G})$ norm.

Chapter 3

GROUPOID RESTRICTION RESULTS

Recall that, for a locally compact group G and a closed subgroup H, the restriction map $A(G) \to A(H)$ is surjective. In the case of $B(G)$, the restriction map at the very least has codomain $B(H)$, though it is not necessarily surjective (see Section 2.3.5). This chapter focuses on examining the analog of this map in the groupoid setting with $B(\mathcal{G})$ and $A(\mathcal{G})$. In Section 3.1, we show that the restriction map $B(\mathcal{G}) \to \mathcal{B}(\mathcal{G}_u^u)$ is a continuous algebra homomorphism. When $\mathcal{G}$ is étale we show a stronger result for the map $R^u_{B(\mathcal{G})}|_{A(\mathcal{G})}$, that is, that the restriction map $A(\mathcal{G}) \to A(\mathcal{G}_u^u)$ is a continuous and surjective algebra homomorphism. When $\mathcal{G}$ is not étale we demonstrate a continuous Banach space homomorphism $A(\mathcal{G}) \to A(\mathcal{G}_u^u)$. In the remainder of the chapter, we demonstrate cases where the restriction map $B(\mathcal{G}) \to B(\mathcal{G}_u^u)$ is or is not surjective. In Section 3.2.1 we discuss the case when $\mathcal{G}$ is a locally compact and transitive groupoid. In this setting we demonstrate the surjectivity of the restriction map on $B(\mathcal{G})$. In Section 3.2.2 we show a similar result for group bundles with a discrete unit space. Finally, in Section 3.2.3 we study a class of groupoids called the "HLS" groupoids after their originators. We show that the restriction map in this case is not necessarily surjective.

3.1 General Results

3.1.1 Subspaces of $\mathcal{G}^u$ and their Measure Structure

This subsection houses elementary structural results we need to discuss the restriction maps on general groupoids.

Lemma 3.1.1. *Let $\mathcal{G}$ be a locally compact, Hausdorff groupoid. Fix $u \in \mathcal{G}^{(0)}$ and equip $\mathcal{G}_v^u$ with the subspace topology. Then $\mathcal{G}_v^u$ is a locally compact, Hausdorff space and $\mathcal{G}_u^u$ is a locally compact, Hausdorff group.*

Proof. Let $\{x_\alpha\}_{\alpha\in I}$ be a net in $\mathcal{G}_v^u$ with $x_\alpha \to x$. Since r is a continuous map, we have $r(x_\alpha) \to r(x)$. At the same time $r(x_\alpha) = u$ for all α, so $r(x) = u$, as $\mathcal{G}$ is Hausdorff[1]. A similar argument can be made with the source map, thus $\mathcal{G}_v^u$ is closed.

Subsets of a Hausdorff space equipped with the subspace topology are themselves Hausdorff. Indeed, if X is a Hausdorff space and $Y \subseteq X$, for distinct points $x, y \in Y$ choose U_x, U_y open in X separating x and y. Then $U_x \cap Y$ and $U_y \cap Y$ are open sets in the subspace topology separating x and y. It is known that any closed (or open) set in a locally compact Hausdorff space is locally compact in the subspace topology ([65, Corollary 29.3]), so we conclude that $\mathcal{G}_v^u$ is locally compact. Proposition 2.1.5 shows that $\mathcal{G}_u^u$ is a group. The continuity of the product and inversion maps on $\mathcal{G}$ is required by Definition 2.1.11, and so $\mathcal{G}_u^u$ is a locally compact groupoid. □

Proposition 3.1.2. *Let $\mathcal{G}$ be a locally compact and Hausdorff groupoid. Fix $u \in \mathcal{G}^{(0)}$ and suppose both $\mathcal{G}_u^v$ and $\mathcal{G}_u^w$ are nonempty. Fix $\gamma_v \in \mathcal{G}_u^v$ and $\gamma_w \in \mathcal{G}_u^w$. Then the map $\varphi_{v,w} : \mathcal{G}_u^u \to \mathcal{G}_v^w$ given by*

$$\varphi_{v,w}(x) = \gamma_w x \gamma_v^{-1} \tag{3.1}$$

is a homeomorphism. Furthermore, $\varphi_{v,v}$ is an isomorphism of locally compact groups.

Proof. First suppose that $x \in \mathcal{G}_u^u$. Since $s(\gamma_w) = u = r(x)$, so $\gamma_w x$ is defined. Similarly, $s(\gamma_w x) = s(x) = u$ while $r(\gamma_v^{-1}) = s(\gamma_v) = u$ so that $\gamma_w x \gamma_v^{-1}$ is defined and belongs to $\mathcal{G}_v^w$. Next, assume that $\varphi_{v,w}(x_1) = \varphi_{v,w}(x_2)$, so $\gamma_w x_1 \gamma_v^{-1} = \gamma_w x_2 \gamma_v^{-1}$. Then by the cancellation property of groupoids (Proposition 2.1.2), $x_1 = x_2$. For $y \in \mathcal{G}_v^w$, setting $x = \gamma_w^{-1} y \gamma_v$ gives that $x \in \mathcal{G}_u^u$ and $\varphi_{v,w}(x) = y$ through a similar source and range analysis. As $\mathcal{G}$ is a topological groupoid, the multiplication and inversion maps are continuous. As $\varphi_{v,w}$ is the repeated composition of these maps, it is continuous. We also have $(\varphi_{v,w})^{-1}(x) = \gamma_w^{-1} x \gamma_v$, which is defined through another source and range analysis. The verification that $(\varphi_{v,w})^{-1}$ is a continuous bijection is identical to that of $\varphi_{v,w}$, so the map is indeed a homeomorphism.

[1] For this argument, it is enough to assume that $\mathcal{G}^{(0)}$ is Hausdorff.

Now assume that $w = v$. As we have shown the map is bijective, we need only demonstrate that $\varphi_{v,v}$ is a group homomorphism. To see this, note

$$\varphi_{v,v}(x)\varphi_{v,v}(y) = (\gamma_v x \gamma_v^{-1})(\gamma_v y \gamma_v^{-1}) = \gamma_v x y \gamma_v^{-1} = \varphi_{v,v}(xy)$$

by groupoid associativity and the identity property of unit space elements. □

In the above proposition, if $v = u$ then γ_u is chosen to be u. When trying to relate $A(\mathcal{G})$ to $A(\mathcal{G}_u^u)$, we will need to formalize how we pass between their L^2 spaces.

Lemma 3.1.3. *Let $\mathcal{G}$ be a locally compact Hausdorff groupoid with a Haar system $\{\lambda^u\}_{u\in\mathcal{G}^{(0)}}$. For $u \in \mathcal{G}^{(0)}$, if λ^u restricted to $\mathcal{G}_u^u$ is non-zero then it is the Haar measure for $\mathcal{G}_u^u$.*

Proof. To prove this statement, we will need to verify that $\lambda^u|_{\mathcal{G}_u^u}$ is both inner and outer regular, finite on compact sets, and left translation invariant (as we've assumed the measure is non-zero). Let $\lambda^u|_{\mathcal{G}_u^u} = \nu$.

Claim 3.1.4. *The measures λ^u and ν are finite on compact sets.*

Proof of Claim 3.1.4: We first establish this for compact subsets of $\mathcal{G}^u$. Let K be a compact subset of $\mathcal{G}^u$ and fix $\varepsilon > 0$. Let U be a precompact open set of $\mathcal{G}^u$ containing K with $\lambda^u(U \setminus K) < \varepsilon$. Apply Lemma A.1.6 to K and U^c to find a continuous and compactly supported function $f : \mathcal{G}^u \to [0,1]$ such that $f|_K = 1$ and $f|_{U^c} = 0$. By Remark A.1.8, we may extend this f to an $\tilde{f} \in C_c^+(\mathcal{G})$. With this extension, by Definition 2.1.12 (2), the function $\tilde{f}^0 : \mathcal{G}^{(0)} \to \mathbb{C}$ defined by $\tilde{f}^0(v) = \int_{\mathcal{G}^v} \tilde{f} d\lambda^v$ is defined for all $v \in \mathcal{G}^{(0)}$. In particular, $\tilde{f}^0(u)$ is finite. However,

$$\lambda^u(K) = \int_K 1 d\lambda^u \leq \int_{\mathcal{G}^u} \tilde{f} d\lambda^u = \tilde{f}^0(u) < \infty$$

and so $\lambda^u(K)$ must be finite. Having shown this for compact sets of $\mathcal{G}^u$, we shift focus to compact sets of $\mathcal{G}_u^u$. Thus, let $C \subseteq \mathcal{G}_u^u$ be compact in $\mathcal{G}_u^u$. By Proposition A.1.3, this implies that C is also compact in $\mathcal{G}^u$ and $\nu(C) = \lambda^u(C) < \infty$.

Claim 3.1.5. *The measure ν is inner regular.*

Proof of Claim 3.1.5: To demonstrate this, we must show that if $O \subseteq \mathcal{G}_u^u$ is open in $\mathcal{G}_u^u$ then

$$\nu(O) = \sup\{\nu(K) : K \subseteq O, K \text{ compact in } \mathcal{G}_u^u\}.$$

We immediately have that $\nu(O) \geq \nu(K)$ for each compact $K \subseteq O$. As O is open in $\mathcal{G}_u^u$, there exists a set $U \subseteq \mathcal{G}^u$ open in $\mathcal{G}^u$ with $O = U \cap \mathcal{G}_u^u$. Fix $\varepsilon > 0$. As λ^u is a regular measure on $\mathcal{G}^u$, there is a compact (in $\mathcal{G}^u$) set $C \subseteq U \subseteq \mathcal{G}^u$ with $\lambda^u(U \setminus C) \leq \varepsilon$. Given that $\mathcal{G}_u^u$ is closed in $\mathcal{G}^u$, by Proposition A.1.3, we have $C \cap \mathcal{G}_u^u$ is compact in $\mathcal{G}_u^u$. Then:

$$\nu(O \setminus (C \cap \mathcal{G}_u^u)) = \nu((U \cap \mathcal{G}_u^u) \setminus (C \cap \mathcal{G}_u^u)) = \nu((U \setminus C) \cap \mathcal{G}_u^u) = \lambda^u((U \setminus C) \cap \mathcal{G}_u^u) \leq \varepsilon$$

so $\nu(O) \leq \nu(C \cap \mathcal{G}_u^u) + \varepsilon$. As ε was arbitrary, we have the reverse inequality and the fact that ν is inner regular.

Claim 3.1.6. *The measure ν is outer regular.*

Proof of Claim 3.1.6: We will show that for any ν-measurable set E of $\mathcal{G}_u^u$ we have

$$\nu(E) = \inf\{\nu(U) : E \subseteq U, U \text{ open in } \mathcal{G}_u^u\}.$$

The right hand side is a trivial upper bound for $\nu(E)$, so we need to show that $\nu(E)$ is an upper bound for the right hand side. Let $E \subseteq \mathcal{G}_u^u$ be a ν-measurable set and U open in $\mathcal{G}_u^u$ such that $E \subseteq U$. Then $U = V \cap \mathcal{G}_u^u$ for V open in $\mathcal{G}^u$. Fix $\varepsilon > 0$. As λ^u is a regular measure on $\mathcal{G}^u$, there is an $O \subseteq \mathcal{G}^u$ open in $\mathcal{G}^u$ such that $\lambda^u(O) \leq \lambda^u(E) + \varepsilon$. Consider the set $O \cap \mathcal{G}_u^u$, which is open in $\mathcal{G}_u^u$. We have

$$\nu(O \cap \mathcal{G}_u^u) = \lambda^u(O \cap \mathcal{G}_u^u) \leq \lambda^u(O) \leq \lambda^u(E) + \varepsilon = \nu(E) + \varepsilon.$$

Since ε was arbitrary, we have that ν is outer regular.

What remains is to show that ν is left translation invariant. As $\mathcal{G}_u^u$ is closed in $\mathcal{G}^u$, if $f \in C_c(\mathcal{G})$ then $f|_{\mathcal{G}_u^u} \in C_c(\mathcal{G}_u^u)$. Definition 2.1.12 (3) then says

$$\int_{\mathcal{G}_u^u} f(xz) d\lambda^u|_{\mathcal{G}_u^u}(z) = \int_{\mathcal{G}_u^u} f(z) d\lambda^u|_{\mathcal{G}_u^u}(z), f \in C_c(\mathcal{G}) \tag{3.2}$$

for $x \in \mathcal{G}_u^u$. Since any $f \in C_c(\mathcal{G}_u^u)$ extends to a function in $C_c(\mathcal{G})$ (Remark A.1.8) we have that Equation (3.2) holds for all $0 \leq f \in C_c(\mathcal{G}_u^u)$ (it actually holds for every

$f \in C_c(\mathcal{G})$ though non-negative functions suffice). [32, Proposition 2.9(b)] shows that any Radon measure on a locally compact group is a Haar measure if Equation (3.2) holds for all $f \in C_c^+(G)$. This implies that $\lambda^u|_{\mathcal{G}_u^u} = \nu$ is the Haar measure for $\mathcal{G}_u^u$.

□

Definition 3.1.7. We will say a groupoid $\mathcal{G}$ with Haar system $\{\lambda^u\}_{u\in\mathcal{G}^{(0)}}$ *satisfies Condition* $(*)$ if for every $u \in \mathcal{G}^{(0)}$ the restriction of λ^u to $\mathcal{G}_u^u$ is non-zero.

Remark 3.1.8.

1. We require Condition $(*)$ to ensure the non-triviality of the spaces $L^2(\mathcal{G}_u^u, \lambda^u|_{\mathcal{G}_u^u})$.

2. Let $X := [0,1]$ with the Lebesgue measure, and set $\mathcal{G} := X \times X$ to be the full equivalence relation groupoid on X. In this groupoid, $\mathcal{G}^{(x,x)} := \mathcal{G}^x = \{(x,y) : y \in [0,1]\}$. Each measure λ^x on $\mathcal{G}^x$ will be just the Lebesgue measure. Then, for $x \in X$ we have $\mathcal{G}_x^x = \{(x,x)\}$ and $\lambda^x(\mathcal{G}_x^x) = 0$ so λ^x does not restrict to the Haar measure of $\mathcal{G}_x^x$.

3. Discrete groupoids, group bundles with open range map, and étale groupoids all satisfy Condition $(*)$. [74, Lemma 1.3] shows that group bundles with open range map have Haar systems, and [91, Theorem 6.9] demonstrates that the Haar system of such group bundles is essentially comprised of the Haar measures of the groups.

4. Recall that for a locally compact and Hausdorff groupoid $\mathcal{G}$, the spaces $\mathcal{G}^u$ and $\mathcal{G}_v^u$ are also locally compact and Hausdorff for each $u, v \in \mathcal{G}^{(0)}$ with $\mathcal{G}_v^u \subseteq \mathcal{G}^u$. This in turn implies that $L^2(\mathcal{G}_v^u, \lambda^u|_{\mathcal{G}_v^u})$ is a subspace of $L^2(\mathcal{G}^u, \lambda^u)$ with the natural embedding: setting $f(x) = 0$ for $x \in \mathcal{G}^u \setminus \mathcal{G}_v^u$. Moreover, $L^2(\mathcal{G}_v^u, \lambda^u|_{\mathcal{G}_v^u})$ is a *closed* subspace. Since the sets $\{\mathcal{G}_v^u\}_{v\in\mathcal{G}^{(0)}}$ are pairwise disjoint as subsets of $\mathcal{G}^u$, we have $L^2(\mathcal{G}_v^u, \lambda^u|_{\mathcal{G}_v^u}) \perp L^2(\mathcal{G}_{v'}^u, \lambda^u|_{\mathcal{G}_{v'}^u})$ for distinct v, v'. This gives us that $L^2(\mathcal{G}^u, \lambda^u) = \oplus_{v:\mathcal{G}_v^u\neq\varnothing} L^2(\mathcal{G}_v^u, \lambda^u|_{\mathcal{G}_v^u})$. Subsequently, when writing $L^2(\mathcal{G}_v^u)$ we will mean $L^2(\mathcal{G}_v^u, \lambda^u|_{\mathcal{G}_v^u})$.

3.1.2 The Restriction map on $B(\mathcal{G})$ and $A(\mathcal{G})$

Lemma 3.1.9. *Let $\mathcal{G}$ be a locally compact, Hausdorff groupoid and let $\mathfrak{H}$ be a $\mathcal{G}$-Hilbert bundle with linear actions $\{\mathcal{T}(x)\}_{x\in\mathcal{G}}$. Then the map $\mathcal{G}_u^u \to \mathcal{U}(\mathfrak{H}_u)$, $x \mapsto \mathcal{T}(x)$ is a continuous unitary representation of $\mathcal{G}_u^u$.*

Proof. By Definition 2.4.14 (2), the mapping $x \mapsto \mathcal{T}(x)$ is a group homomorphism $\mathcal{G}_u^u \to \mathcal{U}(\mathfrak{H}_u)$. Denote this representation by π. To finish the proof, it only remains to show that the map $x \mapsto \pi_{\xi,\eta}(x)$ is continuous for every $\xi, \eta \in \mathfrak{H}_u$. To that end, fix $\xi, \eta \in \mathfrak{H}_u$. By Proposition 2.4.11, there exist bounded continuous sections $\boldsymbol{\xi}, \boldsymbol{\eta}$ such that $\boldsymbol{\xi}(u) = \xi$ and $\boldsymbol{\eta}(u) = \eta$. Definition 2.4.14 requires the map $x \mapsto \mathfrak{H}_{\boldsymbol{\xi},\boldsymbol{\eta}}(x)$ to be continuous on $\mathcal{G}$, thus its restriction to $\mathcal{G}_u^u$ will be continuous. However, $\mathfrak{H}_{\boldsymbol{\xi},\boldsymbol{\eta}}|_{\mathcal{G}_u^u} = \pi_{\xi,\eta}$. □

Theorem 3.1.10. *Let $\mathcal{G}$ be a locally compact, Hausdorff groupoid. For $u \in \mathcal{G}^{(0)}$, the map $R^u_{B(\mathcal{G})} : B(\mathcal{G}) \to B(\mathcal{G}_u^u)$, $\varphi \mapsto \varphi|_{\mathcal{G}_u^u}$ is a norm-decreasing algebra homomorphism.*

Proof. By Lemma 3.1.9, the domain and codomain of $R^u_{B(\mathcal{G})}$ are as claimed. As the algebra operations in both domain and co-domain are pointwise, the mapping will be an algebra homomorphism. Finally, we will show our map is contractive. Suppose $\varphi \in B(\mathcal{G})$, $\varphi = \mathfrak{H}_{\boldsymbol{\xi},\boldsymbol{\eta}}$, and $\varphi|_{\mathcal{G}_u^u} = \pi_{\xi,\eta}$. Then, using the definition of the norm of $B(\mathcal{G}_u^u)$, we have

$$\left\|\varphi|_{\mathcal{G}_u^u}\right\|_{B(\mathcal{G}_u^u)} = \|\pi_{\xi,\eta}\|_{B(\mathcal{G}_u^u)} \leq \|\xi\|\,\|\eta\|\,.$$

By definition, we have $\|\boldsymbol{\xi}\| = \sup_{u\in\mathcal{G}^{(0)}} \|\xi(u)\|$, and so $\|\xi\| \leq \|\boldsymbol{\xi}\|$. Thus

$$\left\|\varphi|_{\mathcal{G}_u^u}\right\|_{B(\mathcal{G}_u^u)} \leq \|\boldsymbol{\xi}\|\,\|\boldsymbol{\eta}\|\,.$$

As this inequality is true for arbitrary representations of φ, it will be true over the infimum, giving $\left\|\varphi|_{\mathcal{G}_u^u}\right\|_{B(\mathcal{G}_u^u)} \leq \|\varphi\|_{B(\mathcal{G})}$ so $R^u_{B(\mathcal{G})}$ is bounded with norm at most 1. □

Remark 3.1.11. We will show in Corollary 3.2.18.1 that $R^u_{B(\mathcal{G})}$ is not necessarily surjective. This is due to the fact that not every representation π of $\mathcal{G}_u^u$ can be seen as the restriction of a $\mathcal{G}$-Hilbert bundle.

We next discuss the restriction of $R^u_{B(\mathcal{G})}$ to $A(\mathcal{G})$.

Theorem 3.1.12. *Let $\mathcal{G}$ be a locally compact, Hausdorff étale groupoid. Let $R^u_{A(\mathcal{G})}$ be the map in Theorem 3.1.10 restricted to $A(\mathcal{G})$. Then $R^u_{A(\mathcal{G})} : A(\mathcal{G}) \to A(\mathcal{G}^u_u)$ is a continuous, norm-decreasing, and surjective algebra homomorphism.*

Proof. Let Λ be the left regular bundle with linear actions $\{L_x\}_{x\in\mathcal{G}}$. As $\mathcal{G}$ is étale, it satisfies Condition $(*)$ (see Definition 3.1.7). We will consider its Haar system to be the collection of counting measures. By Theorem 3.1.10, the collection $\{L_x\}_{x\in\mathcal{G}^u_u}$ forms a continuous unitary representation of $\mathcal{G}^u_u$. Denote this representation by σ. We have $\sigma : \mathcal{G}^u_u \to \mathcal{U}(L^2(\mathcal{G}^u))$ with $(\sigma_x f)(z) = f(x^{-1}z)$. Thus $R^u_{A(\mathcal{G})} : A(\mathcal{G}) \to A_\sigma(\mathcal{G}^u_u)$ is a continuous Banach space homomorphism.

Claim 3.1.13. *$R^u_{A(\mathcal{G})} : A(\mathcal{G}) \to A_\sigma(\mathcal{G}^u_u)$ is surjective.*

Proof of Claim 3.1.13:

1. We first show that $\operatorname{span}\{\sigma_{f,g} : f, g \in L^2(\mathcal{G}^u)\} \subseteq R^u_{A(\mathcal{G})}(A(\mathcal{G}))$. Begin by fixing $\sigma_{f,g}$ with $f, g \in L^2(\mathcal{G}^u)$. By Proposition 2.4.11, there are continuous sections F, G of Λ such that $F^u = f$ and $G^u = g$ and $\|F\|_\Delta = \|f\|_2$, $\|G\|_\Delta = \|g\|_2$. Define the function $\Lambda_{F,G}$. Then $R^u_{A(\mathcal{G})}(\Lambda_{F,G}) = \sigma_{f,g}$. The linearity of $R^u_{A(\mathcal{G})}$ establishes the result for $\operatorname{span}\{\sigma_{f,g} : f, g \in L^2(\mathcal{G}^u)\}$.

2. Next, we show that $A_\sigma(\mathcal{G}^u_u) \subseteq R^u_{A(\mathcal{G})}(A(\mathcal{G}))$. Let $\varphi \in A_\sigma(\mathcal{G}^u_u)$ be arbitrary. By Proposition 2.3.20, we may express $\varphi = \sum_{n=1}^\infty \sigma_{f_n,g_n}$ with $f_n, g_n \in L^2(\mathcal{G}^u)$ for every n. Additionally, we may assume $\|\varphi\|_{A_\sigma(\mathcal{G}^u_u)} = \sum_{n=1}^\infty \|f\|_2 \|g\|_2$. For each σ_{f_n,g_n}, find the preimage Λ_{F_n,G_n} as constructed in Item (1). Note

$$\|\Lambda_{F_n,G_n}\|_{A(\mathcal{G})} \leq \|F_n\|_\Delta \|G_n\|_\Delta = \|f_n\|_2 \|g_n\|_2 .$$

Therefore $\sum_{n=1}^{\infty} \|\Lambda_{F_n,G_n}\|_{A(\mathcal{G})}$ is convergent which implies that $\sum_{n=1}^{\infty} \Lambda_{F_n,G_n} \in A(\mathcal{G})$ ([12, pg. 36]). This convergence and the continuity of $R^u_{A(\mathcal{G})}$ implies that

$$R^u_{A(\mathcal{G})}\left(\sum_{n=1}^{\infty}\Lambda_{F_n,G_n}\right) = R^u_{A(\mathcal{G})}\left(\lim_{k\to\infty}\sum_{n=1}^{k}\Lambda_{F_n,G_n}\right) = \lim_{k\to\infty} R^u_{A(\mathcal{G})}\left(\sum_{n=1}^{k}\Lambda_{F_n,G_n}\right)$$
$$= \lim_{k\to\infty}\sum_{n=1}^{k}\sigma_{f_n,g_n} = \varphi.$$

Claim 3.1.14. *The representation $\sigma \simeq \oplus_{v:\mathcal{G}_v^u\neq\varnothing} L_{\mathcal{G}_u^u}$ where $L_{\mathcal{G}_u^u}$ is the left regular representation of $\mathcal{G}_u^u$.*

Proof of Claim 3.1.14:

1. Fix $v \in \mathcal{G}^{(0)}$ such that $\mathcal{G}_v^u \neq \varnothing$. Choose $f \in L^2(\mathcal{G}_v^u)$. Then $(\sigma_x f)(z) \neq 0$ if and only if $x^{-1}z \in \mathrm{supp}(f) \subseteq \mathcal{G}_v^u$, if and only if $z \in x\mathcal{G}_v^u \subseteq \mathcal{G}_v^u$ so $\sigma_x f \in L^2(\mathcal{G}_v^u)$. Thus the space $L^2(\mathcal{G}_v^u)$ is a closed, nonzero σ-invariant subspace of $L^2(\mathcal{G}^u)$.

2. Since we have assumed Condition $(*)$, $\lambda^u|_{\mathcal{G}_u^u}$ is the Haar measure for $\mathcal{G}_u^u$. This implies that $\sigma^{L^2(\mathcal{G}_u^u)}$ is exactly $L_{\mathcal{G}_u^u}$. We further claim that $\sigma^{L^2(\mathcal{G}_v^u)} \simeq \lambda_{\mathcal{G}_u^u}$ for each v with $\mathcal{G}_v^u \neq \varnothing$.

   For each such v, select a $\gamma_v \in \mathcal{G}_v^u$. Define $U_v : L^2(\mathcal{G}_v^u) \to L^2(\mathcal{G}_u^u)$ by $(U_v f)(z) = f(z\gamma_v)$. Then $(U_v^{-1}f)(z) = f(z\gamma_v^{-1})$ so that U_v is surjective. Secondarily, since the function $\mathcal{G}_u^u \to \mathcal{G}_v^u$ given by $z \mapsto z\gamma_v$ is bijective,

   $$\|U_v f\|_2^2 = \sum_{z\in\mathcal{G}_u^u} |f(z\gamma_v)|^2 = \sum_{z\in\mathcal{G}_v^u} |f(z)|^2 = \|f\|_2^2$$

   so U_v is an isometry. This implies that U_v is a unitary. Since

   $$(U_v \circ \sigma_x^{L^2(\mathcal{G}_v^u)} \circ U_v^{-1})(f)(z) = (\sigma_x^{L^2(\mathcal{G}_v^u)} \circ U_v^{-1})(f)(z\gamma_v) = (U_v^{-1}f)(x^{-1}z\gamma_v)$$
   $$= f(x^{-1}z) = (\sigma^{L^2(\mathcal{G}_u^u)}f)(z)$$

   the operator U_v intertwines $\sigma^{L^2(\mathcal{G}_v^u)}$ and $\sigma^{L^2(\mathcal{G}_u^u)}$. The spaces $L^2(\mathcal{G}_v^u)$ are orthogonal for distinct v, so we have

   $$\sigma = \bigoplus_{v:\mathcal{G}_v^u\neq\varnothing} \sigma^{L^2(\mathcal{G}_v^u)} \simeq \bigoplus_{v:\mathcal{G}_v^u\neq\varnothing} L_{\mathcal{G}_u^u}.$$

Claim 3.1.15. $A_\sigma(\mathcal{G}_u^u) = A(\mathcal{G}_u^u)$

Proof of Claim 3.1.15: By [29, Proposition 5.3.1], since σ is unitarily equivalent to a multiple of $L_{\mathcal{G}_u^u}$, the two representations are quasi-equivalent. [48, Proposition 2.8.12] proves that this implies $A_\sigma(\mathcal{G}_u^u) = A(\mathcal{G}_u^u)$.

