Innovative Solution of a 2-D Elastic Transmission Problem

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Dedicated to Wolfgang L. Wendland
on the occasions of his 70th birthday

Abstract

This paper is concerned with a boundary-field equation approach to a class of boundary value problems exterior to a thin domain. A prototype of this kind of problems is the interaction problem with a thin elastic structure. We are interested in the asymptotic behavior of the solution when the thickness of the elastic structure approaches to zero. In particular, formal asymptotic expansions will be developed, and their rigorous justification will be considered. As will be seen, the construction of these formal expansions hinges on the solutions of a sequence of exterior Dirichlet problems, which can be treated by employing boundary element methods. On the other hand, the justification of the corresponding formal procedure requires an independence on the thickness of the thin domain for the constant in the Korn inequality. It is shown that in spite of the reduction of the dimensionality of the domain under consideration, this class of problems are in general not singular perturbation problems, because of appropriate interface conditions.

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1 Introduction

In this paper we consider a linear model elastic transmission problem posed in the exterior of a thin domain. The thin domain under consideration is an annular region in $\mathbb{R}^2$ with smooth boundaries occupied by a linear isotropic elastic material. The exterior region which imbeds this annular domain is also comprised of an elastic material with different (linear

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and isotropic) elastic properties. The outer boundary is fixed, and the inner boundary is allowed to move as the thickness of the annular region decreases (see 1). We are interested in the asymptotic behavior of the elastic displacement fields both in the interior and exterior of this annular region. In this study the material parameters are fixed, making this analysis different from similar studies in the engineering literature where the contrast between the materials is allowed to vary with the thickness.

In Section 2, we will first study the transmission problem in the case of fixed thickness, and employ boundary integral equations to reduce the (infinite) computational domain to a finite one. A weak formulation is then derived for the resulting nonlocal boundary problem, and the existence and uniqueness of solutions to this formulation are discussed. A key invertibility result is derived here which will be, along with Korn's inequality in a thin domain, very useful for the justification of the subsequent asymptotics.

We present an illustrative example which will motivate our asymptotic study. This example reveals that in this specific situation the displacement fields can be expanded in regular asymptotic series. With these insights, we will then present the general (formal) asymptotic procedure for studying the solutions of the transmission problem in the case of vanishing thickness. Finally, the asymptotic procedure is rigorously justified. It is interesting that the asymptotic procedure, which is applied to a coupled system of equations, actually results in a decoupling of these equations at any given order.

2 The case of fixed thickness, $\epsilon > 0$.

We begin with the transmission problem in the case where the thickness $\epsilon > 0$ of the annular region is fixed. We first describe the transmission problem of interest, and the associated classical uniqueness result. We then truncate the infinite computational domain by means of two integral equations, and introduce a weak formulation for the resulting nonlocal boundary value problem. This weak formulation will be advantageous both for proving existence of solutions, and for leading naturally to a coupled finite element - boundary element scheme. Using Korn's inequality, we establish existence and uniqueness results for our weak formulation; this yields an important invertibility result which will be the key ingredient in the justification of our asymptotic procedure in subsequent sections.

2.1 Formulation of the transmission problem

Let us describe the elastic transmission problem of interest. In what follows, let Ω_ϵ be a thin annular region in the plane bounded by smooth curves Γ_0 and Γ_ϵ, with elastic properties determined by the Lamé coefficients λ_1, μ_1. These coefficients do not change with changing thickness. The unbounded complement of Ω_ϵ is denoted by $\Omega_\infty := \mathbb{R}^2 \setminus \overline{Int(\Gamma_0)}$, and is composed of a linear elastic material with the Lamé parameters λ_2, μ_2. A schematic figure is shown in Figure (1).

We are interested in the 2-dimensional elastic displacement fields $\mathbf{u}_\epsilon \in H^1(\Omega_\epsilon)$ and $\mathbf{U}_\epsilon \in H^1_{loc}(\Omega_\infty)$. We are using the simplified notation $\mathbf{u} \in V$ to mean that each component of the vector-valued function $\mathbf{u}$ belongs to function space V.

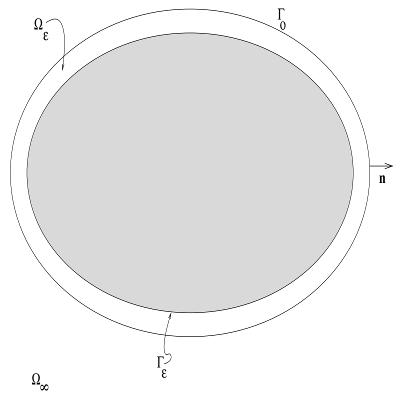

Figure 1: The configuration of the thin domain for fixed thickness. The trace of $\mathbf{u}$ from inside Ω_ϵ on Γ_0 is denoted $\mathbf{u}^-$. All normals are taken to point into the infinite exterior region.

