

## **BERNOULLI AND GEOMETRIC CUSUM CONTROL CHARTS**

### **A Closer Look at the Equivalence of Bernoulli and Geometric CUSUM Control Charts**

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## **A Closer Look at the Equivalence of Bernoulli and Geometric CUSUM Control Charts**

Some researchers have incorrectly concluded that the geometric CUSUM is superior to the Bernoulli CUSUM as a procedure for monitoring a repetitive process, even though the two procedures have been proved to be equivalent for detecting an upward shift in the proportion of nonconforming items. We use an exact Markov-chain-based methodology to re-examine the relationship between geometric CUSUM and Bernoulli CUSUM control charts. Exact methods allow us to differentiate between similar but different-valued quantities that have contributed to some misunderstandings in the literature. We show that for a random-shift model, evaluations of steady-state average number of inspected items until a signal (ANIs) are identical for both geometric CUSUMs and Bernoulli CUSUMs, provided the correct choices of return levels are made. We also show that a steady-state geometric CUSUM based on a fixed-shift model only uses the geometric CUSUM states, while a steady-state geometric CUSUM based on a random-shift model will reach the states of a Bernoulli CUSUM after a long series of zeros. We note that our conclusions are contrary to the published results of other researchers, and we examine these differences in detail.

Keywords: Bernoulli CUSUM; geometric CUSUM; fixed-shift and random-shift models; average run length (ARL); average number inspected (ANI)

### **Introduction**

Since their introduction, control charts (Shewhart, 1931) have had widespread application for improving the quality of manufacturing processes. Page (1954) introduced the CUSUM control chart and showed that it was related to a sequential probability ratio test. Moustakides (1986) proved the optimality of CUSUM control charts for a one-sided detection of a change in distribution. Shewhart charts are CUSUM monitoring schemes where the CUSUM parameter  $k$  equals the Shewhart control limit, and a signal is given when the CUSUM statistic is greater than 0. By Moustakides' (1986) optimality proof, they are optimal one-sided monitoring

procedures for detecting large shifts. Lucas (1973) extended the optimality proof and showed that Shewhart charts are optimal two-sided monitoring procedures for detecting large shifts.

The Shewhart  $p$ -chart is often used for monitoring a manufacturing process where the data can be viewed as a stream of time-ordered Bernoulli 0-1 observations, i.e., conforming or nonconforming. But the Shewhart  $p$ -chart is not effective for monitoring processes when  $p$  is small. Calvin (1983) proposed a variation of the Shewhart  $p$ -chart based on the number of conforming observations until a nonconforming item; this is known as a conforming run length or CRL. Bourke (1991) proposed control charts based on groups of CRLs, including a Run-length CUSUM, now known as a geometric CUSUM. Reynolds & Stoumbos (1999) proposed the Bernoulli CUSUM which is based on the original Bernoulli 0-1 observations.

More recently, a number of researchers have proposed variations of these charts that they claim have “better” performance properties. There is growing evidence, however, that some of these claims may be misleading due to a faulty understanding or incorrect comparison of these charts with the longer-established ones. See Knoth (2016), Knoth et al. (2021a) and Knoth et al. (2021b) for a thorough investigation of some of these methods.

In this we paper we specifically address the confusion between geometric and Bernoulli CUSUM charts for monitoring a proportion of nonconforming items. Chang & Gan (2001, page 796), for example, incorrectly concluded that “*Among all charts, the geometric CUSUM chart is found to be more sensitive in detecting increases in  $p$ , except for very large ones...*”, even though they provided a proof of the equivalence of these two charts in their paper. In a later paper, Chang & Gan (2007) compared several charts using both initial-state and steady-state evaluations of ARL (ANI), and again

incorrectly concluded (page 874) that “*Comparisons of the Shewhart chart with runs rules with geometric CUSUM, Bernoulli CUSUM and synthetic charts showed that the geometric CUSUM chart is the most sensitive in detecting increases in  $p$ .*” In both papers, the comparisons of Bernoulli CUSUM and geometric CUSUM charts were based on the use of a zero headstart for each CUSUM, and choices for CUSUM parameters which gave matching in-control detection performance. But according to their equivalence result, a geometric CUSUM with an initial state of zero is equivalent to a Bernoulli CUSUM with a specific headstart. Thus, their comparison of a geometric CUSUM and a Bernoulli CUSUM was equivalent to a comparison of a Bernoulli CUSUM with a headstart and a Bernoulli CUSUM without a headstart. While similar incorrect comparisons have been made by other authors, we discuss the comparisons made by Chang & Gan because they provided a proof of the equivalence of geometric CUSUMs and Bernoulli CUSUMs.

Subsequently, Szarka & Woodall (2012) discussed the equivalence of Bernoulli CUSUM and geometric CUSUM charts and reasoned that the “*geometric CUSUM has very good properties in a zero-state analysis because of a built-in headstart feature*”. They also noted without numerical evidence “*that the steady-state properties of the Bernoulli CUSUM and geometric CUSUM are the same because the charts can be designed to be mathematically equivalent.*” They provided useful insights into the equivalence of Bernoulli CUSUMs and geometric CUSUMs, but their evaluations were limited by their decision to use simulation rather than exact methods for their investigation. We note that this decision is in accord with their view: “*A steady-state analysis will typically consist of simulating in-control runs, then having a shift occur during monitoring.*” as stated in Szarka & Woodall (2011).

The purpose of this paper is to clarify the relationship between the geometric CUSUM and the Bernoulli CUSUM. Our emphasis is on using exact methods to evaluate steady-state properties of control charts to detect an increase in the proportion of nonconforming items. Since the early 2000s, it has become increasingly accepted that in the comparison of ARL-based detection performance of alternative control charts one can use steady-state evaluations of ARL (e.g., Bourke, 2001a; Szarka & Woodall, 2011; Woodall & Driscoll, 2015). In this paper we argue that in a proper ARL-based approach to comparisons of control charts, one can first identify the parameters of competing schemes that are matched in the sense that the in-control steady-state ARL is the same for the competing schemes. One can then evaluate steady-state ARL levels for any shifts one wishes to guard against, and thereby choose among the competing control charts. (It should be noted that evaluations of steady-state ARL for both in-control and out-of-control states of a process are essential features of many economic models for the design of control charts.) However, the approach described above has not always been followed when newly-proposed control charts were being assessed for their suitability for publication in journals concerned with statistical process control. The simpler approach of matching competing control charts using in-control *initial-state* (or zero-state in the case of CUSUMs) ARL has often been accepted by journal editors. Journals have even accepted articles based on the use of initial-state evaluations for out-of-control conditions, despite the availability of the methodology for the corresponding steady-state evaluations. A consequence of this is that there have been several instances of new control charts appearing in the research journals which have given misleading advice to quality engineers. An example is the Synthetic control chart which was subsequently investigated using steady-state analysis in Woodall & Davis (2002), Bourke (2008), and Knoth (2016).

By using an exact methodology rather than simulation, we are able to achieve additional insights about small differences in the properties that often cannot be seen in simulated results. We are also able to perform a sensitivity analysis of the steady-state distributions with different “return” states, i.e., the reset value of the CUSUM following a false signal. These methods allow us to clarify and correct the comparison of geometric and Bernoulli CUSUMs for monitoring a proportion, and to show that there is little difference between the steady-state distributions for different return states.

