Variational Methods for Boundary Integral Equations: *Theory and Applications*

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Abstract

Variational methods for boundary integral equations deal with the weak formulations of boundary integral equations. Their numerical discretizations are known as the boundary element methods. The later has become one of the most popular numerical schemes in recent years. In this expository paper, we discuss some of the essential features of the methods, their intimate relations with the variational formulations of the corresponding partial differential equations and recent developments with respect to applications in domain composition from both mathematical and numerical points of view.

1 Variational Formulations for Boundary Integral Equations

It is well known that the reduction of boundary value problems to integral equations is by no means a unique process. In spite of many formulations, there are two kinds of boundary integral equations, the first and the second kind boundary integral equations either from the direct or indirect approach. The variational formulations of boundary integral equations in general may depend upon the types of boundary integral equations under consideration. In the following we shall systematically discuss these formulations through simple model problems and present some of the basic theorems and indicate typical ways for obtaining these results which may be employed in general for the variational methods concerning boundary integral equations.

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1.1 Boundary integral equations of the first kind

We begin with the integral equations of the first kind. Our formulation was motivated from the so-called *Fichera method* ([13]). It is a general method of *single–layer potential* introduced by Fichera and his co-workers for treating Dirichlet problems for a large class of elliptic equations of higher order with variable coefficients in the plane (see [4], [5], [23], [6], [1]). To illustrate the idea, and recall some of the basics for the method, let us consider the simplest model problem for the Laplacian in $I\!R^2$

$$\Delta u = 0 \quad \text{in} \quad \Omega, \quad u|_\Gamma = f \quad \text{on} \quad \Gamma. \tag{1}$$

Here Ω is a bounded domain with smooth boundary Γ, and f is the given boundary data satisfying certain regularity condition to be specified. We seek a solution of (1) in the form of single-layer potential

$$u(x) = \int_\Gamma \gamma(x,y)\sigma(y)\, ds_y\ , \quad x \in \Omega \tag{2}$$

where

$$\gamma(x,y) := -\frac{1}{2\pi} log\,|x-y| \tag{3}$$

is the fundamental solution for the Laplace equation $-\Delta u = 0$ and σ is the unknown density function. Then the boundary condition in (1) leads to the integral equation of the first kind for σ

$$V\sigma = f \ \text{ on } \Gamma, \tag{4}$$

where V is the boundary integral operator defined by

$$V\sigma(x) := \int_\Gamma \gamma(x,y)\sigma(y)\, ds_y\ , \quad x \in \Gamma. \tag{5}$$

For smooth f, say $f \in C^{1,\alpha}(\Gamma)$, one may differentiate (4) with respect to arc length along Γ and obtain the integral equation

$$\int_\Gamma \frac{\partial \gamma}{\partial s_x}(x,y)\sigma(y)\, ds_y = \frac{d}{ds_x} f(x) \quad \text{on} \quad \Gamma. \tag{6}$$

This is a singular integral equation with Cauchy-type kernel. Since this is of the first kind, it has index zero in the terminology of [22]. Hence the Fredholm alternative holds. This singular equation then forms the theoretical basis for Fichera's method. However, not only are Cauchy kernels hard to handle numerically, but they produce certain non-uniqueness which must be accounted for. This leads to the following modified Fichera's method. Instead of (2), we now use the representation

$$u(x) = \int_\Gamma \gamma(x,y)\sigma(y) ds_y + \omega\ , \quad x \in \Omega, \tag{7}$$

by adding a unknown constant $\omega \in I\!R$, and consider the modified system of boundary integral equations

$$V\sigma + \omega = f \quad \text{on} \quad \Gamma, \quad \int_\Gamma \sigma \, ds = A, \tag{8}$$

for any given constant $A \in I\!R$. We note that (6) and the first equation of (8) are equivalent and ω is like an integrating constant . The second equation in (8) is a normalization condition. For the interior problem as the present one, the constant A can be chosen arbitrarily, while for the exterior problem , it is usually determined by the condition at infinity. The following classical theorem in the Hölder space has been established in [13].

