# The Inverse Electromagnetic Scattering Problem for a Partially Coated Dielectric

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**Abstract.** We use the linear sampling method to determine the shape and surface conductivity of a partially coated dielectric from a knowledge of the far field pattern of the scattered electromagnetic wave at fixed frequency. A mathematical justification of the method is provided for both the scalar and vector case based on the use of a complete family of solutions. Numerical examples are given for the scalar case.

**Keywords**: Inverse scattering problem, electromagnetic waves, mixed boundary value problems.

## 1. Introduction

The use of dielectrics to coat a perfect conductor in an effort to help hostile objects avoid detection has a long history [14]. More recently, metallic coatings have been used in an effort to make benign dielectric objects look hostile, e.g. coating wooden decoys to make them appear as tanks to radar. In general the obstacle is only partially coated and the extent and the composition of the coating is unknown. Such situations lead to mixed boundary value problems in scattering theory and particular difficulties arise in trying to solve the inverse problem since the boundary conditions on the scattering object are unknown. Inverse scattering problems of this type have been the subject of investigation by us in a series of papers [5]-[8]. In particular, our focus in these papers has been on the inverse problem of determining the shape and surface impedance or surface conductivity from a knowledge of the far field pattern of the scattered electromagnetic wave at fixed frequency.

The inverse scattering problem of determining the shape and surface conductivity of a partially coated dielectric from far field data is considerably more difficult to solve than the complimentary case of a partially coated perfect conductor. This is due to the fact that in case of a coated dielectric the waves can penetrate into the obstacle, thus leading to electromagnetic fields inside the scattering object. Indeed, in our first effort to solve this problem we were only able to provide a mathematical justification of our reconstruction algorithm for the case of the scattering of TE-polarized plane waves by an infinite cylinder [5]. In the present paper we overcome this problem. In particular, a mathematical basis is given for an algorithm that determines the shape and surface conductivity of a partially coated dielectric for both the scalar and vector case. This is accomplished by avoiding the need to solve an interior transmission problem for Maxwell's equations and instead relying on the construction of a special complete family of solutions. Unfortunately, the price paid for the success of this approach is that we can now only handle constant surface conductivities rather than variable surface conductivities as in [5].

The plan of our paper is as follows. In Section 2 we formulate the direct and inverse scattering problem for a dielectric that is partially coated by a highly conductive layer. We then consider the scalar case TM-polarized plane waves and use the linear sampling method [9] to determine the shape and surface conductivity of the scattering object. We then extend our results to the full vector case in $\mathbb{R}^3$ and conclude by providing some numerical examples in the scalar case. In particular, we shall provide an example showing that in the case of very large surface conductivity (e.g. a thick coating) it is possible to determine the portion of the boundary that is coated.

## 2. Formulation of the direct and inverse scattering problem

Let $D \subset \mathbb{R}^3$ be a bounded region with boundary $\Gamma$ such that $D_e := \mathbb{R}^3 \setminus \overline{D}$ is connected. Each simply connected piece of $D$ is assumed to be a Lipschitz curvilinear polyhedron. Moreover we assume that the boundary $\Gamma = \Gamma_1 \cup \Pi \cup \Gamma_2$ is split into two disjoint parts $\Gamma_1$ and $\Gamma_2$ having $\Pi$ as their possible common boundary in $\Gamma$. The domain $D$ is the support of an anisotropic object that is partially coated on a portion $\Gamma_2$ of the boundary by a very thin homogeneous layer of a highly conductive material and the incident field is a time-harmonic electromagnetic plane wave with frequency $\omega$ ($\Gamma_1$ may be the empty set which corresponds to a fully coated obstacle!). The interior electric and magnetic fields $\tilde{E}^{int}$, $\tilde{H}^{int}$, and the exterior electric and magnetic fields $\tilde{E}^{ext}$, $\tilde{H}^{ext}$, satisfy

$$\left\{ \begin{array}{l} \nabla \times \tilde{E}^{ext} - i\omega\mu_0 \tilde{H}^{ext} = 0 \\ \\ \nabla \times \tilde{H}^{ext} + i\omega\epsilon_0 \tilde{E}^{ext} = 0 \end{array} \right. \qquad \text{in } \; D_e \tag{2.1}$$

$$\left\{ \begin{array}{l} \nabla \times \tilde{E}^{int} - i\omega\mu_0 \tilde{H}^{int} = 0 \\ \\ \nabla \times \tilde{H}^{int} + (i\omega\epsilon(x) - \sigma(x))\tilde{E}^{int} = 0 \end{array} \right. \qquad \text{in } \; D \tag{2.2}$$

and on the boundary $\Gamma$

$$\nu \times \tilde{E}^{ext} - \nu \times \tilde{E}^{int} = 0 \qquad \text{on} \quad \Gamma \tag{2.3}$$

$$\nu \times \tilde{H}^{ext} - \nu \times \tilde{H}^{int} = 0 \qquad \text{on} \quad \Gamma_1 \tag{2.4}$$

$$\nu \times \tilde{H}^{ext} - \nu \times \tilde{H}^{int} = \tilde{\eta}\,(\nu \times \tilde{E}^{ext}) \times \nu \qquad \text{on} \quad \Gamma_2. \tag{2.5}$$

The electric permittivity $\epsilon_0$ and magnetic permeability $\mu_0$ of the exterior dielectric medium are positive constants whereas the scatterer has the same magnetic permeability $\mu_0$ as the exterior medium but the electric permittivity $\epsilon$ and conductivity $\sigma$ are real

$3 \times 3$ matrix valued functions. The constant $\tilde{\eta} > 0$ describes the physical properties of the thin coating layer [1]. If we define $\tilde{E}^{(ext,int)} = \frac{1}{\sqrt{\epsilon_0}} E^{(ext,int)}$, $\tilde{H}^{(ext,int)} = \frac{1}{\sqrt{\mu_0}} H^{(ext,int)}$, $k^2 = \epsilon_0 \mu_0 \omega^2$, $N(x) = \frac{1}{\epsilon_0}\left(\epsilon(x) + i\frac{\sigma(x)}{\omega}\right)$, and $\tilde{\eta} = \sqrt{\frac{\mu_0}{\epsilon_0}}\eta$ we obtain the transmission problem

$$\begin{cases} \nabla \times E^{ext} - ikH^{ext} = 0 \\ & \text{in } \; D_e \\ \nabla \times H^{ext} + ikE^{ext} = 0 \end{cases} \tag{2.6}$$

$$\begin{cases} \nabla \times E^{int} - ikH^{int} = 0 \\ & \text{in } \; D \\ \nabla \times H^{int} + ikN(x)E^{int} = 0 \end{cases} \tag{2.7}$$

$$\nu \times E^{ext} - \nu \times E^{int} = 0 \quad \text{on} \quad \Gamma \tag{2.8}$$

$$\nu \times H^{ext} - \nu \times H^{int} = 0 \quad \text{on} \quad \Gamma_1 \tag{2.9}$$

$$\nu \times H^{ext} - \nu \times H^{int} = \eta \left(\nu \times E^{ext}\right) \times \nu \quad \text{on} \quad \Gamma_2, \tag{2.10}$$

where the exterior field $E^{ext}$, $H^{ext}$ is given by

$$E^{ext} = E^i + E^s \tag{2.11}$$

$$H^{ext} = H^i + H^s, \tag{2.12}$$

$E^s$, $H^s$ is the scattered field satisfying the Silver Müller radiation condition

$$\lim_{r\to\infty} (H^s \times x - rE^s) = 0 \tag{2.13}$$

uniformly in $\hat{x} = x/|x|$, $r = |x|$, the incident field $E^i$,$H^i$ is given by

$$\begin{aligned} E^i(x) &:= \frac{i}{k}\nabla \times \nabla \times pe^{ikx\cdot d} = ik(d \times p) \times de^{ikx\cdot d} \\ H^i(x) &:= \nabla \times pe^{ikx\cdot d} \qquad\quad = ikd \times pe^{ikx\cdot d}, \end{aligned} \tag{2.14}$$

the wave number $k$ is positive, $d \in \Omega := \{x \in \mathbb{R}^3 : \; |x| = 1\}$ is a unit vector giving the direction of propagation and $p$ is the polarization vector.

In the following we assume that $N$ is a $3 \times 3$ symmetric matrix-valued function whose entries are in $C^1(\overline{D})$, and $N$ satisfies $\bar{\xi} \cdot \Im(N)\,\xi \geq 0$ and $\bar{\xi} \cdot \Re(N)\,\xi \geq \gamma|\xi|^2$ for all $\xi \in \mathbb{C}^3$ and all $x \in \overline{D}$ where $\gamma$ is a positive constant.

In order to formulate precisely the forward problem we need the following spaces. Letting $(H^s(D))^3$, $(H^s_{loc}(D_e))^3$ and $(H^s(\Gamma))^3$, $s \in \mathbb{R}$, denote the product of the standard Sobolev spaces defined on $D, D_e$ and $\Gamma$ respectively (with the convention $H^0 = L^2$), and

$$\begin{aligned} H(\mathrm{curl}, D) &:= \{u \in (L^2(D))^3 \, : \, \nabla \times u \in (L^2(D))^3\} \\ L^2_t(\Gamma) &:= \{u \in (L^2(\Gamma))^3 \, : \, \nu \cdot u = 0 \quad \text{on} \quad \Gamma\} \\ L^2_t(\Gamma_2) &:= \{u|_{\Gamma_2} : u \in L^2_t(\Gamma)\}, \end{aligned}$$

we introduce the space

$$X(D, \Gamma_2) := \{u \in H(\mathrm{curl}, D) \, : \, \nu \times u|_{\Gamma_2} \in L^2_t(\Gamma_2)\} \tag{2.15}$$

equipped with the norm

$$\|u\|^2_{X(D,\Gamma_2)} = \|u\|^2_{H(curl,D)} + \|\nu \times u\|^2_{L^2(\Gamma_2)}. \tag{2.16}$$

For the exterior domain $D_e$ we define the above spaces in the same way for every $D_e \cap B_R$, with $B_R$ a ball of radius $R$ containing $D$ and denote these spaces by $H_{loc}(\mathrm{curl}, D_e)$ and $X_{loc}(D_e, \Gamma_2)$, respectively. Finally, we introduce the trace space of $X(D, \Gamma_2)$ on $\Gamma$ by

$$Y(\Gamma) := \left\{ h \in (H^{-1/2}(\Gamma))^3 \,:\, \exists\, u \in H_0(\mathrm{curl}, B_R), \quad \begin{array}{l} \nu \times u|_{\Gamma_2} \in L^2_t(\Gamma_2) \\ \text{and} \quad h = \nu \times u|_\Gamma \end{array} \right\}$$

where $H_0(\mathrm{curl}, B_R)$ is the space of functions $u$ in $H(\mathrm{curl}, B_R)$ satisfying $\hat{x} \times u = 0$ on the boundary of $B_R$. As shown in [6] $Y(\Gamma)$ is a Banach space with respect to the norm

$$\|h\|^2_{Y(\Gamma)} := \inf\{\|u\|^2_{H(curl,B_R)} + \|\nu \times u\|^2_{L^2_t(\Gamma_2)}\} \tag{2.17}$$

where the infimum is taken over all functions $u \in H_0(\mathrm{curl}, B_R)$ such that $\nu \times u|_{\Gamma_2} \in L^2_t(\Gamma_2)$ and $h = \nu \times u|_\Gamma$. In fact $Y(\Gamma)$ coincides with $H^{-\frac{1}{2}}_{div}(\Gamma) \cap L^2_t(\Gamma_2)$ where

$$H^{-\frac{1}{2}}_{div}(\Gamma) := \left( u \in (H^{-\frac{1}{2}}(\Gamma))^3, \quad \nu \cdot u = 0, \quad \mathrm{div}_\Gamma\, u \in H^{-\frac{1}{2}}(\Gamma) \right)$$

is the trace space of $\nu \times u|_\Gamma$ for $u \in H_0(\mathrm{curl}, B_R)$ (see [4] and [2], [3] for the case of Lipshitz boundaries). We also recall that the trace space of $(\nu \times u) \times \nu|_\Gamma$ for $u \in H(\mathrm{curl}, B_R)$ is defined by

$$H^{-\frac{1}{2}}_{curl}(\Gamma) := \left( u \in (H^{-\frac{1}{2}}(\Gamma))^3, \quad \nu \cdot u = 0, \quad \mathrm{curl}_\Gamma\, u \in H^{-\frac{1}{2}}(\Gamma) \right),$$

and a duality relation is defined between $H^{-\frac{1}{2}}_{div}(\Gamma)$ and $H^{-\frac{1}{2}}_{div}(\Gamma)$.

