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http://www.aimspress.com/journal/era

ERA, 32(5): 3413–3432.
DOI: 10.3934/era.2024158
Received: 13 April 2024
Revised: 10 May 2024
Accepted: 21 May 2024
Published: 27 May 2024

Research article

A pressure-robust stabilizer-free WG finite element method for the Stokes equations on simplicial grids

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Abstract: A pressure-robust stabilizer-free weak Galerkin (WG) finite element method has been defined for the Stokes equations on triangular and tetrahedral meshes. We have obtained pressure-independent error estimates for the velocity without any velocity reconstruction. The optimal-order convergence for the velocity of the WG approximation has been proved for the L^2 norm and the H^1 norm. The optimal-order error convergence has been proved for the pressure in the L^2 norm. The theory has been validated by performing some numerical tests on triangular and tetrahedral meshes.

Keywords: stabilizer-free; weak Galerkin; finite element; Stokes equations; pressure robust; tetrahedral meshes

1. Introduction

In this manuscript, we detail the development of a new stabilizer-free weak Galerkin (WG) finite element method of any polynomial order in 2D and 3D, on triangular and tetrahedral meshes respectively, for obtaining the solutions of the stationary Stokes equations: Find unknown functions $\mathbf{u}$ (velocity) and p (pressure) such that

$$-\mu\Delta\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega, \tag{1.1}$$

$$\nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega, \tag{1.2}$$

$$\mathbf{u} = 0 \quad \text{on } \partial\Omega, \tag{1.3}$$

where the viscosity $\mu > 0$, and the domain Ω is a polygon or a polyhedron in $\mathbb{R}^d$ $(d = 2, 3)$.

The new pressure-robust, stabilizer-free WG method has a new variational formulation. Find $\mathbf{u}_h \in V_h$ and $p_h \in W_h$ such that

$$(\mu\nabla_w \mathbf{u}_h, \nabla_w \mathbf{v}) + (\nabla_w p_h, \mathbf{v}) = (\mathbf{f}, \mathbf{v}) \quad \forall \mathbf{v} \in V_h, \tag{1.4}$$

$$(\mathbf{u}_h, \nabla_w q) = 0 \qquad \forall q \in W_h, \tag{1.5}$$

where ∇_w denotes weak gradients to be defined in (2.3) and (2.4), and V_h and W_h are P_k and P_{k-1} WG finite element spaces, to be defined in (2.1) and (2.2), respectively. Some studies on the WG methods are as follows: the parabolic equation [1–3], with two-order superconvergence on triangular meshes [4], the convection-diffusion equation [5, 6], 4th order problems [7, 8], with the conforming discontinuous Galerkin formulation [9–11], the Navier-Stokes equations [12–14], with the discrete maximum principle [15], the second order elliptic equations [16, 17], with the energy conservation [18], the Darcy flow [19, 20], the Oseen equations [21], the Stokes equations with pressure-robustness [22–24], the Maxwell equations [25], the Cahn-Hilliard equation [26], the div-curl equations [27], with adaptive refinements [28, 29], the biharmonic equation [30, 31], the biharmonic equation with continuous finite elements [32, 33], the Stokes equations with $H(\text{div})$ elements [34, 35], with two-order superconvergence under the CDG formulation [36, 37], and with two-order superconvergence for the Stokes equations [38].

We note that the moment equation (1.1) is not tested by applying $H(\text{div}, \Omega)$ functions in (1.4), different from most other pressure-robust methods such as those detailed in [22–24], but is tested by using discontinuous polynomials. The pointwise divergence-free property (pressure-robustness) is achieved by introducing pressure face variables, i.e., $p_h = \{p_0, p_b\}$ where $p_b|_e \in P_k(e)$ on every face edge/triangle e of a mesh $\mathcal{T}_h$ (see (2.2) below). The method was previously applied before in hybridized discontinuous Galerkin methods in [39].

It is shown that the WG finite element pair, $V_h - W_h$, is inf-sup stable. The method is shown pressure robust, i.e., both errors of the velocity and the pressure are independent of the pressure p and the viscosity μ in (1.1). This is necessary when μ is small. However, very few methods can achieve this. Under the inf-sup stability, we shall prove quasi-optimal approximation for the velocity in an H^1-norm and in the L^2 norm. The quasi-optimal convergence for pressure in the L^2 norm is also proved. The theory is numerically verified by applying varying degrees of WG finite elements to both triangular and tetrahedral meshes.

2. The WG finite element method

Let $\mathcal{T}_h$ be a mesh on the domain Ω, consisting of conforming shape-regular triangles or shape-regular tetrahedrons. Here we let h_T be the element diameter of $T \in \mathcal{T}_h$, and we let mesh size $h = \max_{T\in\mathcal{T}_h} h_T$.

For $k \geq 1$ and given $\mathcal{T}_h$, the finite element space for velocity is defined by

$$\begin{aligned} V_h = \{\mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\} : \ & \mathbf{v}_0|_T \in [P_k(T)]^d, \ T \in \mathcal{T}_h; \\ & \mathbf{v}_b|_e \in [P_{k+1}(e)]^d, \ e \in \mathcal{E}_h; \ \mathbf{v}_b|_e = \mathbf{0}, \ e \in \mathcal{E}_h \cap \partial\Omega\}, \end{aligned} \tag{2.1}$$

and the finite element space for pressure is defined by

$$\begin{aligned} W_h = \{q_h = \{q_0, q_b\} : \ & q_0|_T \in P_{k-1}(T), \ T \in \mathcal{T}; \ q_b|_e \in P_k(e), \\ & e \in \mathcal{E}_h; (q_0, 1)_{\mathcal{T}_h} + \langle q_b, 1\rangle_{\partial\mathcal{T}_h} = 0\}, \end{aligned} \tag{2.2}$$

where $P_k(e)$ and $P_k(T)$ denote the space of polynomials of degree k or less on the edge/triangle e and triangle/tetrahedron T respectively, $(\cdot,\cdot)_{\mathcal{T}_h} = \sum_{T\in\mathcal{T}_h}(\cdot,\cdot)_T$ and $\langle\cdot,\cdot\rangle_{\partial\mathcal{T}_h} = \sum_{T\in\mathcal{T}_h}\langle\cdot,\cdot\rangle_{\partial T}$.

For a function $\mathbf{v} \in V_h$, the $(k+1)$-degree weak gradient $\nabla_w \mathbf{v}$ is a piecewise polynomial on the mesh $\mathcal{T}_h$, $\nabla_w \mathbf{v}|_T \in [P_{k+1}(T)]^{d\times d}$, such that

$$(\nabla_w \mathbf{v},\ \tau)_T = -(\mathbf{v}_0,\ \nabla\cdot\tau)_T + \langle \mathbf{v}_b,\ \tau\cdot\mathbf{n}\rangle_{\partial T} \quad \forall \tau \in [P_{k+1}(T)]^{d\times d}. \tag{2.3}$$

For a function $q = \{q_0, q_b\} \in W_h$, its weak gradient $\nabla_w q$ is defined as a piecewise vector-valued polynomial such that for each $T \in \mathcal{T}_h$, $\nabla_w q \in [P_k(T)]^d$ satisfies

$$(\nabla_w q, \boldsymbol{\varphi})_T = -(q_0, \nabla\cdot\boldsymbol{\varphi})_T + \langle q_b, \boldsymbol{\varphi}\cdot\mathbf{n}\rangle_{\partial T} \quad \forall \boldsymbol{\varphi} \in [P_k(T)]^d. \tag{2.4}$$

We denote by Π_k the local/element-wise L^2-orthogonal projection onto $[P_k(T)]^j$ where $j = 1,\ d,\ d\times d$ and $T \in \mathcal{T}_h$. Let Π_k^b be a generic edge/face-wise defined L^2 projection onto $[P_k(e)]^j$ for $e \in \partial T$. Define $\mathbf{Q}_h\mathbf{u} = \{\Pi_k\mathbf{u}, \Pi_{k+1}^b\mathbf{u}\} \in V_h$ and $Q_h p = \{\Pi_{k-1}p, \Pi_k^b p\} \in W_h$.