To this point, we have shown that $R^u_{A(\mathcal{G})} : A(\mathcal{G}) \to A(\mathcal{G}_u^u)$ is a continuous and surjective Banach space homomorphism. As $R^u_{A(\mathcal{G})} = R^u_{B(\mathcal{G})}|_{A(\mathcal{G})}$, for $\varphi_1, \varphi_2 \in A(\mathcal{G})$ we have

$$R^u_{A(\mathcal{G})}(\varphi_1\varphi_2) = R^u_{B(\mathcal{G})}(\varphi_1\varphi_2) = R^u_{B(\mathcal{G})}(\varphi_1)R^u_{B(\mathcal{G})}(\varphi_2) = R^u_{A(\mathcal{G})}(\varphi_1)R^u_{A(\mathcal{G})}(\varphi_2)$$

justifying the final assertion that $R^u_{A(\mathcal{G})}$ is a continuous algebra homomorphism with dense range. □

Corollary 3.1.15.1. *$A_\sigma(\mathcal{G}_u^u)$ is a Banach algebra with σ defined as the representation of $\mathcal{G}_u^u$ given by the restriction of the left regular bundle Λ to $\mathcal{G}_u^u$.*

Proof. It will suffice to show that $A_\sigma(\mathcal{G}_u^u)$ is closed under pointwise product. For $f, g \in A_\sigma(\mathcal{G})$, there are $F, G \in A(\mathcal{G})$ with $R^u_{A(\mathcal{G})}(F) = f$ and $R^u_{A(\mathcal{G})}(G) = g$. Then

$$fg = R^u_{A(\mathcal{G})}(F)R^u_{A(\mathcal{G})}(G) = R^u_{A(\mathcal{G})}(FG).$$

As $FG \in A(\mathcal{G})$, $R^u_{A(\mathcal{G})}(FG) \in A_\sigma(\mathcal{G}_u^u)$. □

Remark 3.1.16. The condition that $\mathcal{G}$ is étale is not strictly necessary, though the above theorem may not be true in full generality. We explicitely use the étale condition to guarantee that the maps U_v are isometric. In general, one needs the measure λ^u of the groupoid's Haar system to be both left and right invariant, that is,

$$\int_{\mathcal{G}^u} |f(z\gamma_v)|^2 \, d\lambda^u(z) = \int_{\mathcal{G}^u} |f(z)|^2 \, d\lambda^u(z)$$

for $f \in L^2(\mathcal{G}_v^u)$. This is certainly true for r-discrete groupoids with Haar systems (since the collection of counting measures is a Haar system in this scenario), and will be true for group bundles which have a Haar system for all cases of $u \in \mathcal{G}^{(0)}$ such that $\mathcal{G}^u$ is a unimodular group.

In general, $R^u_{A(\mathcal{G})}$ is not guaranteed to be an algebra homomorphism onto $A_\sigma(\mathcal{G}^u_u)$ or $A(\mathcal{G}^u_u)$. This requires at least $A_\sigma(\mathcal{G}^u_u)$ to be an algebra. However, $R^u_{A(\mathcal{G})}$ can be used to construct a surjective Banach *space* homomorphism $A(\mathcal{G}) \to A(\mathcal{G}^u_u)$. We lose the preservation of the product due to the projection map we must compose with.

Proposition 3.1.17. *Let $\mathcal{G}$ be a locally compact and Hausdorff groupoid with a Haar system which satisfies Condition $(*)$. There exists a projection $P : A_\sigma(\mathcal{G}^u_u) \to A(\mathcal{G}^u_u)$ such that $P \circ R^u_{A(\mathcal{G})}$ is a continuous and surjective Banach space homomorphism.*

Proof. We will use the following two facts:

1. If π_1 and π are two representations of a locally compact group G such that $A_{\pi_1} \subseteq A_\pi$, then there exists a unique decomposition $A_\pi = A_{\pi_1} \oplus A_{\pi_2}$ for a representation π_2 of G ([8, Theorem 3.18]). For any $f \in A_\pi$, we may then uniquely write $f = f_1 + f_2$ with $f_i \in A_{\pi_i}$. Given this decomposition, we can define the projection $P_{\pi_1} : A_\pi \to A_{\pi_1}$ by $P_{\pi_1}(f) = f_1$.

2. Two representations π and υ of a group G are disjoint if $A_\pi \cap A_\upsilon = \{0\}$ ([48, Lemma 2.8.7]). This along with Proposition 2.3.22 demonstrates that the projection P_{π_1} above is continuous.

As in the proof of Theorem 3.1.12, let Λ be the left regular bundle with linear actions $\{L_x\}_{x\in\mathcal{G}}$. $\mathcal{G}$ is assumed to satisfy Condition $(*)$ (see Definition 3.1.7). We denote the representation formed from the collection $\{L_x\}_{x\in\mathcal{G}^u_u}$ by σ. We have $\sigma : \mathcal{G}^u_u \to \mathcal{U}(L^2(\mathcal{G}^u))$ with $(\sigma_x f)(z) = f(x^{-1}z)$. Thus $R^u_{A(\mathcal{G})} : A(\mathcal{G}) \to A_\sigma(\mathcal{G}^u_u)$ is a continuous Banach space homomorphism.

Claim 3.1.13 in Theorem 3.1.12 showed that $R^u_{A(\mathcal{G})}$ is surjective. This argument is independent of the assumption that $\mathcal{G}$ is étale, and so remains true when $\mathcal{G}$ is a locally compact Hausdorff groupoid satisfying Condition $(*)$. We further noted in Claim 3.1.14 of Theorem 3.1.12 that $\sigma^{L^2(\mathcal{G}^u_u)} = L_{\mathcal{G}^u_u}$, the argument again being independent of $\mathcal{G}$ being étale. By the facts listed in the beginning of this proof, this implies that $P_{L_{\mathcal{G}^u_u}} : A_\sigma(\mathcal{G}^u_u) \to A(\mathcal{G}^u_u)$ is a continuous projection. We conclude that $P_{L_{\mathcal{G}^u_u}} \circ R^u_{A(\mathcal{G})}$ is a continuous and surjective Banach space homomorphism $A_\sigma(\mathcal{G}^u_u) \to A(\mathcal{G}^u_u)$. □

Remark 3.1.18. As we had to use the projection onto $P_{L_{\mathcal{G}_u^u}} : A_\sigma(\mathcal{G}_u^u) \to A(\mathcal{G}_u^u)$, we cannot guarantee that $P_{L_{\mathcal{G}_u^u}} \circ R^u_{A(\mathcal{G})}$ respects multiplication. To see this, suppose $f, g \in A_\sigma(\mathcal{G}_u^u)$. By [8, Theorem 3.18], $A_\sigma(\mathcal{G}_u^u) = A(\mathcal{G}_u^u) \oplus A_\pi(\mathcal{G}_u^u)$ for some representation π. Then we may write $f = f_1 + f_2$ and $g = g_1 + g_2$ where $f_1, g_1 \in A(\mathcal{G}_u^u)$ and $f_2, g_2 \in A_\pi(\mathcal{G}_u^u)$. Then $fg = f_1g_1 + f_1g_2 + f_2g_1 + f_2g_2$. As $A(\mathcal{G}_u^u)$ is an ideal in $B(\mathcal{G}_u^u)$ (Proposition 2.3.13), we have that $P(fg) = f_1g_1 + f_1g_2 + f_2g_1$ which is not necessarily $f_1g_1 = (Pf)(Pg)$.

3.2 Surjectivity of the Restriction Map for Certain Important Classes

3.2.1 Transitive Groupoids

A groupoid will be called *transitive* if $\mathcal{G}_u^v \neq \varnothing$ for every pair $u, v \in \mathcal{G}^{(0)}$. It is worth noting $\mathcal{G}_u^u$ is always non-empty, so the additional structure comes into play when $u \neq v$. When $\mathcal{G}$ is a transitive groupoid, evaluation of coefficients of $\mathcal{G}$-Hilbert bundles will only depend on the actions of a subset of $\mathcal{G}$, and Proposition 3.1.2 can be strengthened as we will show.

Remark 3.2.1. Let $\mathcal{G}$ be a transitive groupoid. Fix $u \in \mathcal{G}^{(0)}$ and for each $v \in \mathcal{G}^{(0)}$ fix $\gamma_v \in \mathcal{G}_u^v$ (if $v = u$ then γ_u is chosen to be u). Then the map $\varphi_{v,w}$ of Equation (3.1) is defined for all pairs v, w. This gives us homeomorphisms from $\mathcal{G}_u^u$ to every $\mathcal{G}_w^v$ and isomorphisms to every $\mathcal{G}_v^v$ as in Proposition 3.1.2.

Lemma 3.2.2. *Let $\mathcal{G}$ be a locally compact, transitive groupoid. Let $\mathfrak{H}$ be a $\mathcal{G}$-Hilbert bundle with linear actions $\{\pi(x)\}_{x\in\mathcal{G}}$. Fix a $u \in \mathcal{G}^{(0)}$ and $\gamma_v \in \mathcal{G}_u^v$ as in Remark 3.2.1. Then for every $\boldsymbol{\xi}, \boldsymbol{\eta} \in \Delta_b(\mathfrak{H})$, $\mathfrak{H}_{\boldsymbol{\xi},\boldsymbol{\eta}}$ is uniquely determined by evaluations on $\mathcal{G}_u^u$ and $\{\gamma_v\}_{v\in\mathcal{G}^{(0)}}$ via*

$$\mathfrak{H}_{\boldsymbol{\xi},\boldsymbol{\eta}}(x) = \left\langle \pi(\gamma_w^{-1} x \gamma_v)\pi(\gamma_v^{-1})\boldsymbol{\xi}(s(x)), \pi(\gamma_w)^*\boldsymbol{\eta}(r(x)) \right\rangle$$

where $w = r(x)$ and $v = s(x)$.

Proof. Recall from Equation (3.1) that $\varphi_{v,w} : \mathcal{G}_u^u \to \mathcal{G}_v^w$ is given by $\varphi_{v,w}(x) = \gamma_w x \gamma_v^{-1}$. For $x \in \mathcal{G}_v^w$ we have that $x = \varphi_{v,w}(\gamma_w^{-1} x \gamma_v)$ so that

$$\pi(x) = \pi(\varphi_{v,w}(\gamma_w^{-1} x \gamma_v)) = \pi(\gamma_w(\gamma_w^{-1} x \gamma_v)\gamma_v^{-1}) = \pi(\gamma_w)\pi(\gamma_w^{-1} x \gamma_v)\pi(\gamma_v^{-1})$$

By definition we have $\mathfrak{H}_{\boldsymbol{\xi},\boldsymbol{\eta}}(x) = \langle \pi(x)\boldsymbol{\xi}(s(x)), \eta(r(x))\rangle$, substituting the expression for $\pi(x)$ establishes the result. □

Note that in a transitive groupoid, the range map $r : \mathcal{G}_u \to \mathcal{G}^{(0)}$ is surjective

Theorem 3.2.3. *Let $\mathcal{G}$ be a locally compact, transitive groupoid. Suppose $\gamma : \mathcal{G}^{(0)} \to \mathcal{G}_u$ is a continuous section*[2] *for the range map $r : \mathcal{G}_u \to \mathcal{G}^{(0)}$, i.e., γ is continuous and $r \circ \gamma : \mathcal{G}^{(0)} \to \mathcal{G}^{(0)}$ is the identity map. Then for every $u \in \mathcal{G}^{(0)}$ the map $R^u_{B(\mathcal{G})} : B(\mathcal{G}) \to B(\mathcal{G}^u_u)$ of Theorem 3.1.10 is surjective. In particular $R^u_{B(\mathcal{G})} : B(\mathcal{G}) \to B(\mathcal{G}^u_u)$ is surjective when $\mathcal{G}^{(0)}$ is discrete.*

Proof. Let $f \in B(\mathcal{G}^u_u)$. Then $f = \pi_{\xi_0,\eta_0}$ for some continuous unitary representation π of $\mathcal{G}^u_u$. For each $v \in \mathcal{G}^{(0)}$ set $\mathfrak{H}_v := \mathcal{H}_\pi$. For $v \in \mathcal{G}^{(0)}$ set $\gamma_v = \gamma(v)$ and define $\varphi_{v,w}(x) := \gamma_w x \gamma_v^{-1}$ according to Equation (3.1). Define $\pi(x) := \pi(\varphi^{-1}_{s(x),r(x)}(x))$. As the Hilbert field of $\mathfrak{H}$ is a constant field, the continuous sections are just functions belonging to $C(\mathcal{G}^{(0)}, \mathcal{H}_\pi)$. We claim the map $x \mapsto \pi(x)$ is a groupoid homomorphism. Let $x, y \in \mathcal{G}$ with $s(x) = r(y)$. Then

$$\begin{aligned}\pi(xy) = \pi(\gamma^{-1}_{r(xy)} xy \gamma_{s(xy)}) &= \pi(\gamma^{-1}_{r(x)} xy \gamma_{s(y)}) = \pi(\gamma^{-1}_{r(x)} x \gamma_{s(x)} \gamma^{-1}_{s(x)} y \gamma_{s(y)}) \\ &= \pi(\gamma^{-1}_{r(x)} x \gamma_{s(x)}) \pi(\gamma^{-1}_{r(y)} y \gamma_{s(y)}) = \pi(x)\pi(y).\end{aligned}$$

What remains to show is the continuity of the coefficients $\mathfrak{H}_{\boldsymbol{\xi},\boldsymbol{\eta}}$ for $\boldsymbol{\xi}, \boldsymbol{\eta} \in \Delta_b(\mathfrak{H})$. Let $\{x_\alpha\}_{\alpha \in I}$ be a net converging to x in $\mathcal{G}$ and $\varepsilon > 0$ be given. As the range and source maps $r, s : \mathcal{G} \to \mathcal{G}^{(0)}$ are continuous functions, we have $\{r(x_\alpha)\}_{\alpha \in I}$ and $\{s(x_\alpha)\}_{\alpha \in I}$ converge to $r(x)$ and $s(x)$ respectively. Since $\gamma : \mathcal{G}^{(0)} \to \mathcal{G}_u$ is a continuous function, we further have that $\gamma_{r(x_\alpha)}$ and $\gamma_{s(x_\alpha)}$ converge to $\gamma_{r(x)}$ and $\gamma_{s(x)}$ respectively. This chain of covergence culminates with

$$\varphi^{-1}_{s(x_\alpha),r(x_\alpha)}(x_\alpha) = \gamma^{-1}_{r(x_\alpha)} x_\alpha \gamma_{s(x_\alpha)} \to \gamma^{-1}_{r(x)} x \gamma_{s(x)} = \varphi^{-1}_{s(x),r(x)}(x),$$

[2] *It is an unfortunate coincidence that the terminology for this function matches that of $\mathcal{G}$-Hilbert bundle sections.*

as $x_\alpha \to x$. Let $\{y_\alpha\}_{\alpha\in I} = \left\{\varphi^{-1}_{s(x_\alpha),r(x_\alpha)}(x_\alpha)\right\}_{\alpha\in I}$ and $y = \varphi^{-1}_{s(x),r(x)}(x)$. First suppose $\|\boldsymbol{\eta}(r(x))\| \neq 0$ and $\|\pi(y)\boldsymbol{\xi}(s(x))\| \neq 0$. By Lemma 2.3.4, there exists α_1 and α_2 such that $\|\pi(y_\alpha)\xi(s(x_\alpha)) - \pi(y)\xi(s(x))\| < \frac{\varepsilon}{3\|\eta(r(x))\|}$ and $\|\pi(y_\alpha)\xi(s(x_\alpha)) - \pi(y)\xi(s(x))\| < \frac{\sqrt{\varepsilon}}{9}$ for $\alpha \geq \alpha_1$ and $\alpha \geq \alpha_2$ respectively. As $\boldsymbol{\eta}$ is a continuous section, there are α_3 and α_4 such that $\|\boldsymbol{\eta}(r(x_\alpha)) - \boldsymbol{\eta}(r(x))\| < \frac{\varepsilon}{3\|\pi(y)\xi(s(x))\|}$ and $\|\boldsymbol{\eta}(r(x_\alpha)) - \boldsymbol{\eta}(r(x))\| < 3\sqrt{\varepsilon}$ for $\alpha \geq \alpha_3$ and $\alpha \geq \alpha_4$ respectively. Choose $\alpha_0 \geq \max\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. Then for $\alpha \geq \alpha_0$ we have

$$
\begin{aligned}
&|\mathfrak{H}_{\boldsymbol{\xi},\boldsymbol{\eta}}(x_\alpha) - \mathfrak{H}_{\boldsymbol{\xi},\boldsymbol{\eta}}(x)| = |\langle \pi(y_\alpha)\boldsymbol{\xi}(s(x_\alpha)), \boldsymbol{\eta}(r(x_\alpha))\rangle - \langle \pi(y)\boldsymbol{\xi}(s(x)), \boldsymbol{\eta}(r(x))\rangle| \\
&= \Big| \langle \pi(y_\alpha)\boldsymbol{\xi}(s(x_\alpha)) - \pi(y)\boldsymbol{\xi}(s(x)), \boldsymbol{\eta}(r(x_\alpha)) - \boldsymbol{\eta}(r(x))\rangle \\
&+ \langle \pi(y)\boldsymbol{\xi}(s(x)), \boldsymbol{\eta}(r(x_\alpha)) - \boldsymbol{\eta}(r(x))\rangle + \langle \pi(y_\alpha)\boldsymbol{\xi}(s(x_\alpha)) - \pi(y)\boldsymbol{\xi}(s(x)), \boldsymbol{\eta}(r(x))\rangle \Big| \\
&\leq \|\pi(y_\alpha)\boldsymbol{\xi}(s(x_\alpha)) - \pi(y)\boldsymbol{\xi}(s(x))\| \, \|\boldsymbol{\eta}(r(x_\alpha)) - \boldsymbol{\eta}(r(x))\| \\
&+ \|\pi(y)\boldsymbol{\xi}(s(x))\| \, \|\boldsymbol{\eta}(r(x_\alpha)) - \boldsymbol{\eta}(r(x))\| + \|\pi(y_\alpha)\boldsymbol{\xi}(s(x_\alpha)) - \pi(y)\boldsymbol{\xi}(s(x))\| \, \|\boldsymbol{\eta}(r(x))\| \\
&< \left(\frac{\sqrt{\varepsilon}}{9}\right)\left(3\sqrt{\varepsilon}\right) + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}
$$

If either of $\|\boldsymbol{\eta}(r(x))\|$ and $\|\pi(y)\boldsymbol{\xi}(s(x))\|$ are 0, the result still holds with an easier ε argument. This demonstrates that $\mathfrak{H}_{\boldsymbol{\xi},\boldsymbol{\eta}}$ is continuous, showing that $\mathfrak{H}$ as constructed is a $\mathcal{G}$-Hilbert bundle. Fix $\boldsymbol{\xi}(v) = \xi_0$ and $\boldsymbol{\eta}(v) = \eta_0$ for every $v \in \mathcal{G}^{(0)}$. Then both $\boldsymbol{\xi}, \boldsymbol{\eta} \in \Delta_b(\mathfrak{H})$, and for $x \in \mathcal{G}^u_u$ we have

$$\mathfrak{H}_{\boldsymbol{\xi},\boldsymbol{\eta}}(x) = \langle \mathcal{T}(x)\xi_0, \eta_0\rangle = \langle \pi(x)\xi_0, \eta_0\rangle = \pi_{\xi_0,\eta_0}(x) = f.$$

□

Remark 3.2.4. In light of Theorem 3.2.3, it is natural to ask whether the restriction of $B(\mathcal{G})$ to transitive components yields anything fruitful. We answer this in the negative in a later section (see Corollary 3.2.18.1).

Given a locally compact Hausdorff, transitive groupoid $\mathcal{G}$, Lemma 3.2.2 hints at a relationship between two sets: the collection of $\mathcal{G}$-Hilbert bundles, and representations π of $\mathcal{G}^u_u$ paired with families of unitary mappings $U_\alpha : \mathcal{H}_\pi \to \mathcal{H}_\alpha$. This relationship is clarified in the next result.

Proposition 3.2.5. *Let $\mathcal{G}$ be a transitive locally compact groupoid with discrete unit space $\mathcal{G}^{(0)}$. Then there exists a bijective correspondence between:*

1. *$\mathcal{G}$-Hilbert bundles;*

2. *Unitary representations $\pi : \mathcal{G}_u^u \to \mathcal{U}(\mathcal{H}_\pi)$ with a family of unitaries $\{\mathcal{U}_v\}_{v\in\mathcal{G}^{(0)}}$ where $\mathcal{U}_v : \mathcal{H}_\pi \to \mathcal{H}_v$ is a unitary operator between Hilbert spaces.*

Proof. We describe a surjective map from (1) $\to$ (2) and (2) $\to$ (1). Accepting the axiom of choice, this implies the existence of a bijection between the two sets via the Dual Cantor-Bernstein Theorem[3].

Fix $u \in \mathcal{G}^{(0)}$ and for each $v \in \mathcal{G}^{(0)}$ choose $\gamma_v \in \mathcal{G}_u^v$ (if $v = u$ then γ_u is chosen to be u). Given a $\mathcal{G}$-Hilbert bundle with linear actions $\{\pi(x)\}_{x\in\mathcal{G}}$, the collection $\{\pi(x)\}_{x\in\mathcal{G}_u^u} \cup \{\pi(\gamma_v)\}_{v\in\mathcal{G}^{(0)}}$ is a unitary representation of $\mathcal{G}_u^u$ along with the required family of unitaries.

Conversely, suppose $\pi : \mathcal{G}_u^u \to \mathcal{U}(\mathcal{H}_\pi)$ is a unitary representation of $\mathcal{G}_u^u$ and for each $v \in \mathcal{G}^{(0)}$, $\mathcal{U}_v : \mathcal{H}_\pi \to \mathcal{H}_v$ is a unitary operator. Define the following continuous Hilbert field: for $v \neq u$ set $\mathfrak{H}_v = \mathcal{H}_v$ and $\mathfrak{H}_u = \mathcal{H}_\pi$. Using the notation of Equation (3.1), define $\pi(x) : \mathfrak{H}_{s(x)} \to \mathfrak{H}_{r(x)}$ by

$$\pi(x) = \mathcal{U}_{r(x)}\pi(\varphi_{s(x),r(x)}^{-1}(x))\mathcal{U}_{s(x)}^{-1}.$$

We need to check two conditions to ensure $\mathfrak{H}$ is a $\mathcal{G}$-Hilbert bundle: that the map $x \mapsto \pi(x)$ is a groupoid homomorphism, and the map $x \mapsto \mathfrak{H}_{\boldsymbol{\xi},\boldsymbol{\eta}}$ is continuous for every $\boldsymbol{\xi}, \boldsymbol{\eta} \in \Delta_b(\mathfrak{H})$.

---

[3] The Dual Cantor-Bernstein Theorem states that if X and Y are sets such that there are surjections $f : X \to Y$ and $g : Y \to X$ then there exists a bijection from X to Y. For further details, see [10].

Suppose $(x, y) \in \mathcal{G}^{(2)}$ so $s(x) = r(y)$. Then we have

$$\begin{aligned}
\mathcal{T}(xy) &= \mathcal{U}_{r(xy)}\pi(\varphi^{-1}_{s(xy),r(xy)}(xy))\mathcal{U}^{-1}_{s(xy)} \\
&= \mathcal{U}_{r(x)}\pi(\varphi^{-1}_{s(y),r(x)}(xy))\mathcal{U}^{-1}_{s(y)} \\
&= \mathcal{U}_{r(x)}\pi(\gamma^{-1}_{r(x)}xy\gamma_{s(y)})\mathcal{U}^{-1}_{s(y)} \\
&= \mathcal{U}_{r(x)}\pi(\gamma^{-1}_{r(x)}x\gamma_{s(x)}\gamma^{-1}_{s(x)}y\gamma_{s(y)})\mathcal{U}^{-1}_{s(y)} \\
&= \mathcal{U}_{r(x)}\pi(\gamma^{-1}_{r(x)}x\gamma_{s(x)})\pi(\gamma^{-1}_{s(x)}y\gamma_{s(y)})\mathcal{U}^{-1}_{s(y)} \\
&= \mathcal{U}_{r(x)}\pi(\gamma^{-1}_{r(x)}x\gamma_{s(x)})\mathcal{U}^{-1}_{s(x)}\mathcal{U}_{s(x)}\pi(\gamma^{-1}_{s(x)}y\gamma_{s(y)})\mathcal{U}^{-1}_{s(y)} \\
&= \mathcal{U}_{r(x)}\pi(\gamma^{-1}_{r(x)}x\gamma_{s(x)})\mathcal{U}^{-1}_{s(x)}\mathcal{U}_{r(y)}\pi(\gamma^{-1}_{r(y)}y\gamma_{s(y)})\mathcal{U}^{-1}_{s(y)} \\
&= \mathcal{T}(x)\mathcal{T}(y).
\end{aligned}$$

We require a fair bit of setup to demonstrate the continuity of coefficients. Let $\{x_\alpha\}_{\alpha\in I}$ be a net and suppose $x_\alpha \to x$. Since r and s are continuous, this implies that $\{r(x_\alpha)\}_{\alpha\in I}$ and $\{s(x_\alpha)\}_{\alpha\in I}$ converge to $r(x)$ and $s(x)$ respectively. However, this convergence happens in a discrete space, and nets converge in the discrete topology if and only if they are eventually constant. This implies that there are indices α_1 and α_2 such that $r(x_\alpha)$ and $s(x_\alpha)$ equal $r(x)$ and $s(x)$ for $\alpha \geq \alpha_1$ and $\alpha \geq \alpha_2$ respectively. By Proposition 3.1.2, we have that $\varphi^{-1}_{s(x),r(x)}$ is continuous so $\varphi^{-1}_{s(x),r(x)}(x_\alpha) \to \varphi^{-1}_{s(x),r(x)}(x)$. Thus, since π is a continuous unitary representation, given $\varepsilon > 0$ for some α_3 we have that

$$\left\|\pi(\varphi^{-1}_{s(x),r(x)}(x_\alpha))\mathcal{U}^{-1}_{s(x)}\boldsymbol{\xi}(s(x)) - \pi(\varphi^{-1}_{s(x),r(x)}(x))\mathcal{U}^{-1}_{s(x)}\boldsymbol{\xi}(s(x))\right\| < \frac{\varepsilon}{\left\|\mathcal{U}^{-1}_{r(x)}\boldsymbol{\eta}(r(x))\right\|}$$

for $\alpha \geq \alpha_3$. If $\left\|\mathcal{U}^{-1}_{r(x)}\boldsymbol{\eta}(r(x))\right\| = 0$, make an arbitrary choice for α_3.