The associated stress fields $\underline{\underline{\sigma_1}}(\mathbf{u}_\epsilon)$ and $\underline{\underline{\sigma_2}}(\mathbf{U}_\epsilon)$ are related to the (symmetric) strains $\mathbf{E}(\mathbf{u}_\epsilon)$ and $\mathbf{E}(\mathbf{U}_\epsilon)$ via the linear constitutive relationships

$$\underline{\underline{\sigma_k}}(\mathbf{u}_\epsilon) = \lambda_k(\mathrm{div}(\mathbf{u}_\epsilon))\mathbf{I} + 2\mu_k\mathbf{E}(\mathbf{u}_\epsilon), \qquad \mathbf{E}(\mathbf{u}_\epsilon)_{ij} := \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_i}{\partial x_j}), \qquad k = 1, 2,$$

where tr$M$ denotes the trace of a matrix M. Also note that

$$div\ \underline{\underline{\sigma_k}}(\mathbf{u}_\epsilon) = \mu_k\Delta\mathbf{u} + (\lambda_k + \mu_k)\nabla div\ \mathbf{u}, \qquad k = 1, 2.$$

Let $\mathbf{f} \in L^2(\Omega_\epsilon)$ be the density of a volume force on Ω_ϵ. The equilibrium equation in the annular region then becomes

$$div\ \underline{\underline{\sigma_1}}(\mathbf{u}_\epsilon) = -\mathbf{f} \qquad \text{in } \Omega_\epsilon. \tag{1a}$$

We assume that the inner boundary Γ_ϵ is clamped so that

$$\mathbf{u}_\epsilon = 0 \qquad \text{on } \Gamma_\epsilon \tag{1b}$$

In the exterior, we assume there is no volume force; the balance of forces gives

$$div\ \underline{\underline{\sigma_2}}(\mathbf{U}_\epsilon) = 0 \qquad \text{in } \Omega_\infty. \tag{1c}$$

Across the interface Γ_0 we have the transmission conditions

$$\mathbf{u}_\epsilon = \mathbf{U}_\epsilon + \mathbf{p}_0, \qquad T_1(\mathbf{u}_\epsilon) = T_2(\mathbf{U}_\epsilon) + \mathbf{q}_0, \qquad \text{on } \Gamma_0 \tag{1d}$$

for given data $\mathbf{p}_0 \in H^{1/2}(\Gamma_0), \mathbf{q}_0 \in H^{-1/2}(\Gamma_0)$ (the usual trace spaces). The *tractions* are defined by

$$T_i(\mathbf{u}) := \mathbf{n} \cdot \underline{\underline{\sigma_i}}(\mathbf{u}) = 2\mu_i\frac{\partial}{\partial n}\mathbf{u} + \lambda_i\ (div\ \mathbf{u})\mathbf{n} + \mu_i\mathbf{n} \times \nabla \times \mathbf{u}.$$

In the transmission conditions (1d), the data $\mathbf{p}_0$ describes a possible jump between the displacement fields (non-continuous behavior, disclocations) whereas $\mathbf{q}_0$ characterizes a possible jump between the traction fields, which can appear due to the different material parameters. In order to ensure uniqueness of solutions, we require that $\mathbf{U}_\epsilon$ be a *generalized regular function* in the sense that

$$(\mathbf{U}_\epsilon - \mathbf{w}) = O(1/|x|), \qquad \nabla(\mathbf{U}_\epsilon - \mathbf{w}) = O(1/|x|^2) \tag{1e}$$

where $\mathbf{w}$ is a constant rigid motion. See, for example, [6, 7, 1].

2.2 Reduction to bounded domain

We next use the Betti representation formula for elastic displacements to rewrite the exterior problem (1c,1e) in terms of an integral equation on Γ_0. The solution $\mathbf{U}_\epsilon$ of (1c) satisfies the modified Betti formula for all $x \in \Omega_\infty$:

$$\mathbf{U}_\epsilon(x) = -\int_{\Gamma_0} \mathbf{G}(x,y)T_2(\mathbf{U}_\epsilon)(y) - \mathbf{T}(x,y)\mathbf{U}_\epsilon(y)\, ds_y + \mathbf{a}. \tag{2}$$

Here, $\mathbf{a}$ is a constant vector, and $\mathbf{G}$ is the fundamental tensor:

$$\mathbf{G}(x,y) := \frac{\lambda_2 + 3\mu_2}{4\pi(\lambda_2 + 2\mu_2)} \left\{ -\ln|x-y|\mathbf{I} + \frac{\lambda_2 + \mu_2}{(\lambda_2 + 3\mu_2)} \frac{1}{|x-y|^2}(x-y)(x-y)^T \right\}$$

and

$$\mathbf{T}(x,y) = (T_{2y}(\mathbf{G}(x,y)))^T.$$

The traction T_{2y} operator contains derivatives with respect to the variable y. We note that the radiation condition (1e) will only be satisfied by this representation if the additional compatibility condition holds:

$$\int_{\Gamma_0} T_2(\mathbf{U}_\epsilon)\, ds = \mathbf{0}. \tag{3}$$

To incorporate this compatibility condition in what follows, we define

$$H_0^{-1/2}(\Gamma_0) := \left\{ \chi = (\chi_1, \chi_2) \in H^{-1/2}(\Gamma_0) | \langle \chi_1, 1 \rangle = \langle \chi_2, 1 \rangle = 0 \right\},$$

where $\langle \cdot, \cdot \rangle$ denotes duality pairing between $H^{1/2}(\Gamma_0) \times H^{-1/2}(\Gamma_0)$. Taking limits in (2) as $\Omega_\infty \ni x \to \Gamma_0$, we obtain the integral equation

$$(\frac{1}{2}\mathbf{I} - \mathbf{K})\mathbf{U}_\epsilon^+ + \mathbf{V}\tau = \mathbf{0} \qquad \text{on} \quad \Gamma_0, \tag{4a}$$

where the fields $\mathbf{U}_\epsilon^+$ and $\tau = T_2\mathbf{U}_\epsilon^+$ need to be interpreted as the appropriate traces on Γ_0 coming from Ω_∞. We have used the classical jump relations to arrive at this result. Here, $\mathbf{K}, \mathbf{V}$ are the double and single layer integral operators respectively, (see below).