Theorem 1.1.1 *Given $(f, A) \in C^{1,\alpha}(\Gamma) \times I\!R$, the modified system* (8) *has a unique pair of solutions $(\sigma, \omega) \in C^{0,\alpha}(\Gamma) \times I\!R$.*

Now we consider the weak formulation for (8). To this end, for any $s \in I\!R$, let $H^s(\Gamma)$ denote the trace space. By computing the Fourier symbol of the boundary integral operator V one can show that $V = \psi do_{-1}$ is a pseudo-differential operator of order -1. Hence for smooth Γ, we have the *continuity property* of V, i.e., the mapping

$$V = \psi do_{-1} \ : H^{s-1/2}(\Gamma) \to H^{s+1/2}(\Gamma) \tag{9}$$

is continuous for any $s \in I\!R$. For the weak formulation, we introduce the *boundary sesquilinear form*:

$$\langle \lambda, V\bar{\sigma} \rangle := \int_\Gamma \lambda(x) V\bar{\sigma}(x) \, ds_x, \quad \forall \lambda \in H^{-1/2}(\Gamma), \tag{10}$$

where $\langle \cdot, \cdot \rangle$ denotes the L^2-duality pairing between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$. The variational formulation of (8) now reads: *For given* $(f, A) \in H^{1/2}(\Gamma) \times I\!R$, *find* $(\sigma, \omega) \in H^{-1/2}(\Gamma) \times I\!R$ *such that*

$$\begin{array}{rcl} \langle \lambda, V\bar{\sigma} \rangle + \omega \langle \lambda, 1 \rangle & = & \langle \lambda, \bar{f} \rangle, \quad \forall \lambda \in H^{-1/2}(\Gamma), \\ \langle \sigma, 1 \rangle & = & A. \end{array} \tag{11}$$

We remark that the normalization condition in (11) may also be rewritten in the form

$$\kappa \, \langle \sigma, 1 \rangle = \kappa \, A, \quad \forall \kappa \in I\!R$$

with the test function (constant) $\kappa \in I\!R$. Alternatively, one may replace (11) by the variational equation

$$B((\sigma, \omega), (\lambda, \kappa)) = \ell((\lambda, \kappa)), \quad \forall (\lambda, \kappa)) \in H^{-1/2}(\Gamma) \times I\!R, \tag{12}$$

where the sesquilinear form $B(\cdot, \cdot)$ is defined by

$$B((\sigma, \omega), \lambda, \kappa)) := \langle \lambda, V\bar{\sigma} \rangle + \omega \langle \lambda, 1 \rangle + \kappa \langle \sigma, 1 \rangle,$$

while $\ell(\cdot)$ is the continuous linear functional defined by

$$\ell((\lambda,\kappa)) := \langle \lambda, \bar{f}\rangle + \kappa\ A.$$

Here the product space, $H^{-1/2}(\Gamma)\times I\!R$, is the energy space. Before we can discuss the existence of weak solutions, we need a kind of *Gårding's inequality* for the sesquilinear form $B(\cdot,\cdot)$ on the energy space in the same manner as for weak solution to the partial differential equations. In general, it can be shown that B is not $H^{-1/2}(\Gamma)\times I\!R$ - elliptic, and we can not apply the Lax-Milgram lemma straightforward.

In particular, we observe that for any $\sigma \in H^{-1/2}(\Gamma)$, let

$$v(x) := \int_\Gamma \gamma(x,y)\sigma(y)ds_y, \quad x \in I\!R^2 \setminus \Gamma.$$

Then from the jump relations across Γ of the single-layer potential, we see that

$$V\sigma = v\big|_\Gamma \in H^{1/2}(\Gamma), \quad \sigma = [\frac{\partial u}{\partial n}]\big|_\Gamma \in H^{-1/2}(\Gamma), \quad 0 = [v]\big|_\Gamma \in H^{1/2}(\Gamma).$$

Consequently we have the identity from the generalized first Green formulae,

$$\langle \sigma, V\bar{\sigma}\rangle = \int_\Omega |\nabla\ v|^2 dx + \int_{\Omega^c} |\nabla\ v|^2 dx =: a_\Omega(v,v) + a_{\Omega^c}(v,v). \tag{13}$$

The right hand side of (13) contains two bilinear forms for the corresponding partial differential operator $P = -\Delta$, one in Ω and the other one in the exterior Ω^c. It is this relation in some sense that connects intimately the Gårding inequality of V with that of P. We see that

$$a_\Omega(v,v) = ||v||^2_{H^1(\Omega)} - (Cv,v)_{H^1(\Omega)},$$

where C is a compact operator from $H^1(\Omega)$ into itself. Then again from the generalized first Green formula in Ω, the mapping $\tau : H^1(\Omega,P) \to H^{-1/2}(\Gamma)$ with $v|_\Omega \mapsto \tau\, v = \sigma$ is continuous (Here $H^1(\Omega,P) := \{v \in H^1(\Omega) | Pv \in L^2(\Omega)\} = H^1(\Omega)$, since $Pv = 0$). This takes care of the contribution to the Gårding inequality for V from the interior domain Ω. We would expect, of course, to have the similar results for the exterior domain Ω^c. However, there is a technical difficulty, since v in general is not in $L^2(\Omega^c)$. In order to apply similar arguments to a_{Ω^c}, following [3], we introduce a fixed $C_0^\infty(I\!R^2)$ cut-off function χ with $\chi|_{\bar{\Omega}} = 1$ and $dist(\{x|\chi(x) \neq 1\}, \bar{\Omega}) =: d_0 > 0$, since the generalized first Green formula for Ω^c is valid for $v_c = \chi v$ having compact support. Then with this modification, (13) becomes

$$\langle \sigma, V\bar{\sigma}\rangle = a_\Omega(v,v) + a_{\Omega^c}(v_c,v_c) + \int_{\Omega^c} (Pv_c)\bar{v}_c\ dx. \tag{14}$$

It is easy to see that the last term in (14) defines a compact form on $H^1(\Omega^c \cap \mathrm{supp}\,\chi)$. The following theorem has been proved in [17], [7].