Lemma 2.1. *Let $\boldsymbol{\phi} \in [H_0^1(\Omega)]^d$ and $\psi \in H^1(\Omega)$, then, for $T \in \mathcal{T}_h$*

$$\nabla_w \mathbf{Q}_h \boldsymbol{\phi} = \Pi_{k+1}\nabla\boldsymbol{\phi}, \tag{2.5}$$
$$\nabla_w Q_h \psi = \Pi_k \nabla\psi. \tag{2.6}$$

Proof. Using (2.3) and integration by parts, we obtain the following for any $\tau \in [P_{k+1}(T)]^{d\times d}$:

$$\begin{aligned}(\nabla_w Q_h\boldsymbol{\phi}, \tau)_T &= -(\Pi_k\boldsymbol{\phi}, \nabla\cdot\tau)_T + \langle \Pi_{k+1}^b\boldsymbol{\phi}, \tau\cdot\mathbf{n}\rangle_{\partial T}\\ &= -(\boldsymbol{\phi}, \nabla\cdot\tau)_T + \langle\boldsymbol{\phi}, \tau\cdot\mathbf{n}\rangle_{\partial T}\\ &= (\nabla\boldsymbol{\phi}, \tau)_T = (\Pi_{k+1}\nabla\boldsymbol{\phi}, \tau)_T.\end{aligned}$$

This implies that (2.5) holds. The scalar version, (2.6), is proved in the same manner.

For any function $\varphi \in H^1(T)$, the following trace inequality holds true:

$$\|\varphi\|_e^2 \le C\left(h_T^{-1}\|\varphi\|_T^2 + h_T\|\nabla\varphi\|_T^2\right). \tag{2.7}$$

We define two semi-norms $|\!|\!|\mathbf{v}|\!|\!|$ and $\|\mathbf{v}\|_{1,h}$ for any $\mathbf{v} \in V_h$:

$$|\!|\!|\mathbf{v}|\!|\!|^2 = (\nabla_w\mathbf{v}, \nabla_w\mathbf{v}), \tag{2.8}$$
$$\|\mathbf{v}\|_{1,h}^2 = \sum_{T\in\mathcal{T}_h}\|\nabla\mathbf{v}_0\|_T^2 + \sum_{T\in\mathcal{T}_h} h_T^{-1}\|\mathbf{v}_0 - \mathbf{v}_b\|_{\partial T}^2. \tag{2.9}$$

We also define two semi-norms $|\!|\!|q|\!|\!|$ and $\|q\|_{1,h}$ for any $q \in W_h$:

$$|\!|\!|q|\!|\!|^2 = (\nabla_w q, \nabla_w q), \tag{2.10}$$
$$\|q\|_{1,h}^2 = \sum_{T\in\mathcal{T}_h}\|\nabla q_0\|_T^2 + \sum_{T\in\mathcal{T}_h} h_T^{-1}\|q_0 - q_b\|_{\partial T}^2. \tag{2.11}$$

In fact, $\|\mathbf{v}\|_{1,h}$ is a norm in V_h and $|\!|\!|\cdot|\!|\!|$ is also a norm in V_h, as they have been proved in [4] by applying the norm equivalence as follows:

$$C_1\|\mathbf{v}\|_{1,h} \le |\!|\!|\mathbf{v}|\!|\!| \le C_2\|\mathbf{v}\|_{1,h} \quad \forall \mathbf{v} \in V_h, \tag{2.12}$$

and

$$C_1\|q\|_{1,h} \le |\!|\!|q|\!|\!| \le C_2\|q\|_{1,h} \quad \forall q \in W_h. \tag{2.13}$$

Lemma 2.2. *The following inf-sup conditions hold, for all $q = \{q_0, q_b\} \in W_h$ and $\mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\} \in V_h$:*

$$\sup_{\mathbf{v}\in V_h} \frac{(\mathbf{v}_0, \nabla_w q)}{|||\mathbf{v}|||} \geq \beta \|q_0\|, \tag{2.14}$$

and

$$\sup_{\mathbf{v}\in V_h} \frac{(\mathbf{v}_0, \nabla_w q)}{|||\mathbf{v}|||} \geq \beta h |||q|||, \tag{2.15}$$

where $\beta > 0$ is independent of h and $\mathcal{T}_h$.

Proof. For any given $q = \{q_0, q_b\} \in W_h$, it is known that there exists a function $\tilde{\mathbf{v}} \in H_0(\mathrm{div}, \Omega)$ and $\tilde{\mathbf{v}}|_T \in [P_k(T)]^d$ (see, e.g., [40, (7.4)–(7.6)]) such that

$$\frac{(\nabla \cdot \tilde{\mathbf{v}}, q_0)}{|\tilde{\mathbf{v}}|_{1,h}} \geq C\|q_0\|, \tag{2.16}$$

where

$$|\tilde{\mathbf{v}}|_{1,h}^2 = \sum_{T\in\mathcal{T}_h} (\|\nabla\tilde{\mathbf{v}}\|_T^2 + h_T^{-1}\|[\tilde{\mathbf{v}}]\|_{\partial T}^2).$$

Let

$$\mathbf{v} = \{\tilde{\mathbf{v}}, \tilde{\mathbf{v}}_b\} \in V_h, \quad \text{where } \tilde{\mathbf{v}}_b|_e = \begin{cases} \frac{1}{2}(\tilde{\mathbf{v}}|_{T_1} + \tilde{\mathbf{v}}|_{T_2}) & \text{if } e \in \mathcal{E}_h^0, \\ \mathbf{0} & \text{if } e \in \mathcal{E}_h \cap \partial\Omega, \end{cases}$$

where T_1 and T_2 are the two elements on the two sides of edge/triangle e. For such specially defined $\mathbf{v}$, we have

$$\begin{aligned} \|\mathbf{v}\|_{1,h}^2 &= \sum_{T\in\mathcal{T}_h} (\|\nabla\tilde{\mathbf{v}}\|_T^2 + h_T^{-1}\|\tilde{\mathbf{v}} - \{\tilde{\mathbf{v}}\}\|_{\partial T}^2) \\ &= \sum_{T\in\mathcal{T}_h} (\|\nabla\tilde{\mathbf{v}}\|_T^2 + h_T^{-1}\|[\tilde{\mathbf{v}}]\|_{\partial T}^2) = |\tilde{\mathbf{v}}|_{1,h}^2. \end{aligned} \tag{2.17}$$

It follows from (2.13) and (2.17) that

$$|||\mathbf{v}||| \leq C|\tilde{\mathbf{v}}|_{1,h}. \tag{2.18}$$

By (2.4), we have

$$(\tilde{\mathbf{v}}, \nabla_w q) = \sum_{T\in\mathcal{T}_h} (\langle \tilde{\mathbf{v}} \cdot \mathbf{n}, q_b\rangle_{\partial T} - (\nabla \cdot \tilde{\mathbf{v}}, q_0)_T) = -(\nabla \cdot \tilde{\mathbf{v}}, q_0). \tag{2.19}$$

Combining (2.16), (2.19) and (2.18) implies that

$$\frac{|(\tilde{\mathbf{v}}, \nabla_w q)|}{|||\mathbf{v}|||} = \frac{|(\nabla \cdot \tilde{\mathbf{v}}, q_0)|}{|||\mathbf{v}|||} \geq \frac{|(\nabla \cdot \tilde{\mathbf{v}}, q_0)|}{\|\tilde{\mathbf{v}}\|_{1,h}} \geq \beta\|q_0\|,$$

which implies (2.14).