We could also take the normal derivative of equation (2), and then take limits, to arrive at the integral equation

$$(\frac{1}{2}\mathbf{I} + \mathbf{K}')\tau + \mathbf{W}\mathbf{U}_\epsilon^+ = \mathbf{0}, \qquad \text{on} \quad \Gamma_0. \tag{4b}$$

The integral operator $\mathbf{W}$ is the familiar hypersingular integral operator, while $\mathbf{K}'$ is the adjoint of the double layer operator. We recall some of the main properties of the linear integral operators $\mathbf{K}, \mathbf{K}', \mathbf{V}$ and $\mathbf{W}$ below:

1. $\mathbf{V} : H^{-1/2}(\Gamma_0) \to H^{-1/2}(\Gamma_0)$, where $\mathbf{V}\mu(x) := \langle \mathbf{G}(x, \cdot), \mu \rangle$.
2. $\mathbf{K} : H^{1/2}(\Gamma_0) \to H^{1/2}(\Gamma_0)$, defined by $\mathbf{K}\mu(x) := \langle \mathbf{T}(x, \cdot), \mu \rangle$.
3. $\mathbf{W} : H^{1/2}(\Gamma_0) \to H^{-1/2}(\Gamma_0)$, defined by $\mathbf{W}\mu(x) := -\mathbf{T}_{2,x} \langle \mathbf{T}(x, \cdot), \mu \rangle$.
4. For all $\sigma \in H_0^{-1/2}(\Gamma_0)$, the operator $\mathbf{V}$ is positive definite, that is, there exists a constant γ_v independent of σ such that

$$\langle \sigma, \mathbf{V}\sigma \rangle \geq \gamma_v \|\sigma\|^2_{H^{-1/2}(\Gamma_0)}, \qquad \forall \, \sigma \, \in \, H_0^{-1/2}(\Gamma_0). \tag{5}$$

   Note that since the curve Γ_0 is independent of ϵ, so is the constant γ_v.
5. $\mathbf{W}$ is positive semi-definite, that is, for all $\mathbf{v} \in H^{1/2}(\Gamma_0)$,

$$\langle \mathbf{W}\mathbf{v}, \mathbf{v} \rangle \geq 0. \tag{6}$$

The following well-known result, relating the solutions of these integral equations to the solutions of the exterior problem, is stated here for completeness:

Theorem 2.1 *For any $\mu \in H^{1/2}(\Gamma_0), \chi \in H_0^{-1/2}(\Gamma_0)$, there exists a unique solution $\mathbf{u} \in H^1_{loc}(\Omega_\infty)$ satisfying (1c) and (1e) iff (4a, b) holds with $\mu = \mathbf{u}_\epsilon^+$ and $\chi = \tau_\epsilon$*

(see, for example, [2, 3])

With these integral equations, we are able to reduce the exterior problem for $\mathbf{U}_\epsilon$ to boundary integral equations of $\mathbf{U}_\epsilon^+$ or τ on the interface Γ_0. Once these Cauchy data have been obtained, one can use the Betti representation formula to reconstruct the solution $\mathbf{U}_\epsilon$ in the exterior domain.

Since we have transmission conditions relating $\mathbf{U}_\epsilon^+$ and $\tau_\epsilon = T_2(\mathbf{U}_\epsilon)$ to $\mathbf{u}_\epsilon$ and $T_1(\mathbf{u}_\epsilon)$ respectively, we can write the following equivalent *coupled system* in terms of the unknowns $\mathbf{u}_\epsilon, \tau_\epsilon$:

$$
\begin{aligned}
div \, \underline{\underline{\sigma_1}}(\mathbf{u}_\epsilon) &= -\mathbf{f} \qquad \text{in}\, \Omega_\epsilon, && \text{(7a)}\\
\mathbf{u}_\epsilon &= 0 \qquad \text{on } \Gamma_\epsilon, && \text{(7b)}\\
T_1 \mathbf{u}_\epsilon &= \tau_\epsilon + \mathbf{q}_0 \qquad \text{on } \Gamma_0, && \text{(7c)}\\
(\frac{1}{2}\mathbf{I} - \mathbf{K})(\mathbf{u}_\epsilon^- - \mathbf{p}_0) + \mathbf{V}\tau_\epsilon &= 0 \qquad \text{on } \Gamma_0, && \text{(7d)}\\
(\frac{1}{2}\mathbf{I} + \mathbf{K}')\tau_\epsilon + \mathbf{W}(\mathbf{u}_\epsilon^- - \mathbf{p}_0) &= 0 \qquad \text{on } \Gamma_0. && \text{(7e)}
\end{aligned}
$$

We have included both integral equations; clearly they are not independent and one is sufficient. However, it will prove advantageous from the point of view of analysis to use both. We note here that to obtain τ_ϵ at one point on Γ_0, we need the values of $\mathbf{u}_\epsilon^-$ all around

Γ_0. Conversely, in order to obtain $\mathbf{u}_\epsilon^-$, we need the values of τ_ϵ around Γ_0. The integral equations are thus *non-local* in nature, and hence the coupled system above is a *non-local boundary value problem.* We solve this new problem simultaneously for $\mathbf{u}_\epsilon$ and τ_ϵ, and use the transmission condition to obtain $\mathbf{U}_\epsilon^+$ on Γ_0. This then enables us to use the Betti formula, and finally provides a full solution to the original transmission problem.