Theorem 1.1.2 (a) *There holds the Gårging inequality*

$$Re\, B((\sigma,\omega),(\sigma,\omega)) + \left\{ Re\ <(\sigma, \mathcal{C}_V\sigma)> + c_1|\omega|^2 \right\} \geq c_0 \left(||\sigma||^2_{H^{-1/2}(\Gamma)} + |\omega|^2 \right),$$

for $(\sigma,\omega) \in H^{-1/2}(\Gamma) \times I\!R$, *where* $\mathcal{C}_V : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ *is a compact operator,* $c_0 > 0$ *and* $c_1 \geq 0$ *are constants.* (b) *The corresponding homogeneous variational equation of* (12) *has only the trivial solution.* (c) *As a consequence of* (a) *and* (b), (12) *has a unique pair of solutions* $(\sigma,\omega) \in H^{-1/2}(\Gamma) \times I\!R$.

To extend the single-layer potential approach to the Neumann problem, we consider again the model problem in $I\!R^2$

$$-\Delta u = 0 \quad \text{in} \quad \Omega, \quad \frac{\partial u}{\partial n} = g \quad \text{on} \quad \Gamma \quad \text{with} \quad \int_\Gamma g\, ds = 0. \tag{15}$$

Following [21], we seek a solution in the form

$$u(x) = -\int_\Gamma \frac{\partial}{\partial n_y}\gamma(x,y)\mu(y)ds_y, \quad x \in \Omega. \tag{16}$$

Then the Neumann condition leads to the boundary integral equation of the first kind for the unknown density function μ

$$W\mu = g \quad \text{on} \quad \Gamma. \tag{17}$$

Here W is the so-called *hypersingular boundary integral operator* defined by

$$W\mu(x) := -\frac{\partial}{\partial n_x}\int_\Gamma \frac{\partial}{\partial n_y}\gamma(x,y)\mu(y)ds_y. \tag{18}$$

There is a simple relation between the two boundary integral operators V and W in the present case, namely,

$$-\frac{d}{ds}V\frac{d}{ds} = W. \tag{19}$$

In view of (19), we deduce immediately from Theorem 1.2 the following.

Theorem 1.1.3 *Given* $(g, A_1) \in C^{0,\alpha}(\Gamma) \times I\!R$ *with* $\int_\Gamma g\, ds = 0$, *the system of boundary integral equations*

$$W\mu + \omega_1 = g \quad on \quad \Gamma, \quad \int_\Gamma \mu\, ds = A_1 \tag{20}$$

has a unique pair of solutions $(\mu,\omega_1) \in C^{1,\alpha}(\Gamma) \times I\!R$.

We remark that the boundary integral equation in (20) is equivalent to (17), because of the assumption $\int_\Gamma g\, ds = 0$.

Now we turn to the weak formulation of (20). In contrast to V, W is now a pseudo-differential operator of order $+1$. Hence for smooth Γ, the mapping

$$W = \psi do_{+1} \ : H^{s+1/2}(\Gamma) \to H^{s-1/2}(\Gamma) \tag{21}$$

is continuous. For $\mu \in H^{1/2}(\Gamma)$, and $\nu \in H^{1/2}(\Gamma)$, let

$$\langle W\mu, \bar{\nu}\rangle := \int_\Gamma W\mu(x)\bar{\nu}(x)\, ds_x \tag{22}$$

be the sesquilinear form of W. We then formulate (20) as follows: *For given* $(g, A_1) \in H^{-1/2}(\Gamma) \times I\!R$, *find* $(\mu, \omega_1) \in H^{1/2}(\Gamma) \times I\!R$ *such that*

$$B_1((\mu, \omega_1), (\nu, \kappa)) = \ell((\nu, \kappa)). \tag{23}$$

Here the sesquilinear form $B_1(\cdot, \cdot)$ is defined by

$$B_1((\mu, \omega_1), (\nu, \kappa)) := \langle W\mu, \bar{\nu}\rangle + \omega_1\langle 1, \bar{\nu}\rangle + \kappa\langle 1, \mu\rangle$$

and

$$\ell((\nu, \kappa)) := \langle g, \bar{\nu}\rangle + \kappa\, A_1.$$

Clearly the existence of weak solutions to (23) can now be easily established from Theorem 1.2 with the help of (19). However, in order to illustrate our general unified approach in the absence of (19), we again indicate, in the same manner as for V, some of the typical properties of W which may be shared by the variational method for boundary integral equations of the same type .