Some researchers have assumed a fixed-shift model instead of a random-shift model when comparing the geometric CUSUM chart to other control charts. A fixed-shift model assumes that the shift from an in-control process to an out-of-control process occurs at a nonconforming item, while a random-shift model allows the shift to occur at any item.

Failure to recognize that the geometric CUSUM includes an implicit headstart has contributed to the incorrect conclusion that the geometric CUSUM is superior to the Bernoulli CUSUM. By using exact methods, we are able to evaluate initial-state and steady-state ARL values for these two CUSUMs. We show that when an in-control Bernoulli CUSUM is matched with the equivalent in-control geometric CUSUM, these CUSUMs give identical ARL (ANI) values for out-of-control process shifts. We are also able to show that, for a random-shift process, when the return value for the Bernoulli CUSUM is matched to the corresponding return for the equivalent geometric CUSUM, the calculated Bernoulli CUSUM steady-state values for various out-of-control states exactly match those calculated for the equivalent geometric CUSUM.

Together, the points we address in this article provide important clarifications and understanding of the equivalence of the geometric and Bernoulli CUSUM charts.

## Notation and Equivalence of the Two CUSUMs

Some of the confusion around the geometric CUSUM has been caused by the use of different terminology to describe equivalent or similar concepts. For example, Bourke (1991) introduced the Run-length CUSUM (later renamed “geometric CUSUM”) and described its properties using the term “average number inspected (ANI)” to differentiate between the number of samples required and the number of items required for a control chart to signal. Reynolds & Stoumbos (1999), and later Szarka & Woodall (2012), used the term “average number of observations to signal (ANOS)” to describe the same property. Chang & Gan (2001, 2007) used the equivalent term, “average number of items sampled (ANIS)”. Here, we use the original terminology of ANI that was proposed by Bourke (1991) and note that for the Bernoulli CUSUM, the average run length (ARL) is equivalent to the ANI. For the geometric CUSUM, the ANI is equal to  $ARL(p)/p$ , where  $ARL(p)$  is the average number of CRLs up to the occurrence of a signal and  $1/p$  is the expected number of items in a CRL (geometric) sample.

We denote the in-control and out-of-control nonconformance rates as  $p_0$  and  $p_1$ , where  $p_1 > p_0$ . The Bernoulli CUSUM for detecting an increase in  $p$  can be expressed as:

$$B_i = \max[0, B_{i-1} + X_i - k_B], \quad (1)$$

where  $X_i$  is the  $i^{\text{th}}$  Bernoulli sample. For a conforming sample, we define  $X_i = 0$ , and for a nonconforming sample,  $X_i = 1$ . A signal is given if  $B_i \geq h_B$ , where  $h_B$  is the decision interval. Similarly, the geometric CUSUM for detecting an upper shift in  $p$  can be expressed as:

$$G_j = \max[0, G_{j-1} + k_G - Y_j], \quad (2)$$

where  $k_G = 1/k_B = m$ ,  $Y_j$ , the  $j^{\text{th}}$  CRL, is the number of items up to and including a nonconforming item. A signal is given when  $G_j \geq h_G$ .

The initial values  $B_0$  and  $G_0$  are 0 for a zero-state procedure, or a specified headstart value (Lucas & Crosier, 1982). The use of a headstart can significantly reduce the time to detect an initial out-of-control situation or an ineffective control action, while having much less effect on the in-control ARL. We note that, in the largest-known implementation of CUSUM monitoring procedures, CUSUMs with a headstart have been effectively used at DuPont in more than 100,000 CUSUMs daily.

Chang & Gan (2001, pp. 803-805) proved that “the upper-sided Bernoulli CUSUM chart with parameters  $k_B = 1/m$ ,  $h_B = h$  and headstart  $B_0 = (w + m - 1) / m$  is equivalent to the upper-sided geometric CUSUM chart with parameters  $k_G = m$ ,  $h_G = m \cdot h - m + 1$  and headstart  $G_0 = w$ ”, where  $m$ ,  $h$  and  $w$  are all non-negative integers. (Note that some letters have been changed to make the terminology match ours.) By equivalent, we mean that both CUSUMs will signal in the same place when the same string of Bernoulli samples is obtained. Because both CUSUMS always signal at the same place, their ARL (ANI) properties must be equivalent. But we wish to emphasize that the two CUSUMs as described above are not equivalent in their number of transient states since the Bernoulli CUSUM has  $m-1$  more states. We reemphasize that a zero headstart value for a geometric CUSUM is not equivalent to a zero headstart value for a Bernoulli CUSUM. A geometric CUSUM with a zero headstart value is equivalent to a Bernoulli CUSUM with a headstart of  $(m-1)/m$ .

Using the Markov-chain approach originally described by Brook & Evans (1972) for calculating initial-state properties, and later extended to steady-state evaluations by Crosier (1986) and Lucas & Saccucci (1990), we can calculate the exact ARL and ANI values of both the geometric and Bernoulli CUSUMs. Our procedure is described in Appendix A.

Following this procedure, we evaluated the steady-state ARL and ANI values

for the CUSUM charts by assuming that the CUSUM statistic is reset to a chosen initial-state value whenever a false signal occurs. Some SPC researchers may take the view that steady-state ARL and ANI evaluations are not relevant when the process continues at the in-control level. We disagree with this view and (as mentioned earlier) argue that it is often useful to consider a steady-state evaluation for an in-control process to properly compare competing control schemes. Failure to do so has led some researchers to incorrectly compare worst-case properties for one control chart to non-worst-case properties of other control charts.

We also note that Reynolds & Stoumbos (2000) were the first to publish the Markov-chain approach to evaluate steady-state ARLs for the Bernoulli CUSUM. Subsequently, Bourke (2001a, 2001b) used the Markov chain approach to evaluate the steady-state ARL for both the Bernoulli and geometric CUSUMs. The methodology used by Bourke (2001a), which is also used in this paper, gives evaluations which are in broad agreement with Reynolds and Stoumbos (2000) and are in exact agreement with the evaluations given by the corresponding Markov-chain methodology for the geometric CUSUM which is presented in Bourke (2001b).

### **A Reconsideration of Table 3 in Szarka & Woodall (2012)**

In this section, we revisit the example used by Szarka & Woodall (2012) to illustrate the equivalence of the Bernoulli and geometric CUSUMs and show the advantage of using exact calculations over simulation. In their example, the authors use a Bernoulli CUSUM with parameters  $h_B = 320/61$  and  $k_B = 1/61$  and a geometric CUSUM with corresponding parameters  $h_G = 260$  and  $k_G = 61$ . These CUSUMs were designed for an in-control  $p_0 = 0.01$ , an out-of-control  $p_1 = 0.025$  and an in-control ARL of almost 30,000. Because the value of  $k_B$  is a fraction with denominator  $m = 61$ , the number of

transient states for the Bernoulli CUSUM is equal to  $m \cdot h_B = 320$ . These 320 states can be labeled in different ways. We label the Bernoulli CUSUM states sequentially starting with 0. These labels are equivalent to the numerators of the possible Bernoulli CUSUM headstart values, and the corresponding set of ARL values can be viewed as an ARL vector. Because of Chang & Gan's equivalence theorem, the last 260 elements of the Bernoulli CUSUM ARL vector are also the ANI values of a geometric CUSUM with  $h_G = 260$ ,  $k_G = 61$  for the various headstarts  $0, 1, 2, \dots, 259$ .