2.3 Weak formulation

We now seek weak solutions of the nonlocal boundary value problem derived in the previous subsection. To be more precise, let us first introduce the function space $H_0^1(\Omega_\epsilon) := \{\mathbf{v} \in H^1(\Omega_\epsilon) | \mathbf{v} = \mathbf{0} \text{ on } \mathbf{\Gamma}_\epsilon\}$, and recall that $H_0^{-1/2}(\Gamma_0) := \left\{\chi \in H^{-1/2}(\Gamma_0) | \langle \chi_1, 1\rangle = \langle \chi_2, 1\rangle = 0\right\}$. We now consider the product space

$$\mathcal{H}_\epsilon := \left\{(\mathbf{v}, \chi) \in H_0^1(\Omega_\epsilon) \times H_0^{-1/2}(\Gamma_0)\right\}$$

equipped with the associated product norm

$$\|(\mathbf{v}, \chi)\|_{\mathcal{H}_\epsilon} := (\|\mathbf{v}\|^2_{H^1(\Omega_\epsilon)} + ||\chi||^2_{H^{-1/2}(\Gamma_0)})^{1/2}.$$

Multiplying equations (7a) and (7e) by the test function $\mathbf{v} \in H_0^1(\Omega_\epsilon)$ and its trace on Γ_0, and equation (7d) by $\chi \in H_0^{-1/2}(\Gamma_0)$, integration by parts of the products over Ω_ϵ and Γ_0 respectively then lead to the variational equations

$$\int_{\Omega_\epsilon} \underline{\underline{\sigma_1}}(\mathbf{u}_\epsilon) : \mathbf{E}(\mathbf{v})\, d\Omega_\epsilon - \langle T_1 \mathbf{u}_\epsilon, \mathbf{v}\rangle = \int_{\Omega_\epsilon} \mathbf{f} \cdot \mathbf{v}\, d\Omega_\epsilon, \tag{8a}$$

$$\langle \mathbf{v}, (\frac{1}{2}\mathbf{I} + \mathbf{K}')\tau_\epsilon\rangle + \langle \mathbf{v}, \mathbf{W}(\mathbf{u}_\epsilon - \mathbf{p}_0)\rangle = 0, \tag{8b}$$

$$\langle (\frac{1}{2}\mathbf{I} - \mathbf{K})(\mathbf{u}_\epsilon - \mathbf{p}_0) + \mathbf{V}(\tau_\epsilon), \chi\rangle = 0. \tag{8c}$$

Now, using the transmission conditions (7c) for τ_ϵ, $-\langle T_1(\mathbf{u}_\epsilon), \mathbf{v}\rangle = -\langle \tau_\epsilon + \mathbf{q}_0, \mathbf{v}\rangle$, we can add the equation (8b) to this relationship to obtain

$$\begin{aligned} -\langle T_1(\mathbf{u}_\epsilon), \mathbf{v}\rangle + 0 &= -\langle \tau_\epsilon + \mathbf{q}_0, \mathbf{v}\rangle + \langle \mathbf{v}, (\frac{1}{2}\mathbf{I} + \mathbf{K}')\tau_\epsilon\rangle + \langle \mathbf{v}, \mathbf{W}(\mathbf{u}_\epsilon - \mathbf{p}_0)\rangle \\ &= \langle \mathbf{v}, (-\frac{1}{2}\mathbf{I} + \mathbf{K}')\tau_\epsilon\rangle + \langle \mathbf{v}, \mathbf{W}(\mathbf{u}_\epsilon)\rangle - \langle \mathbf{v}^-, \mathbf{q}_0\rangle - \langle \mathbf{v}^-, \mathbf{W}\mathbf{p}_0\rangle. \end{aligned}$$

We reformulate the problem (8) as one for finding a displacement $\mathbf{u}_\epsilon$ and a traction τ_ϵ. More precisely, the variational formulation of our transmission problem for fixed thickness reads

Definition 2.1 (Problem $\mathbf{P}_\epsilon$) *Find* $(\mathbf{u}_\epsilon, \tau_\epsilon) \in \mathcal{H}_\epsilon$ *such that*

$$\left\{ \begin{aligned} \textstyle\int_{\Omega_\epsilon} \underline{\underline{\sigma_1}}(\mathbf{u}_\epsilon) : \mathbf{E}(\mathbf{v})\, dV + \langle \mathbf{v}^-, \mathbf{W}\mathbf{u}_\epsilon^- + (-\frac{1}{2}\mathbf{I} + \mathbf{K}')\tau_\epsilon\rangle &= \textstyle\int_{\Omega_\epsilon} \mathbf{f} \cdot \mathbf{v}\, dV + \langle \mathbf{v}^-, \mathbf{W}\mathbf{p}_0 + \mathbf{q}_0\rangle \\ \langle (\tfrac{1}{2}\mathbf{I} - \mathbf{K})\mathbf{u}_\epsilon + \mathbf{V}\tau_\epsilon, \chi\rangle &= \langle (\tfrac{1}{2}\mathbf{I} - \mathbf{K})\mathbf{p}_0, \chi\rangle \end{aligned} \right. \tag{9}$$

for all $(\mathbf{v}, \chi) \in \mathcal{H}_\epsilon$*, under the assumption that the given data* $(\mathbf{f}, \mathbf{q}_0) \in L^2(\Omega_\epsilon) \times H^{-1/2}(\Gamma_0)$ *satisfy the compatibility condition*

$$\int_{\Omega_\epsilon} \mathbf{f}\, dV + \int_{\Gamma_0} \mathbf{q}_0\, ds = 0. \tag{10}$$

We note that the assumption on the data is one of the consequences of seeking $\tau_\epsilon \in H_0^{-1/2}(\Gamma_0)$, that is, from the transmission condition (7c), requiring $T_1(\mathbf{u}_\epsilon) - \mathbf{q}_0 = \tau_\epsilon \in H_0^{-1/2}(\Gamma_0)$.