Observe that for any $\mu \in H^{1/2}(\Gamma)$, if

$$w(x) := -\int_\Gamma \frac{\partial}{\partial n_y}\gamma(x, y)\mu(y)ds, \quad x \in I\!R^2 \setminus \Gamma.$$

then we have similar jump relations across Γ of the double-layer potential :

$$W\mu = \frac{\partial w}{\partial n}\big|_\Gamma \in H^{-1/2}(\Gamma), \quad 0 = [\frac{\partial w}{\partial n}]\big|_\Gamma \in H^{-1/2}(\Gamma), \quad \mu = [w]\big|_\Gamma \in H^{1/2}(\Gamma).$$

Again from the generalized first Green formulae we also have the identity

$$\langle W\mu, \bar{\mu}\rangle = a_\Omega(w, w) + a_{\Omega^c}(w, w). \tag{24}$$

Hence with the help of trace theorem, the following theorem can be established by using similar arguments as for (12).

Theorem 1.1.4 (a) *There holds the Gårging inequality*

$$Re\, B_1((\mu,\omega_1),(\mu,\omega_1)) + \left\{ Re\ < \mathcal{C}_W\mu,\mu > + c_1|\omega_1|^2 \right\} \geq c_0 \left(||\mu||^2_{H^{1/2}(\Gamma)} + |\omega_1|^2 \right),$$

for $(\mu,\omega_1) \in H^{1/2}(\Gamma) \times I\!R$, *where* $\mathcal{C}_W : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ *is a compact operator,* $c_0 > 0$ *and* $c_1 \geq 0$ *are constants.* (b) *The corresponding homogeneous variational equation of* (23) *has only the trivial solution.* (c) *As a consequence of* (a) *and* (b), (23) *has a unique pair of solutions* $(\mu,\omega_1) \in H^{1/2}(\Gamma) \times I\!R$.

Up to now we have considered variational formulations of boundary integral equations of the first kind with boundary integral operators of both negative and positive orders. For boundary integral equations of second kind, we generally deal with boundary integral operators of oder zero (that is, they map function spaces into themselves). Hence in the conventional approach, the weak formulations are based on the assumption that both solution and test function spaces are L^2. However, if one is interested in the L^2- duality pairing formulation when the solution space is different from L^2, then the test function space and the solution space can not be the same. In the following, we will show that in case of solution spaces different from L^2, with the help of the boundary integral operators V and W (or similar operators), one may still use the L^2-duality pairing formulations but with the same test and solution function spaces in the same manner as for the first kind integral equations.

1.2 Boundary integral equations of the second kind

We use the same model problem (1) for the Dirichlet problem. If we seek a solution in the form of a double-layer potential,

$$u(x) = -\int_\Gamma \frac{\partial}{\partial n_y}\gamma(x,y)\mu(y)\, ds_y, \quad x \in \Omega, \tag{25}$$

then we arrive at the integral equation of the second kind for μ

$$(\frac{1}{2}I - K)\mu = f \quad \text{on} \quad \Gamma. \tag{26}$$

Here K is the double-layer boundary integral operator defined by

$$K\mu(x) := \int_\Gamma \frac{\partial}{\partial n_y}\gamma(x,y)\mu(y)\, ds_y, \quad x \in \Gamma. \tag{27}$$

For smooth Γ, K is a pseudo-differential operator of order -1 and the mapping

$$\frac{1}{2}I - K = \psi do_0 : H^{s+1/2}(\Gamma) \to H^{s+1/2}(\Gamma) \tag{28}$$

is continuous for any $s \in I\!R$ with a compact operator K on $H^{s+1/2}(\Gamma)$. Hence for given $f \in H^{1/2}(\Gamma)$, our solution space of μ should be $H^{1/2}(\Gamma)$. One weak formulation of (26) based on the L^2-duality pairing between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$ may be as follows. *Given* $f \in H^{1/2}(\Gamma)$, *find* $\mu \in H^{1/2}(\Gamma)$ *such that*

$$\langle W\nu, \overline{(\frac{1}{2}I - K)\mu}\rangle = \langle W\nu, \bar{f}\rangle, \quad \forall \nu \in H^{1/2}(\Gamma). \tag{29}$$

From the mapping property (21) of W, we see that indeed the formulation (29) makes sense. In fact, from the generalized first Green formula for the interior domain Ω, it is easy to verify that the sesquilinear form satisfies the identity

$$\langle W\mu, \overline{(\frac{1}{2}I - K)\mu}\rangle = a_\Omega(u, u)$$

with u represented by (25) for any $\mu \in H^{1/2}(\Gamma)$. Hence it is easy to show that the sesquilinear form satisfies a Gårding inequality by using previous arguments, since K is assumed to be compact. Hence for the existence of solutions, we need only to examine the uniqueness problem.