Expressing the magnetic fields in terms of the electric fields, the direct scattering problem becomes a particular case of the following problem: Given $f \in Y(\Gamma)$, $h \in Y(\Gamma)$, $h_2 \in L^2_t(\Gamma_2)$ find $E^s \in X_{loc}(D_e, \Gamma_2)$, $E^{int} \in X(D, \Gamma_2)$ such that

$$\nabla \times \nabla \times E^s - k^2 E^s = 0 \qquad \text{in} \quad D_e \tag{2.18$a$}$$

$$\nabla \times \nabla \times E^{int} - k^2 N(x) E^{int} = 0 \quad \text{in} \quad D \tag{2.18$b$}$$

$$\nu \times E^s - \nu \times E^{int} = f \quad \text{on} \quad \Gamma \tag{2.18$c$}$$

$$\nu \times (\nabla \times E^s) - \nu \times (\nabla \times E^{int}) = h + \left\{ \begin{array}{ll} 0 & \text{on } \Gamma_1 \\ ik\eta E^s_T + h_2 & \text{on } \Gamma_2 \end{array} \right. \tag{2.18$d$}$$

$$\lim_{r\to\infty} ((\nabla \times E^s) \times x - ikrE^s) = 0 \tag{2.18$e$}$$

where $u_T$ denotes the tangential component $(\nu \times u) \times \nu$. Note that the direct scattering problem corresponds to $f := -\nu \times E^i|_\Gamma$, $h := -\nu \times (\nabla \times E^i)|_\Gamma$, and $h_2 := ik\eta\, E^i_T|_{\Gamma_2}$. The following theorem concerning the well-posedness of the above problem was proved in [8].

**Theorem 2.1** *The transmission problem (2.18a)-(2.18e) has a unique solution $E^{int} \in X(D,\Gamma_2)$, $E^s \in X_{loc}(D_e,\Gamma_2)$ which satisfies*

$$\|E^{int}\|_{X(D,\Gamma_2)} + \|E^s\|_{X(B_R\setminus\overline{D},\Gamma_2)} \leq C\left(\|f\|_{Y(\Gamma)} + \|h\|_{Y(\Gamma)} + \|h_2\|_{L^2_t(\Gamma_2)}\right) \tag{2.19}$$

*for some positive constant $C$ depending on $R$ but not on $f$, $h$ and $h_2$.*

It is known [10] that the radiating solution $E^s$ to (2.18$a$)-(2.18$e$) has the asymptotic behavior

$$E^s(x) = \frac{e^{ik|x|}}{|x|}\left\{E_\infty(\hat{x}) + O\left(\frac{1}{|x|}\right)\right\} \tag{2.20}$$

as $|x| \to \infty$, where $E_\infty$ is defined on the unit sphere $\Omega$ and is known as the *electric far field pattern*. In the case of incident plane waves given by (2.14) the electric far field pattern depends on the incident direction and polarization which will be indicated by $E_\infty(\hat{x}) = E_\infty(\hat{x}, d, p)$.

The *inverse scattering problem* we are concern with is to determine $D$ and $\eta$ from a knowledge of the electric far field pattern $E_\infty(\hat{x}, d, p)$ of the scattered field $E^s$ for $\hat{x}$, $-d \in \Omega_0$, where $\Omega_0$ is a subset of the unit sphere $\Omega$, and three linearly independent polarizations $p$. Note that no a priori knowledge of the amount of coating is required.

## 3. The scalar case

First we assume the scatterer is an infinitely long cylinder with axis in the $z$-direction and assume that the incident electromagnetic field is a plane wave propagating in the direction perpendicular to the cylinder. Let the bounded domain $D \subset \mathbb{R}^2$ with Lipschitz boundary $\Gamma$ be the cross section of the cylinder such that the exterior domain $D_e := \mathbb{R}^2 \setminus \overline{D}$ is connected. We denote by $\nu$ the outward unit normal to $\Gamma$ defined almost everywhere on $\Gamma$. Again the boundary $\Gamma$ is split into $\Gamma = \Gamma_1 \cup \Pi \cup \Gamma_2$. Here $\Gamma_1$ corresponds to the uncoated part and $\Gamma_2$ corresponds to the coated part. We assume that the dielectric is orthotropic, i.e. the matrix $N$ is of the form

$$N = \begin{pmatrix} n_{11} & n_{12} & 0 \\ n_{21} & n_{22} & 0 \\ 0 & 0 & n \end{pmatrix}.$$

If we consider incident waves such that the electric field is polarized parallel to the z axis, then the electric fields have only a component in the $z$ direction, i.e.

$$E^i = (0,\ 0,\ u^i), \qquad E^{int}_0 = (0,\ 0,\ v), \qquad E^s = (0,\ 0,\ u^s).$$

Then the direct scattering problem for the electric field reads: Given $f \in H^{\frac{1}{2}}(\Gamma)$, $h_1 \in H^{-\frac{1}{2}}(\Gamma_1)$ and $h_2 \in H^{-\frac{1}{2}}(\Gamma_2)$ find $v \in H^1(D)$ and $u^s \in H^1_{loc}(D_e)$ such that

$$\Delta u^s + k^2\, u^s = 0 \qquad \text{in} \quad D_e \tag{3.1$a$}$$

$$\Delta v + k^2 n(x)\, v = 0 \qquad \text{in} \quad D \tag{3.1$b$}$$

$$v - u^s = f \qquad \text{on} \quad \Gamma \tag{3.1c}$$

$$\frac{\partial v}{\partial \nu} - \frac{\partial u^s}{\partial \nu} = h_1 \qquad \text{on} \quad \Gamma_1 \tag{3.1d}$$

$$\frac{\partial v}{\partial \nu} - \frac{\partial u^s}{\partial \nu} = ik\eta\, u^s + h_2 \qquad \text{on} \quad \Gamma_2 \tag{3.1e}$$

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0 \tag{3.1f}$$

where $r = |x|$, $n \in C^1(\overline{D})$, $\Re(n) > 0$ and $\Im(n) \geq 0$. Here $H^1(D)$, $H^1_{loc}(D_e)$ are the usual Sobolev spaces, $H^{\frac{1}{2}}(\Gamma)$ and $H^{-\frac{1}{2}}(\Gamma)$ are the corresponding trace space and its dual respectively, and

$$H^{\frac{1}{2}}(\Gamma_0) := \{u|_{\Gamma_0} : u \in H^{\frac{1}{2}}(\Gamma)\}$$

for $\Gamma_0 \subset \Gamma$.

Due to the compact embedding of $H^{\frac{1}{2}}(\Gamma_2)$ into $L^2(\Gamma_2)$, the problem (3.1*a*)-(3.1*f*) can be seen as a compact perturbation of the same problem for $\eta = 0$ which is a particular case of the problem considered in [12]. Therefore the Fredholm alternative can be applied to (3.1*a*)-(3.1*f*). In particular, to prove the existence of a solution to (3.1*a*)-(3.1*f*) it suffices to show only the uniqueness.

**Lemma 3.1** *The problem (3.1a)-(3.1f) has at most one solution.*

**Proof**. Applying Green's formula in $D$ and $D_e \cap B_R$ to $v, \overline{v}$ and $u^s, \overline{u}^s$ respectively, where $u^s, v$ is a solution corresponding to $f = h_1 = h_2 = 0$ we obtain

$$\int_D (|\nabla v|^2 - k^2 n(x)|v|)\, dy = \int_\Gamma \overline{v}\frac{\partial v}{\partial \nu}\, ds$$
$$= ik\eta \int_{\Gamma_2} |u^s|\, ds + \int_{B_R} \overline{u^s}\frac{\partial u^s}{\partial \nu}\, ds - \int_{D_e \cap B_R} (|\nabla u^s|^2 - k^2|u^s|)\, dy.$$

Hence

$$\Im\left(\int_{B_R} u^s \frac{\partial \overline{u}^s}{\partial \nu}\, ds\right) \geq 0$$

and the uniqueness follows from Rellich's lemma and the unique continuation principle [10].

Summarizing the above we have the following result:

**Theorem 3.2** *The problem (3.1a)-(3.1f) has a unique solution* $v \in H^1(D)$, $u^s \in H^1_{loc}(D_e)$ *which satisfies*

$$\|v\|_{H^1(D)} + \|u^s\|_{H^1(B_R\setminus\overline{D})} \leq C\left(\|f\|_{H^{\frac{1}{2}}(\Gamma)} + \|h_1\|_{H^{-\frac{1}{2}}(\Gamma_1)} + \|h_2\|_{H^{-\frac{1}{2}}(\Gamma_2)}\right) \tag{3.2}$$

*for some positive constant* $C$ *depending on* $R$ *but not on* $f$, $h_1$ *and* $h_2$.

*3.1. The inverse problem*

We now consider the scattering problem (3.1*a*)-(3.1*f*) corresponding to incident plane waves, i.e. with $f := e^{ikx\cdot d}|_\Gamma$, $h_1 := \frac{\partial e^{ikx\cdot d}}{\partial \nu}|_{\Gamma_1}$ and $h_2 := \left(\frac{\partial e^{ikx\cdot d}}{\partial \nu} + ik\eta e^{ikx\cdot d}\right)\Big|_{\Gamma_2}$, where $d \in \Omega := \{x \in \mathbb{R}^2 : |x| = 1\}$ denotes the incident direction.
The corresponding scattered field $u^s$ has the asymptotic behavior [10]

$$u^s(x) = \frac{e^{ikr}}{\sqrt{r}} u_\infty(\hat{x}, d) + O(r^{-3/2}), \quad r \to \infty \tag{3.3}$$

where $u_\infty(\hat{x}, d)$ is defined on the unit circle $\Omega$ and is called the *far field pattern* of the radiating solution $u^s$. The inverse problem we consider here is to determine *both* the shape of the scattering object $D$ and the surface conductivity $\eta$ from a knowledge of the far field pattern $u_\infty(\hat{x}, d)$ for all incident plane waves $u^i := e^{ikx\cdot d}$, $d \in \Omega$, and all observation directions $\hat{x} \in \Omega$ (Note that it suffices to know the far field pattern corresponding to all $d \in \Omega_1 \subset \Omega$ and all $\hat{x} \in \Omega_2 \subset \Omega$ [9]; of particular interest is the case $d = -\hat{x} \in \Omega_0 \subset \Omega$).