Next we shall derive (2.15). For any $\mathbf{v} = \{\mathbf{v}_0, \mathbf{0}\} \in V_h$ and $\tau \in [P_{k+1}(T)]^{d\times d}$, we have the following by (2.3), (2.7) and the inverse inequalities:

$$(\nabla_w \mathbf{v}, \tau)_T = -(\mathbf{v}_0, \nabla\cdot\tau)_T = (\nabla\mathbf{v}_0, \tau)_T - \langle \mathbf{v}_0, \tau\cdot\mathbf{n}\rangle_{\partial T}$$
$$\leq \|\nabla\mathbf{v}_0\|_T\|\tau\|_T + Ch_T^{-1/2}\|\mathbf{v}_0\|_{\partial T}\|\tau\|_T,$$

which implies that

$$|\!|\!|\mathbf{v}|\!|\!| \leq Ch^{-1}\|\mathbf{v}_0\|. \tag{2.20}$$

It follows from (2.20) that for any $\mathbf{v} = \{\mathbf{v}_0, \mathbf{0}\} \in V_h$

$$\frac{|(\mathbf{v}_0, \nabla_w q)|}{|\!|\!|\mathbf{v}|\!|\!|} \geq Ch\frac{|(\mathbf{v}_0, \nabla_w q)|}{\|\mathbf{v}_0\|}.$$

Then we have

$$\sup_{\mathbf{v}\in V_h}\frac{|(\mathbf{v}_0, \nabla_w q)|}{|\!|\!|\mathbf{v}|\!|\!|} \geq Ch\sup_{\mathbf{v}\in V_h}\frac{|(\mathbf{v}_0, \nabla_w q)|}{\|\mathbf{v}_0\|} \geq \beta h\|\nabla_w q\|,$$

which implies (2.15). This completes the proof of the lemma.

Lemma 2.3. *There is a unique solution for the WG finite element equations (1.4) and (1.5).*

Proof. We only need to show that zero is the unique solution of (1.4) and (1.5) if $\mathbf{f} = \mathbf{0}$. We let $\mathbf{f} = \mathbf{0}$ and $\mathbf{v} = \mathbf{u}_h$ in (1.4) and $q = p_h$ in (1.5). By summing the two equations, we get

$$(\nabla_w \mathbf{u}_h,\ \nabla_w \mathbf{u}_h) = 0.$$

It implies that $\nabla_w \mathbf{u}_h = 0$ on T. By (2.12), we also obtain that $\|\mathbf{u}_h\|_{1,h} = 0$. Thus, $\mathbf{u}_h = \mathbf{0}$.

Since $\mathbf{u}_h = \mathbf{0}$ and $\mathbf{f} = \mathbf{0}$, (1.4) is reduced to $(\mathbf{v}_0, \nabla_w p_h) = 0$ for any $\mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\} \in V_h$. Then the inf-sup conditions (2.15) and (2.14) imply that $\nabla_w p_h = 0$ and $p_0 = 0$. By (2.13), we have that $\|p_h\|_{1,h} = 0$ and $p_0 = p_b = 0$ on ∂T. We have proved the lemma.

3. Error equations

To derive the equations that the errors satisfy, we introduce $\mathbf{e}_h = \mathbf{Q}_h\mathbf{u} - \mathbf{u}_h$ and $\epsilon_h = Q_h p - p_h$.

Lemma 3.1. *[41, Theorem 3.1] For $\tau \in [H^{k+2}(\Omega)]^d$, a quasi-projection Π_h can be defined such that $\Pi_h\tau \in [H(\mathrm{div}, \Omega)]^d$, $\Pi_h\tau|_T \in [P_{k+1}(T)]^{d\times d}$ and for $\mathbf{v}_0 \in [P_k(T)]^d$,*

$$(\nabla\cdot\tau,\ \mathbf{v}_0)_T = (\nabla\cdot\Pi_h\tau,\ \mathbf{v}_0)_T, \tag{3.1}$$
$$-(\nabla\cdot\tau,\ \mathbf{v}_0) = (\Pi_h\tau,\ \nabla_w\mathbf{v}), \tag{3.2}$$
$$\|\Pi_h\tau - \tau\| \leq Ch^{k+2}|\tau|_{k+2}, \tag{3.3}$$

where $|\cdot|_{k+2}$ is the semi-H^{k+2} Sobolev norm on the space.

Lemma 3.2. *[41, Theorem 3.1] Let $\tau \in H^{k+1}(\Omega)$. A quasi-projection π_h can be defined such that $\pi_h\tau \in H(\text{div},\Omega)$, $\pi_h\tau|_T \in [P_k(T)]^d$ and for $q_0 \in P_{k-1}(T)$,*

$$
\begin{aligned}
(\nabla\cdot\tau,\ q_0)_T &= (\nabla\cdot\pi_h\tau,\ q_0)_T, && (3.4)\\
-(\nabla\cdot\tau,\ q_0) &= (\pi_h\tau,\ \nabla_w q), && (3.5)\\
\|\pi_h\tau-\tau\| &\le Ch^{k+1}|\tau|_{k+1}, && (3.6)
\end{aligned}
$$

where $|\cdot|_{k+1}$ is the semi-H^{k+1} Sobolev norm on the space.

Lemma 3.3. *For any $\mathbf{v}\in V_h$ and $q \in W_h$, the following error equations hold true:*

$$
\begin{aligned}
(\mu\nabla_w\mathbf{e}_h,\ \nabla_w\mathbf{v}) + (\nabla_w\epsilon_h,\ \mathbf{v}_0) &= e_1(\mathbf{u},\mathbf{v}), && (3.7)\\
(\mathbf{e}_0,\ \nabla_w q) &= e_2(\mathbf{u},q), && (3.8)
\end{aligned}
$$

where

$$
\begin{aligned}
e_1(\mathbf{u},\ \mathbf{v}) &= \mu(\Pi_{k+1}\nabla\mathbf{u}-\Pi_h\nabla\mathbf{u},\nabla_w\mathbf{v}), && (3.9)\\
e_2(\mathbf{u},q) &= (\Pi_k\mathbf{u}-\pi_h\mathbf{u},\nabla_w q). && (3.10)
\end{aligned}
$$

Proof. First, we test (1.1) by applying $\mathbf{v}_0$ with $\mathbf{v}=\{\mathbf{v}_0,\mathbf{v}_b\}\in V_h$ to obtain

$$-(\mu\Delta\mathbf{u},\ \mathbf{v}_0) + (\nabla p,\ \mathbf{v}_0) = (\mathbf{f},\ \mathbf{v}_0). \quad (3.11)$$

It follows from (3.2) and (2.5) that

$$-(\mu\nabla\cdot\nabla\mathbf{u},\ \mathbf{v}_0) = (\mu\Pi_h\nabla\mathbf{u},\ \nabla_w\mathbf{v}) = (\mu\nabla_w\mathbf{Q}_h\mathbf{u},\ \nabla_w\mathbf{v}) - e_1(\mathbf{u},\mathbf{v}). \quad (3.12)$$

It follows from (2.6) that

$$(\nabla p,\ \mathbf{v}_0) = (\Pi_k\nabla p,\ \mathbf{v}_0) = (\nabla_w Q_h p,\ \mathbf{v}_0). \quad (3.13)$$

We substitute (3.12) and (3.13) into (3.11) to obtain

$$(\mu\nabla_w\mathbf{Q}_h\mathbf{u},\nabla_w\mathbf{v}) + (\mu\nabla_w Q_h p,\ \mathbf{v}_0) = (\mathbf{f},\mathbf{v}_0) + e_1(\mathbf{u},\mathbf{v}). \quad (3.14)$$

We subtract (3.14) from (1.4) to get

$$(\mu\nabla_w\mathbf{e}_h,\nabla_w\mathbf{v}) + (\mu\nabla_w\epsilon_h,\ \mathbf{v}_0) = e_1(\mathbf{u},\mathbf{v}) \quad \forall\mathbf{v}\in V_h. \quad (3.15)$$

Multiplying (1.2) by $q=\{q_0,q_b\}\in W_h$, by applying (3.2), it follows that

$$0 = (\nabla\cdot\mathbf{u},\ q_0) = -(\pi_h\mathbf{u},\nabla_w q) = -(\Pi_k\mathbf{u},\nabla_w q) + e_2(\mathbf{u},q), \quad (3.16)$$

which implies that

$$(\Pi_k\mathbf{u},\nabla_w q) = e_2(\mathbf{u},q). \quad (3.17)$$

The difference between (3.17) and (1.5) implies (3.8). We have proved the lemma.

4. Quasi-optimal finite element solutions

We shall first prove the optimal order error estimates of the $|\!|\!|\cdot|\!|\!|$ norm for the velocity $\mathbf{u}_h$, and of the L^2 norm for the pressure p_h.