Table 1 in this paper shows the ARL values for the zero-state Bernoulli CUSUM considered in Table 3 of Szarka & Woodall (2012), as well as for four different headstart values. We have numbered the columns in Table 1 from [1] to [5]. We use the additional notation [1]~SW[4] and [2]~SW[1] to indicate that columns [1] and [2] in our Table 1, which are evaluated using exact methods, are related to columns [4] and [1] of Table 3 of Szarka & Woodall (2012). We note that column SW[1] was based on simulation while column SW[4] was calculated by Reynolds & Stoumbos (1999) using exact methods and included in Table 3 of Szarka & Woodall (2012).

The five columns in Table 1 display a subset of the 320 possible starting values for the CUSUM and can be described as:

- A zero-state Bernoulli CUSUM;
- A Bernoulli CUSUM with headstart of  $(m-1)/m = 60/61$  (which is equivalent to a geometric zero-state CUSUM with  $k_G = 61$  and  $h_G = 260$ );
- A Bernoulli CUSUM with headstart of  $h_B/2 = 160/61$ ;
- A Bernoulli CUSUM with headstart of  $190/61$  (which is equivalent to a geometric CUSUM with an  $h_G/2 = 130$  headstart);
- A Bernoulli CUSUM with an extreme headstart of  $319/61$ .

Supplemental Table S1 contains the ARLs for the entire set of 320 possible initial values for the Bernoulli CUSUM. Each row of Table S1 shows, for a given value of  $p$ , the ARL (or equivalently ANI) values of a Bernoulli CUSUM. The last 260 rows are also the ANI values for the equivalent geometric CUSUMs with headstarts of 0, 1, 2, ... 259. Examination of Tables 1 and S1 show that a small headstart makes little difference in the in-control properties, but it significantly improves the out-of-control properties. Because a headstart results in a relatively small worsening of in-control properties and a relatively large improvement in out-of-control properties, adding a headstart to a CUSUM improves the apparent performance. However, that apparent improvement may be overstated because the headstart properties only apply at startup or after a restart following a corrective action. Once enough time has elapsed since startup or the most recent corrective action, the CUSUM will reset to zero, and the zero-state ARL better describes the CUSUM performance until the next corrective action. The merit of steady-state analysis is that it provides a meaningful average of a range of initial-state performances. These tables also show that Bernoulli CUSUM headstart values larger than  $(m-1)/m$  should usually be considered since a larger headstart will more quickly detect an initial out-of-control situation while only having a small impact on the in-control ARL. A headstart of  $h/2$ , that we also use and recommend, was almost always used in the large CUSUM implementation at DuPont that we discussed earlier.

As expected, our calculated Bernoulli zero-state column in Table 1 exactly matches the corresponding calculated Bernoulli zero-state column in Table 3 of Szarka & Woodall (2012), column [4]. Columns [1] to [3] in Table 3 of Szarka & Woodall (2012) show approximate results obtained from simulation, so some differences will be seen which we will explore later. Comparing our zero-state and 60/61 headstart columns in Table 1, we see that, with this small headstart, there is only a small change

in the  $ARL_{IC}$  ( $ANI_{IC}$ ) values (from 29,248.55 to 29,148.55), but a larger percentage change in  $ARL_{OOC}$  ( $ANI_{OOC}$ ). It is interesting that the difference between these two columns is exactly  $1/p$ , the expected number of items in a CRL. Chang & Gan (2001, p. 805) proved this relationship for  $h \leq 2 - (1/m)$ . An intuitive explanation of the general relationship is provided in Appendix B.

### *Steady-State Sensitivity Analysis*

In this section we perform a steady-state sensitivity analysis by calculating the steady-state probability vector using each of the five states in Table 1 as the return state following a false signal. The steady-state probability vector contains the probability that the CUSUM statistic is in a given state for a particular choice of return-level (following a false signal). These steady-state vectors are shown in Table S2. For completeness, the sixth column in Table S2 gives the geometric CUSUM steady-state probabilities for a fixed-shift process using a return to the zero state. We note that all six steady-state vectors are similar in that they all have a probability mass of approximately 0.4 in the first element with the remaining probability spread over the remaining 319 elements for the Bernoulli CUSUM or 259 elements for the geometric CUSUM. Figure 1 shows a graph of the values of the Bernoulli steady-state vector with a zero-return versus state number. The shape of the steady-state vector is interesting, but we are not aware of any meaningful implications of the shape.

The steady-state ARL (ANI) value is obtained by summing the product of the corresponding entries of the steady-state vector and the appropriate ARL (ANI) vector. Table 2 gives the Bernoulli CUSUM steady-state ARL (ANI) values for the five return values considered in Table 1. The sixth column in Table 2 gives geometric CUSUM steady-state ANI values for a fixed-shift process using a return to the zero state. Note

that the five columns using different return states for the CUSUMs all show quite similar steady-state ARL (ANI) results. That is, the steady-state ARL evaluations are only slightly affected by the choice of return state. The steady-state  $ARL_{IC}$  ( $ANI_{IC}$ ) values are within a few percent of the zero-state values shown in Table 1, while the out-of-control values can differ by as much as 15%. The difference between the steady-state ARL (ANI) value when a large return value is used and the corresponding initial-state ARL (ANI) value for a large headstart, on the other hand, can be quite sizable. For example, the initial-state  $ARL_{IC}$  ( $ANI_{IC}$ ) value for a Bernoulli CUSUM with the extreme headstart is 11,863.60 while the steady-state  $ARL_{IC}$  ( $ANI_{IC}$ ) of 28,884.02 is much larger. This demonstrates that when a large headstart is used, steady-state performance is often not achieved due to an early signal.

We also calculated steady-state random-shift ANI values for the four geometric CUSUMs with  $k = 61$ ,  $h = 260$ , and with return values (0, 100, 130, 259) using the procedure of Bourke (2001b). These geometric CUSUMs correspond to Bernoulli CUSUMs with return values (60/61, 160/61, 190/61 and 319/61). We found that the ANI values calculated using the method in Bourke (2001b) exactly match the results shown in our Table 2, which were calculated using a separate Markov-chain methodology. By demonstrating exact agreement between corresponding steady-state evaluations for the Bernoulli and geometric CUSUMs, we have provided evidence that two quite separate Markov-chain-based methodologies (Crosier, 1986; Lucas and Saccucci, 1990; and Bourke, 2001b) for such evaluations are both correctly implemented. See Appendix A for a detailed description of the approach we used to model the steady-state distribution, including how we restarted the CUSUM statistic after an out-of-control signal.

