# BASIS PROPERTIES OF TRACES
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# OF
# SPHERICAL-SEPARABLE SOLUTIONS
# OF THE HELMHOLTZ EQUATION

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# Basis Properties of Traces and Normal Derivatives of Spherical-Separable Solutions of the Helmholtz Equation*

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**Abstract.** The classical solutions of the Helmholtz equation resulting from the separation-of-variables procedure in spherical coördinates are frequently used in one way or another to approximate other solutions. In particular, traces and/or normal derivatives of certain sequences of these spherical-separable solutions are commonly used as trial- and test-functions in Galerkin procedures for the approximate solution of boundary-operator problems arising from the reformulation of exterior or interior boundary-value problems and set on the boundary $\Gamma$ of the domain where a solution is wanted. While the *completeness* properties of these traces and normal derivatives in the usual Hilbert space $L_2(\Gamma)$ are well known, their *basis* properties are not. We show that such sequences of traces or normal derivatives of the outgoing spherical-separable solutions form bases for $L_2(\Gamma)$ only when $\Gamma$ is a sphere centered at the pole of the spherical solutions; corresponding results are given for the entire solutions, accounting for the possibility of an interior eigenvalue. We identify other Hilbert spaces, connected with the far-field pattern, for which these functions do provide bases. We apply the results to discuss some aspects of the Waterman schemes for approximate solution of scattering problems (the so-called "$T$-matrix method"), including the previous article of KRISTENSSON, RAMM, AND STRÖM (*J. Math. Phys.* **24** (1983), 2619–2631) on the convergence of such methods.

## CONTENTS

## 0. Introduction.

Boundary-value problems for the Helmholtz equation

$$\Delta u + \kappa^2 u = 0 \qquad\qquad (0.1)$$

in either an interior domain $\Omega_-$ in $\mathbb{R}^3$ or the corresponding exterior domain $\Omega_+ := \mathbb{R}^3 \setminus \overline{\Omega}_-$ (with a radiation condition) are frequently replaced by equivalent boundary-operator problems set on the boundary $\Gamma := \partial\Omega_- = \partial\Omega_+$. In the construction of approximate solutions for the reformulated problems, trial and/or test functions are sometimes generated from the traces and normal derivatives on $\Gamma$ of certain countable families of *spherical-wave functions*, *i.e.*, those special solutions of (0.1) that result from separating the variables in spherical coördinates. In the analysis and validation of such approximation schemes, it is frequently enough to know that one or more of these families of traces and normal derivatives have the properties of linear independence and "completeness" in an appropriate Hilbert space of functions defined on the boundary $\Gamma$; as usual, by *completeness* we mean that the linear span of the family is dense in the Hilbert space. For definiteness, let us suppose that the Hilbert space of interest is the familiar space $L_2(\Gamma)$, comprising those complex functions on $\Gamma$ with square-integrable moduli. Since the completeness properties of the traces and normal derivatives of such collections of spherical wave-functions are already fairly well known in the latter space, in many cases there is no difficulty in verifying the required hypothesis.

However, in other instances one needs to know—or finds it convenient to assume—that such a countable collection of traces or normal derivatives of spherical-separable solutions actually enjoys the much stronger property of forming a *basis* for $L_2(\Gamma)$. For example, one strategy consists in hypothesizing the required basis property, proving the desired convergence result, and later attempting to identify domains $\Omega_-$ for which the basis property—and so also the convergence proof—holds.

Moreover, it is not uncommon to find unsubstantiated and even erroneous claims arising from confusion about the basis property in the heuristic sorts of arguments that are sometimes used to motivate experimentation with one or another algorithm. For example, a number of papers on the WATERMAN [21] schemes for approximate solution of problems in obstacle scattering contain statements that are at best misleading and at worst simply incorrect, owing to tacit claims about the validity of certain infinite-series expansions. There, the difficulty sometimes arises also from confusion about the very meanings of the completeness property and the basis property; it is common to find the former mistakenly assumed either to imply or to be synonymous with the latter, so that a number of writers have incorrectly asserted the existence of series representations in terms of spherical-wave traces or normal derivatives merely on the strength of the completeness property. In such cases, practically every ensuing "conclusion" is either incorrect or holds only for certain geometries. The effect of all this has been the propagation of confusion in the literature on "$T$-matrix methods" to an extent that is now extremely difficult to overcome.

In any event, a definitive statement concerning the basis properties of the traces and normal derivatives of the spherical-wave functions in $L_2(\Gamma)$ has apparently been lacking to this point. In fact, it seems that the basis questions have been settled only for the simplest case, in which the underlying domain $\Omega_-$ is a ball centered at the pole of the spherical-separable solutions; there, the traces and normal derivatives form *orthogonal families* in $L_2(\Gamma)$, so that the basis property does turn out to be implied by the completeness property. Consequently, in view of the previous remarks, it is not surprising to find that this gap has led to an accumulation of open questions and conjectures

2

about the motivation and heuristic justification, to say nothing of the convergence properties, of various approximate-solution schemes that employ the spherical-wave functions.

As the first main result of this note, it is proven that certain sequences of traces and normal derivatives of the outgoing spherical-wave functions form bases for $L_2(\Gamma)$ only in the classically well-known case in which the boundary $\Gamma$ is a sphere centered at the pole; the appropriately qualified corresponding assertions for the regular spherical-wave functions are also established. Convergence proofs that have relied on the basis property in $L_2(\Gamma)$ must then be interpreted accordingly, since they will be known to apply only in the spherical-domain setting. On the other hand, we identify a weaker sense in which the outgoing functions do form bases, *viz.*, with respect to an inner product that is intimately connected with the far-field patterns of outgoing solutions of (0.1). Interspersed with these results, we remark on a few of their implications for previous work concerning the Waterman schemes for approximate solution of scattering problems, in [21] and [13].

## 1. Notations; statement of the main basis result.

This section is devoted to the introduction of some notation, a review of the simplest definitions and results about bases in Hilbert spaces, and a statement of the $L_2(\Gamma)$-basis properties that we shall establish.

Throughout, we suppose $\Omega_-$ to be a bounded and connected regularly open subset of $\mathbb{R}^3$ for which the corresponding exterior domain $\Omega_+ := \mathbb{R}^3 \setminus \overline{\Omega}_-$ is also connected. The common boundary $\Gamma := \partial\Omega_- = \partial\Omega_+$ we take to be of class $C^{2,\alpha}$, and we denote by $\mathbf{n}$ the continuous unit-normal field for $\Gamma$ that is oriented toward the exterior domain $\Omega_+$. We suppose in Sections 1–5 that the "wavenumber" parameter $\kappa$ in (0.1) is complex and nonzero, with nonnegative imaginary part; however, in Sections 6 and 7 we shall require $\kappa$ to be real and positive. The trace and normal derivative on $\Gamma$ of an appropriate function $u$ defined either in $\Omega_-$ or in $\Omega_+$ we indicate by $u|_\Gamma$ and $u_{,\mathbf{n}}$, respectively.

To introduce the two families of spherical-separable solutions of (0.1) that we study here, we begin by fixing a point $\mathcal{O} \in \Omega_-$ to serve as the pole of a spherical coördinate system for $\mathbb{R}^3$. The first family $\{\, V_{lm}^{\kappa\mathcal{O}} \mid m = -l, \ldots, l, \quad l = 0, 1, 2, \ldots \,\}$ comprises outgoing solutions of (0.1) in $\mathbb{R}^3 \setminus \{\mathcal{O}\}$ (and so must have singularities at $\mathcal{O}$); in particular, each function is a solution of (0.1) in all of $\Omega_+$. As always, a solution $u$ of (0.1) in an exterior domain is said to be *outgoing* if it satisfies the Sommerfeld radiation condition

$$\lim_{\varrho\to\infty} \varrho\{\hat{\mathbf{e}}\cdot\operatorname{grad} u(\mathcal{O} + \varrho\,\hat{\mathbf{e}}) - i\kappa u(\mathcal{O} + \varrho\,\hat{\mathbf{e}})\} = 0 \qquad \text{uniformly for} \quad \hat{\mathbf{e}} \in \Sigma_1, \tag{1.1}$$

with $\Sigma_1$ denoting throughout the surface of the unit ball in $\mathbb{R}^3$. Specifically, using the spherical Hankel functions of the first kind $h_l^{(1)}$, for the indicated pairs $(l, m)$ we set

$$V_{lm}^{\kappa\mathcal{O}}(\mathbf{x}) := \sqrt{2}\, h_l^{(1)}\big(\kappa|\mathbf{x} - \mathcal{O}|\big)\widehat{Y}_{lm}\Big(\frac{\mathbf{x} - \mathcal{O}}{|\mathbf{x} - \mathcal{O}|}\Big) \qquad \text{for} \quad \mathbf{x} \neq \mathcal{O};$$

here, the spherical-surface harmonics $\{\, \widehat{Y}_{lm} \mid m = -l, \ldots, l, \quad l = 0, 1, 2, \ldots \,\}$ with $\widehat{Y}_{lm}$ of order $l$, are chosen to form a complete and orthonormal set in the usual Hilbert space $L_2(\Sigma_1)$ associated with the Lebesgue measure on the unit sphere. For definiteness, we take

$$\widehat{Y}_{lm}(\hat{\mathbf{e}}) := \left\{\frac{2l+1}{2\pi(1+\delta_{0m})}\frac{(l-|m|)!}{(l+|m|)!}\right\}^{\frac{1}{2}} P_l^{|m|}(\cos\vartheta_{\hat{\mathbf{e}}})\begin{cases} \sin m\varphi_{\hat{\mathbf{e}}} & \text{if } m < 0 \\[2mm] \cos m\varphi_{\hat{\mathbf{e}}} & \text{if } m \geq 0 \end{cases} \qquad \text{whenever} \quad \hat{\mathbf{e}} \in \Sigma_1,$$

with $(\vartheta_{\hat{\mathbf{e}}}, \varphi_{\hat{\mathbf{e}}})$ indicating the usual spherical coördinates of $\hat{\mathbf{e}}$ relative to a fixed coördinate system having pole at the origin. $P_l^m$ denotes the associated Legendre function of order $m$ and degree $l$ "on the cut," as it is defined in, *e.g.*, [1]. One then checks easily that each $\widehat{Y}_{lm}$ is of unit norm in $L_2(\Sigma_1)$, owing to the indicated choice of normalizing coefficients.

The second family $\{\, \operatorname{Reg} V_{lm}^{\kappa\mathcal{O}} \mid m = -l, \ldots, l, \quad l = 0, 1, 2, \ldots \,\}$ uses instead the spherical Bessel functions of the first kind $j_l$, and so consists of entire solutions of (0.1); we put

$$\operatorname{Reg} V_{lm}^{\kappa\mathcal{O}}(\mathbf{x}) := \sqrt{2}\, j_l\big(\kappa|\mathbf{x} - \mathcal{O}|\big)\widehat{Y}_{lm}\Big(\frac{\mathbf{x} - \mathcal{O}}{|\mathbf{x} - \mathcal{O}|}\Big) \qquad \text{for} \quad \mathbf{x} \neq \mathcal{O},$$

and set the value of $\operatorname{Reg} V_{lm}^{\kappa\mathcal{O}}(\mathcal{O})$ to ensure continuity at the pole. When we assume that $\kappa$ is real we shall have $\operatorname{Reg} V_{lm}^{\kappa\mathcal{O}} = \operatorname{Re} V_{lm}^{\kappa\mathcal{O}}$, since we are using a family of real spherical harmonics.

We choose once and for all a bijection $n \mapsto \left(l^{\star}(n), m^{\star}(n)\right)$ carrying the positive integers onto the set $\left\{ (l,m) \mid m = -l, \ldots, l, \quad l = 0, 1, 2, \ldots \right\}$. Throughout, $\left\{ V_n^{\kappa \mathcal{O}} \right\}_{n=1}^{\infty}$ then denotes the resultant single-indexing of the family $\left\{ V_{lm}^{\kappa \mathcal{O}} \right\}$, $i.e.$, we set $V_n^{\kappa \mathcal{O}} := V_{l^{\star}(n) m^{\star}(n)}^{\kappa \mathcal{O}}$ for each positive integer $n$. Of course, we use the same single-index notation for the entire solutions.

For the convenience of the reader, we review the definition and most fundamental facts concerning bases in the Hilbert-space setting; we have no need for the more general developments in a Banach space. One can consult, $e.g.$, YOUNG [23] or MARTI [16] for more details and complete proofs of the results cited. Throughout, $\left(H, (\,\cdot\,, \,\cdot\,)_H\right)$ denotes a separable complex Hilbert space that we suppose to be of infinite dimension. Since there is confusion in some of the literature on applications concerning the distinction between the completeness and basis properties, we give the definitions of both terms.

**Definitions.** Let $(f_n)_{n=1}^{\infty}$ be a sequence from $H$. Then $(f_n)_{n=1}^{\infty}$ is *complete in $H$* iff its linear span

$$\mathrm{sp}\left\{f_n\right\}_{n=1}^{\infty} := \left\{ \sum_{j=1}^{N} c_j f_j \,\middle|\, N \text{ a positive integer}, \quad c_j \in \mathbb{C} \quad \text{for} \quad j = 1, \ldots, N \right\}$$

is dense in $H$, $i.e.$, iff for any $h \in H$ there exists a sequence from $\mathrm{sp}\left\{f_n\right\}_{n=1}^{\infty}$ converging to $h$ in the norm of $H$. The sequence $(f_n)_{n=1}^{\infty}$ is a *Schauder basis* for $H$ iff there corresponds to each $h \in H$ a unique sequence of scalars $\left(\Lambda_n(h)\right)_{n=1}^{\infty}$ such that the representation

$$h = \sum_{n=1}^{\infty} \Lambda_n(h) f_n$$

holds with convergence in the norm $\|\cdot\|_H$.

Alternately, one can characterize a complete sequence as one whose orthogonal complement in $H$ is the trivial subspace. Henceforth, the term "basis" shall mean "Schauder basis." Obviously, a basis $(f_n)_{n=1}^{\infty}$ for $H$ forms a linearly independent set and is complete in $H$. However, the basis property is far stronger than the completeness property; in particular, the examples here will show that the completeness property does not imply the basis property.

Now let $(f_n)_{n=1}^{\infty}$ be a basis for $H$. One can show that each member of the associated family $\left(\Lambda_n\right)_{n=1}^{\infty}$ of (linear) *coefficient functionals* is bounded; it follows that there is a unique sequence $(f_n^{\star})_{n=1}^{\infty}$ in $H$ such that $\Lambda_n(h) = (h, f_n^{\star})_H$ for each $n$ and $h \in H$; the representations therefore appear as

$$h = \sum_{n=1}^{\infty} (h, f_n^{\star})_H f_n \qquad \text{for each} \quad h \in H.$$

It is easy to see that $(f_n)_{n=1}^{\infty}$ and $(f_n^{\star})_{n=1}^{\infty}$ form a biorthonormal pair, $i.e.$, $(f_m, f_n^{\star})_H = \delta_{mn}$ for all $m$ and $n$, while it can be shown that $(f_n^{\star})_{n=1}^{\infty}$ also comprises a basis for $H$.

It is also useful to recall the fundamental property of "minimality," which is a type of strengthening of the property of linear independence.

**Definition.** A sequence $(f_n)_{n=1}^{\infty}$ from $H$ is *minimal in $H$* iff, for each $m$, $f_m$ does not belong to the closure $\overline{\mathrm{sp}}\left\{ f_n \mid n \neq m \right\}$ of the linear span of the other elements of the sequence.

It is important to expose the intimate connection between the property of minimality and the existence of a biorthonormal sequence:

**Lemma 1.1.** *Let $(f_n)_{n=1}^\infty$ be a sequence from the separable, infinite-dimensional Hilbert space $H$.*

  *(i.)* *$(f_n)_{n=1}^\infty$ possesses a biorthonormal sequence in $H$ iff it is minimal in $H$.*

  *(ii.)* *Let $(f_n)_{n=1}^\infty$ be minimal in $H$. There is precisely one biorthonormal sequence for $(f_n)_{n=1}^\infty$ in $H$ iff $(f_n)_{n=1}^\infty$ is also complete in $H$.*

*Proof*: (*i*). Let $(f_n)_{n=1}^\infty$ be minimal in $H$. Choose any positive integer $m$: the closed span $\mathcal{M}_m := \overline{\mathrm{sp}}\{\, f_n \mid n \neq m \,\}$ is a proper subspace of $H$, since it does not contain $f_m$. With $Q_m$ denoting the $H$-orthoprojector onto the orthogonal complement $H \ominus \mathcal{M}_m$, let $f_m^\star := \left\| Q_m f_m \right\|_H^{-2} Q_m f_m$. Then $\left(f_n, f_m^\star\right)_H = 1$ for $n = m$, but $= 0$ for $n \neq m$. Clearly, this implies the existence of a biorthonormal sequence for $(f_n)_{n=1}^\infty$ in $H$. Conversely, suppose that $(f_n)_{n=1}^\infty$ has a biorthonormal sequence $(f_n^\star)_{n=1}^\infty$ but is not minimal, so that there is some $m$ and a sequence $\left(F_n^{(m)}\right)_{n=1}^\infty$ from $\mathrm{sp}\{\, f_n \mid n \neq m \,\}$ with $F_n^{(m)} \to f_m$ in the norm of $H$ as $n \to \infty$. Then we should find that $\left(f_m, f_m^\star\right)_H = \lim_{n\to\infty}\left(F_n^{(m)}, f_m^\star\right)_H = 0$, contradicting the equality $\left(f_m, f_m^\star\right)_H = 1$. We conclude that the existence of a biorthonormal sequence implies the property of minimality.

(*ii*). By the first statement, $(f_n)_{n=1}^\infty$ possesses a biorthonormal sequence $(f_n^\star)_{n=1}^\infty$. Suppose that $(f_n)_{n=1}^\infty$ is not complete in $H$. Then there is a nonzero $f_0^\star$ in the orthogonal complement $H \ominus \overline{\mathrm{sp}}\left\{f_n\right\}_{n=1}^\infty$, and the sequence $(f_n^\star + f_0^\star)_{n=1}^\infty$ is clearly biorthonormal to $(f_n)_{n=1}^\infty$ and distinct from $(f_n^\star)_{n=1}^\infty$. Conversely, let $(f_n)_{n=1}^\infty$ be complete in $H$. If $(f_n^{\star\star})_{n=1}^\infty$ is also biorthonormal to $(f_n)_{n=1}^\infty$, then we find for each $m$ that $\left(f_n, f_m^\star - f_m^{\star\star}\right)_H = 0$ for all $n$, implying that $f_m^\star = f_m^{\star\star}$. $\quad\square$

Lemma 1.1 implies, in particular, that a basis possesses the property of minimality; it is well known—and the examples here will show—that minimality is certainly not sufficient to ensure the basis property, however.

It is easy to check that the image-sequence $(Lf_n)_{n=1}^\infty$ of a basis $(f_n)_{n=1}^\infty$ for $H$ under a bijection $L \in \mathcal{B}(H)$ (the collection of all bounded linear operators from $H$ to itself) is also a basis, with biorthonormal sequence $(L^{*-1} f_n^\star)_{n=1}^\infty$ ($L^*$ denoting the Hilbert-space adjoint of $L$). Bases $(f_n)_{n=1}^\infty$ and $(Lf_n)_{n=1}^\infty$ related in this way are said to be *equivalent*. A basis that is equivalent to an orthonormal basis is termed a *Riesz basis*.

Our first principal aim is a complete description of the basis properties of the four sequences $\left(V_{n,\mathbf{n}}^{\kappa\mathcal{O}}\right)_{n=1}^\infty$, $\left(V_n^{\kappa\mathcal{O}}\big|_\Gamma\right)_{n=1}^\infty$, $\left(\mathrm{Reg}\, V_{n,\mathbf{n}}^{\kappa\mathcal{O}}\right)_{n=1}^\infty$, and $\left(\mathrm{Reg}\, V_n^{\kappa\mathcal{O}}\big|_\Gamma\right)_{n=1}^\infty$ in the familiar Hilbert space $H^0(\Gamma) \equiv \left(L_2(\Gamma), (\,\cdot\,,\,\cdot\,)_0\right)$, comprising those (equivalence classes of) complex measurable functions defined $\lambda_\Gamma$-a.e. and having moduli square-integrable with respect to the Lebesgue measure $\lambda_\Gamma$ on $\Gamma$, the inner product being given by

$$(f,g)_0 := \int_\Gamma f\,\overline{g}\,d\lambda_\Gamma;$$

throughout, an overbar indicates complex conjugation.

For the latter sequences, we can only rarely assert the lack of the basis property as a consequence of a lack of the necessary completeness or minimality. To see that this is so, we shall review the (well-known) completeness properties and describe the (perhaps less-familiar) minimality properties of each of the four sequences. In preparation, we recall that a complex number $\mu$ is termed a *Dirichlet* [respectively, *Neumann*] *eigenvalue for* $-\Delta$ *in* $\Omega_-$ iff there exists a nonzero $v \in C^2(\Omega_-) \cap C(\overline{\Omega}_-)$ [respectively, $C^2(\Omega_-) \cap C^1(\overline{\Omega}_-)$] such that $-\Delta v = \mu v$ in $\Omega_-$ and $v|_\Gamma = 0$ [respectively, $v_{,\mathbf{n}} = 0$]. It is well known that the collection of all Dirichlet [respectively, Neumann] eigenvalues for $-\Delta$ in $\Omega_-$ is

a countably infinite and unbounded set of positive [respectively, nonnegative] real numbers without a limit point.

The following statements are implied by the results of Section 3.

**Proposition 1.1.** *Recall the regularity hypotheses imposed on* $\Omega_-$. *Let* $\mathcal{O} \in \Omega_-$.

(*i*.) *Each of the sequences* $\left(V_{n,\mathbf{n}}^{\kappa\mathcal{O}}\right)_{n=1}^{\infty}$ *and* $\left(V_n^{\kappa\mathcal{O}}\big|_{\Gamma}\right)_{n=1}^{\infty}$ *is complete and minimal in* $H^0(\Gamma)$.

(*ii*.) *The sequence* $\left(\operatorname{Reg}V_{n,\mathbf{n}}^{\kappa\mathcal{O}}\right)_{n=1}^{\infty}$, $\left[\text{respectively, } \left(\operatorname{Reg}V_n^{\kappa\mathcal{O}}\big|_{\Gamma}\right)_{n=1}^{\infty}\right]$ *is complete in* $H^0(\Gamma)$ *iff* $\kappa^2$ *is not a Neumann [respectively, Dirichlet] eigenvalue for* $-\Delta$ *in* $\Omega_-$.

(*iii*.) *The sequence* $\left(\operatorname{Reg}V_{n,\mathbf{n}}^{\kappa\mathcal{O}}\right)_{n=1}^{\infty}$, $\left[\text{respectively, } \left(\operatorname{Reg}V_n^{\kappa\mathcal{O}}\big|_{\Gamma}\right)_{n=1}^{\infty}\right]$ *is minimal in* $H^0(\Gamma)$ *iff* $\kappa^2$ *is not a Neumann [respectively, Dirichlet] eigenvalue for* $-\Delta$ *in* $\Omega_-$.

When $\Omega_-$ is a ball $B_R(\mathcal{O})$ of radius $R > 0$ centered at $\mathcal{O}$ each of the four sequences is orthogonal, so the basis property in that case depends upon the completeness property and the nonvanishing of all elements of the sequence. We show that this latter already-familiar case is the only one in which the sequences form bases, by establishing

**Theorem 1.1.** *Recall the regularity hypotheses placed on* $\Omega_-$. *Let* $\mathcal{O} \in \Omega_-$.

(*i*.) *The sequences* $\left(V_{n,\mathbf{n}}^{\kappa\mathcal{O}}\right)_{n=1}^{\infty}$ *and* $\left(V_n^{\kappa\mathcal{O}}\big|_{\Gamma}\right)_{n=1}^{\infty}$ *are bases for* $H^0(\Gamma)$ *iff* $\Omega_-$ *is a ball centered at* $\mathcal{O}$.

(*ii*.) *Suppose that* $\kappa^2$ *is not a Neumann [respectively, Dirichlet] eigenvalue for* $-\Delta$ *in* $\Omega_-$: *the sequence* $\left(\operatorname{Reg}V_{n,\mathbf{n}}^{\kappa\mathcal{O}}\right)_{n=1}^{\infty}$ $\left[\text{respectively, } \left(\operatorname{Reg}V_n^{\kappa\mathcal{O}}\big|_{\Gamma}\right)_{n=1}^{\infty}\right]$ *is a basis for* $H^0(\Gamma)$ *iff* $\Omega_-$ *is a ball centered at* $\mathcal{O}$.

(*iii*.) *Suppose that* $\kappa^2$ *is a Neumann [respectively, Dirichlet] eigenvalue for* $-\Delta$ *in* $\Omega_-$:

(*a*.) *the sequence* $\left(\operatorname{Reg}V_{n,\mathbf{n}}^{\kappa\mathcal{O}}\right)_{n=1}^{\infty}$ $\left[\text{respectively, } \left(\operatorname{Reg}V_n^{\kappa\mathcal{O}}\big|_{\Gamma}\right)_{n=1}^{\infty}\right]$ *is not a basis for its closed span* $\mathcal{R}\left(I - \overline{D_\kappa^*}\right) \subsetneqq H^0(\Gamma)$ $\left[\text{respectively, } \mathcal{R}\left(I + D_\kappa\right) \subsetneqq H^0(\Gamma)\right]$;

(*b*.) *if* $\Omega_-$ *is a ball centered at* $\mathcal{O}$, *then precisely one element of the sequence is zero; if this element is deleted, the remaining sequence is a basis for its closed span.*

The proof of Theorem 1.1, which is surprisingly easy, is given in Section 4, following the review of some necessary preliminaries in Section 2 and the discussion of the completeness and minimality properties in Section 3. In Section 5, we remark on the immediate implications of the general lack of the basis property for the applicability of the results developed in [13] to the question of convergence of the first scheme of WATERMAN [21] for approximate solution of problems of the scattering of time-harmonic acoustic waves by an obstacle. In contrast to the situation in $H^0(\Gamma)$, we show in Section 6 that the sequences of traces and normal derivatives of the outgoing spherical-separable solutions do possess the basis property in certain larger Hilbert spaces that are intimately connected with the far-field patterns of the outgoing solutions of the Helmholtz equation in $\Omega_+$. Finally, in Section 7 we point out a connection between the present observations and the second scheme of WATERMAN [21], by verifying in the case of an ellipsoidal boundary $\Gamma$ the equality $\mathcal{QT} = -\operatorname{Re}\mathcal{Q}$, which evidently has been frequently claimed as satisfied in general by the transition matrix $\mathcal{T}$.

## 2. Background information on solutions of the Helmholtz equation.

Since we rely on them continually in the subsequent developments, we quickly review here without proof the elementary facts about the structure of solutions of (0.1) in $\Omega_-$ and in $\Omega_+$, their integral representations, and the associated integral operators in spaces of functions on $\Gamma$. We maintain the notation of Section 1 and the regularity conditions imposed there on $\Omega_-$, although for most of the present section one need not require that $\Omega_-$ be connected.