It is not difficulty to see that the homogeneous equation of (29) has non-trivial solutions, $\mu = constants$, although the corresponding homogeneous equation of the original equation (26) has only the trivial solution. As usual, we may augment (29) by adding a normalization condition and consider a modified system such as

$$\begin{aligned} \langle W\nu, \overline{(\frac{1}{2}I - K)\mu}\rangle + \omega_2\langle \nu, 1\rangle &= \langle W\nu, \bar{f}\rangle, \quad \forall \nu \in H^{1/2}(\Gamma) \\ \int_\Gamma \mu\, ds &= 0, \end{aligned} \tag{30}$$

where $\omega_2 \in I\!R$ is an unknown constant. We note that the variational equation (29) and that of (30) are equivalent. We now summarize the main results in the following theorem.

Theorem 1.2.1 (a) *For given* $f \in H^{1/2}(\Gamma)$, *the system* (30) *has a unique pair of solutions* $(\mu, \omega_2) \in H^{1/2}(\Gamma) \times I\!R$. (b) *The unique solution of* (26) *is of the form*

$$\mu = \mu_0 - c,$$

where μ_0 *is the unique solution of* (30) *and* c *is the constant defined by*

$$c = -\frac{1}{m(\Gamma)} \int_\Gamma (f + K\mu_0) ds, \quad m(\Gamma) = \int_\Gamma ds.$$

We remark that the constant c in the theorem plays the same role as an integrating constant in the Fichera method, since one may rewrite the variational equation of (30) in the form

$$\langle \nu, W\overline{(\frac{1}{2}I - K)\mu}\rangle + \omega_2\langle \nu, 1\rangle = \langle \nu, W\bar{f}\rangle, \quad \forall \nu \in H^{1/2}(\Gamma),$$

and both f and $f + c$ lead to the same right hand side of (29) and (30).

We now turn our attention to the Neumann problem (15). We seek a solution u in the form of a single-layer potential

$$u(x) = \int_\Gamma \gamma(x, y)\sigma(y)\, ds_y\ , \quad x \in \Omega. \tag{31}$$

This leads to the second kind boundary integral equation for the density function σ

$$(\frac{1}{2}I + K')\sigma = g \quad \text{on} \quad \Gamma, \tag{32}$$

where the boundary integral operator K' is the transpose of K defined by

$$K'\sigma(x) := \int_\Gamma \frac{\partial}{\partial n_x}\gamma(x, y)\sigma(y)\, ds_y, \quad x \in \Gamma. \tag{33}$$

Again for smooth Γ, K' is a pseudo-differential operator of order -1 and we have the mapping property that

$$\frac{1}{2}I + K' = \psi do_0 : H^{s-1/2}(\Gamma) \to H^{s-1/2}(\Gamma) \tag{34}$$

is continuous for any $s \in I\!R$. In an analog to (29), given $g \in H^{-1/2}(\Gamma)$ with $\int_\Gamma g\, ds = 0$, we say that $\sigma \in H^{-1/2}(\Gamma)$ *is a weak solution of* (32) *if it satisfies the variational equation*

$$\langle(\frac{1}{2}I + K')\sigma, \overline{V\lambda}\rangle = \langle g, \overline{V\lambda}\rangle, \quad \forall\lambda \in H^{-1/2}(\Gamma). \tag{35}$$

Again, the boundary sesquilinear form in (35) is related to the u in (31) through

$$\langle(\frac{1}{2}I + K')\sigma, \overline{V\sigma}\rangle = a_\Omega(u, u). \tag{36}$$

We notice that the solution of (32) is not *unique* and we must modify the variational equation (35) in order to take into account of the non-uniqueness. To avoid of repeating the arguments, we collect the results in the following.

Theorem 1.2.2 *For given $g \in H^{-1/2}(\Gamma)$ satisfying $\int_\Gamma g\, ds = 0$, the variational equation* (35) *has a unique solution $\sigma \in H^{-1/2}(\Gamma)$ subject to the constrain*

$$\int_\Gamma \sigma ds = 0. \tag{37}$$

The system of equations (35) and (37) *is equivalent to the modified system consisting of the variational equation*

$$\langle(\frac{1}{2}I + K')\sigma, \overline{V\lambda}\rangle + \omega_3\langle 1, \bar{\lambda}\rangle = \langle g, \overline{V\lambda}\rangle, \quad \forall\lambda \in H^{-1/2}(\Gamma) \tag{38}$$

together with (37)*, which has a unique solution pair $(\sigma, \omega_3) \in H^{-1/2}(\Gamma) \times I\!R$.*

We remark that the unique solution of (35) and (37) (or (38) and (37)) is a solution of (32) and the corresponding u given by (31) will be a solution of the Neumann problem (2), although the latter is unique up to a constant.