Our method for determining $D$ and $\eta$ is based on the construction of a special complete set of functions in $H^{\frac{1}{2}}(\Gamma) \times H^{-\frac{1}{2}}(\Gamma)$. In this construction certain values of the wave number $k$ will play a special role. In particular, values of $k$ for which

$$\begin{cases} \Delta w + k^2 w = 0 \\ \Delta v + k^2 n(x) v = 0 \end{cases} \quad \text{in} \quad D \tag{3.4a}$$

$$v - w = 0 \quad \text{on} \quad \Gamma \tag{3.4b}$$

$$\frac{\partial v}{\partial \nu} - \frac{\partial w}{\partial \nu} = 0 \quad \text{on} \quad \Gamma_1 \tag{3.4c}$$

$$\frac{\partial v}{\partial \nu} - \frac{\partial w}{\partial \nu} = ik\eta\, w \quad \text{on} \quad \Gamma_2 \tag{3.4d}$$

has a non trivial solution $u \in H^1(D)$ and $v \in H^1(D)$ such that $\Delta v \in L^2(D)$ are called *transmission eigenvalues.*

**Lemma 3.3** *If $\Im(n(x_0)) > 0$ at $x_0 \in D$ then transmission eigenvalues do not exist.*

**Proof**. Let $v, w$ be a solution to (3.4*a*)-(3.4*d*). Applying Green's theorem to $v, \overline{v}$, making use of the boundary conditions and again applying Green's theorem to $w, \overline{w}$ we obtain

$$ik^2 \int_D \Im(n)|v|^2\, dy = \int_\Gamma \left( v\frac{\partial \overline{v}}{\partial \nu} - \overline{v}\frac{\partial v}{\partial \nu}\right) ds = -2ik\eta \int_{\Gamma_2} |w|^2\, ds.$$

Hence

$$\int_D \Im(n)|v|^2\, dy = 0 \qquad \text{and} \qquad \int_{\Gamma_2} |w|^2\, ds = 0$$

Since $\Im(n) > 0$ in a small ball $B_{x_0} \subset D$, from the first equality we obtain that $v = 0$ in $B_{x_0}$, whence by unique continuation $v \equiv 0$ in $D$. From the boundary conditions and the integral representation formula $w$ also vanishes in $D$.

Note that in the case when $\Im(n) = 0$ and $\Gamma_2 = \Gamma$ (fully coated obstacle) the transmission eigenvalues form a subset of the *Dirichlet eigenvalues* for

$$\begin{aligned} &\Delta u + k^2 n(x) u = 0 \qquad && \text{in} \quad D \\ &u = 0 && \text{on} \quad \Gamma. \end{aligned} \tag{3.5}$$

We now recall that a Herglotz wave function with kernel $g \in L^2(\Omega)$ is an entire solution of the Helmholtz equation defined by

$$v_g(x) = \int\limits_{\Omega} e^{-ikx\cdot d} g(d) ds(d), \qquad x \in \mathbb{R}^2. \tag{3.6}$$

The following theorem plays an important role in the analysis of the inverse problem. We consider the space of distributions

$$V(D) := \left\{ v \in H^1(D) : \, \Delta v + k^2 n(x) v = 0 \quad \text{in} \quad D \right\}$$

and define the subset of $H^{\frac{1}{2}}(\Gamma) \times H^{-\frac{1}{2}}(\Gamma)$ by

$$\mathcal{W} := \left\{ \left( v_g - v, \, \frac{\partial}{\partial \nu}(v_g - v) + ik\tilde{\eta} v_g \right) : \, g \in L^2(\Omega), \, v \in V(D) \right\}$$

where $\tilde{\eta} = \eta$ on $\Gamma_2$, $\tilde{\eta} = 0$ on $\Gamma_1$, and $v_g$ is a Herglotz wave function with kernel $g$.

**Theorem 3.4** *Suppose that $k$ is neither a Dirichlet eigenvalue nor a transmission eigenvalue. Then $\mathcal{W}$ is dense in $H^{\frac{1}{2}}(\Gamma) \times H^{-\frac{1}{2}}(\Gamma)$.*

**Proof**. Let $\varphi \in H^{-\frac{1}{2}}(\Gamma)$ and $\psi \in H^{\frac{1}{2}}(\Gamma)$ be such that

$$\int\limits_{\Gamma} (v_g - v)\varphi \, ds + \int\limits_{\Gamma} \left[ \frac{\partial}{\partial \nu}(v_g - v) + ik\tilde{\eta} v_g \right] \psi \, ds = 0 \tag{3.7}$$

for all $g \in L^2(\Omega)$ and $v \in V(D)$. Setting first $v = 0$ and interchanging the order of integrations we obtain

$$\int\limits_{\Omega} g(\hat{x}) \int\limits_{\Gamma} \left\{ \varphi(y) e^{-iky\cdot\hat{x}} + \psi(y) \left[ \frac{\partial e^{-iky\cdot\hat{x}}}{\partial \nu} + ik\tilde{\eta} e^{-iky\cdot\hat{x}} \right] \right\} ds(y) ds(\hat{x}) = 0$$

for all $g \in L^2(\Omega)$ and hence

$$\int\limits_{\Gamma} \varphi(y) e^{-iky\cdot\hat{x}} \, ds(y) + \int\limits_{\Gamma} \psi(y) \left[ \frac{\partial e^{-iky\cdot\hat{x}}}{\partial \nu} + ik\tilde{\eta} e^{-iky\cdot\hat{x}} \right] ds(y) = 0.$$

The left hand side of the above expression is the far field pattern of the potential

$$u(x) = \int\limits_{\Gamma} \varphi(y) \Phi(x, y) \, ds(y) + \int\limits_{\Gamma} \psi(y) \frac{\partial \Phi(x, y)}{\partial \nu} \, ds(y) + ik\tilde{\eta} \int\limits_{\Gamma} \psi(y) \Phi(x, y) \, ds(y)$$

where $\Phi(x, z) := \frac{i}{4} H_0^{(1)}(k|x - z|)$ with $H_0^{(1)}$ being a Hankel function of the first kind of order zero. Note that $u$ is in $H^1(D)$ and $H^1_{loc}(D_e)$, and $u$ satisfies the Helmholtz equation in $D$ and $D_e$. Therefore we conclude that $u(x) = 0$ in $D_e$. Using the jump relations across the boundary of the single and double layer potential [16] we then obtain

$$\psi = -u^-, \qquad \varphi = \frac{\partial u^-}{\partial \nu} + ik\tilde{\eta}u^- \qquad \text{on} \qquad \Gamma \tag{3.8}$$

where the superscript $-$ indicates that the limit is obtained by approaching the boundary $\Gamma$ from $D$.

Next we set $g = 0$ in (3.7), use (3.8) and Green's formula to obtain

$$\begin{aligned} 0 &= \int_\Gamma \left[ u^- \frac{\partial v}{\partial \nu} - v \left( \frac{\partial u^-}{\partial \nu} + ik\tilde{\eta}u^- \right) \right] ds \\ &= k^2 \int_D (1-n)uv\, dx - ik\tilde{\eta} \int_\Gamma uv\, ds \end{aligned} \tag{3.9}$$

Now let $w \in H^1(D)$ be unique the solution of

$$\begin{aligned} &\Delta w + k^2 n\, w = k^2(1-n)\, u && \text{in} \quad D \\ &w = 0 && \text{on} \quad \Gamma. \end{aligned}$$

Applying Green's theorem and (3.9) yield

$$\int_\Gamma v \frac{\partial w}{\partial \nu}\, ds = k^2 \int_D (1-n)uv\, dx = ik\tilde{\eta} \int_\Gamma uv\, ds$$

hence

$$\int_\Gamma v \left( \frac{\partial w}{\partial \nu} - ik\tilde{\eta}u \right) ds = 0$$

for all $v \in V(D)$ which implies that

$$\frac{\partial w}{\partial \nu} - ik\tilde{\eta}u = 0 \qquad \text{on} \quad \Gamma. \tag{3.10}$$

Note that for any $f \in H^{\frac{1}{2}}(\Gamma)$ one can find a unique $v \in V(D)$ such that $v|_\Gamma = f$ since $k$ is not a Dirichlet eigenvalue. We observe that $u$ and $\tilde{w} = u + w$ satisfy

$$\left\{ \begin{aligned} &\Delta u + k^2\, u = 0 \\ &\Delta \tilde{w} + k^2 n(x) \tilde{w} = 0 \end{aligned} \right. \qquad \text{in} \quad D \tag{3.11}$$

$$\tilde{w} - u = 0 \qquad \text{on} \quad \Gamma \tag{3.12}$$

$$\frac{\partial \tilde{w}}{\partial \nu} - \frac{\partial u}{\partial \nu} = 0 \qquad \text{on} \quad \Gamma_1 \tag{3.13}$$

$$\frac{\partial \tilde{w}}{\partial \nu} - \frac{\partial u}{\partial \nu} = ik\eta u \qquad \text{on} \quad \Gamma_2 \tag{3.14}$$

whence $u = \tilde{w} = 0$ provided that $k$ is not a transmission eigenvalue. Therefore (3.8) implies $\varphi = \psi = 0$ which proves the result.

The following lemma is a technical tool we need for solving the inverse problem. We define the closed subset $H(\Gamma)$ of $H^{\frac{1}{2}}(\Gamma) \times H^{-\frac{1}{2}}(\Gamma)$ by

$$H(\Gamma) = \left\{ \left( u|_\Gamma, \frac{\partial u}{\partial \nu} + ik\tilde{\eta}u \Big|_\Gamma \right) : u \in H^1(D), \;\; \Delta u + k^2 u = 0 \right\}.$$

Then we consider the bounded operator $\mathcal{B} : H(\Gamma) \to L^2(\Omega)$ which maps $f := u|_\Gamma$, $h_1 := \frac{\partial u}{\partial \nu}\big|_{\Gamma_1}$, $h_2 := \frac{\partial u}{\partial \nu} + ik\eta u\big|_{\Gamma_2}$ onto the far field pattern $u_\infty$ of the solution $(v, u^s)$ to $(3.1a)$-$(3.1f)$ with boundary data $(f, h_1, h_2)$.

**Lemma 3.5** *The operator $\mathcal{B} : H(\Gamma) \to L^2(\Omega)$ is compact, injective and has dense range providing that $k$ is neither a Dirichlet eigenvalue nor a transmission eigenvalue.*

**Proof**. Compactness is a simple consequence of the fact that $\mathcal{B}$ can be seen as a composition of the continuous solution operator to $(3.1a)$-$(3.1f)$ with the compact operator which maps a radiating solution to its far field (c.f. [10]).
Next we show the injectivity of $\mathcal{B}$. Let $\mathcal{B}(\varphi, \psi) = 0$ where $\varphi = u|_\Gamma$ and $\psi = \frac{\partial u}{\partial \nu} + ik\tilde{\eta}u\big|_\Gamma$ for $u \in H^1(D)$ such that $\Delta u + k^2 u = 0$, and let $(v, u^s)$ be the solution to $(3.1a)$-$(3.1f)$ corresponding to this boundary data. Hence $u^s \equiv 0$ for $x \in \mathbb{R}^2 \setminus \overline{D}$. This implies that $v$ satisfies

$$\Delta v + k^2 n\, v = 0 \quad \text{in} \quad D, \qquad v = \varphi \quad \text{and} \quad \frac{\partial v}{\partial \nu} = \psi \quad \text{on} \quad \Gamma.$$

Hence $(v, u)$ solves (3.11)-(3.14) and since $k$ is not a transmission eigenvalue we have that $v \equiv 0$ and consequently $\varphi = \psi \equiv 0$.
Finally it remains to show that $\mathcal{B}(H^1(D))$ is dense in $L^2(\Omega)$. It is known [10] that the far field patterns of radiating spherical wave functions

$$H_n^{(1)}(kr)e^{\pm in\phi}, \qquad\qquad n = 0, 1, \ldots$$

where $H_n^{(1)}$ are the Hankel functions of the first kind of order $n$, form a complete set in $L^2(\Omega)$. For a arbitrarily small $\epsilon$ and fixed $n$ from Theorem 3.4 we can find a Herglotz function $v_{g_\epsilon}$ and a $v_\epsilon \in V(D)$ such that

$$\begin{cases} v_\epsilon - H_n^{(1)}(kr)e^{\pm in\phi} = v_{g_\epsilon} + \alpha_\epsilon \\[2ex] \dfrac{\partial v_\epsilon}{\partial \nu} - \left( \dfrac{\partial}{\partial \nu} + ik\tilde{\eta} \right) H_n^{(1)}(kr)e^{\pm in\phi} = \dfrac{\partial v_{g_\epsilon}}{\partial \nu} + ik\tilde{\eta}v_{g_\epsilon} + \beta_\epsilon \end{cases} \tag{3.15}$$

on $\Gamma$ where

$$\|\alpha_\epsilon\|_{H^{\frac{1}{2}}(\Gamma)} \leq \epsilon \qquad \|\beta_\epsilon\|_{H^{-\frac{1}{2}}(\Gamma)} \leq \epsilon.$$

We observe that $v_\epsilon$ and $H_n^{(1)}(kr)e^{\pm in\phi}$ solve the transmission problem $(3.1a)$-$(3.1f)$ with data given by the right hand side of (3.15). Hence the far field pattern $u_\infty^\epsilon$ of the scattered field $u_\epsilon^s$ corresponding to $v_{g_\epsilon}$ as incident wave approximate the far field pattern of $H_n^{(1)}(kr)e^{\pm in\phi}$ with discrepancy $\epsilon$ because the far field pattern depends continuously on the scattered wave which on the other hand depends continuously on the data. Noticing that $u_\infty^\epsilon$ is in the range of $\mathcal{B}$ proves the lemma.