Let $\varepsilon > 0$, and $\boldsymbol{\xi}, \boldsymbol{\eta} \in \Delta_b(\mathfrak{H})$ be given, and $\alpha_1, \alpha_2, \alpha_3$ as above. Let $\alpha_0 \geq \alpha_i$,

$1 \leq i \leq 3$. Then, for $\alpha \geq \alpha_0$ we have

$$\begin{aligned}
&|\mathfrak{H}_{\boldsymbol{\xi},\boldsymbol{\eta}}(x_\alpha) - \mathfrak{H}_{\boldsymbol{\xi},\boldsymbol{\eta}}(x)| = |\langle \mathcal{T}(x_\alpha)\boldsymbol{\xi}(s(x_\alpha)), \boldsymbol{\eta}(r(x_\alpha))\rangle - \langle \mathcal{T}(x)\boldsymbol{\xi}(s(x)), \boldsymbol{\eta}(r(x))\rangle| \\
&= \Big| \Big\langle \mathcal{U}_{r(x_\alpha)}\pi(\varphi^{-1}_{s(x_\alpha),r(x_\alpha)}(x_\alpha))\mathcal{U}^{-1}_{s(x_\alpha)}\boldsymbol{\xi}(s(x_\alpha)), \boldsymbol{\eta}(r(x_\alpha))\Big\rangle \\
&- \Big\langle \mathcal{U}_{r(x)}\pi(\varphi^{-1}_{s(x),r(x)}(x))\mathcal{U}^{-1}_{s(x)}\boldsymbol{\xi}(s(x)), \boldsymbol{\eta}(r(x))\Big\rangle \Big|.
\end{aligned}$$

As $\alpha \geq \alpha_1$ and α_2, we have this as

$$\begin{aligned}
|\mathfrak{H}_{\boldsymbol{\xi},\boldsymbol{\eta}}(x_\alpha) - \mathfrak{H}_{\boldsymbol{\xi},\boldsymbol{\eta}}(x)| =& \Big| \Big\langle \pi(\varphi^{-1}_{s(x),r(x)}(x_\alpha))\mathcal{U}^{-1}_{s(x)}\boldsymbol{\xi}(s(x)), \mathcal{U}^{-1}_{r(x)}\boldsymbol{\eta}(r(x))\Big\rangle \\
&- \Big\langle \pi(\varphi^{-1}_{s(x),r(x)}(x))\mathcal{U}^{-1}_{s(x)}\boldsymbol{\xi}(s(x)), \mathcal{U}^{-1}_{r(x)}\boldsymbol{\eta}(r(x))\Big\rangle \Big| \\
\leq& \Big\|\mathcal{U}^{-1}_{r(x)}\boldsymbol{\eta}(r(x))\Big\| \cdot \Big\|\pi(\varphi^{-1}_{s(x),r(x)}(x_\alpha))\mathcal{U}^{-1}_{s(x)}\boldsymbol{\xi}(s(x)) - \pi(\varphi^{-1}_{s(x),r(x)}(x))\mathcal{U}^{-1}_{s(x)}\boldsymbol{\xi}(s(x))\Big\| \\
&< \varepsilon
\end{aligned}$$

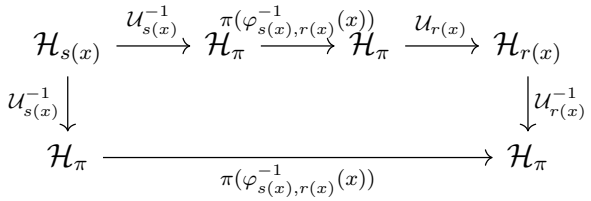
as $\alpha \geq \alpha_3$, so $\mathfrak{H}_{\boldsymbol{\xi},\boldsymbol{\eta}}$ is continuous and $\mathfrak{H}$ is a $\mathcal{G}$-Hilbert bundle. □

Corollary 3.2.5.1. *Let $\mathcal{G}$ be a transitive locally compact groupoid with discrete unit space $\mathcal{G}^{(0)}$. Then any $\mathcal{G}$-Hilbert bundle is unitarily equivalent to a $\mathcal{G}$-Hilbert bundle with a constant Hilbert field.*

Proof. Let $\mathfrak{H}$ be a $\mathcal{G}$-Hilbert bundle of a transitive locally compact groupoid with $\mathcal{G}^{(0)}$ discrete. Let the groupoid action be given by $\{\mathcal{T}(x)\}_{x\in\mathcal{G}}$ By Proposition 3.2.5, $\mathcal{T}(x) = \mathcal{U}^{-1}_{s(x)}\pi(\varphi^{-1}_{s(x),r(x)}(x))\mathcal{U}_{r(x)}$ for a unitary representation $\pi : \mathcal{G}^u_u \to \mathcal{H}_\pi$ and a family of unitaries $\mathcal{U}_v : \mathcal{H}_\pi \to \mathfrak{H}_v$.

Let $\mathfrak{G}$ be the constant Hilbert field (see Example 2.4.3) with $\mathfrak{G}_v := \mathcal{H}_\pi$ for every $v \in \mathcal{G}^{(0)}$. Define $\varrho(x) : \mathcal{H}_\pi \to \mathcal{H}_\pi$ by $\varrho(x) = \pi(\varphi^{-1}_{s(x),r(x)}(x))$. Then $\mathfrak{G}$ with linear actions $\{\varrho(x)\}_{x\in\mathcal{G}}$ is a $\mathcal{G}$-Hilbert bundle.

For each $v \in \mathcal{G}^{(0)}$ define $T_v : \mathfrak{H}_v \to \mathcal{H}_\pi$ by $T_u := \mathcal{U}^{-1}_u$. Clearly each T_v is a unitary operator, and for every $x \in \mathcal{G}$ the following diagram commutes



So the collection of operators $\{T_v\}_{v\in\mathcal{G}^{(0)}}$ intertwines $\mathfrak{H}$ and $\mathfrak{G}$. □

3.2.2 Group Bundles with Discrete Unit Space

Let X be an arbitrary set. For each $x \in X$, let G_x be a locally compact Hausdorff group. The *group bundle* formed by the collection $\{G_x\}_{x\in X}$ is defined to be the groupoid $\mathcal{G} = \cup_{x\in X} \{x\} \times G_x$ with partial product $(x, g)(x, h) = (x, gh)$ and inversion $(x, g)^{-1} = (x, g^{-1})$. As each G_x embeds into $\mathcal{G}$, when the distinction is clear we simply regard $\mathcal{G}$ as the disjoint union of groups and omit the first coordinate. In this event, if $\zeta = (x, g) \in \mathcal{G}$ we will equivalently say $\zeta \in G_x$. Further note that we can identify the unit space of $\mathcal{G}$ with X via $(x, e_{G_x}) \mapsto x$. For this section, we assume that the group bundle has been equipped with a topology such that $\mathcal{G}$ is locally compact, Hausdorff, and $\mathcal{G}^{(0)}$ is discrete.

Proposition 3.2.6. *$\mathcal{G}$ possesses a Haar system.*

Proof. Lemma 1.3 of [74] states that a group bundle will have a Haar system exactly when the range map is open. As $\mathcal{G}^{(0)}$ is discrete, every set is open, thus the range map is open. Theorem 6.9 of [91] shows that the collection of Haar measures on the groups G_x is a Haar system on $\mathcal{G}$. □

The main result of this section concerns the restriction map for this class of groupoid.

Proposition 3.2.7. *Let $\mathcal{G}$ be a locally compact and Hausdorff group bundle with discrete unit space. For $x \in X$, let $R^x_{B(\mathcal{G})} : B(\mathcal{G}) \to B(G_x)$ be the restriction map of Theorem 3.1.10. Then $R^x_{B(\mathcal{G})}$ is surjective.*

Proof. Let π^x be a unitary representation of G_x and ξ_x, η_x be fixed vectors in $\mathcal{H}_{\pi^x}$. For $y \neq x$, fix $\pi^y : G_y \to \mathcal{U}(\mathcal{H}_{\pi^y})$ to be a unitary representation of $\mathcal{G}_y$ and choose ξ_y, η_y fixed unit vectors in $\mathcal{H}_{\pi^y}$.

Set $\mathfrak{H}_z := \mathcal{H}_{\pi^z}$ for each $z \in X$. Further define $\pi((z, g_z)) := \pi^z(g_z)$. We claim that $\mathfrak{H}$ is a $\mathcal{G}$-Hilbert bundle. As the groupoid elements are only composable if they belong to the same group, and each π^z is a group homomorphism, we have that π is a groupoid homomorphism. The proof that each coefficient $\mathfrak{H}_{\boldsymbol{\xi},\boldsymbol{\eta}}$ is continuous is very similar to the proof of Proposition 3.2.5 since for a net $\{x_\alpha\}_{\alpha\in I}$ the convergences of $\{r(x_\alpha)\}_{\alpha\in I}$ and $\{s(x_\alpha)\}_{\alpha\in I}$ are in the discrete space $\mathcal{G}^{(0)}$. If we choose bounded continuous sections $\boldsymbol{\xi}(z) = \xi_z$ and $\boldsymbol{\eta}(z) = \eta_z$, via Proposition 2.4.11 we have that $R^x_{B(\mathcal{G})}\mathfrak{H}_{\boldsymbol{\xi},\boldsymbol{\eta}} = \pi^x_{\xi_x,\eta_x}$ and the map is surjective. □

3.2.3 HLS Groupoids

In this section, we study a particular class of group bundles called the HLS[4] groupoid, originally appearing in [42]. We begin by presenting the construction given by [90]. Let G be a discrete group. An *approximating sequence* for G is a sequence $\{K_n\}_{n\in\mathbb{N}}$ of subgroups of G with the following properties:

1. Each K_n is a normal subgroup of G of finite index;
2. The sequence is nested: $K_n \supseteq K_{n+1}$ for all n;
3. The intersection of the sequence is trivial: $\bigcap_n K_n = \{e_G\}$.

An *approximated group* is a pair $(G, \{K_n\})$ consisting of a discrete group together with a fixed approximating sequence. As each K_n is normal and finite index, the group G/K_n is defined and finite. Set $G_n := G/K_n$ and $\pi_n : G \to G_n$ the canonical quotient map, $G_\infty := G$ and π_∞ the identity map. We will define $\mathbb{N}_* = \mathbb{N} \cup \{\infty\}$ to be the one point compactification of $\mathbb{N}$.

[4] HLS stands for Higson, Lafforgue, and Skandalis, the authors who originated the construction.

The following lemma is a generalization of [90, Lemma 2.8], where a similar result for the group $\mathbb{F}_2$ is proved. This lemma shows that there is a large class of groups G which can produce an HLS groupoid. A *residually finite* group is one in which if $x \neq e_G$ then there exists a homomorphism φ from G to a finite group such that $\varphi(x) \neq e$.

Lemma 3.2.8. *Let G be a residually finite and finitely generated group. Then G has an approximating sequence.*

Proof. Let Γ be any finite group. As G is finitely generated, any homomorphism $G \to \Gamma$ is uniquely determined by its actions on the finite number of generators. Thus there is a finite number of homomorphisms $G \to \Gamma$, enumerated (not necessarily uniquely) by mapping each generator of G to each element of Γ. Since there are a finite number of distinct groups of size at most n (up to isomorphism), the set $\{\varphi : G \to \Gamma \text{ a group homomorphism, } |\Gamma| \leq n\}$ is finite. Define

$$K_n := \bigcap \{\ker \varphi | \varphi : G \to \Gamma \text{ a group homomorphism, } |\Gamma| \leq n\}.$$

As the kernel of a homomorphism to a finite group, each $\ker \varphi$ is of finite index. The intersection of a finite number of subgroups of finite index is again a subgroup of finite index, so $[G : K_n] < \infty$. As G is residually finite, the intersection of all the K_n is trivial. Finally, the sequence is clearly nested. □

Definition 3.2.9. [90, Definition 2.1] The HLS groupoid associated to an approximated group $(G, \{K_n\}_{n\in\mathbb{N}})$ is the group bundle

$$\mathcal{G} := \sqcup_{n\in\mathbb{N}_*} \{n\} \times G_n.$$

Equip $\mathcal{G}$ with the topology generated by the following open sets:

1. The set $\{(n, g)\}$ for each $n \in \mathbb{N}$ and $g \in G_n$;
2. The set $\text{ray}_n(g) := \{(m, \pi_m(g)) : m \in \mathbb{N}_*, m \geq n\}$ for fixed $g \in G$ and $n \in \mathbb{N}$.

We refer to sets of the second type as *rays*, although [3] refers to these sets as tails. We also refer to the subgroupoid $\mathcal{G} \setminus G$ as the *discrete part* of $\mathcal{G}$, as the restriction of the topology of $\mathcal{G}$ to this set is discrete. A point $(\infty, x) \in \mathcal{G}$ will be called a *point at infinity*.

Remark 3.2.10. HLS groupoids provide interesting examples in the context of amenability. Recall that a group is amenable if and only if the reduced and full C^*-algebras of the group coincide (see Proposition 2.2.13). In the groupoid setting, there are at least two notions of amenability (topological and measurewise, Definitions 2.2.8 and 3.3.1 respectively in [5]), and either definition implies the equality of the full and reduced C^*-algebras ([5, Proposition 6.1.8]). [90, Lemma 2.4] demonstrates that the converse is not true using the HLS groupoid associated to the group $\mathbb{F}_2$ and approximating sequence as given in Lemma 3.2.8.

Proposition 3.2.11. *Let $\mathcal{G}$ be the HLS groupoid associated to the approximated group $(G, \{K_n\}_{n\in\mathbb{N}})$. Then the following are true:*

1. *Each ray is open and compact.*
2. *A set $K \subseteq \mathcal{G}$ is compact if and only if K consists of a finite number of sets $S_n \subseteq ray_n(x)$ with $(\infty, x) \in S$ and a finite number of points not contained in any S_n.*
3. *$\mathcal{G}$ is locally compact, Hausdorff and étale. If G is countable, then $\mathcal{G}$ is second countable.*

Proof. 1. The rays form part of the topological base, so they are by definition open sets. To show compactness, consider an arbitrary open cover $\{U_\alpha\}_{\alpha\in I}$ of $\mathrm{ray}_n(x)$. Then at least one set in the open cover must contain (∞, x), say U_{α_x}. Since the collection in Definition 3.2.9 is a base for the topology of $\mathcal{G}$, there exists an $n' \in \mathbb{N}$ such that $\mathrm{ray}_{n'}(x) \subseteq U_{\alpha_x}$. If $n' \leq n$, then $\mathrm{ray}_n(x) \subseteq \mathrm{ray}_{n'}(x)$ and we have extracted a finite subcover. Otherwise, there is a finite number of points

$(n, \pi_n(x)), (n+1, \pi_{n+1}(x)), \ldots (n'-1, \pi_{n'-1}(x))$ in $\text{ray}_n(x) \setminus \text{ray}_{n'}(x)$. Choose open sets $U_{\alpha_n}, \ldots, U_{\alpha_{n'-1}}$ from $\{U_\alpha\}_{\alpha \in I}$ containing these points. Then $\{U_{\alpha_i}\}_{i=n}^{n'-1} \cup \{U_{\alpha_x}\}$ is a finite subcover.

2. To show that any set of the described form is compact, it suffices to show a single $S \subseteq \text{ray}_n(x)$ with $(\infty, x) \in S$ is compact. The proof of this is similar to the proof that $\text{ray}_n(x)$ is compact; given any open cover of S, there must be at least one U containing (∞, x) and some $\text{ray}_{n'}(x) \subseteq U$. Taking this U and open sets containing the finite number of points of S not covered by U produces the finite subcover. Conversely, any compact set K can intersect only a finite number of distinct rays. The set K cannot intersect any $\text{ray}_n(x)$ infinitely many times (in say $\{(n_i, \pi_i(x))\}_{i \in \mathbb{N}}$) without containing (∞, x), as we could take the sets $\{(n_i, \pi_i(x))\}_{i \in \mathbb{N}}$ as part of an open cover which would not admit a finite subcover. Thus, K intersects finitely many rays $\text{ray}_1(x_1), \ldots, \text{ray}_n(x_n)$, in sets S_i which contain the points $\{(\infty, x_i)\}$. Finally, K cannot contain infinitely many points outside of the sets S_i as we could use those singleton sets to form part of an open cover which would not admit a finite subcover.

3. To see that $\mathcal{G}$ is locally compact, note that an arbitrary point $(n, \pi_n(x))$ for $n \in \mathbb{N}_*$ is contained in $\text{ray}_1(x)$ which is both open and compact. Next, choose distinct points (n, x) and (m, y) in $\mathcal{G}$. If both $n, m < \infty$ then these points are separated by the singletons $\{(n, x)\}$ and $\{(m, y)\}$. If only one of n, m is ∞, without loss of generality n, then the two points are separated by the open sets $\{(m, y)\}$ and $\text{ray}_{m+1}(x)$. Finally, if $n = m = \infty$, we claim that we can choose an N such that $\text{ray}_N(x) \cap \text{ray}_N(y) = \varnothing$. This is because if $x^{-1}y \neq e_G$, then $x^{-1}y \notin \cap K_n$, and so there is at least one N such that $x^{-1}y \notin K_N$. Since $K_{n+1} \subseteq K_n$, $x^{-1}y \notin K_n$ for any $n \geq N$ which implies $\text{ray}_N(x) \cap \text{ray}_N(y) = \varnothing$ so $\mathcal{G}$ is Hausdorff.

   Next, we show that $\mathcal{G}$ is étale which amounts to showing that the range map $r : \mathcal{G} \to \mathcal{G}$ is a local homeomorphism. Any point $(n, \pi_n(x))$ is contained in the

open set $\mathrm{ray}_n(x)$, and $r(\mathrm{ray}_n(x)) = \mathrm{ray}_n(e_G)$ which is open in $\mathcal{G}$. It is clear that the range map restricted to $\mathrm{ray}_n(x)$ is a homeomorphism.

Now assume that G is countable. To show that $\mathcal{G}$ is second countable, we must argue that the base for its topology is countable. As G was assumed to be countable, $\{\{(n,g)\} : n \in \mathbb{N}, g \in G\} \cup \{\mathrm{ray}_n(x) : n \in \mathbb{N}, x \in G\}$ is countable and so the topological base is countable.

□

As $\mathcal{G}$ is Hausdorff and each $\mathrm{ray}_n(x)$ is compact, by [81, Theorem 2.4] they are also closed.

Remark 3.2.12. Let $\mathcal{G}$ be the HLS groupoid associated to the approximated group $(G, \{K_n\}_{n\in\mathbb{N}})$. Consider the set $S := \mathrm{ray}_n(x) \cup \mathrm{ray}_m(y)$ for $x \neq y \in G$. We can find an N such that $\mathrm{ray}_N(x) \cap \mathrm{ray}_N(y) = \varnothing$ (see the proof of Proposition 3.2.11, Item (3)) and $T := \{z : z \in S \setminus (\mathrm{ray}_N(x) \cup \mathrm{ray}_N(y))\}$ is finite. Then

$$S = \mathrm{ray}_N(x) \sqcup \mathrm{ray}_N(y) \sqcup T.$$

The particular structure of HLS groupoids allows us to construct explicit preimages for the restriction map $R^n_{A(\mathcal{G})} : A(\mathcal{G}) \to A(G_n)$.

Theorem 3.2.13. *Let $\mathcal{G}$ be the HLS groupoid associated to the approximated group $(G, \{K_n\}_{n\in\mathbb{N}})$. Then for each $n \in \mathbb{N}^*$ the map $R^n_{A(\mathcal{G})} : A(\mathcal{G}) \to A(G_n)$ is surjective.*

Proof. Define $G_\infty := G$, and $q_n : G \to G/K_n$ the canonical quotient map. If $n < \infty$, $R^n_{A(\mathcal{G})}$ is immediately surjective as any $f \in \ell^2(G_n)$ is also an element of $C_c(\mathcal{G})$. We will then focus on the map $R^\infty_{A(\mathcal{G})}$ and show that every $L_{f,g} \in A(G_\infty)$ with $f, g \in \ell^2(G_\infty)$ has a pre-image under $R^\infty_{A(\mathcal{G})}$.

To begin, fix $f \in \ell^2(G_\infty)$, assume that $f \geq 0$ pointwise. For $n \in \mathbb{N}$ define the following functions:

$$\begin{aligned} f_n &: G/K_n \to \mathbb{C} \\ f_n(xK_n) &:= \|f|_{xK_n}\|_2 . \end{aligned} \tag{3.3}$$

Next define

$$F : \mathcal{G} \to \mathbb{C} \tag{3.4}$$

$$F((n, q_n(x))) := \begin{cases} f_n(xK_n) & n < \infty \\ f(x) & n = \infty. \end{cases}$$

Claim 3.2.14. *If* $\operatorname{supp}(f)$ *is compact, then* $\operatorname{supp}(F)$ *is compact.*

Proof of Claim 3.2.14: Since $\operatorname{supp}(f) \subseteq G_\infty$ is compact and G_∞ is discrete, $\operatorname{supp}(f) = \{z_1, \ldots, z_m\}$ for some $m \in \mathbb{N}$. Suppose for $n \in \mathbb{N}$, $F((n, q_n(x)) > 0$. We then have $\|f|_{xK_n}\| > 0$ and so $xK_n \cap \operatorname{supp}(f) \neq \varnothing$. From this we see $q_n(x) = q_n(z_i)$ for some z_i which implies $(n, q_n(x)) \in \operatorname{ray}_n(z_i)$ showing $\operatorname{supp}(F)$ is compact by Proposition 3.2.11.

Claim 3.2.15. *F is continuous on $\mathcal{G}$, and the rate of convergence is uniform for all $x \in G_\infty$. That is, for all $x \in G_\infty$ and every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for every $n \geq N$*

$$f(x)^2 \leq f((n, q_n(x)))^2 \leq f(x)^2 + \varepsilon.$$

Proof of Claim 3.2.15: Let $x \in G_\infty$ and $\varepsilon > 0$ be arbitrary. First note that

$$F((n, q_n(x)))^2 = f_n(xK_n)^2 = \|f|_{xK_n}\|_2^2 = \sum_{z \in K_n} |f(xz)|^2 = |f(x)|^2 + \sum_{z \in K_n \setminus \{e_{G_\infty}\}} |f(xz)|^2 . \tag{3.5}$$

Since $\|f\|_2^2 < \infty$, we have that $\sum_{z \in K_n \setminus \{e_{G_\infty}\}} |f(xz)|^2 \leq \sum_{z \in G_\infty \setminus \{e_{G_\infty}\}} < \infty$. This implies that there is a finite set $E \subseteq G_\infty \setminus \{e_{G_\infty}\}$ such that

$$\sum_{z \in K_n \setminus (E \cup \{e_{G_\infty}\})} |f(xz)|^2 < \varepsilon, \quad \forall x \in G_\infty, \forall n \in \mathbb{N}.$$

Let $E = \{y_1, \ldots, y_m\}$. As $\{K_n\}_{n \in \mathbb{N}}$ is a nested sequence, $\cap_{n \in \mathbb{N}} K_n = \{e_{G_\infty}\}$, and $y_i \neq e_{G_\infty}$ there exist N_i such that for all $n \geq N_i$, $y_i \notin K_n$. Set $N = \max\{N_i : 1 \leq i \leq m\}$. Then for $n \geq N$, we have

$$\sum_{z \in K_n \setminus \{e_{G_\infty}\}} |f(xz)|^2 = \sum_{z \in K_n \setminus (E \cup \{e_{G_\infty}\})} |f(xz)|^2 < \varepsilon. \tag{3.6}$$

Putting Equations 3.5 and 3.6 together, we have

$$f(x)^2 \leq F((n, q_n(x)))^2 \leq f(x)^2 + \varepsilon, \quad \forall x \in G_\infty.$$

Since f was assumed to be pointwise positive, we conclude that for every $n \geq N$ and $x \in G_\infty$, $|f(x) - F((n, q_n(x)))| < \varepsilon$.

By Remark A.2.5, given an arbitrary $f \in \ell^2(G_\infty)$, we may decompose f into $f = f_1 - f_2 + i(f_3 - f_4)$ where $f_i \geq 0$ pointwise for $1 \leq i \leq 4$ by taking the positive and negative parts of the real and imaginary parts of f. For each f_i, construct F_i as in Equation (3.4) and define

$$\begin{aligned} F &: \mathcal{G} \to \mathbb{C} \\ F &= F_1 - F_2 + i(F_3 - F_4) \end{aligned} \tag{3.7}$$

If we assume f is continuous, then f_i is continuous for each i. By Claim 2, we have that each F_i is continuous and so F is continuous.

Taken together with Claim 3.2.14 and Claim 3.2.15, this construction demonstrates that if we take $f \in C_c(G_\infty)$ then $F \in C_c(\mathcal{G})$.

Claim 3.2.16. *The function F as defined for general $f \in \ell^2(G_\infty)$ is a continuous section of the left regular bundle Λ.*

Proof of Claim 3.2.16: First assume that $f \geq 0$.

1. Note that

$$\|F^n\|_2^2 = \sum_{xK_n \in G_n} |f_n(xK_n)|^2 = \sum_{xK_n \in G_n} \|f|_{xK_n}\|_2^2 = \|f\|_2^2 .$$

   Constant functions are always continuous, and so the map $n \mapsto \|F^n\|$ is continuous.

2. We show that F is locally close to the fundamental sections of Λ, $C_c(\mathcal{G})$. Let $\varepsilon > 0$ be given, and choose $h \in C_c(G_\infty)$ with $\|h - f\|_2 < \varepsilon$. Construct H from h via Equation (3.4). By Claims 1 and 2, $H \in C_c(\mathcal{G})$. Then:

$$\|F^n - H^n\|_2^2 = \sum_{xK_n \in G_n} \left| \|h|_{xK_n}\| - \|f|_{xK_n}\| \right|^2 \leq \sum_{xK_n \in G_n} \|h|_{xK_n} - f|_{xK_n}\|^2 = \|h - f\|_2^2 .$$

This implies $\|F^n - H^n\|_2 < \varepsilon$.

We will now show analogs of Items (1) and (2) for general f. To that end, consider a general $f \in \ell^2(G_\infty)$, $f = f_1 - f_2 + i(f_3 - f_4)$, $f_i \geq 0$, and $F = F_1 - F_2 + i(F_3 - F_4)$. From Claim 3.2.15, we have $F_i((n, q_n(x)) \to F_i((\infty, x))$ for all $x \in G_\infty$ so that $F((n, q_n(x)) \to f(x)$.

3. We first claim that $\limsup_{n\to\infty} \|F^n\|_2 \leq \|f\|_2^2$. To show this, for $n \in \mathbb{N}$ let E_n be a complete set of representatives for cosets in G_n, and suppose $|G_n| = |E_n|$. We have

$$\begin{aligned}
\|F^n\|_2^2 &= \sum_{x \in E_n} |F((n, q_n(x)))|^2 \\
&= \sum_{x \in E_n} |F_1((n, q_n(x))) - F_2((x, q_n(x))) + i(F_3((n, q_n(x)))) - F_4((n, q_n(x))))|^2 \\
&= \sum_{x \in E_n} \left[(\|f_1|_{xK_n}\| - \|f_2|_{xK_n}\|)^2 + (\|f_3|_{xK_n}\| - \|f_4|_{xK_n}\|)^2 \right] \\
&\leq \sum_{x \in E_n} \|f_1|_{xK_n}\|^2 + \|f_2|_{xK_n}\|^2 + \|f_3|_{xK_n}\|^2 + \|f_4|_{xK_n}\|^2 \\
&= \sum_{x \in E_n} \|(f_1 - f_2)|_{xK_n}\|^2 + \|(f_3 - f_4)|_{xK_n}\|^2
\end{aligned}$$

where the last equality holds as the supports of f_1 and f_2 are disjoint, as are the supports of f_3 and f_4. We conclude

$$\|F^n\|_2^2 \leq \sum_{x \in E_n} \|((f_1 - f_2) + i(f_3 - f_4))\,|_{xK_n}\|^2 = \|f\|_2^2 .$$

This shows that $\limsup_{n\to\infty} \|F^n\|_2^2 \leq \|f\|_2^2$.

Next, we show that $\liminf_{n\to\infty} \|F^n\|_2^2 \geq \|f\|_2^2$ which will demonstrate that the map which sends $n \to \|F^n\|$ is continuous.