2.4 The main a priori result

The existence and uniqueness of solutions to the problem $(\mathbf{P}_\epsilon)$ will be established easily via the Lax-Milgram lemma. To this end, define the bounded, continuous bilinear form $\mathcal{A}_\epsilon$ from $\mathcal{H}_\epsilon \times \mathcal{H}_\epsilon \to \mathbb{R}$ as

$$\mathcal{A}_\epsilon\left((\mathbf{u}, \sigma), (\mathbf{v}, \chi)\right) := \int_{\Omega_\epsilon} \underline{\underline{\sigma_1}}(\mathbf{u}) : \mathbf{E}(\mathbf{v})\, d\Omega_\epsilon + \langle \mathbf{v}, \mathbf{W}\mathbf{u} + (-\frac{1}{2}\mathbf{I} + \mathbf{K}')\sigma\rangle + \langle (\frac{1}{2}\mathbf{I} - \mathbf{K})\mathbf{u} + \mathbf{V}(\sigma), \chi\rangle \tag{11}$$

We note that this corresponds to adding both equations in $(\mathbf{P}_\epsilon)$. Let

$$\mathcal{F}_\epsilon(\mathbf{v}, \chi) := \int_{\Omega_\epsilon} \mathbf{f} \cdot \mathbf{v}\, dV + \langle \mathbf{v}^-, \mathbf{W}\mathbf{p}_0 + \mathbf{q}_0\rangle + \langle (\frac{1}{2}\mathbf{I} - \mathbf{K})\mathbf{p}_0, \chi\rangle.$$

It is not hard to see that $\mathcal{F}_\epsilon$ is a continuous linear functional on $\mathcal{H}_\epsilon$. The problem $\mathbf{P}_\epsilon$ can then be rewritten as: *Find* $(\mathbf{u}_\epsilon, \tau_\epsilon) \in \mathcal{H}_\epsilon$ *such that for all* $(\mathbf{v}, \chi) \in \mathcal{H}_\epsilon$,

$$\mathcal{A}_\epsilon\left((\mathbf{u}_\epsilon, \tau), (\mathbf{v}, \chi)\right) = \mathcal{F}_\epsilon(\mathbf{v}, \chi).$$

Theorem 2.2 *The bilinear form* $\mathcal{A}_\epsilon$ *is* $\mathcal{H}_\epsilon$*-elliptic; in particular, there is a thickness* $\epsilon_0 > 0$ *such that for all* $\epsilon \in (0, \epsilon_0)$ *there exists a constant* α *independent of* ϵ *such that*

$$\alpha \|(\mathbf{u}, \sigma)\|^2_{\mathcal{H}_\epsilon} \le \mathcal{A}_\epsilon\left((\mathbf{u}, \sigma), (\mathbf{u}, \sigma)\right) \quad \forall (\mathbf{u}, \sigma) \in \mathcal{H}_\epsilon. \tag{12}$$

Proof. Setting $(\mathbf{v}, \chi) = (\mathbf{u}, \sigma)$ in the definition of $\mathcal{A}_\epsilon$. we obtain

$$\begin{aligned} \mathcal{A}_\epsilon\left((\mathbf{u}, \sigma), (\mathbf{u}, \sigma)\right) &= \int_{\Omega_\epsilon} \underline{\underline{\sigma_1}}(\mathbf{u}) : \mathbf{E}(\mathbf{u})\, d\Omega_\epsilon + \langle \mathbf{u}, \mathbf{W}\mathbf{u}\rangle + \langle \mathbf{V}\sigma, \sigma\rangle \\ &\ge C_1 \|\mathbf{E}(\mathbf{u})\|^2_{L^2(\Omega_\epsilon)} + \gamma_1 \|\sigma\|^2_{H_0^{-1/2}(\Gamma_0)}, \end{aligned}$$

where we have used the ellipticity of the strain-stress relation. At this stage, we invoke Korn's inequality, which states that

$$\int_{\Omega_\epsilon} \mathbf{E}(\mathbf{u}) : \mathbf{E}(\mathbf{u})\, d\Omega_\epsilon \ge C \|\mathbf{u}\|^2_{H^1(\Omega_\epsilon)}, \tag{13}$$

where the constant C is independent of ϵ as can be seen as follows. We consider an arbitrary element $\mathbf{u} \in H_0^1(\Omega_\epsilon)$ and its zero-extension $\tilde{\mathbf{u}}$ on Ω_0. The domain Ω_0 is a fixed annular region