### *Comparison of Simulated and Calculated Vectors*

In this section, we compare the simulated results shown in Table 3 of Szarka & Woodall (2012) with the calculated results in our Tables 1 and 2. The ANI values for the zero-state geometric CUSUM shown in Table 3, column [1], of Szarka & Woodall (2012) match, within expected simulation error, our calculated values for the zero-state Bernoulli CUSUM with a headstart of 60/61 shown in Table 1. This is expected since a geometric CUSUM with a return of 0 is equivalent to a Bernoulli CUSUM with a return of 60/61, based on the equivalence results of Chang & Gan (2001). The ANI values for the steady-state fixed-shift geometric CUSUM shown in Table 3 of Szarka & Woodall (2012), column [2], match our corresponding values in Table 2, column [6], except for the in-control ANI values. Additionally, the steady-state random-shift geometric CUSUM ANI values shown in Table 3 of Szarka & Woodall (2012), column [3], match our corresponding calculated values for the steady-state Bernoulli CUSUM with a headstart of 60/61 in Table 2 except for the in-control ANI values.

We now consider the large differences between Szarka & Woodall's simulated values and our calculated ANI values. Table 3 of Szarka & Woodall (2012) displays (for the in-control state) the same simulated ANI value of 29,201.6 for the zero-state, steady-state fixed-shift, and steady-state random-shift geometric CUSUMs (columns [1], [2], and [3]). The values in columns [2] and [3] (in Table 3 of Szarka & Woodall (2012)) appear to be based on the simulated results for the zero-state evaluation shown in column [1]. Our calculated values differ because we are evaluating three different CUSUM properties that have similar, but different, ANI values. To be explicit, we calculate an in-control ANI value of 29,148.55 for the initial-state Bernoulli CUSUM with a headstart of 60/61, using Equation (4) in Appendix A and a Bernoulli CUSUM  $R$  matrix, and we get the same ANI value when we use Equation (4) with a geometric

CUSUM  $R$  matrix and a zero headstart. We get an in-control ANI value of 28,419.40 for the steady-state fixed-shift geometric CUSUM by using the geometric CUSUM steady-state vector and multiplying by 100 (because  $1/p = 100$ ). We calculate an in-control ANI value of 28,979.72 for the steady-state Bernoulli CUSUM with a return of 60/61 in two equivalent ways, one using a steady-state calculation based on a Bernoulli  $R$  matrix and a return to the 60/61 state, and the other using the method of Bourke (2001b). For designing monitoring procedures, we note that ARL or ANI values seldom, if ever, need to be recorded with more than three significant figures. However, here we record our ARL and ANI values to two decimal places (resulting in higher precision for many table entries than using three significant figures) because we are displaying small differences between simulated and exact results. Our three exact ANI values (29,148.55; 28,419.40; 28,979.72) differ, in some cases considerably, from the single repeated value (29,201.6) reported in Table 3 of Szarka & Woodall for these three different quantities. The extent of the differences between our exact values and the simulated values in Szarka & Woodall (2012) may be judged in terms of the number of standard errors as follows: 1.8, 26.1, and 7.4, respectively.

We specifically note that the exact out-of-control ANI values for the case where  $p = 1.0$  are 5.0 for the zero-state geometric CUSUM, 4.2 for the steady-state fixed-shift geometric CUSUM, and 5.2 for the steady-state random-shift geometric CUSUM. Because the ANI value of 5.2 is larger than any element of the geometric CUSUM when  $p = 1.0$ , it would appear that this value is incorrect. However, when the geometric CUSUM is simulated for a random-shift process, the states of a Bernoulli CUSUM are reached after a long string of zeros. The ARL (ANI) vector for a Bernoulli CUSUM with a shift to  $p = 1.0$  is equal to  $ARL_{320}$  ( $\underline{ANI}_{320}$ ) = ( $\underline{6}_{20}$ ,  $\underline{5}_{60}$ ,  $\underline{4}_{60}$ ,  $\underline{3}_{60}$ ,  $\underline{2}_{60}$ ,  $\underline{1}_{60}$ ), so a steady-state ANI for the geometric CUSUM greater than 5 is possible. This explains

why an ANI value of 5.2 was simulated by Szarka & Woodall (2012) and calculated by us.

### **A Reconsideration of Tables 6 and 8 in Chang & Gan (2007)**

In the Introduction, we mentioned the conclusion drawn in Chang & Gan (2007, p. 874) that “*the geometric CUSUM chart is the most sensitive in detecting increases in  $p$ .*” Their evidence in support of this conclusion was provided in Tables 6 and 8 of that paper, and we now reconsider some evaluations presented in those tables. We first use Table 6 to give an explicit example of how the comparison of zero-state and headstart ANIs lead to that incorrect conclusion.

Table 3 in this paper contains in-control and out-of-control Bernoulli CUSUM ARL values for two of the cases considered in Table 6 of Chang & Gan (2007). These ARL values are in columns [2] and [3] of our Table 3. We label these columns [2]=CG[2] and [3]=CG[5] to indicate that they contain the ANIs from Table 6 of Chang & Gan (2007). Because of the Chang & Gan (2001) equivalence theorem, the Bernoulli CUSUMs with a headstart that are shown in columns [2] and [4] (of our Table 3) are also equivalent to geometric CUSUMs with parameters  $(k_G=1195, h_G=827, \text{headstart}=0)$  and  $(k_G=1195, h_G=893, \text{headstart}=0)$ , respectively. Chang and Gan compared the results for a zero-state Bernoulli CUSUM, which we provide in column [3], to a zero-state geometric CUSUM, which we provide in column [2], by “matching” their in-control ANIs to approximately 30,000. But this zero-state geometric CUSUM is equivalent to a Bernoulli CUSUM with a headstart of 1194/1195. The authors concluded that the geometric CUSUM was superior because the out-of-control ANI of 1,514 for the Bernoulli CUSUM given in column [3] is larger than the out-of-control ANI value of 1,018 for the geometric CUSUM given in column [2]. However, the authors were not

evaluating comparable CUSUMs because an initial-state Bernoulli CUSUM with a headstart of 0 is a worst-case CUSUM and does not have an equivalent geometric CUSUM.

We now reconsider the steady-state evaluations presented in Table 8 of Chang & Gan (2007) where we found errors in their simulation results. The geometric CUSUM with  $k_G = 1195$ ,  $h_G = 822$ , with a zero headstart, was proposed to detect a shift from  $p_0 = 0.0003$  to  $p_1 = 0.0018$ , while the Bernoulli CUSUM with  $k = 1/1195$ ,  $h = 2000/1195$ , with a zero headstart, was proposed as an alternative scheme to detect this shift. Chang & Gan simulated the results of Table 8 using 500,000 simulation replications at each value of  $p$  to obtain their results. In Table 4 of this paper, we provide the simulation values taken from Table 8 of Chang & Gan (2007) for the above-mentioned CUSUMs, together with the corresponding exact evaluations using the Markov-chain-based methodology described in this paper. For convenience in referring to Table 4, we have numbered the columns from [1] to [5]. As done previously, we use the additional notation [1]=CG[2] and [5]=CG[5] to indicate that columns [1] and [5] in our Table 4 contains the simulation evaluations from columns [2] and [5] from Table 8 of Chang & Gan (2007). The corresponding Markov-chain-based evaluations (that we calculated) are given in columns [2] and [4]. We also carried out the Markov-chain-based evaluation of steady-state ANI for the equivalent Bernoulli CUSUM, and the values are identical with those for the geometric CUSUM in column [2] of Table 4.