Consider $\|f\|_2^2 = \sum_{x\in G_\infty} |f(x)|^2$. For every $\varepsilon > 0$, there exists a finite set E such that $\sum_{x\notin E} |f(x)|^2 < \varepsilon$. We claim that there is an $N' \in \mathbb{N}$ such that for every $n \geq N'$, $q_n(x) \neq q_n(y)$ for every $x, y \in E$ with $x \neq y$. Note that the set $\{x^{-1}y : x, y \in E, x \neq y\}$ is a finite set not including e_{G_∞}. We also know that the sets $\{K_n\}_{n\in\mathbb{N}}$ are nested and $\cap K_n = \{e_{G_\infty}\}$, so for every $z \neq e_{G_\infty}$ there is an N_z such that for $n \geq N_z$, $z \notin K_n$. Let $N' = \max\{N_{x^{-1}y} : x, y \in F\}$. So for $n \geq N'$, $x^{-1}y \notin K_n$ and $q_n(x) \neq q_n(y)$. Consequently, for this N' and $n \geq N'$, we can assume that the set of representatives of G_n contains E. So, we have for every $n \geq N'$

$$\|f\|_2^2 - \varepsilon \leq \sum_{x\in E} |f(x)|^2 \leq \sum_{\{x_1,\ldots,x_{|G_n|}\}} |f(x_i)|^2 . \tag{3.8}$$

Let $N'' = \max\{N, N'\}$. For every $n \geq N''$, we have

$$\sum_{\{x_1,\ldots,x_{|G_n|}\}} |f(x_i)|^2 \leq \sum_{\{x_1,\ldots,x_{|G_n|}\}} |F((n, q_n(x)))|^2 = \|F^n\|_2^2 . \tag{3.9}$$

Putting together Equations 3.8 and 3.9, we get $\|f\|_2^2 \leq \liminf_{n\to\infty} \|F^n\|_2^2$.

Next, given $\varepsilon > 0$ we show that there is an $H \in C_c(\mathcal{G})$ such that $\|F - H\|_\Delta < \varepsilon$. Choose $h_i \in C_c^+(G_\infty)$ such that $\|f_i - h_i\|_2 < \frac{\varepsilon}{4}$. Define $h = h_1 - h_2 + i(h_3 - h_4)$ so that $h \in C_c(G_\infty)$ and define H from h as we have for F. We then have $\|F_i^n - H_i^n\|_2 < \frac{\varepsilon}{4}$. Then

$$\begin{aligned}\|F^n - H^n\|_2 &= \|(F_1^n - H_1^n) - (F_2^n - H_2^n) + i(F_3^n - H_3^n) - i(F_4^n - H_4^n)\|_2 \\ &\leq \sum_{i=1}^4 \|F_i^n - H_i^n\|_2 < \varepsilon.\end{aligned}$$

We have now demonstrated that for $f \in \ell^2(G_\infty)$, constructing F as in Equation (3.7) yields a continuous section of Λ with $F^\infty = f$. Let $L_{f,g} \in A(G_\infty)$ with $f, g \in \ell^2(G_\infty)$. Construct F, G as in Equation (3.7). Then $\Lambda_{F,G} \in A(\mathcal{G})$ and $R^\infty_{A(\mathcal{G})}(\Lambda_{F,G}) = L_{f,g}$. Proposition 2.3.16 shows that $A(G_\infty)$ is exactly the collection of such $L_{f,g}$, and so $R^\infty_{A(\mathcal{G})}$ is surjective. □

This construction is particularly interesting as it leads to the following corollary.

Corollary 3.2.16.1. *If $\mathcal{G}$ is an HLS groupoid corresponding to the approximated group $(G, \{K_n\}_{n\in\mathbb{N}})$, then $R^n_{A(\mathcal{G})} : A(\mathcal{G}) \to A(G_n)$ is a quotient map.*

Proof. Let $u \in A(G_n)$ be arbitrary. There exists $f, g \in \ell^2(G_n)$ such that $u = L_{f,g}$ and $\|u\| = \|f\|_2 \|g\|_2$ (Proposition 2.3.8). Let F, G be the sections constructed for f, g as in Theorem 3.2.13. We claim that $\|\Lambda_{F,G}\|_{A(\mathcal{G})} = \|f\|_2 \|g\|_2$. Note first that

$$\|\Lambda_{F,G}\|_{A(\mathcal{G})} \leq \|F\|_\Delta \|G\|_\Delta = \left(\sup_{m\in\mathbb{N}_*} \|F|_{G_m}\|_2\right)\left(\sup_{m\in\mathbb{N}_*} \|G|_{G_m}\|_2\right) = \|f\|_2 \|g_2\| .$$

By Theorem 3.1.12,

$$\|f\|_2 \|g\|_2 = \|u\|_{A(G_n)} = \left\|R^n_{A(\mathcal{G})}(\Lambda_{F,G})\right\|_{A(G_n)} \leq \|\Lambda_{F,G}\|_{A(\mathcal{G})} .$$

so $\|\Lambda_{F,G}\|_{A(\mathcal{G})} = \|f\|_2 \|g\|_2$. □

Suppose G is a discrete group with an approximating sequence $\{K_n\}_{n\in\mathbb{N}}$. Let $\mathcal{G}$ be the HLS groupoid associated to this pair, and $\mathfrak{H}$ a $\mathcal{G}$-Hilbert bundle with linear actions $\{\mathcal{T}(x)\}_{x\in\mathcal{G}}$. For $n \in \mathbb{N}_*$, denote by π^n the representation obtained from restricting $\mathfrak{H}$ to the fiber G_n (Lemma 3.1.9) and let q_n be the canonical quotient map $G_\infty \to G_n$. For details of weak containment of representations, see Section A.3.

Proposition 3.2.17. *For the groupoid $\mathcal{G}$ given above, the representation π^∞ is weakly contained in $\{\pi^n \circ q_n : n \in \mathbb{N}\}$.*

Proof. We proceed via Proposition A.3.5 whereby we will demonstrate that for any vector $u \in \mathcal{H}_{\pi^\infty}$, there is a sequence $\{\psi_n\}_{n\in\mathbb{N}}$ such that:

- $\psi_n \to \pi^\infty_{u,u}$ uniformly on all compact subsets $K \subseteq G_\infty$;
- Each ψ_m is a linear combination of elements of $\{(\pi^n \circ q_n)_{\xi_n,\xi_n} : \xi_n \in \mathcal{H}_{\pi^n}, n \in \mathbb{N}\}$.

We first collect a few facts:

1. As $\mathcal{G}_\infty$ is a discrete group, if $K \subseteq G_\infty$ is compact then it is finite.

2. If a sequence of functions converges pointwise on a finite set, then it converges uniformly.

These imply that it will suffice to provide a sequence ψ_n which converges pointwise to $\pi^\infty_{u,u}$. Recall that, as an HLS groupoid, $\mathcal{G}$ consists only of points of the form $(n, q_n(x))$ for some $x \in G_\infty$ and $n \in \mathbb{N}_*$. Fix $z \in G_\infty$. For $\mathfrak{H}_{\boldsymbol{\xi},\boldsymbol{\eta}}$ to be continuous at (∞, z) we must have

$$\langle \pi^n(q_n(z))\boldsymbol{\xi}(n), \boldsymbol{\eta}(n)\rangle \to \langle \pi^\infty(z)\boldsymbol{\xi}(\infty), \eta(\infty)\rangle\,,$$

since $\{(n, q_n(z))\}_{n\in\mathbb{N}_*}$ converges to (∞, z).

With this in mind, let K be a compact set of G_∞ with $z \in K$. Fix $\boldsymbol{\xi} \in \Delta_b(\mathfrak{H})$ with $\boldsymbol{\xi}(\infty) = u$ (whose existence is guaranteed by Proposition 2.4.11). Define $\psi_m(x) := \mathfrak{H}_{\boldsymbol{\xi},\boldsymbol{\xi}}(x) = \langle \pi^m(q_m(x))\boldsymbol{\xi}(x), \boldsymbol{\xi}(x)\rangle$. Then $\psi_n(x) = (\pi^n \circ q_n)_{\boldsymbol{\xi}(n),\boldsymbol{\xi}(n)}$ and $\psi_n(z) \to \pi^\infty_{u,u}(z)$. □

The following definitions are contained in [56]. Let G be a residually finite group. Set $\widetilde{G}$ to be the collection of (equivalence classes of) unitary representations of G and $\widetilde{G}_F$ the set of representations which factor through a finite quotient. A group will *have property FD* if for every $\pi \in \tilde{G}$, π is weakly contained in $\oplus_{\sigma\in\widetilde{G}_F}\sigma$. Via [31], this is equivalent to saying $\widetilde{G}_F$ is dense in $\widetilde{G}$ in the Fell topology.

Note that any representation π of G_n defines a representation on G via composition with the quotient map. In the context of an HLS groupoid, setting $\widetilde{G}_n := \left\{\pi \circ q_n : \pi \in \widetilde{G}_n\right\}$ yields the following theorem when paired with Proposition 3.2.17.

Theorem 3.2.18. *Let $\mathcal{G}$ be an HLS groupoid and $R^\infty_{B(\mathcal{G})} : B(\mathcal{G}) \to B(G_\infty)$ the restriction map of Theorem 3.1.10. If $R^\infty_{B(\mathcal{G})}$ is surjective, then $\cup_{n\in\mathbb{N}}\widetilde{G}_n$ is dense in $\widetilde{G}_\infty$. In particular, G_∞ has property FD.*

Proof. Let $\pi \in \widetilde{G}_\infty$. We will show that π is weakly contained in $\cup\widetilde{G}_n$. As $R^\infty_{B(\mathcal{G})}$ is surjective, every representation π of G_∞ can be seen as π^∞ for some $\mathcal{G}$-Hilbert bundle. To see this, let $\xi, \eta \in \mathcal{H}_\pi$ and consider the coefficient function $\pi_{\xi,\eta} \in B(G_\infty)$. As $R^\infty_{B(\mathcal{G})}$ is surjective, there is some $\mathcal{G}$-Hilbert bundle $\mathfrak{H}$ with linear actions $\{\mathcal{T}(x)\}_{x\in\mathcal{G}}$ and sections $\boldsymbol{\xi}, \boldsymbol{\eta} \in \Delta_b(\mathfrak{H})$ such that $R^u_{B(\mathcal{G})}(\mathfrak{H}_{\boldsymbol{\xi},\boldsymbol{\eta}}) = \pi_{\xi,\eta}$. Let π^∞ denote the representation formed by $\{\mathcal{T}(x)\}_{x\in G_\infty}$ so that $R^u_{B(\mathcal{G})}(\mathfrak{H}_{\boldsymbol{\xi},\boldsymbol{\eta}}) = \pi^\infty_{\boldsymbol{\xi}(\infty),\boldsymbol{\eta}(\infty)} = \pi_{\xi,\eta}$. By Proposition 3.2.17, π^∞ is weakly contained in $\cup_{n\in\mathbb{N}}\widetilde{G}_n$. □

Corollary 3.2.18.1. *There exist HLS groupoids for which $R^{\infty}_{B(\mathcal{G})}$ is not surjective.*

Proof. By Proposition 3.2.17, the collection of finitely factoring representations will weakly contain every $\pi \in \widetilde{G}_{\infty}$. If G_{∞} does not have Property FD, there is at least one representation π not weakly contained in $\widetilde{G}_F$, and so $R^{\infty}_{B(\mathcal{G})}$ is not surjective. □

For a specific example of an HLS groupoid for which $R^{\infty}_{B(\mathcal{G})}$ is not surjective, take $G_{\infty} = \mathbb{F}_2$ and the approximating sequence $\{K_n\}$ the one developed in Lemma 3.2.8. If $n < \infty$ we have $R^{n}_{B(\mathcal{G})} : B(\mathcal{G}) \to B(G_n)$ is surjective, as $\mathcal{G} \setminus G_{\infty}$ is a group bundle with discrete unit space.

3.2.4 More on HLS Groupoids

Conjecture 3.2.19. *Proposition 3.2.11 gives a complete description of the compact sets of an HLS groupoid. We suspect that this description can be used to provide a description of the elements of $A(\mathcal{G})$.*

If $f \in C_c(\mathcal{G})$ with $f((x,\infty)) \neq 0$ has support of a subset of a ray, it must have support the entire ray, sans finitely many points. By the definition of limits, after some $N \in \mathbb{N}$ we have $|f(n, \pi_n(x))| > 0$ for $n \geq N$.

Moving towards a description of $A(\mathcal{G})$, note that one Haar system on an HLS groupoid will be the counting measures over each group in the bundle. Recall that for each $n \in \mathbb{N}$ the group G_n is finite. With this, we can give a description of the coefficients of the left regular bundle Λ. For $\boldsymbol{\xi}, \boldsymbol{\eta} \in C_c(\mathcal{G})$ and $x \in \mathcal{G}$

$$\Lambda_{\boldsymbol{\xi},\boldsymbol{\eta}}(x) = \left\langle L_x \boldsymbol{\xi}^{s(x)}, \boldsymbol{\eta}^{r(x)} \right\rangle = \sum_{z \in \mathcal{G}^{r(x)}} \boldsymbol{\xi}^{r(x)}(x^{-1}z)\overline{\boldsymbol{\eta}^{r(x)}(z)} = \left(\overline{\boldsymbol{\eta}^{r(x)}} * \widetilde{\boldsymbol{\xi}^{r(x)}} \right)(x) \tag{3.10}$$

where $*$ denotes convolution of functions, $\check{\boldsymbol{\xi}}(x) = \boldsymbol{\xi}(x^{-1})$, and we use the fact that $s(x)$ and $r(x)$ coincide for group bundles. With this, we show that the two definitions of the Fourier algebra of groupoids coincide in this case.

Proposition 3.2.20. *If $\mathcal{G}$ is an HLS groupoid, then $A(\mathcal{G}) = \overline{B(\mathcal{G}) \cap C_c(\mathcal{G})}^{\|\cdot\|_{B(\mathcal{G})}}$.*

Proof. Let $\mathcal{G}$ be the HLS groupoid associated to the approximated group $(G, \{K_n\}_{n\in\mathbb{N}})$. We first show that $A_c(\mathcal{G}) = C_c(\mathcal{G})$. By Remark 2.4.25 we have that $A_c(\mathcal{G}) \subseteq C_c(\mathcal{G})$. For the reverse containment, let $x \in G_\infty$, $n \in \mathbb{N}_*$, and $f \in C_c(\mathcal{G})$ such that $\text{supp}(f) \subseteq \mathbb{1}_{\text{ray}_n(x)}$. Let $g = \mathbb{1}_{\text{ray}_n(e_{G_\infty})}$. Then, for $m \in \mathbb{N}_*$ and $z \in G_\infty$ we have

$$\begin{aligned}\Lambda_{g,f}((m, q_m(z))) &= \sum_{(m',q_{m'}(y))\in\mathcal{G}} g((m, q_m(z^{-1}))(m', q_{m'}(y)))f((m', q_{m'}(y))) \\ &= \sum_{y\in G_m} g((m, q_m(z^{-1}y)))f((m, q_m(y))).\end{aligned}$$

This sum will be zero if $m < n$. Otherwise, by the definition of g, for the sum to be nonzero we must have $q_m(y) = q_m(z)$. Thus this sum will be $f((m, q_m(z)))$ exactly when $(m, q_m(z)) \in \text{supp}(f)$ and 0 otherwise, i.e.

$$\Lambda_{g,f}((m, q_m(z))) = \begin{cases} 0 & m < n \\ f((m, q_m(z))) & m \geq n \end{cases}$$

so $\Lambda_{g,f} = f$ and $f \in A_{cf}(\mathcal{G})$. Any $f \in C_c(\mathcal{G})$ will have its support contained in finitely many rays, and so $C_c(\mathcal{G}) \subseteq A_c(\mathcal{G})$. We conclude

$$\overline{B(\mathcal{G}) \cap C_c(\mathcal{G})}^{\|\cdot\|_{B(\mathcal{G})}} = \overline{B(\mathcal{G}) \cap A_c(\mathcal{G})}^{\|\cdot\|_{B(\mathcal{G})}} = \overline{A_c(\mathcal{G})}^{\|\cdot\|_{B(\mathcal{G})}} = A(\mathcal{G}).$$

□

We may also bound the Fourier norm of those elements which are compactly supported.

Proposition 3.2.21. *If $\mathcal{G}$ is an HLS groupoid and $f \in C_c(\mathcal{G})$ is supported on n disjoint rays, then $\|f\|_{A(\mathcal{G})} \leq n \|f\|_\infty$.*

Proof. We first calculate $\left\|\mathbb{1}_{\text{ray}_n(x)}\right\|_\Delta$:

$$\sup_{m\in\mathbb{N}_*} \left\|\mathbb{1}_{\text{ray}_n(x)}|_{G_m}\right\|_2 = \sup_{m\in\mathbb{N}_*} \left\|\mathbb{1}_{\{(m,q_m(x))\}}\right\|_2 = 1.$$

Given this,

$$\left\|\mathbb{1}_{\text{ray}_n(x)}\right\|_{A(\mathcal{G})} \leq \left\|\Lambda_{\mathbb{1}_{\text{ray}_n(e_{G_\infty})},\mathbb{1}_{\text{ray}_n(x)}}\right\|_{A(\mathcal{G})} \leq \left\|\mathbb{1}_{\text{ray}_n(e_{G_\infty})}\right\|_\Delta \left\|\mathbb{1}_{\text{ray}_n(x)}\right\|_\Delta = 1.$$

In $A(\mathcal{G})$, $\mathbb{1}^2_{\mathrm{ray}_n(x)} = \mathbb{1}_{\mathrm{ray}_n(x)}$ and so $1 \leq \left\|\mathbb{1}_{\mathrm{ray}_n(x)}\right\|_{A(\mathcal{G})}$ so $\left\|\mathbb{1}_{\mathrm{ray}_n(x)}\right\|_{A(\mathcal{G})} = 1$.

Let $f \in C_c(\mathcal{G})$ be supported on disjoint $\mathrm{ray}_{m_i}(x_i)$, $i \in [n]$. We have that

$$\|f\|_{A(\mathcal{G})} = \left\|\sum_{i\in[n]} f_i\right\|_{A(\mathcal{G})}$$

where $f_i = f \cdot \mathbb{1}_{\mathrm{ray}_{m_i}(x_i)} \in A(\mathcal{G})$ so

$$\begin{aligned}\|f\|_{A(\mathcal{G})} &\leq \sum_{i\in[n]} \|f_i\|_{A(\mathcal{G})} = \sum_{i\in[n]} \left\|\Lambda_{\mathbb{1}_{\mathrm{ray}_{m_i}(e_{G_\infty})}, f_i}\right\| \leq \sum_{i\in[n]} \|f_i\|_\Delta \left\|\mathbb{1}_{\mathrm{ray}_{m_i}(e_{G_\infty})}\right\|_\Delta = \sum_{i\in[n]} \|f_i\|_\Delta \\ &\leq n \|f\|_\infty .\end{aligned}$$

The last inequality is due to the fact that

$$\|f_i\|_\Delta = \sup_{m\in\mathbb{N}_*} \|f_i|_{G_m}\|_2 = \sup_{m\in\mathbb{N}_*} |f(m, q_m(x_i))| \leq \|f\|_\infty .$$

□

Chapter 4

APPLICATIONS: BANACH ALGEBRAIC PROPERTIES OF THE FOURIER AND FOURIER-STIELTJES ALGEBRAS

4.1 Hereditary Properties

A property P of a Banach Algebra A will be called *hereditary* if it is preserved under continuous homomorphisms with dense range, i.e. if A has property P, and there is a continuous algebra homomorphism $\theta : A \to B$ such that $\overline{\theta(A)} = B$ then B has property P. We showed in Theorem 3.1.12 that $R^u_{A(\mathcal{G})} : A(\mathcal{G}) \to A(\mathcal{G}^u_u)$ is a continuous, surjective algebra homomorphism for any $u \in \mathcal{G}^{(0)}$ when $\mathcal{G}$ is étale. The following subsections apply this to obtain a number of results regarding Banach algebraic properties of $A(\mathcal{G})$ and $B(\mathcal{G})$.

4.1.1 Bounded Approximate Identities

Recall that a Banach algebra A does not necessarily have a multiplicative identity, that is, an element e such that $e * x = x * e = x$ for every $x \in A$.

Example 4.1.1. Let X be a locally compact, Hausdorff space and let $A := C_0(X)$, the set of complex valued continuous functions on X which vanish at infinity. Equip A with pointwise multiplication and addition of functions along with the norm $\|f\|_{\sup} = \sup_{x \in X} |f(x)|$. It can be easily deduced that A is in fact a Banach algebra. It is easy to see that A has a multiplicative identity if and only if X is compact.

If X is compact, then the constant function 1 belongs to A and serves as a multiplicative identity. Conversely, suppose there is some function $e(x) \in A$ such that $e(x)f(x) = f(x)$ for every $f \in A$. Fix an $x_0 \in X$. As X is locally compact, we have that x_0 is contained in some compact set K. As K is compact and Hausdorff, it is normal, and we may use Urysohn's lemma (Lemma A.1.6) to find a continuous

f such that $f(x_0) = 1$ and f vanishes on the boundary of K. We may also extend f to a continuous function $\overline{f}$ defined on all of X which vanishes outside of K. Then $\overline{f}(x_0)e(x_0) = 1 = \overline{f}(x_0)$ so $e(x_0) = 1$. As x_0 was arbitrary, e must be the constant function 1 from which we can easily deduce that X must be compact.

Example 4.1.2. If a locally compact group G is discrete, then its group algebra $L^1(G)$ will possess a multiplicative identity (the identity being the point mass at the group identity). If G is non-discrete, then $L^1(G)$ fails to have an identity.

When a Banach algebra fails to have a multiplicative identity, it may have a weaker construct known as a bounded approximate identity.

Recall Definition 2.2.5, that an approximate identity of a normed algebra A is a net of elements $\{\psi_\lambda\}_{\lambda \in I}$ such that $\|\psi_\lambda \cdot x - x\| \to_\lambda 0$ and $\|x \cdot \psi_\lambda - x\| \to_\lambda 0$ for every $x \in A$. If $\sup\{\|\psi_\lambda\| : \lambda \in I\} < \infty$ then the net is a bounded approximate identity.

Kaniuth and Lau present the construction for the bounded approximate identity for $L^1(G)$ in a remark shortly after Proposition 1.3.4 of [48], which we include here.

Example 4.1.3. Let G be a locally compact and Hausdorff group with Haar measure μ. Let $\mathcal{U}$ be the collection of all open sets containing e_G. For each $U \in \mathcal{U}$, use Lemma A.1.6 to choose an $f_U \in C_c^+(U)$ with $\int_G f d\mu = 1$. Clearly the net $\{f_U\}_{U \in \mathcal{U}} \subseteq L^1(G)$ is bounded. Furthermore, for $g \in L^1(G)$, as $U \to \{e_G\}$ we have $\|f * g - g\| \to 0$ as well as $\|g * f - g\| \to 0$.

Recall from Proposition 2.2.13 that $A(G)$ for a locally compact group G has a bounded approximate identity if and only if G is amenable (as in Definition 2.2.12). We will shortly present the converse direction of the proof (and construct the bounded approximate identity). To do this, we will need two facts. The first fact is that if G is amenable then the trivial representation is weakly contained in the left regular (Proposition 2.2.13). The second is given by the following proposition.

Proposition 4.1.4. *[48, Proposition 1.8.6] Let G be a locally compact group, and suppose $\{u_\beta\}_{\beta\in I} \subseteq P(G)$[1] is a net which converges uniformly on compact subsets of G. Then*

$$\lim_\beta \|(u_\beta - u)v\|_{A(G)} = 0$$

for every $v \in A(G)$.

Example 4.1.5. [48, Theorem 2.7.2] Suppose G is a locally compact and Hausdorff group. Further suppose that G is amenable. We will construct the bounded approximate identity for $A(G)$. As G is amenable, the trivial representation is weakly contained in L. Thus, given any compact $K \subseteq G$ and $\varepsilon > 0$, there is a function $u_{K,\varepsilon} \in P(G)$ with $u_{K,\varepsilon} = L_{f,f}$ for some $f \in L^2(G)$ and $|u_{K,\varepsilon}(x) - 1| \leq \varepsilon$ for every $x \in K$. By the density of $C_c(G)$ in $L^p(G)$, we can freely assume that each $u_{K,\varepsilon}$ has compact support.

Next, impose an ordering on the set of pairs (K, ε) by $(K, \varepsilon) \geq (K', \varepsilon')$ if $K \subseteq K'$ and $\varepsilon \leq \varepsilon'$. This forms a directed set $I = \{(K, \varepsilon) : K \subseteq G, K \text{ compact}, \varepsilon > 0\}$. With this definition, the net $\{u_\alpha\}_{\alpha\in I}$ satisfies the conditions of Proposition 4.1.4 with $u = \mathbb{1}_G$, and therefore forms an approximate identity for $A(G)$. Since $u_\alpha(e_G) \to 1$, if we substitute each u_α with $u_\alpha(e_G)^{-1}u_\alpha$, we have $1 = u_\alpha(e_G) = \|u_\alpha\|_{A(G)}$, so the net is actually a bounded approximate identity.

We will now demonstrate that existence of a bounded approximate identity is a hereditary property.

Proposition 4.1.6. *Let A, B be Banach algebras and let A have a bounded approximate identity. Suppose $\varphi : A \to B$ is a continuous algebra homomorphism with dense range. Then B has a bounded approximate identity.*

Proof. Let $\{\psi_\lambda\}_{\lambda\in I}$ be a bounded approximate identity in A. Since φ is continuous, $\{\varphi(\psi_\lambda)\}_{\lambda\in I}$ is bounded in B (say by M). First, considering $b = \varphi(a)$ for some $a \in A$, we have

$$\lim_\lambda \|\varphi(\psi_\lambda) \cdot b - b\|_B = \lim_\lambda \|\varphi(\psi_\lambda \cdot a) - \varphi(a)\|_B \leq \lim_\lambda \|\varphi\| \, \|\psi_\lambda \cdot a - a\|_A = 0.$$

[1] *See Section 2.3.2 for the definition of functions of positive type.*

Next, let b be an arbitrary element of B and fix $\varepsilon > 0$. Since $\varphi(A)$ is dense in B we may find an element $a \in A$ with $\|\varphi(a) - b\|_B < \frac{\varepsilon}{2(M+1)}$. As $\{\psi_\lambda\}_{\lambda \in I}$ is a bounded approximate identity in A, there is a λ_0 such that for $\lambda \geq \lambda_0$ $\|\psi_\lambda \cdot a - a\| < \frac{\varepsilon}{2\|\varphi\|}$. Then, for $\lambda \geq \lambda_0$:

$$\begin{aligned}
\|\varphi(\psi_\lambda) \cdot b - b\|_B &= \|\varphi(\psi_\lambda) \cdot (b - \varphi(a)) + \varphi(\psi_\lambda) \cdot \varphi(a) - \varphi(a) + \varphi(a) - b\|_B \\
&\leq \|\varphi(\psi_\lambda) \cdot (b - \varphi(a))\|_B + \|\varphi(\psi_\lambda) \cdot \varphi(a) - \varphi(a)\|_B + \|\varphi(a) - b\|_B \\
&\leq M \|b - \varphi(a)\|_B + \|\varphi\| \, \|\psi_\lambda \cdot a - a\|_A + \|\varphi(a) - b\|_B \\
&= (M+1) \|\varphi(a) - b\|_B + \|\varphi\| \, \|\psi_\lambda \cdot a - a\|_A < \varepsilon.
\end{aligned}$$

□

Given this, we have the following corollary to Theorem 3.1.12.