The method we will use here to determine $D$ and $\eta$ is based on solving the *far field equation*

$$(Fg)(\hat{x}) = \gamma e^{-ik\hat{x}\cdot z} \qquad g \in L^2(\Omega), \quad z \in D \tag{3.16}$$

where $\gamma = \frac{e^{i\pi/4}}{\sqrt{8\pi k}}$ and $F : L^2(\Omega) \to L^2(\Omega)$ is the *far field operator* given by

$$Fg(\hat{x}) := \int_\Omega u_\infty(\hat{x}, d) g(d)\, ds(d). \tag{3.17}$$

Note that $\gamma e^{-ik\hat{x}\cdot z}$ is the far field pattern of the fundamental solution $\Phi(x, z) := \frac{i}{4} H_0^{(1)}(k|x - z|)$ to the Helmholtz equation in $\mathbb{R}^2$ with $H_0^{(1)}$ being a Hankel function of the first kind of order zero.

**Theorem 3.6** *Assume that $k$ is neither a transmission eigenvalue nor a Dirichlet eigenvalue. Then we have:*

*(i) For $z \in D$ and every $\epsilon > 0$ there exists a solution $g_\epsilon^z \in L^2(\Omega)$ of the inequality*

$$\|Fg_\epsilon^z - \gamma e^{-ik\hat{x}\cdot z}\|_{L^2(\Omega)} < \epsilon$$

*such that*

$$\lim_{z\to\Gamma} \|g_\epsilon^z\|_{L^2(\Omega)} = \infty \quad \textit{and} \quad \lim_{z\to\Gamma} \|v_{g_\epsilon^z}\|_{H^1(D)} = \infty,$$

*where $v_{g_\epsilon^z}$ is the Herglotz wave function with kernel $g_\epsilon^z$.*

*(ii) For $z \in \mathbb{R}^2 \setminus \overline{D}$ and every $\epsilon > 0$ and $\delta > 0$ there exists a solution $g_{\epsilon,\delta}^z =\in L^2(\Omega)$ of the inequality*

$$\|Fg_{\epsilon,\delta}^z - \gamma e^{-ik\hat{x}\cdot z}\|_{L^2(\Omega)} < \epsilon + \delta$$

*such that*

$$\lim_{\delta\to 0} \|g_{\epsilon,\delta}^z\|_{L^2(\Omega)} = \infty \quad \textit{and} \quad \lim_{\delta\to 0} \|v_{g_{\epsilon,\delta}^z}\|_{H^1(D)} = \infty,$$

*where $v_{g_{\epsilon,\delta}^z}$ is the Herglotz wave function with kernel $g_{\epsilon,\delta}^z$.*

**Proof**. Let $z \in D$. Given $\epsilon > 0$, from Theorem 3.4 there exists $v_{g_\epsilon^z}$ with $g_\epsilon := g_\epsilon^z \in L^2(\Omega)$ and $v_\epsilon \in V(D)$ such that

$$\left\{\begin{array}{l} v_\epsilon - \Phi(\cdot,\, z) = v_{g_\epsilon} + \alpha_\epsilon \\[2ex] \dfrac{\partial v_\epsilon}{\partial \nu} - \dfrac{\partial \Phi(\cdot,\, z)}{\partial \nu} - ik\tilde{\eta}\Phi(\cdot,\, z) = \dfrac{\partial v_{g_\epsilon}}{\partial \nu} + ik\tilde{\eta} v_{g_\epsilon} + \beta_\epsilon \end{array}\right. \quad \text{on } \Gamma \tag{3.18}$$

where

$$\|\alpha_\epsilon\|_{H^{\frac{1}{2}}(\Gamma)} \le \epsilon \qquad \|\beta_\epsilon\|_{H^{-\frac{1}{2}}(\Gamma)} \le \epsilon.$$

We note that $Fg_\epsilon$ is the far field pattern of the solution to (3.1$a$)-(3.1$f$) with $f := v_{g_\epsilon}|_\Gamma$, $h := \frac{\partial v_{g_\epsilon}}{\partial \nu}|_{\Gamma_1}$ and $g := \left(\frac{\partial v_{g_\epsilon}}{\partial \nu} + ik\eta v_{g_\epsilon}\right)\Big|_{\Gamma_2}$ and $\gamma e^{-ik\hat{x}\cdot z}$ is the far field pattern of $\Phi(\cdot,\, z)$ which together with $v_\epsilon$ solves (3.1$a$)-(3.1$f$) with boundary data the right hand side of (3.18). Hence from the estimate (3.2) and the fact that the far field pattern depends continuously on the scattered field we obtain

$$\|Fg_\epsilon - \gamma e^{-ik\hat{x}\cdot z}\|_{L^2(\Omega)} < \epsilon.$$

Now let $z$ approach the boundary $\Gamma$ from inside. Using (3.2) for the solution $v_\epsilon$ and $\Phi(\cdot,\, z)$ of the transmission problem with the transmission condition (3.18), we obtain

$$\|v_\epsilon\|_{H^1(D)}+\|\Phi(\cdot,\, z)\|_{H^1(B_R\setminus\overline{D})} \le C\left(\|v_{g_\epsilon}\|_{H^{\frac{1}{2}}(\Gamma)} + \left\|\frac{\partial v_{g_\epsilon}}{\partial \nu}\right\|_{H^{-\frac{1}{2}}(\Gamma)} + \epsilon\right) \le \tilde{C}\left(\|v_{g_\epsilon}\|_{H^1(D)} + \epsilon\right)$$

where $C, \tilde{C}$ are two positive constants independent of $z$ and $\epsilon$. Since $\epsilon$ is fixed we finally have that

$$\|v_{g_\epsilon(\cdot,z)}\|_{H^1(D)} \to \infty \quad \text{and} \quad \|g_\epsilon(\cdot,\, z)\|_{L^2(\Omega)} \to \infty \quad \text{as} \quad z \to \Gamma.$$

Next let us consider $z \in \mathbb{R}^2 \setminus \overline{D}$. For these points $\gamma e^{-ik\hat{x}\cdot z}$ does not belong to the range of the operator $\mathcal{B}$ defined in Lemma 3.5 because $\Phi(\cdot, z)$ is not an $H^1$-solution to the Helmholtz equation in the exterior of $D$. But, from Lemma 3.5, using Tikhonov regularization, we can construct a regularized solution of the equation

$$\mathcal{B}(\varphi, \psi) = \gamma e^{-ik\hat{x}\cdot z}. \tag{3.19}$$

In particular, if $(\varphi_z^\alpha, \psi_z^\alpha) = \left(u_z^\alpha|_\Gamma,\, \frac{\partial u_z^\alpha}{\partial \nu} + ik\tilde{\eta}u_z^\alpha\Big|_\Gamma\right) \in H(\Gamma)$ with $u_z^\alpha \in H^1(D)$ being a solution of the Helmholtz equation is a regularized solution of (3.19) corresponding to the regularization parameter $\alpha$ chosen by a regular regularization strategy (e.g. the Morozov discrepancy principle [10]), we have

$$\|\mathcal{B}((\varphi_z^\alpha, \psi_z^\alpha)) - \gamma e^{-ik\hat{x}\cdot z}\|_{L^2(\Omega)} < \delta, \tag{3.20}$$

for an arbitrary small but fixed $\delta > 0$, and

$$\lim_{\alpha\to 0}\left(\|\varphi_z^\alpha\|_{H^{\frac{1}{2}}(\partial D)} + \|\psi_z^\alpha\|_{H^{-\frac{1}{2}}(\partial D)}\right) = \lim_{\alpha\to 0}\|u_z^\alpha\|_{H^1(D)} = \infty. \tag{3.21}$$

Note that in this case we have that $\alpha \to 0$ as $\delta \to 0$. Then the second part of the theorem follows from the fact that $u_z^\alpha$ can be approximated arbitrarily closely in the $H^1(D)$ norm by a Herglotz wave function $v_g$ [11], from the continuity of $\mathcal{B}$ and the fact that $Fg = \mathcal{B}(v_g|_\Gamma,\, \frac{\partial v_g}{\partial \nu} + ik\tilde{\eta}v_g\Big|_\Gamma)$. This ends the proof.

Theorem 3.6 provides a mathematical justification of the *linear sampling method* for determining $D$. In particular the boundary of $D$ is characterized as the set of points where $\|g\|_{L^2(\Omega)}$ becomes large for $g \in L^2(\Omega)$ being the (regularized) solution of the far field equation (3.16).

Having reconstructed $D$, we next provide a formula for determining the surface conductivity $\eta$.

**Lemma 3.7** *Assume that $k$ is neither a Dirichlet eigenvalue nor a transmission eigenvalue. For any point $z$ in $D$ and an arbitrary small $\epsilon > 0$ we have that*

$$\left|\int_D \Im(n)|\nabla v_\epsilon^z|^2\, dx + k\eta\int_{\Gamma_2} |v_{g_\epsilon^z} + \Phi(\cdot,\, z)|^2\, ds + 2k\pi|\gamma|^2 + \Im\left(v_{g_\epsilon^z}(z)\right)\right| \le C\epsilon$$

*where $v_\epsilon^z \in V(D)$ and the Herglotz wave function $v_{g_\epsilon^z}$ with $g_\epsilon^z \in L^2(\Omega)$ are such that $v_{g_\epsilon^z} - v_\epsilon^z$ approximates $\Phi(\cdot\,,z)$ in the $H^{\frac{1}{2}}(\Gamma)$ norm with discrepancy $\epsilon$, and $\left(\frac{\partial}{\partial\nu} + ik\tilde{\eta}\right)(v_{g_\epsilon^z} - v_\epsilon^z)$ approximates $\left(\frac{\partial}{\partial\nu} + ik\tilde{\eta}\right)\Phi(\cdot\,,z)$ in the $H^{-\frac{1}{2}}(\Gamma)$ norm with discrepancy $\epsilon$ and $C > 0$ is a constant independent of $\epsilon$.*