Lemma 4.1. *Let $\mathbf{u} \in [H^{k+1}(\Omega)]^d$, $\mathbf{v} \in V_h$ and $q \in W_h$. The following estimates hold:*

$$|e_1(\mathbf{u},\ \mathbf{v})| \leq C\mu h^k |\mathbf{u}|_{k+1} |\!|\!|\mathbf{v}|\!|\!|, \tag{4.1}$$

$$|e_2(\mathbf{u},\ q)| \leq Ch^{k+1} |\mathbf{u}|_{k+1} |\!|\!|q|\!|\!|, \tag{4.2}$$

where $e_1(\cdot,\cdot)$ and $e_2(\cdot,\cdot)$ have been defined in (3.9) *and* (3.10)*, respectively.*

Proof. By the Cauchy-Schwarz inequality, the definitions of Π_{k+1} and Π_h, we compute

$$\begin{aligned} |e_1(\mathbf{u},\ \mathbf{v})| &= \mu|(\Pi_{k+1}\nabla\mathbf{u} - \Pi_h\nabla\mathbf{u}, \nabla_w\mathbf{v})| \\ &\leq C\mu h^k |\mathbf{u}|_{k+1} |\!|\!|\mathbf{v}|\!|\!|. \end{aligned}$$

Similarly, we have

$$\begin{aligned} |e_2(\mathbf{u},\ q)| &= |(\pi_h\mathbf{u} - \Pi_k\mathbf{u}, \nabla_w q)| \\ &\leq Ch^{k+1}|\mathbf{u}|_{k+1} || |\!|\!|q|\!|\!|. \end{aligned}$$

We have proved the lemma.

Theorem 4.1. *Let $(\mathbf{u}, p) \in ([H^{k+1}(\Omega)]^d \cap [H_0^1(\Omega)]^d) \times (H^k(\Omega) \cap L_0^2(\Omega))$ be the solutions of (1.1)–(1.3). Let $(\mathbf{u}_h, p_h) \in V_h \times W_h$ be the solutions of (1.4) and (1.5). Then, the following error estimates hold true:*

$$|\!|\!|\mathbf{Q}_h\mathbf{u} - \mathbf{u}_h|\!|\!| \leq Ch^k |\mathbf{u}|_{k+1}, \tag{4.3}$$

$$h|\!|\!|Q_h p - p_h|\!|\!| + ||\Pi_{k-1}p - p_0|| \leq C\mu h^k |\mathbf{u}|_{k+1}. \tag{4.4}$$

Proof. It follows from (3.7) that for any $\mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\} \in V_h$, by (4.1), we have

$$\begin{aligned} |(\nabla_w\epsilon_h,\ \mathbf{v}_0)| &= |(\mu\nabla_w\mathbf{e}_h,\ \nabla_w\mathbf{v}) - e_1(\mathbf{u}, \mathbf{v})| \\ &\leq C\mu(|\!|\!|\mathbf{e}_h|\!|\!| + h^k|\mathbf{u}|_{k+1})|\!|\!|\mathbf{v}|\!|\!|. \end{aligned} \tag{4.5}$$

Then applying the estimate (4.5) and (2.15) yields

$$h|\!|\!|\epsilon_h|\!|\!| \leq C\mu(|\!|\!|\mathbf{e}_h|\!|\!| + h^k|\mathbf{u}|_{k+1}). \tag{4.6}$$

By letting $\mathbf{v} = \mathbf{e}_h$ in (3.7) and $q = \epsilon_h$ in (3.8), and by using (3.8), we have

$$|\!|\!|\mathbf{e}_h|\!|\!|^2 = |\mu^{-1}e_1(\mathbf{u}, \mathbf{e}_h) - e_2(\mathbf{u}, \epsilon_h)|.$$

It then follows from (4.1), (4.2) and (4.6) that

$$\begin{aligned} |\!|\!|\mathbf{e}_h|\!|\!|^2 &\leq Ch^k|\mathbf{u}|_{k+1}|\!|\!|\mathbf{e}_h|\!|\!| + Ch^k|\mathbf{u}|_{k+1}h|\!|\!|\epsilon_h|\!|\!| \\ &\leq Ch^k|\mathbf{u}|_{k+1}|\!|\!|\mathbf{e}_h|\!|\!| + Ch^{2k}|\mathbf{u}|_{k+1}^2, \end{aligned}$$

which implies (4.3). The estimate (4.4) follows from (4.5), (4.3) and the inf-sup conditions (2.14) and (2.15). We have proved the theorem.

We shall derive next the optimal-order convergence for velocity in the L^2 norm by using the duality argument. Recall that $\mathbf{e}_h = \{\mathbf{e}_0, \mathbf{e}_b\} = \mathbf{Q}_h\mathbf{u} - \mathbf{u}_h$ and $\epsilon_h = Q_h p - p_h$. Consider the problem of seeking $(\boldsymbol{\psi}, \xi)$ such that

$$\begin{aligned} -\mu\Delta\boldsymbol{\psi} + \nabla\xi &= \mathbf{e}_0 \quad &\text{in } \Omega, \qquad (4.7)\\ \nabla\cdot\boldsymbol{\psi} &= 0 \quad &\text{in } \Omega, \qquad (4.8)\\ \boldsymbol{\psi} &= 0 \quad &\text{on } \partial\Omega. \qquad (4.9) \end{aligned}$$

Assume that the duality problem given by (4.7)–(4.9) has the $\mathbf{H}^2(\Omega)\times H^1(\Omega)$-regularity property that the solution $(\boldsymbol{\psi}, \xi) \in \mathbf{H}^2(\Omega)\times H^1(\Omega)$ and the following a priori estimate holds true:

$$\mu\|\boldsymbol{\psi}\|_2 + \|\xi\|_1 \leq C\|\mathbf{e}_0\|. \qquad (4.10)$$

We need the following lemma first.

Lemma 4.2. *For any $\mathbf{v} \in V_h$ and $q \in W_h$, the following equations hold true:*

$$\begin{aligned} (\mu\nabla_w\mathbf{Q}_h\boldsymbol{\psi},\ \nabla_w\mathbf{v}) + (\nabla_w Q_h\xi,\ \mathbf{v}_0) &= (\mathbf{e}_0, \mathbf{v}_0) + e_3(\boldsymbol{\psi}, \mathbf{v}), \qquad (4.11)\\ (\Pi_k\boldsymbol{\psi}, \nabla_w q) &= e_4(\boldsymbol{\psi}, q), \qquad (4.12) \end{aligned}$$

where $\boldsymbol{\psi}$ and ξ are defined in (4.7)*, and*

$$\begin{aligned} e_3(\boldsymbol{\psi},\ \mathbf{v}) &= \langle\mu(\nabla\boldsymbol{\psi} - \Pi_{k+1}\nabla\boldsymbol{\psi})\cdot\mathbf{n},\ \mathbf{v}_0 - \mathbf{v}_b\rangle_{\partial\mathcal{T}_h},\\ e_4(\boldsymbol{\psi},\ q) &= \langle(\boldsymbol{\psi} - \Pi_k\boldsymbol{\psi})\cdot\mathbf{n}, q_0 - q_b\rangle_{\partial\mathcal{T}_h}. \end{aligned}$$

Proof. Testing (4.7) by applying $\mathbf{v}_0$ with $\mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\} \in V_h$ gives

$$-(\mu\Delta\boldsymbol{\psi},\ \mathbf{v}_0) + (\nabla\xi,\ \mathbf{v}_0) = (\mathbf{e}_0,\ \mathbf{v}_0). \qquad (4.13)$$

By performing integration by parts, and setting $\langle\nabla\boldsymbol{\psi}\cdot\mathbf{n}, \mathbf{v}_b\rangle_{\partial\mathcal{T}_h} = 0$, we derive

$$-(\Delta\boldsymbol{\psi},\ \mathbf{v}_0) = (\nabla\boldsymbol{\psi}, \nabla\mathbf{v}_0)_{\mathcal{T}_h} - \langle\nabla\boldsymbol{\psi}\cdot\mathbf{n}, \mathbf{v}_0 - \mathbf{v}_b\rangle_{\partial\mathcal{T}_h}. \qquad (4.14)$$