Corollary 4.1.6.1. *Let $\mathcal{G}$ be a locally compact, Hausdorff étale groupoid. If $\mathcal{G}_u^u$ is non-amenable for some $u \in \mathcal{G}^{(0)}$, then $A(\mathcal{G})$ does not contain a bounded approximate identity.*

Proof. Leptin's result described in Proposition 2.2.13 (4) shows the equivalence between amenability of a group and existence of a bounded approximate identity in the Fourier Algebra. As existence of a bounded approximate identity is hereditary by Proposition 4.1.6, and $R^u_{A(\mathcal{G})} : A(\mathcal{G}) \to A(\mathcal{G}_u^u)$ is a surjective algebra homomorphism, non-existence in $A(\mathcal{G}_u^u)$ implies non-existence in $A(\mathcal{G})$. □

Question 4.1.7. *Let $\mathcal{G}$ be an HLS groupoid for the approximated group $(G, \{K_n\}_{n \in \mathbb{N}})$. Suppose G is amenable. Can then the construction given in Example 4.1.5 be adapted to construct a bounded approximate identity for $A(\mathcal{G})$?*

4.1.2 Amenability

Recall that a group is amenable if there is a left invariant mean on $L^\infty(G)$ (see Definition 2.2.12), and by Proposition 2.2.13 a group is amenable if and only if its group algebra $L^1(G)$ is amenable as a Banach algebra.

The concept, definition, and study of Banach algebra amenability originated from Johnson in [45] and was originally phrased in terms of the cohomology of $L^1(G)$, though we present the definitions without that lens. To begin, a *module* refers to the generalization of the notion of a vector space wherein the scalar field is replaced with a ring which acts on the underlying space. The module action is required to satisfy the same axioms scalar multiplication does.

Definition 4.1.8. [6, Section 1.2] Let M be an abelian group and R a ring. M is defined to be a *module over* R each $r \in R$ defines a function $r : M \to M$, $x \mapsto r \cdot x$ such that

1. $(r+s) \cdot x = r \cdot x + s \cdot x$ for $r, s \in R$, $x \in M$;
2. $r \cdot (x+y) = r \cdot x + r \cdot y$ for $r \in R$, $x, y \in M$;
3. $(rs) \cdot x = r \cdot (s \cdot x)$ for $r, s \in R$, $x \in M$;
4. $1_R \cdot x = x$ for multiplicative identity $1_R \in R$, $x \in M$.

If R, S are rings, we may also form (R, S) *bimodules*, which consist of a vector space with ring actions from R on the left and S on the right (obeying the analogs of the four axioms above). In the cases of a single ring, the module is called left or right (depending on the orientation of the action). Let A be a Banach algebra. A Banach space E which is also a left A-module is called a *left Banach A-module* ([59, Definition 2.1.1]) if there is a constant C such that $\|a \cdot x\|_E \leq C \|a\|_A \|x\|_E$ for all $a \in A$ and $x \in E$. A right Banach A-module is defined similarly, and a general Banach A-module is both a left and right Banach A-module.

Definition 4.1.9. Let A be a Banach algebra and E a Banach A-module. A linear map $D : A \to E$ is called a *derivation* if

$$D(ab) = a \cdot D(b) + (D(a)) \cdot b$$

for every $a, b \in A$. If D is also bounded as a linear operator it is called a *bounded derivation*. For a fixed $x \in E$, define the derivation $\mathrm{ad}_x(a) := a \cdot x - x \cdot a$. Derivations of this form are called *inner*.

[59, Exercise 2.1.1] provides further details of Banach A-modules and the linear duals of these spaces.

Proposition 4.1.10. *[59, Exercise 2.1.1] Let A be a Banach algebra. Then the following is true:*

1. *If E is a left Banach A-module, then E^* is a right Banach A-module via the action*
$$(\varphi \cdot a)(x) = \varphi(a \cdot x).$$
2. *If E is a right Banach A-module, then E^* is a left Banach A-module via the action*
$$(a \cdot \varphi)(x) = \varphi(x \cdot a).$$
3. *If E is a Banach A-module then E^* is as well with module actions given above.*

Remark 4.1.11. These may be generalized to the case where we are given two Banach algebras A, B and a continuous homomorphism $\varphi : A \to B$. If E is a Banach B-bimodule, then E is a Banach A-bimodule. For $x \in E$ and $a \in A$, define $a \cdot x := \varphi(a) \cdot x$ and $x \cdot a := x \cdot \varphi(a)$.

The dual spaces of Banach A-modules as in Proposition 4.1.10 are known as *dual Banach A-modules*. Given this background, we may now define amenable Banach algebras. The definition as originally phrased is Definition 2.1.9 of [59].

Definition 4.1.12. A Banach algebra A is *amenable* if every bounded derivation from A into a dual Banach module is inner.

Amenable Banach algebras always possess a bounded approximate identity ([59, Proposition 2.2.1]). Amenability of Banach algebras was further generalized in [9]. A

Banach algebra A is said to be *weakly amenable* if every bounded derivation $D : A \to A^*$ is inner.

Lemma 4.1.13. *Let A be a commutative Banach algebra ($ab = ba$ for every pair of elements a, b). Then A is weakly amenable precisely when every bounded derivation (Definition 4.1.9) from $A \to A^*$ is identically 0.*

Proof. Suppose A is weakly amenable and let $D : A \to A^*$ be a bounded derivation. By definition, D is inner, so there is some $f \in A^*$ such that $D(a) = a \cdot f - f \cdot a$ for $a \in A$. For $a' \in A$, we then have

$$D(a') = (a \cdot f - f \cdot a)(a') = f(a'a - aa') = 0$$

since A is commutative. Conversely, if every bounded derivation from $A \to A^*$ is 0, then $D(a) = a \cdot 0 - 0 \cdot a$ where 0 is the 0 functional, and so D is inner. □

The study of the weak amenability of the Fourier algebra was initiated by [33] and [35], continued by [22], [23], and concluded in [55].

In [35, Theorem 3.3], the authors show that if the connected component of the identity of a locally compact group is abelian, then the group's Fourier algebra is weakly amenable. In [22], Choi and Ghandehari use the structure theory of Lie groups to show that if G is a semisimple, connected Lie group then $A(G)$ fails to be weakly amenable. They achieve this in part due to restrictions of homomorphisms to subgroups, which further motivates our interest in restriction style results. In [23], the same authors verify that the Fourier algebra for 1-connected, non-abelian nilpotent Lie groups also fails to be weakly amenable. Following this, Lee, Ludwig, Samei, and Spronk show in [53] that $A(\mathcal{G})$ for a connected Lie group G is weakly amenable only if G is abelian. They achieve this by studying the anti-diagonal in $G \times G$, and ultimately conclude that when G is a Lie group $A(\mathcal{G})$ is weakly amenable if and only if the connected component of the identity is abelian. This line of study concluded with [55] where Losert, using the approach of Choi and Ghandehari, shows that $A(G)$ is weakly amenable if and

only if the connected component of the identity is abelian for any connected locally compact group, generalizing the result [53] past Lie groups.

Amenability of Banach algebras possesses the same hereditary feature as possession of a bounded approximate identity: if A is an amenable Banach algebra, and $\Upsilon : A \to B$ is a continuous algebra homomorphism with dense range, then B is also amenable. This is shown in [59, Proposition 2.3.1]. Weak amenability does not possess the same property in full generality, but does when the algebras are commutative.

Proposition 4.1.14. *Let A, B be commutative Banach algebras with A weakly amenable. Suppose $\varphi : A \to B$ is a continuous algebra homomorphism with dense range. Then B is weakly amenable.*

Proof. Let D be a bounded derivation of B into B^*. The map $\Upsilon := \varphi^* \circ D \circ \varphi$ is a bounded derivation $A \to A^*$. To show this, we must demonstrate that $\Upsilon(xy) = x \cdot \Upsilon(y) + (\Upsilon(x)) \cdot y$ for every pair $x, y \in A$. As $\Upsilon : A \to A^*$, we know that for an $a \in A$

$$(x \cdot \Upsilon(y))(a) = \Upsilon(y)(ax) \quad ((\Upsilon(x) \cdot y))(a) = \Upsilon(x)(ya)$$

by Proposition 4.1.10. Then, for $a \in A$

$$\begin{aligned}
\Upsilon(xy)(a) &= (\varphi^* \circ D \circ \varphi)(xy)(a) = D(\varphi(xy))(\varphi(a)) = D(\varphi(x)\varphi(y))(\varphi(a)) \\
&= (\varphi(x) \cdot D(\varphi(y)) + (D(\varphi(x))) \cdot \varphi(y))\,(\varphi(a)) \\
&= D(\varphi(y))(\varphi(ax)) + D(\varphi(x))(\varphi(ya)) \\
&= \Upsilon(y)(ax) + ((\Upsilon(x) \cdot y))(a) = \Upsilon(x)(ya) = (x \cdot \Upsilon(y) + (\Upsilon(x)) \cdot y)(a)
\end{aligned}$$

showing that $\Upsilon(xy) = x \cdot \Upsilon(y) + (\Upsilon(x)) \cdot y$. The map Υ is also the composition of bounded linear maps and so it is a bounded linear map itself, so $\Upsilon : A \to A^*$ is indeed a bounded derivation.

As A is weakly amenable, by Lemma 4.1.13 Υ is identically 0. We will show that this implies that D is identically 0, which, by the same lemma suffices to show that B is weakly amenable. We first consider how D acts on elements in the range of φ. We see

$$D(\varphi(a))(\varphi(a')) = (\varphi^* \circ D \circ \varphi)(a)(a') = (\Upsilon)(a)(a') = 0.$$

As the range of φ is dense, given $b \in B$ and $\varepsilon > 0$, there is an $a' \in A$ such that $\|\varphi(a') - b\|_B < \frac{\varepsilon}{\|D(\varphi(a))\|}$. Therefore:

$$|D(\varphi(a))(b) - D(\varphi(a))(\varphi(a'))| = |D(\varphi(a))(b - \varphi(a'))| \leq \|D(\varphi(a))\| \, \|b - \varphi(a')\|_B < \varepsilon$$

and so $D(\varphi(a)) = 0$ for every $a \in A$. Since D is continuous and 0 on a dense subset, D is 0 everywhere. □

Given this, we also have a number of corollaries to Theorem 3.1.12.

Corollary 4.1.14.1. *Let $\mathcal{G}$ be a locally compact, Hausdorff étale groupoid. If $A(\mathcal{G}_u^u)$ is not (weakly) amenable for some $u \in \mathcal{G}^{(0)}$, then $A(\mathcal{G})$ is not (weakly) amenable.*

Proof. That amenability is a hereditary property can be seen in [59, Proposition 2.3.1]. That weak amenability is a hereditary property of commutative Banach algebras was shown in Proposition 4.1.14. Thus $A(\mathcal{G}_u^u)$ failing to be (weakly) amenable implies the failure of $A(\mathcal{G})$ to be (weakly) amenable. □

Corollary 4.1.14.2. *Let $\mathcal{G}$ be a locally compact, Hausdorff étale groupoid. Then, if $\mathcal{G}_u^u$ contains a closed subgroup H such that $A(H)$ is not weakly amenable, then $A(\mathcal{G})$ is not weakly amenable.*

Proof. [35, Lemma 3.1] shows that if $\mathcal{G}_u^u$ is a locally compact group with a closed subgroup H, then the weak amenability of $A(\mathcal{G}_u^u)$ implies the weak amenability of $A(H)$. If $A(H)$ is not weakly amenable, neither is $A(\mathcal{G}_u^u)$ and by Corollary 4.1.14.1 $A(\mathcal{G})$ is not weakly amenable. □

Definition 4.1.15. A Banach algebra A is said to be *contractible* if every bounded derivation $D : A \to E$ where E is a Banach A-bimodule is inner.

The following lemma shows that contractibility is a hereditary property. A proof can also be found in [27, Proposition 2.8.64].

Lemma 4.1.16. *Let A and B be Banach algebras with $\varphi : A \to B$ a continuous homomorphism with dense range. If A is contractible, then B is contractible.*

Proof. Suppose E is a Banach B-bimodule, and let $D : B \to E$ be a bounded derivation. Then $D \circ \varphi : A \to E$ is a bounded derivation with module actions given by Remark 4.1.11 (i.e., $a \cdot x = \varphi(a) \cdot x$ and $x \cdot a = x \cdot \varphi(a)$). Let $\Upsilon := D \circ \varphi$. Since A is contractible, Υ is inner, and so there is some $x \in E$ such that $(\Upsilon)(a) = a \cdot x - x \cdot a = \varphi(a) \cdot x - x \cdot \varphi(a)$. If $\|D\| = 0$, we are done as D is inner. Otherwise, for $b \in B$, there is an $a \in A$ such that $\|\varphi(a) - b\|_B < \frac{\varepsilon}{\|D\|}$. Then

$$\|D(\varphi(a)) - D(b)\| \leq \|D\| \, \|\varphi(a) - b\|_B < \varepsilon$$

and so D is inner. □

Corollary 4.1.16.1. *Let $\mathcal{G}$ be a locally compact, Hausdorff étale groupoid. Then, if $\mathcal{G}_u^u$ is not contractible for some $u \in \mathcal{G}^{(0)}$, then $A(\mathcal{G})$ is not contractible.*

Proof. Lemma 4.1.16 shows that contractability of Banach algebras is a hereditary property, and so non-contractability of $A(\mathcal{G}_u^u)$ implies the non-contractability of $A(\mathcal{G})$. □

4.2 Box Sum Decomposition

Let $\mathcal{G}$ be the group bundle formed from the collection of locally compact, Hausdorff groups $\{G_t\}_{t \in T}$ (see Example 2.1.10). Recall that we may identify $\mathcal{G}^{(0)}$ with T as $r((t, g)) = (t, e_{G_t})$ and for $x \in \mathcal{G}$ we use $x \in G_t$ interchangably with $x = (t, g_t)$ for some $g_t \in G_t$. Equip $\mathcal{G}$ with a locally compact and Hausdorff topology such that $\mathcal{G}^{(0)}$ is discrete. For each $t \in T$, let $u_t \in B(G_t)$. Using these, define the function

$$\boxplus_{t \in T} u_t : \mathcal{G} \to \mathbb{C} \tag{4.1}$$

which sends an element $x \in \mathcal{G}$ to $u_t(x)$ if $x \in G_t$. Given this, we will define

$$\ell^\infty - \boxplus_{t \in T} B(G_t) := \left\{ \boxplus_{t \in T} u_t : \sup_{t \in T} \|u_t\|_{B(\mathcal{G})} < \infty \right\}$$

with componentwise scalar multiplication, addition, and multiplication. We equip $\ell^\infty - \boxplus_{t \in T} B(G_t) = B(\mathcal{G})$ with the norm $\|\boxplus_{t \in T} u_t\| := \sup_{t \in T} \|u_t\|_{B(\mathcal{G})}$, and in the next proposition, we will show that $\ell^\infty - \boxplus_{t \in T} B(G_t) = B(\mathcal{G})$ as sets.

Proposition 4.2.1. *Let $\mathcal{G}$ be a group bundle formed from the collection of locally compact, Hausdorff groups $\{G_t\}_{t\in T}$. Suppose $\mathcal{G}$ is locally compact and Hausdorff, and $\mathcal{G}^{(0)}$ is discrete. Then $B(\mathcal{G}) = \boxplus_{t\in T} B(G_t)$ as sets.*

Proof. Choose $\omega \in B(\mathcal{G})$. Then $\omega = \mathfrak{H}_{\boldsymbol{\xi},\boldsymbol{\eta}}$ for some $\mathcal{G}$-Hilbert bundle $\mathfrak{H}$ with linear action $\{\mathcal{T}(x)\}_{x\in\mathcal{G}}$ and $\boldsymbol{\xi},\boldsymbol{\eta} \in \Delta_b(\mathfrak{H})$. Note that $\omega(x) = \langle \mathcal{T}(x)\boldsymbol{\xi}(s(t)), \boldsymbol{\eta}(r(t))\rangle$. By Lemma 3.1.9, for each $t \in T$ the collection of $\{\mathcal{T}(x)\}_{x\in\mathcal{G}_t^t}$ is a unitary representation of $\mathcal{G}_t^t = \mathcal{G}^t = G_t$. Let π^t be this representation, and set ξ_t, η_t to be $\boldsymbol{\xi}(s(t))$ and $\boldsymbol{\eta}(r(t))$ respectively. Then, for $x \in G_t$,

$$\mathfrak{H}_{\boldsymbol{\xi},\boldsymbol{\eta}}(x) = \langle \mathcal{T}(x)\boldsymbol{\xi}(t), \boldsymbol{\eta}(t)\rangle = \pi^t_{\xi_t,\eta_t}(x)$$

so $\mathfrak{H}_{\boldsymbol{\xi},\boldsymbol{\eta}} = \boxplus_{t\in T}\pi^t_{\xi_t,\eta_t}$. To complete the proof, note that

$$\sup_{t\in T}\left\|\pi^t_{\xi_t,\eta_t}\right\|_{B(G_t)} \leq \sup_{t\in T}\|\xi_t\|_{\mathcal{H}_{\pi^t}}\|\eta_t\|_{\mathcal{H}_{\pi^t}} \leq \sup_{t\in T}\|\boldsymbol{\xi}\|_\Delta\|\boldsymbol{\eta}\|_\Delta < \infty$$

and so $B(\mathcal{G}) \subseteq \ell^\infty - \boxplus_{t\in T} B(G_i)$.

Conversely, choose $\boxplus_{t\in T} u_t \in \ell^\infty - \boxplus_{t\in T} B(G_i)$. For each u_t, choose $\pi^t_{\xi_t,\eta_t} \in B(G_t)$ so $u_t = \pi^t_{\xi_t,\eta_t}$ and $\|u\|_{B(G_t)} = \|\xi_t\|\,\|\eta_t\|$[2]. Without loss of generality, we may assume $\|\eta_t\| = 1$ for each $t \in T$. For $t \in T$, set $\mathfrak{H}_t := \mathcal{H}_{\pi^t}$, $\boldsymbol{\xi}(t) = \xi_t$ and $\boldsymbol{\eta}(t) = \eta_t$. We claim $\boldsymbol{\xi},\boldsymbol{\eta} \in \Delta_b(\mathfrak{H})$. The sections are continuous as $\mathcal{G}^{(0)}$ was assumed discrete. As $\|\eta_t\| = 1$ for every t, $\|\boldsymbol{\eta}\|_\Delta = 1$. We also have

$$\|\boldsymbol{\xi}\|_\Delta = \sup_{t\in T}\|\xi_t\| = \sup_{t\in T}\|\xi_t\|\,\|\eta_t\| = \sup_{t\in T}\|u_t\| < \infty.$$

Define $\mathcal{T}((t,g)) = \pi^t(g)$. The map $(t,g) \mapsto \mathcal{T}(g)$ is clearly a groupoid homomorphism, as elements are only composable when they belong to the same G_t and each π^t is a group homomorphism. To see that $\mathfrak{H}_{\boldsymbol{\xi},\boldsymbol{\eta}}$ is continuous, let $\varepsilon > 0$ and suppose that $\{(t_\alpha, g_\alpha)\}_{\alpha\in I}$ is a net converging to (t,g) in $\mathcal{G}$. As $\mathcal{G}^{(0)}$ is homeomorphic to T via the map $(s, e_{G_s}) \mapsto s$, we have that $t_\alpha \to t$ in T. Then, since r is a continuous map and $\mathcal{G}^{(0)}$ is discrete, there are α_1, α_2 respectively such that if $\alpha \geq \max\{\alpha_1, \alpha_2\}$, we have

[2] Such a representation is guaranteed by Proposition 2.3.8.

$\boldsymbol{\xi}(t_\alpha) = \boldsymbol{\xi}(t)$ and $\boldsymbol{\eta}(t_\alpha) = \boldsymbol{\eta}(t)$ as $t_\alpha \to t$ in a discrete space if and only if t_α is eventually constant. Then, for $\alpha \geq \max\{\alpha_1, \alpha_2\}$:

$$\mathfrak{H}_{\boldsymbol{\xi},\boldsymbol{\eta}}((t_\alpha, g_\alpha)) = \langle \mathcal{T}((t_\alpha, g_\alpha))\boldsymbol{\xi}(t_\alpha), \boldsymbol{\eta}(t_\alpha)\rangle = \left\langle \mathcal{T}^t(g_\alpha)\boldsymbol{\xi}(t), \boldsymbol{\eta}(t)\right\rangle = \pi^t_{\xi_t,\eta_t}(g_\alpha).$$

As π^t is a continuous unitary representation, for sufficiently large α we have

$$\left\|\pi^t_{\xi_t,\eta_t}(g_\alpha) - \pi^t_{\xi_t,\eta_t}(g)\right\| < \varepsilon,$$

so $\mathfrak{H}_{\boldsymbol{\xi},\boldsymbol{\eta}}$ is continuous. We then have $\boxplus_{t\in T}\pi^t_{\xi_t,\eta_t} = \mathfrak{H}_{\boldsymbol{\xi},\boldsymbol{\eta}} \in B(\mathcal{G})$ and $B(\mathcal{G}) = \ell^\infty - \boxplus_{t\in T} B(G_i)$. □

From this proposition we deduce something a bit stronger and more familiar. Define

$$\ell^\infty - \oplus_{t\in T} B(G_t) = \left\{ f : T \to \prod_{t\in T} B(G_t) \middle| f(t) \in B(G_t), \sup_t \|f(t)\| < \infty \right\}$$

and equip this space with the norm

$$\|f\|_{\ell^\infty} = \sup_t \|f(t)\|_{B(G_t)}.$$

Proposition 4.2.2. *Let $\mathcal{G}$ be the group bundle formed from the collection of locally compact, Hausdorff groups $\{G_t\}_{t\in T}$. Suppose $\mathcal{G}$ is locally compact and Hausdorff, and $\mathcal{G}^{(0)}$ is discrete. Then $B(\mathcal{G}) \simeq \ell^\infty - \oplus_{t\in T} B(G_t)$ is an isometry.*

Proof. Define the map $\Phi : B(\mathcal{G}) \to \ell^\infty - \oplus_{t\in T} B(G_t)$ by $\varphi \mapsto (R^t_{B(\mathcal{G})}\varphi)_{t\in T}$. It's clear that Φ is a linear surjection via Proposition 4.2.1. We show the injectivity of Φ by showing it has trivial kernel. If $\Phi\varphi$ is the all 0 vector, then $R^t_{B(\mathcal{G})}\varphi$ is identically 0 for each $t \in T$, so φ must be identically 0.

We claim that Φ is isometric. To that end, choose $\varphi \in B(\mathcal{G})$. Then

$$\|\Phi\varphi\|_{\ell^\infty} = \sup_{t\in T} \left\|R^t_{B(\mathcal{G})}\varphi\right\| \leq \sup_{t\in T} \|\varphi\|_{B(\mathcal{G})} = \|\varphi\|_{B(\mathcal{G})}$$

since Theorem 3.1.10 shows that each $R^t_{B(\mathcal{G})}$ is norm-decreasing. This shows Φ is bounded with norm at most 1. Via Proposition 4.2.1, we know that $\varphi = \boxplus_{t\in T}\pi^t_{\xi_t,\eta_t}$. This gives us that

$$\begin{aligned}
\|\varphi\|_{B(\mathcal{G})} = \inf_{\varphi=\mathfrak{H}_{\boldsymbol{\xi},\boldsymbol{\eta}}} \|\boldsymbol{\xi}\|\,\|\boldsymbol{\eta}\| &= \inf_{\varphi=\mathfrak{H}_{\boldsymbol{\xi},\boldsymbol{\eta}}} \left(\sup_{t\in T}\|\boldsymbol{\xi}(t)\|\cdot\sup_{t\in T}\|\boldsymbol{\eta}(t)\|\right) \\
&\geq \inf_{\varphi=\mathfrak{H}_{\boldsymbol{\xi},\boldsymbol{\eta}}} \left(\sup_{t\in T}\|\boldsymbol{\xi}(t)\|\,\|\boldsymbol{\eta}(t)\|\right) \\
&\geq \sup_{t\in T}\left(\inf_{\varphi=\mathfrak{H}_{\boldsymbol{\xi},\boldsymbol{\eta}}} \|\boldsymbol{\xi}(t)\|\,\|\boldsymbol{\eta}(t)\|\right) \qquad (4.2) \\
&= \sup_{t\in T}\left\|\pi^t_{\boldsymbol{\xi}(t),\boldsymbol{\eta}(t)}\right\| \qquad (4.3) \\
&= \|\Phi\varphi\|_{\ell^\infty}
\end{aligned}$$

where Inequality (4.2) results from [78, Corollary 3.8.4], and Equality (4.3) from the fact that $\varphi = \mathfrak{H}_{\boldsymbol{\xi},\boldsymbol{\eta}}$ ranges over all coefficient function representations of $u_t = \pi^t_{\xi,\eta}$ so that Φ is an isometry. □

Corollary 4.2.2.1. *Let $\mathcal{G} = \sqcup_{t\in T}G_t$ be a locally compact and Hausdorff group bundle such that $\mathcal{G}^{(0)}$ is discrete. The map $\Phi : B(\mathcal{G}) \to \ell^\infty - \oplus_{t\in T}B(G_t)$ is an algebra homomorphism.*

Proof. Proposition 4.2.2 shows that Φ is an isometric isometry. To see that it also preserves products, note that

$$\begin{aligned}
\Phi(uv) = \left(R^t_{B(\mathcal{G})}(uv)\right)_{t\in T} = \left(R^t_{B(\mathcal{G})}(u)R^t_{B(\mathcal{G})}(v)\right)_{t\in T} &= \left(R^t_{B(\mathcal{G})}(u)\right)\left(R^t_{B(\mathcal{G})}(v)\right)_{t\in T} \\
&= \Phi(u)\Phi(v).
\end{aligned}$$

□

We may apply this result to show an isometric isomorphism $A(\mathcal{G}) \simeq c_0 - \oplus_{t\in T}A(G_t)$ where $c_0 - \oplus_{t\in T}A(G_t)$ is a subset of $\ell^\infty - \oplus_{t\in T}B(G_t)$ where the map $t \mapsto \|f(t)\|$ vanishes at infinity, and $f : T \to \prod_{t\in T}A(G_t)$ with $f(t) \in A(G_t)$.