**Proof**. From Theorem 3.4, for given $\epsilon > 0$ there exists $v_\epsilon^z \in V(D)$ and a Herglotz wave function $v_{g_\epsilon^z}$ with $g_\epsilon^z \in L^2(\Omega)$ such that

$$\begin{cases} v_\epsilon^z - \Phi(\cdot,\,z) = v_{g_\epsilon^z} + \alpha_\epsilon \\ \dfrac{\partial v_\epsilon^z}{\partial\nu} - \dfrac{\partial\Phi(\cdot,\,z)}{\partial\nu} - ik\tilde{\eta}\Phi(\cdot,\,z) = \dfrac{\partial v_{g_\epsilon^z}}{\partial\nu} + ik\tilde{\eta}v_{g_\epsilon^z} + \beta_\epsilon \end{cases} \tag{3.22}$$

on $\Gamma$ where

$$\|\alpha_\epsilon\|_{H^{\frac{1}{2}}(\Gamma)} \le \epsilon \qquad \|\beta_\epsilon\|_{H^{-\frac{1}{2}}(\Gamma)} \le \epsilon. \tag{3.23}$$

Applying Green's formula we obtain

$$\int_\Gamma \left(v_\epsilon^z \frac{\partial\overline{v}_\epsilon^z}{\partial\nu} - \overline{v}_\epsilon^z\frac{\partial v_\epsilon^z}{\partial\nu}\right) ds = 2i\int_D \Im(n)|\nabla v_\epsilon^z|^2\,dx. \tag{3.24}$$

On the other hand using (3.22) we have that

$$\begin{aligned} \int_\Gamma \left(v_\epsilon^z \frac{\partial\overline{v}_\epsilon^z}{\partial\nu} - \overline{v}_\epsilon^z\frac{\partial v_\epsilon^z}{\partial\nu}\right) ds &= \int_\Gamma \left(\alpha_\epsilon\frac{\partial\overline{\beta}_\epsilon}{\partial\nu} - \overline{\alpha}_\epsilon\frac{\partial\beta_\epsilon}{\partial\nu}\right) ds \\ + \int_\Gamma \left(\alpha_\epsilon\frac{\partial\overline{v}_\epsilon^z}{\partial\nu} - \overline{v}_\epsilon^z\frac{\partial\beta_\epsilon}{\partial\nu}\right) ds &+ \int_\Gamma \left(v_\epsilon^z\frac{\partial\overline{\beta}_\epsilon}{\partial\nu} - \overline{\alpha}_\epsilon\frac{\partial v_\epsilon^z}{\partial\nu}\right) ds \\ + \int_\Gamma \left(W_\epsilon^z\frac{\partial\overline{W}_\epsilon^z}{\partial\nu} - \overline{W}_\epsilon^z\frac{\partial W_\epsilon^z}{\partial\nu}\right) ds &- 2ik\eta\int_{\Gamma_2}|W_\epsilon^z|^2\,ds \end{aligned} \tag{3.25}$$

where $W_\epsilon^z := (v_{g_\epsilon^z} + \Phi(\cdot,\,z))$. Note that due to (3.23) first three integrals in the right hand side of (3.25) become in absolute value smaller then $C\epsilon$, where the positive constant $C$ does not depend on $\epsilon$. Next, using again Green's formula, we obtain

$$\begin{aligned} &\int_\Gamma \left(W_\epsilon^z\frac{\partial\overline{W}_\epsilon^z}{\partial\nu} - \overline{W}_\epsilon^z\frac{\partial W_\epsilon^z}{\partial\nu}\right) ds \\ &= \int_\Gamma \left(\Phi(\cdot,z)\frac{\partial\overline{\Phi(\cdot,z)}}{\partial\nu} - \overline{\Phi(\cdot,z)}\frac{\partial\Phi(\cdot,z)}{\partial\nu}\right) ds \\ &+ \int_\Gamma \left(v_{g_\epsilon^z}\frac{\partial\overline{\Phi(\cdot,z)}}{\partial\nu} - \overline{\Phi(\cdot,z)}\frac{\partial v_{g_\epsilon^z}}{\partial\nu}\right) ds \\ &+ \int_\Gamma \left(\Phi(\cdot,z)\frac{\partial\overline{v}_{g_\epsilon^z}}{\partial\nu} - \overline{v}_{g_\epsilon^z}\frac{\partial\Phi(\cdot,z)}{\partial\nu}\right) ds \end{aligned}$$

Green's theorem applied to the radiating solution $\Phi(\cdot, z)$ of the Helmholtz equation in $D_e$ implies that

$$\int_\Gamma \left( \Phi(\cdot, z) \frac{\partial \overline{\Phi(\cdot, z)}}{\partial \nu} - \overline{\Phi(\cdot, z)} \frac{\partial \Phi(\cdot, z)}{\partial \nu} \right) ds = -2ik \int_\Omega \Phi_\infty(\cdot, z) \overline{\Phi_\infty(\cdot, z)} ds = -4ik\pi|\gamma|^2$$

and from the representation formula for $v_{g_\epsilon^z}$ we have that

$$\int_\Gamma \left( W_\epsilon^z \frac{\partial \overline{W}_\epsilon^z}{\partial \nu} - \overline{W}_\epsilon^z \frac{\partial W_\epsilon^z}{\partial \nu} \right) ds = -4ik\pi|\gamma|^2 - 2i\Im(v_{g_\epsilon^z}(z)). \tag{3.26}$$

Finally, combining (3.24), (3.25) and (3.26) yields the result.

**Theorem 3.8** *Let $z$ be a fixed point in $D$, $\Im(n) = 0$ and assume that $k$ is neither a Dirichlet eigenvalue nor a transmission eigenvalue. Then for every $\epsilon > 0$ there exists a Herglotz wave function $v_{g_\epsilon^z}$ with kernel $g_\epsilon^z \in L^2(\Omega)$ an approximate solution of the far field equation (3.16) such that*

$$\left| \eta + \frac{2k\pi|\gamma|^2 + \Im\left(v_{g_\epsilon^z}(z)\right)}{k\|(v_{g_\epsilon^z} + \Phi(\cdot, z))\|^2_{L^2(\Gamma_2)}} \right| \le \epsilon. \tag{3.27}$$

**Proof**. This theorem is a direct consequence of Theorem 3.6 and Lemma 3.7.

Note that since in practice $\Gamma_2$ is unknown, in general we must replace $\Gamma_2$ by $\Gamma$ in (3.27). In this case (3.27) only yields a *lower bound* for the unknown parameter $\eta$.

## 4. The vector case

Now we turn to the *inverse problem* for the vector case. Given the incident plane wave $E^i = ik(d \times p) \times d\, e^{ikx\cdot d}$ and the corresponding electric far field pattern $E_\infty(\hat{x}, d, p)$ for $\hat{x}, d$ in the unit sphere $\Omega$ and three linearly independent polarizations $p$, determine $D$ and $\eta$.

The analysis of this inverse problems follows the lines of the inverse problem in the scalar case treated in Section 3. We define *Maxwell eigenvalues* to be the values of $k$ for which

$$\begin{aligned} &\nabla \times \nabla \times E + k^2 N(x) E = 0 && \text{in} \quad D \\ &\nu \times E = 0 && \text{on} \quad \Gamma, \end{aligned}$$

has a nontrivial solution, and *transmission eigenvalues* the values of $k$ for which

$$\left\{ \begin{array}{l} \nabla \times \nabla \times E_0 - k^2 E_0 = 0 \\ \nabla \times \nabla \times E - k^2 N(x) E = 0 \end{array} \right. \qquad \text{in} \quad D \tag{4.1$a$}$$

$$\nu \times E - \nu \times E_0 = 0 \qquad \text{on} \quad \Gamma \tag{4.1$b$}$$

$$\nu \times (\nabla \times E) - \nu \times (\nabla \times E_0) = 0 \qquad \text{on} \quad \Gamma_1 \tag{4.1$c$}$$

$$\nu \times (\nabla \times E) - \nu \times (\nabla \times E_0) = -ik\eta(\nu \times E_0) \times \nu \qquad \text{on} \quad \Gamma_2. \tag{4.1$d$}$$

has a nontrivial solution. Note that if $\bar{\xi} \cdot \Im(N)\,\xi\, > 0$ at a point $x_0 \in D$ Maxwell eigenvalues and transmission eigenvalues do not exist.

We now define an *electromagnetic Herglotz pair* to be a pair of vector fields of the form

$$E_g(x) = \int_\Omega e^{ikx\cdot d} g(d)ds(d), \qquad H_g(x) = \frac{1}{ik}\nabla \times E_g(x) \tag{4.2}$$

where $g \in L^2_t(\Omega)$. It is easy to see that $\nabla \times \nabla \times E_g - k^2 E_g = 0$. Next we consider the vector space

$$E(D) := \{E \in X(D,\Gamma_2),\, \nabla \times E \in X(D,\Gamma_2),\, \nabla \times \nabla \times E - k^2 N(x)E = 0 \text{ in } D\}$$

and define the subset of $\mathcal{Y}(\Gamma) := H^{-\frac{1}{2}}_{div}(\Gamma) \times Y(\Gamma_1) \times L^2_t(\Gamma_2)$ by

$$\mathcal{E} := \{\nu \times (E_g - E),\, \nu \times \nabla \times (E_g - E)|_{\Gamma_1},\, \nu \times \nabla \times (E_g - E) - ik\eta(\nu \times E_g) \times \nu|_{\Gamma_2}\}$$

for all $E \in E(D)$, $g \in L^2_t(\Gamma)$, $E_g$ the electric field of the electromagnetic Herglotz pair with kernel $g$, where $Y(\Gamma_1) := \{h|_{\Gamma_1} :\ h \in Y(\Gamma)\}$.

**Theorem 4.1** *Suppose that $k$ is neither a Maxwell eigenvalue nor a transmission eigenvalue. Then $\mathcal{E}$ is dense in $\mathcal{Y}(\Gamma)$.*

Let $\varphi \in H^{-\frac{1}{2}}_{curl}(\Gamma)$ and $\psi \in H^{-\frac{1}{2}}_{curl}(\Gamma) \cap L^2_t(\Gamma_2)$ such that

$$\int_\Gamma \nu\times(E_g - E)\cdot\varphi\, ds + \int_\Gamma \nu\times\nabla\times(E_g - E)\psi\, ds - \int_{\Gamma_2} ik\eta(E_g)_T\cdot\psi\, ds = 0 \tag{4.3}$$

for all $g \in L^2_t(\Omega)$ and $E \in E(D)$. Note that $\varphi \in H^{-\frac{1}{2}}_{curl}(\Gamma)'$ and $\psi|_{\Gamma_1} \in Y(\Gamma_1)'$ (see [6], Section 2.2 for the characterization of the dual space $Y(\Gamma_1)'$). The first and the second integral in (4.3) is understood in the sense of duality paring between $H^{-\frac{1}{2}}_{div}(\Gamma)$ and $H^{-\frac{1}{2}}_{div}(\Gamma)$ while the third integral containing $\eta$ is understood in the $L^2_t(\Gamma_2)$ sense. Setting first $E = 0$ in (4.3) and interchanging the order of integrations we obtain

$$0 = \hat{x} \times \left\{ \int_\Gamma (\varphi \times \nu)e^{-iky\cdot\hat{x}}\, ds + ik\,\hat{x} \times \int_\Gamma (\psi \times \nu)e^{-iky\cdot\hat{x}}\, ds \right. \tag{4.4}$$
$$\left. -ik\eta \int_{\Gamma_2} [(\nu \times \psi) \times \nu]e^{-iky\cdot\hat{x}}\, ds \right\} \times \hat{x}$$