It is very noticeable that the simulation evaluations in column [1] differ considerably from the Markov-chain-based evaluations in column [2]. In order to demonstrate the likely closeness of simulation evaluations (based on 500,000 replications) to the exact evaluations in column [2], we repeated the simulations for this geometric CUSUM. Our simulation results are given in column [3], and one can

immediately see their closeness to the exact values in column [2]. It appears that the reported simulation evaluations (in column [1] of Table 4) for this geometric CUSUM as given in Table 8 of Chang & Gan (2007) are in error. This error in simulation has contributed to the incorrect conclusion in Chang & Gan (2007) that “*the geometric CUSUM chart is the most sensitive in detecting increases in  $p$ .*” We also note the differences the authors observed between the Bernoulli CUSUM and geometric CUSUM were minor compared to the simulation problems described above. There are differences between steady-state ARLs using zero and  $(m-1)/m$  returns. These differences were also small in our steady-state sensitivity study.

There are also simulation evaluations in Table 8 of Chang & Gan (2007) for a second geometric CUSUM with parameters  $k_G = 3094$ ,  $h_G = 5025$ , with a zero headstart. Again, we found that these simulation evaluations differed considerably from the corresponding exact evaluations. Table 8 also contains ANIS evaluations for three Bernoulli CUSUMs. We selected the CUSUM with parameters  $k_B = 1/1195$  and  $h_B = 2000/1195$ . The simulation evaluations for this Bernoulli CUSUM from Table 8 are given in column [5] of Table 4 in this paper. The corresponding Markov-chain-based evaluations are given in column [4], and one can see that the simulation evaluations in column [5] are relatively close to the column [4] figures.

### **The Two Counts in $m$ Observations Monitoring Procedure**

Here we discuss two examples of the 2-in- $m$  monitoring procedure provided by Lucas (1989). This discussion enables one to see more explicitly why evaluating a geometric CUSUM with a fixed shift requires using only states from a geometric CUSUM, while evaluating a geometric CUSUM with a random shift requires using the states of a Bernoulli CUSUM with an  $(m-1)/m$  return. When an  $(m-1)/m$  headstart is used, a 2-in-

$m$  monitoring procedure will signal when a single count is observed before the  $m^{\text{th}}$  observation or when two counts are observed within  $m$  observations. When the headstart is zero, the 2-in- $m$  monitoring procedure will not signal on the first count.

We note that the 2-in- $m$  monitoring procedure is a Bernoulli CUSUM with  $h_B = 1.0$ ,  $k_B = 1/m$ . We note that one of the  $m$  values shown in Table 4 is  $m = 61$  to match the  $m$  value used in Table 3 of Szarka & Woodall (2012), while the other  $m$  value used is the  $m = 25$  value that was discussed in Lucas (1989). Both monitoring procedures are for  $p_0 = 0.01$ . The  $m = 25$  scheme is appropriate for detecting an order-of-magnitude shift to  $p_1 = 0.10$  while the  $m = 61$  scheme is for detecting a shift to  $p_1 = 0.025$ . The 2-in- $m$  procedure with an  $(m-1)/m$  headstart is also a geometric CUSUM with  $h_G = 1$ ,  $k_G = m$ , a geometric Shewhart chart, and a CCC-1 chart (termed  $RL_1$  in Bourke, 1991). Hereafter we will use the terminology geometric CUSUM to describe all these procedures that are 2-in- $m$  procedures with an  $(m-1)/m$  headstart. When geometric CUSUM procedures are discussed, only the headstart properties are usually shown. As a result, these charts appear to perform better than the Bernoulli CUSUM because the headstart properties appear to be better than the zero-state properties. We emphasize that the results are incomplete and misleading when only headstart properties are shown because the headstart properties may not describe the properties after the CUSUM has been operating for some time.

Table 5 shows the ARL (ANI) values of the 2-in- $m$  monitoring procedures described above. The first column shows zero-state Bernoulli CUSUM ARL values, while the second column shows the Bernoulli CUSUM ARL values for a headstart equal to  $(m-1)/m$ . Recall that the Bernoulli CUSUM with a headstart of  $(m-1)/m$  is equivalent to the geometric CUSUM with a headstart of 0. Because there is only one transient state for this geometric CUSUM chart, the ANI values shown in the second

column of Table 5 are also the steady-state ANI values for the fixed-shift geometric CUSUM control chart. Therefore, for the 2-in- $m$  case it is obvious that a geometric CUSUM with a fixed shift uses only the states of a geometric process.

Comparing the zero-state and headstart columns shows the advantage of a headstart at startup or when restarting after an ineffective control action. For example, using a headstart of 60/61 for the 2-in-61 Bernoulli CUSUM decreases the zero-state  $ARL_{IC}$  ( $ANI_{IC}$ ) from 320.83 to 220.83, an increase of 31% in false alarms, while reducing the zero-state  $ANI_{OOC}$  ( $ARL_{OOC}$ ) by nearly 50% for values of  $p$  greater than 0.10. Similarly, using a headstart of 24/25 for the 2-in-25 Bernoulli CUSUM decreases the  $ANI_{IC}$  ( $ARL_{IC}$ ) from 566.59 to 466.59, a decrease of approximately 17%, while reducing the  $ANI_{OOC}$  ( $ARL_{OOC}$ ) by nearly 50% for values of  $p$  greater than 0.10. We note again that this apparently better performance for CUSUMs with a headstart has caused some researchers to claim that a CRL procedure, such as a geometric CUSUM or a CCC- $r$  procedure, should be used for monitoring “high quality processes,” (i.e., processes where  $p$  values are small) when they can be no better than a Bernoulli CUSUM. Exposing incorrect claims of monitoring superiority is currently an active research area. See, for example, Bourke (2008), Knoth (2016), Knoth et al. (2021a) and Knoth et al. (2021b).

Table 5 also contains the steady-state ARLs for the 2-in-61 Bernoulli CUSUM with returns equal to 0 and 60/61 and the 2-in-25 Bernoulli CUSUM with returns equal to 0 and 24/25. We again note the very similar steady-state ARLs for these two quite different return values. Examination of Table 5 shows that the steady-state ARLs are smaller than the corresponding zero-state Bernoulli ARLs ( $ANIs$ ) with a headstart of 0, but larger than the corresponding values with a headstart of  $(m-1)/m$ . In other words, steady-state Bernoulli CUSUM ARL ( $ANI$ ) values are larger than the  $ANI$  values for

the geometric CUSUM. Steady-state ARLs (ANIs) must be smaller than the zero-state values because zero-state ARL (ANI) values are worst-case values; they are the first and largest element in an ARL (ANI) vector.

Because there is a single transient state, the ANI values for the geometric Shewhart chart with  $LCL = m$  are also zero-state geometric CUSUM values with  $k_G = m$  and  $h_G = 1$  and steady-state fixed-shift geometric CUSUM ANI values. We again observe that, if results for a geometric CUSUM chart with a fixed-shift are desired, the results are obtained by using only the states from a geometric process. For a geometric CUSUM with a random-shift process, a consecutive run of more than  $m$  zeros before the shift occurs indicates that the zero state for a Bernoulli CUSUM has been reached.