By integration by parts, and given(2.3) and (2.5), we have

$$\begin{aligned} (\nabla\boldsymbol{\psi}, \nabla\mathbf{v}_0)_{\mathcal{T}_h} &= (\Pi_{k+1}\nabla\boldsymbol{\psi}, \nabla\mathbf{v}_0)_{\mathcal{T}_h}\\ &= -(\mathbf{v}_0, \nabla\cdot(\Pi_{k+1}\nabla\boldsymbol{\psi}))_{\mathcal{T}_h} + \langle\mathbf{v}_0, \Pi_{k+1}\nabla\boldsymbol{\psi}\cdot\mathbf{n}\rangle_{\partial\mathcal{T}_h}\\ &= (\Pi_{k+1}\nabla\boldsymbol{\psi}, \nabla_w\mathbf{v})_{\mathcal{T}_h} + \langle\mathbf{v}_0 - \mathbf{v}_b, \Pi_{k+1}\nabla\psi\cdot\mathbf{n}\rangle_{\partial\mathcal{T}_h}\\ &= (\nabla_w\mathbf{Q}_h\boldsymbol{\psi}, \nabla_w\mathbf{v})_{\mathcal{T}_h} + \langle\mathbf{v}_0 - \mathbf{v}_b, \Pi_{k+1}\nabla\boldsymbol{\psi}\cdot\mathbf{n}\rangle_{\partial\mathcal{T}_h}. \qquad (4.15) \end{aligned}$$

Combining (4.14) and (4.15) gives

$$-(\mu\Delta\boldsymbol{\psi},\ \mathbf{v}_0) = (\mu\nabla_w\mathbf{Q}_h\boldsymbol{\psi}, \nabla_w v) - e_3(\boldsymbol{\psi}, \mathbf{v}). \qquad (4.16)$$

By applying the definition of Π_k, (2.6), and (3.13), we obtain

$$(\nabla\xi,\ \mathbf{v}_0) = (\Pi_k\nabla\xi,\ \mathbf{v}_0) = (\nabla_w Q_h\xi,\ \mathbf{v}_0). \qquad (4.17)$$

Combining (4.16) and (4.17) with (4.13) yields (4.11).

Testing (4.8) by applying q_0 with $q = \{q_0, q_b\} \in W_h$ gives

$$(\nabla \cdot \boldsymbol{\psi},\ q_0) = 0. \tag{4.18}$$

By applying integration by parts, we obtain

$$\begin{aligned}
(\nabla \cdot \boldsymbol{\psi},\ q_0) &= -(\Pi_k\boldsymbol{\psi}, \nabla q_0)_{\mathcal{T}_h} + \langle \boldsymbol{\psi} \cdot \mathbf{n}, q_0 - q_b\rangle_{\partial\mathcal{T}_h} \\
&= (\nabla \cdot \Pi_k\boldsymbol{\psi}, q_0)_{\mathcal{T}_h} - \langle \Pi_k\boldsymbol{\psi} \cdot \mathbf{n}, q_0\rangle_{\partial\mathcal{T}_h} + \langle \boldsymbol{\psi} \cdot \mathbf{n}, q_0 - q_b\rangle_{\partial\mathcal{T}_h} \\
&= -(\Pi_k\boldsymbol{\psi}, \nabla_w q) - \langle \Pi_k\boldsymbol{\psi} \cdot \mathbf{n}, q_0 - q_b\rangle_{\partial\mathcal{T}_h} + \langle \boldsymbol{\psi} \cdot \mathbf{n}, q_0 - q_b\rangle_{\partial\mathcal{T}_h} \\
&= -(\Pi_k\boldsymbol{\psi}, \nabla_w q) + e_4(\boldsymbol{\psi}, q).
\end{aligned} \tag{4.19}$$

Combining (4.18) and (4.19) yields

$$(\Pi_k\boldsymbol{\psi}, \nabla_w q) = e_4(\boldsymbol{\psi}, q). \tag{4.20}$$

We have proved the lemma.

By the same argument as that for (4.16), (3.7) has another form, i.e.,

$$(\mu\nabla_w\mathbf{e}_h,\ \nabla_w\mathbf{v}) + (\nabla_w\epsilon_h,\ \mathbf{v}_0) = e_3(\mathbf{u}, \mathbf{v}), \tag{4.21}$$

$$(\mathbf{e}_0, \nabla_w q) = e_4(\mathbf{u}, q). \tag{4.22}$$

Letting $\mathbf{v} = \mathbf{Q}_h\boldsymbol{\psi}$ and $q = Q_h\xi$ in (4.21) and (4.22), we obtain

$$(\mu\nabla_w\mathbf{e}_h,\ \nabla_w\mathbf{Q}_h\boldsymbol{\psi}) + (\nabla_w\epsilon_h,\ \Pi_k\boldsymbol{\psi}) = e_3(\mathbf{u}, \mathbf{Q}_h\boldsymbol{\psi}), \tag{4.23}$$

$$(\mathbf{e}_0, \nabla_w Q_h\xi) = e_4(\mathbf{u}, Q_h\xi). \tag{4.24}$$

Letting $\mathbf{v} = \mathbf{e}_h$ and $q = \epsilon_h$ in (4.11) and (4.12), we have

$$(\mu\nabla_w\mathbf{Q}_h\boldsymbol{\psi},\ \nabla_w\mathbf{e}_h) + (\nabla_w Q_h\xi,\ \mathbf{e}_0) = (\mathbf{e}_0, \mathbf{e}_0) + e_3(\boldsymbol{\psi}, \mathbf{e}_h), \tag{4.25}$$

$$(\Pi_k\boldsymbol{\psi}, \nabla_w\epsilon_h) = e_4(\boldsymbol{\psi}, \epsilon_h), \tag{4.26}$$

By applying (4.26), (4.23) becomes

$$(\mu\nabla_w\mathbf{Q}_h\boldsymbol{\psi},\ \nabla_w\mathbf{e}_h) = e_3(\mathbf{u}, \mathbf{Q}_h\boldsymbol{\psi}) + e_4(\boldsymbol{\psi}, \epsilon_h). \tag{4.27}$$

Theorem 4.2. *Let* $(\mathbf{u}, p) \in ([H^{k+1}(\Omega)]^d \cap [H_0^1(\Omega)]^d) \times (H^k(\Omega) \cap L_0^2(\Omega))$ *be the solutions of (1.1)–(1.3). Let* $(\mathbf{u}_h, p_h) \in V_h \times W_h$ *denote the unique solutions of (1.4) and (1.5). With the condition (4.10), the following error bound holds:*

$$\|\Pi_k\mathbf{u} - \mathbf{u}_0\| \le Ch^{k+1}|\mathbf{u}|_{k+1}. \tag{4.28}$$

Proof. Letting $\mathbf{v} = \mathbf{e}_h$ in (4.11) yields

$$\|\mathbf{e}_0\|^2 = (\mu\nabla_w Q_h\boldsymbol{\psi},\ \nabla_w\mathbf{e}_h)_{\mathcal{T}_h} - (\mathbf{e}_0, \nabla_w Q_h\xi) + e_3(\boldsymbol{\psi}, \mathbf{e}_h). \tag{4.29}$$

By applying (4.27) and (4.24), (4.29) becomes

$$\|\mathbf{e}_h\|^2 = e_3(\mathbf{u}, \mathbf{Q}_h\boldsymbol{\psi}) + e_4(\boldsymbol{\psi}, \epsilon_h) + e_3(\boldsymbol{\psi}, \mathbf{e}_h) + e_4(\mathbf{u}, Q_h\xi). \tag{4.30}$$