**Traces and normal derivatives on $\Gamma$.** For an appropriate complex function $u$ defined in a one-sided neighborhood of $\Gamma$, the trace $u|_\Gamma$ and normal derivative $u_{,\mathbf{n}}$ on $\Gamma$ are taken in the *normal-$L_2$ sense* (our terminology); *cf.*, *e.g.*, [11], [4]. To recall the definitions, for any $s > 0$ let $\mathbf{N}_s : \Gamma \to \mathbb{R}^3$ be defined by $\mathbf{N}_s(\mathbf{x}) := \mathbf{x} + s\mathbf{n}(\mathbf{x})$ for each $\mathbf{x} \in \Gamma$. Then the regularity of $\Omega_-$ implies that there is some $s_o > 0$ such that $\mathbf{N}_s(\Gamma) \subset \Omega_+$ whenever $0 < s < s_o$. Thus, if $u$ is any function defined in $\Omega_+$ the composition $u \circ \mathbf{N}_s$ is defined on $\Gamma$ for all sufficiently small positive $s$. Now suppose that $u \in C(\Omega_+)$: we say that $u$ *has a trace on $\Gamma$ in the normal-$L_2$ sense* iff there exists $u|_\Gamma \in H^0(\Gamma)$ such that $\lim_{s \to 0^+} \left\| u \circ \mathbf{N}_s - u|_\Gamma \right\|_0 = 0$, in which case $u|_\Gamma$ is termed *the normal-$L_2$ trace of $u$ on $\Gamma$*. Similarly, now supposing that $u \in C^1(\Omega_+)$, we say that $u$ *has a normal derivative on $\Gamma$ in the normal-$L_2$ sense* iff there exists $u_{,\mathbf{n}} \in H^0(\Gamma)$ such that $\lim_{s \to 0^+} \left\| \mathbf{n} \cdot (\mathrm{grad}\, u) \circ \mathbf{N}_s - u_{,\mathbf{n}} \right\|_0 = 0$, in which case we call $u_{,\mathbf{n}}$ *the normal-$L_2$ normal derivative of $u$ on $\Gamma$*. When $u_{,\mathbf{n}}$ exists we say that $u$ *is $L_2$-regular at $\Gamma$*; it is not difficult to show that $u|_\Gamma$ also exists when $u_{,\mathbf{n}}$ exists. Obviously, one can make entirely analogous definitions for appropriate functions defined in $\Omega_-$; we suppose that this has been done.

**Spaces of regular solutions.** We introduce the collection $W_+(\Omega_+; \kappa)$ of outgoing solutions of the Helmholtz equation in $\Omega_+$ that are $L_2$-regular at $\Gamma$,

$$W_+(\Omega_+; \kappa) := \left\{ u \in C^2(\Omega_+) \;\middle|\; (0.1) \text{ holds in } \Omega_+, (1.1) \text{ holds, and } u \text{ is } L_2\text{-regular at } \Gamma \right\},$$

and its counterpart $W_-(\Omega_-; \kappa)$ for $\Omega_-$,

$$W_-(\Omega_-; \kappa) := \left\{ u \in C^2(\Omega_-) \;\middle|\; (0.1) \text{ holds in } \Omega_- \text{ and } u \text{ is } L_2\text{-regular at } \Gamma \right\}.$$

For $u$ in either $W_+(\Omega_+; \kappa)$ or $W_-(\Omega_-; \kappa)$ we refer to $u|_\Gamma$ and $u_{,\mathbf{n}}$ as, respectively, *the Dirichlet data* and *the Neumann data of $u$*.

**Single- and double-layer potentials.** We use the fundamental solution for the Helmholtz operator $\Delta + \kappa^2$ that is given by, for each $\mathbf{x} \in \mathbb{R}^3$,

$$E_{\mathbf{x}}^\kappa(\mathbf{y}) := -\frac{e^{\mathrm{i}\kappa|\mathbf{y}-\mathbf{x}|}}{2\pi|\mathbf{y}-\mathbf{x}|} \qquad \text{for} \quad \mathbf{y} \in \mathbb{R}^3 \setminus \{\mathbf{x}\}.$$

By $E_{\mathbf{x},\mathbf{n}}^\kappa(\mathbf{y})$ we mean the normal derivative $\mathbf{n}(\mathbf{y}) \cdot \mathrm{grad}\, E_{\mathbf{x}}^\kappa(\mathbf{y})$ at the point $\mathbf{y} \in \Gamma$, $\mathbf{y} \neq \mathbf{x}$. With $E^\kappa$, we define *the single-layer potential $V_\kappa\{h\}$* and *the double-layer potential $W_\kappa\{h\}$ with density $h \in H^0(\Gamma)$* to be the complex functions in $\Omega_- \cup \Omega_+$ given by

$$V_\kappa\{h\}(\mathbf{x}) := \int_\Gamma E_{\mathbf{x}}^\kappa h \, d\lambda_\Gamma \qquad \text{and} \qquad W_\kappa\{h\}(\mathbf{x}) := \int_\Gamma E_{\mathbf{x},\mathbf{n}}^\kappa h \, d\lambda_\Gamma \qquad \text{for} \quad \mathbf{x} \in \Omega_- \cup \Omega_+.$$

It is almost always more convenient to work with the restrictions of the potentials to the interior and exterior domains $\Omega_-$ and $\Omega_+$, *i.e.*, with $V_\kappa^\pm\{h\} := V_\kappa\{h\}|\Omega_\pm$ and $W_\kappa^\pm\{h\} := W_\kappa\{h\}|\Omega_\pm$. Then

one finds that $V_\kappa^\pm\{h\} \in W_\pm(\Omega_\pm; \kappa)$ for each $h \in H^0(\Gamma)$. In fact, for any such $h$ it is well known that we can set

$$S_\kappa h(\mathbf{x}) := \int_\Gamma E_\mathbf{x}^\kappa h\, d\lambda_\Gamma \qquad \text{and} \qquad D_\kappa h(\mathbf{x}) := \int_\Gamma E_{\mathbf{x},\mathbf{n}}^\kappa h\, d\lambda_\Gamma \qquad \text{for} \quad \lambda_\Gamma\text{-a.a. } \mathbf{x} \in \Gamma$$

to get elements $S_\kappa h$ and $D_\kappa h$ of $H^0(\Gamma)$, and that the linear operators $S_\kappa$, $D_\kappa : H^0(\Gamma) \to H^0(\Gamma)$ thereby defined are compact. In this notation, KERSTEN [11] shows that the traces and normal derivatives of the exterior and interior single-layer potentials are given by

$$V_\kappa^\pm\{h\}\big|_\Gamma = S_\kappa h \qquad \text{and} \qquad V_\kappa^\pm\{h\}_{,\mathbf{n}} = (\pm I + \overline{D_\kappa^*})h \quad \text{for each} \quad h \in H^0(\Gamma); \tag{2.1}$$

here, $D_\kappa^*$ denotes the adjoint of $D_\kappa$ in $H^0(\Gamma)$ while the *conjugate* $\overline{L}$ of a linear operator $L$ from one space of complex functions to another is defined by setting $\overline{L}g := \overline{L\overline{g}}$ for $g \in \mathcal{D}(\overline{L}) := \big\{ g \mid \overline{g} \in \mathcal{D}(L) \big\}$ (so, for example, $S_\kappa^* = \overline{S}_\kappa$). On the other hand, the restrictions $W_\kappa^\pm\{h\}$ do not possess normal derivatives at $\Gamma$ for every $h \in H^0(\Gamma)$, although they do have traces, which are given by (again, relying on [11])

$$W_\kappa^\pm\{h\}\big|_\Gamma = (\mp I + D_\kappa)h \quad \text{for} \quad h \in H^0(\Gamma). \tag{2.2}$$

For a given $h \in H^0(\Gamma)$ it is shown further in [11] that $W_\kappa^+\{h\}$ and $W_\kappa^-\{h\}$ either both possess or both fail to possess a normal-$L_2$ normal derivative at $\Gamma$; by defining $\mathcal{D}(T_\kappa)$ to be the linear manifold of elements of $H^0(\Gamma)$ for which these restrictions do have such normal derivatives, we simply set

$$T_\kappa h := W_\kappa^+\{h\}_{,\mathbf{n}} \quad \big(= W_\kappa^-\{h\}_{,\mathbf{n}}\big) \qquad \text{for every} \quad h \in \mathcal{D}(T_\kappa),$$

to get an operator $T_\kappa : \big\{\mathcal{D}(T_\kappa) \subset H^0(\Gamma)\big\} \to H^0(\Gamma)$ that is densely defined (since $\mathcal{D}(T_\kappa)$ turns out to contain $C^{1,\alpha}(\Gamma)$, $0 < \alpha \leq 1$) and unbounded, but closed. In fact, one can show that the domain $\mathcal{D}(T_\kappa^*)$ of the adjoint $T_\kappa^*$ coincides with $\mathcal{D}(T_\kappa)$ and that $T_\kappa^* = \overline{T}_\kappa$. At any rate, the inclusions $W_\kappa^\pm\{h\} \in W_\pm(\Omega_\pm; \kappa)$ do hold whenever the density $h$ is chosen from $\mathcal{D}(T_\kappa)$.

**The Neumann-to-Dirichlet-data operator for $\Omega_+$ and $\kappa$.** With the conditions imposed on $\Omega_-$ and $\kappa$, we know that the linear operation $u \mapsto u_{,\mathbf{n}}$ is bijective from $W_+(\Omega_+; \kappa)$ onto $H^0(\Gamma)$; *cf.* [4], where the result is established by building on the classical developments in [2]. That is, corresponding to each $g \in H^0(\Gamma)$ there is precisely one $\mathfrak{u}_g \in W_+(\Omega_+; \kappa)$ satisfying (0.1), (1.1), and the boundary condition $\mathfrak{u}_{g,\mathbf{n}} = g$; this just expresses the existence and uniqueness result for the Neumann/radiation problem with $H^0(\Gamma)$-data for the Helmholtz operator in $\Omega_+$. It follows that an operator $A_\kappa : H^0(\Gamma) \to H^0(\Gamma)$ is defined by setting

$$A_\kappa g := \mathfrak{u}_g\big|_\Gamma \qquad \text{for each} \quad g \in H^0(\Gamma).$$

Thus, $A_\kappa$ maps the Neumann data $u_{,\mathbf{n}}$ to the corresponding Dirichlet data $u\big|_\Gamma$ for each $u \in W_+(\Omega_+; \kappa)$. The operator $A_\kappa$ turns out to be compact in $H^0(\Gamma)$, while its adjoint is given by $A_\kappa^* = \overline{A}_\kappa$. Further, $-i\kappa A_\kappa$ is "strictly dissipative," *i.e.*,

$$\mathrm{Im}\left(\kappa A_\kappa g, g\right)_0 < 0 \qquad \text{whenever} \quad g \in H^0(\Gamma), \quad g \neq 0. \tag{2.3}$$

Then each of $A_\kappa$ and $A_\kappa^*$ is injective and has dense range in $H^0(\Gamma)$. We find further that $\mathcal{D}(T_\kappa) = \mathcal{R}(A_\kappa)$. Since it is easy to show that $\mathcal{D}(T_\kappa)$ is closed under conjugation by checking that it coincides with $\mathcal{D}(T_0)$, we conclude that the range $\mathcal{R}(A_\kappa)$ is also closed under complex conjugation, while the

relation $A_\kappa^* = \overline{A}_\kappa$ implies then that $\mathcal{R}(A_\kappa) = \mathcal{R}(A_\kappa^*)$. Except for the latter argument, a development of these properties of $A_\kappa$ is given in [4], by following a line of reasoning somewhat different from that traced here.

**Integral representations.** With an extension of the classical Divergence Theorem to the present setting in which traces and normal derivatives are taken in the normal-$L_2$ sense, one obtains integral representations of the familiar forms for elements of both $W_-(\Omega_-; \kappa)$ and $W_+(\Omega_+; \kappa)$, in terms of their Dirichlet and Neumann data on $\Gamma$. Thus, for $u^+ \in W_+(\Omega_+; \kappa)$ we find the pair of equalities

$$\left.\begin{array}{c} u^+(\mathbf{x}) \\ 0 \end{array}\right\} = \frac{1}{2} \int_\Gamma \left\{ E_{\mathbf{x}}^\kappa u_{,\mathbf{n}}^+ - E_{\mathbf{x},\mathbf{n}}^\kappa u^+|_\Gamma \right\} d\lambda_\Gamma = \frac{1}{2}\left\{ V_\kappa^\pm \{u_{,\mathbf{n}}^+\}(\mathbf{x}) - W_\kappa^\pm \{u^+|_\Gamma\}(\mathbf{x}) \right\}, \quad \mathbf{x} \in \Omega_\pm, \quad (2.4)$$

while for $u^- \in W_-(\Omega_-; \kappa)$ we have

$$\left.\begin{array}{c} 0 \\ u^-(\mathbf{x}) \end{array}\right\} = -\frac{1}{2} \int_\Gamma \left\{ E_{\mathbf{x}}^\kappa u_{,\mathbf{n}}^- - E_{\mathbf{x},\mathbf{n}}^\kappa u^-|_\Gamma \right\} d\lambda_\Gamma = -\frac{1}{2}\left\{ V_\kappa^\pm \{u_{,\mathbf{n}}^-\}(\mathbf{x}) - W_\kappa^\pm \{u^-|_\Gamma\}(\mathbf{x}) \right\}, \quad \mathbf{x} \in \Omega_\pm. \quad (2.5)$$

The representations here clearly imply that the traces $u^+|_\Gamma$ and $u^-|_\Gamma$ lie in $\mathcal{D}(T_\kappa)$ whenever $u^+ \in W_+(\Omega_+; \kappa)$ and $u^- \in W_-(\Omega_-; \kappa)$. Thus, for example, we conclude that $E_{\mathbf{y}}^\kappa|_\Gamma \in \mathcal{D}(T_\kappa)$ for each $\mathbf{y} \in \Omega_- \cup \Omega_+$.

By using the operator $A_\kappa$ we can modify these integral representations to forms that are sometimes more useful. First, we observe that the solution $\mathfrak{u}_g \in W_+(\Omega_+; \kappa)$ of the exterior Neumann/radiation problem with data $g \in H^0(\Gamma)$ has the integral representation

$$\mathfrak{u}_g(\mathbf{x}) = \frac{1}{2} \int_\Gamma \left\{ E_{\mathbf{x}}^\kappa g - E_{\mathbf{x},\mathbf{n}}^\kappa A_\kappa g \right\} d\lambda_\Gamma = \frac{1}{2} \int_\Gamma \left\{ E_{\mathbf{x}}^\kappa - A_\kappa E_{\mathbf{x},\mathbf{n}}^\kappa \right\} g \, d\lambda_\Gamma \qquad \text{for each} \quad \mathbf{x} \in \Omega_+, \quad (2.6)$$

the second equality holding in view of the relation $A_\kappa^* = \overline{A}_\kappa$. Meanwhile, the solution $\mathfrak{v}_f \in W_+(\Omega_+; \kappa)$ of the Dirichlet/radiation problem with data-function $f \in \mathcal{R}(A_\kappa)$ is given by

$$\mathfrak{v}_f(\mathbf{x}) = \frac{1}{2} \int_\Gamma \left\{ E_{\mathbf{x}}^\kappa A_\kappa^{-1} f - E_{\mathbf{x},\mathbf{n}}^\kappa f \right\} d\lambda_\Gamma = \frac{1}{2} \int_\Gamma \left\{ A_\kappa^{-1} E_{\mathbf{x}}^\kappa|_\Gamma - E_{\mathbf{x},\mathbf{n}}^\kappa \right\} f \, d\lambda_\Gamma \qquad \text{for each} \quad \mathbf{x} \in \Omega_+, \quad (2.7)$$

since $E_{\mathbf{x}}^\kappa|_\Gamma \in \mathcal{D}(T_\kappa) = \mathcal{R}(A_\kappa)$ for $\mathbf{x} \in \Omega_+$. Note that the first integral in (2.7) is not defined if $f$ is not in $\mathcal{R}(A_\kappa)$; indeed, the Dirichlet/radiation problem in $\Omega_+$ cannot have a solution in $W_+(\Omega_+; \kappa)$ unless the data $f$ belong to $\mathcal{R}(A_\kappa)$. However, the second integral in (2.7) makes sense for any $f \in H^0(\Gamma)$ and we show at the end of this section that it provides a solution of the Dirichlet problem for such $f$.

Further, we find, in contrast to the situation for $W_+(\Omega_+; \kappa)$, that each element of $W_-(\Omega_-; \kappa)$ can always be represented by either an interior single-layer or an interior double-layer potential. In fact, since $A_\kappa E_{\mathbf{x},\mathbf{n}}^\kappa = E_{\mathbf{x}}^\kappa|_\Gamma$ when $\mathbf{x} \in \Omega_-$, from the integral representation of (2.5) for $u^- \in W_-(\Omega_-; \kappa)$ we obtain both

$$u^-(\mathbf{x}) = -\frac{1}{2} \int_\Gamma \left\{ E_{\mathbf{x}}^\kappa u_{,\mathbf{n}}^- - E_{\mathbf{x},\mathbf{n}}^\kappa u^- \right\} d\lambda_\Gamma = -\frac{1}{2} \int_\Gamma E_{\mathbf{x},\mathbf{n}}^\kappa \left\{ A_\kappa u_{,\mathbf{n}}^- - u^- \right\} d\lambda_\Gamma, \qquad (2.8)$$

and

$$u^-(\mathbf{x}) = -\frac{1}{2} \int_\Gamma E_{\mathbf{x}}^\kappa|_\Gamma \left\{ u_{,\mathbf{n}}^- - A_\kappa^{-1} u^-|_\Gamma \right\} d\lambda_\Gamma, \qquad \text{for each} \quad \mathbf{x} \in \Omega_-. \qquad (2.9)$$

10

Moreover, one can show that, if $h \in H^0(\Gamma)$ [respectively, $h \in \mathcal{R}(A_\kappa)$] with $V_\kappa^-\{h\} = 0$ [respectively, $W_\kappa^-\{h\} = 0$], then $h = 0$. That is, the densities figuring in the representations by interior single- and double-layer potentials are unique (which need not be the case for representations by exterior potentials, depending upon whether $\kappa^2$ is an interior Dirichlet [respectively, Neumann] eigenvalue). For example, suppose that $h \in \mathcal{R}(A_\kappa)$ and $W_\kappa^-\{h\} = 0$. Then $(I + D_\kappa)h = 0$ and $W_\kappa^+\{h\}_{,\mathbf{n}} = W_\kappa^-\{h\}_{,\mathbf{n}} = 0$. The latter equality implies that $W_\kappa^+\{h\} = 0$, by the uniqueness theorem for the exterior Neumann/radiation problem (with the normal derivatives taken in the normal $L_2$-sense). Therefore, also $(I - D_\kappa)h = 0$, whence $h = 0$. To show that $h$ must vanish if $h \in H^0(\Gamma)$ and $V_\kappa^-\{h\} = 0$, one relies on the uniqueness theorem for the exterior Dirichlet/radiation problem with traces taken in the normal $L_2$-sense; *cf*. [9].

**Operator relations.** There are important commutation-type relations connecting the operators $S_\kappa$, $D_\kappa$, and $T_\kappa$; *cf*., *e.g.*, [4]. Here, we need to cite several relations of the same sort between these three operators and $A_\kappa$. Directly from the defining property of the operator $A_\kappa$ and the trace and normal-derivative expressions for the exterior layer potentials, we find $A_\kappa\big(I + \overline{D_\kappa^*}\big) = S_\kappa$ and $A_\kappa T_\kappa = -(I - D_\kappa)\big|\mathcal{R}(A_\kappa)$. One can also show that $D_\kappa A_\kappa = A_\kappa \overline{D_\kappa^*}$ and $T_\kappa A_\kappa = -I + \overline{D_\kappa^*}$. Therefore, we have $A_\kappa\big(I + \overline{D_\kappa^*} + \zeta T_\kappa\big) = \big(S_\kappa - \zeta(I - D_\kappa)\big)\big|\mathcal{R}(A_\kappa)$ and $\big(I + D_\kappa + \zeta T_\kappa\big)A_\kappa = S_\kappa - \zeta\big(I - \overline{D_\kappa^*}\big)$ for any $\zeta \in \mathbb{C}$. Now, if $\operatorname{Im}\zeta \neq 0$ and $\operatorname{Im}\zeta \operatorname{Im}\kappa^2 \geq 0$, it is well known that the operators $I + D_\kappa + \zeta T_\kappa$ and $I + \overline{D_\kappa^*} + \zeta T_\kappa$ are bijective from $\mathcal{R}(A_\kappa)$ onto $H^0(\Gamma)$, so we obtain the representations

$$A_\kappa = \big(S_\kappa - \zeta(I - D_\kappa)\big)\big(I + \overline{D_\kappa^*} + \zeta T_\kappa\big)^{-1} = \big(I + D_\kappa + \zeta T_\kappa\big)^{-1}\big(S_\kappa - \zeta(I - \overline{D_\kappa^*})\big). \tag{2.10}$$

For the same $\zeta$, $S_\kappa - \zeta\big(I - \overline{D_\kappa^*}\big)$ and $S_\kappa - \zeta(I - D_\kappa)$ are also known to be bijective on $H^0(\Gamma)$.

From these relations we can establish a result that is of use in several instances:

**Lemma 2.1.** *Let $\zeta$ denote any complex number such that $\operatorname{Im}\zeta \neq 0$ and $\operatorname{Im}\zeta \operatorname{Im}\kappa^2 \geq 0$. If $u^-$ denotes any element of $W_-(\Omega_-; \kappa)$, then*

$$u^-_{,\mathbf{n}} - A_\kappa^{-1}u^-\big|_\Gamma = -2\big(S_\kappa - \zeta(I - \overline{D_\kappa^*})\big)^{-1}\big\{u^-\big|_\Gamma + \zeta u^-_{,\mathbf{n}}\big\}. \tag{2.11}$$

*Proof*: Let $u^- \in W_-(\Omega_-; \kappa)$. From the integral representation in (2.5) for $u^-$ and the trace and normal-derivative expressions for the interior layer-potentials, one can easily verify that $S_\kappa u^-_{,\mathbf{n}} = (-I + D_\kappa)u^-\big|_\Gamma$ and $(I + \overline{D_\kappa^*})u^-_{,\mathbf{n}} = T_\kappa u^-\big|_\Gamma$, so

$$\big(S_\kappa - \zeta(I - \overline{D_\kappa^*})\big)u^-_{,\mathbf{n}} = \big(I + D_\kappa + \zeta T_\kappa\big)u^-\big|_\Gamma - 2\big\{u^-\big|_\Gamma + \zeta u^-_{,\mathbf{n}}\big\},$$

while the second expression for $A_\kappa$ given in (2.10) shows that $A_\kappa^{-1} = \big(S_\kappa - \zeta(I - \overline{D_\kappa^*})\big)^{-1}\big(I + D_\kappa + \zeta T_\kappa\big)$. Upon combining these, we get (2.11). ☐

We end this section by verifying the earlier claim that the solution $\mathfrak{v}_g$ of the exterior Dirichlet/radiation problem with data $g \in H^0(\Gamma)$ is given by the second form appearing in (2.7),

$$\mathfrak{v}_g(\mathbf{x}) = \frac{1}{2}\int_\Gamma \big\{A_\kappa^{-1}E_\mathbf{x}^\kappa\big|_\Gamma - E_{\mathbf{x},\mathbf{n}}^\kappa\big\}g\, d\lambda_\Gamma \qquad \text{for each} \quad \mathbf{x} \in \Omega_+.$$

To see this, first let $\zeta \in \mathbb{C}$ be as in Lemma 2.1. Then the operator $\big(S_\kappa - \zeta(I - \overline{D_\kappa^*})\big)$ is a bijection of $H^0(\Gamma)$ onto itself, and it is easy to check directly that the radiating solution appearing on the right in

$$\mathfrak{v}_g(\mathbf{x}) = \int_\Gamma \big\{E_\mathbf{x}^\kappa + \zeta E_{\mathbf{x},\mathbf{n}}^\kappa\big\}\big(S_\kappa - \zeta(I - D_\kappa)\big)^{-1}g\, d\lambda_\Gamma, \qquad \text{for each} \quad \mathbf{x} \in \Omega_+, \tag{2.12}$$

has trace $\mathfrak{v}_g\big|_\Gamma = g$, so that (2.12) is correct. Consequently, by appealing to (2.11) after noting that $E_\mathbf{x}^\kappa$ gives an element of $W_-(\Omega_-; \kappa)$ when $\mathbf{x} \in \Omega_+$ and that $S_\kappa$ is self-conjugate-adjoint, we find

$$\mathfrak{v}_g(\mathbf{x}) = \int_\Gamma \Big\{\big(S_\kappa - \zeta(I - \overline{D_\kappa^*})\big)^{-1}\big\{E_\mathbf{x}^\kappa\big|_\Gamma + \zeta E_{\mathbf{x},\mathbf{n}}^\kappa\big\}\Big\}g\, d\lambda_\Gamma = \frac{1}{2}\int_\Gamma \big\{A_\kappa^{-1}E_\mathbf{x}^\kappa\big|_\Gamma - E_{\mathbf{x},\mathbf{n}}^\kappa\big\}g\, d\lambda_\Gamma, \ \mathbf{x} \in \Omega_+,$$

as claimed.

## 3. Completeness and minimality properties.

In this section, we recall the completeness properties of the four sequences of traces and normal derivatives of spherical-wave functions and establish their minimality properties, all in $H^0(\Gamma)$. In particular, the results here imply those cited in Proposition 1.1.

Throughout, by $R_{\mathcal{O}}^-$ and $R_{\mathcal{O}}^+$ we denote the radii of, respectively, the inscribed and the circumscribed spheres for $\Omega_-$ that are centered at $\mathcal{O}$, $i.e.$, $R_{\mathcal{O}}^- := \min_{\mathbf{x}\in\Gamma}|\mathbf{x}-\mathcal{O}|$, $R_{\mathcal{O}}^+ := \max_{\mathbf{x}\in\Gamma}|\mathbf{x}-\mathcal{O}|$.