2 Domain Decomposition and The Steklov-Poincaré Operators

In recent years, because of the advances of multiprocessor computer architecture, the method of domain decomposition has become a particularly attractive technique of solving large scale boundary value problems. One of the most important features in domain decomposition approach is to compute interface values between subdomains efficiently through appropriate interface operators from both theoretical and numerical approximation points of view.

There is a natural connection between these interface operators, which are generally known as *the Steklov-Poincaré operators*, and suitable boundary integral operators employed in boundary element methods. In fact, one can show that these Steklov-Poincaré operators can be expressed explicitly by boundary integral operators and the corresponding integral equations are readily treated by boundary element methods in parallel processing. Hence it is in scientific computing that, we believe, the boundary integral methods will play a most significant role in the future.

2.1 Basic concepts in domain decomposition

We now use the same simple model problem (1) to illustrate the basic idea of the method of domain decomposition. For ease of reading, we recast the problem as follows:

$$\Delta u = 0 \quad \text{in} \quad \Omega, \quad u|_\Gamma = f \quad \text{on} \quad \Gamma. \tag{39}$$

The method begins with partitioning Ω into subdomains and the goal here is to solve the problem (39) only on the subdomains. For simplicity, suppose that Ω is partitioned into two subdomains Ω_1 and Ω_2 by an interface Γ_{12}. We assume that Γ_{12} is at least continuously differentiable and divides Ω in such a way that the boundaries $\Gamma_k = \partial\Omega_k, k = 1, 2$ of the resulting subdomains are Lipschitz. For example, Γ_{12} might be an open line segment partitioning Ω. If we denote by $u_k = u|_{\Omega_k}, k = 1, 2$, the restrictions of u on Ω_k, then the *subproblems* are defined by

$$\begin{array}{rcl} \Delta u_k & = & 0 \text{ in } \Omega_k, \\ u_k = \mu \text{ on } \Gamma_{12} & \text{and} & u_k = f \text{ on } \Gamma_k \cap \Gamma, \ k = 1, 2. \end{array} \tag{40}$$

Here $\mu = u|_{\Gamma_{12}}$ is unknown. However, for any given μ, the subproblems can be solved in parallel, and the normal derivatives of the solutions, $\frac{\partial u_k}{\partial n_k}, k = 1, 2$ generally will not satisfy the *compatibility condition*,

$$\frac{\partial u_1}{\partial n_1} = -\frac{\partial u_2}{\partial n_2} \quad \text{on } \Gamma_{12} \tag{41}$$

unless $\mu = u|_{\Gamma_{12}}$ exactly (recall that all the normals n_k are exterior with respect to the domain Ω_k). In other words, to solve (31) by the method of domain decomposition

reduces to seek the Dirichlet data μ on the interface Γ_{12} such that, when (40) is solved, the normal derivatives $\frac{\partial u_k}{\partial n_k}$ of the resulting solutions have the same values on Γ_{12}, that is, $\frac{\partial u_k}{\partial n_k}$ satisfy (41). Equivalently, this amounts to solving μ from (41), since the solutions of (40) and their normal derivatives depend on μ as well as on f. Now if we separate the dependence of u_k according to

$$u_k = u_k(\mu) + u_k(f), \tag{42}$$

then we may rewrite (41) in the form

$$(S_1 + S_2)\mu = g \quad \text{on } \Gamma_{12}. \tag{43}$$

In equation (43), $g := -\left(\frac{\partial u_1(f)}{\partial n_1} + \frac{\partial u_2(f)}{\partial n_2}\right)\Big|_{\Gamma_{12}}$ is a known function in terms of the normal derivatives of $u_k(f)$ on Γ_{12}, and we have introduced the operator S_k,

$$S_k : \mu \longrightarrow \frac{\partial u_k}{\partial n_k}(\mu), \tag{44}$$

which is referred to as *the Steklov–Poincaré operator*, a Dirichlet–Neumann map. As will be seen, S_k can be explicitly represented in terms of boundary integral operators.