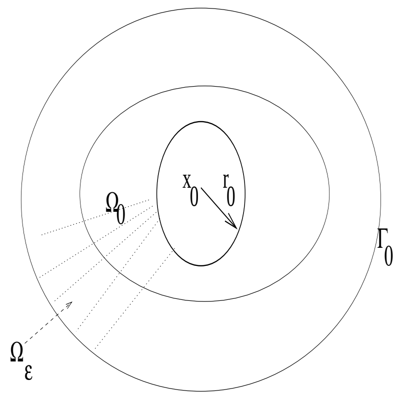

Figure 2: Geometry for proof of Korn's inequality

with the boundary $\partial\Omega_0 = \Gamma_0 \cup \{x \in \mathbb{R}^2 : |x - x_0| = r_0\}$, which contains all $\Omega_\epsilon, \epsilon \leq \epsilon_0$, in its interior and $x_0 \in \Omega_0 \setminus \overline{\Omega}_{\epsilon_0}$ is a fixed point (see Figure (2). Then in [12], it is proved that there is a constant $C = C(\Omega_0)$ such that

$$\int_{\Omega_0} \mathbf{E}(\tilde{\mathbf{u}}) : \mathbf{E}(\tilde{\mathbf{u}}) dx \geq C\|\tilde{\mathbf{u}}\|^2_{H^1(\Omega_0)}$$

for the set of zero-extensions $\tilde{\mathbf{u}}$ of elements $\mathbf{u} \in H^1_0(\Omega_\epsilon)$. Now

$$\int_{\Omega_\epsilon} \mathbf{E}(\mathbf{u}) : \mathbf{E}(\mathbf{u})\, d\Omega_\epsilon = \int_{\Omega_\epsilon} \mathbf{E}(\tilde{\mathbf{u}}) : \mathbf{E}(\tilde{\mathbf{u}}) d\Omega_\epsilon = \int_{\Omega_0} \mathbf{E}(\tilde{\mathbf{u}}) : \mathbf{E}(\tilde{\mathbf{u}}) dx \geq C\|\tilde{\mathbf{u}}\|^2_{H^1(\Omega_0)} = C\|\tilde{\mathbf{u}}\|^2_{H^1(\Omega_\epsilon)},$$

which is desired inequality (13). The $\mathcal{H}_\epsilon$-ellipticity of $\mathcal{A}_\epsilon$, i.e., (12), then follows with $\alpha := min\{C_1 C, \gamma_1)\}$ independent of ϵ.

This ellipticity result, the continuity of $\mathcal{F}_\epsilon$ and the Lax-Milgram lemma guarantees the existence of a unique solution to the problem $(\mathbf{P}_\epsilon)$. We note here again the use of Korn's inequality in a thin region which enabled us to determine a coercivity constant C independent of ϵ.

We end this section with a brief summary of results thus far. We have derived a weak formulation for the elastic-elastic transmission problem in the case when the thickness of the annular region, ϵ, is fixed. We have proved existence and uniqueness of solutions to this variational problem; in particular, we have shown that the norm of the inverse is bounded.

We shall now investigate the asymptotic behavior of the solutions as $\epsilon \to 0+$. However, a simple, tractable example where the details can be explicitly computed can provide much insight into asymptotic behavior, and will motivate our choice of asymptotic expansion. We therefore digress to present such an example.

3 An illustrative example

In this section, we consider the elastic transmission problem in the case of a special geometry: that of concentric circles. We also make the assumption of a constant body force density

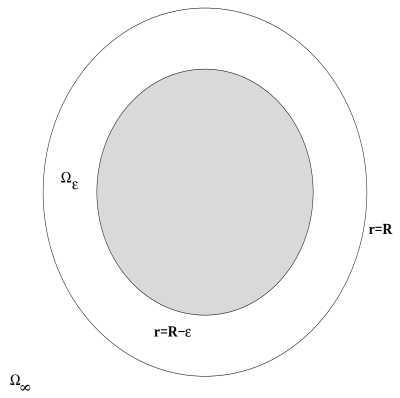

Figure 3: The configuration of the thin domain for the model problem

$\mathbf{f}$; this assumption can be relaxed to include smooth forces, but the consequent analysis even for the simple geometry is complicated. The simple annular geometry will allow us to explicitly compute the solution of the classical transmission problem using Fourier series. We will notice that the coefficients of the series will contain the thickness parameter $\epsilon > 0$.

3.1 A model problem

In this section, let Γ_0 denote a circle centered at the origin, with radius $R > 0$. Let Γ_ϵ be a circle centered at the origin with radius $R - \epsilon$, where $0 < \epsilon << R$. The annular region between Γ_ϵ and Γ_0 is called Ω_ϵ. This geometry is illustrated in Figure (3). We recall that the classical transmission problem is to find $\mathbf{u}_\epsilon$ and $\mathbf{U}_\epsilon$ such that

$$\begin{aligned} div\ \underline{\underline{\sigma_1}}(\mathbf{u}_\epsilon) &= \mathbf{f} \qquad \text{in } \Omega_\epsilon, && (14a)\\ \mathbf{u}_\epsilon &= \mathbf{0} \qquad \text{on } \Gamma_\epsilon, && (14b)\\ div\ \underline{\underline{\sigma_2}}(\mathbf{U}_\epsilon) &= \mathbf{0} \qquad \text{in } \Omega_\infty, && (14c)\end{aligned}$$

where

$$div\ \underline{\underline{\sigma_i}}(\mathbf{u}) = \mu_i \Delta \mathbf{u} + (\lambda_i + \mu_i)\nabla div\ \mathbf{u}$$

and where $\mathbf{f}$ = constant describes the body force density. Across the interface Γ_0 we have the transmission conditions

$$\mathbf{u}_\epsilon = \mathbf{U}_\epsilon + \mathbf{p}, \qquad T_1(\mathbf{u}_\epsilon) = T_2(\mathbf{U}_\epsilon) + \mathbf{q} \qquad \text{on } \Gamma_\epsilon \tag{14d}$$

for given data $\mathbf{p} \in H^{1/2}(\Gamma_0), \mathbf{q} \in H^{-1/2}(\Gamma_0)$. We also have the behavior at infinity given by

$$(\mathbf{u} - \mathbf{w}) = O(1/|x|), \qquad \nabla(\mathbf{u} - \mathbf{w}) = O(1/|x|^2), \tag{14e}$$

where $\mathbf{w}$ is a constant rigid motion. Recall that the material properties λ_i, μ_i are independent of $\epsilon > 0$. We note that for a planar strain problem the components of the (symmetric) stress tensor can be written in terms of the Airy stress function ϕ (in polar coordinates) (see, e.g., [9, 5]).