Next we shall estimate all of the terms on the right hand side of (4.30). Using the Cauchy-Schwarz inequality, the trace inequality (2.7), and the definition of Π_{k+1}, we obtain

$$\begin{aligned}
|e_3(\mathbf{u}, \mathbf{Q}_h\boldsymbol{\psi})| &\leq \mu \left|\langle(\nabla\mathbf{u} - \Pi_{k+1}\nabla\mathbf{u})\cdot\mathbf{n},\ \Pi_k\boldsymbol{\psi} - \Pi_{k+1}^b\boldsymbol{\psi}\rangle_{\partial\mathcal{T}_h}\right| \\
&\leq \mu\left(\sum_{T\in\mathcal{T}_h}\|\nabla\mathbf{u} - \Pi_{k+1}\nabla\mathbf{u}\|_{\partial T}^2\right)^{1/2}\left(\sum_{T\in\mathcal{T}_h}\|\Pi_k\boldsymbol{\psi} - \boldsymbol{\psi}\|_{\partial T}^2\right)^{1/2} \\
&\leq C\mu h^{k+1}|\mathbf{u}|_{k+1}|\boldsymbol{\psi}|_2. \qquad (4.31)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
|e_4(\mathbf{u}, Q_h\xi)| &\leq \left|\langle(\mathbf{u} - \Pi_k\mathbf{u})\cdot\mathbf{n},\ \Pi_{k-1}\xi - \Pi_k^b\xi\rangle_{\partial\mathcal{T}_h}\right| \\
&\leq \left(\sum_{T\in\mathcal{T}_h}\|\mathbf{u} - \Pi_k\mathbf{u}\|_{\partial T}^2\right)^{1/2}\left(\sum_{T\in\mathcal{T}_h}\|\Pi_{k-1}\xi - \xi\|_{\partial T}^2\right)^{1/2} \\
&\leq Ch^{k+1}|\mathbf{u}|_{k+1}|\xi|_1. \qquad (4.32)
\end{aligned}$$

Using the Cauchy-Schwarz inequality and the trace inequalities, applying (2.12) and (4.3), we obtain

$$\begin{aligned}
|e_3(\boldsymbol{\psi}, \mathbf{e}_h)| &\leq \mu\left|\langle(\nabla\boldsymbol{\psi} - \Pi_{k+1}\nabla\boldsymbol{\psi})\cdot\mathbf{n},\ \mathbf{e}_0 - \mathbf{e}_b\rangle_{\partial\mathcal{T}_h}\right| \\
&\leq \mu\left(\sum_{T\in\mathcal{T}_h}h_T\|\nabla\boldsymbol{\psi} - \Pi_{k+1}\nabla\boldsymbol{\psi}\|_{\partial T}^2\right)^{1/2}\left(\sum_{T\in\mathcal{T}_h}h_T^{-1}\|\mathbf{e}_0 - \mathbf{e}_b\|_{\partial T}^2\right)^{1/2} \\
&\leq C\mu h|\boldsymbol{\psi}|_2|\!|\!|\mathbf{e}_h|\!|\!| \\
&\leq C\mu h^{k+1}|\mathbf{u}|_{k+1}|\boldsymbol{\psi}|_2. \qquad (4.33)
\end{aligned}$$

By (4.4), we have

$$\begin{aligned}
|e_4(\boldsymbol{\psi}, \epsilon_h)| &\leq \left|\langle(\boldsymbol{\psi} - \Pi_k\boldsymbol{\psi})\cdot\mathbf{n},\ \epsilon_0 - \epsilon_b\rangle_{\partial\mathcal{T}_h}\right| \\
&\leq \left(\sum_{T\in\mathcal{T}_h}\|\boldsymbol{\psi} - \Pi_k\boldsymbol{\psi}\|_{\partial T}^2\right)^{1/2}\left(\sum_{T\in\mathcal{T}_h}\|\epsilon_0 - \epsilon_b\|_{\partial T}^2\right)^{1/2} \\
&\leq Ch|\boldsymbol{\psi}|_2 h|\!|\!|\epsilon_h|\!|\!| \\
&\leq C\mu h^{k+1}|\mathbf{u}|_{k+1}|\boldsymbol{\psi}|_2. \qquad (4.34)
\end{aligned}$$

Substituting all the four bounds above into (4.30), we get

$$\|\mathbf{e}_h\|^2 \leq Ch^{k+1}|\mathbf{u}|_{k+1}(\mu\|\boldsymbol{\psi}\|_2 + \|\xi\|_1).$$

By applying this inequality and the regularity condition (4.10), (4.28) is proved.

5. Numerical test

In the first example in 2D, we have chosen the domain $\Omega = (0,1)\times(0,1)$ for the Stokes equations (1.1)–(1.3). We have chosen an $\mathbf{f}$ (depending on μ) in (1.1) such that the exact solution of (1.1)–(1.3) is as follows (independent of μ):

$$\mathbf{u} = \begin{pmatrix} (2y - 6y^2 + 4y^3)(x^2 - 2x^3 + x^4) \\ -(2x - 6x^2 + 4x^3)(y^2 - 2y^3 + y^4) \end{pmatrix}, \qquad p = -2x^3 + 3x^2 - x. \tag{5.1}$$

We computed the solution (5.1) on triangular grids shown, as in Figure 1 for the P_k-WG/P_{k-1}-WG mixed finite elements for $k = 1, 2, 3, 4$, and 5. The results are listed in Tables 1–5. As shown, the optimal order of convergence has been achieved for all solutions in all norms. From the data, we can see the method is pressure-robust and the error is independent of the viscosity μ.

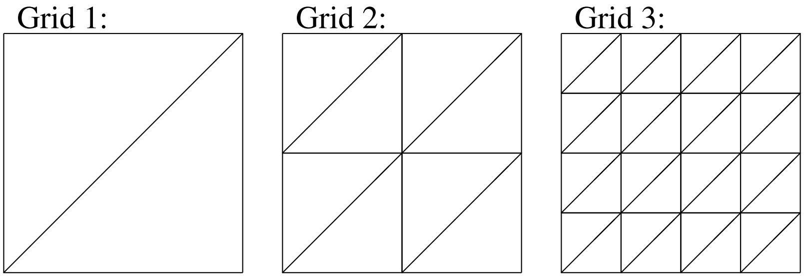

Figure 1. The uniform triangular meshes used to compute the results in Tables 1–5.

Table 1. The order of convergence and the error results given by the P_1/P_0 WG element for the solution (5.1) for the Figure 1 meshes.

Grid	$\|\mathbf{u}-\mathbf{u}_h\|_0$	$O(h^r)$	$\|\!\|\!\|\mathbf{u}-\mathbf{u}_h\|\!\|\!\|$	$O(h^r)$	$\|\Pi_{k-1}p - p_0\|_0$	$O(h^r)$
	By the P_1-WG/P_0-WG elements, $\mu = 1$ in (1.1).					
5	0.5113E-03	1.94	0.2717E-01	0.99	0.1927E-01	1.00
6	0.1285E-03	1.99	0.1357E-01	1.00	0.9706E-02	0.99
7	0.3209E-04	2.00	0.6769E-02	1.00	0.4871E-02	0.99
	By the P_1-WG/P_0-WG elements, $\mu = 10^{-6}$ in (1.1).					
5	0.5113E-03	1.94	0.2717E-01	0.99	0.1929E-07	1.00
6	0.1285E-03	1.99	0.1357E-01	1.00	0.9700E-08	0.99
7	0.3209E-04	2.00	0.6769E-02	1.00	0.4881E-08	0.99

Table 2. The order of convergence and the error results given by the P_2/P_1 WG element for the solution (5.1) for the Figure 1 meshes.

Grid	$\|\mathbf{u}-\mathbf{u}_h\|_0$	$O(h^r)$	$\|\!\|\mathbf{u}-\mathbf{u}_h\|\!\|$	$O(h^r)$	$\|\Pi_{k-1}p-p_0\|_0$	$O(h^r)$
	By the P_2-WG/P_1-WG elements, $\mu = 1$ in (1.1).					
4	0.9758E-04	3.03	0.6913E-02	1.91	0.1433E-01	1.13
5	0.1185E-04	3.04	0.1747E-02	1.98	0.4625E-02	1.63
6	0.1458E-05	3.02	0.4356E-03	2.00	0.1301E-02	1.83
	By the P_2-WG/P_1-WG elements, $\mu = 10^{-6}$ in (1.1).					
4	0.9758E-04	3.03	0.6913E-02	1.91	0.1435E-07	1.13
5	0.1185E-04	3.04	0.1747E-02	1.98	0.4643E-08	1.63
6	0.1458E-05	3.02	0.4356E-03	2.00	0.1345E-08	1.79

Table 3. The order of convergence and the error results given by the P_3/P_2 WG element for the solution (5.1) for the Figure 1 meshes.