We begin with the outgoing solutions, for which we introduce two sequences of fundamental importance. In Section 2, we pointed out that each element of $W_-(\Omega_-;\kappa)$ has both an interior double-layer and an interior single-layer representation, with the required density in each case being unique. Accordingly, for each positive integer $n$ there are unique elements $U_n^{\kappa\mathcal{O}}$ and $W_n^{\kappa\mathcal{O}}$ in $H^0(\Gamma)$ such that

$$\operatorname{Reg}V_n^{\kappa\mathcal{O}}(\mathbf{x}) = \frac{\mathrm{i}}{\kappa}W_\kappa^-\big\{U_n^{\kappa\mathcal{O}}\big\}(\mathbf{x}) = \frac{\mathrm{i}}{\kappa}V_\kappa^-\big\{W_n^{\kappa\mathcal{O}}\big\}(\mathbf{x}) \qquad \text{for each} \quad \mathbf{x}\in\Omega_-. \tag{3.1}$$

Explicitly, upon referring to (2.8) and (2.9) we find that

$$\left.\begin{aligned} U_n^{\kappa\mathcal{O}} &= -\tfrac{\mathrm{i}\kappa}{2}\big\{\operatorname{Reg}V_n^{\kappa\mathcal{O}}|_\Gamma - A_\kappa\operatorname{Reg}V_{n,\mathbf{n}}^{\kappa\mathcal{O}}\big\} \\ W_n^{\kappa\mathcal{O}} &= -\tfrac{\mathrm{i}\kappa}{2}\big\{A_\kappa^{-1}\operatorname{Reg}V_n^{\kappa\mathcal{O}}|_\Gamma - \operatorname{Reg}V_{n,\mathbf{n}}^{\kappa\mathcal{O}}\big\} \end{aligned}\right\} \quad \text{for} \quad n = 1,\,2,\,3,\,\dots\,.$$

As in the general case, it is clear that $U_n^{\kappa\mathcal{O}} = A_\kappa W_n^{\kappa\mathcal{O}}$; in particular, $U_n^{\kappa\mathcal{O}} \in \mathcal{R}(A_\kappa)$. From the latter representations, we observe that $U_n^{\kappa\mathcal{O}}$ is the trace on $\Gamma$ of the "total field" in the time-harmonic scattering of the incident acoustic wave with complex amplitude $-\frac{\mathrm{i}\kappa}{2}\operatorname{Reg}V_n^{\kappa\mathcal{O}}$ by a hard obstacle occupying $\overline{\Omega}_-$, while $-W_n^{\kappa\mathcal{O}}$ has the corresponding interpretation as the normal derivative of the total field in the scattering of the same wave by a soft obstacle in $\overline{\Omega}_-$. From (3.1), with (2.1) and (2.2) we get

$$\left.\begin{aligned} \tfrac{\mathrm{i}}{\kappa}(I + D_\kappa)U_n^{\kappa\mathcal{O}} &= \tfrac{\mathrm{i}}{\kappa}S_\kappa W_n^{\kappa\mathcal{O}} = \operatorname{Reg}V_n^{\kappa\mathcal{O}}|_\Gamma \\ \tfrac{\mathrm{i}}{\kappa}T_\kappa U_n^{\kappa\mathcal{O}} &= -\tfrac{\mathrm{i}}{\kappa}(I - \overline{D_\kappa^*})W_n^{\kappa\mathcal{O}} = \operatorname{Reg}V_{n,\mathbf{n}}^{\kappa\mathcal{O}} \end{aligned}\right\} \quad \text{for} \quad n = 1,\,2,\,3,\,\dots\,. \tag{3.2}$$

The functions $U_n^{\kappa\mathcal{O}}$ and $W_n^{\kappa\mathcal{O}}$ arise quite naturally in other contexts, as well. For example, they appear in the construction of series expansions of outgoing waves in terms of $\big(V_n^{\kappa\mathcal{O}}\big)_{n=1}^\infty$. To see this, we first recall the well-known expansion of the fundamental solution

$$E_\mathbf{x}^\kappa(\mathbf{y}) = -\mathrm{i}\kappa\sum_{l=0}^\infty\sum_{m=-l}^l V_{lm}^{\kappa\mathcal{O}}(\mathbf{x})\operatorname{Reg}V_{lm}^{\kappa\mathcal{O}}(\mathbf{y}), \qquad \text{for} \quad 0 \le |\mathbf{y}-\mathcal{O}| < |\mathbf{x}-\mathcal{O}|, \tag{3.3.1}$$

which follows from the addition theorems for the spherical Bessel functions and the Legendre polynomials. The convergence properties of the series in (3.3.1) and others appearing in the sequel, which are to be interpreted as limits of the form $\lim_{N\to\infty}\sum_{m=0}^N\sum_{m=-l}^l \alpha_{lm}$, follow from the fundamental and general results on series representations of solutions of the Helmholtz equation in spherical annuli centered at $\mathcal{O} \in \mathbb{R}^3$ (including balls and the complements of the closures of balls) that are developed in the book of VEKUA[20]. Thus, for fixed $\mathbf{x} \ne \mathcal{O}$ the function $\mathbf{y} \mapsto E_\mathbf{x}^\kappa(\mathbf{y})$ is a solution of the Helmholtz equation in the ball $B_{|\mathbf{x}-\mathcal{O}|}(\mathcal{O})$, and the expansion (3.3.1) converges absolutely and uniformly on every closed subset of the ball; partial derivatives of $E_\mathbf{x}^\kappa$ of any order may be computed through term-by-term differentiation, each derived series having the same convergence properties as

the original. For fixed $\mathbf{x}$ the function $\mathbf{y} \mapsto E_{\mathbf{x}}^{\kappa}(\mathbf{y})$ is an outgoing solution of the Helmholtz equation in the complement of the closure of the ball $B_{|\mathbf{x}-\mathcal{O}|}(\mathcal{O})$ if $\mathbf{x} \neq \mathcal{O}$ or everywhere except at $\mathcal{O}$ if $\mathbf{x} = \mathcal{O}$; interchange of $\mathbf{x}$ and $\mathbf{y}$ on the right in (3.3.1) produces a series converging absolutely and uniformly on every closed subset of the exterior of the ball or point, and all partial derivatives may be computed term-by-term in the manner already described for the first case.

Owing to the absolute convergence of the series in (3.3.1), it may be "unbracketed" and rearranged (*cf.*, *e.g.*, KNOPP[12, §3.6, Theorem 2]) to yield

$$E_{\mathbf{x}}^{\kappa}(\mathbf{y}) = -\mathrm{i}\kappa \sum_{n=1}^{\infty} V_n^{\kappa\mathcal{O}}(\mathbf{x}) \operatorname{Reg} V_n^{\kappa\mathcal{O}}(\mathbf{y}), \qquad \text{for} \quad 0 \le |\mathbf{y} - \mathcal{O}| < |\mathbf{x} - \mathcal{O}|; \qquad (3.3.2)$$

recall the single-indexing $n \mapsto \left(l^{\star}(n), m^{\star}(n)\right)$ already fixed. This expansion can be used in (2.6) when $|\mathbf{x} - \mathcal{O}| > R_{\mathcal{O}}^{+}$; the uniform-convergence properties of the series and the boundedness of the operator $A_{\kappa}$ in $H^0(\Gamma)$ permit all the necessary operations to be performed term-by-term, and we get for the solution $\mathfrak{u}_g$ of the exterior Neumann/radiation problem with $H^0(\Gamma)$-data $g$

$$\mathfrak{u}_g(\mathbf{x}) = \sum_{n=1}^{\infty} \left(g, \overline{U_n^{\kappa\mathcal{O}}}\right)_0 V_n^{\kappa\mathcal{O}}(\mathbf{x}) \qquad \text{for} \quad |\mathbf{x} - \mathcal{O}| > R_{\mathcal{O}}^{+}. \qquad (3.4)$$

Thus, knowledge of the sequence $\left(U_n^{\kappa\mathcal{O}}\right)_{n=1}^{\infty}$ permits construction of the coefficients in the series expansion for $\mathfrak{u}_g$ in terms of the outgoing spherical-separable solutions, which converges at least outside the circumscribing sphere centered at the pole $\mathcal{O}$. Similarly, knowledge of $\left(W_n^{\kappa\mathcal{O}}\right)_{n=1}^{\infty}$ allows one to generate the expansion coefficients for the solution of the exterior Dirichlet/radiation problem with data in $\mathcal{R}(A_{\kappa})$.

Further connections between these sequences are established in

**Proposition 3.1.** *Recall that the bounded domain $\Omega_{-} \subset \mathbb{R}^3$ is regularly open and connected, with $C^{2,\alpha}$-regular boundary $\Gamma$, while $\Omega_{+}$ is connected and $\mathcal{O} \in \Omega_{-}$.*

(*i*.) *Each of the sequences $\left(V_{n,\mathbf{n}}^{\kappa\mathcal{O}}\right)_{n=1}^{\infty}$ and $\left(\overline{U_n^{\kappa\mathcal{O}}}\right)_{n=1}^{\infty}$ is complete and minimal in $H^0(\Gamma)$; the pair is biorthonormal in $H^0(\Gamma)$.*

(*ii*.) *Each of the sequences $\left(V_n^{\kappa\mathcal{O}}\big|_{\Gamma}\right)_{n=1}^{\infty}$ and $\left(\overline{W_n^{\kappa\mathcal{O}}}\right)_{n=1}^{\infty}$ is complete and minimal in $H^0(\Gamma)$; the pair is biorthonormal in $H^0(\Gamma)$.*

*Proof*: (*i*). Although the completeness of $\left(V_{n,\mathbf{n}}^{\kappa\mathcal{O}}\right)_{n=1}^{\infty}$ in $H^0(\Gamma)$ is well known (*cf.*, *e.g.*, [17]), we shall re-establish the result here to indicate a modification that is necessary for the argument given in [3]. If $g$ is a *continuous* complex function on $\Gamma$ such that

$$\left(g, V_{n,\mathbf{n}}^{\kappa\mathcal{O}}\right)_0 = 0 \qquad \text{for} \quad n = 1, 2, \ldots , \qquad (3.5)$$

then the reasoning of [3], employing classical results for potentials with continuous densities, shows that $g = 0$. But essentially the same argument serves to yield the completeness of $\left(V_{n,\mathbf{n}}^{\kappa\mathcal{O}}\right)_{n=1}^{\infty}$ in $H^0(\Gamma)$, provided one appeals to properties of the layer potentials with densities in $H^0(\Gamma)$ and the uniqueness theorem for the exterior Neumann/radiation problem with data in $H^0(\Gamma)$. Thus, supposing that $g \in H^0(\Gamma)$ satisfies (3.5), we construct the interior double-layer potential $W_{\kappa}^{-}\{\overline{g}\}$ with density $\overline{g}$. For any $\mathbf{x}$ lying in the inscribed ball $B_{R_{\mathcal{O}}^{-}}(\mathcal{O})$ the expansion of $E_{\mathbf{x}}^{\kappa}$ obtained by switching the arguments $\mathbf{x}$ and $\mathbf{y}$ in (3.3) can be differentiated term-by-term, inserted into the definition of $W_{\kappa}^{-}\{\overline{g}\}(\mathbf{x})$, and integrated term-by-term; with (3.5), we get $W_{\kappa}^{-}\{\overline{g}\}(\mathbf{x}) = 0$. Since $\Omega_{-}$

13

is connected, the real-analyticity of $W_\kappa^-\{\overline{g}\}$ implies that it must therefore vanish in all of $\Omega_-$. We conclude that $\overline{g}$ is in $\mathcal{R}(A_\kappa)$ with $W_\kappa^-\{\overline{g}\}_{,\mathbf{n}} = 0$, and so also $W_\kappa^+\{\overline{g}\}_{,\mathbf{n}} = 0$. The uniqueness theorem for the exterior Neumann/radiation problem with data in $H^0(\Gamma)$ then says that $W_\kappa^+\{\overline{g}\}$ vanishes in all of $\Omega_+$. Therefore, we find that

$$2\overline{g} = (I + D_\kappa)\overline{g} - (-I + D_\kappa)\overline{g} = W_\kappa^-\{\overline{g}\}\big|_\Gamma - W_\kappa^+\{\overline{g}\}\big|_\Gamma = 0,$$

which finishes the proof of the completeness of $\left(V_{n,\mathbf{n}}^{\kappa\mathcal{O}}\right)_{n=1}^\infty$.

To prove the completeness of $\left(\overline{U_n^{\kappa\mathcal{O}}}\right)_{n=1}^\infty$ in $H^0(\Gamma)$, suppose that $g \in H^0(\Gamma)$ with $\left(g, \overline{U_n^{\kappa\mathcal{O}}}\right)_0 = 0$ for every $n$. Then (3.4) says that $\mathfrak{u}_g$ must vanish outside $B_{R_\mathcal{O}^+}(\mathcal{O})$; the real-analyticity of $\mathfrak{u}_g$ then requires that it vanish in all of $\Omega_+$ since the latter is connected, so $g = \mathfrak{u}_{g,\mathbf{n}} = 0$.

The biorthonormality claimed for the pair of sequences is proven by using the cited properties of $A_\kappa$ and Green's Theorem to transform to integration over the surface of a ball $B_R(\mathcal{O})$ and exploit the orthonormality of the spherical harmonics $\{\widehat{Y}_{lm}\}$ in $H^0(\Sigma_1)$:

$$\left(V_{lm,\mathbf{n}}^{\kappa\mathcal{O}}, \overline{U_{kn}^{\kappa\mathcal{O}}}\right)_0 = -\frac{\mathrm{i}\kappa}{2}\int_\Gamma V_{lm,\mathbf{n}}^{\kappa\mathcal{O}}\left\{\operatorname{Reg}V_{kn}^{\kappa\mathcal{O}} - A_\kappa \operatorname{Reg}V_{kn,\mathbf{n}}^{\kappa\mathcal{O}}\right\}d\lambda_\Gamma$$

$$= -\frac{\mathrm{i}\kappa}{2}\int_\Gamma\left\{V_{lm,\mathbf{n}}^{\kappa\mathcal{O}}\operatorname{Reg}V_{kn}^{\kappa\mathcal{O}} - V_{lm}^{\kappa\mathcal{O}}\operatorname{Reg}V_{kn,\mathbf{n}}^{\kappa\mathcal{O}}\right\}d\lambda_\Gamma$$

$$= -(\mathrm{i}\kappa)\kappa R^2\left\{h_l^{(1)\prime}(\kappa R)j_l(\kappa R) - h_l^{(1)}(\kappa R)j_l'(\kappa R)\right\}\delta_{(k,n)}^{(l,m)};$$

the Wronskian appearing within the braces here has the value $\mathrm{i}/(\kappa R)^2$ ([1]), whence the biorthonormality results. According to Lemma 1.1, it now follows also that each of the sequences $\left(V_{n,\mathbf{n}}^{\kappa\mathcal{O}}\right)_{n=1}^\infty$ and $\left(\overline{U_n^{\kappa\mathcal{O}}}\right)_{n=1}^\infty$ is minimal in $H^0(\Gamma)$.

$(ii)$. The completeness of $\left(V_n^{\kappa\mathcal{O}}\big|_\Gamma\right)_{n=1}^\infty$ in $H^0(\Gamma)$ follows directly from that of $\left(V_{n,\mathbf{n}}^{\kappa\mathcal{O}}\right)_{n=1}^\infty$ and properties of $A_\kappa$. In fact, if $g \in H^0(\Gamma)$ and $\left(g, V_n^{\kappa\mathcal{O}}\big|_\Gamma\right)_0 = 0$ for every $n$, then $\left(A_\kappa^* g, V_{n,\mathbf{n}}^{\kappa\mathcal{O}}\right)_0 = 0$ for every $n$, since $V_n^{\kappa\mathcal{O}}\big|_\Gamma = A_\kappa V_{n,\mathbf{n}}^{\kappa\mathcal{O}}$. By statement $(i)$, we must have $A_\kappa^* g = 0$, so $g = 0$, since $A_\kappa^*$ is injective.

Next, we show that $\left(W_n^{\kappa\mathcal{O}}\right)_{n=1}^\infty$ is complete in $H^0(\Gamma)$; the argument here requires a somewhat deeper result. In Section 2, we noted that the operator $\left(S_\kappa - \zeta(I - \overline{D_\kappa^*})\right)$ is a bijection of $H^0(\Gamma)$ onto itself whenever $\zeta \in \mathbb{C}$ with $\operatorname{Im}\zeta \neq 0$ and $\operatorname{Im}\zeta\operatorname{Im}\kappa^2 \geq 0$. With $\zeta$ denoting any such complex number, let us take $v = \operatorname{Reg}V_n^{\kappa\mathcal{O}}$ in the statement of Lemma 2.1 to produce

$$W_n^{\kappa\mathcal{O}} = \frac{\mathrm{i}\kappa}{2}\left\{\operatorname{Reg}V_{n,\mathbf{n}}^{\kappa\mathcal{O}} - A_\kappa^{-1}\operatorname{Reg}V_n^{\kappa\mathcal{O}}\big|_\Gamma\right\}$$

$$= -\mathrm{i}\kappa\left(S_\kappa - \zeta(I - \overline{D_\kappa^*})\right)^{-1}\left\{\operatorname{Reg}V_n^{\kappa\mathcal{O}}\big|_\Gamma + \zeta\operatorname{Reg}V_{n,\mathbf{n}}^{\kappa\mathcal{O}}\right\} \qquad \text{for} \quad n = 1, 2, \dots. \tag{3.6}$$

The claimed completeness of $\left(W_n^{\kappa\mathcal{O}}\right)_{n=1}^\infty$ will clearly be implied by (3.6) and the following statement:

**Lemma 3.1.** *If $\zeta \in \mathbb{C}$ with $\operatorname{Im}\zeta \neq 0$ and $\operatorname{Im}\zeta\operatorname{Im}\kappa^2 \geq 0$, then $\left(\operatorname{Reg}V_n^{\kappa\mathcal{O}}\big|_\Gamma + \zeta\operatorname{Reg}V_{n,\mathbf{n}}^{\kappa\mathcal{O}}\right)_{n=1}^\infty$ is a complete sequence in $H^0(\Gamma)$.*

The proof of Lemma 3.1 is given following the completion of the proof of Proposition 3.1.

Finally, the biorthonormality of the pair $\left(V_n^{\kappa\mathcal{O}}\big|_\Gamma\right)_{n=1}^\infty$ and $\left(\overline{W_n^{\kappa\mathcal{O}}}\right)_{n=1}^\infty$ follows from that of the pair $\left(V_{n,\mathbf{n}}^{\kappa\mathcal{O}}\right)_{n=1}^\infty$ and $\left(\overline{U_n^{\kappa\mathcal{O}}}\right)_{n=1}^\infty$, since $V_n^{\kappa\mathcal{O}}\big|_\Gamma = A_\kappa V_{n,\mathbf{n}}^{\kappa\mathcal{O}}$ and $\overline{W_n^{\kappa\mathcal{O}}} = \overline{A_\kappa^{-1}U_n^{\kappa\mathcal{O}}} = A_\kappa^{*-1}\overline{U_n^{\kappa\mathcal{O}}}$. $\quad\Box$

*Proof of Lemma* 3.1: Fix $\zeta \in \mathbb{C}$ with $\mathrm{Im}\,\zeta \neq 0$ and $\mathrm{Im}\,\zeta\,\mathrm{Im}\,\kappa^2 \geq 0$. Suppose that $g \in H^0(\Gamma)$ satisfies $\left(g, \mathrm{Reg}\,V_n^{\kappa\mathcal{O}}\big|_\Gamma + \zeta\,\mathrm{Reg}\,V_{n,\mathbf{n}}^{\kappa\mathcal{O}}\right)_0 = 0$ for each positive integer $n$. Construct the corresponding element $v_g^+ \in W_+(\Omega_+; \kappa)$ by setting $v_g^+(\mathbf{x}) := \int_\Gamma \left\{E_{\mathbf{x}}^\kappa + \zeta E_{\mathbf{x},\mathbf{n}}^\kappa\right\} \overline{g}\,d\lambda_\Gamma$ for $\mathbf{x} \in \Omega_+$. Choosing any $\mathbf{x}$ with $\left|\mathbf{x} - \mathcal{O}\right| > R_\mathcal{O}^+$, *i.e.*, lying outside the ball circumscribing $\Omega_-$ and centered at $\mathcal{O}$, we can insert the expansion of (3.3) for $E_{\mathbf{x}}^\kappa(\mathbf{y})$, $\mathbf{y} \in \Gamma$, into the definition of $v_g^+$ and differentiate and integrate term-by-term; with the orthogonality assumption, we find that $v_g^+(\mathbf{x}) = 0$. The real-analyticity of $v_g^+$ then implies that $v_g^+$ vanishes in all of the connected exterior domain $\Omega_+$. Therefore, $\left(S_\kappa + \zeta(-I + D_\kappa)\right)\overline{g} = v_g^+\big|_\Gamma = 0$. Since $S_\kappa - \zeta(I - D_\kappa)$ is an isomorphism of $H^0(\Gamma)$ when $\zeta$ has the indicated properties, we conclude that $\overline{g} = 0$. $\quad\square$

Now we turn to the entire solutions, reviewing first their completeness properties. In contrast to the situation for the outgoing solutions, the value of $\kappa$ is here decisive.

It is convenient to introduce some additional notation. If $\kappa^2$ is a Dirichlet [respectively, Neumann] eigenvalue for $-\Delta$ in $\Omega_-$, by $\mathsf{D}_\kappa(\Omega_-)$ [respectively, $\mathsf{N}_\kappa(\Omega_-)$] we indicate the (finite-dimensional) complex-linear space comprising all of the corresponding eigenfunctions in $\Omega_-$ with the zero-function adjoined; if $\kappa^2$ is not such an eigenvalue, then $\mathsf{D}_\kappa(\Omega_-)$ [respectively, $\mathsf{N}_\kappa(\Omega_-)$] shall denote simply the trivial subspace. Since we are supposing here that the boundary $\Gamma$ is of class $C^{2,\alpha}$, we can appeal to the results on the Dirichlet and oblique-derivative problems in GILBARG AND TRUDINGER [8] to conclude that the elements of $\mathsf{D}_\kappa(\Omega_-)$ and $\mathsf{N}_\kappa(\Omega_-)$ are in $C^{2,\alpha}(\overline{\Omega}_-)$; clearly, they are then $L_2$-regular at $\Gamma$, and so belong to $W_-(\Omega_-; \kappa)$.

**Proposition 3.2.** *Recall that the bounded domain $\Omega_- \subset \mathbb{R}^3$ is regularly open and connected, with $C^{2,\alpha}$-regular boundary $\Gamma$, while $\Omega_+$ is connected and $\mathcal{O} \in \Omega_-$.*

> (*i.*) *The closed spans in $H^0(\Gamma)$ of the traces and normal derivatives on $\Gamma$ of the elements of the sequence $\left(\mathrm{Reg}\,V_n^{\kappa\mathcal{O}}\right)_{n=1}^\infty$ are given by*

$$\overline{\mathrm{sp}}\left\{\mathrm{Reg}\,V_n^{\kappa\mathcal{O}}\big|_\Gamma\right\}_{n=1}^\infty = \mathcal{R}\left(I + D_\kappa\right) = H^0(\Gamma) \ominus \mathcal{N}\left(I + \overline{D_\kappa^*}\right) \tag{3.7}$$

> *and*

$$\overline{\mathrm{sp}}\left\{\mathrm{Reg}\,V_{n,\mathbf{n}}^{\kappa\mathcal{O}}\right\}_{n=1}^\infty = \mathcal{R}\left(I - \overline{D_\kappa^*}\right) = H^0(\Gamma) \ominus \mathcal{N}\left(I - D_\kappa\right). \tag{3.8}$$

> (*ii.*) *The null spaces appearing in (i) are also expressed as*

$$\mathcal{N}\left(I + \overline{D_\kappa^*}\right) = \mathcal{N}\left(S_\kappa\right) = \left\{\,v_{,\mathbf{n}} \mid v \in \mathsf{D}_\kappa(\Omega_-)\,\right\} \tag{3.9}$$

> *and*

$$\mathcal{N}\left(I - D_\kappa\right) = \mathcal{N}\left(T_\kappa\right) = \left\{\,v\big|_\Gamma \mid v \in \mathsf{N}_\kappa(\Omega_-)\,\right\}. \tag{3.10}$$

> (*iii.*) *The sequence $\left(\mathrm{Reg}\,V_n^{\kappa\mathcal{O}}\big|_\Gamma\right)_{n=1}^\infty$ [respectively, $\left(\mathrm{Reg}\,V_{n,\mathbf{n}}^{\kappa\mathcal{O}}\right)_{n=1}^\infty$] is complete in $H^0(\Gamma)$ iff $\kappa^2$ is not a Dirichlet [respectively, Neumann] eigenvalue for $-\Delta$ in $\Omega_-$.*

*Proof*: (*i*). To establish the first equalities in (3.7) and (3.8) we use the following trivial observation (a proof of which we may omit):

**Lemma 3.2.** *Let $L : B_1 \to B_2$ be a bounded linear operator from the Banach space $B_1$ into the Banach space $B_2$. If $\left(f_n\right)_{n=1}^\infty$ is a complete sequence in $B_1$, then $\overline{\mathcal{R}(L)} = \overline{\mathrm{sp}}\left\{Lf_n\right\}_{n=1}^\infty$.*

Now, the ranges $\mathcal{R}\left(I + D_\kappa\right)$ and $\mathcal{R}\left(I - \overline{D_\kappa^*}\right)$ are always closed in $H^0(\Gamma)$ (since $D_\kappa$ and $\overline{D_\kappa^*}$ are compact in that space). Therefore, with Lemma 3.2, the first equality in (3.7) follows from the completeness of $\left(U_n^{\kappa\mathcal{O}}\right)_{n=1}^\infty$ in $H^0(\Gamma)$ and the equalities $(I + D_\kappa)U_n^{\kappa\mathcal{O}} = -\mathrm{i}\kappa\mathrm{Reg}\,V_n^{\kappa\mathcal{O}}\big|_\Gamma$, $n = 1, 2,$

..., from $(3.2)_1$; the first equality in (3.8) follows from the completeness of $\left(W_n^{\kappa\mathcal{O}}\right)_{n=1}^{\infty}$ in $H^0(\Gamma)$ and the equalities $(I - \overline{D_\kappa^*})W_n^{\kappa\mathcal{O}} = i\kappa\mathrm{Reg}\,V_{n,\mathbf{n}}^{\kappa\mathcal{O}}$ of $(3.2)_2$. The second equality in (3.7) will follow if we show that $\mathcal{N}\left(I + \overline{D_\kappa^*}\right) = \mathcal{N}\left(I + D_\kappa^*\right)$, since the orthogonal complement of the null space of a bounded operator on a Hilbert space is the closure of the range of its adjoint. Similarly, the second equality in (3.8) will follow by showing that $\mathcal{N}\left(I - D_\kappa\right) = \mathcal{N}\left(I - \overline{D}_\kappa\right)$. But the equalities $\mathcal{N}\left(I + \overline{D_\kappa^*}\right) = \mathcal{N}\left(I + D_\kappa^*\right)$ and $\mathcal{N}\left(I - D_\kappa\right) = \mathcal{N}\left(I - \overline{D}_\kappa\right)$ will follow from (3.9) and (3.10), respectively. For example, since $\mathcal{N}\left(I + D_\kappa^*\right)$ is precisely the set of conjugates of the elements of $\mathcal{N}\left(I + \overline{D_\kappa^*}\right)$, the equality $\mathcal{N}\left(I + \overline{D_\kappa^*}\right) = \mathcal{N}\left(I + D_\kappa^*\right)$ will follow by showing that $\mathcal{N}\left(I + \overline{D_\kappa^*}\right)$ is closed under conjugation. But it is easy to see that $\left\{\, v_{,\mathbf{n}} \mid v \in \mathrm{D}_\kappa(\Omega_-) \,\right\}$ is closed under conjugation for every (pertinent) value of $\kappa$, for, if $\kappa^2$ is not a Dirichlet eigenvalue for $-\Delta$ in $\Omega_-$, then the set is just the trivial subspace, while if $\kappa^2$ is such an eigenvalue, then it is real, whence the claim is true by the definition of $\mathrm{D}_\kappa(\Omega_-)$. Therefore, (3.9) will indeed imply that $\mathcal{N}\left(I + \overline{D_\kappa^*}\right) = \mathcal{N}\left(I + D_\kappa^*\right)$. Similarly, the equality $\mathcal{N}\left(I - D_\kappa\right) = \mathcal{N}\left(I - \overline{D}_\kappa\right)$ will follow from (3.10). Thus, the proof of $(i)$ will be complete once $(ii)$ has been established (without use of $(i)$!).