$$\underline{\underline{\sigma}} = \begin{pmatrix} \tau_{rr} & \tau_{r\theta} \\ \tau_{r\theta} & \tau_{\theta\theta} \end{pmatrix}, \qquad \tau_{rr} = r^{-2}\frac{\partial^2 \phi}{\partial \theta^2} + r^{-1}\frac{\partial \phi}{\partial r}, \qquad \tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right), \tau_{\theta,\theta} = \frac{\partial^2 \phi}{\partial r^2}.$$

The static equilibrium equations for the stress then reduce to the following equation for the Airy stress function :

$$\nabla^4 \phi = C\ \nabla \cdot \mathbf{f} \tag{15}$$

with $C = \frac{2\nu-1}{1-\nu}$ in the case of plane strain, and $C = \nu - 1$ for plane stress problems. Here ν denotes the Poession ratio.

Returning to our model problem, suppose the stresses in Ω_ϵ are written in terms of the Airy stress function ϕ, and the stresses in the exterior region Ω_∞ are written in terms of the Airy stress function ψ. We then have

$$\nabla^4 \phi = 0 \quad \text{in } \Omega_\epsilon, \qquad \nabla^4 \psi = 0 \quad \text{in } \Omega_\infty. \tag{16}$$

That is, both ϕ and ψ satisfy the biharmonic equation, since we have constant force density in Ω_ϵ. We also need to rewrite the boundary condition on Γ_ϵ and the interface conditions on Γ_0 in terms of the Airy stress functions.

We seek a solution of the biharmonic equation exterior to a circle such that the stresses at infinity are bounded and the displacement is a generalized regular displacement in the sense of (1e). We also require that the displacements and stresses are single-valued (see [11]). A solution of the biharmonic equation satisfying these requirements is given by the so-called Michell solution ([10]):

$$\begin{aligned} \psi(r,\theta) &= b_o \ln r + A_0\theta + (b_1 r \ln r + \frac{c_1}{r} + A_1 r\theta)\exp(i\theta) \\ &\quad + \sum_{n=2}^{\infty}(c_n r^{-n} + d_n r^{2-n})\exp(in\theta). \end{aligned}$$

Here, the constants $b_0, c_0, A_0, b_1, c_1, A_1, c_n, d_n$ need to be located by matching the interface conditions on Γ_0. In particular, we note that b_1 and A_1 must be related in order to insure the single-valuedness of displacements, via the relation

$$b_1 = -\frac{1-\nu_2}{2} i A_1,$$

where ν_2 is determined via the elastic properties in the exterior region.

On the other hand, the solution of the biharmonic equation in an annular region is given by

$$\begin{aligned} \phi(r,\theta) &= \alpha_0 + \beta_0 \ln r + \gamma_0 r^2 + \eta_0\theta + \left(\alpha_1 r + \frac{\gamma_1}{r} + \delta_1 r^3 + \beta_1 r \ln r + \eta_1 r\theta\right)\exp(i\theta) \\ &\quad + \sum_{n=2}^{\infty}\left(\alpha_n r^n + \beta_n r^{n+2} + \gamma_n r^{-n} + \delta_n r^{2-n}\right)\exp(in\theta). \end{aligned}$$

Here again β_1, η_1 are related in order to obtain single-value displacements, and

$$\beta_1 = -\frac{1-\nu_1}{2} i\eta_1.$$

We assume that the data $\mathbf{p}, \mathbf{q}$ are sufficiently smooth, and in fact

$$\mathbf{p} = \sum_{n=0}^{\infty} \mathbf{p}_n \exp in\theta, \qquad \mathbf{q} = \sum_{n=0}^{\infty} \mathbf{q}_n \exp in\theta.$$

Since we are primarily interested in displacements and tractions, it is worth noting that with the Michell solution, the displacement $\mathbf{u}_\epsilon = u_r\hat{e}_r + u_\theta\hat{e}_\theta$ in the annular region is given by

$$\begin{aligned}
u_r &= \frac{1}{E_1}\left[-\frac{\beta_0}{r}(1+\nu_1) + 2\gamma_0(1-\nu_1)r\right] \\
&+ \frac{1}{E_1}\left[\beta_1(1-\nu_1)\ln(r) + \frac{\gamma_1}{r^2}(1+\nu_1) + \delta_1 r^2(1-\nu_1) - 2\delta_1\nu_1 r^2 + 2i\eta_1\ln(r)\right]\exp i\theta \\
&+ \frac{1}{E_1}\sum_{n=2}^{\infty}\left[\alpha_n n(1+\nu_1)r^{n-1} + \beta_n\left(n-2+n\nu_1+2\nu_1\right)r^{n+1}\right. \\
&\qquad \left. -\gamma_n n(1+\nu_1)r^{-n-1} - \delta_n\left(n+2+\nu_1 n - 2\nu_1\right)r^{1-n}\right]\exp(in\theta), \\
u_\theta &= \frac{1}{E_1}\left[\beta_1(1-\nu_1)(1-\ln r) + \gamma_1\frac{1+\nu_1}{r^2} + \delta_1(5+\nu_1)r^2 + 2i\eta_1(\ln r + \nu_1)\right] i\exp(i\theta) \\
&+ \frac{1}{E_1}\sum_{n=2}^{\infty}\left[\alpha_n n(1+\nu_1)r^{n-1} + \beta_n\left(n(1+\nu_1)+4\right)r^{n+1}\right. \\
&\qquad \left. -\gamma_n n(1+\nu_1)r^{-n-1} - \delta_n\left(n(1+\nu_1)-4\right)r^{1-n}\right] i\exp(in\theta).
\end{aligned}$$