### **Comments about CCC- $r$ Charts**

In this section, we make a few observations about CCC- $r$  charts. As previously mentioned, Calvin (1983) first introduced the idea of using CRL information. Bourke (1991) showed that the CCC-1 ( $RL_1$ ) chart performs poorly in detecting small shifts in  $p$ . This is because the CCC-1 chart uses data in a single CRL, and there is no carry-forward of this information. As a result, Bourke (1991) proposed the use of a moving sum of two CRLs, which he referred to as the  $RL_2$  chart. Xie et al. (1998) extended CCC charts to CCC- $r$  charts which are based on the cumulative count of items until the  $r^{th}$  nonconforming item is observed. An obvious way to detect smaller shifts is to use a CUSUM procedure. CUSUM procedures have optimality properties as proved by Moustakides (1986). We recommend using CUSUM procedures over other control chart procedures when it is important to detect small shifts because no other monitoring procedure has ever been shown to be superior to an appropriately-designed CUSUM chart.

As with the geometric CUSUM, there is a “hidden headstart” aspect of CCC- $r$  procedures. For any CCC- $r$  procedure, a string of zeros greater than or equal to the LCL will lead to their “worst-case” properties that we have never seen shown. For a CCC- $r$  procedure, the worst-case scenerio will occur with a probability of at least  $(1-p)^{LCL-1}$  (the probability of not getting a nonconforming sample before the LCL.) When this occurs, the time to detect a shift approximately doubles because the first set of  $r$ -counts merely resets the procedure, and the second set of  $r$ -counts is needed to signal. For the CCC-1 chart, Table 5 explicitly shows this result because the ANI values for  $p \geq 0.05$  with an  $(m-1)/m$  headstart are about half the zero-state ANIs. When worst-case properties are not shown, a procedure can appear to perform better than it will in practice.

## Summary

We have taken a closer look at the claim that the geometric CUSUM is superior to the Bernoulli CUSUM. Using exact methods, we found simulation errors in Chang & Gan (2007), and to a considerably lesser extent in Szarka & Woodall (2012), that have contributed to the misunderstanding of the equivalence of a geometric CUSUM and a Bernoulli CUSUM. Chang & Gan’s errors appear to be due to a faulty simulation. Szarka & Woodall, on the other hand, recorded the same in-control ANI values for zero-state, steady-state fixed-shift, and steady-state random-shift geometric CUSUMs, as well as for zero-state and steady-state random-shift Bernoulli CUSUMs. Using exact calculations, we were able to clearly differentiate between these quantities, which are similar but not identical.

Using exact methods, we also showed that when the return value for the Bernoulli CUSUM is matched to the corresponding return for the geometric CUSUM, the calculated Bernoulli steady-state ARL (ANI) values exactly match those calculated

for a geometric CUSUM. However, our sensitivity analysis of the steady-state distributions with different “return” states showed that there is little difference between the steady-state distributions for different return states. A steady-state geometric CUSUM with a fixed-shift uses only the states of the geometric CUSUM. A steady-state geometric CUSUM with a random-shift will reach the states of a Bernoulli CUSUM. We reiterate that for a process subject to a random shift, a geometric CUSUM with a zero reset-value (following each false signal) is equivalent to a corresponding Bernoulli CUSUM with a reset value of  $(m-1)/m$ . We argue that a random-shift model is generally more applicable than a fixed-shift model, which assumes that a process change can only occur at a nonconforming item.

We also agree with others that it can be misleading to only show headstart results, a practice that has been used in some control chart comparisons. Comparing the zero-state geometric CUSUM (which is equivalent to a Bernoulli CUSUM with a headstart of  $(m-1)/m$ ) to a zero-state Bernoulli CUSUM, gives misleading comparisons. We also note that the Bernoulli CUSUM’s headstart of  $(m-1)/m$  is usually smaller than the recommended headstart of  $h/2$  (Lucas & Crosier, 1982).

We conclude by noting that while there is an equivalence between the Bernoulli and geometric CUSUM charts, the choice of chart may depend on the particular application. The geometric CUSUM may be more usable to monitor an upward shift in the proportion nonconforming since the CUSUM statistic only has to be updated when a nonconforming item occurs. In a highly-conforming process, this will require far fewer updates to the control statistic compared to the Bernoulli CUSUM. This advantage would be especially large if one is manually updating or reviewing the control statistic. Another feature of the geometric CUSUM is that the use of Markov-chain methodology can accommodate both

fixed shift and random shift processes. On the other hand, the Bernoulli CUSUM may be simpler to use with automatic data collection, or if one is interested in detecting process changes in both directions.

### Appendix A: Details of the Markov Chain Approach

The Markov chain approach uses the probability transition matrix,  $\mathbf{P}$ , to calculate the ARL properties of the CUSUM charts. The probability transition matrix can be represented as:

$$\mathbf{P} = \begin{pmatrix} \mathbf{R} & (\mathbf{I} - \mathbf{R})\underline{\mathbf{1}} \\ \underline{\mathbf{0}}^T & 1 \end{pmatrix}, \quad (3)$$

where the element value of 1 in the last row and column indicates the absorbing or signal state and  $\mathbf{R}$  is the matrix for the transient states. The ARL vector is given by:

$$\underline{ARL} = (\mathbf{I} - \mathbf{R})^{-1}\underline{\mathbf{1}}. \quad (4)$$

The  $\mathbf{R}$  matrix for the Bernoulli CUSUM was given by Bourke (2001a) and Pollard et al. (2018), and the  $\mathbf{R}$  matrix for the version of the geometric CUSUM where the CRL does not include the final nonconforming item was given by Bourke (1991). For completeness, the  $\mathbf{R}$  matrix for the geometric CUSUM where the observation,  $Y_j$ , in equation (2) includes the final nonconforming item is given below:

State	0	1	...	(k-1)	(k)	(k+1)	...	(h-1)
<b>0</b>	$P[Y \geq k]$	$P[Y = k-1]$	...	$P[Y = 1]$	0	0	...	0
<b>1</b>	$P[Y \geq k+1]$	$P[Y = k]$	...	$P[Y = 2]$	$P[Y = 1]$	0	...	0
<b>2</b>	$P[Y \geq k+2]$	$P[Y = k+1]$	...	$P[Y = 3]$	$P[Y = 2]$	$P[Y = 1]$	...	0
<b>3</b>	$P[Y \geq k+3]$	$P[Y = k+2]$	...	$P[Y = 4]$	$P[Y = 3]$	$P[Y = 2]$	...	0
<b>⋮</b>	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
<b>(h-k-1)</b>	$P[Y \geq h-1]$	$P[Y = h-2]$	...	$P[Y = h-k]$	$P[Y = h-k-1]$	$P[Y = h-k-2]$	...	0
<b>(h-k)</b>	$P[Y \geq h]$	$P[Y = h-1]$	...	$P[Y = h-k+1]$	$P[Y = h-k]$	$P[Y = h-k-1]$	...	$P[Y = 1]$
<b>(h-k+1)</b>	$P[Y \geq h+1]$	$P[Y = h]$	...	$P[Y = h-k+2]$	$P[Y = h-k+1]$	$P[Y = h-k]$	...	$P[Y = 2]$
<b>⋮</b>	...	...	...	...	...	...	...	...
<b>(h-1)</b>	$P[Y \geq k+h-1]$	$P[Y = k+h-2]$	...	$P[Y = h]$	$P[Y = h-1]$	$P[Y = h-2]$	...	$P[Y = k]$

where  $P(Y = i) = p \cdot (1 - p)^{i-1}$  and  $P(Y \geq i) = (1 - p)^{i-1}$ . This transition matrix refers to the case of  $(h - 1) \geq k$ .