Proposition 4.2.3. *Let $\mathcal{G}$ be the group bundle formed from the collection of locally compact, Hausdorff groups $\{G_t\}_{t\in T}$. Equip $\mathcal{G}$ with a locally compact and Hausdorff topology such that $\mathcal{G}^{(0)}$ is discrete. Then $A(\mathcal{G}) \simeq c_0 - \oplus_{t\in T}A(G_t)$ is an isometry.*

Proof. The map $\Phi : B(\mathcal{G}) \to \ell^\infty - \oplus_{t\in T} B(G_t)$ of Proposition 4.2.2 is an isometric isometry. By Theorem 3.1.12, $R^t_{A(\mathcal{G})} : A(\mathcal{G}) \to A(G_t)$. Begin by considering $A_{cf}(\mathcal{G})$ (see Definition 2.4.24). Note that we may similarly define $A_{cf}(G_t)$ and $A_c(G_t)$ to be the sets of coefficient functions $L^{G_t}_{f,g}$ with $f, g \in C_c(G_t)$ and the algebra generated by $A_{cf}(G_t)$ respectively. Take $u \in A_{cf}(\mathcal{G})$. Then $u = \sum_{i=1}^n \Lambda_{f_i,g_i}$ with $f_i, g_i \in C_c(\mathcal{G})$. Note $\mathcal{G}^{(0)}$ is discrete, there is a finite set $F \subseteq T$ such that $R^t_{A(\mathcal{G})}(u)$ is nonzero. Moreover, each $f_i|_{G_t} \in C_c(\mathcal{G}_t)$, and so $R^t_{A(\mathcal{G})}(\Lambda_{f_i,g_i}) \in A_{cf}(G_t)$. This demonstrates

$$\Phi(A_{cf(\mathcal{G})}) = \oplus_{t\in T} A_{cf}(G_t).$$

As Φ is an algebra homomorphism by Corollary 4.2.2.1, we have

$$\Phi(A_c(\mathcal{G})) = \oplus_{t\in T} A_c(G_t).$$

Then, by definition of $A(\mathcal{G})$ and the fact that Φ is an isometry, we have

$$\Phi(A(\mathcal{G})) = \Phi(\overline{A_c(\mathcal{G})}) = \overline{\Phi(A_c(\mathcal{G})} = \overline{\oplus_{t\in T} A_c(G_t)} = \overline{\oplus_{t\in T} \overline{A_c(G_t)}} = \overline{\oplus_{t\in T} A(G_t)}.$$

Thus, $\Phi : A(\mathcal{G}) \to \overline{\oplus_{t\in T} A(G_t)} = c_0 - \oplus_{t\in T} A(G_t)$ is an isometric algebra homomorphism. □

By Proposition 2.3.17 if G is a compact group, then $A(G)$ and $B(G)$ coincide. This is true for groupoids at least in the event that the group bundles in this section are formed from a finite union. In the infinite case, the union of compact groups need not be compact, via the simple example of $\sqcup_{i\in\mathbb{N}}\mathbb{T}$. Otherwise, we have

$$B(\mathcal{G}) = \boxplus_{i=1}^n B(G_i) = \boxplus_{i=1}^n A(G_i) = A(\mathcal{G}).$$

We would be remiss if we did not mention the appearance of Propositions 4.2.2 and 4.2.3 in a thesis by Marti Perez [58, Example 6.1.1]. This thesis arrives at the results as corollaries of its larger theorems, while we tackle them more directly. [58, Example 6.1.1] applies [58, Proposition 5.4.1] which establishes the result for groupoids such that each transitive component is both open and closed. Perez' thesis proceeds using a slightly different definition of $A(\mathcal{G})$, one built by considering the left regular bundle with infinite multiplicity.

4.3 Gelfand Spectrum

Let G be a locally compact group. The Gelfand Spectrum of $A(G)$, denoted $\sigma(A(G))$, is defined to be the set of all non-zero multiplicative linear functionals on $A(G)$.

Proposition 4.3.1. *[13, Proposition 2.16.3] Let A be a Banach algebra, and let φ be a multiplicative linear functional on A. Then φ is continuous, and $\|\varphi\| \leq 1$.*

In the group case, it has been shown that $\sigma(A(G))$ can be identified with G itself via point evaluations.

Proposition 4.3.2. *[31, Theorem 3.34] Let G be a locally compact group. For each $\zeta \in G$, let $\chi_\zeta : A(G) \to \mathbb{C}$ be the function given by $\chi_\zeta(u) = u(\zeta)$. Then the map $\Phi : \zeta \mapsto \chi_\zeta$ is a homeomorphism $G \to \sigma(A(G))$.*

Proof. We outline this proof in a number of steps so that we may later contrast this with the groupoid case.

1. By Proposition 2.3.12, we have that the map Φ is a homeomorphism from G to a closed subset of $\sigma(A(G))$[3]. It is also clear that Φ is injective.

2. What remains to show is that any $\chi \in \sigma(A(G))$ is of the form χ_ζ for some $\zeta \in G$. To that end, let $\chi \in \sigma(A(G))$ and for the sake of contradiction assume that $\chi \neq \chi_\zeta$ for every $\zeta \in G$. Then, for each $\zeta \in G$, there is some $u_\zeta \in A(G)$ such that $\chi(u_\zeta) = 1$, but $\chi_\zeta(u_\zeta) = 0$. If $u_\zeta(\zeta) = 0$, this is shown by scaling. Otherwise, if $u_\zeta(\zeta) = 0$ implies $\chi(u_\zeta) = 0$, then $\ker \chi_\zeta \subseteq \ker \chi$, so $\chi = c\chi_\zeta$, a contradiction.

3. [48, Lemma 2.3.7] states: if $a \in G$ and $f \in A(G)$ such that $f(a) = 0$, then, given $\varepsilon > 0$ there is an $h \in C_c(G) \cap A(G)$ vanishing in a neighborhood of a such that $\|h - f\|_{A(G)} < \varepsilon$. This implies that every $v \in A(G)$ vanishing at ζ is the limit of a sequence $\{v_n\}_n$ in $A(G)$ such that each v_n vanishes in some neighborhood of ζ.

[3] Eymard further cites [77, Theorem 3.2.4]. We will discuss this in more detail in Proposition 4.3.3.

Thus, given $\varepsilon > 0$ we can find a $\tilde{u}_\zeta$ such that $\tilde{u}_\zeta$ vanishes in a neighborhood V_ζ of ζ and $\|u_\zeta - \tilde{u}_\zeta\|_{A(G)} < \varepsilon$.

4. Since $A(G) \cap C_c(G)$ is dense in $A(G)$, there is some $u_0 \in C_c(G) \cap A(G)$ such that $\chi(u_0) = 1$. Choose $\zeta_1, \ldots, \zeta_n \in \text{supp}(u_0)$ such that

$$\text{supp}(u_0) \subseteq \cup_{j=1}^n V_{\zeta_j}$$

5. Define $u := u_0 \tilde{u}_{\zeta_1} \ldots \tilde{u}_{\zeta_n} \in A(G)$. Then $u(\zeta) = 0$ for every $\zeta \in G$ but

$$\chi(u) = \chi(u_0) \prod_{j=1}^n \chi(\tilde{u}_{\zeta_j}) = 1$$

a contradiction. Thus, we must have $\chi = \chi_\zeta$ for some $\zeta \in G$.

6. The proof concludes by noting that $A(G)$ is uniformly dense in $C_0(G)$ ([48, Corollary 2.3.5 (ii)]), thus the topology on G coincides with the w^* topology in $\sigma_{A(G)}$. This is because $A(G) \cap C_c(G)$ is a self-adjoint sub algebra, and by [48, Proposition 2.3.2] separates the points of G.

□

This proof is as it is presented in [48], while the original proof due to [31] requires the following theorem appearing in [77]. We present this to prove a similar claim about $\sigma(A(\mathcal{G}))$ when $\mathcal{G}$ is a locally compact, Hausdorff group bundle with discrete unit space.

Let Ω represent a commutative algebra and Φ_Ω the class of non-zero homomorphisms of Ω into $\mathbb{C}$. The space Φ_Ω clearly contains the Gelfand spectrum. A function space strongly separates the points of its domain if, given distinct x, y there exists a function f with $f(x) \neq f(y)$. The natural embedding is one of point evaluation, $x \in \Omega$ embeds as the function of point evaluation at x.

Proposition 4.3.3. *[77, Theorem 3.2.4] Let Ω be locally compact and let $\mathfrak{C}$ be a (real or complex) subalgebra of $C_0(\mathfrak{C})$. Then, if $\mathfrak{C}$ strongly separates the points of Ω, the natural embedding of $\Omega \to \Phi_\mathfrak{C}$ is a homeomorphism which takes Ω onto a closed subset of $\Phi_\mathfrak{C}$.*

In the context of Eymard's result, Ω is taken to be a locally compact group, and $\mathfrak{C} = A(G)$. That $A(G)$ belongs to $C_0(G)$ can be seen in [48, Corollary 2.3.5 (i)] and the fact that $A(G)$ strongly separates points is a consequence of [48, Proposition 2.3.2].

Proposition 4.3.4. *Let $\mathcal{G} := \sqcup_{t\in T} G_t$ be a locally compact and Hausdorff group bundle, and suppose $\mathcal{G}^{(0)}$ is discrete. Then $\sigma(A(\mathcal{G}))$ is homeomorphic to $\mathcal{G}$.*

Proof. Note that since $\mathcal{G}$ is locally compact, and Hausdorff, so too is G_t for each $t \in T$. Let $\Omega = \mathcal{G}$, and $\mathfrak{C} = A(\mathcal{G})$. [70, Proposition 14] gives us that $A(\mathcal{G}) \subseteq C_0(\mathcal{G})$, and Proposition 16 (ii) of the same source gives that $A(\mathcal{G})$ strongly separates points. Thus, the natural embedding is a homeomorphism into $\sigma(A(\mathcal{G}))$. What remains is to demonstrate that any multiplicative, linear functional on $A(\mathcal{G})$ must correspond to point evaluation for a given groupoid element.

Choose $\Phi \in \sigma(A(\mathcal{G}))$. Note by Proposition 4.2.1 and Theorem 3.1.12, any $\varphi \in A(\mathcal{G})$ has the form $\varphi = \boxplus_{t\in T}\varphi_t$ where $\varphi_t \in A(G_t)$. We can also view each $A(G_t)$ embedded in $A(\mathcal{G})$ by considering f_t as $\tilde{f}_t$, the box sum with f_t as component t and the zero function elsewhere. Choose a $\varphi \in A(\mathcal{G})$ such that $\Phi(\varphi) \neq 0$. Let $\varphi = \boxplus_{t\in T}\varphi_t$. We claim that Φ cannot be supported on more than 1 element $\tilde{\varphi}_t \in T$. For the sake of contradiction, assume that $\Phi(\tilde{\varphi}_p)\Phi(\tilde{\varphi}_q) \neq 0$ for $p, q \in T$ and $p \neq q$. Then $\Phi(\tilde{\varphi}_p\tilde{\varphi}_q) \neq 0$. However, $\tilde{\varphi}_p\tilde{\varphi}_q$ is the zero function, and so $\Phi(\tilde{\varphi}_p\tilde{\varphi}_q) = 0$, a contradiction. Thus Φ is nonzero for at most one $\tilde{\varphi}_t$. Next, assume that for every $t \in T$, $\Phi(\varphi_t) = 0$. We will show that this implies $\Phi(\varphi) = 0$ which shows that Φ is nonzero for exactly one $\tilde{\varphi}_t$.

If $\varphi \in A_{cf}(\mathcal{G})$, by Proposition 4.2.3 there are only finitely many nonzero φ_t and we may write $\varphi = \sum \tilde{\varphi}_t$. Then we have

$$\Phi(\varphi) = \Phi\left(\sum \tilde{\varphi}_t\right) = \sum \Phi(\varphi) = 0.$$

Similarly, if $\varphi \in A_c(\mathcal{G})$ we are done, as we may express $\varphi = \sum_{i=1}^{n}\prod_{j=1}^{m} u_{i,j} \in C_c(\mathcal{G})$ with each $u_{i,j} \in C_c(\mathcal{G}) \cap A_{cf}(\mathcal{G})$. Again invoking Proposition 4.2.3, this implies there

are finitely many nonzero φ_t and

$$\Phi(\varphi) = \Phi\left(\sum_{i=1}^{n}\prod_{j=1}^{m} u_{i,j}\right) = \sum_{i=1}^{n}\prod_{j=1}^{m}\Phi(u_{i,j}) = 0.$$

Now, let $\varphi \in A(\mathcal{G})$. For the sake of contradiction, assume that $\Phi(\varphi) \neq 0$. Without loss of generality we may assume $\Phi(\varphi) = 1$. Since $A_{cf}(\mathcal{G})$ spans a dense sub-algebra, there is a $u \in A_{cf}(\mathcal{G})$ such that $\Phi(u) \neq 0$. If no such u existed, then by the continuity, linearity, and multiplicativity of Φ, we would have

$$\Phi(\varphi) = \Phi\left(\lim_{n,m}\sum_{i=1}^{n}\prod_{j=1}^{m} u_{i,j}\right) = \lim_{n,m}\sum_{i=1}^{n}\prod_{j=1}^{m}\Phi(u_{i,j}) = 0.$$

Scale u so $\Phi(u) = 1$. Then, we have that $\varphi u \in C_c(\mathcal{G})$ and so only finitely many $(\varphi u)_t$ can be nonzero. $\Phi(\varphi u) = 1$. At the same time, $(\varphi u)_t = \varphi_t u_t$ which implies $\Phi(\varphi_t u_t) = \Phi(\varphi_t)\Phi(u_t) = 0$. This is a contradiction, as this implies that $\Phi(\varphi u) = 0$.

Thus, Φ is nonzero for exactly one $\tilde{\varphi}_t$. This gives us that Φ restricts to a nonzero, multiplicative linear functional on $A(G_t)$ for some t. By Proposition 4.3.2, this means that $\Phi|_{A(G_t)} = \chi_g$ for some $g \in G_t$, and $\Phi|_{A(G_s)} = 0$ for $s \neq t$. We conclude that $\Phi = \chi_{(t,g)}$, completing the proof. □

Paterson comments in [70] that he does not know whether or not $\sigma_{A(\mathcal{G})}$ is homeomorphic to $\mathcal{G}$ for an arbitrary groupoid. We additionally question whether it is, given that we have a positive result for a class of group bundles. A possible sketch of the groupoid case would mimic the steps shown in the proof of Proposition 4.3.2. Step 1 is shown by invoking [70, Proposition 16] which states that for every compact subset $K \subseteq \mathcal{G}$ and $K \subseteq U$ open, there is an $f \in A_{cf}(\mathcal{G})$ with $f \in C_c(U)$, $f(K) = \{1\}$, and $0 \leq f \leq 1$, and so the map $x \mapsto \chi_x$ must be injective. Step 2 is identical. By definition, $A(\mathcal{G})$ By definition, $A(\mathcal{G}) \cap C_c(\mathcal{G})$ contains $A_{cf}(\mathcal{G})$, and this is a dense subset of $A(\mathcal{G})$. In light of this, Step 4 will be the same, and the definition/contradiction needed in Step 5 is identical. This proof would also conclude with the same arguments, as $A(\mathcal{G}) \cap C_c(\mathcal{G})$ is a self-adjoint sub-algebra under the involution $\Lambda_{f,g}^* = \Lambda_{g,f}$ (extended via linearity and density) and separates points by [70, Proposition 16].

The missing development is the lemma used in Step 3, which we pose as a question. With the above outline, a positive answer of the following will show this result for arbitrary locally compact groupoids with Haar systems.

Question 4.3.5. *Let $\mathcal{G}$ be a locally compact and Hausdorff groupoid. Suppose that, for $x \in \mathcal{G}$ and $u \in A(\mathcal{G})$, $u(x) = 0$. Then, given $\varepsilon > 0$, does there exist an open $U_{x,\varepsilon}$ containing x and a function $u_\varepsilon \in A(\mathcal{G})$ such that $\|u - u_\varepsilon\|_{A(\mathcal{G})} < \varepsilon$ and $u(U_{x,\varepsilon}) = \{0\}$.*

Chapter 5

STRUCTURE OF $A(\mathcal{G})$

5.1 Compact Groupoids

In [32, Chapter 5], the author develops harmonic analysis (including Schur's lemma and the orthogonality relations) over compact groups in order to present the Peter-Weyl theorem, that every representation of a compact group decomposes into the direct sum of irreducible representations. The paper [4] attempts to generalize these results to the locally compact and compact groupoid setting in order to show groupoid versions of these results. We dedicated part of our efforts to these results as one can use them along with the Fourier coefficient spaces A_π (Definition 2.3.19) of [8] to show

$$\|u\|_{A(G)} = \sum_{\pi \in \hat{G}} d_\pi \, \|\pi(u)\|_1$$

where $\|\cdot\|_1$ is the tracial class norm of finite matrices and G is a compact group. Our hope was to use the results of [4] to achieve an analogous formula for compact groupoids. The formula itself is still an open question, and in attempting to verify Amini's results we have noticed several discrepancies, some of which seem to be of a serious nature. The paper [14] also raises other concerns in the lack of treatment of the continuity condition of sections. [14] also develops groupoid versions of Schur's lemma and the Peter-Weyl theorem, though to arrive at these results they impose stricter conditions on the $\mathcal{G}$-Hilbert bundles: they require a condition they refer to as "internally irreducible", that is, the representations formed on the isotropy groups by $\mathcal{G}$-Hilbert bundles are all irreducible. To avoid the possible ambiguities in verifying the results of [4], we have chosen to restrict our focus when needed to group bundles with discrete unit space. We show internal irreducibility for this class of groupoid in Proposition 5.1.3.

When $\mathcal{G}$ is compact, and we impose that $\mathcal{G}^{(0)}$ is discrete, the continuity of the range map implies that $\mathcal{G}^{(0)}$ is finite. Under these assumptions, the set of continuous sections Γ of any $\mathcal{G}$-Hilbert bundle $\mathfrak{H}$ will just be the collection of functions $f : \mathcal{G}^{(0)} \to \sqcup_{i\in\mathbb{N}}\mathfrak{H}_i$ such that $f(i) \in \mathfrak{H}_i$. Recall that for group bundles, we use $x \in G_i$ and $(i, x) \in \mathcal{G}$ interchangeably.

Lemma 5.1.1. *Let $\mathcal{G} := \sqcup_{i=1}^n G_i$ be a group bundle with $\mathcal{G}^{(0)}$ discrete. Then there exists a bijective correspondence between:*

1. *$\mathcal{G}$-Hilbert bundles;*

2. *Collections of representations $\{\pi^i : G_i \to \mathcal{U}(\mathcal{H}_{\pi^i})\}_{i=1}^n$.*

Proof. Given a $\mathcal{G}$-Hilbert bundle $\mathfrak{H}$ with linear actions $\{\mathcal{T}(x)\}_{x\in\mathcal{G}}$, for each i the set $\{\mathcal{T}(x)\}_{x\in G_i}$ forms a unitary representation $\pi^i : G_i \to \mathcal{U}(\mathfrak{H}_i)$. Conversely, suppose for each i that π^i is a representation of G_i. Let $\mathfrak{H}_i = \mathcal{H}_{\pi^i}$, and set $\mathcal{T}((i, x)) = \pi^i(x)$. As groupoid elements are only composable when they belong to the same group, and each π^i is a group representation, $\mathcal{T}(x)$ satisfies the required homomorphism property. If $\boldsymbol{\xi}, \boldsymbol{\eta} \in \Delta_b(\mathfrak{H})$, then $\mathfrak{H}_{\boldsymbol{\xi},\boldsymbol{\eta}} = \boxplus_{i=1}^n \pi^i_{\boldsymbol{\xi}(i),\boldsymbol{\eta}(i)}$, which is continuous by Proposition 4.2.1, and so $\mathfrak{H}$ defines a $\mathcal{G}$-Hilbert bundle. □

Definition 5.1.2. [4, Definition 2.7] Let $\mathfrak{H}$ be a $\mathcal{G}$-Hilbert bundle. A Hilbert field $\mathfrak{M} = \{\mathfrak{M}_u\}_{u\in\mathcal{G}^{(0)}}$ is called a *sub-bundle* if $\mathfrak{M}_u \leq \mathfrak{H}_u$ is a closed subspace for every $u \in \mathcal{G}^{(0)}$. A sub-bundle is said to be *invariant* if $\mathcal{T}(x)\mathfrak{M}_{s(x)} \leq \mathfrak{M}_{r(x)}$ for every $x \in \mathcal{G}$. A $\mathcal{G}$-Hilbert bundle is called *irreducible* if, for every sub-bundle $\mathfrak{M}$ and $u \in \mathcal{G}^{(0)}$, either $\mathfrak{M}_u = 0$ or $\mathfrak{H}_u$.

Proposition 5.1.3. *Let $\mathcal{G} = \sqcup_{i=1}^n G_i$ be a group bundle with $\mathcal{G}^{(0)}$ discrete. Then a $\mathcal{G}$-Hilbert bundle is irreducible if and only if the representation on G_i formed by $\{\mathcal{T}(x)\}_{x\in G_i}$ is irreducible for every $i \in \mathcal{G}^{(0)}$.*

Proof. We instead show that a $\mathcal{G}$-Hilbert bundle $\mathfrak{H}$ is reducible if and only if for some $i \in \mathcal{G}^{(0)}$ the representation formed on G_i is reducible. First assume that $\mathfrak{H}$ is a $\mathcal{G}$-Hilbert

bundle with associated representations $\pi^i : G_i \to \mathcal{U}(\mathcal{H}_{\pi^i})$, and the representation π^j of G_j is reducible. Then $\mathcal{H}_{\pi^j}$ has a non-trivial, proper invariant subspace M. Let $\mathfrak{H}'$ be the $\mathcal{G}$-Hilbert bundle corresponding to the representations $\pi^1, \ldots, (\pi^j)^M, \ldots, \pi^n$ of Lemma 5.1.1. Then $\mathfrak{H}'$ is a non-trivial, proper invariant sub-bundle of $\mathfrak{H}$, so $\mathfrak{H}$ is reducible.

If $\mathfrak{H}$ is a reducible $\mathcal{G}$-Hilbert bundle, then there is a sub-bundle with $\mathfrak{M}_i \leq \mathfrak{H}_i$ with $\mathcal{T}(x)\mathfrak{M}_{s(x)} \leq \mathfrak{M}_{r(x)}$ for each $x \in \mathcal{G}$ and at least one $\mathfrak{M}_i$ is a non-trivial, proper subspace of $\mathfrak{H}_i$. The collection $\{\mathcal{T}(x)\}_{x \in G_i}$ forms a representation $\pi : G_i \to \mathcal{U}(\mathfrak{H}_i)$. This representation is reducible as $\mathfrak{M}_i$ is a non-trivial invariant subspace. □

The converse direction in Proposition 5.1.3 will be true for general locally compact groupoids. The direct implication will need more sophisticated tools. The distinguishing feature of the group bundle case is that $\mathcal{G}_i^j = \varnothing$ for $i \neq j$. If one $\mathcal{G}_i^j$ is non-empty, then $\mathcal{T}(x)\mathcal{T}(y)$ need not always be defined if we simply reduce the Hilbert space $\mathfrak{H}_i$ to $\mathfrak{M}_i$.

Once sub-bundles and irreducibility have been defined, the next step is to recall the notion of unitary equivalence of $\mathcal{G}$-Hilbert bundles of Definition 2.4.16. Two bundles are unitarily equivalent if there is a collection of unitary operators $T_u : \mathfrak{H}_u \to \mathfrak{H}'_u$ which intertwine their linear actions, i.e. if the following diagram commutes for every $x \in \mathcal{G}$.

$$\begin{array}{ccc} \mathfrak{H}_{s(x)} & \xrightarrow{\mathcal{T}_x} & \mathfrak{H}_{r(x)} \\ \Big\downarrow{\scriptstyle T_{s(x)}} & & \Big\downarrow{\scriptstyle T_{r(x)}} \\ \mathfrak{H}'_{s(x)} & \xrightarrow[\varrho_x]{} & \mathfrak{H}'_{r(x)} \end{array}$$

Following the notation of the group case, define $C(\mathfrak{H}, \mathfrak{H}')$ to be the vector space of intertwining operator collections under pointwise addition. We will omit the second argument in the event the two $\mathcal{G}$-Hilbert bundles coincide. For a sub-bundle $\mathfrak{M}$ of a $\mathcal{G}$-Hilbert bundle $\mathfrak{H}$, define the perpendicular bundle $\mathfrak{M}^\perp$ as the $\mathcal{G}$-Hilbert bundle with

$(\mathfrak{M}^{\perp})_u = (\mathfrak{M}_u)^{\perp}$. The continuous sections in general are those which map $\mathcal{G}^{(0)}$ into $\sqcup_{u\in\mathcal{G}^{(0)}}\mathfrak{M}_u^{\perp}$, which is a subspace of the continuous sections of $\mathfrak{H}$.

Proposition 5.1.4. *[4, Lemma 2.9] Let $\mathfrak{H}$ be a $\mathcal{G}$-Hilbert bundle of a group bundle with $\mathcal{G}^{(0)}$ discrete, and $\mathfrak{M}$ a sub-bundle. If $\mathfrak{M}$ is invariant, so too is $\mathfrak{M}^{\perp}$.*

Proof. Let $v \in \mathfrak{M}^{\perp}_{s(x)}$ and $w \in \mathfrak{M}_{r(x)}$. Then

$$\langle \pi(x)v, w\rangle = \left\langle v, \pi\left(x^{-1}\right) w\right\rangle = 0$$

since $\pi\left(x^{-1}\right) w \in \mathfrak{M}_{s(x)}$ by the invariance of $\mathfrak{M}$. Thus $\pi(x)v \in \mathfrak{M}^{\perp}_{r(x)}$. □

Proposition 5.1.5. *[4, Lemma 2.12] Let $\mathfrak{H}$ be a $\mathcal{G}$-Hilbert bundle of a group bundle with $\mathcal{G}^{(0)}$ discrete and let $\mathfrak{M}$ be a sub-bundle. Define $P_i : \mathfrak{H}_i \to \mathfrak{M}_i$ to be the canonical projection map. Then $\mathfrak{M}$ is invariant if and only if $\{P_i\}_{i\in\mathcal{G}^{(0)}} \in C(\mathfrak{H})$.*

Proof. First assume that $\mathfrak{M}$ is invariant. Choose a $v \in \mathfrak{M}_{s(x)}$. Then:

$$P_{r(x)}\pi(x)v = \pi(x)v = \pi(x)P_{s(x)}v$$

so $\{P_i\}_{i\in\mathcal{G}^{(0)}}$ intertwines $\mathfrak{H}$. Conversely, assume that $\{P_i\}_{i\in\mathcal{G}^{(0)}} \in C(\mathfrak{H})$. Then, for every $x \in \mathcal{G}$, $\pi(x)P_{s(x)} = P_{r(x)}\pi(x)$. Fix $v \in \mathfrak{M}_{s(x)}$. Then we have

$$\pi(x)v = \pi(x)P_{s(x)}v = P_{r(x)}\pi(x)v \in \mathfrak{M}_{r(x)}$$

so $\mathfrak{M}$ is invariant. □

With these results, following the developmental paths of [32] and [4], one would prove a groupoid version of Schur's lemma, as in [4, Theorem 2.14]. The group case is presented in [32, Theorem 3.5] and states that a unitary representation π of a locally compact group G is irreducible if and only if $C(\pi) = \{cI_{\mathcal{H}_\pi} : c \in \mathbb{C}\}$, and two irreducible representations π_1 and π_2 are unitarily equivalent if and only if $C(\pi_1, \pi_2)$ is 1-dimensional. [4, Theorem 2.14] makes the same claim in the groupoid case, though we will arrive at a different conclusion. The error does not seem to be about the assumption.

Lemma 5.1.6. *Let $\mathcal{G} := \sqcup_{i=1}^{n} G_i$ be a group bundle with $\mathcal{G}^{(0)}$ discrete and $\mathfrak{H}$, $\mathfrak{G}$ two $\mathcal{G}$-Hilbert bundles with linear actions $\{\mathcal{T}(x)\}$ and $\{\varrho(x)\}$ respectively. Let π^i be the representation formed on G_i by $\mathfrak{H}$ as in Lemma 5.1.1, and σ^i the same for $\mathfrak{G}$. Then $\{T_i\}_{i\in\mathcal{G}^{(0)}} \in C(\mathfrak{H},\mathfrak{G})$ if and only if $T_i \in C(\pi^i,\sigma^i)$ for every i.*

Proof. First suppose $\{T_i\}_{i\in\mathcal{G}^{(0)}} \in C(\mathfrak{H},\mathfrak{G})$. Then each $T_i : \mathfrak{H}_i \to \mathfrak{G}_i$ and $T_{s(x)}\mathcal{T}(x) = \varrho(x)T_{r(x)}$ for every $x \in \mathcal{G}$. As $\mathcal{G}$ is a group bundle, the source and range maps coincide, so if $s(x) = r(x) = j$ we immediately see $T_j\pi^j(x) = \sigma^j(x)T_j$ so $T_j \in C(\pi^j,\sigma^j)$.