$(ii)$. The first equality in (3.9) is clearly implied by the operator relation $A_\kappa(I + \overline{D_\kappa^*}) = S_\kappa$ and the fact that $A_\kappa$ is injective. Further, the first equality in (3.10) will result from the relation $(I - D_\kappa)\big|\mathcal{R}(A_\kappa) = -A_\kappa T_\kappa$, provided we have shown that $\mathcal{N}\left(I - D_\kappa\right) \subset \mathcal{R}\left(A_\kappa\right)$. To verify the latter inclusion, we observe first from the relation $D_\kappa A_\kappa = A_\kappa \overline{D_\kappa^*}$ that $\mathcal{R}\left(A_\kappa\right)$ is invariant under $D_\kappa$ and, moreover, that the resultant operator $D_\kappa\big|\mathcal{R}(A_\kappa) : \mathcal{R}\left(A_\kappa\right) \to \mathcal{R}\left(A_\kappa\right)$ is compact when the range $\mathcal{R}\left(A_\kappa\right)$ is equipped with the inner product $(f,h) \mapsto \left(A_\kappa^{-1}f, A_\kappa^{-1}h\right)_0$. Since $\mathcal{R}\left(A_\kappa\right)$ is then densely (and compactly) imbedded in $H^0(\Gamma)$, we infer from Theorem IV of LAX [14] that $\mathcal{N}\left(I - D_\kappa\right) \subset \mathcal{R}\left(A_\kappa\right)$.

The equality of the first and third members of (3.9) and of (3.10) is established in [2] for the classical case, set in $C(\Gamma)$. Since $D_\kappa$ and $\overline{D_\kappa^*}$ are compact in $C(\Gamma)$ as well as in $H^0(\Gamma)$, we can again appeal to the results of LAX [14] to conclude that the null spaces $\mathcal{N}\left(I + \overline{D_\kappa^*}\right)$ and $\mathcal{N}\left(I - D_\kappa\right)$ on $H^0(\Gamma)$ are actually contained in $C(\Gamma)$, and so rely on [2] to complete the proof of $(ii)$. However, we shall supply a direct argument here.

Accordingly, to establish the second equality in (3.9), suppose first that $v$ is an interior Dirichlet eigenfunction for $-\Delta$. Then the integral representation of $v$ reduces to $v = -\frac{1}{2}V_\kappa^-\{v_{,\mathbf{n}}\}$, whence $S_\kappa v_{,\mathbf{n}} = -2v\big|_\Gamma = 0$. On the other hand, for $f \in \mathcal{N}\left(S_\kappa\right)$ let us construct the interior single-layer $w := -\frac{1}{2}V_\kappa^-\{f\}$. Then $w$ is in $\mathrm{D}_\kappa(\Omega_-)$ and we get $w_{,\mathbf{n}} = -\frac{1}{2}(-I + \overline{D_\kappa^*})f = \frac{1}{2}(I - \overline{D_\kappa^*})f + \frac{1}{2}(I + \overline{D_\kappa^*})f = f$, since also $f \in \mathcal{N}\left(I + \overline{D_\kappa^*}\right)$.

The proof of the second equality in (3.10) is analogous. If $v$ is a Neumann eigenfunction for $-\Delta$ in $\Omega_-$, then its integral representation is just $v = \frac{1}{2}W_\kappa^-\{v\big|_\Gamma\}$, whence $T_\kappa v\big|_\Gamma = 2v_{,\mathbf{n}} = 0$. To prove the reversed inclusion, let $f \in \mathcal{N}\left(T_\kappa\right)$ and form the interior double-layer potential $w := \frac{1}{2}W_\kappa^-\{f\}$. Then $w \in \mathrm{N}_\kappa(\Omega_-)$ and we compute $w\big|_\Gamma = \frac{1}{2}(I + D_\kappa)f = \frac{1}{2}(I + D_\kappa)f + \frac{1}{2}(I - D_\kappa)f = f$, since $f \in \mathcal{N}\left(I - D_\kappa\right)$, as well.

$(iii)$. This conclusion is an immediate consequence of $(i)$ and $(ii)$.  $\square$

The minimality properties of the traces and normal derivatives of the sequence of entire solutions also depend upon the value of $\kappa$, as the next result shows.

**Proposition 3.3.** *The sequence $\left(\mathrm{Reg}\,V_n^{\kappa\mathcal{O}}\big|_\Gamma\right)_{n=1}^{\infty}$ $\left[\text{respectively, } \left(\mathrm{Reg}\,V_{n,\mathbf{n}}^{\kappa\mathcal{O}}\right)_{n=1}^{\infty}\right]$ is minimal in $H^0(\Gamma)$ iff $\kappa^2$ is not a Dirichlet [respectively, Neumann] eigenvalue for $-\Delta$ in $\Omega_-$; if $\kappa^2$ is not such an*

*eigenvalue, the corresponding biorthonormal sequence is* $\left(\overline{-i\kappa(I+\overline{D_\kappa^*})^{-1}V_{n,\mathbf{n}}^{\kappa\mathcal{O}}}\right)_{n=1}^\infty$ *[respectively,*
$\left(\overline{i\kappa(I-D_\kappa)^{-1}V_n^{\kappa\mathcal{O}}|_\Gamma}\right)_{n=1}^\infty$*].*

*Proof*: We use the characterization of minimality given in Lemma 1.1.*i*. Consider first the sequence $\left(\operatorname{Reg}V_n^{\kappa\mathcal{O}}|_\Gamma\right)_{n=1}^\infty$, which is, according to $(3.2)_1$, just $\left(\frac{i}{\kappa}(I+D_\kappa)U_n^{\kappa\mathcal{O}}\right)_{n=1}^\infty$. Suppose that this sequence possesses a biorthonormal sequence $\left(w_n\right)_{n=1}^\infty$ in $H^0(\Gamma)$. Then $\left(-\frac{i}{\overline{\kappa}}(I+D_\kappa^*)w_n\right)_{n=1}^\infty$ must be biorthonormal to $\left(U_n^{\kappa\mathcal{O}}\right)_{n=1}^\infty$, which we know already to be minimal and complete, with unique biorthonormal sequence $\left(\overline{V_{n,\mathbf{n}}^{\kappa\mathcal{O}}}\right)_{n=1}^\infty$. Therefore, we must have

$$\frac{i}{\overline{\kappa}}\left(I+\overline{D_\kappa^*}\right)\overline{w}_n = V_{n,\mathbf{n}}^{\kappa\mathcal{O}} \qquad \text{for}\quad n=1,\,2,\,\dots\,,$$

implying, in particular, that the range $\mathcal{R}\left(I+\overline{D_\kappa^*}\right)$ is dense; since this same range is closed, it must be all of $H^0(\Gamma)$, so that the null space $\mathcal{N}\left(I+\overline{D_\kappa^*}\right)$ is the trivial subspace. Therefore, $\kappa^2$ is not an interior Dirichlet eigenvalue for $-\Delta$. Conversely, assuming that $\kappa^2$ is not a Dirichlet eigenvalue for $-\Delta$ in $\Omega_-$, it is easy to check that $\left(\overline{-i\kappa(I+\overline{D_\kappa^*})^{-1}V_{n,\mathbf{n}}^{\kappa\mathcal{O}}}\right)_{n=1}^\infty$ is (well-defined and) biorthonormal to $\left(\operatorname{Reg}V_n^{\kappa\mathcal{O}}|_\Gamma\right)_{n=1}^\infty$ (and so must be the unique biorthonormal sequence in that case), so the latter sequence is minimal in $H^0(\Gamma)$.

Next, we turn to $\left(\operatorname{Reg}V_{n,\mathbf{n}}^{\kappa\mathcal{O}}\right)_{n=1}^\infty$, or, in view of $(3.2)_2$, the sequence $\left(-\frac{i}{\kappa}(I-\overline{D_\kappa^*})W_n^{\kappa\mathcal{O}}\right)_{n=1}^\infty$. Assume that $\left(w_n\right)_{n=1}^\infty$ is biorthonormal to the latter sequence in $H^0(\Gamma)$; then $\left(\frac{i}{\overline{\kappa}}(I-\overline{D}_\kappa)w_n\right)_{n=1}^\infty$ must be biorthonormal to $\left(W_n^{\kappa\mathcal{O}}\right)_{n=1}^\infty$. Reasoning essentially as before, we must have

$$-\frac{i}{\overline{\kappa}}\left(I-D_\kappa\right)\overline{w}_n = V_n^{\kappa\mathcal{O}}|_\Gamma \qquad \text{for}\quad n=1,\,2,\,\dots\,,$$

which shows that the range $\mathcal{R}\left(I-D_\kappa\right)$ must be dense in $H^0(\Gamma)$, and therefore coincides with $H^0(\Gamma)$. Then $\mathcal{N}\left(I-D_\kappa\right)=\{0\}$, whence we conclude that $\kappa^2$ is not an interior Neumann eigenvalue for $-\Delta$, completing the proof of the second equivalence in one direction. On the other hand, if $\kappa^2$ is not a Neumann eigenvalue for $-\Delta$ in $\Omega_-$, then the sequence $\left(\overline{i\kappa(I-D_\kappa)^{-1}V_n^{\kappa\mathcal{O}}|_\Gamma}\right)_{n=1}^\infty$ is well-defined, and a simple calculation shows that it is biorthonormal to $\left(\operatorname{Reg}V_{n,\mathbf{n}}^{\kappa\mathcal{O}}\right)_{n=1}^\infty$ (and so is the unique such sequence); it follows that $\left(\operatorname{Reg}V_{n,\mathbf{n}}^{\kappa\mathcal{O}}\right)_{n=1}^\infty$ is minimal in $H^0(\Gamma)$ in that case.  $\square$

## 4. Proof of Theorem 1.1.

We begin the proof of Theorem 1.1 by showing how statements (*ii*) and (*iii*) of the theorem will follow from assertion (*i*) and/or the results of the preceding two sections.

The following statement clearly implies that Theorem 1.1.*ii* is true if Theorem 1.1.*i* is true.

**Lemma 4.1.** (*i.*) *Suppose that* $\kappa^2$ *is not a Dirichlet eigenvalue for* $-\Delta$ *in* $\Omega_-$. *Then* $\left(\operatorname{Reg} V_n^{\kappa\mathcal{O}}\big|_\Gamma\right)_{n=1}^\infty$ *is a basis for* $H^0(\Gamma)$ *iff* $\left(V_{n,\mathbf{n}}^{\kappa\mathcal{O}}\right)_{n=1}^\infty$ *is a basis for* $H^0(\Gamma)$.

(*ii.*) *Suppose that* $\kappa^2$ *is not a Neumann eigenvalue for* $-\Delta$ *in* $\Omega_-$. *Then* $\left(\operatorname{Reg} V_{n,\mathbf{n}}^{\kappa\mathcal{O}}\right)_{n=1}^\infty$ *is a basis for* $H^0(\Gamma)$ *iff* $\left(V_n^{\kappa\mathcal{O}}\big|_\Gamma\right)_{n=1}^\infty$ *is a basis for* $H^0(\Gamma)$.

*Proof*: (*i*). Since $\kappa^2$ is not a Dirichlet eigenvalue for $-\Delta$ in $\Omega_-$, the operator $I + \overline{D_\kappa^*}$ is invertible and the sequence $\left(\operatorname{Reg} V_n^{\kappa\mathcal{O}}\big|_\Gamma\right)_{n=1}^\infty$ is complete and minimal in $H^0(\Gamma)$, with the corresponding unique biorthonormal sequence $\left(\overline{-i\kappa(I + \overline{D_\kappa^*})^{-1} V_{n,\mathbf{n}}^{\kappa\mathcal{O}}}\right)_{n=1}^\infty$. It follows that either *each* of the three sequences $\left(\operatorname{Reg} V_n^{\kappa\mathcal{O}}\big|_\Gamma\right)_{n=1}^\infty$, $\left((I + \overline{D_\kappa^*})^{-1} V_{n,\mathbf{n}}^{\kappa\mathcal{O}}\right)_{n=1}^\infty$, and $\left(V_{n,\mathbf{n}}^{\kappa\mathcal{O}}\right)_{n=1}^\infty$ is a basis or *none* is.

(*ii*). Since $\kappa^2$ is not a Neumann eigenvalue for $-\Delta$ in $\Omega_-$, the operator $I - D_\kappa$ is invertible and the sequence $\left(\operatorname{Reg} V_{n,\mathbf{n}}^{\kappa\mathcal{O}}\right)_{n=1}^\infty$ is complete and minimal in $H^0(\Gamma)$, with the corresponding unique biorthonormal sequence $\left(\overline{i\kappa(I - D_\kappa)^{-1} V_n^{\kappa\mathcal{O}}\big|_\Gamma}\right)_{n=1}^\infty$. Therefore, just as in the proof of the first statement, either *each* of the three sequences $\left(\operatorname{Reg} V_{n,\mathbf{n}}^{\kappa\mathcal{O}}\right)_{n=1}^\infty$, $\left((I - D_\kappa)^{-1} V_n^{\kappa\mathcal{O}}\big|_\Gamma\right)_{n=1}^\infty$, and $\left(V_n^{\kappa\mathcal{O}}\big|_\Gamma\right)_{n=1}^\infty$ is a basis or *none* is. $\quad\square$

Now, considering Theorem 1.1.*iii*, suppose that $\kappa^2$ *is* a Neumann [respectively, Dirichlet] eigenvalue for $-\Delta$ in $\Omega_-$. We showed in Proposition 3.3 that the sequence $\left(\operatorname{Reg} V_{n,\mathbf{n}}^{\kappa\mathcal{O}}\right)_{n=1}^\infty$ [respectively, $\left(\operatorname{Reg} V_n^{\kappa\mathcal{O}}\big|_\Gamma\right)_{n=1}^\infty$] is not minimal in $H^0(\Gamma)$ in that case, so neither can it be minimal in the Hilbert space comprising its closed span $\mathcal{R}\left(I - \overline{D_\kappa^*}\right)$ [respectively, $\mathcal{R}\left(I + D_\kappa\right)$] (which is, of course, here a proper subspace of $H^0(\Gamma)$); in particular, it cannot then form a basis for that closed span. Thus, Theorem 1.1.*iii.a* is correct. Now suppose further that $\Omega_-$ is a ball centered at $\mathcal{O}$. In this case, the sequence in question is orthogonal in $H^0(\Gamma)$, whence it is clear that the sequence remaining after deletion of the zero-elements of $\left(\operatorname{Reg} V_{n,\mathbf{n}}^{\kappa\mathcal{O}}\right)_{n=1}^\infty$ [respectively, $\left(\operatorname{Reg} V_n^{\kappa\mathcal{O}}\big|_\Gamma\right)_{n=1}^\infty$] will be a basis for the closed span of the sequence. According to the results on the zeros of the Bessel functions given in WATSON [24], there is precisely one such zero-element in the sequence. This completes the proof of Theorem 1.1.*iii.b*.

Finally, we turn to the verification of Theorem 1.1.*i*, for which we require two further simple results in preparation. First, it is important to examine the behavior of the series in (3.3.2) when the positions of $\mathbf{x}$ and $\mathbf{y}$ are switched:

**Lemma 4.2.** *The infinite series* $\sum_{n=1}^\infty \operatorname{Reg} V_n^{\kappa\mathcal{O}}(\mathbf{y}) V_n^{\kappa\mathcal{O}}(\mathbf{x})$ *diverges whenever* $|\mathbf{y} - \mathcal{O}| > |\mathbf{x} - \mathcal{O}| > 0$.

*Proof*: Pick $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^3$ with $|\mathbf{y} - \mathcal{O}| > |\mathbf{x} - \mathcal{O}|$; denote by $(\varrho_\mathbf{x}, \vartheta_\mathbf{x}, \varphi_\mathbf{x})$ and $(\varrho_\mathbf{y}, \vartheta_\mathbf{y}, \varphi_\mathbf{y})$ their spherical coördinates in a system with pole at $\mathcal{O}$. For the proof, we show merely that the sequence of terms of the series in question is not a null sequence, by producing a subsequence that is not null. The subsequence that we choose is given in terms of the bijection $n \mapsto \left(l^\star(n), m^\star(n)\right)$ as $\left(\alpha_{n_k} := \operatorname{Reg} V_{n_k}^{\kappa\mathcal{O}}(\mathbf{y}) V_{n_k}^{\kappa\mathcal{O}}(\mathbf{x})\right)_{k=1}^\infty$, in which $(n_k)_{k=1}^\infty$ is the increasing sequence such that $m^\star(n_k) = 0$ for each $k$. Every nonnegative integer appears exactly once in the corresponding subsequence

$\left(l_k := l^\star(n_k)\right)_{k=1}^\infty$, so that $l_k \to \infty$ as $k \to \infty$. Explicitly, we find $\alpha_{n_k} = \left(\alpha_k^1 + i\alpha_k^2\right)/2\pi$ for each $k \geq 1$, in which

$$\alpha_k^1 := \left(2l_k + 1\right)j_{l_k}(\kappa\varrho_\mathbf{x})j_{l_k}(\kappa\varrho_\mathbf{y})P_{l_k}(\cos\vartheta_\mathbf{x})P_{l_k}(\cos\vartheta_\mathbf{y}),$$

$$\alpha_k^2 := \left(2l_k + 1\right)y_{l_k}(\kappa\varrho_\mathbf{x})j_{l_k}(\kappa\varrho_\mathbf{y})P_{l_k}(\cos\vartheta_\mathbf{x})P_{l_k}(\cos\vartheta_\mathbf{y}),$$

with $y_l$ denoting the spherical Bessel function of the second kind and order $l$. The asymptotic behavior of the Bessel functions $J_\nu$ and $Y_\nu$ for fixed argument and large positive order can be determined from their simplest representations. Thus, one finds (*cf.*, *e.g.*, [1], [24])

$$\left.\begin{array}{l} J_\nu(z) \sim \dfrac{1}{\sqrt{2\pi\nu}}\left(\dfrac{ez}{2\nu}\right)^\nu \\[3mm] Y_\nu(z) \sim -\sqrt{\dfrac{2}{\pi\nu}}\left(\dfrac{ez}{2\nu}\right)^{-\nu} \end{array}\right\} \qquad \text{for} \quad z \quad \text{fixed} \quad \text{and} \quad \nu \to \infty \quad \text{through positive values.} \qquad (4.1)$$

First, $(4.1)_1$ and the bound $|P_l(x)| \leq 1$, holding for every $x \in [-1, 1]$ and every Legendre polynomial $P_l$, certainly imply that $\alpha_k^1 \to 0$ as $k \to \infty$. Further, by also using $(4.1)_2$ we find that

$$\left(2l + 1\right)y_l(\kappa\varrho_\mathbf{x})j_l(\kappa\varrho_\mathbf{y}) \sim -\frac{1}{\kappa\varrho_\mathbf{x}}\left(\frac{\varrho_\mathbf{y}}{\varrho_\mathbf{x}}\right)^l \qquad \text{as} \quad l \to \infty.$$

We recall the exact values $P_l(\pm 1) = (\pm 1)^l$ and the asymptotic relation

$$P_l(\cos\vartheta) \sim \sqrt{\frac{2}{\pi l \sin\vartheta}}\sin\left(\frac{(2l + 1)\vartheta}{2} + \frac{\pi}{4}\right) \qquad \text{as} \quad l \to \infty, \qquad \text{uniformly for} \quad 0 < \delta \leq \vartheta \leq \pi - \delta$$

(with any positive $\delta$) from, *e.g.*, [15]. Now, if $\vartheta_\mathbf{x}$ and $\vartheta_\mathbf{y}$ lie in $(0, \pi)$, it is apparent that the sequence $\left(\left|\sin\left(\frac{(2l_k + 1)\vartheta_\mathbf{x}}{2} + \frac{\pi}{4}\right)\sin\left(\frac{(2l_k + 1)\vartheta_\mathbf{y}}{2} + \frac{\pi}{4}\right)\right|\right)_{k=1}^\infty$ does not converge to zero and so has a subsequence with a positive lower bound; the other possible positions of $\vartheta_\mathbf{x}$ and $\vartheta_\mathbf{y}$ can be checked similarly. By combining these facts, from the decisive hypothesis that $\varrho_\mathbf{y} > \varrho_\mathbf{x}$ we see that the sequence $\left(\alpha_k^2\right)_{k=1}^\infty$ is in fact unbounded. Therefore, the sequence $\left(\alpha_{n_k}\right)_{k=1}^\infty$ cannot have limit zero. □

Let us also check an intuitively apparent geometric fact:

**Lemma 4.3.** *Recall that $\Omega_- \subset \mathbb{R}^3$ is (nonvoid) bounded, open, and connected, with $\Omega_+ := \mathbb{R}^3 \setminus \overline{\Omega_-}$ also connected, and $\mathcal{O} \in \Omega_-$. If $\Omega_-$ is a ball, suppose that $\mathcal{O}$ is not its center. Then there exist $\mathbf{x} \in \Omega_+$ and $\mathbf{y} \in \Omega_-$ satisfying $|\mathbf{y} - \mathcal{O}| > |\mathbf{x} - \mathcal{O}|$.*

*Proof*: The argument uses the continuity of the map $\mathbf{x} \mapsto |\mathbf{x} - \mathcal{O}|$ and the compactness of $\Gamma$. Let $\mathbf{y}_{\min}$ be a point of $\Gamma$ nearest $\mathcal{O}$ and $\mathbf{y}_{\max}$ a point of $\Gamma$ farthest from $\mathcal{O}$; then the radii of the inscribed and circumscribed spheres for $\Gamma$ centered at $\mathcal{O}$ are given by $R_\mathcal{O}^- = |\mathbf{y}_{\min} - \mathcal{O}|$ and $R_\mathcal{O}^+ = |\mathbf{y}_{\max} - \mathcal{O}|$, respectively. Since $R_\mathcal{O}^- = R_\mathcal{O}^+$ iff $\Omega_-$ is a ball centered at $\mathcal{O}$, the hypotheses imply that $R_\mathcal{O}^- < R_\mathcal{O}^+$. Thus, setting $\varepsilon := \left(R_\mathcal{O}^+ - R_\mathcal{O}^-\right)/2$, it is clear that we can find an $\mathbf{x} \in \Omega_+$ with $|\mathbf{y}_{\min} - \mathbf{x}| < \varepsilon$ and a $\mathbf{y} \in \Omega_-$ with $|\mathbf{y}_{\max} - \mathbf{y}| < \varepsilon$. But then $\mathbf{x}$ and $\mathbf{y}$ possess the required property:

$$|\mathbf{x} - \mathcal{O}| \leq |\mathbf{x} - \mathbf{y}_{\min}| + |\mathbf{y}_{\min} - \mathcal{O}| < \varepsilon + R_\mathcal{O}^- = R_\mathcal{O}^+ - \varepsilon < |\mathbf{y}_{\max} - \mathcal{O}| - |\mathbf{y} - \mathbf{y}_{\max}| \leq |\mathbf{y} - \mathcal{O}|. \qquad □$$

*Proof of Theorem 1.1.i*: The sufficiency of the condition follows from a familiar argument: if $\Omega_-$ is a ball centered at $\mathcal{O}$, each of the two sequences in question comprises nonzero elements and is orthogonal and complete in $H^0(\Gamma)$, whence it forms a basis for that space.

We shall prove the necessity first for $\left(V_{n;\mathbf{n}}^{\kappa\mathcal{O}}\right)_{n=1}^{\infty}$. Accordingly, let us suppose that this sequence forms a basis for $H^0(\Gamma)$; we shall show that $\Omega_-$ must then be a ball centered at $\mathcal{O}$. Under the assumption, we must have

$$g = \sum_{n=1}^{\infty} \left(g, \overline{U_n^{\kappa\mathcal{O}}}\right)_0 V_{n;\mathbf{n}}^{\kappa\mathcal{O}} \qquad \text{for every} \quad g \in H^0(\Gamma). \tag{4.2}$$

In turn, we can use (4.2) in (2.6) and, owing to the $H^0(\Gamma)$-convergence of the series, operate term-by-term to get an expansion for the solution $\mathfrak{u}_g$ of the Neumann/radiation problem with data $g$ that converges throughout $\Omega_+$:

$$\mathfrak{u}_g(\mathbf{x}) = \frac{1}{2}\int_\Gamma \left\{E_{\mathbf{x}}^\kappa - A_\kappa E_{\mathbf{x},\mathbf{n}}^\kappa\right\} g\, d\lambda_\Gamma = \sum_{n=1}^{\infty}\left(g, \overline{U_n^{\kappa\mathcal{O}}}\right)_0 \left(\frac{1}{2}\int_\Gamma \left\{E_{\mathbf{x}}^\kappa - A_\kappa E_{\mathbf{x},\mathbf{n}}^\kappa\right\} V_{n;\mathbf{n}}^{\kappa\mathcal{O}}\, d\lambda_\Gamma\right)$$

$$= \sum_{n=1}^{\infty}\left(g, \overline{U_n^{\kappa\mathcal{O}}}\right)_0 V_n^{\kappa\mathcal{O}}(\mathbf{x}) \qquad \text{for} \quad \mathbf{x}\in\Omega_+, \quad g\in H^0(\Gamma), \tag{4.3}$$

the final equality following from (2.6) with $g$ replaced by $V_{n;\mathbf{n}}^{\kappa\mathcal{O}}$. In particular, (4.3) must hold whenever we take $g = E_{\mathbf{y},\mathbf{n}}^\kappa$ for any $\mathbf{y}\in\Omega_-$, in which case $\mathfrak{u}_g$ is simply the restriction $E_{\mathbf{y}}^\kappa\big|_{\Omega_+}$. Moreover, for such a choice we can explicitly compute the expansion coefficients in (4.3), by remembering that $A_\kappa^* = \overline{A_\kappa}$ and $A_\kappa E_{\mathbf{y},\mathbf{n}}^\kappa = E_{\mathbf{y}}^\kappa\big|_\Gamma$, and using $(2.5)_2$:

$$\left(E_{\mathbf{y},\mathbf{n}}^\kappa, \overline{U_n^{\kappa\mathcal{O}}}\right)_0 = -\frac{\mathrm{i}\kappa}{2}\int_\Gamma E_{\mathbf{y},\mathbf{n}}^\kappa\left\{\operatorname{Reg}V_n^{\kappa\mathcal{O}} - A_\kappa \operatorname{Reg}V_{n;\mathbf{n}}^{\kappa\mathcal{O}}\right\} d\lambda_\Gamma$$

$$= -\frac{\mathrm{i}\kappa}{2}\int_\Gamma \left\{E_{\mathbf{y},\mathbf{n}}^\kappa \operatorname{Reg}V_n^{\kappa\mathcal{O}} - E_{\mathbf{y}}^\kappa\big|_\Gamma \operatorname{Reg}V_{n;\mathbf{n}}^{\kappa\mathcal{O}}\right\} d\lambda_\Gamma = -\mathrm{i}\kappa\operatorname{Reg}V_n^{\kappa\mathcal{O}}(\mathbf{y}) \qquad \text{for} \quad \mathbf{y}\in\Omega_-.$$

Therefore, (4.3) shows that

$$E_{\mathbf{y}}^\kappa(\mathbf{x}) = -\mathrm{i}\kappa\sum_{n=1}^{\infty}\operatorname{Reg}V_n^{\kappa\mathcal{O}}(\mathbf{y})V_n^{\kappa\mathcal{O}}(\mathbf{x}) \qquad \text{for every} \quad \mathbf{y}\in\Omega_- \quad \text{and} \quad \mathbf{x}\in\Omega_+. \tag{4.4}$$

But this implies that $\Omega_-$ is a ball centered at $\mathcal{O}$, for, otherwise we could find a $\mathbf{y}_o\in\Omega_-$ and an $\mathbf{x}_o\in\Omega_+$ such that $|\mathbf{y}_o-\mathcal{O}| > |\mathbf{x}_o-\mathcal{O}|$ (Lemma 4.3), contradicting (4.4), since the series appearing there would diverge for $\mathbf{x} = \mathbf{x}_o$ and $\mathbf{y} = \mathbf{y}_o$ (Lemma 4.2).