The right hand side of the above expression is the far field pattern of the following electric and magnetic dipole distribution defined by

$$P(x) = \frac{1}{k^2}\nabla \times \nabla \times \int_\Gamma (\varphi(y) \times \nu)\Phi(x,y)\, ds_y - \nabla \times \int_\Gamma (\psi(y) \times \nu)\Phi(x,y)\, ds_y$$
$$- i\frac{\eta}{k}\nabla \times \nabla \times \int_{\Gamma_2} [(\nu \times \psi(y)) \times \nu]\Phi(x,y)\, ds_y \tag{4.5}$$

where $\Phi(x,y) := \frac{1}{4\pi}\frac{e^{ik|x-y|}}{|x-y|}$. Therefore we conclude that $P(x) = 0$ in $D_e := \mathbb{R}^3 \setminus \overline{D}$. Hence taking the limit of $P(x)$ as $x \to \Gamma$ from both sides we obtain

$$\nu \times P^- = -\nu \times \psi \qquad\qquad \nu \times \nabla \times P^- - ik\tilde{\eta}(\nu \times P^-) \times \nu = \nu \times \varphi$$

on $\Gamma$, where the superscript - indicates the limit obtained by approaching the boundary $\Gamma$ from $D$, and $\tilde{\eta} = 0$ on $\Gamma_1$ and $\tilde{\eta} = \eta$ on $\Gamma_2$. We remark that $P(x)$ and $\operatorname{curl} P(x)$ are both square integrable in any compact subset of $D$ and $D_e$. Furthermore, since $\varphi \times \nu$ and $\psi \times \nu$ are in $H_{div}^{-\frac{1}{2}}(\Gamma)$, the potentials over $\Gamma$ in (4.5) and the corresponding jump relations are well defined from potential theory for single layer potentials with $H^{-\frac{1}{2}}$ densities [16], while the jump relations for the potential over $\Gamma_2$ with $L^2$ density is interpreted in the sense of the $L^2$ limit ([10] p.172). Next, setting $E_g = 0$ in (4.3), using the expressions for $\varphi$ and $\psi$ and Green's formula together with a parallel surfaces argument (see [13]) we obtain

$$\begin{aligned} 0 &= \int_\Gamma \left[(\nu \times P^-) \cdot \nabla \times E - (\nu \times E) \cdot \nabla \times P^- - ik\tilde{\eta}(\nu \times E) \cdot (\nu \times P^-)\right] ds \\ &= k^2 \int_D P^- \cdot (I - N)E \, dx - ik\eta \int_{\Gamma_2} (\nu \times E) \cdot (\nu \times P^-) \, ds. \end{aligned} \tag{4.6}$$

Note that $P \in L^2(D)$. Now let $F \in H(\text{curl}, D)$ be the unique solution (c.f. [17]) of

$$\begin{aligned} &\nabla \times \nabla \times F - k^2 N F = k^2 (I - N) P \qquad &&\text{in} \quad D \\ &\nu \times F = 0 &&\text{on} \quad \Gamma. \end{aligned}$$

Using the vector Green formula for $E$ and $F$, from (4.6) we obtain

$$\int_D (\nu \times E) \cdot \nabla \times F \, ds = -k^2 \int_D P^- \cdot (I - N) E \, dx = -ik\eta \int_{\Gamma_2} (\nu \times E) \cdot (\nu \times P^-) \, ds.$$

Hence

$$\int_\Gamma (\nu \times E) \cdot \left[\nabla \times F + ik\tilde{\eta}(\nu \times P^-)\right] ds = 0$$

for all $E \in E(D)$ whence

$$\nu \times \nabla \times F + ik\tilde{\eta}(\nu \times P^-) \times \nu = 0 \qquad \text{on} \quad \Gamma$$

since $k$ is not a Maxwell eigenvalue. Now we observe that $P$ and $\tilde{E} = P + F$ satisfy

$$\begin{aligned} &\left\{ \begin{array}{c} \nabla \times \nabla \times P - k^2 P = 0 \\ \nabla \times \nabla \times \tilde{E} - k^2 N(x) \tilde{E} = 0 \end{array} \right. && \text{in} \quad D \\ &\nu \times \tilde{E} - \nu \times P = 0 && \text{on} \quad \Gamma \\ &\nu \times (\nabla \times \tilde{E}) - \nu \times (\nabla \times P) = 0 && \text{on} \quad \Gamma_1 \\ &\nu \times (\nabla \times \tilde{E}) - \nu \times (\nabla \times P) = -ik\eta(\nu \times P) \times \nu && \text{on} \quad \Gamma_2 \end{aligned} \tag{4.7}$$

which implies that $P = \tilde{E} = 0$ in $D$ provided $k$ is not a transmission eigenvalue. Therefore $\varphi = \psi = 0$ which proves the theorem. We remark that, in order to conclude that $P = \tilde{E} = 0$ in $D$, the $H(\text{curl}, D_h)$-regularity of $P$, where $\overline{D}_h \subset D$ allows us to apply the vector Green's formula in any compact subset $D_h$ of $D$ and then take the limit of the surface integrals since the boundary relations in (4.7) hold in the $L^2$-limit sense (see Lemma 3.1 for a similar proof in the scalar case).

Next we define the *far field operator* $F : L^2_t(\Omega) \to L^2_t(\Omega)$ by

$$(Fg)(\hat{x}) := \int\limits_{\Omega} E_\infty(\hat{x}, d, g(d))ds(d), \qquad \hat{x} \in \Omega \quad \text{and} \quad g \in L^2_t(\Omega) \tag{4.8}$$

and look for solutions $g \in L^2_t(\Omega)$ of the *far field equation*

$$(Fg)(\hat{x}) := E_{e,\infty}(\hat{x}, z, q) \tag{4.9}$$

where

$$E_{e,\infty}(\hat{x}, z, q) = \frac{ik}{4\pi}(\hat{x} \times q) \times \hat{x}\, e^{-ik\hat{x}\cdot z}$$

is the electric far field pattern of the electric dipole with polarization $q$ given by

$$E_e(x, z, q) := \frac{i}{k}\nabla_x \times \nabla_x \times\ q\,\Phi(x, z)$$

with $\Phi(x, z) := \frac{1}{4\pi}\frac{e^{ik|x-z|}}{|x-z|}$. We can now prove the following theorem.

**Theorem 4.2** *Assume that $k$ is neither a transmission eigenvalue nor a Maxwell eigenvalue and let $F$ be the far field operator corresponding to (2.6)-(2.13). Then we have:*

*(i) For $z \in D$ and every $\epsilon > 0$ there exists a solution $g^z_\epsilon \in L^2_t(\Omega)$ of the inequality*

$$\|Fg^z_\epsilon - E_{e,\infty}(\cdot,\, z, q)\|_{L^2_t(\Omega)} < \epsilon$$

*such that*

$$\lim_{z\to\Gamma} \|g^z_\epsilon\|_{L^2_t(\Omega)} = \infty \quad \textit{and} \quad \lim_{z\to\Gamma} \|E_{g^z_\epsilon}\|_{X(D,\Gamma_2)} = \infty,$$

*where $E_{g^z_\epsilon}$ is the electric field of the electromagnetic Herglotz pair with kernel $g^z_\epsilon$.*

*(ii) For $z \in \mathbb{R}^3 \setminus \overline{D}$ and every $\epsilon > 0$ and $\delta > 0$ there exists a solution $g^z_{\epsilon,\delta} \in L^2_t(\Omega)$ of the inequality*

$$\|Fg^z_{\epsilon,\delta} - E_{e,\infty}(\cdot,\, z, q)\|_{L^2(\Omega)} < \epsilon + \delta$$

*such that*

$$\lim_{\delta\to 0} \|g^z_{\epsilon,\delta}\|_{L^2_t(\Omega)} = \infty \quad \textit{and} \quad \lim_{\delta\to 0} \|E_{g^z_{\epsilon,\delta}}\|_{X(D,\Gamma_2)} = \infty,$$

*where $E_{g^z_{\epsilon,\delta}}$ is the electric field of the electromagnetic Herglotz pair with kernel $g^z_{\epsilon,\delta}$.*

**Proof**. First one needs to prove a similar result as in Lemma 3.5 for the operator $\mathcal{B}$ which in this case maps $f := \nu \times E|_\Gamma$, $h := \nu \times (\nabla \times E)|_\Gamma$ and $g := ik\eta(\nu \times E) \times \nu|_{\Gamma_2}$, where $E \in X(D, \Gamma_2)$ satisfies $\nabla \times \nabla \times E - k^2E = 0$, onto the electric far field pattern of the corresponding solution of (2.18$a$)-(2.18$e$). This can be done exactly in the same way as in Lemma 3.5 by making use of the result of Theorem 4.1 and using the divergence free vector spherical wave functions [10]. Then the result of the theorem follows in the same way as the proof of Theorem 3.6 which uses the analog of the approximation property proved in Theorem 4.1 for the vector case. For the case of $z \in \mathbb{R}^3 \setminus \overline{D}$ the approximation property proved in Theorem 2.5 of [6] is also needed.

This theorem establishes the theoretical basis of the linear sampling method for reconstructing the domain $D$. We now use the approximate solution $g^z$ for $z \in D$ of the far field equation (4.9) to give an approximation for the surface conductivity $\eta$. To this end we need the following lemma.

**Lemma 4.3** *Assume that $k$ is neither a Maxwell eigenvalue nor a transmission eigenvalue. For any point $z$ in $D$ and a arbitrary small $\epsilon > 0$ we have that*

$$\left| \int_D \overline{E}^z_\epsilon \cdot \Im(N) E^z_\epsilon \, dx + k\eta \int_{\Gamma_2} |\nu \times (E_{g^z_\epsilon} + E_e(\cdot, z, q)|^2 \, ds + \frac{k^2}{6\pi} \|q\|^2 - k\Re\left(E_{g^z_\epsilon}(z)\right) \right| \leq C\epsilon \tag{4.10}$$

*where $E^z_\epsilon \in E(D)$ and the electric Herglotz wave function $E_{g^z_\epsilon}$ with $g^z_\epsilon \in L^2_t(\Omega)$ are such that $\left[\nu \times (E_{g^z_\epsilon} - E^z_\epsilon),\ \nu \times \nabla \times (E_{g^z_\epsilon} - E^z_\epsilon) - ik\tilde{\eta}(\nu \times E_{g^z_\epsilon}) \times \nu\right]$ approximates $[\nu \times E_e(\cdot, z, q),\ \nu \times \nabla \times E_e(\cdot, z, q) - ik\tilde{\eta}(\nu \times E_e(\cdot, z, q)) \times \nu]$ in the $\mathcal{Y}(\Gamma)$ norm with discrepancy $\epsilon$ and $C > 0$ is a constant independent of $\epsilon$.*

**Proof**. From Theorem 4.1, for given $\epsilon > 0$, there exists a $E^z_\epsilon \in E(D)$ and a electromagnetic Herglotz pair with electric field $E_{g^z_\epsilon}$ and kernel $g^z_\epsilon \in L^2_t(\Omega)$ such that

$$\begin{cases} \nu \times E^z_\epsilon - E_e(\cdot, z, q)) = E_{g^z_\epsilon} + \alpha_\epsilon \\ \\ \nu \times \nabla \times (E^z_\epsilon - E_e(\cdot, z, q)) + ik\tilde{\eta}(\nu \times E_e(\cdot, z, q)) \times \nu \\ = \nu \times \nabla \times E_{g^z_\epsilon} - ik\tilde{\eta}(\nu \times E_{g^z_\epsilon}) \times \nu + \beta_\epsilon \end{cases} \tag{4.11}$$

on $\Gamma$ where

$$\| (\alpha_\epsilon, \beta_\epsilon) \|_{\mathcal{Y}(\Gamma)} < \epsilon. \tag{4.12}$$

Applying the vector Green's formula to $E^z_\epsilon$ and $\overline{E}^z_\epsilon$ in $D$ (see [17] for the case of $H(\mathrm{curl}, D)$ functions) we obtain

$$\int_\Gamma \left(\nu \times E^z_\epsilon \cdot \mathrm{curl}\, \overline{E^z_\epsilon} - \nu \times \overline{E^z_\epsilon} \cdot \mathrm{curl}\, E^z_\epsilon\right) \, ds = \int_D \overline{E}^z_\epsilon \cdot \Im(N) E^z_\epsilon \, dx. \tag{4.13}$$