Conversely, assume $T_i \in C(\pi^i,\sigma^i)$ for every i. If $s(x) = j$ then $T_j\pi^j(x) = \sigma^j(x)T_j$ is equivalent to $T_{s(x)}\mathcal{T}(x) = \varrho(x)T_{r(x)}$ so $\{T_i\}_{i\in\mathcal{G}^{(0)}} \in C(\mathfrak{H},\mathfrak{G})$. □

Corollary 5.1.6.1. *Let $\mathcal{G} := \sqcup_{i=1}^{n} G_i$ be a group bundle with $\mathcal{G}^{(0)}$ discrete. Let $\mathfrak{H}$ and $\mathfrak{G}$ be two $\mathcal{G}$-Hilbert bundles, with associated representations $\{\pi^i\}_{i=1}^n$ and $\{\sigma^i\}_{i=1}^n$ respectively. Then $\mathfrak{H} \simeq \mathfrak{G}$ if and only if $\pi^i \simeq \sigma^i$ for $1 \le i \le n$.*

Proof. If $\mathfrak{H} \simeq \mathfrak{G}$, then there is a unitary operator collection $\{T_i : \mathfrak{H}_i \to \mathfrak{G}_i\}_{i=1}^n$ and by Lemma 5.1.6 each $T_i \in C(\pi^i,\sigma^i)$ so $\pi^i \simeq \sigma^i$. Conversely, if for each i there is a unitary $T_i \in C(\pi^i,\sigma^i)$, then $\{T_i\}_{i=1}^n \in C(\mathfrak{H},\mathfrak{G})$ and so $\mathfrak{H} \simeq \mathfrak{G}$. □

We now have enough tools to state and prove a version of Schur's lemma for group bundles with $\mathcal{G}^{(0)}$ discrete.

Proposition 5.1.7. *Let $\mathcal{G} := \sqcup_{i=1}^{n} G_i$ be a group bundle with $\mathcal{G}^{(0)}$ discrete, and let $\mathfrak{H},\mathfrak{H}^1,\mathfrak{H}^2$ be $\mathcal{G}$-Hilbert bundles with $\mathfrak{H}^1,\mathfrak{H}^2$ irreducible. Then:*

1. *$\mathfrak{H}$ is irreducible if and only if $C(\mathfrak{H})$ consists of operator bundles of the form $\{c_i I_{\mathfrak{H}_i} : c_i \in \mathbb{C}\}_{i\in\mathcal{G}^{(0)}}$;*
2. *$\mathfrak{H}^1 \simeq \mathfrak{H}^2$ if and only if $C(\mathfrak{H}^1,\mathfrak{H}^2)$ is n-dimensional.*

Proof. 1. First suppose that $\mathfrak{H}$ is irreducible, and choose $\{T_i\}_{i\in\mathcal{G}^{(0)}} \in C(\mathfrak{H})$. By Lemma 5.1.1 we have that $\mathfrak{H}$ can be seen as the collection of representations

$\{\pi^i : G_i \to \mathcal{U}(\mathfrak{H}_i)\}_{i\in\mathcal{G}^{(0)}}$, and by Proposition 5.1.3, we have that each π^i is irreducible. By Lemma 5.1.6, each $T_i \in C(\pi^i)$ and so $T_i = c_i I_{\mathfrak{H}_i}$ by Schur's lemma for groups.

Conversely, suppose $C(\mathfrak{H})$ consists of operator bundles of the form $\{c_i I_{\mathfrak{H}_i}\}_{i\in\mathcal{G}^{(0)}}$. Then, by Lemma 5.1.6, for each i we have that $C(\pi^i)$ is only comprised of scalar operators, and so each π^i is irreducible again by Schur's lemma for groups. By Proposition 5.1.3, $\mathfrak{H}$ is irreducible.

2. Suppose $\{T_i\}_{i\in\mathcal{G}^{(0)}} \in C(\mathfrak{H}^1, \mathfrak{H}^2)$. Let π^i and σ^i be the representations formed on G_i by $\mathfrak{H}^1$ and $\mathfrak{H}^2$ respectively. By Proposition 5.1.3, each π^i and σ^i are irreducible, and by Lemma 5.1.6, $T_i \in C(\pi^i, \sigma^i)$ for each $i \in \mathcal{G}^{(0)}$. By Schur's lemma for groups, each $C(\pi^i, \sigma^i)$ is 1-dimensional, so there exist unitary operators $B_i : \mathfrak{H}^1_i \to \mathfrak{H}^2_i$ such that $T_i = c_i B_i$. For $i, j \in \mathcal{G}^{(0)}$ define $\{S_{i,j}\}_{j\in\mathcal{G}^{(0)}}$ by $S_{i,j} = B_i$ if $i = j$ and 0 otherwise. Then $\left\{\{S_{i,j}\}_{j\in\mathcal{G}^{(0)}}\right\}_{i\in\mathcal{G}^{(0)}}$ is a basis for $C(\mathfrak{H}_1, \mathfrak{H}_2)$ and has size n.

□

Corollary 5.1.7.1. *Let $\mathcal{G} := \sqcup_{i=1}^n G_i$ be a group bundle with $\mathcal{G}^{(0)}$ discrete and each G_i abelian. If $\mathfrak{H}$ is an irreducible $\mathcal{G}$-Hilbert bundle, then for each i we have* $\dim(\mathfrak{H}_i) = 1$.

Proof. For each G_i, fix an x_i. As each G_i is abelian, we have $\{\mathcal{T}(x_i)\}_{i\in\mathbb{N}} \in C(\mathfrak{H})$ and so by Proposition 5.1.7 each $\mathcal{T}(x_i) = c_i I_{\mathfrak{H}_i}$. This implies that each 1-dimensional subspace of $\mathfrak{H}_i$ is invariant, so $\dim(\mathfrak{H}_i) = 1$. □

From here, following the classical development, one would begin to pivot towards showing results that rely on the compactness of the group(oid). The next major proposition that is typically shown is that the Hilbert spaces of irreducible representations are finite dimensional, and that each representation decomposes into the direct sum of irreducible representations. [4, Theorem 2.16] makes these claims about $\mathcal{G}$-Hilbert bundles and their Hilbert space fibers, however, we will demonstrate an example of an

reducible $\mathcal{G}$-Hilbert bundle which does not decompose. To show this, we need a fibered version of the operator used to handle the analogous result in the group case.

Proposition 5.1.8. *[4, Lemma 2.15] Let $\mathfrak{H}$ be a $\mathcal{G}$-Hilbert bundle of a compact groupoid $\mathcal{G}$ with linear actions $\{\pi(x)\}_{x\in\mathcal{G}}$. Fix $\boldsymbol{\xi} \in \Delta_b(\mathfrak{H})$ with $\|\boldsymbol{\xi}(u)\|_{\mathfrak{H}_u} = 1$ for every $u \in \mathcal{G}^{(0)}$. For $u \in \mathcal{G}^{(0)}$ define $T_u : \mathfrak{H}_u \to \mathfrak{H}_u$ by*

$$T_u v = \int_{\mathcal{G}^u} \langle v, \pi(x)\boldsymbol{\xi}(s(x))\rangle \, \pi(x)\boldsymbol{\xi}(s(x)) d\lambda^u(x).$$

Then each T_u is positive, non-zero, compact, and $\{T_u\}_{u\in\mathcal{G}^{(0)}} \in C(\mathfrak{H})$.

Proof. Proceeding in order, we have:

1. The inner product

$$\langle T_u v, v\rangle = \int_{\mathcal{G}^u} \langle v, \pi(x)\boldsymbol{\xi}(s(x))\rangle \, \langle \pi(x)\boldsymbol{\xi}(s(x)), v\rangle \, d\lambda^u(x) \geq 0$$

   so T_u is positive.

2. When $v = \boldsymbol{\xi}(u)$, the real valued function $f(x) = |\langle \boldsymbol{\xi}(u), \pi(x)\boldsymbol{\xi}(s(x))\rangle|^2$ satisfies $f(u) = 1$, and f is continuous. Thus, we may find an open neighborhood where the integrand is bounded below by $\frac{1}{2}$, so T_u is non-zero.

3. Since $\mathcal{G}^u$ is compact, the map $x \mapsto \pi(x)\boldsymbol{\xi}(s(x))$ is uniformly continuous on $\mathcal{G}^u$, so there exists a finite disjoint collection of subsets of $\mathcal{G}^u$ $\{E_i\}$ and $x_i \in E_i$ such that $\|\pi(x)\boldsymbol{\xi}(s(x)) - \pi(x)\boldsymbol{\xi}(s(x_j))\| < \frac{\varepsilon}{2}$ for $x \in E_j$. Then we have $\|\langle v, \pi(x)\boldsymbol{\xi}(s(x))\rangle - \langle v, \pi(x)\boldsymbol{\xi}(s(x_j))\rangle\| \leq \frac{\varepsilon}{2}(2\|v\|)$ for $x \in E_j$. Setting $T_u^\varepsilon v$ to be $\sum_j \lambda^u(E_j) \langle v, \pi(x_j)\boldsymbol{\xi}(s(x_j))\rangle \, \pi(x_j)\boldsymbol{\xi}(s(x_j))$ gives that $\|T_u v - T_u^\varepsilon v\| \leq \varepsilon \|v\|$. Each T_u^ε is finite rank, so T_u must be compact (as the limit of finite rank operators).

4. To demonstrate that $\{T_u\}_{u\in\mathcal{G}^{(0)}}$ intertwines, note

$$\begin{aligned}
T_{r(y)}\pi(y)v &= \int_{\mathcal{G}^{r(y)}} \langle \pi(y)v, \pi(x)\boldsymbol{\xi}(s(x))\rangle\, \pi(x)\boldsymbol{\xi}(s(x))d\lambda^{r(y)}(x) \\
&= \int_{\mathcal{G}^{r(y)}} \left\langle v, \pi\left(y^{-1}x\right)\boldsymbol{\xi}(s(x))\right\rangle \pi(x)\boldsymbol{\xi}(s(x))d\lambda^{r(y)}(x) \\
&= \int_{\mathcal{G}^{s(y)}} \langle v, \pi(x)\boldsymbol{\xi}(s(x))\rangle\, \pi(yx)\boldsymbol{\xi}(s(x))d\lambda^{s(y)}(x) \\
&= \pi(y)T_{s(y)}v.
\end{aligned}$$

□

Proposition 5.1.9. *Let $\mathcal{G} := \sqcup_{i=1}^{n} G_i$ be a compact group bundle of compact groups with $\mathcal{G}^{(0)}$ discrete and $\mathfrak{H}$ a $\mathcal{G}$-Hilbert bundle. If $\mathfrak{H}$ is irreducible, then for each $i \in \mathcal{G}^{(0)}$, $\mathfrak{H}_i$ is finite-dimensional.*

Proof. Let $\{T_i\}_{i\in\mathcal{G}^{(0)}}$ be as defined in Proposition 5.1.8. By Proposition 5.1.7, since $\mathfrak{H}$ is irreducible, each $T_i = c_i I_{\mathfrak{H}_i}$. As T_i is compact for each i, we must have that $\mathfrak{H}_i$ is finite-dimensional. □

If we take the dimension of a $\mathcal{G}$-Hilbert bundle to be $\sup\left\{\dim \mathfrak{H}_u : u \in \mathcal{G}^{(0)}\right\}$, Proposition 5.1.9 implies that every irreducible $\mathcal{G}$-Hilbert bundle of a compact group bundle of finitely many compact groups is finite-dimensional. This is not true with an infinite number of compact groups, as we can take G_i to be any finite discrete group with an irreducible representation of dimension n. We now present, in contrast to the compact group case, a reducible $\mathcal{G}$-Hilbert bundle of a compact groupoid which does not decompose into the direct sum of $\mathcal{G}$-Hilbert bundles. This refutes [4, Theorem 2.16] which asserts that every continuous $\mathcal{G}$-Hilbert bundle of a compact groupoid decomposes into the direct sum of irreducible $\mathcal{G}$-Hilbert bundles.

Example 5.1.10. Let $\mathcal{G} := \mathbb{T}\sqcup\mathbb{T}$. Let $\mathfrak{H}$ be the $\mathcal{G}$-Hilbert bundle with $\mathfrak{H}_1 = \mathbb{C}$, $\mathfrak{H}_2 = \mathbb{C}^2$ and linear actions $\pi((1,\theta)) = \theta^2$, $\pi((2,\theta)) = \begin{bmatrix} \theta & 0 \\ 0 & \theta^3 \end{bmatrix}$. Then $\mathfrak{H}$ is certainly reducible, as the bundles given by $\mathfrak{M}_1 = \mathbb{C}$, $\mathfrak{M}_2 = \mathbb{C} \oplus 0$ and $\mathfrak{M}'_1 = \mathbb{C}$, $\mathfrak{M}'_2 = 0 \oplus \mathbb{C}$ are both

proper, invariant sub-bundles. However, $\mathfrak{H}$ does *not* decompose into the direct sum of invariant sub-bundles. Recall that for $\mathcal{G}$-Hilbert bundles $\mathfrak{G}$ and $\mathfrak{K}$ with linear actions $\{\varrho(x)\}$ and $\{\tau(x)\}$ respectively, their direct sum has Hilbert field $\{\mathfrak{G}_u \oplus \mathfrak{K}_u\}$ and linear actions $\{\varrho(x) \oplus \tau(x)\}$. Therefore, any direct sum of $\mathcal{G}$-Hilbert bundles of $\mathcal{G} = \mathbb{T} \sqcup \mathbb{T}$ must have Hilbert space fibers with dimension at least 2. However, $\dim(\mathfrak{H}_1) = 1$, so it cannot be expressed as the direct sum of *any* $\mathcal{G}$-Hilbert bundles, let alone irreducible ones.

The natural development past this point would be to show a groupoid version of Schur's orthogonality relations. [4] proceeds by considering a compact groupoid $\mathcal{G}$ possessing a Haar system $\{\lambda^u\}_{u\in\mathcal{G}^{(0)}}$ and a coefficient $\mathfrak{H}_{\boldsymbol{\xi},\boldsymbol{\eta}}(x) = \langle \mathcal{T}(x)\boldsymbol{\xi}(s(x)), \boldsymbol{\eta}(r(x))\rangle$ of a $\mathcal{G}$-Hilbert bundle $\mathfrak{H}$, and attempting to decompose this by fixing bases for $\mathfrak{H}_{s(x)}$ and $\mathfrak{H}_{r(x)}$. However, these decompositions still retain a dependency on the variable x, and so Amini considers the restrictions of these functions to $\mathcal{G}_u^v$, the linear span of which is we denote $\mathcal{E}_{u,v}^{\mathfrak{H}}$. Such restrictions fix the range and source maps, removing this dependency. In [4, Theorem 3.6], the author presents orthogonality relations and claims that $\mathcal{E}_{u,v}^{\mathfrak{H}} \perp \mathcal{E}_{u,v}^{\mathfrak{H}'}$ as subspaces of $L^2(\mathcal{G}_u^v)$ when $\mathfrak{H} \not\simeq \mathfrak{H}'$. We claim this is not the case.

Example 5.1.11. Consider $\mathcal{G} := \mathbb{T} \sqcup \mathbb{T}$. The irreducible representations of $\mathbb{T}$ are categorized by [32, Theorem 4.5] by $\chi_n : \mathbb{T} \to \mathbb{C}$ with $\chi_n(\theta) = \theta^n$. By Lemma 5.1.1 and Proposition 5.1.3, we may define an irreducible $\mathcal{G}$-Hilbert bundle by choosing irreducible representations for each group in the bundle. Let us form $\mathfrak{H}$ by choosing χ_1 for the first copy of $\mathbb{T}$, and χ_3 for the second. Similarly, define $\mathfrak{G}$ by choosing χ_1 and χ_2. By Corollary 5.1.6.1, $\mathfrak{H} \not\simeq \mathfrak{G}$ as $\chi_3 \not\simeq \chi_2$, however, every $f \in \mathcal{E}_{1,1}^{\mathfrak{H}}$ is a coefficient of χ_1, as is every $g \in \mathcal{E}_{1,1}^{\mathfrak{H}'}$. If we let $0 \neq f = g$, we have $\langle f, f\rangle > 0$ so $\mathcal{E}_{1,1}^{\mathfrak{H}} \not\perp \mathcal{E}_{1,1}^{\mathfrak{H}'}$.

We still see a form of the orthogonality relations when the associated representations are *all* not unitarily equivalent, however, in this case we are just restating the orthogonality relations for groups on a fiberwise basis.

5.2 Coefficient Spaces $A_{\mathfrak{H}}$

We begin this section by recalling the definition of Fourier coefficient spaces (Definition 2.3.19). Let G be a locally compact group, and π a continuous unitary representation. The Fourier coefficient space associated to π is given by

$$A_\pi(G) := \overline{\text{span}}\left\{\pi_{\xi,\eta} : \xi, \eta \in \mathcal{H}_\pi\right\}$$

the closure taken with respect to the $B(G)$ norm.

Definition 5.2.1. Let $\mathcal{G}$ be a locally compact groupoid, and $\mathfrak{H}$ a $\mathcal{G}$-Hilbert bundle. Define the Fourier space of $\mathfrak{H}$ to be

$$A_{\mathfrak{H}}(\mathcal{G}) := \overline{\text{span}}\left\{\mathfrak{H}_{\boldsymbol{\xi},\boldsymbol{\eta}} : \boldsymbol{\xi}, \boldsymbol{\eta} \in \Delta_b(\mathfrak{H})\right\}$$

the closure taken with respect to the $B(\mathcal{G})$ norm.

Question 5.2.2. *In the group case, we have that $A_L(G) = A(G)$ for the left regular representation L. Is this necessarily true for $A_\Lambda(\mathcal{G})$ and $A(\mathcal{G})$ for the left regular bundle? If not, is there a $\mathcal{G}$-Hilbert bundle such that $A_{\mathfrak{H}}(\mathcal{G}) = A(\mathcal{G})$?*

Some authors work with the left regular bundle with multiplicity (see [76]). It is possible that the $\mathcal{G}$-Hilbert bundle needed in Question 5.2.2 is the left regular bundle Λ with infinite multiplicity. Recall that, for a locally compact group G and a function $u \in B(\mathcal{G})$, the representation and pair of vectors which achieve the norm in $B(G)$ given by Proposition 2.3.8 are referred to as the *Eymard representation* and *Eymard vectors*. For $u \in B(\mathcal{G})$ for a locally compact groupoid $\mathcal{G}$, if there is a $\mathcal{G}$-Hilbert bundle $\mathfrak{H}$ and bounded sections $\boldsymbol{\xi}, \boldsymbol{\eta}$ such that $u = \mathfrak{H}_{\boldsymbol{\xi},\boldsymbol{\eta}}$ and $\|u\|_{B(\mathcal{G})} = \|\boldsymbol{\xi}\|_\Delta \|\boldsymbol{\eta}\|_\Delta$, $\mathfrak{H}$ will be referred to as an *Eymard $\mathcal{G}$-Hilbert bundle* for u, and $\boldsymbol{\xi}, \boldsymbol{\eta}$ as *Eymard sections*.

Proposition 5.2.3. *Let $\mathcal{G} := \sqcup_{i=1}^n G_i$ be a group bundle. Every $u \in B(\mathcal{G})$ possesses an Eymard $\mathcal{G}$-Hilbert bundle along with Eymard sections.*

Proof. By Proposition 4.2.2, we have $B(\mathcal{G})$ is isometrically isomorphic to $\ell^\infty\text{-}\oplus_{i=1}^n B(G_i)$, and so for $u \in B(\mathcal{G})$ we have

$$\|u\|_{B(\mathcal{G})} = \max\left\{\left\|R^i_{B(\mathcal{G})} u\right\|_{B(G_i)} : 1 \leq i \leq n\right\}.$$

Given that $R^i_{B(\mathcal{G})}u \in B(G_i)$, there is a unitary representation π^i of G_i and vectors $\xi_i, \eta_i \in \mathcal{H}_{\pi^i}$ such that $R^i_{B(\mathcal{G})}u = \pi^i_{\xi_i,\eta_i}$ and $\left\|R^i_{B(\mathcal{G})}u\right\|_{B(\mathcal{G})} = \|\xi_i\|\,\|\eta_i\|$. Without loss of generality, we may assume $\|\eta_i\| = 1$ for each i. Define $\mathfrak{H}$ to be the $\mathcal{G}$-Hilbert bundle corresponding to $\{\pi^i\}_{i=1}^n$ as in Lemma 5.1.1, and set $\boldsymbol{\xi}(i) = \xi_i$, $\boldsymbol{\eta}(i) = \eta_i$. Then $\boldsymbol{\xi}, \boldsymbol{\eta} \in \Delta_b(\mathfrak{H})$, $u = \mathfrak{H}_{\boldsymbol{\xi},\boldsymbol{\eta}}$, and

$$\begin{aligned}\|\boldsymbol{\xi}\|_\Delta \|\boldsymbol{\eta}\|_\Delta = \left(\max_i \|\xi_i\|\right)\left(\max_i \|\eta_i\|\right) = \max_i \|\xi_i\|\,\|\eta_i\| = \max_i \left\{\left\|R^i_{B(\mathcal{G})}u\right\|_{B(G_i)}\right\} \\ = \|u\|_{B(\mathcal{G})}\,.\end{aligned}$$

□

Question 5.2.4. *For every locally compact groupoid $\mathcal{G}$ and every $u \in B(\mathcal{G})$, does there exist an Eymard $\mathcal{G}$-Hilbert bundle and Eymard sections?*

Lemma 5.2.5. *Let $X_1, \ldots, X_n$ be normed metric spaces. Then $\overline{\ell^\infty - \oplus_{i=1}^n X_i} = \ell^\infty - \oplus_{i=1}^n \overline{X_i}$.*

Proof. Choose $x \in \overline{\ell^\infty - \oplus_{i=1}^n X_i}$. Then x is the limit of a sequence of terms $(x_i^m)_{i=1}^n$ and

$$\|x - (x_i^m)_{i=1}^n\| \to_m 0.$$

This implies that $\|x_i - x_i^m\|_{X_i} \to_m 0$, so $x \in \ell^\infty - \oplus_{i=1}^n \overline{X_i}$. The proof of the converse is nearly identical, just read in reverse. □

Proposition 5.2.6. *Let $\mathcal{G} := \sqcup_{i=1}^n G_i$ be a group bundle with $\mathcal{G}^{(0)}$ discrete. Let $\mathfrak{H}$ be a $\mathcal{G}$-Hilbert bundle with associated representations given by Lemma 5.1.1 $\{\pi^i\}_{i=1}^n$. Then $A_{\mathfrak{H}} \simeq \ell^\infty - \oplus_{i=1}^n A_{\pi^i}$.*

Proof. By Proposition 4.2.2, we have that

$$\operatorname{span}\left\{\mathfrak{H}_{\boldsymbol{\xi},\boldsymbol{\eta}} : \boldsymbol{\xi}, \boldsymbol{\eta} \in \Delta_b(\mathfrak{H})\right\} \simeq \ell^\infty - \oplus_{i=1}^n \operatorname{span}\left\{\pi^i_{\boldsymbol{\xi}(i),\eta(i)}\right\}.$$

By Lemma 5.2.5, we have

$$\begin{aligned} A_{\mathfrak{H}} = \overline{\operatorname{span}\left\{\mathfrak{H}_{\boldsymbol{\xi},\boldsymbol{\eta}} : \boldsymbol{\xi},\boldsymbol{\eta} \in \Delta_b(\mathfrak{H})\right\}} &\simeq \overline{\ell^\infty - \oplus_{i=1}^n \operatorname{span}\left\{\pi^i_{\boldsymbol{\xi}(i),\eta(i)}\right\}} \\ &= \ell^\infty - \oplus_{i=1}^n \overline{\operatorname{span}\left\{\pi^i_{\boldsymbol{\xi}(i),\eta(i)}\right\}} \\ &= \ell^\infty - \oplus_{i=1}^n A_{\pi^i}. \end{aligned}$$

□

Proposition 5.2.7. *Let $\mathcal{G}$ be the group bundle formed from the collection of compact, Hausdorff groups $\{G_t\}_{t\in T}$. Equip $\mathcal{G}$ with a locally compact and Hausdorff topology such that $\mathcal{G}^{(0)}$ is discrete. Then, for $u \in A(\mathcal{G})$, we have*

$$\|u\|_{A(\mathcal{G})} = \sup_{t\in T}\left\{\sum_{\pi\in\hat{G}_t} d_\pi \left\|\pi\left(R^t_{A(\mathcal{G})}u\right)\right\|_1\right\}.$$

Proof. By Proposition 4.2.3, we have that $A(\mathcal{G}) \simeq c_0 \oplus_{t\in T} A(G_t)$ via the map $u \mapsto (R^t_{A(\mathcal{G})}u)_{t\in T}$. As each G_t is a compact group, we have

$$\left\|R^t_{A(\mathcal{G})}u\right\|_{A(G_t)} = \sum_{\pi\in\hat{G}_t}\left\|\pi\left(R^t_{A(\mathcal{G})}u\right)\right\|_1.$$

□

As stated earlier, our motivation was to show compact groupoid versions of Propositions 2.3.20-2.3.22, as the combination of these propositions in the compact group case leads to the formula

$$\|u\|_{A(G)} = \sum_{\pi\in\hat{G}} d_\pi \|\pi(f)\|_1. \tag{5.1}$$

A great deal of the original proofs for these propositions relies on structure that we either have yet to develop in the groupoid case, or is non-existent, such as the dual nature between $B(G)$, $C^*(G)$ and $A(G)$ with $VN(G)$. These may be circumvented in the compact group case via arguments involving Schur orthogonality and the Peter-Weyl theorem, and most of the propositions in sequence follow from the previous. A lack of these in the general groupoid case has hindered our progress. We present a partial result in extending these propositions and pose the following questions.