Grid	$\|\mathbf{u}-\mathbf{u}_h\|_0$	$O(h^r)$	$\|\!\|\mathbf{u}-\mathbf{u}_h\|\!\|$	$O(h^r)$	$\|\Pi_{k-1}p-p_0\|_0$	$O(h^r)$
	By the P_3-WG/P_2-WG elements, $\mu = 1$ in (1.1).					
4	0.6255E-05	3.92	0.5836E-03	2.89	0.1448E-02	2.58
5	0.3879E-06	4.01	0.7294E-04	3.00	0.2066E-03	2.81
6	0.2392E-07	4.02	0.9022E-05	3.02	0.2751E-04	2.91
	By the P_3-wG/P_2-WG elements, $\mu = 10^{-6}$ in (1.1).					
4	0.6255E-05	3.92	0.5836E-03	2.89	0.1489E-08	2.54
5	0.3879E-06	4.01	0.7294E-04	3.00	0.3668E-09	2.02
6	0.2392E-07	4.02	0.9022E-05	3.02	0.3366E-09	0.12

Table 4. The order of convergence and the error results given by the P_4/P_3 WG element for the solution (5.1) for the Figure 1 meshes.

Grid	$\|\mathbf{u}-\mathbf{u}_h\|_0$	$O(h^r)$	$\|\!\|\mathbf{u}-\mathbf{u}_h\|\!\|$	$O(h^r)$	$\|\Pi_{k-1}p-p_0\|_0$	$O(h^r)$
	By the P_4-WG/P_3-WG elements, $\mu = 1$ in (1.1).					
3	0.1291E-04	4.63	0.6964E-03	3.67	0.6446E-03	3.47
4	0.4222E-06	4.93	0.4501E-04	3.95	0.4543E-04	3.83
5	0.1321E-07	5.00	0.2814E-05	4.00	0.2983E-05	3.93
	By the P_4-WG/P_3-WG elements, $\mu = 10^{-6}$ in (1.1).					
3	0.1291E-04	4.63	0.6964E-03	3.67	0.8212E-09	3.12
4	0.4222E-06	4.93	0.4501E-04	3.95	0.4223E-09	0.96
5	0.1321E-07	5.00	0.2814E-05	4.00	0.4401E-09	0.00

Table 5. The order of convergence and the error results given by the P_5/P_4 WG element for the solution (5.1) for the Figure 1 meshes.

Grid	$\|\mathbf{u}-\mathbf{u}_h\|_0$	$O(h^r)$	$\|\!\|\mathbf{u}-\mathbf{u}_h\|\!\|$	$O(h^r)$	$\|\Pi_{k-1}p-p_0\|_0$	$O(h^r)$
	By the P_5-WG/P_4-WG elements, $\mu=1$ in (1.1).					
2	0.5642E-04	4.37	0.1814E-02	3.60	0.9773E-03	3.55
3	0.9738E-06	5.86	0.6161E-04	4.88	0.3342E-04	4.87
4	0.1546E-07	5.98	0.1952E-05	4.98	0.1069E-05	4.97
	By the P_5-WG/P_4-WG elements, $\mu=10^{-6}$ in (1.1).					
2	0.5642E-04	4.37	0.1814E-02	3.60	0.1211E-08	3.24
3	0.9738E-06	5.86	0.6161E-04	4.88	0.3831E-09	1.66
4	0.1546E-07	5.98	0.1953E-05	4.98	0.5148E-09	0.00

We note that for some high level grids the computer round-off error was found to exceed the truncation error when $\mu=10^{-6}$, as described in Tables 3–5.

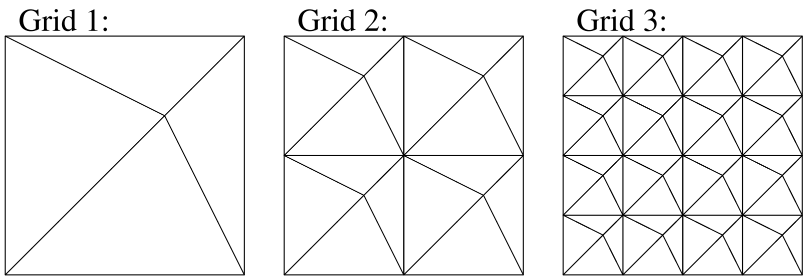

Figure 2. The triangular meshes used to compute the results in Tables 6–10.

We computed the 2D solution (5.1) again for slightly perturbed triangular meshes, as illustrated in Figure 2 by employing the P_k-WG/P_{k-1}-WG mixed finite elements for $k=1,2,3,4$ and 5. The computational results are listed in Tables 6–10. The quasi-optimal convergence has been achieved for all solutions in all norms.

Table 6. The order of convergence and the error results given by the P_1/P_0 WG element for the solution (5.1) for the Figure 2 meshes.

Grid	$\|\mathbf{u}-\mathbf{u}_h\|_0$	$O(h^r)$	$\|\!\|\mathbf{u}-\mathbf{u}_h\|\!\|$	$O(h^r)$	$\|\Pi_{k-1}p-p_0\|_0$	$O(h^r)$
	By the P_1-WG/P_0-WG elements, $\mu=1$ in (1.1).					
4	0.7627E-03	1.85	0.3264E-01	0.96	0.1809E-01	1.28
5	0.1954E-03	1.96	0.1639E-01	0.99	0.7991E-02	1.18
6	0.4904E-04	1.99	0.8189E-02	1.00	0.3835E-02	1.06
	By the P_1-WG/P_0-WG elements, $\mu=10^{-6}$ in (1.1).					
4	0.7627E-03	1.85	0.3264E-01	0.96	0.1808E-07	1.28
5	0.1954E-03	1.96	0.1639E-01	0.99	0.7999E-08	1.18
6	0.4904E-04	1.99	0.8189E-02	1.00	0.3846E-08	1.06

Table 7. The order of convergence and the error results given by the P_2/P_1 WG element for the solution (5.1) for the Figure 2 meshes.

Grid	$\|\|\mathbf{u}-\mathbf{u}_h\|\|_0$	$O(h^r)$	$\|\|\|\mathbf{u}-\mathbf{u}_h\|\|\|$	$O(h^r)$	$\|\|\Pi_{k-1}p-p_0\|\|_0$	$O(h^r)$
	By the P_2-WG/P_1-WG elements, $\mu=1$ in (1.1).					
4	0.2225E-04	2.95	0.2167E-02	1.94	0.2011E-02	1.04
5	0.2784E-05	3.00	0.5434E-03	2.00	0.6969E-03	1.53
6	0.3480E-06	3.00	0.1355E-03	2.00	0.2017E-03	1.79
	By the P_2-WG/P_1-WG elements, $\mu=10^{-6}$ in (1.1).					
4	0.2225E-04	2.95	0.2167E-02	1.94	0.2031E-08	1.03
5	0.2784E-05	3.00	0.5434E-03	2.00	0.7851E-09	1.37
6	0.3480E-06	3.00	0.1355E-03	2.00	0.3701E-09	1.08

Table 8. The order of convergence and the error results given by the P_3/P_2 WG element for the solution (5.1) for the Figure 2 meshes.

Grid	$\|\|\mathbf{u}-\mathbf{u}_h\|\|_0$	$O(h^r)$	$\|\|\|\mathbf{u}-\mathbf{u}_h\|\|\|$	$O(h^r)$	$\|\|\Pi_{k-1}p-p_0\|\|_0$	$O(h^r)$
	By the P_3-WG/P_2-WG elements, $\mu=1$ in (1.1).					
3	0.1339E-04	3.62	0.9788E-03	2.69	0.2254E-02	1.84
4	0.8544E-06	3.97	0.1239E-03	2.98	0.3795E-03	2.57
5	0.5341E-07	4.00	0.1541E-04	3.01	0.5423E-04	2.81
	By the P_3-WG/P_2-WG elements, $\mu=10^{-6}$ in (1.1).					
3	0.1339E-04	3.62	0.9788E-03	2.69	0.2311E-08	1.80
4	0.8544E-06	3.97	0.1239E-03	2.98	0.4983E-09	2.21
5	0.5341E-07	4.00	0.1541E-04	3.01	0.3679E-09	0.44

Table 9. The order of convergence and the error results given by the P_4/P_3 WG element for the solution (5.1) for the Figure 2 meshes.