The steady-state ARL vector is calculated using the procedure of Crosier (1986), Lucas & Saccucci (1990), and Bourke (2001a). Following their procedure, the  $\mathbf{P}$  matrix is made ergodic by moving the 1 in the absorbing state of the last row to the column corresponding to the desired return state following a false signal. The steady-state vector,  $\underline{\mathbf{s}}$ , is obtained by solving the equation  $\mathbf{P} \cdot \underline{\mathbf{s}} = \underline{\mathbf{s}}$ , subject to the constraint that the entries of  $\underline{\mathbf{s}}$  sum to 1. A modified steady-state vector that includes only the transient states is obtained by deleting the entry associated with the absorbing state and adjusting the remaining entries by multiplying by  $1/(1-s_A)$ , where  $s_A$  is the entry of the steady-state vector corresponding to the absorbing state.

Knoth (2021) recently summarized known results about steady-state distributions. There have been two major approaches to calculating the steady-state distribution. The approach used here is referred to as the cyclic approach. The other approach is referred to as the conditional approach. We agree with Taylor (1968) who first described the two approaches and noted that the cyclic approach “would have been closer to the way control charts are actually used.” It models how we have conducted monitoring procedures when the process is in-control. Knoth’s mathematical approach shows that the steady state distribution, with a return to the  $i^{\text{th}}$  state, could be more simply obtained as:  $\underline{\mathbf{s}}_i = \underline{\mathbf{e}}_i \cdot (\mathbf{I} - \mathbf{R})^{-1}$ , where  $\underline{\mathbf{e}}_i = (0, \dots, 0, 1, 0, \dots, 0)$ . Although Knoth argued that “For whatever reason, Crosier (1986), Lucas & Saccucci (1990), and Champ (1992) propose slightly more involved methods that deliver, fortunately, the same result,” we note that the use of our model-building approach is more intuitive to a practitioner and is what led us to conduct the sensitivity study that examines the (small) differences between steady-state results for returns to different headstart values. We

also note that it is not clear how more than a single steady-state distribution could be obtained using the conditional steady-state approach. Therefore, it is not clear how different steady-state ARLs would be obtained when different headstarts were used.

### **Appendix B: Explanation of the Observed Difference of $1/p$**

In this section, we provide an explanation of the observed difference of  $(1/p)$  between the in-control ARLs that we observed in Table 1 and Table 4 for a Bernoulli CUSUMs with headstarts 0 and  $(m-1)/m$ . We denote the in-control zero-state ARL for a Bernoulli CUSUM as  $ARL(0, p)$  and the in-control ARL for Bernoulli CUSUM with a headstart of  $(m-1)/m$  as  $ARL((m-1)/m, p)$ .

Consider the Bernoulli CUSUM with parameters  $k_B = 1/m$  and  $h_B$  with an initial value,  $B_0 = 0$ :

$$B_j = \text{Max}[0, B_{j-1} + X_j - k_B].$$

The CUSUM values will remain at zero until we encounter  $X = 1$ . Initially, let us suppose that  $X_1 = 1$ . Then we have,

$$B_1 = \text{Max}[0, 0 + 1 - 1/m] = (m-1)/m.$$

Thus, for all those sequences of Bernoulli observations that begin with  $X_1 = 1$ , the ARL after the first observation ( $X_1 = 1$ ) is the same as for a Bernoulli CUSUM starting with a headstart of  $(m-1)/m$ . Note that this is also the ANI for a geometric CUSUM with a zero headstart.

Next, we abandon the supposition that  $X_1 = 1$ . The average number of CUSUM evaluations needed until the first  $X = 1$  occurs is  $1/p$ . All of these CUSUM evaluations remain at zero, for a CUSUM starting out from zero, until  $X = 1$  occurs. Hence, the difference between  $ARL(0, p)$  and  $ARL((m-1)/m, p)$  is equal to  $1/p$ .

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### **List of Tables and Figures**

Table 1. Bernoulli CUSUM ARL/ANIs for Various Headstarts

Table 2. Bernoulli CUSUM Steady-State ARL/ANI Values for a Random Shift and Geometric CUSUM Steady-State ANI Values for a Fixed Shift

Table 3. Initial-State Geometric CUSUM/Bernoulli CUSUM ARL/ANI Comparison

Table 4. Evaluations of Steady-State ARL/ANI for a Geometric CUSUM and a Bernoulli CUSUM

Table 5. ARLs / ANIs for the 2-in-m Procedure

Figure 1: Bernoulli Steady-State Vector (Return State = 1)

**Table 1. Bernoulli CUSUM ARL/ANIs for Various Headstarts\***

$(k_B=1/61 \text{ and } h_B=320/61)$					
Initial-State Headstart**					
	0	60/61	160/61	190/61	319/61
$p$	[1]~SW[4]	[2]~SW[1]	[3]	[4]	[5]
<b>0.010</b>	<b>29248.55</b>	<b>29148.55</b>	<b>27879.93</b>	<b>26820.66</b>	<b>11863.60</b>
0.015	2847.19	2780.52	2327.49	2090.88	590.23
0.020	951.73	901.73	664.21	568.51	133.64
<b>0.025</b>	<b>526.59</b>	<b>486.59</b>	<b>335.01</b>	<b>281.60</b>	<b>66.32</b>
0.030	359.50	326.17	217.75	182.05	44.71
0.040	219.24	194.24	127.00	105.94	28.20
0.050	157.79	137.79	89.68	74.80	21.15
0.060	123.32	106.65	69.37	57.91	17.13
0.070	101.23	86.94	56.57	47.34	14.48
0.080	85.82	73.32	47.72	40.12	12.59
0.090	74.44	63.33	41.20	34.88	11.15
0.100	65.68	55.68	36.19	30.92	10.02
0.150	41.17	34.50	22.06	20.06	6.67
0.200	30.19	25.19	15.67	15.00	5.00
0.300	20.00	16.67	10.05	10.00	3.33
0.500	12.00	10.00	6.00	6.00	2.00
0.750	8.00	6.67	4.00	4.00	1.33
1.000	6.00	5.00	3.00	3.00	1.00

\* Column numbers in this table are shown in brackets. The tilde symbol (~) is used to indicate that a column in our table is related to a column in Table 3 of Szarka & Woodall (2012). ARL (ANI) values for the in-control and out-of-control design parameters are shown in bold text.

\*\* Columns with headstarts equal to 60/61, 160/61, 190/61, and 319/61 equivalently give ANI values for a geometric CUSUM with headstarts equal to 0, 100, 130 and 259, respectively.