On the other hand, using (4.11) and defining $W^z_\epsilon := E_{g^z_\epsilon} + E_e(\cdot, z, q)$, we have that

$$\begin{aligned} &\int_\Gamma \left(\nu \times E^z_\epsilon \cdot \mathrm{curl}\, \overline{E^z_\epsilon} - \nu \times \overline{E^z_\epsilon} \cdot \mathrm{curl}\, E^z_\epsilon\right) \, ds \\ &= \int_\Gamma \left(\nu \times W^z_\epsilon \cdot \mathrm{curl}\, \overline{W^z_\epsilon} - \nu \times \overline{W^z_\epsilon} \cdot \mathrm{curl}\, W^z_\epsilon\right) \, ds \\ &\quad - 2ik\eta \int_{\Gamma_2} |(\nu \times W^z_\epsilon) \times \nu|^2 \, ds + R^z_\epsilon \end{aligned} \tag{4.14}$$

where $|R^z_\epsilon| \leq C\epsilon$ for a positive constant $C$ independent of $\epsilon$. Again using the vector Green's formula, the integral representation formula and connecting the radiating solution $E_e(\cdot,\, z, q)$ to its far field pattern as in [7] Theorem 3.1, we obtain

$$\int_\Gamma \left(\nu \times W^z_\epsilon \cdot \operatorname{curl} \overline{W^z_\epsilon} - \nu \times \overline{W^z_\epsilon} \cdot \operatorname{curl} W^z_\epsilon\right) ds \tag{4.15}$$
$$= -\frac{ik^2}{3\pi}\|q\|^2 + ikq \cdot \left[E_{g^z_\epsilon}(z) + \overline{E}_{g^z_\epsilon}(z)\right]$$

Finally, combining (4.13), (3.25) and (4.15), we have the following theorem:

**Theorem 4.4** *Let $z$ be a fixed point in $D$, $\Im(N) = 0$ and assume that $k$ is neither a Maxwell eigenvalue nor a transmission eigenvalue. Then for every $\epsilon > 0$ there exists an electromagnetic Herglotz pair $E_{g^z_\epsilon}, H_{g^z_\epsilon}$ with kernel $g^z_\epsilon \in L^2_t(\Omega)$ an approximate solution of the far field equation (4.9) such that*

$$\left| \eta + \frac{\frac{k}{6\pi}\|q\|^2 - \Re\left(E_{g^z_\epsilon}(z)\right)}{\|\nu \times (E_{g^z_\epsilon} + E_e(\cdot,\, z, q))\|^2_{L^2_t(\Gamma_2)}} \right| \leq \epsilon. \tag{4.16}$$

**Proof**. This theorem is a direct consequence of Theorem 4.2 and Lemma 4.3.

We note that the remark following Theorem 3.8 also applies in the present case of equation (4.16).

## 5. Numerical examples

We shall now present some simple numerical examples in two dimensions to demonstrate the performance of our proposed scheme for solving the inverse problem. The choice of scatterers and general approach is similar to that in [5] where details of the method may be found. Briefly, we choose a particular scatterer and surface conductivity. Then we predict the far field pattern using a cubic finite element method terminated by a perfectly matched layer. After the addition of noise (the method for doing this and noise level are the same as in [5]), the approximate far field data is used to reconstruct the boundary of the scatterer using a discrete approximation to $1/\|g\|_{L^2(\Omega)}$ as an indicator function. This involves choosing a particular contour value $C$ so that the reconstructed scatterer is given by the curve where $1/\|g\|_{L^2(\Omega)} = C$. We shall comment more on the choice of $C$ shortly. Once the boundary of the scatterer has been approximated, we use (3.27) to approximate the surface conductivity $\eta$. As in [5] the relevant norm on $\Gamma_2$ is computed using the trapezoidal rule with 100 integration points and we use the single point $z = (0, 0)^T$.

The two simple scatterers considered here are an ellipse and a rectangle shown in Fig. 1. Using the far field pattern for these figures we can compute the reconstruction of $\eta$. In the upcoming sections we show results computed using 21 values of the exact conductivity $\eta$ between 0 and 2 and for 41 cutoffs $C$ between 0.2 and 0.8. In all cases we use 61 incoming waves uniformly distributed in $[0, 2\pi]$ and 61 data points at the same angles.

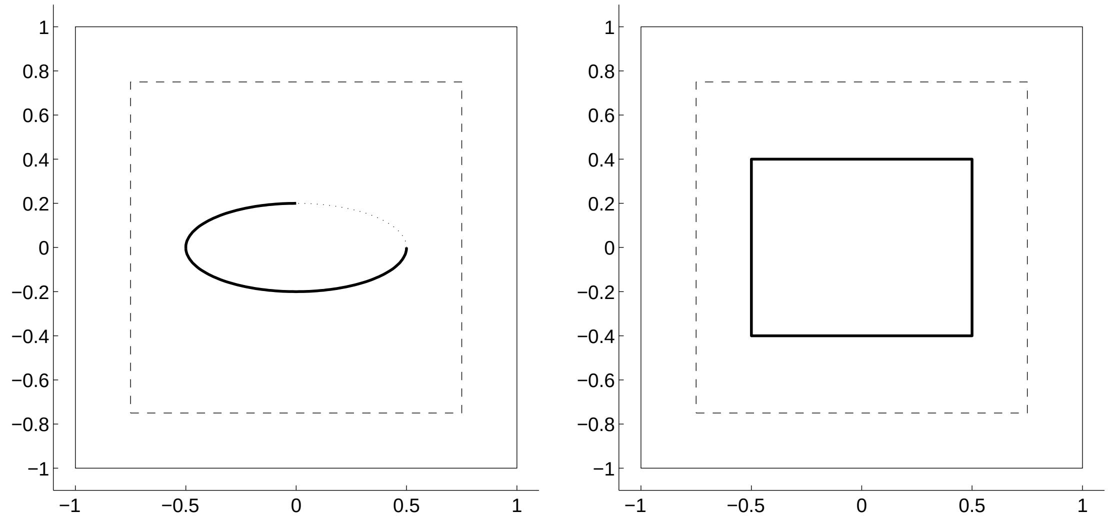
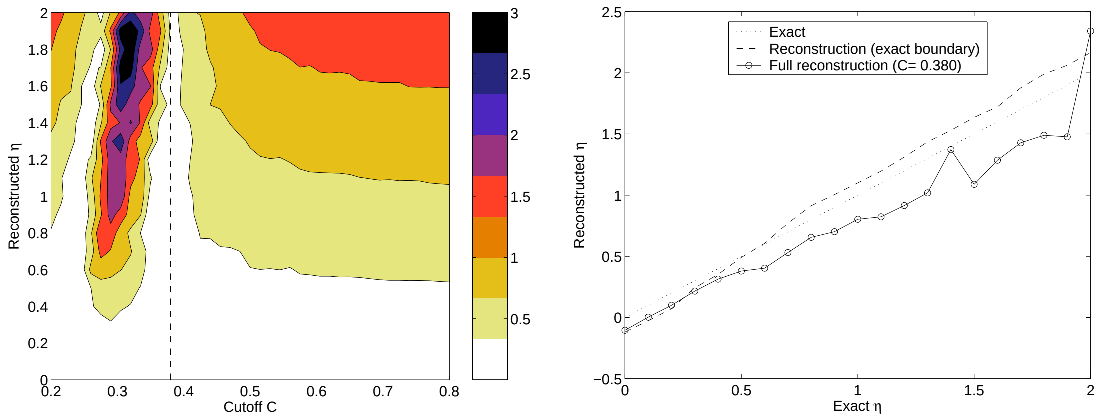



**Figure 1.** The two simple scatterers used in this paper. For the ellipse we mark a dark line the portion of the boundary used for the partially coated results. In both panels the dashed line shows the start of the PML layer and the entire figure is the region meshed using the finite element method for the forward problem.

*5.1. Scattering by an ellipse*

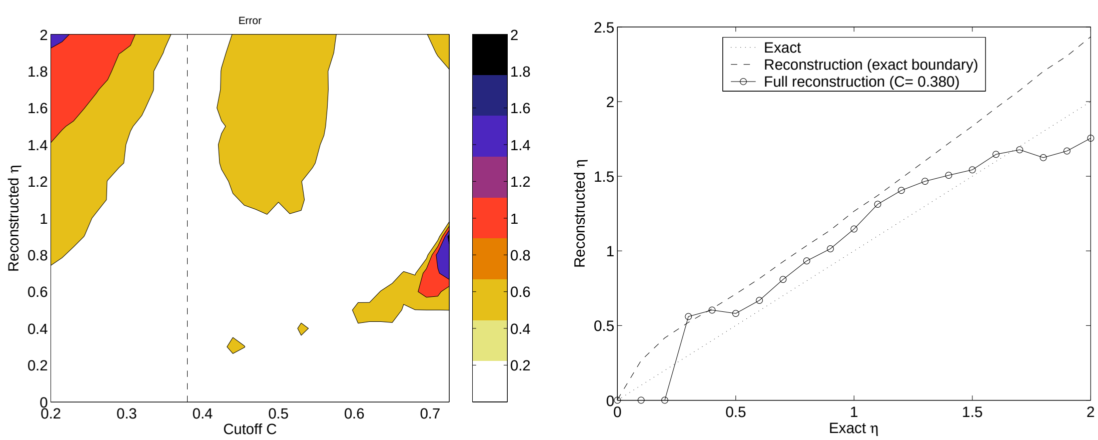
We start by reporting results for a fully coated (i.e. $\eta$ is constant on the entire boundary) ellipse. We investigate the reconstruction of $\eta$ for a range of $\eta$, for various choices of cutoff $C$ and for two wave numbers. We start with $k = 5$. In Fig. 2(a) we show a contour map of the error in the reconstruction as a function of the exact $\eta$ and cutoff $C$. We would like to have a single choice of cutoff independent of the exact value of $\eta$ for the reconstruction. and in the figure we mark a reasonable choice (the choice $C = 0.380$ minimizes the infinity norm of the reconstruction over the range indicated). This choice lies in a valley of the error surface for almost its whole length. In Fig. 2(b) we show the reconstruction of $\eta$ for the chosen cutoff value as the exact value of $\eta$ varies. We also show the reconstruction using (3.27) but with the exact boundary. Using the exact boundary gives a much better reconstruction so, in this case, the error in the reconstruction of $\eta$ is mostly due to the preliminary reconstruction of the boundary of the ellipse by the linear sampling method. This example illustrates a problem with using a combination of the linear sampling method and (3.27) to reconstruct $\eta$: the method is sensitive to the choice of the cutoff $C$. In [5] we found a heuristic criterion for choosing the cutoff which worked well in the examples we have tried for the method of that paper, but no such heuristic is obvious for the method in this paper.

For $k = 10$ the behavior of the algorithm is broadly similar to the case when $k = 5$. In this case the far field data is less accurate since we used the same finite element grid for solving the forward problem for both wave numbers. Results are shown in Fig. 3. There is still a choice of the cutoff $C$ that provides a reasonable reconstruction throughout the range of $\eta$. The value of the optimal cutoff is $C = 0.395$ which is roughly that used when




(a) Error in the reconstruction (b) Reconstruction for $C = 0.38$.

**Figure 2.** Here we show results for reconstructing the surface conductivity of a fully coated ellipse at $k = 5$. The boundary is shown in Fig. 1. In the left panel we show the error in the reconstructed conductivity $\eta$ as a function of the exact $\eta$ and the cut off $C$. The dotted line at $C = 0.38$ marks the optimal choice. Using this choice of cutoff gives the results in the right hand panel where we also show the results of using (3.27) knowing the exact boundary. Most of the error in the reconstruction of $\eta$ results from the reconstruction of the boundary of the ellipse by the linear sampling method.