Grid	$\|\|\mathbf{u}-\mathbf{u}_h\|\|_0$	$O(h^r)$	$\|\|\|\mathbf{u}-\mathbf{u}_h\|\|\|$	$O(h^r)$	$\|\|\Pi_{k-1}p-p_0\|\|_0$	$O(h^r)$
	By the P_4-WG/P_3-WG elements, $\mu=1$ in (1.1).					
2	0.2795E-04	3.98	0.1327E-02	3.12	0.3037E-02	2.70
3	0.8864E-06	4.98	0.8282E-04	4.00	0.2700E-03	3.49
4	0.2742E-07	5.01	0.5105E-05	4.02	0.1973E-04	3.77
	By the P_4-WG/P_3-WG elements, $\mu=10^{-6}$ in (1.1).					
2	0.2795E-04	3.98	0.1327E-02	3.12	0.3134E-08	2.66
3	0.8864E-06	4.98	0.8282E-04	4.00	0.5525E-09	2.50
4	0.2742E-07	5.01	0.5105E-05	4.02	0.4121E-09	0.42

Table 10. The order of convergence and the error results given by the P_5/P_4 WG element for the solution (5.1) for the Figure 2 meshes.

Grid	$\|\mathbf{u}-\mathbf{u}_h\|_0$	$O(h^r)$	$\|\!\|\mathbf{u}-\mathbf{u}_h\|\!\|$	$O(h^r)$	$\|\Pi_{k-1}p-p_0\|_0$	$O(h^r)$
	By the P_5-WG/P_4-WG elements, $\mu = 1$ in (1.1).					
2	0.2375E-05	6.00	0.1322E-03	5.02	0.2879E-03	4.38
3	0.3664E-07	6.02	0.4074E-05	5.02	0.1069E-04	4.75
4	0.1017E-08	5.17	0.4051E-06	3.33	0.3607E-06	4.89
	By the P_5-WG/P_4-WG elements, $\mu = 10^{-6}$ in (1.1).					
2	0.2375E-05	6.00	0.1322E-03	5.02	0.6884E-09	3.14
3	0.3664E-07	6.02	0.4075E-05	5.02	0.4504E-09	0.61
4	0.1013E-08	5.18	0.4030E-06	3.34	0.5125E-09	0.00

In the third test, we performed 3D numerical computation on domain $\Omega = (0,1)\times(0,1)\times(0,1)$. We chose an $\mathbf{f}$ in (1.1) such that we would have the following exact solution

$$\mathbf{u} = \begin{pmatrix} -2^{10}(x-1)^2x^2(y-1)^2y^2(z-3z^2+2z^3) \\ 2^{10}(x-1)^2x^2(y-1)^2y^2(z-3z^2+2z^3) \\ 2^{10}[(x-3x^2+2x^3)(y^2-y)^2-(x^2-x)^2(y-3y^2+2y^3)](z^2-z)^2 \end{pmatrix}, \qquad (5.2)$$
$$p = -10(3y^2-2y^3-y).$$

The 3D meshes are illustrated in Figure 3. The computational results are listed in Tables 11–13.

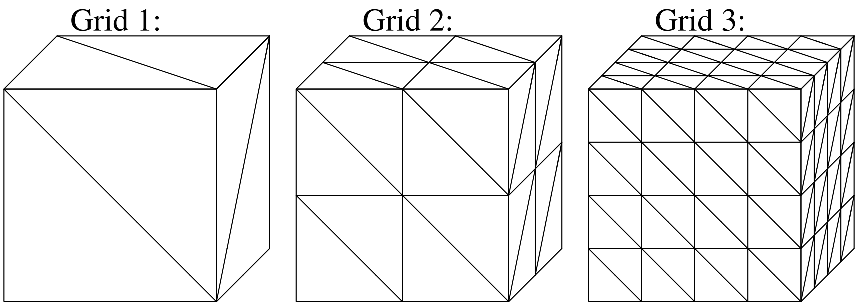

Figure 3. The tetrahedral meshes used to compute the results in Tables 11–13.

Table 11. The order of convergence and the error results given by the P_1/P_0 WG finite element for the solution (5.2) for the Figure 3 meshes.

Grid	$\|\|\mathbf{u}-\mathbf{u}_h\|\|_0$	$O(h^r)$	$\|\|\|\mathbf{u}-\mathbf{u}_h\|\|\|$	$O(h^r)$	$\|\|\Pi_{k-1}p-p_0\|\|_0$	$O(h^r)$
	By the P_1-WG/P_0-WG elements, $\mu=1$ in (1.1).					
3	0.366E-01	1.20	0.696E+00	0.73	0.412E+00	2.62
4	0.975E-02	1.91	0.355E+00	0.97	0.812E-01	2.34
5	0.242E-02	2.01	0.180E+00	0.98	0.145E-01	2.49
	By the P_1-WG/P_0-WG elements, $\mu=10^{-3}$ in (1.1).					
3	0.428E-01	1.03	0.763E+00	0.67	0.553E-03	2.42
4	0.108E-01	1.98	0.362E+00	1.08	0.134E-03	2.05
5	0.249E-02	2.12	0.180E+00	1.00	0.196E-04	2.77

Table 12. The order of convergence and the error results given by the P_2/P_1 WG finite element for the solution (5.2) for the Figure 3 meshes.

Grid	$\|\|\mathbf{u}-\mathbf{u}_h\|\|_0$	$O(h^r)$	$\|\|\|\mathbf{u}-\mathbf{u}_h\|\|\|$	$O(h^r)$	$\|\|\Pi_{k-1}p-p_0\|\|_0$	$O(h^r)$
	By the P_2-WG/P_1-WG elements, $\mu=1$ in (1.1).					
3	0.618E-02	2.90	0.231E+00	1.83	0.145E+00	1.78
4	0.639E-03	3.27	0.498E-01	2.21	0.258E-01	2.49
5	0.476E-04	3.75	0.117E-01	2.09	0.540E-03	5.58
	By the P_2-WG/P_1-WG elements, $\mu=10^{-3}$ in (1.1).					
3	0.615E-02	2.91	0.209E+00	1.97	0.135E-03	1.91
4	0.685E-03	3.17	0.473E-01	2.14	0.232E-04	2.54
5	0.474E-04	3.85	0.117E-01	2.02	0.716E-06	5.02

Table 13. The order of convergence and the error results given by the P_3/P_2 WG finite element for the solution (5.2) for the Figure 3 meshes.

Grid	$\|\|\mathbf{u}-\mathbf{u}_h\|\|_0$	$O(h^r)$	$\|\|\|\mathbf{u}-\mathbf{u}_h\|\|\|$	$O(h^r)$	$\|\|\Pi_{k-1}p-p_0\|\|_0$	$O(h^r)$
	By the P_3-WG/P_2-WG elements, $\mu=1$ in (1.1).					
3	0.644E-03	3.89	0.376E-01	2.80	0.257E-01	3.70
4	0.351E-04	4.20	0.480E-02	2.97	0.896E-03	4.84
5	0.198E-05	4.15	0.618E-03	2.96	0.465E-04	4.27
	By the P_3-WG/P_2-WG elements, $\mu=10^{-3}$ in (1.1).					
3	0.708E-03	3.71	0.382E-01	2.70	0.314E-04	3.58
4	0.507E-04	3.80	0.484E-02	2.98	0.140E-05	4.48
5	0.210E-05	4.60	0.618E-03	2.97	0.398E-07	5.14

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

Yan Yang is supported in part by the Program of Sichuan National Applied Mathematics Center, No. 2023-KFJJ-01-001.

Conflict of interest

The authors declare there are no conflicts of interest.

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