**Table 2. Bernoulli CUSUM Steady-State ARL/ANI Values for a Random Shift and Geometric CUSUM Steady-State ANI Values for a Fixed Shift\***

$p$	Bernoulli $k_B=1/61; h_B=320/61$					Geometric $h_G=260;$ $k_G=61$
	Return Values**					Return Value
	0	60/61	160/61	190/61	319/61	0
	[1]~SW[5]	[2]~SW[3]	[3]	[4]	[5]	[6]~SW[2]
<b>0.010</b>	<b>28980.64</b>	<b>28979.72</b>	<b>28969.68</b>	<b>28962.79</b>	<b>28884.02</b>	<b>28419.40</b>
0.015	2752.72	2752.40	2749.42	2747.73	2735.11	2582.73
0.020	896.99	896.81	895.27	894.48	889.53	803.93
<b>0.025</b>	<b>488.21</b>	<b>488.08</b>	<b>487.08</b>	<b>486.59</b>	<b>483.75</b>	<b>424.84</b>
0.030	329.93	329.83	329.10	328.75	326.79	281.99
0.040	198.90	198.83	198.36	198.14	196.95	166.71
0.050	142.24	142.19	141.84	141.68	140.82	118.02
0.060	110.72	110.68	110.40	110.28	109.61	91.31
0.070	90.64	90.61	90.38	90.28	89.73	74.45
0.080	76.72	76.69	76.50	76.41	75.95	62.83
0.090	66.49	66.47	66.30	66.22	65.82	54.33
0.100	58.65	58.63	58.48	58.41	58.06	47.84
0.150	36.83	36.82	36.73	36.68	36.46	29.87
0.200	26.94	26.93	26.86	26.83	26.67	21.81
0.300	17.66	17.65	17.61	17.59	17.48	14.30
0.500	10.48	10.47	10.45	10.44	10.37	8.49
0.750	6.95	6.95	6.93	6.92	6.88	5.63
1.000	5.20	5.19	5.18	5.18	5.14	4.21

\* Column numbers in this table are shown in brackets. The tilde symbol (~) is used to indicate that a column in our table is related to a column in Table 3 of Szarka & Woodall (2012). ARL (ANI) values for the in-control and out-of-control design parameters are shown in bold text.

\*\* Bernoulli CUSUM columns with return values equal to 60/61, 160/61, 190/61, and 319/61 equivalently give geometric steady-state random-shift ANI values with returns equal to 0, 100, 130, and 259, respectively.

**Table 3. Initial-State Geometric CUSUM/Bernoulli CUSUM ARL/ANI Comparison\***

	Bernoulli ( $h_B=2087/1195$ , headstart=0)	Bernoulli ( $h_B=2087/1195$ , headstart=1194/1195)	Bernoulli ( $h_B=2021/1195$ , headstart=0)	Bernoulli ( $h_B=2021/1195$ , headstart=1194/1195)
	Geometric No Equivalent	Geometric ( $h_G=893$ , headstart=0)	Geometric No Equivalent	Geometric ( $h_G=827$ , headstart=0)
$p$	[1]	[2]=CG[2]**	[3]=CG[5]**	[4]
0.0003	33354	30021	30001	26668
0.0018	1574	1018	1514	958

\* For all Bernoulli designs,  $k_B = 1/1195$  and for all geometric designs,  $k_G = 1195$ .  
Column numbers in this table are shown in brackets.

\*\* The equal symbol (=) is used to indicate that the results displayed in a column are reproduced from a column in Table 6 of Chang & Gan (2007).

**Table 4. Evaluations of Steady-State ARL/ANI for a Geometric CUSUM and a Bernoulli CUSUM\***

$p$	Geometric $k_G = 1195, h_G = 822$			Bernoulli $k_B = 1/1195, h_B = 2000/1195$	
	C&G Table 8	Markov chain (return = 0)	New Simulation (500,000 reps)	Markov chain (return = 0)	C&G Table 8
	[1]=CG[2]	[2]	[3]	[4]	[5]=CG[5]
<b>0.00030</b>	<b>28730</b>	<b>29408.3</b>	<b>29429</b>	<b>28759.6</b>	<b>28614</b>
0.00036	20187	20696.4	20711	20288.2	20151
0.00048	11641	12056.6	12044	11864.3	11815
0.00060	7721	8051.1	8053	7945.7	7934
0.00096	3442	3639.0	3635	3610.7	3592
0.00150	1731	1854.5	1854	1846.8	1850
<b>0.00180</b>	<b>1331</b>	<b>1438.8</b>	<b>1440</b>	<b>1434.3</b>	<b>1434</b>
0.00200	1149	1248.8	1250	1245.6	1247
0.00400	482	528.3	528	528.2	527
0.00800	220	242.9	243	243.5	243
0.01000	173	191.7	192	192.4	192

\* Column numbers in this table are shown in brackets. The equal symbol (=) is used to indicate that the results displayed in a column are reproduced from a column in Table 8 of Chang & Gan (2007).

**Table 5. ARLs / ANIs for the 2-in- $m$  Procedure\***

$p$	$k=1/61, h=1, m=61$				$k=1/25, h=1, m=25$			
	Initial-State		Steady-State		Initial-State		Steady-State	
	Headstart		Return		Headstart		Return	
	0	60/61**	0	60/61**	0	24/25**	0	24/25**
	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]
<b>0.010</b>	<b>320.83</b>	<b>220.83</b>	<b>301.90</b>	<b>293.32</b>	<b>566.59</b>	<b>466.59</b>	<b>556.69</b>	<b>554.57</b>
0.015	178.49	111.82	165.37	159.43	285.80	219.14	279.09	277.65
0.020	121.18	71.18	110.98	106.36	180.13	130.13	175.01	173.91
<b>0.025</b>	<b>91.21</b>	<b>51.21</b>	<b>82.77</b>	<b>78.95</b>	127.84	87.84	123.67	122.78
0.030	73.05	39.72	65.80	62.51	97.61	64.28	94.08	93.32
0.040	52.36	27.36	46.61	44.01	65.03	40.03	62.29	61.70
0.050	40.97	20.97	36.15	33.97	48.25	28.25	45.99	45.51
0.060	33.75	17.08	29.58	27.69	38.21	21.55	36.28	35.86
0.070	28.76	14.47	25.07	23.40	31.61	17.32	29.90	29.53
0.080	25.08	12.58	21.78	20.28	26.95	14.45	25.42	25.09
0.090	22.26	11.15	19.26	17.90	23.51	12.40	22.11	21.82
<b>0.100</b>	20.02	10.02	17.27	16.03	<b>20.87</b>	<b>10.87</b>	<b>19.58</b>	<b>19.30</b>
0.150	13.33	6.67	11.41	10.54	13.47	6.80	12.53	12.33
0.200	10.00	5.00	8.52	7.85	10.02	5.02	9.27	9.11
0.300	6.67	3.33	5.66	5.20	6.67	3.33	6.13	6.02
0.500	4.00	2.00	3.38	3.11	4.00	2.00	3.66	3.59
0.750	2.67	1.33	2.25	2.07	2.67	1.33	2.43	2.38
1.000	2.00	1.00	1.69	1.55	2.00	1.00	1.82	1.79

\* ARL (ANI) values for the in-control and out-of-control design parameters are shown in bold text.

\*\* A Bernoulli CUSUM with a headstart =  $(m-1)/m$  is also a fixed-shift geometric CUSUM with a headstart = 0, a CCC-1 (RL1), and a Geometric Shewhart chart. We usually describe all of these charts as a geometric CUSUM. A steady-state Bernoulli CUSUM with a return of  $(m-1)/m$  is also a steady-state random-shift geometric CUSUM with a return of zero.

**Figure 1: Bernoulli Steady-State Vector (Return State = 1)**