We remark that the technique of reducing the original problem in Ω to the solutions of decoupled problems in subdomains Ω_k and to the equation on Γ_{12} for the interface unknown is usually called the *domain decomposition* or *substructuring*. Of central importance in domain decomposition is how to solve (43) efficiently. In practice, equation (43) is solved by some kind of iterative scheme and in each iteration, this amounts to solving the subproblems (40) with the updating data on the interface. In what follows, we shall refer to (43) and the alike as the *Steklov–Poincaré operator equation*. In the next section, we will discuss the weak formulation of the Steklov–Poincaré operator equation, and in particular, we will represent the Steklov–Poincaré operators by using the boundary integral operators introduced in the previous sections.

2.2 A variational formulation of the Steklov-Poincaré operator equation

To avoid being too special to lose the generality, we first modify our previous setting from 2 subdomains to N subdomains. To be more precise, let

$$\overline{\Omega} = \bigcup_{i=1}^{N} \overline{\Omega}_i \tag{45}$$

be a given domain decomposition of Ω into N non–overlapping subdomains Ω_i with Lipschitz boundaries $\Gamma_i = \partial\Omega_i$. We denote $\Gamma_{ij} = \Gamma_i \cap \Gamma_j$, $i, j = 1, \ldots N$, the interface

boundaries. The *skeleton* Γ_S of the domain decomposition is defined by

$$\Gamma_S := \bigcup_{i=1}^{N} \Gamma_i \;=\; (\bigcup_{i,j=1}^{N} \Gamma_{ij}) \bigcup \Gamma. \tag{46}$$

As before, the restriction of u to Γ_i is denoted by $u_i = u|_{\Omega_i}, i = 1 \cdots N$. Then the original boundary value problem (39) for u is equivalent to the collection of transmission problems for the restriction u_i defined by

$$\Delta u_i = 0 \quad \text{in } \Omega_i, \qquad u_i = f \quad \text{on } \Gamma_i \cap \Gamma \tag{47}$$

together with the transmission conditions

$$u_i = u_j, \quad \frac{\partial u_i}{\partial n_i} = -\frac{\partial u_j}{\partial n_j} \quad \text{on } \Gamma_{ij}. \tag{48}$$

We note that the second transmission condition in (48) corresponding to the compatibility condition (41). Before we formulate precisely the subproblems (cf.(40)), and derive the corresponding Steklov–Poincaré operator equations (cf.(43)), we need some functions. In what follows, let $H^{1/2}(\Gamma_S)$ be the trace space of $H^1(\Omega)$ defined by

$$H^{1/2}(\Gamma_S) := \left\{ \mu = u|_{\Gamma_S} \;\middle|\; u \in H^1(\Omega) \right\}$$

equipped with the norm

$$||\mu||_{H^{1/2}(\Gamma_S)} := \left\{ \sum_{i=1}^{N} ||u_{|\Gamma_i}||^2_{H^{1/2}(\Gamma_i)} \right\}^{1/2}. \tag{49}$$

Now let $\mu \in H^{1/2}(\Gamma_S)$ with $\mu = f$ on Γ. We consider the local boundary value problems for $v_i \in H^1(\Omega_i), i = 1, \cdots N$ defined by

$$\Delta v_i = 0 \quad \text{in } \Omega_i, \qquad v_i|_{\Gamma_i} = \mu \quad \text{on } \Gamma_i \tag{50}$$

and define the Steklov-Poincaré operators S_i by

$$S_i \mu := \frac{\partial v_i}{\partial n_i} \quad \text{on } \Gamma_i. \tag{51}$$

Clearly, problems (50) will be the corresponding subproblems, if we require that μ satisfies the compatibility conditions

$$(S_i + S_j)\mu = 0 \quad \text{on } \Gamma_{ij}. \tag{52}$$

In this case, it is easy to see from the definition of μ that $v_i \equiv u_i, i = 1, \cdots N$. From now on, problems (50) with v_i replaced by u_i are referred to the *subproblems* for the restrictions u_i, and (51), the corresponding Steklov–Poincaré operator equations.

We are now in a position to formulate the variational equations for (52). For given $f \in H^{1/2}(\Gamma)$, let $\widetilde{f} \in H^{1/2}(\Gamma_S)$ be an arbitrary but fixed extension of f satisfying $\widetilde{f}|_\Gamma = f$ on Γ and let

$$H^{1/2}_\Gamma(\Gamma_S) := \left\{ \nu \in H^{1/2}(\Gamma_S) \,\middle|\, \nu|_\Gamma = 0 \right\}. \tag{53}$$

Then we have the variational formulation of (52): *Find* $\mu \in H^{1/2}(\Gamma_S)$ *such that*

$$\mu - \widetilde{f} \in H^{1/2}_\Gamma(\Gamma_S) \quad \textit{and} \quad \sum_{i=1}^{N} \int_{\Gamma_i} S_i\mu \cdot \nu \, ds = 0, \quad \forall\, \nu \in H^{1/2}_\Gamma(\Gamma_S). \tag{54}$$