The proof of the necessity for $\left(V_n^{\kappa\mathcal{O}}\big|_\Gamma\right)_{n=1}^{\infty}$ is analogous to that just given for $\left(V_{n;\mathbf{n}}^{\kappa\mathcal{O}}\right)_{n=1}^{\infty}$. Suppose that the former sequence is a basis for $H^0(\Gamma)$, so that we must have

$$g = \sum_{n=1}^{\infty}\left(g, \overline{W_n^{\kappa\mathcal{O}}}\right)_0 V_n^{\kappa\mathcal{O}}\big|_\Gamma \qquad \text{for every} \quad g\in H^0(\Gamma).$$

Now, we can use such an expansion in (2.7) and operate term-by-term to get a representation for the solution $\mathfrak{v}_f$ of the exterior Dirichlet/radiation problem with data $f\in\mathcal{R}\left(A_\kappa\right)$ converging throughout $\Omega_+$:

$$\mathfrak{v}_f(\mathbf{x}) = \frac{1}{2}\int_\Gamma \left\{A_\kappa^{-1}E_{\mathbf{x}}^\kappa\big|_\Gamma - E_{\mathbf{x},\mathbf{n}}^\kappa\right\} f\, d\lambda_\Gamma = \sum_{n=1}^{\infty}\left(f, \overline{W_n^{\kappa\mathcal{O}}}\right)_0 \left(\frac{1}{2}\int_\Gamma\left\{A_\kappa^{-1}E_{\mathbf{x}}^\kappa\big|_\Gamma - E_{\mathbf{x},\mathbf{n}}^\kappa\right\}V_n^{\kappa\mathcal{O}}\, d\lambda_\Gamma\right)$$

$$= \sum_{n=1}^{\infty}\left(f, \overline{W_n^{\kappa\mathcal{O}}}\right)_0 V_n^{\kappa\mathcal{O}}(\mathbf{x}) \qquad \text{for} \quad \mathbf{x}\in\Omega_+, \quad f\in\mathcal{R}\left(A_\kappa\right). \tag{4.5}$$

20

In particular, (4.5) holds whenever we take $f = E_{\mathbf{y}}^{\kappa}\big|_{\Gamma}$ with $\mathbf{y} \in \Omega_-$, in which case $\mathfrak{v}_f$ is simply the restriction $E_{\mathbf{y}}^{\kappa}\big|_{\Omega_+}$. Noting that $A_\kappa^{-1} E_{\mathbf{y}}^{\kappa}\big|_{\Gamma} = E_{\mathbf{y},\mathbf{n}}^{\kappa}$ for such $\mathbf{y}$, the expansion coefficients in (4.5) are again found to be

$$\left(E_{\mathbf{y}}^{\kappa}\big|_{\Gamma}, \overline{W_n^{\kappa\mathcal{O}}}\right)_0 = \left(E_{\mathbf{y}}^{\kappa}\big|_{\Gamma}, A_\kappa^{*-1}\overline{U_n^{\kappa\mathcal{O}}}\right)_0 = \left(E_{\mathbf{y},\mathbf{n}}^{\kappa}, \overline{U_n^{\kappa\mathcal{O}}}\right)_0 = -\mathrm{i}\kappa\mathrm{Reg}\, V_n^{\kappa\mathcal{O}}(\mathbf{y}).$$

Therefore, again we come to (4.4), which again forces the conclusion that $\Omega_-$ is a ball centered at $\mathcal{O}$.

This finishes the proof of ($i$). Since we have already indicated how the proofs of statements ($ii$) and ($iii$) follow, the proof of Theorem 1.1 is complete. $\qquad\square$

## 5. An application: studying a previous result on $T$-matrix methods.

Here we point out the implications of Theorem 1.1 for the previous work of KRISTENSSON, RAMM, AND STRÖM [13] on the convergence of certain "$T$-matrix methods" in the approximate solution of problems of time-harmonic scattering by obstacles; the developments of [13] appear in a more accessible form in the book of RAMM [18]. In particular, we want to show that the results of [13] do not suffice to substantiate the first method proposed in WATERMAN [21].

We shall begin by establishing the existence and essential mapping property of the "transition matrix." We restrict our attention to the case in which a "sound-hard" obstacle occupies the closure of $\Omega_-$, since all of the central points can be made within that setting; other boundary conditions can be treated in a similar fashion. Let $u^\iota$ be a given *incident field*, by which we mean a solution of the Helmholtz equation (0.1) in an open set $\Omega_\iota$ containing the closure of $\Omega_-$. The corresponding *scattered field* in $\Omega_+$ is then the unique element $u^\sigma \in W_+(\Omega_+; \kappa)$ with the Neumann data given by

$$u^\sigma_{,\mathbf{n}} = -u^\iota_{,\mathbf{n}} \qquad \text{on} \quad \Gamma. \tag{5.1}$$

According to (3.4), $u^\sigma$ has the expansion

$$u^\sigma(\mathbf{x}) = -\sum_{n=1}^\infty \left(u^\iota_{,\mathbf{n}}, \overline{U_n^{\kappa\mathcal{O}}}\right)_0 V_n^{\kappa\mathcal{O}}(\mathbf{x}) = \sum_{n=1}^\infty \sigma_n V_n^{\kappa\mathcal{O}}(\mathbf{x}) \qquad \text{(at least) for} \quad |\mathbf{x} - \mathcal{O}| > R_\mathcal{O}^+, \tag{5.2}$$

in which the *scattered-field expansion coefficients* are given by $\sigma_n := -\left(u^\iota_{,\mathbf{n}}, \overline{U_n^{\kappa\mathcal{O}}}\right)_0$; for brevity, we are omitting an indication of the dependence of these coefficients upon the various parameters, such as the choice of $\mathcal{O} \in \Omega_-$. Note that the series in (5.2) has the convergence and term-by-term differentiability properties recounted in Section 3.

From (2.5) and (3.3), it is clear that the incident field $u^\iota$ possesses an expansion

$$u^\iota(\mathbf{x}) = \sum_{l=0}^\infty \sum_{m=-l}^l \iota_{lm} \operatorname{Reg} V_{lm}^{\kappa\mathcal{O}}(\mathbf{x}) = \sum_{n=1}^\infty \iota_n \operatorname{Reg} V_n^{\kappa\mathcal{O}}(\mathbf{x}) \qquad \text{for} \quad |\mathbf{x} - \mathcal{O}| < R_\iota, \tag{5.3}$$

for some positive $R_\iota \geq R_\mathcal{O}^-$; again, the convergence properties of the series here are just those already listed in Section 3. In the special circumstance in which the incident-field domain $\Omega_\iota$ contains the circumscribing ball $B_{R_\mathcal{O}^+}(\mathcal{O})$, *i.e.*, when $R_\mathcal{O}^+ < R_\iota$ (so that the "sources" of the incident field are "not too near" the obstacle), we shall say that the incident field $u^\iota$ is $(\Omega_-, \mathcal{O})$-*regular*. Then, for an $(\Omega_-, \mathcal{O})$-regular $u^\iota$ we may compute $u^\iota_{,\mathbf{n}}$ term-by-term from (5.3) and find the scattered-field expansion coefficients expressed as

$$\sigma_n := -\left(u^\iota_{,\mathbf{n}}, \overline{U_n^{\kappa\mathcal{O}}}\right)_0 = -\sum_{m=1}^\infty \left(\operatorname{Reg} V_{m,\mathbf{n}}^{\kappa\mathcal{O}}, \overline{U_n^{\kappa\mathcal{O}}}\right)_0 \iota_m, \qquad n = 1, 2, \ldots. \tag{5.4}$$

The array $\mathcal{T}^{\kappa\mathcal{O}} := \left\{-\left(\operatorname{Reg} V_{m,\mathbf{n}}^{\kappa\mathcal{O}}, \overline{U_n^{\kappa\mathcal{O}}}\right)_0\right\}_{m,n=1}^\infty$ emerging in this manipulation is generally called the *transition matrix*, or $T$-*matrix*, since it contains the information necessary to transform the (known) incident-field expansion coefficients $\left(\iota_n\right)_{n=1}^\infty$ into the desired scattered-field expansion coefficients $\left(\sigma_n\right)_{n=1}^\infty$ for *any* incident field $u^\iota$ satisfying the indicated hypothesis; the array depends on only $\Omega_-$, $\kappa$, $\mathcal{O}$, and the particular boundary condition characterizing the material of the scattering obstacle. (More correctly, $\mathcal{T}^{\kappa\mathcal{O}}$ should be called, say, the "acoustic hard-scattering spherical transition matrix

for $\Omega_-$, $\kappa$, and $\mathcal{O}$," but here we shall continue to employ the abbreviated term, since we are examining just one case and there is no chance of confusion.)

Since the $T$-matrix $\mathcal{T}^{\kappa\mathcal{O}}$ serves as a sort of Neumann-Green function, it is clearly worthwhile to study methods for computing it. Of course, a direct and explicit construction of $\mathcal{T}^{\kappa\mathcal{O}}$ requires knowledge of the $U_n^{\kappa\mathcal{O}}$, which are themselves to be found through the solution of "canonical" Neumann/radiation problems in $\Omega_+$, so that one must resort to an approximation scheme in all but the simplest geometries. Two schemes are proposed in WATERMAN [21] which can be regarded as intended for approximation of $\mathcal{T}^{\kappa\mathcal{O}}$. We shall be concerned in this section with just the first of these and, as noted, with just that for the hard-scattering problem. Waterman's first algorithm is based on the relations between the incident- and scattered-field expansion coefficients and the trace of the *total field* $u^\tau := u^\iota + u^\sigma$ (defined in $\Omega_\iota \cap \Omega_+$), *viz.*,

$$\left.\begin{array}{l} \left(u^\tau\big|_\Gamma, \overline{V_{n,\mathbf{n}}^{\kappa\mathcal{O}}}\right)_0 = \dfrac{2\mathrm{i}}{\kappa}\iota_n \\[2ex] \sigma_n = \dfrac{\mathrm{i}\kappa}{2}\left(u^\tau\big|_\Gamma, \overline{\operatorname{Reg} V_{n,\mathbf{n}}^{\kappa\mathcal{O}}}\right)_0 \end{array}\right\} \qquad \text{for} \quad n = 1,\, 2,\, \dots; \tag{5.5}$$

these are derived in [21], and follow readily from the relations (2.4) and (2.5) cited here; *cf.*, also, (5.4). The strategy aims at exploiting $(5.5)_1$ to generate an approximation to $u^\tau\big|_\Gamma$, which is then to be used to produce approximations for the desired coefficients $\sigma_n$ on the basis of $(5.5)_2$; as a by-product, a finite array purporting to be an "approximate $T$-matrix" can be identified. Observe that the completeness of $\left(V_{n,\mathbf{n}}^{\kappa\mathcal{O}}\right)_{n=1}^\infty$ in $H^0(\Gamma)$ guarantees that $u^\tau\big|_\Gamma$ is the unique element of $H^0(\Gamma)$ satisfying $(5.5)_1$, so the corresponding moment problem is in fact uniquely solvable for any $\kappa$; this is one of the most attractive features of the approach.

Thus, one begins by selecting in $H^0(\Gamma)$ a sequence of trial functions appropriate for construction of a convergent linear approximation to $u^\tau\big|_\Gamma$. Such a family should be not only linearly independent but also at least complete in $H^0(\Gamma)$, since the collection of total-field traces contains all of the $U_n^{\kappa\mathcal{O}}$, and so is dense in $H^0(\Gamma)$. Strangely, Waterman chose the traces $\left(\operatorname{Reg} V_n^{\kappa\mathcal{O}}\big|_\Gamma\right)_{n=1}^\infty$ to serve as his sequence of trial functions, even though he apparently knew that this sequence is not complete in $H^0(\Gamma)$ when $\kappa^2$ is an interior Dirichlet eigenvalue. In any event, one attempts to produce, at least for all sufficiently large $N$, a linear combination

$$\left[u^\tau\big|_\Gamma\right]_N := \sum_{m=1}^{N} \alpha_m^N \operatorname{Reg} V_m^{\kappa\mathcal{O}}\big|_\Gamma$$

in which the coefficients are to be determined by the linear system $(cf.\ (5.5)_1)$

$$\sum_{m=1}^{N} \left(\operatorname{Reg} V_m^{\kappa\mathcal{O}}\big|_\Gamma, \overline{V_{n,\mathbf{n}}^{\kappa\mathcal{O}}}\right)_0 \alpha_m^N = \frac{2\mathrm{i}}{\kappa}\iota_n, \qquad n = 1,\, \dots,\, N; \tag{5.6}$$

subsequently, intended approximations $\left\{\sigma_n^N\right\}_{n=1}^N$ to the first $N$ scattered-field expansion coefficients are to be constructed by $(cf.\ (5.5)_2)$

$$\sigma_n^N := \frac{\mathrm{i}\kappa}{2}\left(\left[u^\tau\big|_\Gamma\right]_N, \overline{\operatorname{Reg} V_{n,\mathbf{n}}^{\kappa\mathcal{O}}}\right)_0 = \frac{\mathrm{i}\kappa}{2}\sum_{m=1}^{N}\left(\operatorname{Reg} V_m^{\kappa\mathcal{O}}\big|_\Gamma, \overline{\operatorname{Reg} V_{n,\mathbf{n}}^{\kappa\mathcal{O}}}\right)_0 \alpha_m^N, \qquad n = 1,\, \dots,\, N, \tag{5.7}$$

whence prospective approximations $u_N^\sigma$ to the scattered field $u^\sigma$ in $\Omega_+$ are to be generated by

$$u_N^\sigma(\mathbf{x}) := \sum_{m=1}^{N} \sigma_n^N V_n^{\kappa\mathcal{O}}(\mathbf{x}), \qquad \text{for} \quad \mathbf{x} \in \Omega_+. \tag{5.8}$$

Based on this motivation, the first Waterman algorithm for the acoustic hard-scattering problem entails two steps:

**(W.I.1)  establish viability**: show that the $N \times N$ matrix $\mathcal{Q}_N^{\kappa\mathcal{O}}$ figuring in (5.6), with elements

$$\mathcal{Q}_{mn}^{\kappa\mathcal{O}} := \left( \operatorname{Reg} V_m^{\kappa\mathcal{O}} \big|_\Gamma, \overline{V_{n;\mathbf{n}}^{\kappa\mathcal{O}}} \right)_0, \tag{5.9}$$

is invertible for all $N$ greater than some $N_0$;

**(W.I.2)  establish convergence**: show that the resultant sequence $\left( u_N^\sigma \right)_{N=N_0}^{\infty}$ constructed from (5.8), with the coefficients $\left( \sigma_n^N \right)_{n=1}^{N}$ obtained from (5.6) and (5.7) for $N > N_0$, converges in some manner to the unique solution $u^\sigma$.

When this program goes through, "approximate $T$-matrices" can be identified. That is, when the inverse $\left( \mathcal{Q}_N^{\kappa\mathcal{O}} \right)^{-1}$ exists for $N > N_0$, (5.6) and (5.7) will give

$$\sigma_n^N = \sum_{q=1}^{N} \left\{ -\sum_{m=1}^{N} \left( \mathcal{Q}_N^{\kappa\mathcal{O}} \right)_{qm}^{-1} \left( \operatorname{Reg} V_m^{\kappa\mathcal{O}} \big|_\Gamma, \overline{\operatorname{Reg} V_{n;\mathbf{n}}^{\kappa\mathcal{O}}} \right)_0 \right\} \iota_q, \qquad n = 1, \ldots, N, \tag{5.10}$$

displaying the approximate scattered-field coefficients as images of the (first $N$) incident-field coefficients under the operation of an $N \times N$ "$N^{\text{th}}$ approximate $T$-matrix" $\mathcal{T}_N^{\kappa\mathcal{O}}$, the general element $\left( \mathcal{T}_N^{\kappa\mathcal{O}} \right)_{nq}$ of which appears within the braces in (5.10). That is, since it is easy to see that the matrix $\operatorname{Reg} \mathcal{Q}_N^{\kappa\mathcal{O}}$ with elements $\left( \operatorname{Reg} V_m^{\kappa\mathcal{O}} \big|_\Gamma, \overline{\operatorname{Reg} V_{n;\mathbf{n}}^{\kappa\mathcal{O}}} \right)_0$ is symmetric, we get

$$\mathcal{T}_N^{\kappa\mathcal{O}} = -\operatorname{Reg} \mathcal{Q}_N^{\kappa\mathcal{O}} \left\{ \left( \mathcal{Q}_N^{\kappa\mathcal{O}} \right)^{-1} \right\}^{\mathsf{T}} = \left\{ -\left( \mathcal{Q}_N^{\kappa\mathcal{O}} \right)^{-1} \operatorname{Reg} \mathcal{Q}_N^{\kappa\mathcal{O}} \right\}^{\mathsf{T}}, \tag{5.11}$$

the superscript "$\mathsf{T}$" indicating "transpose."

While the Waterman schemes attracted much numerical experimentation and heuristic argument, little progress was made in answering the fundamental questions concerning the viability and convergence of the algorithms. KRISTENSSON, RAMM, AND STRÖM [13] (*cf*., also, the reorganization in RAMM [18]) study one approach to the construction of a convergence proof, through a more general formulation than that set up in [21], by considering the general exterior Neumann/radiation problem (not just the scattering problem) and initially permitting more flexibility in the choices of the trial- and (radiating-wave) test-function sequences. We shall indicate how this latter formulation does specialize to cover the first Waterman algorithm when the trial and test functions are constructed appropriately from the spherical-wave functions, but that the convergence theorem established in [13] does not generally apply to that case, because the hypotheses require at least that the test-function sequence form a basis (in fact, a Riesz basis) for $H^0(\Gamma)$.

We begin by describing the approximation scheme of [13], not in the full generality arranged there, but just for (*a*) the special selections of the trial and test functions that lead to the first

Waterman algorithm and $(b)$ the acoustic hard-scattering problem, *i.e.*, with the special Neumann data corresponding to an incident field, as in (5.1). The relations

$$\left(u^\sigma\big|_\Gamma, \overline{V_{n,\mathbf{n}}^{\kappa\mathcal{O}}}\right)_0 = -\left(u^\iota_{,\mathbf{n}}, \overline{V_n^{\kappa\mathcal{O}}}\big|_\Gamma\right)_0, \qquad \text{for} \quad n = 1, 2, \dots, \tag{5.12}$$

are exploited in [13] for construction of approximations to the trace $u^\sigma\big|_\Gamma$ of the scattered field; (5.12) is simply the form implied by the second Green identity for the radiating solutions $u^\sigma$ and $V_n^{\kappa\mathcal{O}}$ of the Helmholtz equation in $W_+(\Omega_+; \kappa)$, with account taken of the Neumann condition (5.1). To proceed parallel to the developments in [13], one chooses $\left(\operatorname{Reg} V_n^{\kappa\mathcal{O}}\big|_\Gamma\right)_{n=1}^\infty$ as trial-function sequence and seeks to construct an approximation to $u^\sigma\big|_\Gamma$ in the form

$$\left[u^\sigma\big|_\Gamma\right]_N := \sum_{m=1}^N \widetilde{\sigma}_m^N \operatorname{Reg} V_m^{\kappa\mathcal{O}}\big|_\Gamma$$

in which the coefficients $\left\{\widetilde{\sigma}_n^N\right\}_{n=1}^N$ are to be determined by a linear system with the same coefficient matrix $\mathcal{Q}_N^{\kappa\mathcal{O}}$ as that in (5.6) (*cf.* (5.12)),

$$\sum_{m=1}^N \left(\operatorname{Reg} V_m^{\kappa\mathcal{O}}\big|_\Gamma, \overline{V_{n,\mathbf{n}}^{\kappa\mathcal{O}}}\right)_0 \widetilde{\sigma}_m^N = -\sum_{m=1}^N \left(\operatorname{Reg} V_{m,\mathbf{n}}^{\kappa\mathcal{O}}, \overline{V_n^{\kappa\mathcal{O}}}\big|_\Gamma\right)_0 \iota_m \qquad n = 1, \dots, N; \tag{5.13}$$

here, in anticipation of a result on continuous dependence on the data, the actual data-function $u^\iota_{,\mathbf{n}}$ has been replaced on the right in (5.13) by the approximation $\sum_{m=1}^N \iota_m \operatorname{Reg} V_{m,\mathbf{n}}^{\kappa\mathcal{O}}$ (as always, we suppose that $u^\iota$ is $(\Omega_-, \mathcal{O})$-regular). Now, another application of Green's theorem shows that

$$\left(\operatorname{Reg} V_m^{\kappa\mathcal{O}}\big|_\Gamma, \overline{V_{n,\mathbf{n}}^{\kappa\mathcal{O}}}\right)_0 = \left(\operatorname{Reg} V_{m,\mathbf{n}}^{\kappa\mathcal{O}}, \overline{V_n^{\kappa\mathcal{O}}}\big|_\Gamma\right)_0 + \frac{2\mathrm{i}}{\kappa}\delta_{mn} \qquad \text{for} \quad m, n = 1, 2, \dots,$$

whence a summation yields

$$\sum_{m=1}^N \left(\operatorname{Reg} V_m^{\kappa\mathcal{O}}\big|_\Gamma, \overline{V_{n,\mathbf{n}}^{\kappa\mathcal{O}}}\right)_0 \iota_m = \sum_{m=1}^N \left(\operatorname{Reg} V_{m,\mathbf{n}}^{\kappa\mathcal{O}}, \overline{V_n^{\kappa\mathcal{O}}}\big|_\Gamma\right)_0 \iota_m + \frac{2\mathrm{i}}{\kappa}\iota_n \qquad \text{for} \quad n = 1, \dots, N. \tag{5.14}$$

By addition, from (5.13) and (5.14) we obtain the system (5.6) figuring in the first Waterman scheme, with $\alpha_m^N$ identified as $\widetilde{\sigma}_m^N + \iota_m$. That is, the system in (5.6) has unique solution iff the same is true of that in (5.13); when the systems are uniquely solvable, their respective solutions are related by $\alpha_n^N = \widetilde{\sigma}_n^N + \iota_n$, $n = 1, \dots, N$. Therefore, viability and convergence results for the first Waterman scheme will indeed follow from corresponding results for the scheme of KRISTENSSON, RAMM, AND STRÖM [13], an appropriate statement about continuous dependence on the data in the latter, and facts about the convergence of the series expansion of the incident field, under the important *proviso* that the hypotheses imposed on the trial and test functions in [13] are fulfilled by the appropriate traces and normal derivatives of the spherical-wave functions.

With this connection established, we can summarize the convergence result of [13] and so verify that, in view of Theorem 1.1, the conditions required there are (almost always) too stringent for application in establishing the viability and convergence of the first Waterman scheme. Theorem IV.3 and the succeeding Propositions IV.1 and IV.2 of RAMM [18] give conditions sufficient to ensure that system (5.13) have unique solution for all sufficiently large $N$ and the traces of the resultant fields constructed as in (5.8) converge in $H^0(\Gamma)$ to the trace of the unique solution of the hard-scattering problem. Specifically, the conditions of Theorem IV.3 require that $\left(V_{n,\mathbf{n}}^{\kappa\mathcal{O}}\right)_{n=1}^\infty$ be a Riesz basis for

$H^0(\Gamma)$, that $\left(\operatorname{Reg} V_n^{\kappa\mathcal{O}}\big|_\Gamma\right)_{n=1}^\infty$ be linearly independent and complete in $H^0(\Gamma)$, and, in the sense of the usual order relation for self-adjoint linear operators on the Hilbert space $\ell_2$ of complex sequences with square-summable moduli, that the Gram matrix of $\left(\operatorname{Reg} V_n^{\kappa\mathcal{O}}\big|_\Gamma\right)_{n=1}^\infty$ be less than or equal to the product $\mathcal{Q}^{\kappa\mathcal{O}^*}\mathcal{Q}^{\kappa\mathcal{O}}$, in which $\mathcal{Q}^{\kappa\mathcal{O}}$ denotes the infinite matrix with elements $\mathcal{Q}_{mn}^{\kappa\mathcal{O}}$, as in (5.9) (which will induce a bounded linear operator on $\ell_2$ under the other hypotheses imposed). Meanwhile, Proposition IV.3 of [18] provides conditions under which one is assured not only of viability and convergence but also stability of those results under sufficiently small perturbations of the matrices and the righthand sides of the finite-dimensional linear systems; $cf$. [18, Proposition IV.4]. The sufficient conditions there require that $both$ $\left(V_{n,\mathbf{n}}^{\kappa,\mathcal{O}}\right)_{n=1}^\infty$ and $\left(\operatorname{Reg} V_n^{\kappa\mathcal{O}}\big|_\Gamma\right)_{n=1}^\infty$ be Riesz bases for $H^0(\Gamma)$ and that the $infimum$ of the set of smallest eigenvalues of the matrices $\mathcal{Q}_N^{\kappa\mathcal{O}^*}\mathcal{Q}_N^{\kappa\mathcal{O}}$, $N \geq N_0$, be positive. In any event, in view of the (Riesz-) basis requirements imposed, the present Theorem 1.1 implies that the sufficient conditions of [13] hold for the first Waterman scheme only when $\Omega_-$ is a ball centered at $\mathcal{O}$ and $\kappa^2$ is not a Dirichlet eigenvalue for $-\Delta$ in $\Omega_-$ ($i.e.$, in just the setting in which one can already establish the viability and convergence of the scheme directly from the properties of the spherical-wave functions).

Therefore, while the analysis of [13] does provide conditions under which $some$ "$T$-matrix schemes" can be substantiated, these do not include those of WATERMAN [21], which have evidently been the ones most frequently implemented numerically.