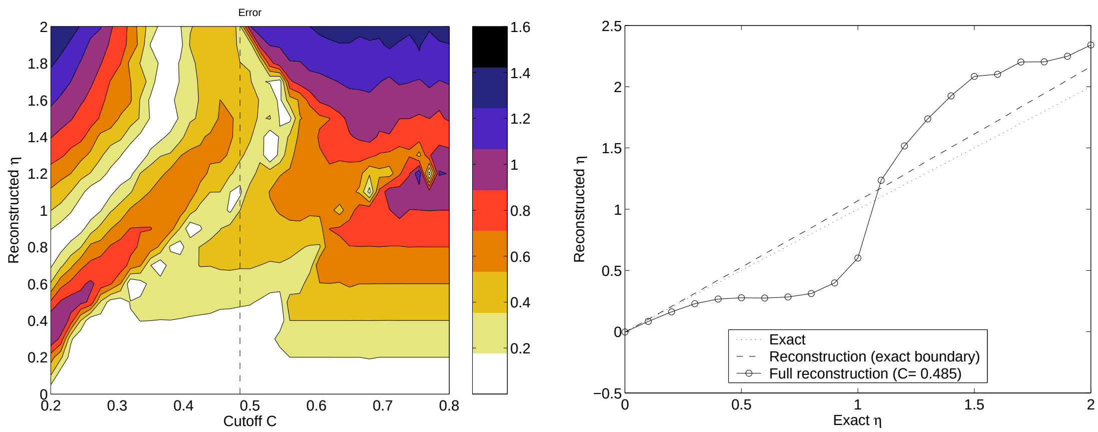

(a) Error in the reconstruction (b) Reconstruction for $C = 0.38$.

**Figure 3.** Here we show results for reconstructing the surface conductivity of a fully coated ellipse at $k = 10$. See the caption to Fig. 2 for an explanation.

$k = 5$ so that a single choice of cutoff could be used for both wave numbers. Oddly the reconstruction using the exact boundary overestimates $\eta$ throughout the range, perhaps due to the less accurate far field data.

### *5.2. Scattering by an rectangle*

Next we investigate the reconstruction of the fully coated rectangle. As in the previous subsection, $\eta$ is constant on the entire surface of the rectangle and we perform the




(a) Error in the reconstruction (b) Reconstruction for $C = 0.485$.

**Figure 4.** Here we show results for reconstructing the surface conductivity of a fully coated rectangle at $k = 5$. See the caption to Fig. 2 and the text for an explanation.

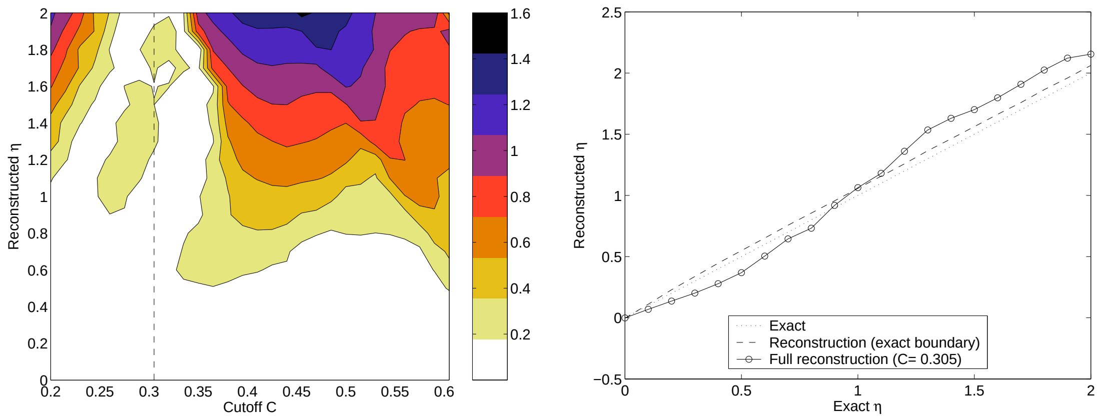
reconstruction for the same choice of wave numbers and other parameters as in the previous subsection. In [5] we found that the rectangle presented a more difficult reconstruction problem than the ellipse, and that is also the case here. When $k = 5$ we show the results of reconstructing $\eta$ in Fig. 4. In the left hand panel we see that there is no longer a choice of cutoff $C$ that provides a small reconstruction error for all $\eta$ (i.e. lies in a valley of the error surface). We have marked $C = 0.485$ as a dashed line and this minimizes the maximum norm error in the reconstruction over the range of $\eta$ here. In the right hand panel we show the reconstruction of $\eta$ using the optimal cutoff as well as a reconstruction using the exact boundary. Clearly the overall error in the reconstruction is almost entirely due to the preliminary reconstruction of the boundary of the scatterer.

One way to improve the reconstruction of a scatterer by the linear sampling method is to increase the wave number. In Fig. 5 we show the results of reconstructing the surface conductivity of the rectangle when $k = 10$. Note that in this case we used a finite element mesh of half the mesh size compared to that used when $k = 5$. The reconstruction of $\eta$ is now improved. Unfortunately the optimal cutoff $C$ is now markedly smaller than for the case of the ellipse or the rectangle at $k = 5$ indicating that the cutoff needs to be chosen depending both on the scatterer and the wave number.

### *5.3. Partially coated scatterer*

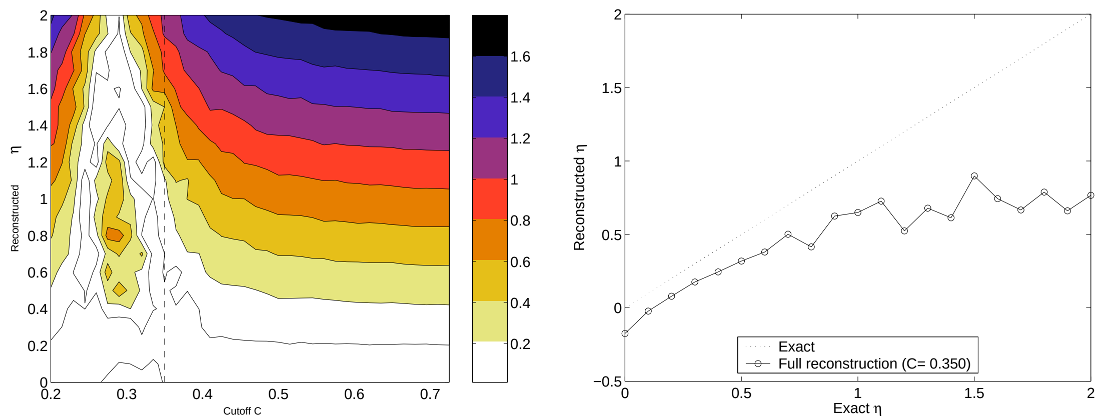
Here we show some results for applying (3.27) when $\Gamma_2$ is a proper subset of $\Gamma$. In particular we use the partially coated subset of the boundary of the ellipse shown in Fig. 1. The results are shown in Fig. 6. As expected, the reconstruction under estimates the true value of $\eta$ when $\Gamma \neq \partial D$.

If the surface conductivity $\eta$ is large, equation (3.18) suggests that for any $z$ inside the scatterer $v_{g^z} + \Phi \approx 0$ on the coated portion of $\Gamma$ (i.e. where $\eta$ is large).




(a) Error in the reconstruction (b) Reconstruction for $C = 0.305$.

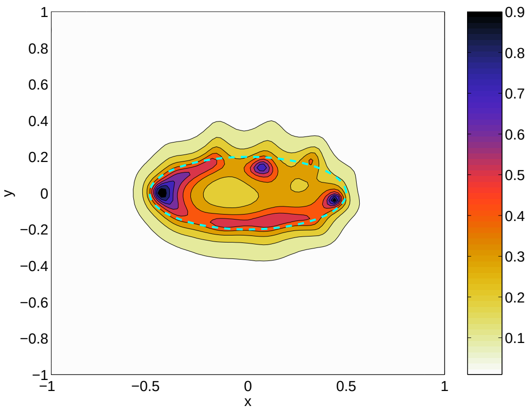
**Figure 5.** Here we show results for reconstructing the surface conductivity of a fully coated rectangle at $k = 10$. See the caption to Fig. 2 and the text for an explanation.



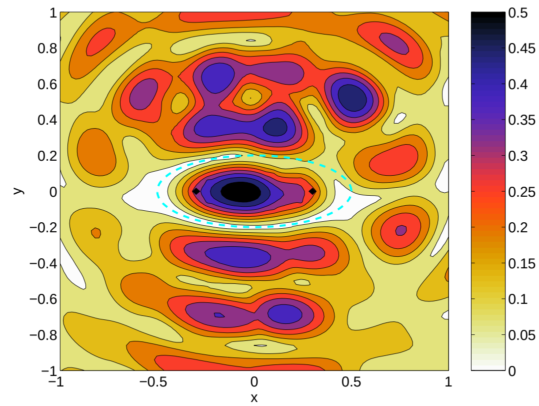

(a) Error in the reconstruction (b) Reconstruction for $C = 0.35$.

**Figure 6.** Here we show results for reconstructing the surface conductivity of a partially coated ellipse when $k = 5$. See the caption to Fig. 2 and the text for an explanation.

Thus we should be able to use this quantity as an indicator for $\Gamma_2$. Numerical experiments (not shown here) show that the choice $z = (0,0)^T$ used for computing $\eta$ does not work well to determine this conducting boundary. Instead we compute $v_{g^z} + \Phi$ for $z$ closer to the reconstructed boundary. For the ellipse we choose $z = (0.3\cos(\pi(j-1)/10), 0.1\sin(\pi(j-1)/10))^T$, $j = 1, \cdots, 10$ and plot a contour map of the average value of $|v_{g^z} + \Phi|$. In Fig. 7(a) we show the linear sampling method reconstruction of the partially coated ellipse when $k = 10$ and $\eta = 20$ on the coated portion of the boundary. In the right hand panel we show the average value of $|v_{g^z} + \Phi|$ using the ten choices of $z$ mentioned previously. Clearly the minimum of this function does pick out the conducting portion of the boundary. Fig. 7(a) clearly shows that some feature of the boundary changes in the upper right quadrant of the reconstruction.



(a) Reconstruction of $\partial D$



(b) Reconstruction for $\Gamma_2$

**Figure 7.** Here we reconstruct the boundary $\Gamma_2$ of the ellipse shown in Fig. 1 when $\eta = 20$ on $\Gamma_2$ and $k = 5$. The left panel shows the indicator function $1/\|g\|_{L^2(\Omega)}$ from the basic linear sampling method. The right panel shows the average value of $|v_{g^z} + \Phi|$ using 10 choices of $z$ within the ellipse (see text). The conducting portion of the boundary is correctly located as a minimum of $|v_{g^z} + \Phi|$. In both cases the dashed line shows the exact boundary $\partial D$.

But the use of $v_{g^z} + \Phi$ gives an indicator for the portion of the boundary where the conductivity is high, thus giving a more precise statement about the nature of the domain.

## 6. Conclusion

We have shown how the linear sampling method may be used to reconstruct both the shape and surface conductivity of an object. In particular we have provided a mathematical justification for the method for the full Maxwell system. Numerical results in two dimensions show that provided the shape of the scatterer is reconstructed sufficiently accurately, the proposed method can provide a reliable reconstruction of the surface conductivity. The main difficulty with the method (in the case of TM-polarization) is that it is necessary to make an accurate choice of the cutoff parameter in the shape reconstruction problem. In future we shall test the method in three dimensions uising multiple polarizations and try to develop heuristic techniques for choosing an appropriate cutoff parameter.

## Acknowledgments

We gratefully acknowledge the support of our research by the Air Force Office of Scientific Research under grants F49620-02-1-0071 and F49620-02-1-0353.

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