An alternative formulation is *to find* $\widetilde{\mu} \in H^{1/2}_\Gamma(\Gamma_S)$ *satisfying the variational equation*

$$\sum_{i=1}^{N} \int_{\Gamma_i} S_i\widetilde{\mu} \cdot \nu \, ds, = -\sum_{i=1}^{N} \int_{\Gamma_i} S_i\widetilde{f} \cdot \nu \, ds, \quad \forall\, \nu \in H^{1/2}_\Gamma(\Gamma_S). \tag{55}$$

In order to express the Steklov-Poincaré operator in terms of boundary integral operators, we now take a *dètour* and consider the reduction of a typical subproblem for u_i to boundary integral equations. We will use the direct approach based on the Green representation formula for the solution

$$u_i(x) = \int_{\Gamma_i} \gamma(x,y)\sigma_i(y)ds_y - \int_{\Gamma_i} \frac{\partial \gamma}{\partial n_y}(x,y)\mu(y)ds_y, \quad x \in \Omega_i \tag{56}$$

with $\mu := u_i|_{\Gamma_i}$ and $\sigma_i := \frac{\partial u_i}{\partial n}\Big|_{\Gamma_i}$. Here and in the sequel we adopt the same conventions and notations in Section 1 except with a subscript i when necessary in order to emphasize on the dependence of the subdomains. By the standard arguments, we arrive at the following two boundary integral equations for the Cauchy data μ and σ_i

$$\mu = V_i\sigma_i + \left(\frac{1}{2}I - K_i\right)\mu \quad \text{on} \quad \Gamma_i, \tag{57}$$

$$\sigma_i = \left(\frac{1}{2}I + K_i'\right)\sigma_i + W_i\mu \quad \text{on} \quad \Gamma_i. \tag{58}$$

We remark that for the Lipschitz boundary Γ_i, the mapping properties of the boundary integral operators $V_i, W_i, (\frac{1}{2}I - K_i)$ and $(\frac{1}{2}I + K_i')$ from (9), (21), (28) and (34) remain valid for $|s| \le 1/2$ (see, e.g. [2]). Without loss of generality, we may assume that V_i is $H^{-1/2}(\Gamma_i)$–*elliptic*, that is, there exists a constant $c > 0$ such that

$$\langle \sigma_i, V_i\sigma_i \rangle \ge c||\sigma_i||^2_{H^{-1/2}(\Gamma_i)}, \forall \sigma_i \in H^{-1/2}(\Gamma_i) \tag{59}$$

(see, e.g. [17],[7], if $dia\,\Omega_i < 1$). Hence from (57), we obtain

$$\sigma_i = V_i^{-1}\left(\frac{1}{2}I + K_i\right)\mu,$$

and substituting it into (58) yields the expression

$$\sigma_i = S_i\mu,$$

where $S_i = \psi do_{+1}$ is the local *Steklov-Poincaré operator* explicitly defined by

$$S_i\mu := \left(\frac{1}{2}I + K_i'\right)V_i^{-1}\left(\frac{1}{2}I + K_i\right)\mu + W_i\mu. \tag{60}$$

With this explicit expression of S_i in terms of the boundary integral operators given in (60), we obtain in the following one of the main results in domain decomposition.

Theorem 2.2.1 *The boundary bilinear form*

$$\langle S\mu, \nu\rangle := \sum_{i=1}^{N}\int_{\Gamma_i} S_i\mu\cdot\nu\; ds$$

is bounded in $H^{1/2}(\Gamma_S)$ and $H_\Gamma^{1/2}(\Gamma_S)$–elliptic. Consequently, there is a unique solution $\mu \in H^{1/2}(\Gamma_S)$ to the variational equation (54).

We remark that the $H_\Gamma^{1/2}(\Gamma_S)$ – ellipticity of the bilinear form follows easily from the representation of S_i in (60). In particular, note that the contribution from first term is non-negative, and hence can be neglected. The result then follows immediately from the second term.

3 Concluding Remarks

We comment that although our presentation here deals with only simple model problems, yet we believe that it does cover all the essential features for the variational methods for boundary integral equations as well as their application to the fruitful research area, the domain decomposition. In closing the paper, we include here some additional relevant references concerning general results with respect the variational methods for boundary integral equations, their applications to mathematical physics, mechanics as well as some recent developments in domain decomposition via boundary integral equations. We refer interested readers to the references [20], [3], [24] (for variational methods concerning boundary integral equations in general), [8],[9], [10] (for applications to linear and nonlinear problems in fluid mechanics and elasticity), [12], [11] (for acoustic and electromagnetic scattering), and [18], [19], [15], [14], [16] (for domain decomposition via boundary integral equations), to name a few.

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