## 6. Basis results in other spaces; connections with the far-field pattern.

From this point on, we require that $\kappa$ be real and positive. Under this hypothesis, there are still further important connections between the sequences $\left(V_n^{\kappa\mathcal{O}}\right)_{n=1}^\infty$ and $\left(U_n^{\kappa\mathcal{O}}\right)_{n=1}^\infty$ and between $\left(V_n^{\kappa\mathcal{O}}\big|_\Gamma\right)_{n=1}^\infty$ and $\left(W_n^{\kappa\mathcal{O}}\right)_{n=1}^\infty$, which we explain in this section. It turns out that these sequences do form orthogonal bases for spaces of distributions on $\Gamma$ that are intimately related to the far-field patterns of radiating-wave amplitudes in $\Omega_+$. In these developments, we are led naturally to the polar decomposition of the far-field pattern operators.

As usual, we deal with two cases, the "Neumann" and the "Dirichlet."

**6.1. The Neumann setting.** We introduce the operator $B_\kappa$ in $H^0(\Gamma)$ as a multiple of the "imaginary part" of $A_\kappa$:

$$B_\kappa := \frac{\mathrm{i}\kappa}{4}\left(A_\kappa - A_\kappa^*\right).$$

Obviously, $B_\kappa$ is compact and self-adjoint; it follows also from (2.3) that $B_\kappa$ is positive-definite, *i.e.*, that $\left(B_\kappa g, g\right)_0 > 0$ for each nonzero $g \in H^0(\Gamma)$. In particular, $B_\kappa$ is injective, so the range $\mathcal{R}\left(B_\kappa\right)$ is dense in $H^0(\Gamma)$. From the relation $A_\kappa^* = \overline{A}_\kappa$ we find that $B_\kappa$ is also self-conjugate, *i.e.*, that $\overline{B}_\kappa = B_\kappa$, so it is clear that $\mathcal{R}\left(B_\kappa\right)$ is closed under complex conjugation. The properties of $B_\kappa$ ensure that it has a compact, self-adjoint, injective square root $B_\kappa^{\frac{1}{2}}$, *i.e.*, such that $B_\kappa = B_\kappa^{\frac{1}{2}} B_\kappa^{\frac{1}{2}}$. A surprising mapping property of $B_\kappa$ is established in

**Proposition 6.1.** *Let $\kappa$ be real and positive.*

*(i.)* $B_\kappa$ *maps each* $V_{n,\mathbf{n}}^{\kappa\mathcal{O}}$ *to the corresponding* $\overline{U_n^{\kappa\mathcal{O}}}$:

$$B_\kappa V_{n,\mathbf{n}}^{\kappa\mathcal{O}} = \overline{U_n^{\kappa\mathcal{O}}}, \qquad for \quad n = 1,\,2,\,3,\,\dots. \tag{6.1}$$

*(ii.)* $\left(B_\kappa^{\frac{1}{2}} V_{n,\mathbf{n}}^{\kappa\mathcal{O}}\right)_{n=1}^\infty$ *is an orthonormal basis for* $H^0(\Gamma)$.

*Proof*: $(i)$. In general, for $g \in H^0(\Gamma)$ we compute

$$B_\kappa g = \frac{\mathrm{i}\kappa}{4}\left\{A_\kappa g + A_\kappa \overline{g} - A_\kappa^* g - \overline{A_\kappa^* g}\right\} = -\frac{\mathrm{i}\kappa}{2}\left\{\mathrm{Re}\left(A_\kappa^* g\right) - A_\kappa\left(\mathrm{Re}\,g\right)\right\};$$

in particular, by choosing $g = \overline{V_{n,\mathbf{n}}^{\kappa\mathcal{O}}}$ and recalling that $\kappa$ is now real, we get

$$B_\kappa \overline{V_{n,\mathbf{n}}^{\kappa\mathcal{O}}} = -\frac{\mathrm{i}\kappa}{2}\left\{\mathrm{Re}\left(\overline{V_n^{\kappa\mathcal{O}}}\big|_\Gamma\right) - A_\kappa\left(\mathrm{Re}\,\overline{V_{n,\mathbf{n}}^{\kappa\mathcal{O}}}\right)\right\} = -\frac{\mathrm{i}\kappa}{2}\left\{\mathrm{Reg}\,V_n^{\kappa\mathcal{O}}\big|_\Gamma - A_\kappa\,\mathrm{Reg}\,V_{n,\mathbf{n}}^{\kappa\mathcal{O}}\right\} = U_m^{\kappa\mathcal{O}}.$$

Finally, we recall that $B_\kappa$ is self-conjugate.

$(ii)$. Since $\left(V_{n,\mathbf{n}}^{\kappa\mathcal{O}}\right)_{n=1}^\infty$ and $\left(\overline{U_n^{\kappa\mathcal{O}}}\right)_{n=1}^\infty$ form a biorthonormal pair in $H^0(\Gamma)$, the orthonormality of $\left(B_\kappa^{\frac{1}{2}} V_{n,\mathbf{n}}^{\kappa\mathcal{O}}\right)_{n=1}^\infty$ in $H^0(\Gamma)$ follows directly from the self-adjointness of $B_\kappa^{\frac{1}{2}}$ and (6.1):

$$\left(B_\kappa^{\frac{1}{2}} V_{m,\mathbf{n}}^{\kappa\mathcal{O}}, B_\kappa^{\frac{1}{2}} V_{n,\mathbf{n}}^{\kappa\mathcal{O}}\right)_0 = \left(V_{m,\mathbf{n}}^{\kappa\mathcal{O}}, B_\kappa V_{n,\mathbf{n}}^{\kappa\mathcal{O}}\right)_0 = \left(V_{m,\mathbf{n}}^{\kappa\mathcal{O}}, \overline{U_n^{\kappa\mathcal{O}}}\right)_0 = \delta_{mn} \qquad for\ all \quad m,\,n.$$

The completeness of $\left(B_\kappa^{\frac{1}{2}} V_{n,\mathbf{n}}^{\kappa\mathcal{O}}\right)_{n=1}^\infty$ in $H^0(\Gamma)$ is an immediate consequence of the completeness of $\left(V_{n,\mathbf{n}}^{\kappa\mathcal{O}}\right)_{n=1}^\infty$ in $H^0(\Gamma)$ and the injectivity of $B_\kappa^{\frac{1}{2}}$. $\quad\square$

**Remark.** Generally, we shall use without comment results following by a simple conjugation argument from those explicitly proven. For example, it is easy to see that $\left(B_\kappa^{\frac{1}{2}} \overline{V_{n,\mathbf{n}}^{\kappa\mathcal{O}}}\right)_{n=1}^\infty$ is also an orthonormal basis for $H^0(\Gamma)$.

It is useful to verify that $B_\kappa^{\frac{1}{2}}$ inherits two more of the properties of $B_\kappa$:

**Lemma 6.1.** $(i.)$ $\mathcal{R}\big(B_\kappa^{\frac{1}{2}}\big) \subset \mathcal{R}\big(A_\kappa\big)$.

$(ii.)$ $B_\kappa^{\frac{1}{2}}$ is self-conjugate, i.e., $\overline{B_\kappa^{\frac{1}{2}}} = B_\kappa^{\frac{1}{2}}$. In particular, $\mathcal{R}\big(B_\kappa^{\frac{1}{2}}\big)$ is closed under conjugation.

*Proof*: $(i)$. Let $f = B_\kappa^{\frac{1}{2}} \widetilde{f} \in \mathcal{R}\big(B_\kappa^{\frac{1}{2}}\big)$. We shall show that the interior double-layer $W_\kappa^-\{f\}$ with density $f$ is the restriction to $\Omega_-$ of an entire solution of the Helmholtz equation, i.e., of a solution of $(0.1)$ in all of $\mathbb{R}^3$. From this it will follow that $W_\kappa^-\{f\}$ is $L_2$-regular at $\Gamma$, so $f \in \mathcal{D}(T_\kappa)$ and $T_\kappa f := W_\kappa^-\{f\}_{,\mathbf{n}}$ exists; by recalling the characterization $\mathcal{D}(T_\kappa) = \mathcal{R}\big(A_\kappa\big)$, we will finally be able to conclude that $f \in \mathcal{R}\big(A_\kappa\big)$, which will imply that $(i)$ is true. To check that our first assertion here is correct, we use the expansion of the fundamental solution-value $E_\mathbf{x}^\kappa(\mathbf{y})$ that is implied by $(3.3.2)$ for $\mathbf{x}$ lying in the inscribed ball $B_{R_\mathcal{O}^-}(\mathcal{O})$ for $\Omega_-$ centered at $\mathcal{O}$ and $\mathbf{y}$ lying outside the ball, along with the convergence properties of that expansion, to derive

$$W_\kappa^-\{f\}(\mathbf{x}) = -\mathrm{i}\kappa \sum_{n=1}^\infty \big(f, \overline{V_{n,\mathbf{n}}^{\kappa\mathcal{O}}}\big)_0 \mathrm{Reg}\, V_n^{\kappa\mathcal{O}}(\mathbf{x}) = -\mathrm{i}\kappa \sum_{n=1}^\infty \big(\widetilde{f}, B_\kappa^{\frac{1}{2}} \overline{V_{n,\mathbf{n}}^{\kappa\mathcal{O}}}\big)_0 \mathrm{Reg}\, V_n^{\kappa\mathcal{O}}(\mathbf{x}), \quad |\mathbf{x} - \mathcal{O}| < R_\mathcal{O}^-.$$

$$(6.2)$$

Now, since $\big(B_\kappa^{\frac{1}{2}} \overline{V_{n,\mathbf{n}}^{\kappa\mathcal{O}}}\big)_{n=1}^\infty$ is an orthonormal basis for $H^0(\Gamma)$, the sequence $\big(\big(\widetilde{f}, B_\kappa^{\frac{1}{2}} \overline{V_{n,\mathbf{n}}^{\kappa\mathcal{O}}}\big)_0\big)_{n=1}^\infty$ belongs to $\ell_2$; we shall show that this implies the convergence of the series in $(6.2)$ in all of $\mathbb{R}^3$ to a solution of $(0.1)$. To that end, we fix any positive $R$ and let $\Gamma_R$ denote the boundary $\partial B_R(\mathcal{O})$ of the ball of radius $R$ centered at $\mathcal{O}$; the normal derivative on $\Gamma_R$ of an appropriate function $u$ we indicate by $u_{,\mathbf{n}_R}$. Since it is easy to check that $|j_n(x)| \leq 1$ and $|j_n'(x)| \leq \frac{3}{2}$ for all $n \geq 0$ and $x \geq 0$, it follows that both of $f_R := -\mathrm{i}\kappa \sum_{n=1}^\infty \big(\widetilde{f}, B_\kappa^{\frac{1}{2}} \overline{V_{n,\mathbf{n}}^{\kappa\mathcal{O}}}\big)_0 \mathrm{Reg}\, V_n^{\kappa\mathcal{O}}\big|_{\Gamma_R}$ and $g_R := -\mathrm{i}\kappa \sum_{n=1}^\infty \big(\widetilde{f}, B_\kappa^{\frac{1}{2}} \overline{V_{n,\mathbf{n}}^{\kappa\mathcal{O}}}\big)_0 \mathrm{Reg}\, V_{n,\mathbf{n}_R}^{\kappa\mathcal{O}}$ converge in $L_2(\Gamma_R)$, since each is a series in the orthonormal basis $\big(R^{-1}\widehat{Y}_n\big)_{n=1}^\infty$ for $L_2(\Gamma_R)$ with coefficients in $\ell_2$. Therefore, we can construct a solution $w_R$ of the Helmholtz equation in the ball $B_R(\mathcal{O})$ by setting

$$w_R(\mathbf{x}) := -\frac{1}{2} \int_{\Gamma_R} \big\{ E_\mathbf{x}^\kappa g_R - E_{\mathbf{x},\mathbf{n}_R}^\kappa f_R \big\} \, d\lambda_{\Gamma_R} \qquad \text{for} \quad |\mathbf{x} - \mathcal{O}| < R.$$

Upon inserting the series representations for $f_R$ and $g_R$ and integrating term-by-term, we get

$$w_R(\mathbf{x}) = -\mathrm{i}\kappa \sum_{n=1}^\infty \big(\widetilde{f}, B_\kappa^{\frac{1}{2}} \overline{V_{n,\mathbf{n}}^{\kappa\mathcal{O}}}\big)_0 \Big\{ -\frac{1}{2} \int_{\Gamma_R} \big\{ E_\mathbf{x}^\kappa \mathrm{Reg}\, V_{n,\mathbf{n}_R}^{\kappa\mathcal{O}} - E_{\mathbf{x},\mathbf{n}_R}^\kappa \mathrm{Reg}\, V_n^{\kappa\mathcal{O}} \big\} \, d\lambda_{\Gamma_R} \Big\}, \quad |\mathbf{x} - \mathcal{O}| < R;$$

by using the counterpart of $(2.5)$ with $\Gamma_R$ replacing $\Gamma$, we recognize the expression within the large brackets as $\mathrm{Reg}\, V_n^{\kappa\mathcal{O}}(\mathbf{x})$, so the latter series coincides with that in $(6.2)$ if $|\mathbf{x} - \mathcal{O}| < R_\mathcal{O}^-$. Therefore, when $R$ is sufficiently large, $w_R$ provides an extension of $W_\kappa^-\{f\}$ to the ball $B_R(\mathcal{O})$. Since $R$ was arbitrary and $w_{R'}$ extends $w_R$ if $R' > R$, the proof of $(i)$ is complete.

$(ii)$. It is well known that $B_\kappa^{\frac{1}{2}}$ can be constructed as the strong limit in $\mathcal{B}\big(H^0(\Gamma)\big)$ of a recursively defined sequence of operators that are polynomials in $B_\kappa$; the details can be found in, e.g., [19]. An inspection of the development there reveals that the coefficients of the polynomial operators are all real. Since $B_\kappa$ is self-conjugate, it follows immediately from this observation that $B_\kappa^{\frac{1}{2}}$ possesses the same property. It is then clear that $f \in \mathcal{R}\big(B_\kappa^{\frac{1}{2}}\big)$ iff $\overline{f} \in \mathcal{R}\big(B_\kappa^{\frac{1}{2}}\big)$. $\quad\square$

The definition

$$(f, g)_-^\mathcal{N} := \big(B_\kappa^{\frac{1}{2}} f, B_\kappa^{\frac{1}{2}} g\big)_0, \qquad \text{for} \quad f, g \in L_2(\Gamma),$$

28

provides an inner product for $L_2(\Gamma)$. In view of the compactness of $B_\kappa$ in $H^0(\Gamma)$, it is clear that this inner product is strictly weaker than $(\cdot\,,\cdot)_0$ and, denoting by $H_{\mathcal{N}}^-(\Gamma)$ the completion of the pre-Hilbert space $\left(L_2(\Gamma),(\cdot\,,\cdot)_-^{\mathcal{N}}\right)$, that the natural injection of $H^0(\Gamma)$ into $H_{\mathcal{N}}^-(\Gamma)$ is compact. It is also useful to introduce the Hilbert space $H_{\mathcal{N}}^+(\Gamma)$ obtained by equipping $\mathcal{R}\!\left(B_\kappa^{\frac{1}{2}}\right)$ with the inner product given by

$$(f,g)_+^{\mathcal{N}} := \left(B_\kappa^{-\frac{1}{2}}f, B_\kappa^{-\frac{1}{2}}g\right)_0, \qquad \text{for} \quad f,\, g \in \mathcal{R}\!\left(B_\kappa^{\frac{1}{2}}\right);$$

$H_{\mathcal{N}}^-(\Gamma)$ can be identified as the antidual of $H_{\mathcal{N}}^+(\Gamma)$. It is easy to see that in these new structures we get

**Corollary 6.1.** $\left(V_{n;\mathbf{n}}^{\kappa\mathcal{O}}\right)_{n=1}^{\infty}$ *is an orthonormal basis for* $H_{\mathcal{N}}^-(\Gamma)$; $\left(\overline{U_n^{\kappa\mathcal{O}}}\right)_{n=1}^{\infty}$ *is an orthonormal basis for* $H_{\mathcal{N}}^+(\Gamma)$.

*Proof*: This follows directly from Proposition 6.1, the definitions of $H_{\mathcal{N}}^-(\Gamma)$ and $H_{\mathcal{N}}^+(\Gamma)$, and the completeness of $\left(V_{n;\mathbf{n}}^{\kappa\mathcal{O}}\right)_{n=1}^{\infty}$ in $H^0(\Gamma)$. $\square$

In passing, let us point out that we can use the latter statement in conjunction with relation $(3.2)_1$ to establish another basis property for a certain class of domains $\Omega_-$. Specifically, we have a condition sufficient to imply that the functions in $\mathcal{R}\!\left(B_\kappa^{\frac{1}{2}}\right)$ can be represented by $H_{\mathcal{N}}^+(\Gamma)$-convergent infinite-series expansions in the elements of the sequence $\left(\operatorname{Reg} V_n^{\kappa\mathcal{O}}\big|_\Gamma\right)_{n=1}^{\infty}$:

**Proposition 6.2.** *Suppose that* $\mathcal{R}\!\left(B_\kappa^{\frac{1}{2}}\right)$ *is invariant under* $D_\kappa$ *and the restriction* $D_\kappa \big| \mathcal{R}\!\left(B_\kappa^{\frac{1}{2}}\right)$ *is compact when regarded as acting in* $H_{\mathcal{N}}^+(\Gamma)$; *these conditions are fulfilled when* $\Gamma$ *is, for example, ellipsoidal. If* $\kappa^2$ *is not a Dirichlet eigenvalue for* $-\Delta$ *in* $\Omega_-$, *then* $\left(\operatorname{Reg} V_n^{\kappa\mathcal{O}}\big|_\Gamma\right)_{n=1}^{\infty}$ *is a Riesz basis for* $H_{\mathcal{N}}^+(\Gamma)$.

*Proof*: Assuming that $\kappa^2$ is not a Dirichlet eigenvalue for $-\Delta$ in $\Omega_-$, we know that $I + D_\kappa$ is injective in $H^0(\Gamma)$, so the restriction $(I + D_\kappa)\big|\mathcal{R}\!\left(B_\kappa^{\frac{1}{2}}\right)$ acting in $H_{\mathcal{N}}^+(\Gamma)$ is also injective, and is therefore an isomorphism of $H_{\mathcal{N}}^+(\Gamma)$, under the compactness hypothesis. Since $\left(U_n^{\kappa\mathcal{O}}\right)_{n=1}^{\infty}$ is clearly also an orthonormal basis for $H_{\mathcal{N}}^+(\Gamma)$, it now follows from $(3.2)_1$ that $\left(\operatorname{Reg} V_n^{\kappa\mathcal{O}}\big|_\Gamma\right)_{n=1}^{\infty}$ is a Riesz basis for $H_{\mathcal{N}}^+(\Gamma)$. Finally, to verify that the hypotheses hold when $\Gamma$ is ellipsoidal, consider the condition

$$\mathcal{R}\!\left(B_\kappa^{\frac{1}{2}}\right) \text{ is invariant under } D_\kappa \quad \text{and} \quad B_\kappa^{-\frac{1}{2}} D_\kappa B_\kappa^{\frac{1}{2}} \in \mathcal{B}\!\left(H^0(\Gamma)\right) \text{ is compact;} \qquad (\text{C.2})'$$

the designation $(\text{C.2})'$ is used here to maintain consistent notation with [6], where the condition is shown to be implied by the equality

$$D_\kappa B_\kappa = B_\kappa \overline{D_\kappa^*}. \tag{C.2.2}$$

In turn, (C.2.2) is shown in [6] to hold at least whenever $\Gamma$ is ellipsoidal. But now it is simple to check that $(\text{C.2})'$ implies the compactness of the restriction $D_\kappa \big| \mathcal{R}\!\left(B_\kappa^{\frac{1}{2}}\right)$ in $H_{\mathcal{N}}^+(\Gamma)$. Indeed, suppose that $(\text{C.2})'$ holds and let $(f_n)_{n=1}^{\infty}$ be a weakly null sequence from $H_{\mathcal{N}}^+(\Gamma)$, so that $\left(B_\kappa^{-\frac{1}{2}}f_n, B_\kappa^{-\frac{1}{2}}g\right)_0 \to 0$ for every $g \in \mathcal{R}\!\left(B_\kappa^{\frac{1}{2}}\right)$; it follows, in particular, that $\left(B_\kappa^{-\frac{1}{2}}f_n\right)_{n=1}^{\infty}$ is weakly null in $H^0(\Gamma)$. Therefore,

$$\left\|D_\kappa f_n\right\|_+^{\mathcal{N}} = \left\|B_\kappa^{-\frac{1}{2}} D_\kappa f_n\right\|_0 = \left\|\left(B_\kappa^{-\frac{1}{2}} D_\kappa B_\kappa^{\frac{1}{2}}\right) B_\kappa^{-\frac{1}{2}} f_n\right\|_0 \to 0 \qquad \text{as} \quad n \to \infty,$$

showing that $D_\kappa \big| \mathcal{R}\!\left(B_\kappa^{\frac{1}{2}}\right)$ is compact in $H_{\mathcal{N}}^+(\Gamma)$. (It is just as easy to verify the reversed implication, so that the two conditions are in fact equivalent.) $\square$

We can establish a connection between the inner product $(\cdot\,,\cdot)_-^{\mathcal{N}}$ and far-field patterns of elements of $W_+(\Omega_+;\kappa)$, although we do not know now whether the full completion $H_{\mathcal{N}}^-(\Gamma)$ has

an alternate characterization in terms of something that is more readily interpreted physically. Accordingly, we shall first quickly recall the basic facts about the far-field pattern of a radiating wave. Let $u$ be a solution of (0.1) in $\Omega_+$ that is also outgoing, $i.e.$, satisfies (1.1). Then, corresponding to the fixed point $\mathcal{O}$, there is a unique complex function $u_\infty^{\mathcal{O}}$ defined on the unit sphere $\Sigma_1$ of $\mathbb{R}^3$ such that

$$u\big(\mathcal{O} + \varrho\,\hat{\mathbf{e}}\big) = \frac{e^{i\kappa\varrho}}{\varrho} u_\infty^{\mathcal{O}}(\hat{\mathbf{e}}) + O\left(\frac{1}{\varrho^2}\right) \qquad \text{as} \quad \varrho \to \infty, \qquad \text{uniformly for} \quad \hat{\mathbf{e}} \in \Sigma_1.$$

$u_\infty^{\mathcal{O}}$ is termed *the far-field pattern of $u$ with respect to $\mathcal{O}$*; the pertinent developments can be found in, $e.g.$, [2]. We define an operator $\Phi_{\mathcal{N}}^{\kappa\mathcal{O}} : H^0(\Gamma) \to H^0(\Sigma_1)$, the *Neumann far-field-pattern operator with respect to $\mathcal{O}$*, according to

$$\Phi_{\mathcal{N}}^{\kappa\mathcal{O}} g := (\mathfrak{u}_g)_\infty^{\mathcal{O}} \qquad \text{for} \quad g \in H^0(\Gamma),$$

$i.e.$, $\Phi_{\mathcal{N}}^{\kappa\mathcal{O}}$ maps $g \in H^0(\Gamma)$ to the far-field pattern of the corresponding unique $\mathfrak{u}_g \in W_+(\Omega_+;\kappa)$ with Neumann data $g$. To get an integral representation for $\Phi_{\mathcal{N}}^{\kappa\mathcal{O}}$, we use the obvious formula

$$u_\infty^{\mathcal{O}}(\hat{\mathbf{e}}) = \lim_{\varrho \to \infty} \left(\frac{e^{i\kappa\varrho}}{\varrho}\right)^{-1} u\big(\mathcal{O} + \varrho\,\hat{\mathbf{e}}\big) \qquad \text{for each} \quad \hat{\mathbf{e}} \in \Sigma_1$$

in conjunction with the integral representation in (2.6) for the solution $\mathfrak{u}_g$ of the exterior Neumann/radiation problem with data $g \in H^0(\Gamma)$ to write

$$\Phi_{\mathcal{N}}^{\kappa\mathcal{O}} g(\hat{\mathbf{e}}) = -\frac{1}{4\pi} \int_\Gamma \left\{ \mathrm{e}_{\hat{\mathbf{e}}}^{\kappa\mathcal{O}} g - \mathrm{e}_{\hat{\mathbf{e}};\mathbf{n}}^{\kappa\mathcal{O}} A_\kappa g \right\} d\lambda_\Gamma = -\frac{1}{4\pi} \int_\Gamma \left\{ \mathrm{e}_{\hat{\mathbf{e}}}^{\kappa\mathcal{O}} - A_\kappa \mathrm{e}_{\hat{\mathbf{e}};\mathbf{n}}^{\kappa\mathcal{O}} \right\} g\, d\lambda_\Gamma \qquad \text{for each} \quad \hat{\mathbf{e}} \in \Sigma_1,$$

(6.3)

in which we have employed the notation $\mathrm{e}_{\hat{\mathbf{e}}}^{\kappa\mathcal{O}}$ for the complex amplitude of a certain plane wave propagating in the direction $-\hat{\mathbf{e}}$:

$$\mathrm{e}_{\hat{\mathbf{e}}}^{\kappa\mathcal{O}}(\mathbf{y}) := e^{-i\kappa\hat{\mathbf{e}}\cdot(\mathbf{y}-\mathcal{O})} \qquad \text{for} \quad \mathbf{y} \in \mathbb{R}^3, \quad \hat{\mathbf{e}} \in \Sigma_1.$$

From the first form in (6.3), it is clear that $\Phi_{\mathcal{N}}^{\kappa\mathcal{O}}$ is a compact operator. The well-known expansion of a plane wave in terms of spherical harmonics now takes the particular form

$$\mathrm{e}_{\hat{\mathbf{e}}}^{\kappa\mathcal{O}}(\mathbf{y}) = \frac{4\pi}{\sqrt{2}} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \mathrm{i}^{-l} \widehat{Y}_{lm}(\hat{\mathbf{e}}) \mathrm{Reg}\, V_{lm}^{\kappa\mathcal{O}}(\mathbf{y}), \qquad \text{for} \quad \mathbf{y} \in \mathbb{R}^3, \quad \hat{\mathbf{e}} \in \Sigma_1, \tag{6.4}$$

as one can check by using the addition theorems for the spherical Bessel functions and the Legendre functions, and accounting for the normalizing constants used here. The convergence properties of the series permit its insertion into (6.3) and the performance of operations term-by-term to get

$$\Phi_{\mathcal{N}}^{\kappa\mathcal{O}} g(\hat{\mathbf{e}}) = \frac{\sqrt{2}}{\kappa} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \big(g, \overline{U_{lm}^{\kappa\mathcal{O}}}\big)_0 \mathrm{i}^{-l-1} \widehat{Y}_{lm}(\hat{\mathbf{e}})$$

$$= \frac{\sqrt{2}}{\kappa} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \big(B_\kappa^{\frac{1}{2}} g, B_\kappa^{\frac{1}{2}} V_{lm;\mathbf{n}}^{\kappa\mathcal{O}}\big)_0 \mathrm{i}^{-l-1} \widehat{Y}_{lm}(\hat{\mathbf{e}})$$

$$= \frac{\sqrt{2}}{\kappa} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \big(g, V_{lm;\mathbf{n}}^{\kappa\mathcal{O}}\big)_-^{\mathcal{N}} \mathrm{i}^{-l-1} \widehat{Y}_{lm}(\hat{\mathbf{e}}) \qquad \text{for} \quad \hat{\mathbf{e}} \in \Sigma_1, \qquad g \in H^0(\Gamma). \tag{6.5}$$

Clearly, the second equality in (6.5) says that $\Phi_{\mathcal{N}}^{\kappa\mathcal{O}}$ has the factorization

$$\Phi_{\mathcal{N}}^{\kappa\mathcal{O}} = \frac{\sqrt{2}}{\kappa}\Psi_{\mathcal{N}}^{\kappa\mathcal{O}}B_{\kappa}^{\frac{1}{2}}, \tag{6.6}$$

in which the operator $\Psi_{\mathcal{N}}^{\kappa\mathcal{O}} : H^0(\Gamma) \to H^0(\Sigma_1)$ is defined by

$$\Psi_{\mathcal{N}}^{\kappa\mathcal{O}}g := \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(g, B_{\kappa}^{\frac{1}{2}}V_{lm;\mathbf{n}}^{\kappa\mathcal{O}}\right)_0 \mathrm{i}^{-l-1}\widehat{Y}_{lm} \qquad \text{for} \quad g \in H^0(\Gamma), \tag{6.7}$$

and is clearly unitary, *i.e.*, an isometric isomorphism, in view of the orthonormal bases appearing for $H^0(\Gamma)$ and $H^0(\Sigma_1)$. But then, since $(\sqrt{2}/\kappa)B_{\kappa}^{\frac{1}{2}}$ is self-adjoint, we recognize (6.6) as giving the polar decomposition of $\Phi_{\mathcal{N}}^{\kappa\mathcal{O}}$. We recap this development in

**Proposition 6.3.** *The polar decomposition of the Neumann far-field-pattern operator* $\Phi_{\mathcal{N}}^{\kappa\mathcal{O}}$ : $H^0(\Gamma) \to H^0(\Sigma_1)$ *is given by* (6.6), *in which the unitary operator* $\Psi_{\mathcal{N}}^{\kappa\mathcal{O}} : H^0(\Gamma) \to H^0(\Sigma_1)$ *is defined by* (6.7).