Proposition 5.2.8. *Let $\mathcal{G}$ be a compact groupoid and $\mathfrak{H}$ a $\mathcal{G}$-Hilbert bundle. If there exist $(\boldsymbol{\xi}_n)_{n\in\mathbb{N}}$ and $(\boldsymbol{\eta}_n)_{n\in\mathbb{N}} \subseteq \Delta_b(\mathfrak{H})$ with $\sum_{n\in\mathbb{N}} \|\boldsymbol{\xi}_n\|_\Delta \cdot \|\boldsymbol{\eta}_n\|_\Delta < \infty$, then $u = \sum_{n\in\mathbb{N}} \mathfrak{H}_{\boldsymbol{\xi}_n,\boldsymbol{\eta}_n} \in A_{\mathfrak{H}}$.*

Proof. [12, pg. 36] shows that a series $\sum_{n\in\mathbb{N}} x_n$ in a Banach space converges if it is absolutely convergent. We clearly have

$$\sum_{n\in\mathbb{N}} \left\|\mathfrak{H}_{\boldsymbol{\xi}_n,\boldsymbol{\eta}_n}\right\|_{A(\mathcal{G})} \leq \sum_{n\in\mathbb{N}} \|\boldsymbol{\xi}_n\|_\Delta \cdot \|\boldsymbol{\eta}_n\|_\Delta < \infty$$

and so $u = \sum_{n\in\mathbb{N}} \mathfrak{H}_{\boldsymbol{\xi}_n,\boldsymbol{\eta}_n} \in A_{\mathfrak{H}}$. □

Question 5.2.9. *Let $\mathcal{G}$ be a compact groupoid and $\mathfrak{H}$ an irreducible $\mathcal{G}$-Hilbert bundle. For every $u \in A_{\mathfrak{H}}$ such that u has an Eymard $\mathcal{G}$-Hilbert bundle, does there exist $\{\boldsymbol{\xi}_n\}_{n\in\mathbb{N}}$ and $\{\boldsymbol{\eta}_n\}_{n\in\mathbb{N}} \subseteq \Delta_b(\mathfrak{H})$ such that $u = \sum_{n\in\mathbb{N}} \mathfrak{H}_{\boldsymbol{\xi}_n,\boldsymbol{\eta}_n}$ and $\|u\|_{A(\mathcal{G})} = \sum_{n\in\mathbb{N}} \|\boldsymbol{\xi}_n\|_\Delta \cdot \|\boldsymbol{\eta}_n\|_\Delta$?*

Question 5.2.10. *Let $\mathcal{G}$ be a compact groupoid and let $\mathfrak{H}, \mathfrak{G}$ be two $\mathcal{G}$-Hilbert bundles. Under what conditions do we have $A_{\mathfrak{H}} \cap A_{\mathfrak{G}} = \{0\}$?*

Question 5.2.11. *Let $\{\mathfrak{H}^i\}_{i\in I}$ be a family of representations, and set $\mathfrak{H} = \oplus_{i\in I}\mathfrak{H}^i$. Assume that each $u \in A_{\mathfrak{H}}$ has an Eymard $\mathcal{G}$-Hilbert bundle. Is it true that $A_{\mathfrak{H}}$ is precisely the functions of $B(\mathcal{G})$ which can be written $u = \sum_{i\in I} u_i$ with $u_i \in A_{\mathfrak{H}^i}$ and $\sum_{i\in I} \|u_i\|_{A(\mathcal{G})} < \infty$?*

Proposition 5.2.12. *Given Proposition 5.2.8 and assuming Question 5.2.9 is true, then Question 5.2.11 is true.*

Proof. That such sums belong to $A_{\mathfrak{H}}$ is clear as each $A_{\mathfrak{H}^i} \leq A_{\mathfrak{H}}$. Conversely, let $u \in A_{\mathfrak{H}}$. Then, by Question 5.2.9, there are sequences of bounded sections $\{\boldsymbol{\xi}_n\}_{n\in\mathbb{N}}, \{\boldsymbol{\eta}_n\}_{n\in\mathbb{N}}$ such that $u = \sum \mathfrak{H}_{\boldsymbol{\xi}_n,\boldsymbol{\eta}_n}$. As $\mathfrak{H} = \oplus\mathfrak{H}^i$, we may further decompose $\boldsymbol{\xi}_n = \sum_i \boldsymbol{\xi}_{n,i}$ and $\boldsymbol{\eta}_n = \sum_i \boldsymbol{\eta}_{n,i}$ with $\|\boldsymbol{\xi}_n\|^2 = \sum_i \|\boldsymbol{\xi}_{n,i}\|^2$ (similarly for $\boldsymbol{\eta}_n$). Thus, each $\mathfrak{H}_{\boldsymbol{\xi}_n,\boldsymbol{\eta}_n}$ can be written as $\sum_i \mathfrak{H}^i_{\boldsymbol{\xi}_{n,i},\boldsymbol{\eta}_{n,i}}$. Next, note

$$\sum_i \|\boldsymbol{\xi}_{n,i}\| \cdot \|\boldsymbol{\xi}_{n,i}\| \leq \left(\sum_i \|\boldsymbol{\xi}_{n,i}\|^2\right)^{1/2} \left(\sum_i \|\boldsymbol{\eta}_{n,i}\|^2\right)^{1/2} < \infty$$

which implies

$$\sum_n \sum_i \left\|\boldsymbol{\xi}_{n,i}\right\| \cdot \left\|\boldsymbol{\eta}_{n,i}\right\| \leq \sum_n \|\boldsymbol{\xi}_n\| \cdot \|\boldsymbol{\eta}_n\| < \infty.$$

Thus, $u(x) = \sum_i \sum_n \mathfrak{H}^i_{\boldsymbol{\xi}_{n,i},\boldsymbol{\eta}_{n,i}}(x)$ and $u_i := \sum_n \mathfrak{H}_{\boldsymbol{\xi}_{n,i},\boldsymbol{\eta}_{n,i}}$ which also establishes the inequality. □

Question 5.2.13. *Let $\mathcal{G}$ be a compact group. If $\mathfrak{H}$ and $\mathfrak{G}$ are two $\mathcal{G}$-Hilbert bundles with $\mathfrak{H} \nsim \mathfrak{G}$, does it follow that $A_{\mathfrak{H}\oplus\mathfrak{G}} = A_{\mathfrak{H}} \oplus A_{\mathfrak{G}}$?*

Question 5.2.14. *Let $(\mathfrak{H}^i)_{i\in I}$ be pairwise non-unitarily equivalent $\mathcal{G}$-Hilbert bundles of a compact groupoid $\mathcal{G}$ with $\mathfrak{G} = \oplus\mathfrak{H}_i$. Then for $u \in A_{\mathfrak{G}}$, is there is a unique decomposition $u = \sum_i u_i$ with $u_i \in A_{\mathfrak{H}^i}$ and $\|u\| = \sum_i \|u_i\|$?*

Question 5.2.15. *Let $\mathcal{G}$ be a compact groupoid. Is there an suitably generalized version of the operator $\pi(f)$ that allows one to write a formula which reduces to Equation (5.1) when $\mathcal{G}$ is a group?*

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Appendix

SUPPLEMENTAL CONTENT

A.1 Topological Results

This piece of the appendix contains several topological results and definitions we will intermittently use. The results may be sourced from most suitable graduate topology books, such as [65]. For a set X, let $\mathcal{P}(X)$ denote the *power set of* X, which is the set of all subsets of X.

Definition A.1.1. Let X be a set, and $\tau \subseteq \mathcal{P}(X)$. If:

1. $\varnothing, X \in \tau$;
2. If $\{U_\alpha\}_{\alpha \in I} \subseteq \tau$ then $\bigcup_{\alpha \in I} U_\alpha \in \tau$;
3. If $\{U_1, \ldots, U_n\} \subseteq \tau$ then $\bigcap_{i=1}^{n} U_i \in \tau$,

then τ is called a *topology* on X, and the pair (X, τ) is called a *topological space*.

The sets contained in τ are called the *open sets* of X. It is often difficult to exactly describe every open set of a topology. Instead, we give a basis of open sets from which the rest of the topology is determined. If (X, τ) is a topological space, a set $\mathcal{B} \subseteq \tau$ is a *basis for the topology* if every open set $U \in \tau$ can be written as the union of some sub-family of $\mathcal{B}$.

Example A.1.2.

1. Let X be any set. The *discrete topology* is given by the basis consisting of every singleton point. In this topology, it can be shown that every set is open.
2. Let $X = \mathbb{R}$. The standard topology used on $\mathbb{R}$ is given by the basis of open intervals, that is, $\mathcal{B} := \{(a, b) : a, b \in \mathbb{R}, a \leq b\}$.

3. Let (X, τ) be a topological space and $Y \subseteq X$. The *subspace topology* on Y is given by the open sets $U \cap Y$ where U is open in X.

If (X, τ) is a topological space, a set $S \subseteq X$ is called *closed* if $S^c \in \tau$, that is, if S^c is open. A point $x \in X$ is called a *limit point* of a set S if, for every open set $U \in \tau$ such that $x \in U$, $U \cap S$ contains a point of S distinct from x. An alternative characterization of closed sets is that a set S is closed if it contains all its limit points.

Given the notion of limit points, we can define sequences (and nets). A sequence in a topological space X is a function $f : \mathbb{N} \to X$, though we can denote a sequence as we do for real sequences: $\{x_n\}_{n\in\mathbb{N}}$. A sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x in X if, for every open neighborhood U of x, there exists an $N \in \mathbb{N}$ such that $x_n \in U$ for $n \geq N$. Nets are simply a generalization of a sequence being a function from a directed set into a topological space. A directed set is a non-empty set A along with a transitive and reflexive relation (denoted $\leq$) with the added condition that for every $a, b \in A$ there is a $c \in A$ with $a \leq c$ and $b \leq c$.

An open cover of a set S is a collection of open sets $\{U_\alpha\}_{\alpha\in I}$ such that $S \subseteq \cup_{\alpha\in I} U_\alpha$. A subcover is any subset of an open cover. A set $K \subseteq X$ is called *compact* if every open cover of K admits a finite subcover. A topological space X will be called *locally compact* if for every $x \in X$, there exists an open set U and a compact set K such that $x \in U \subseteq K$. A *Hausdorff* space is a topological space where points can be separated via open sets, i.e. $x \neq y$ implies the existence of U, V open and disjoint such that $x \in U$, $y \in V$.

Proposition A.1.3. *Let (X, τ) be a topological space, and $Y \subseteq X$ a subset equipped with the subspace topology. Then a set K is compact in Y if and only if it is compact in X.*

Proof. Suppose K is compact in Y. Then every open cover of K in Y admits a finite subcover. Let $\{U_\alpha\}_{\alpha\in I}$ be an open cover of K in X. Then $\{U_\alpha \cap Y\}_{\alpha\in I}$ is an open cover of K in Y which by assumption admits a finite subcover $\{U_i \cap Y\}_{1\leq i\leq n}$. Thus $\{U_i \cap Y\}_{1\leq i\leq n}$ is a finite subcover of K in X, so K is compact in X.

Conversely, suppose K is compact in X and let $\{V_\alpha\}_{\alpha\in I}$ be an open cover of K in Y. Then every $V_\alpha = U_\alpha \cap Y$ for some open subset of X U_α. This implies that $\{U_\alpha\}_{\alpha\in I}$ is an open cover of K in X, which admits a finite subcover. Intersecting this finite subcover with Y gives a finite subcover of K in Y. □

A topological space is called σ-compact if it can be expressed as the union of a countable number of compact sets ([81, Definition 2.16]). Additionally, if X is a topological space with a basis $\mathcal{B}$, if $\mathcal{B}$ is countable then X is called *second countable*.

A *normal* space is a topological space X that satisfies the following property: if A and B are two disjoint closed subsets of X, then there exists open $A \subseteq U$ and $B \subseteq V$ with U and V disjoint.

Proposition A.1.4. *Every compact Hausdorff space is normal.*

Proof. Let A and B be two disjoint closed subsets of X. For the moment, assume $B = \{b\}$. For each $a \in A$, we may find disjoint open sets U_a and V_a separating a and b. A is compact as a closed subset of a compact space. Since $\cup_{a\in A} U_a$ is an open cover of A, we may find a finite subcover $\cup_{i=1}^n U_{a_i}$. Letting $U := \cup_{i=1}^n U_{a_i}$ and $V := \cap_{i=1}^n V_{a_i}$ concludes the proof in this case.

Next let B be arbitrary. For each $b \in B$ apply the above argument to find two open disjoint open sets $A \subseteq U_b$ and $b \in V_b$ separating b from A. As B is compact and clearly a subset of $\cup V_b$, we may extract a finite subcover $\cup_{i=1}^n V_{b_i}$. Taking $U = \cap_{i=1}^n U_{b_i}$ and $V = \cup_{i=1}^n V_{b_i}$ establishes the claim. □

Lemma A.1.5. *Let X be a locally compact space. Let K be a compact subset of X and D open such that $K \subseteq D$. Then there exists an open set E with $\overline{E}$ compact and $K \subseteq E \subseteq \overline{E} \subseteq D$.*

Proof. As in the previous proposition, first assume that $K = \{x\}$. Take any compact set N containing x. In the induced topology, N is compact and Hausdorff thus normal. In particular, we may apply the previous proposition to the sets $A := \{x\}$ and $B :=$

$N \setminus D$ to produce two disjoint sets $U, V \subseteq N$ which are open in the induced topology with $x \in U$, $B \subseteq V$.

As these are open in the induced topology, there exists U_0 and V_0 open in X with $U = U_0 \cap N$ and $V = V_0 \cap N$. Let $E := \text{Int}(U)$. By definition $x \in E$ and E is open. As $E \subseteq U \subseteq N$, $\overline{E} \subseteq \overline{N} = N$, giving the compactness of E.

Finally, we have the containments

$$E \cap V_0 \subseteq U \cap V_0 \subseteq N \cap U_0 \cap V_0 = U \cap V = \varnothing$$

so that $E \subseteq V_0^c$. Since V_0^c is closed, $\overline{E} \subseteq V_0^c$. Combining this with the fact that $\overline{E} \subseteq N$

$$\overline{E} \subseteq N \cap V_0^c \subseteq N \cap (N \setminus D)^c = N \cap D \subseteq D.$$

With the singleton case in hand, assume K is an arbitrary compact set. For each $x \in K$, find E_x such that $K \subseteq E_x \subseteq \overline{E_x} \subseteq D$. By compactness, we may cover K with finitely many of the E_x. Set $E = \cup_{i=1}^n E_{x_i}$. Then we have

$$K \subseteq E \subseteq \overline{E} \subseteq \cup_{i=1}^n \overline{E_{x_i}} \subseteq D.$$

□

Topological spaces were introduced partly to generalize the notion of continuity. Let (X, τ_X) and (Y, τ_Y) be two topological spaces. A function $f : X \to Y$ is *continuous* if $f^{-1}(V) \in \tau_X$ for every $V \in \tau_Y$ (if the preimage of every open set is open). Equivalently, f is continuous at a point x if for every net $\{x_\alpha\}_{\alpha \in I}$ which converges to x in X, the net $\{f(x_\alpha)\}_{\alpha \in I}$ converges to $f(x)$ in Y.

A *partition of unity* of a topological space X is a set R of continuous functions from X to $[0, 1]$ such that for every point in x: (i) There is a neighborhood of x such that all but finitely many functions of R are 0, and (ii) $\sum_{f \in R} f(x) = 1$ for every $x \in X$. If the space is locally compact, given any open cover $\{U_i\}_{i \in I}$ there exists a partition of unity $\{\rho_j\}_{j \in J}$ (J possibly distinct from I) such that each ρ_j has compact support, and for each $j \in J$, $\operatorname{supp} \rho_j \subseteq U_i$ for some i. There is another formulation where the space need not be locally compact, in which case the partition of unity is indexed over

the same set, without necessarily requiring the functions to be compactly supported. When the space is compact, both conditions are satisfied.

Given a space is both locally compact and Hausdorff, a finite open cover always has a continuous partition of unity subordinate to it, meaning $\operatorname{supp} \rho_j \subseteq U_j$ for each $j \in J$.

Lemma A.1.6 (Urysohn's Lemma for locally compact spaces)**.** *Let X be a locally compact space, and let $K, F \subseteq X$ be two disjoint sets, with K compact and F closed. Then there exists a continuous function $f : X \to [0, 1]$ such that $f|_K = 1$ and $f|_F = 0$.*

Proof. Apply Lemma 1 to the pair K and F^c to find an open E with $\overline{E}$ compact with $K \subseteq E \subseteq \overline{E} \subseteq F^c$. We may apply this again to K and E to find another open G with $\overline{G}$ compact such that $K \subseteq G \subseteq \overline{G} \subseteq E$.

Restrict attention to $\overline{E}$, a normal space as it is compact and Hausdorff. Applying Urysohn's general lemma, there is a continuous function $g : \overline{E} \to [0, 1]$ with $g|K = 1$ and $g|_{E \setminus G} = 0$. Define

$$f(x) := \begin{cases} g(x) & x \in \overline{E} \\ 0 & x \in \overline{E}^c. \end{cases}$$

As $f|_E = g|_E$, $f|_E$ is continuous. Furthermore, $f|_{\overline{G}^c} = 0$. Given this, we have two open sets $E, \overline{G}^c$ with $E \cup \overline{G}^c = X$ with $f|_E$ and $f|_{\overline{G}^c}$ continuous. Then f is continuous with the remaining properties obvious. □

Proposition A.1.7 (Tietz Extension Theorem)**.** *If X is a normal space, A a closed subset of X, and $f : A \to \mathbb{R}$ is continuous, then there exists a continuous extension of f to X, that is, there is a map $F : X \to \mathbb{R}$, continuous on X, with $F(a) = f(a)$ for every $a \in A$. Moreover, F may be chosen such that $\sup_A \{|f(a)|\} = \sup_X \{|F(x)|\}$.*

Remark A.1.8. The Tietz Extension Theorem can be extended to $f : A \to \mathbb{R}^n$ by applying the theorem to the components of f. Since $\mathbb{R}^2$ is homeomorphic to $\mathbb{C}$, this lets us apply the Tietz extension theorem to complex valued functions.

A.2 Measure Theory and the Haar Measure

Measures are set functions defined on σ-algebras and are designed to give a meaningful way to discuss the "size" of sets. This appendix section provides a background into measure theory and integration as well as the Haar measure of a group. The notes here were developed using [Chapter 3, [41]], [Section 2.2,[32]] as well as various sections of [89]. Let X be a set, and $\mathcal{F}$ a family of subsets of X.

Definition A.2.1. The family $\mathcal{F}$ is called a σ-algebra (or a σ-field) if

1. $X \in \mathcal{F}$;
2. If $A \in \mathcal{F}$ then $A^c \in \mathcal{F}$;
3. Let I be a countable index set. If, for all $i \in I$, $A_i \in \mathcal{F}$ then $\cup_i A_i \in \mathcal{F}$.

As a remark, using Item 2 and Item 3 in the above definition we can show that $\mathcal{F}$ must also be closed under countable intersection. It can also be shown that the intersection of any number of σ-algebras is again a σ-algebra. Given a collection B of subsets of X, define the σ-algebra generated by B to be the intersection over all σ-algebras containing B. If X is a topological space, the most common σ-algebra on X is the *Borel σ-algebra*, which is the σ-algebra generated by the open sets. Thus the Borel σ-algebra also contains all closed sets. A pair $(X, \mathcal{F})$ where $\mathcal{F}$ is a σ-algebra of X is called a *measurable space*.

If Y is a closed subset of X, and X is a topological space equipped with the Borel σ-algebra $\mathcal{B}_X$, then the Borel σ-algebra on Y is a sub-algebra of $\mathcal{B}_X$.

Definition A.2.2. [41, pg. 118] If $(X, \mathcal{F})$ is a measurable space, a function $\mu : \mathcal{F} \to \mathbb{R}_+ \cup \{\infty\}$ is called a countably additive measure if

1. $\mu(\varnothing) = 0$
2. $\mu\left(\bigcup_{i=1}^{\infty} \mu(A_i)\right) = \sum_{i=1}^{\infty} \mu(A_i)$ for all pairwise disjoint collections $\{A_i\}_{i=1}^{n} \subseteq \mathcal{F}$.

We will refer to countably additive measures simply as *measures*.

Example A.2.3. Let X be any set, and $\mathcal{F}$ any σ-algebra on X. Define $\mu(S) := |S|$ for $S \in \mathcal{F}$. This is called the *counting measure*, or the *discrete measure*.

Example A.2.4. Let $(X, \mathcal{B}_X)$ be a measurable space equipped with the Borel σ-algebra. Let Y be a closed subset of X, and equip Y with its Borel σ-algebra $\mathcal{B}_Y$. If μ is a measure on X, define the *restricted measure* on Y to be $\mu|_{\mathcal{B}_Y}$.

A function $f : X \mapsto \mathbb{R}_+ \cup \{\infty\}$ will be called *measurable* if $f^{-1}(U) \in \mathcal{F}$ for every open set U. A complex-valued function will be measurable if its real and imaginary parts are both measurable. If X is equipped with the Borel σ-algebra (or any other σ-algebra containing the open sets) then any continuous function is measurable.

A theory of integration can be developed for measurable functions on a measurable space $(X, \mathcal{F})$ equipped with measure μ. We first define integrals of characteristic functions by setting $\int_G \mathbb{1}_S(x) d\mu(x) = \mu(S)$. Next, we extend this definition linearly to functions of the form $\sum_{k=1}^n c_k \mathbb{1}_{S_k}$, called *simple functions*.

From this, if f is non-negative and measurable, and $E \in \mathcal{F}$, we set

$$\int_E f(x) d\mu(x) := \sup \left\{ \int_E g(x) d\mu(x) : 0 \leq g \leq f, g \text{ simple} \right\}.$$

Remark A.2.5. Given a function $f : X \to \mathbb{C}$, we may express it as a linear combination of real valued functions via $\mathrm{Re}(f) := \frac{f+\overline{f}}{2}$ and $\mathrm{Im}(f) := \frac{f-\overline{f}}{2i}$. Then $f = \mathrm{Re}(f) + i\mathrm{Im}(f)$. It is then immediate that f is continuous if and only if both $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are continuous. Following this, we may write any real valued functions as the sum of pointwise non-negative functions by defining $f^+ := \max\{f, 0\}$ and $f^- := \max\{-f, 0\}$. Then $f = f^+ - f^-$. We may apply this to the real and imaginary parts above to produce a decomposition of any complex valued function into the sum of four real valued, non-negative functions. Noting $|f| = f^+ + f^-$, $f^+ = \frac{f+|f|}{2}$ and $f^- = \frac{|f|-f}{2}$, we see that if f is continuous, then so are f^+ and f^-.

Since every $f : X \to \mathbb{C}$ can be written as a linear combination of functions with non-negative values, we can define $\int f(x) d\mu(x)$. When the measure is understood, we write $\int f(x) dx$.

Definition A.2.6. Let $(X, \mathcal{F})$ be a measurable space such that $\mathcal{F}$ contains all open sets. A measure μ on X will be called:

1. *Inner regular* if $\mu(U) = \sup\{\mu(K) : K \text{ compact}, K \subseteq U\}$ for every open U;
2. *Outer regular* if $\mu(S) = \inf\{\mu(U) : U \text{ open}, S \subseteq U\}$ for all $S \in \mathcal{F}$;
3. *Regular* if it is both inner and outer regular;
4. *Radon* if it is regular and finite on all compact sets.

From here we will only consider locally compact groups as our base set X and $\mathcal{F}$ their Borel σ-algebra. As groups possess a product, we can define left and right translates of a set S by an element $x \in G$ by $xS := \{xs : s \in S\}$ and $Sx := \{sx : s \in S\}$ (we may similarly define XS and SX for a set X). A measure μ on a locally compact group is *left (resp. right) translation invariant* if $\mu(xS) = \mu(S)$ (or $\mu(Sx) = \mu(S)$) for all Borel subsets $S \subseteq G$ and $x \in G$. All locally compact groups possess a non-zero translation invariant Radon measure, called a *Haar measure* [Theorem 2.10, [32]].

Moreover the Haar measure for a locally compact group is unique up to multiplication by a positive constant. If G is non-abelian, we must make a choice between left or right translation invariance. Typically authors elect to let the Haar measure μ be left invariant. If μ is the Haar measure of a locally compact group, setting $\mu_x(E) = \mu(Ex)$ also produces a Haar measure. Since μ is unique up to multiplication by a positive constant, there is some $\Delta_G(x)$ such that $\mu_x = \Delta_G(x)\mu$.

This is the *modular function* of the group, a continuous homomorphism from G to the positive real numbers under multiplication. For a more detailed description, see [32, Section 2.2]. In one sense, the modular function measures the failure of the left invariant Haar measure to also be right invariant. Clearly, if G is abelian, then Δ_G is identically 1. Groups where the modular function is always 1 are called *unimodular*. Abelian and compact groups are always unimodular. The modular function provides several change of variable formulas in integration, the main two being $d\mu(x^{-1}) = \Delta(x^{-1})d\mu(x)$ and $d\mu(xy) = \Delta(y)d\mu(x)$.

A.3 C^*-Algebras and Weak Containment of Representations

A Banach space A equipped with an involution $* : A \to A$ such that $(x+y)^* = x^* + y^*$, $(\lambda x)^* = \overline{\lambda} x^*$ and $(xy)^* = y^* x^*$ is called a *Banach $*$-algebra*.

Definition A.3.1. A *C^*-algebra* is a Banach $*$-algebra whose norm satisfies the inequality $\|x^* x\| = \|x^*\| \, \|x\|$.

Example A.3.2.

1. The space $M_n(\mathbb{C})$ equipped with the operator norm obtained by viewing $A \in M_n(\mathbb{C})$ as a linear operator from $\mathbb{C}^n \to \mathbb{C}^n$ is a C^* algebra. Its involution is given by the conjugate transpose.

2. The set $B(\mathcal{H})$ of bounded operators on a Hilbert space under the operator norm is a C^* algebra, with its involution given by the adjoint operator. Every C^* algebra can be represented as $B(\mathcal{H})$ for some Hilbert Space $\mathcal{H}$ as per the Gelfand-Naimark Theorem (see [88, Theorem 9.14]).

Given any locally compact group, we may complete $C_c(G)$ with respect to an appropriate C^*-norm to form a C^*-algebra. Taking the norm

$$\|f\|_{full} := \sup \{\|\pi(f)\| : \pi \text{ is a representation of } G\}$$

where $\pi(f)$ is the induced representation of $C_c(G)$ acting on $\mathcal{H}_\pi$, the completion gives $C^*_{full}(G)$. If we restrict the norm above to only consider the left regular representation, we form the reduced norm and the completion gives the reduced group C^*-algebra $C^*_{red}(G)$.

As with groups, it will be helpful to study representations of C^*-algebras. A $*$-representation of a C^*-algebra A is a homomorphism $\pi : A \to B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ such that $\pi(a^*) = \pi(a)^*$. From this definition we could study much of the same consequences as group representation theory enjoys. However, our goals lie elsewhere, in a more general notion known as weak containment.

The original definition of weak containment, is in terms of representations of $C*$-algebras. Let A be a C^* algebra, with $\pi : A \to B(\mathcal{H})$ a $*$-representation. Define $\ker \pi := \{a \in A : \pi(a) = 0\}$.

Definition A.3.3. [48, Definition 1.7.2] Let S and T be sets of representations of a C^*-algebra A. We say S *is weakly contained in* T, denoted $S \prec T$, if $\cap_{\tau \in T} \ker \tau \subseteq \cap_{\sigma \in S} \ker \sigma$.

If S consists of a single representation π we write $\pi \prec T$. The collections S and T are called *weakly equivalent* if $S \prec T$ and $T \prec S$. For a given representation π of a C^*-algebra A, and $\xi \in \mathcal{H}_\pi$, the function $\phi_\xi(a) = \pi_{\xi,\xi}(a)$ is *associated to* π [pg 30, [48]]. A function is associated to a set of representations S if it is associated to some representation in S. For a locally compact group G, the collection of non-degenerate $*$-representations of $C^*(G)$ and the set of (classes of) continuous unitary representations of G exist in bijective correspondence. Via this correspondence, we can extend the notion of weak containment to unitary representations of groups.

Lemma A.3.4. *Let π be a unitary representation of a locally compact group G, and S a set of unitary representations of G. Then $\pi \prec S$ if and only if $\pi \prec \oplus_{\sigma \in S} \sigma$.*

Proof. The proof is shown exactly by the observation that $\ker \oplus_{\sigma \in S} \sigma = \cap_{\sigma \in S} \ker \sigma$, which is immediately deduced from the fact that an $f \in C^*(G)$ belongs to $\ker \oplus_{\sigma \in S} \sigma$ if and only if $(\oplus_{\sigma \in S} \sigma) f = 0$, which is true if and only if $\sigma(f) = 0$ for every $\sigma \in S$. □

Arsac later provided an equivalent condition for weak containment in the group setting, without having to rely on passing through to the group C^*-algebra. A collection of these results can be found translated in [48].

Proposition A.3.5. *[48, Proposition 1.7.6] Let S be a set of representations of a locally compact group G, and π a representation of G. Then $\pi \prec S$ if and only if for any function φ associated to π, there exists a net $\{\psi_\alpha\}_{\alpha \in I}$ of linear combinations of functions associated to S such that $\psi_\alpha \to \varphi$ uniformly on every compact subset of G.*

For a discrete group, compact subsets are finite, and so uniform convergence is equivalent to pointwise convergence.