Meanwhile, although $(\kappa/\sqrt{2})\Phi_{\mathcal{N}}^{\kappa\mathcal{O}} : H^0(\Gamma) \to H^0(\Sigma_1)$ is compact and injective, with dense range, if we regard this operator instead as densely defined in $H_{\mathcal{N}}^-(\Gamma)$, *i.e.*, as $(\kappa/\sqrt{2})\Phi_{\mathcal{N}}^{\kappa\mathcal{O}} : \{L_2(\Gamma) \subset H_{\mathcal{N}}^-(\Gamma)\} \to H^0(\Sigma_1)$, then the third equality in (6.5) shows that in this setting we have an isometry with dense range, once again because of the orthonormal bases appearing (this time for $H_{\mathcal{N}}^-(\Gamma)$ and $H^0(\Sigma_1)$), and so the operator has extension to a unitary operator taking $H_{\mathcal{N}}^-(\Gamma)$ onto $H^0(\Sigma_1)$. That is, we compute from either (6.5) or (6.6)

$$\frac{\kappa^2}{2}\left(\Phi_{\mathcal{N}}^{\kappa\mathcal{O}}f, \Phi_{\mathcal{N}}^{\kappa\mathcal{O}}g\right)_{H^0(\Sigma_1)} = \left(f, g\right)_-^{\mathcal{N}} \qquad \text{for} \quad f \text{ and } g \in L_2(\Gamma),$$

which extends to hold for $f$ and $g \in H_{\mathcal{N}}^-(\Gamma)$. Immediately, we get an expression of a weak sort of continuous dependence on the data in the exterior Neumann/radiation problem, in the form of the following characterization of approximation of the far-field pattern in $H^0(\Sigma_1)$, but expressed in terms of approximation of the Neumann data on the boundary $\Gamma$:

**Proposition 6.4.** *Let* $g \in H^0(\Gamma)$. *A sequence* $(g_n)_{n=1}^{\infty}$ *from* $H^0(\Gamma)$ *converges to* $g$ *in the norm of* $H_{\mathcal{N}}^-(\Gamma)$ *iff the corresponding sequence* $\left((\mathfrak{u}_{g_n})_{\infty}^{\mathcal{O}} = \Phi_{\mathcal{N}}^{\kappa\mathcal{O}}g_n\right)_{n=1}^{\infty}$ *of far-field patterns converges to* $(\mathfrak{u}_g)_{\infty}^{\mathcal{O}}$ *in the norm of* $H^0(\Sigma_1)$.

**Remarks.** (1.) This characterization is used in [6] to show that, for a certain class of shapes including the ellipsoids, the far-field patterns of approximations generated from the second Waterman scheme will converge in $H^0(\Sigma_1)$ to that of the desired solution.

(2.) It is also easy to show that the sequence $\left((u_N^{\sigma})_{\infty}^{\mathcal{O}}\right)_{N=N_0}^{\infty}$ of far-field patterns, with each $u_N^{\sigma}$ as in (5.8), will converge to the far-field pattern of the scattered field $u^{\sigma}$ iff the sequence $\left((\sigma_n^N)_{n=1}^{N}\right)_{N=N_0}^{\infty}$ of coefficient sequences (each extended by zero) converges to the sequence $\left(\sigma_n := -(u_{;\mathbf{n}}^{\iota}, \overline{U_n^{\kappa\mathcal{O}}})_0\right)_{n=1}^{\infty}$ of expansion coefficients for $u^{\sigma}$ in the norm of $\ell_2$. A corresponding statement holds for the more general exterior Neumann/radiation problem with any data $g \in H^0(\Gamma)$.

**6.2. The Dirichlet setting.** The corresponding developments in the "Dirichlet setting" are more delicate. We begin by identifying the counterpart of the operator $B_{\kappa}$. Since $\mathcal{R}(A_{\kappa}^*) = \mathcal{R}(A_{\kappa})$, we can define an operator $C_{\kappa} : \{\mathcal{R}(A_{\kappa}) \subset H^0(\Gamma)\} \to H^0(\Gamma)$ densely in $H^0(\Gamma)$ by setting

$$C_{\kappa}f := \frac{\mathrm{i}\kappa}{4}\left(A_{\kappa}^{*-1} - A_{\kappa}^{-1}\right)f \qquad \text{for each} \quad f \in \mathcal{R}(A_{\kappa});$$

$\mathcal{R}(B_\kappa) \subset \mathcal{R}(A_\kappa)$, so it is clear that we can also write

$$C_\kappa f = A_\kappa^{*\,-1} B_\kappa A_\kappa^{-1} f = A_\kappa^{-1} B_\kappa A_\kappa^{*\,-1} f \qquad \text{for each} \quad f \in \mathcal{R}(A_\kappa).$$

In the following collection of first properties, we show that $C_\kappa$ is symmetric, while its adjoint $C_\kappa^*$ is also symmetric and defined on all of $H^0(\Gamma)$, and is therefore bounded and self-adjoint. That is, $C_\kappa$ is bounded; its bounded extension to all of $H^0(\Gamma)$ is its adjoint $C_\kappa^* = C_\kappa^{**}$, which is self-adjoint.

**Lemma 6.2.** (*i.*) *The operator $C_\kappa$ is injective, with range*

$$\mathcal{R}(C_\kappa) = \mathcal{R}(A_\kappa^{-1} B_\kappa) = \left\{ \, f \in H^0(\Gamma) \,\Big|\, A_\kappa f \in \mathcal{R}(B_\kappa) \, \right\}$$
$$= \mathcal{R}(A_\kappa^{*\,-1} B_\kappa) = \left\{ \, f \in H^0(\Gamma) \,\Big|\, A_\kappa^* f \in \mathcal{R}(B_\kappa) \, \right\}$$

*and inverse given by*

$$C_\kappa^{-1} f = A_\kappa^* B_\kappa^{-1} A_\kappa f = A_\kappa B_\kappa^{-1} A_\kappa^* f \qquad \text{for each} \quad f \in \mathcal{R}(C_\kappa).$$

(*ii.*) $C_\kappa$ *is symmetric, i.e.,* $C_\kappa \subset C_\kappa^*$.

(*iii.*) *The adjoint $C_\kappa^*$ is symmetric with domain $\mathcal{D}(C_\kappa^*) = H^0(\Gamma)$, so $C_\kappa^*$ is bounded and self-adjoint. In fact, the operators $A_\kappa^{-1} B_\kappa$, $A_\kappa^{*\,-1} B_\kappa$, $A_\kappa^{-1} B_\kappa^{\frac{1}{2}}$, and $A_\kappa^{*\,-1} B_\kappa^{\frac{1}{2}}$ are all defined on $H^0(\Gamma)$ and bounded, with $\left(A_\kappa^{-1} B_\kappa\right)^*$ and $\left(A_\kappa^{*\,-1} B_\kappa\right)^*$ mapping $H^0(\Gamma)$ into $\mathcal{R}(B_\kappa^{\frac{1}{2}}) \subset \mathcal{R}(A_\kappa)$, and*

$$C_\kappa^* = A_\kappa^{-1}\left(A_\kappa^{-1} B_\kappa\right)^* = \left(A_\kappa^{-1} B_\kappa^{\frac{1}{2}}\right)\left(A_\kappa^{-1} B_\kappa^{\frac{1}{2}}\right)^* = A_\kappa^{*\,-1}\left(A_\kappa^{*\,-1} B_\kappa\right)^* = \left(A_\kappa^{*\,-1} B_\kappa^{\frac{1}{2}}\right)\left(A_\kappa^{*\,-1} B_\kappa^{\frac{1}{2}}\right)^*.$$

(*iv.*) $C_\kappa$ *is bounded and essentially self-adjoint, i.e., its closure $C_\kappa^{**} = C_\kappa^*$, which is, in this case, its continuous extension to all of $H^0(\Gamma)$, is self-adjoint.*

*Proof*: (*i*). Either of the alternate characterizations $C_\kappa f = A_\kappa^{*\,-1} B_\kappa A_\kappa^{-1} f = A_\kappa^{-1} B_\kappa A_\kappa^{*\,-1} f$, $f \in \mathcal{R}(A_\kappa)$, shows that $C_\kappa$ is injective. Routine checking will verify the remaining assertions of (*i*).

(*ii*). Let $g \in \mathcal{D}(C_\kappa) = \mathcal{R}(A_\kappa)$; it is easy to see that $(C_\kappa f, g)_0 = (f, C_\kappa g)_0$ for every $f \in \mathcal{D}(C_\kappa)$, *i.e.*, that $g \in \mathcal{D}(C_\kappa^*)$ and $C_\kappa^* g = C_\kappa g$. Thus, $C_\kappa \subset C_\kappa^*$.

(*iii*). Since $\mathcal{R}(B_\kappa) \subset \mathcal{R}(B_\kappa^{\frac{1}{2}}) \subset \mathcal{R}(A_\kappa) = \mathcal{R}(A_\kappa^*)$, each of $A_\kappa^{-1} B_\kappa$, $A_\kappa^{*\,-1} B_\kappa$, $A_\kappa^{-1} B_\kappa^{\frac{1}{2}}$, and $A_\kappa^{*\,-1} B_\kappa^{\frac{1}{2}}$ is defined on all of $H^0(\Gamma)$; since each is also closed, it is in $\mathcal{B}(H^0(\Gamma))$. Therefore, we have $\left(A_\kappa^{-1} B_\kappa\right)^* = \left(A_\kappa^{-1} B_\kappa^{\frac{1}{2}} B_\kappa^{\frac{1}{2}}\right)^* = B_\kappa^{\frac{1}{2}}\left(A_\kappa^{-1} B_\kappa^{\frac{1}{2}}\right)^*$, with a similar result for $\left(A_\kappa^{*\,-1} B_\kappa\right)^*$, showing that each of these operators has range in $\mathcal{R}(B_\kappa^{\frac{1}{2}}) \subset \mathcal{R}(A_\kappa)$, which implies that both $A_\kappa^{-1}\left(A_\kappa^{-1} B_\kappa\right)^*$ and $A_\kappa^{*\,-1}\left(A_\kappa^{*\,-1} B_\kappa\right)^*$ are also in $\mathcal{B}(H^0(\Gamma))$. Moreover, we compute

$$\left(C_\kappa f, g\right)_0 = \left(A_\kappa^{-1} B_\kappa A_\kappa^{*\,-1} f, g\right)_0 = \left(A_\kappa^{*\,-1} f, (A_\kappa^{-1} B_\kappa)^* g\right)_0 = \left(f, A_\kappa^{-1}(A_\kappa^{-1} B_\kappa)^* g\right)_0$$

$$\text{whenever} \quad f \in \mathcal{D}(C_\kappa) = \mathcal{R}(A_\kappa) \quad \text{and} \quad g \in H^0(\Gamma),$$

showing that $C_\kappa^* = A_\kappa^{-1}\left(A_\kappa^{-1} B_\kappa\right)^* \in \mathcal{B}(H^0(\Gamma))$. Since

$$A_\kappa^{-1}\left(A_\kappa^{-1} B_\kappa\right)^* = A_\kappa^{-1}\left(A_\kappa^{-1} B_\kappa^{\frac{1}{2}} B_\kappa^{\frac{1}{2}}\right)^* = \left(A_\kappa^{-1} B_\kappa^{\frac{1}{2}}\right)\left(A_\kappa^{-1} B_\kappa^{\frac{1}{2}}\right)^*,$$

32

while the latter operator is clearly self-adjoint, we conclude that $C_\kappa^*$ is self-adjoint. Now the remaining statements either follow immediately or are proven analogously.

$(iv)$. These statements follow directly from $(ii)$ and $(iii)$. $\qquad\square$

Now we can show that the operator $C_\kappa^*$ performs in the Dirichlet setting the same rôle as $B_\kappa$ in the Neumann setting:

**Proposition 6.5.** $(i.)$ $C_\kappa^*$ maps each $V_n^{\kappa\mathcal{O}}\big|_\Gamma$ to the corresponding $\overline{W_n^{\kappa\mathcal{O}}}$:

$$C_\kappa^* V_n^{\kappa\mathcal{O}}\big|_\Gamma = \overline{W_n^{\kappa\mathcal{O}}}, \qquad \text{for} \quad n = 1,\,2,\,3,\,\ldots\,. \tag{6.8}$$

$(ii.)$ $C_\kappa^*$ is injective and positive, and so possesses a self-adjoint, injective square root $C_\kappa^{*\frac{1}{2}}$.

$(iii.)$ Both $C_\kappa^*$ and $C_\kappa^{*\frac{1}{2}}$ are self-conjugate; in particular, their ranges are closed under complex conjugation.

$(iv.)$ $\big(C_\kappa^{*\frac{1}{2}} V_n^{\kappa\mathcal{O}}\big|_\Gamma\big)_{n=1}^\infty$ is an orthonormal basis for $H^0(\Gamma)$.

*Proof*: $(i)$. This follows by direct computation. In fact, we can always write

$$C_\kappa^* V_n^{\kappa\mathcal{O}}\big|_\Gamma = C_\kappa V_n^{\kappa\mathcal{O}}\big|_\Gamma = \frac{\mathrm{i}\kappa}{4}\Big\{A_\kappa^{*-1} V_n^{\kappa\mathcal{O}}\big|_\Gamma - V_{n,\mathbf{n}}^{\kappa\mathcal{O}}\Big\}$$

$$= \frac{\mathrm{i}\kappa}{4}\Big\{A_\kappa^{*-1}\big(V_n^{\kappa\mathcal{O}}\big|_\Gamma + \overline{V_n^{\kappa\mathcal{O}}}\big|_\Gamma\big) - \big(V_{n,\mathbf{n}}^{\kappa\mathcal{O}} + \overline{V_{n,\mathbf{n}}^{\kappa\mathcal{O}}}\big)\Big\} = \frac{\mathrm{i}\kappa}{2}\Big\{A_\kappa^{*-1}\mathrm{Re}\,V_n^{\kappa\mathcal{O}}\big|_\Gamma - \mathrm{Re}\,V_{n,\mathbf{n}}^{\kappa\mathcal{O}}\Big\};$$

by recalling that $\kappa$ is real, we get

$$C_\kappa^* V_n^{\kappa\mathcal{O}}\big|_\Gamma = \frac{\mathrm{i}\kappa}{2}\Big\{A_\kappa^{*-1}\mathrm{Reg}\,V_n^{\kappa\mathcal{O}}\big|_\Gamma - \mathrm{Reg}\,V_{n,\mathbf{n}}^{\kappa\mathcal{O}}\Big\} = \overline{W_n^{\kappa\mathcal{O}}}, \qquad n = 1,\,2,\,3,\,\ldots\,.$$

$(ii)$. We just showed that the range $\mathcal{R}\big(C_\kappa^*\big)$ contains the span $\mathrm{sp}\,\big\{\overline{W_n^{\kappa\mathcal{O}}}\big\}_{n=1}^\infty$, which is dense in $H^0(\Gamma)$; since $C_\kappa^*$ is self-adjoint, it is therefore injective. Meanwhile, the nonnegativity of $C_\kappa^*$ follows directly from, say, its representation $C_\kappa^* = \big(A_\kappa^{-1}B_\kappa^{\frac{1}{2}}\big)\big(A_\kappa^{-1}B_\kappa^{\frac{1}{2}}\big)^*$, i.e., we have $\big(C_\kappa^* f, f\big)_0 \geq 0$ whenever $f \in H^0(\Gamma)$. Now the existence of the self-adjoint square root $C_\kappa^{*\frac{1}{2}}$ follows; since $C_\kappa^*$ is injective, $C_\kappa^{*\frac{1}{2}}$ has the same property. (Now it is also clear that $\big(C_\kappa^* f, f\big)_0 = 0$ only for $f = 0$.)

$(iii)$. The equality $\overline{C_\kappa^*} = C_\kappa^*$ follows directly from any of the representations given for $C_\kappa^*$ in Lemma 6.2, by recalling that $B_\kappa$ is self-conjugate and $A_\kappa^* = \overline{A_\kappa}$. Then the self-conjugacy of $C_\kappa^{*\frac{1}{2}}$ follows from that of $C_\kappa^*$ by the same reasoning used in the case of $B_\kappa$, in Lemma 6.1.$ii$.

$(iv)$. Statement $(i)$ implies that $\big(C_\kappa^{*\frac{1}{2}} V_n^{\kappa\mathcal{O}}\big|_\Gamma\big)_{n=1}^\infty$ is orthonormal in $H^0(\Gamma)$, while the completeness of the sequence in $H^0(\Gamma)$ follows from that of $\big(V_n^{\kappa\mathcal{O}}\big|_\Gamma\big)_{n=1}^\infty$ and the fact that $C_\kappa^{*\frac{1}{2}}$ is self-adjoint and injective. $\qquad\square$

Now we can summarize for the Dirichlet setting the constructions analogous to those already described for the far-field patterns in the Neumann setting. We begin by introducing the Hilbert spaces $H_{\mathrm{D}}^+(\Gamma)$ and $H_{\mathrm{D}}^-(\Gamma)$: $H_{\mathrm{D}}^+(\Gamma)$ is the range $\mathcal{R}\big(C_\kappa^{*\frac{1}{2}}\big)$ equipped with the inner product $(\cdot\,,\,\cdot)_+^{\mathrm{D}}$ defined by $(f,g)_+^{\mathrm{D}} := \big(C_\kappa^{*-\frac{1}{2}}f, C_\kappa^{*-\frac{1}{2}}g\big)_0$ for $f$ and $g \in \mathcal{R}\big(C_\kappa^{*\frac{1}{2}}\big)$; $H_{\mathrm{D}}^-(\Gamma)$ is the completion of $L_2(\Gamma)$ under the inner product $(\cdot\,,\,\cdot)_-^{\mathrm{D}}$ given by $(f,g)_-^{\mathrm{D}} := \big(C_\kappa^{*\frac{1}{2}}f, C_\kappa^{*\frac{1}{2}}g\big)_0$ for $f$ and $g \in L_2(\Gamma)$. One can show that $H_{\mathrm{D}}^-(\Gamma)$ is a realization of the antidual of $H_{\mathrm{D}}^+(\Gamma)$. Moreover, directly from Proposition 6.5 and the completeness of $\big(V_n^{\kappa\mathcal{O}}\big|_\Gamma\big)_{n=1}^\infty$ in $H^0(\Gamma)$ we get the counterpart of Corollary 6.1:

**Corollary 6.2.** $\left(V_n^{\kappa\mathcal{O}}\big|_\Gamma\right)_{n=1}^\infty$ is an orthonormal basis for $H_{\mathrm{D}}^-(\Gamma)$; $\left(\overline{W_n^{\kappa\mathcal{O}}}\right)_{n=1}^\infty$ is an orthonormal basis for $H_{\mathrm{D}}^+(\Gamma)$.

Corresponding to Proposition 6.2, we obtain a further basis property under appropriate circumstances, this time for $\left(\mathrm{Reg}\,V_{n,\mathbf{n}}^{\kappa\mathcal{O}}\right)_{n=1}^\infty$ in $H_{\mathrm{D}}^+(\Gamma)$, by combining Corollary 6.2 and $(3.2)_2$:

**Proposition 6.6.** Suppose that $\mathcal{R}\left(C_\kappa^{*\frac{1}{2}}\right)$ is invariant under $\overline{D_\kappa^*}$ and the restriction $\overline{D_\kappa^*}\big|\mathcal{R}\left(C_\kappa^{*\frac{1}{2}}\right)$ is compact when regarded as acting in $H_{\mathrm{D}}^+(\Gamma)$; these conditions are fulfilled when $\Gamma$ is, for example, ellipsoidal. If $\kappa^2$ is not a Neumann eigenvalue for $-\Delta$ in $\Omega_-$, then $\left(\mathrm{Reg}\,V_{n,\mathbf{n}}^{\kappa\mathcal{O}}\right)_{n=1}^\infty$ is a Riesz basis for $H_{\mathrm{D}}^+(\Gamma)$.

*Proof*: Suppose that $\kappa^2$ is not a Neumann eigenvalue for $-\Delta$ in $\Omega_-$, so that $I - \overline{D_\kappa^*}$ is injective in $H^0(\Gamma)$. The restriction $(I - \overline{D_\kappa^*})\big|\mathcal{R}\left(C_\kappa^{*\frac{1}{2}}\right)$ is therefore injective in $H_{\mathrm{D}}^+(\Gamma)$, and so gives an isomorphism of the latter space, in view of the compactness hypothesis on the restriction $\overline{D_\kappa^*}\big|\mathcal{R}\left(C_\kappa^{*\frac{1}{2}}\right)$. Now $(3.2)_2$ shows that $\left(\mathrm{Reg}\,V_{n,\mathbf{n}}^{\kappa\mathcal{O}}\right)_{n=1}^\infty$ is a Riesz basis for $H_{\mathrm{D}}^+(\Gamma)$.

Finally, just as in the proof of Proposition 6.2, one can check that the following condition implies that $\mathcal{R}\left(C_\kappa^{*\frac{1}{2}}\right)$ is invariant under $\overline{D_\kappa^*}$ and the restriction $\overline{D_\kappa^*}\big|\mathcal{R}\left(C_\kappa^{*\frac{1}{2}}\right)$ is compact in $H_{\mathrm{D}}^+(\Gamma)$:

$$\mathcal{R}\left(C_\kappa^{*\frac{1}{2}}\right) \text{ is invariant under } \overline{D_\kappa^*} \quad \text{and} \quad C_\kappa^{*-\frac{1}{2}}\overline{D_\kappa^*}C_\kappa^{*\frac{1}{2}} \in \mathcal{B}\left(H^0(\Gamma)\right) \text{ is compact} \qquad (\mathrm{C}.3)'$$

(in fact, the two are equivalent). We shall sketch a proof of the claim that $(\mathrm{C}.3)'$ holds whenever $\Gamma$ is an ellipsoid, which will complete the proof of the Proposition. We already cited the relation $(\mathrm{C}.2.2)$ as holding for ellipsoidal $\Gamma$; let us show that it implies the equality

$$\overline{D_\kappa^*}C_\kappa^* = C_\kappa^* D_\kappa. \qquad (\mathrm{C}.3.2)$$

Suppose then that $(\mathrm{C}.2.2)$ is true. Since we noted in Section 2 that we always have $D_\kappa A_\kappa = A_\kappa \overline{D_\kappa^*}$, $(\mathrm{C}.2.2)$ implies that now $D_\kappa A_\kappa^* = A_\kappa^* \overline{D_\kappa^*}$ must hold, as well, in view of the definition of $B_\kappa$. By exploiting these relations, we find

$$\overline{D_\kappa^*}C_\kappa^* f = \overline{D_\kappa^*}C_\kappa f = \overline{D_\kappa^*}A_\kappa^{-1}B_\kappa A_\kappa^{*-1}f$$

$$= A_\kappa^{-1}D_\kappa B_\kappa A_\kappa^{*-1}f = A_\kappa^{-1}B_\kappa \overline{D_\kappa^*}A_\kappa^{*-1}f$$

$$= A_\kappa^{-1}B_\kappa A_\kappa^{*-1}D_\kappa f = C_\kappa D_\kappa f = C_\kappa^* D_\kappa f \qquad \text{for} \quad f \in \mathcal{R}\left(A_\kappa\right);$$

this result then extends to hold on all of $H^0(\Gamma)$, *i.e.*, to give $(\mathrm{C}.3.2)$. (And the reversed implication can be proven in a similar manner, so that $(\mathrm{C}.2.2)$ and $(\mathrm{C}.3.2)$ are equivalent.) Now one can fill in the reasoning showing that $(\mathrm{C}.3.2)$ implies $(\mathrm{C}.3)'$, by following along parallel to the analogous argument given in [6] for the "Neumann case." That is, one first shows that $(\mathrm{C}.3)'$ is equivalent to the condition that $D_\kappa$ is compact when regarded as densely defined on $L_2(\Gamma) \subset H_{\mathrm{D}}^-(\Gamma)$ and mapping into $H_{\mathrm{D}}^-(\Gamma)$ and then appeals to results of LAX [14] to show that the latter condition is implied by $(\mathrm{C}.3.2)$. $\quad\square$

**Remark.** In [6] it is shown that condition $(\mathrm{C}.2.2)$ is equivalent to the symmetry

$$\left(V_{m,\mathbf{n}}^{\kappa\mathcal{O}}, \mathrm{Reg}\,V_n^{\kappa\mathcal{O}}\big|_\Gamma\right)_0 = \left(V_{n,\mathbf{n}}^{\kappa\mathcal{O}}, \mathrm{Reg}\,V_m^{\kappa\mathcal{O}}\big|_\Gamma\right)_0 \qquad \text{for all} \quad m \text{ and } n = 1,\,2,\,3,\,\ldots\,; \qquad (\mathrm{C}.2.2)'$$

a corresponding argument will show that condition $(\mathrm{C}.3.2)$ is equivalent to the symmetry

$$\left(V_m^{\kappa\mathcal{O}}\big|_\Gamma, \mathrm{Reg}\,V_{n,\mathbf{n}}^{\kappa\mathcal{O}}\right)_0 = \left(V_n^{\kappa\mathcal{O}}\big|_\Gamma, \mathrm{Reg}\,V_{m,\mathbf{n}}^{\kappa\mathcal{O}}\right)_0 \qquad \text{for all} \quad m \text{ and } n = 1,\,2,\,3,\,\ldots \qquad (\mathrm{C}.3.2)'$$

($cf.$ [21]).

Just as for the previous case, we define *the Dirichlet far-field-pattern operator* $\Phi_{\mathcal{D}}^{\kappa\mathcal{O}} : H^0(\Gamma) \to H^0(\Sigma_1)$ *with respect to* $\mathcal{O}$ by setting $\Phi_{\mathcal{D}}^{\kappa\mathcal{O}}g := \left(\mathfrak{v}_g\right)_{\infty}^{\mathcal{O}}$ for each $g \in H^0(\Gamma)$, with $\mathfrak{v}_g$ denoting, as usual, the solution of the exterior Dirichlet/radiation problem for the data $g$. From the expression given in (2.12), we compute $\left(\mathfrak{v}_g\right)_{\infty}^{\mathcal{O}}$ and then appeal to (2.11) of Lemma 2.1 to get an integral representation for the operator $\Phi_{\mathcal{D}}^{\kappa\mathcal{O}}$:

$$
\begin{aligned}
\Phi_{\mathcal{D}}^{\kappa\mathcal{O}}g(\hat{\mathbf{e}}) &= -\frac{1}{2\pi}\int_{\Gamma}\left\{\mathrm{e}_{\hat{\mathbf{e}}}^{\kappa\mathcal{O}} + \zeta\mathrm{e}_{\hat{\mathbf{e}};\mathbf{n}}^{\kappa\mathcal{O}}\right\}\left(S_{\kappa} - \zeta(I - D_{\kappa})\right)^{-1}g\,d\lambda_{\Gamma} \\
&= -\frac{1}{2\pi}\int_{\Gamma}\left\{\left(S_{\kappa} - \zeta(I - \overline{D_{\kappa}^{*}})\right)^{-1}\left\{\mathrm{e}_{\hat{\mathbf{e}}}^{\kappa\mathcal{O}}\big|_{\Gamma} + \zeta\mathrm{e}_{\hat{\mathbf{e}};\mathbf{n}}^{\kappa\mathcal{O}}\right\}\right\}g\,d\lambda_{\Gamma} \\
&= -\frac{1}{4\pi}\int_{\Gamma}\left\{A_{\kappa}^{-1}\mathrm{e}_{\hat{\mathbf{e}}}^{\kappa\mathcal{O}}\big|_{\Gamma} - \mathrm{e}_{\hat{\mathbf{e}};\mathbf{n}}^{\kappa\mathcal{O}}\right\}g\,d\lambda_{\Gamma} \qquad \text{for each}\quad \hat{\mathbf{e}} \in \Sigma_1;
\end{aligned}
\tag{6.9}
$$

from the first form in (6.9) it is clear that $\Phi_{\mathcal{D}}^{\kappa\mathcal{O}}$ is compact. Once again using the expansion (6.4) for $\mathrm{e}_{\hat{\mathbf{e}}}^{\kappa\mathcal{O}}$, we get

$$
\begin{aligned}
\Phi_{\mathcal{D}}^{\kappa\mathcal{O}}g(\hat{\mathbf{e}}) &= \frac{\sqrt{2}}{\kappa}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(g, \overline{W_{lm}^{\kappa\mathcal{O}}}\right)_0 \mathrm{i}^{-l-1}\widehat{Y}_{lm}(\hat{\mathbf{e}}) \\
&= \frac{\sqrt{2}}{\kappa}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(C_{\kappa}^{*\frac{1}{2}}g, C_{\kappa}^{*\frac{1}{2}}V_{lm}^{\kappa\mathcal{O}}\big|_{\Gamma}\right)_0 \mathrm{i}^{-l-1}\widehat{Y}_{lm}(\hat{\mathbf{e}}) \\
&= \frac{\sqrt{2}}{\kappa}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(g, V_{lm}^{\kappa\mathcal{O}}\big|_{\Gamma}\right)_{-}^{\mathrm{D}} \mathrm{i}^{-l-1}\widehat{Y}_{lm}(\hat{\mathbf{e}}) \qquad \text{for}\quad \hat{\mathbf{e}} \in \Sigma_1, \qquad g \in H^0(\Gamma).
\end{aligned}
\tag{6.10}
$$

The second equality in (6.10) yields the factorization

$$
\Phi_{\mathcal{D}}^{\kappa\mathcal{O}} = \frac{\sqrt{2}}{\kappa}\Psi_{\mathcal{D}}^{\kappa\mathcal{O}}C_{\kappa}^{*\frac{1}{2}},
\tag{6.11}
$$

with the isometric isomorphism $\Psi_{\mathcal{D}}^{\kappa\mathcal{O}} : H^0(\Gamma) \to H^0(\Sigma_1)$ defined by

$$
\Psi_{\mathcal{D}}^{\kappa\mathcal{O}}g := \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(g, C_{\kappa}^{*\frac{1}{2}}V_{lm}^{\kappa\mathcal{O}}\big|_{\Gamma}\right)_0 \mathrm{i}^{-l-1}\widehat{Y}_{lm} \qquad \text{for}\quad g \in H^0(\Gamma).
\tag{6.12}
$$

Clearly, (6.11) is the polar decomposition of $\Phi_{\mathcal{D}}^{\kappa\mathcal{O}}$:

**Proposition 6.7.** *Equality* (6.11) *gives the polar decomposition of the Dirichlet far-field-pattern operator* $\Phi_{\mathcal{D}}^{\kappa\mathcal{O}} : H^0(\Gamma) \to H^0(\Sigma_1)$, *with the unitary operator* $\Psi_{\mathcal{D}}^{\kappa\mathcal{O}} : H^0(\Gamma) \to H^0(\Sigma_1)$ *defined by* (6.12).

Earlier, we showed directly only that $C_{\kappa}^{*}$ is bounded, but now from (6.11) and the compactness of $\Phi_{\mathcal{D}}^{\kappa\mathcal{O}}$ we infer:

**Corollary 6.3.** $C_{\kappa}^{*}$ *is compact in* $H^0(\Gamma)$.

Meanwhile, if we regard the operator $(\kappa/\sqrt{2})\Phi_{\mathcal{D}}^{\kappa\mathcal{O}}$ as $(\kappa/\sqrt{2})\Phi_{\mathcal{D}}^{\kappa\mathcal{O}} : \left\{L_2(\Gamma) \subset H_{\mathrm{D}}^{-}(\Gamma)\right\} \to H^0(\Sigma_1)$, *i.e.*, as densely defined in $H_{\mathrm{D}}^{-}(\Gamma)$, then the third equality in (6.10) shows that it is an

isometry with dense range, and so has extension to a unitary operator taking $H_{\mathrm{D}}^-(\Gamma)$ onto $H^0(\Sigma_1)$. That is, we have

$$\frac{\kappa^2}{2}\big(\Phi_{\mathcal{D}}^{\kappa\mathcal{O}}f, \Phi_{\mathcal{D}}^{\kappa\mathcal{O}}g\big)_{H^0(\Sigma_1)} = \big(f,g\big)_-^{\mathrm{D}} \qquad \text{for} \quad f \text{ and } g \in L_2(\Gamma),$$

which extends to hold for $f$ and $g \in H_{\mathrm{D}}^-(\Gamma)$. Consequently, we obtain a characterization of approximation of the far-field pattern in $H^0(\Sigma_1)$, expressed in terms of approximation of the Dirichlet data on the boundary $\Gamma$:

**Proposition 6.8.** *Let* $g \in H^0(\Gamma)$. *A sequence* $\big(g_n\big)_{n=1}^{\infty}$ *from* $H^0(\Gamma)$ *converges to* $g$ *in the norm of* $H_{\mathrm{D}}^-(\Gamma)$ *iff the corresponding sequence* $\big((\mathfrak{v}_{g_n})_\infty^{\mathcal{O}} = \Phi_{\mathcal{D}}^{\kappa\mathcal{O}}g_n\big)_{n=1}^{\infty}$ *of far-field patterns converges to* $(\mathfrak{v}_g)_\infty^{\mathcal{O}}$ *in the norm of* $H^0(\Sigma_1)$.

## 7. An application: establishing "$\mathcal{Q}\mathcal{T} = -\operatorname{Re}\mathcal{Q}$."

In Section 5, we remarked on the first algorithm proposed by WATERMAN [21]; here, we comment on the second scheme suggested there. We continue to suppose that $\kappa$ is now real and positive. For the most part we restrict attention to the problem of acoustic scattering by a hard obstacle, so that we deal with the Neumann/radiation problem in $\Omega_+$ with special data of the form $g = -u^\iota{}_{,\mathbf{n}}$, where the incident field $u^\iota$ is again assumed to be $(\Omega_-, \mathcal{O})$-regular, *i.e.*, is a solution of (0.1) in a ball $B_{R_\iota}(\mathcal{O})$ with $R_\iota > R_\mathcal{O}^+$.

According to (5.11), the first Waterman algorithm proposes the formation of an $N \times N$ $N^{\text{th}}$ approximate $T$-matrix $\mathcal{T}_N^{\kappa\mathcal{O}}$ satisfying

$$\mathcal{Q}_N^{\kappa\mathcal{O}}\big(\mathcal{T}_N^{\kappa\mathcal{O}}\big)^{\mathsf{T}} = -\operatorname{Re}\mathcal{Q}_N^{\kappa\mathcal{O}}, \qquad \text{for all sufficiently large } N, \tag{7.1}$$

in which the elements of the $N \times N$ matrix $\mathcal{Q}_N^{\kappa\mathcal{O}}$ are given in (5.9). The *upshot* of the further heuristic argument in [21] is that, if (7.1) yields a sequence converging to the transition matrix, then so also should the prescription

$$\mathcal{Q}_N^{\kappa\mathcal{O}}\widetilde{\mathcal{T}}_N^{\kappa\mathcal{O}} = -\operatorname{Re}\mathcal{Q}_N^{\kappa\mathcal{O}}, \qquad \text{for all sufficiently large } N, \tag{7.2.1}$$

since it is well known that the actual transition matrix $\mathcal{T}^{\kappa\mathcal{O}}$ is symmetric; this symmetry can be easily verified directly from the definition given in Section 5, by using Green's Theorem and the property $A_\kappa^* = \overline{A}_\kappa$:

$$\mathcal{T}_{nm}^{\kappa\mathcal{O}} := -\big(\operatorname{Reg}V_{m,\mathbf{n}}^{\kappa\mathcal{O}}, \overline{U_n^{\kappa\mathcal{O}}}\big)_0 = \frac{\mathrm{i}\kappa}{2}\int_\Gamma \operatorname{Reg}V_{m,\mathbf{n}}^{\kappa\mathcal{O}}\Big\{\operatorname{Reg}V_n^{\kappa\mathcal{O}}|_\Gamma - A_\kappa \operatorname{Reg}V_{n,\mathbf{n}}^{\kappa\mathcal{O}}\Big\}\,d\lambda_\Gamma$$

$$= \frac{\mathrm{i}\kappa}{2}\int_\Gamma \Big\{\operatorname{Reg}V_m^{\kappa\mathcal{O}}|_\Gamma \operatorname{Reg}V_{n,\mathbf{n}}^{\kappa\mathcal{O}} - \big(A_\kappa \operatorname{Reg}V_{m,\mathbf{n}}^{\kappa\mathcal{O}}\big)\operatorname{Reg}V_{n,\mathbf{n}}^{\kappa\mathcal{O}}\Big\}\,d\lambda_\Gamma = -\big(\operatorname{Reg}V_{n,\mathbf{n}}^{\kappa\mathcal{O}}, \overline{U_m^{\kappa\mathcal{O}}}\big)_0 = \mathcal{T}_{mn}^{\kappa\mathcal{O}}.$$

More explicitly, the equality in (7.2.1) is

$$\sum_{j=1}^N \big(\operatorname{Reg}V_m^{\kappa\mathcal{O}}\big|_\Gamma, \overline{V_{j,\mathbf{n}}^{\kappa\mathcal{O}}}\big)_0 \big(\widetilde{\mathcal{T}}_N^{\kappa\mathcal{O}}\big)_{jn} = -\big(\operatorname{Reg}V_m^{\kappa\mathcal{O}}\big|_\Gamma, \operatorname{Reg}V_{n,\mathbf{n}}^{\kappa\mathcal{O}}\big)_0, \qquad \text{for} \quad m, n = 1, \ldots, N. \tag{7.2.2}$$

Thus, based upon (7.2), the two steps of the second Waterman algorithm for the acoustic hard-scattering problem, with $u^\sigma$ denoting the scattered field corresponding to the $(\Omega_-, \mathcal{O})$-regular incident field, comprise

**(W.II.1)** **establish viability**: this step coincides with (W.I.1);

**(W.II.2)** **establish convergence**: show that the sequence $\big(\tilde{u}_N^\sigma\big)_{N=N_0}^\infty$ constructed by

$$\tilde{u}_N^\sigma(\mathbf{x}) := \sum_{m=1}^N \tilde{\sigma}_n^N V_n^{\kappa\mathcal{O}}(\mathbf{x}), \qquad \text{for} \quad \mathbf{x} \in \Omega_+, \qquad \text{for} \quad N > N_0, \tag{7.3}$$

in which the coefficients $\big(\tilde{\sigma}_n^N\big)_{n=1}^N$ are obtained from

$$\tilde{\sigma}_n^N := \sum_{m=1}^N \big(\widetilde{\mathcal{T}}_N^{\kappa\mathcal{O}}\big)_{nm}\iota_m, \qquad \text{for} \quad n = 1, \ldots, N \tag{7.4}$$

(the incident-field expansion coefficients being as in (5.3)), with $\widetilde{\mathcal{T}}_N^{\kappa\mathcal{O}}$ determined by (7.2), converges in some manner to the unique solution $u^\sigma$.

Of course, when the scheme is viable, the coefficients $\left(\tilde{\sigma}_n^N\right)_{n=1}^N$ in (7.4) can be determined directly from

$$\sum_{n=1}^N \left(\mathrm{Reg}\, V_m^{\kappa\mathcal{O}}\big|_\Gamma, \overline{V_{n,\mathbf{n}}^{\kappa\mathcal{O}}}\right)_0 \tilde{\sigma}_n^N = -\sum_{n=1}^N \left(\mathrm{Reg}\, V_m^{\kappa\mathcal{O}}\big|_\Gamma, \mathrm{Reg}\, V_{n,\mathbf{n}}^{\kappa\mathcal{O}}\right)_0 \iota_n, \qquad \text{for} \quad m,\, n = 1,\, \ldots,\, N.$$

We already indicated, in Remark (1) following Proposition 6.4, the extent to which the scheme (W.II) is justified in [6].

Evidently, the heuristic motivation given in [21] for this second algorithm seems to derive from a claim that the actual transition matrix $\mathcal{T}^{\kappa\mathcal{O}}$ satisfies the infinite system

$$\sum_{j=1}^\infty \left(\mathrm{Reg}\, V_m^{\kappa\mathcal{O}}\big|_\Gamma, \overline{V_{j,\mathbf{n}}^{\kappa\mathcal{O}}}\right)_0 \mathcal{T}_{jn}^{\kappa\mathcal{O}} = -\left(\mathrm{Reg}\, V_m^{\kappa\mathcal{O}}\big|_\Gamma, \mathrm{Reg}\, V_{n,\mathbf{n}}^{\kappa\mathcal{O}}\right)_0, \qquad \text{for} \quad m,\, n = 1,\, 2,\, \ldots,$$

*i.e.*, recalling the expressions for the $\mathcal{T}_{jn}^{\kappa\mathcal{O}}$, that the relations

$$\sum_{j=1}^\infty \left(\mathrm{Reg}\, V_m^{\kappa\mathcal{O}}\big|_\Gamma, \overline{V_{j,\mathbf{n}}^{\kappa\mathcal{O}}}\right)_0 \left(-\mathrm{Reg}\, V_{n,\mathbf{n}}^{\kappa\mathcal{O}}, \overline{U_j^{\kappa\mathcal{O}}}\right)_0 = -\left(\mathrm{Reg}\, V_m^{\kappa\mathcal{O}}\big|_\Gamma, \mathrm{Reg}\, V_{n,\mathbf{n}}^{\kappa\mathcal{O}}\right)_0, \qquad \text{for} \quad m,\, n = 1,\, 2,\, \ldots,$$

(7.5)

hold for the current $\Omega_-$ and $\kappa$. When this is the case, (7.2) and the succeeding recipe can be viewed as a *formal* application of the classical *abscnittsmethode* for approximate solution of an infinite system of linear equations in a space of sequences (or infinite matrices); *cf.*, *e.g.*, HELLINGER AND TOEPLITZ [7] or KANTOROVICH AND KRYLOV [10] (in the latter work, the term "reduction of order" is used). Here, we consider merely the validity of (7.5), in particular, showing that it holds when the boundary $\Omega_-$ is an ellipsoid. The "argument" of [21] purporting to establish (7.5) is completely formal and constitutes no verification, so it is rather surprising to find that (7.5) does turn out to be true in at least some cases. Nevertheless, the relation has been subsequently cited without question to such an extent that the impression of its general validity is probably now indelible.

To be sure, the question of the validity of (7.5) may be entirely peripheral to a justification of the second Waterman scheme in some cases. That is, even if (7.5) fails for a certain geometry, one can still study the steps (W.II.1) and (W.II.2) in their own right; the correctness of (7.5) for some $\Omega_-$ just signals that one *might* approach the examination of the second Waterman scheme in that instance as an application of the *abscnittsmethode*, and so constitutes just a small step in a full program. In fact, the crucial issues would then hinge on the properties of the operator between appropriate sequence spaces that is induced by the infinite matrix $\mathcal{Q}^{\kappa\mathcal{O}}$. Let us just note here without proof that one can carry through such an approach successfully for a class of boundaries $\Gamma$ that includes ellipsoids, at least when $\kappa^2$ is not an interior Dirichlet eigenvalue for $-\Delta$, since then $\mathcal{Q}^{\kappa\mathcal{O}}$ induces an isomorphism in the sequence space $\ell_2$ admitting application of a Bubnov-Galerkin scheme that coincides with the *abscnittsmethode*. We omit this development, since it appears to afford no results beyond those given in [6].

**Proposition 7.1.** *The relation* (7.5) *holds whenever the collection* $\left(\mathrm{Reg}\, V_n^{\kappa\mathcal{O}}\big|_\Gamma\right)_{n=1}^\infty$ *of traces belongs to the range* $\mathcal{R}\left(B_\kappa^{\frac{1}{2}}\right)$ *of the square root of the operator* $B_\kappa$. *This condition obtains when* $\mathcal{R}\left(B_\kappa^{\frac{1}{2}}\right)$ *is invariant under* $D_\kappa$, *and so holds, in particular, when the boundary* $\Gamma$ *is an ellipsoid.*

38

*Proof*: Since the sequence $\big(B_\kappa^{\frac{1}{2}} V_{n,\mathbf{n}}^{\kappa\mathcal{O}}\big)_{n=1}^\infty$ is an orthonormal basis for $H^0(\Gamma)$, it is clear that

$$(f,g)_0 = \sum_{j=1}^\infty \big(f, B_\kappa^{\frac{1}{2}} V_{j,\mathbf{n}}^{\kappa\mathcal{O}}\big)_0 \big(B_\kappa^{\frac{1}{2}} V_{j,\mathbf{n}}^{\kappa\mathcal{O}}, g\big)_0 \qquad \text{whenever} \quad f,\, g \in H^0(\Gamma).$$

If we also have $g \in \mathcal{R}\big(B_\kappa^{\frac{1}{2}}\big)$, it follows that

$$(f,g)_0 = \big(B_\kappa^{\frac{1}{2}} f, B_\kappa^{-\frac{1}{2}} g\big)_0 = \sum_{j=1}^\infty \big(B_\kappa^{\frac{1}{2}} f, B_\kappa^{\frac{1}{2}} V_{j,\mathbf{n}}^{\kappa\mathcal{O}}\big)_0 \big(B_\kappa^{\frac{1}{2}} V_{j,\mathbf{n}}^{\kappa\mathcal{O}}, B_\kappa^{-\frac{1}{2}} g\big)_0$$

$$= \sum_{j=1}^\infty \big(f, \overline{U_j^{\kappa\mathcal{O}}}\big)_0 \big(V_{j,\mathbf{n}}^{\kappa\mathcal{O}}, g\big)_0, \qquad \text{for} \quad f \in H^0(\Gamma), \quad g \in \mathcal{R}\big(B_\kappa^{\frac{1}{2}}\big). \tag{7.6}$$

Now, when we know that each $\operatorname{Reg} V_m^{\kappa\mathcal{O}}\big|_\Gamma$ belongs to $\mathcal{R}\big(B_\kappa^{\frac{1}{2}}\big)$, we can take $g = \operatorname{Reg} V_m^{\kappa\mathcal{O}}\big|_\Gamma$ and $f = -\operatorname{Reg} V_{n,\mathbf{n}}^{\kappa\mathcal{O}}$ in (7.6) to get (7.5).

Further, the inclusion $\big\{\operatorname{Reg} V_n^{\kappa\mathcal{O}}\big|_\Gamma\big\}_{n=1}^\infty \subset \mathcal{R}\big(B_\kappa^{\frac{1}{2}}\big)$ clearly follows from $(3.2)_1$ in case $\mathcal{R}\big(B_\kappa^{\frac{1}{2}}\big)$ is invariant under $D_\kappa$. Finally, we already indicated that (C.2.2) implies that $D_\kappa$ maps $\mathcal{R}\big(B_\kappa^{\frac{1}{2}}\big)$ into itself and holds when $\Gamma := \partial\Omega_-$ is an ellipsoid. $\quad\square$

The counterpart to (7.5) for the acoustically soft obstacle, *i.e.*, for the "Dirichlet setting," is

$$\sum_{j=1}^\infty \big(\operatorname{Reg} V_{m,\mathbf{n}}^{\kappa\mathcal{O}}, \overline{V_j^{\kappa\mathcal{O}}}\big|_\Gamma\big)_0 \big(-\operatorname{Reg} V_n^{\kappa\mathcal{O}}\big|_\Gamma, \overline{W_j^{\kappa\mathcal{O}}}\big)_0 = -\big(\operatorname{Reg} V_{m,\mathbf{n}}^{\kappa\mathcal{O}}, \operatorname{Reg} V_n^{\kappa\mathcal{O}}\big|_\Gamma\big)_0, \qquad m,\, n = 1,\, 2,\, \dots,$$

$$\tag{7.7}$$

which we can establish under an analogous set of conditions:

**Proposition 7.2.** *The relation* (7.7) *holds whenever the collection* $\big(\operatorname{Reg} V_{n,\mathbf{n}}^{\kappa\mathcal{O}}\big)_{n=1}^\infty$ *of normal derivatives belongs to the range* $\mathcal{R}\big(C_\kappa^{*\frac{1}{2}}\big)$ *of the square root of the operator* $C_\kappa^*$. *This condition obtains when* $\mathcal{R}\big(C_\kappa^{*\frac{1}{2}}\big)$ *is invariant under* $\overline{D_\kappa^*}$, *and so holds, in particular, when the boundary* $\Gamma$ *is an ellipsoid.*

*Proof*: The proof is constructed by following along the argument in the proof of Proposition 7.1, *mutatis mutandis*. By recalling that the sequence $\big(C_\kappa^{*\frac{1}{2}} V_n^{\kappa\mathcal{O}}\big|_\Gamma\big)_{n=1}^\infty$ is an orthonormal basis for $H^0(\Gamma)$, we conclude that

$$(f,g)_0 = \big(C_\kappa^{*\frac{1}{2}} f, C_\kappa^{*-\frac{1}{2}} g\big)_0 = \sum_{j=1}^\infty \big(C_\kappa^{*\frac{1}{2}} f, C_\kappa^{*\frac{1}{2}} V_j^{\kappa\mathcal{O}}\big|_\Gamma\big)_0 \big(C_\kappa^{*\frac{1}{2}} V_j^{\kappa\mathcal{O}}\big|_\Gamma, C_\kappa^{*-\frac{1}{2}} g\big)_0$$

$$= \sum_{j=1}^\infty \big(f, \overline{W_j^{\kappa\mathcal{O}}}\big)_0 \big(V_j^{\kappa\mathcal{O}}\big|_\Gamma, g\big)_0, \qquad \text{for} \quad f \in H^0(\Gamma), \quad g \in \mathcal{R}\big(C_\kappa^{*\frac{1}{2}}\big).$$

If each $\operatorname{Reg} V_{m,\mathbf{n}}^{\kappa\mathcal{O}}$ belongs to $\mathcal{R}\big(C_\kappa^{*\frac{1}{2}}\big)$, we can take $g = \operatorname{Reg} V_{m,\mathbf{n}}^{\kappa\mathcal{O}}$ and $f = -\operatorname{Reg} V_n^{\kappa\mathcal{O}}\big|_\Gamma$ in the latter equality to get (7.7).

The inclusion $\big\{\operatorname{Reg} V_{n,\mathbf{n}}^{\kappa\mathcal{O}}\big\}_{n=1}^\infty \subset \mathcal{R}\big(C_\kappa^{*\frac{1}{2}}\big)$ is implied by $(3.2)_2$ if $\mathcal{R}\big(C_\kappa^{*\frac{1}{2}}\big)$ is invariant under $\overline{D_\kappa^*}$. But we already indicated that (C.3.2) implies that $\overline{D_\kappa^*}$ maps $\mathcal{R}\big(C_\kappa^{*\frac{1}{2}}\big)$ into itself and holds when $\Gamma$ is an ellipsoid. $\quad\square$

Finally, we should also point out that the validity of an appropriate form of the so-called "Rayleigh hypothesis" for the given $\Omega_-$ will imply (7.5) and (7.7). To explain this just for the

Neumann setting, suppose that $\Omega_-$ has the property that the expansion (5.2) of the scattered field in fact converges outside the *inscribed* ball $B_{R_{\mathcal{O}}^-}(\mathcal{O})$ whenever the incident field $u^\iota$ is one of the regular solutions $\mathrm{Reg}\,V_n^{\kappa\mathcal{O}}$; evidently, assuming that the term-by-term operations are justified, we should then also find

$$-\mathrm{Reg}\,V_{n,\mathbf{n}}^{\kappa\mathcal{O}} = \sum_{j=1}^\infty \left(-\mathrm{Reg}\,V_{n,\mathbf{n}}^{\kappa\mathcal{O}}, \overline{U_j^{\kappa\mathcal{O}}}\right)_0 V_{j,\mathbf{n}}^{\kappa\mathcal{O}}, \qquad \text{for} \quad n = 1,\,2,\,3,\,\dots\,.$$

Of course, we already know the latter expansions to hold in a very weak sense, as the generalized Fourier expansions of the $-\mathrm{Reg}\,V_{n,\mathbf{n}}^{\kappa\mathcal{O}}$ in the orthonormal basis $\left(V_{n,\mathbf{n}}^{\kappa\mathcal{O}}\right)_{n=1}^\infty$ for $H_{\mathcal{N}}^-(\Gamma)$; if the convergence is actually in $H^0(\Gamma)$, the relations (7.5) clearly follow from this alternate argument. We can sketch an analogous derivation for (7.7). However, it is apparently not known whether there exist any $\Omega_-$ for which the "Rayleigh hypothesis" in the form used here actually obtains.

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