while the traction T_1 is given by

$$\begin{aligned}
T_1 &= \frac{\beta_0}{r^2} + 2\gamma_0 + \left(\frac{\beta_1}{r} - \frac{2\gamma_1}{r^3} + 2\delta_1 r + i\frac{2\eta_1}{r}\right) expi\theta \\
&\quad - \sum_{n=2}^{\infty}\left[\alpha_n n(n-1)r^{n-2} + \beta_n(n+1)(n-2)r^n\right. \\
&\qquad \left. +\gamma_n n(n+1)r^{-n-2} + \delta_n(n-1)(n+2)r^{-n}\right]\exp(in\theta).
\end{aligned}$$

Similarly, the displacement in the exterior, $\mathbf{U}_\epsilon = U_r\hat{e}_r + U_\theta\hat{e}_\theta$ is given by

$$\begin{aligned}
U_r &= \frac{1}{E_2}\left[-\frac{b_0}{r}(1+\nu_2)\right] + \frac{1}{E_2}\left[\frac{c_1}{r^2}(1+\nu_2)\right]\exp i\theta \\
&+ \frac{1}{E_2}\sum_{n=2}^{\infty}\left[-c_n n(1+\nu_2)r^{-n-1} - d_n\left(n+2+\nu_2 n - 2\nu_2\right)r^{1-n}\right]\exp(in\theta), \\
U_\theta &= \frac{1}{E_2}\left[b_1(1-\nu_2)(1-\ln r) + c_1\frac{1+\nu_2}{r^2} + 2iA_1(\ln r + \nu_2)\right] i\exp(i\theta) \\
&+ \frac{1}{E_2}\sum_{n=2}^{\infty}\left[-c_n n(1+\nu_2)r^{-n-1} - d_n\left(n(1+\nu_2)-4\right)r^{1-n}\right] i\exp(in\theta).
\end{aligned}$$

with the traction being given by

$$T_2 \;=\; \frac{b_0}{r^2} - 2\frac{c_1}{r^3}\exp(i\theta) - \sum_{n=2}^{\infty}\left(c_n n(n+1)r^{-n-2} + d_n(n-1)(n+2)r^{-n}\right)\exp(in\theta)$$

By matching $u_r(R-\epsilon,\theta)=0, u_\theta(R-\epsilon,\theta)=0, u_r(R,\theta)=U_r(R,\theta)+p_r, u_\theta(R,\theta)=U_\theta(R,\theta)+p_\theta, T_1(R)=T_2(R)+q_r$, we can locate the coefficients $\beta_0,\gamma_0,\beta_1,\gamma_1$, etc.

We then write $r=R+\epsilon\xi$, $\xi\in(-1,0)$ and examine the behavior for small ϵ of $u_r(R-\epsilon\xi,\theta)$, $u_\theta(R-\epsilon\xi,\theta)$ (we used the symbolic software MAPLE for this computation). Since the calculation leads to fairly messy expressions, we present here only the expansion for u_r up to $O(\epsilon)$.

$$u_r(R-\epsilon\xi,\theta) = 0+$$
$$\epsilon\left[\frac{(-1+xi)}{R}\frac{(E_2p_{r0}+q_0R+q_0R\nu_2)}{(1+\nu_2)}(-1+\nu_1^2)+(coeff_1)\exp(i\theta)+\sum_{n=2}^{\infty}(coeff_n)\exp(in\theta)\right]$$
$$u_\theta(R-\epsilon\xi,\theta) = 0+\epsilon(coeff2)_1+\epsilon^2(coeff2)_2+...$$

where the value of $(coeff_1)$ is computed by MAPLE. Despite its apparent algebraic complexity, it can be verified that $(coeff_1)$ is a bounded term independent of ϵ. Similar computations for coefficients up to and including second order were performed; in each case, the resultant terms, though messy, are bounded, and independent of ϵ. These detailed calculations are not presented here in the interests of brevity. The most important observation is that at leading order $\mathbf{u}_0=\mathbf{0}$, and $\mathbf{u}_\epsilon$ can be expanded in a regular asymptotic series in ϵ.

Similarly, if we examine the traction $T_2(R)$, we obtain a regular asymptotic series of the form,

$$T_2(R) \;=\; 2\frac{E_2p_{r0}E_1R}{2E_1R^2+2E_1R^2nu_2}+O(\epsilon),$$

where the subsequent terms again yield a regular asymptotic sequence. Again, it was verified that the coefficients do not depend on ϵ, and are bounded.

4 An asymptotic procedure for solutions of $(\mathbf{P}_\epsilon)$

In this section we use the insights gained from the model problem to analyze the asymptotic behavior of solutions to problem $(\mathbf{P}_\epsilon)$ in the limit as $\epsilon\to 0$. Recall that the variational problem under consideration is : *Find* $(\mathbf{u}_\epsilon,\tau)\in\mathcal{H}_\epsilon$ *such that*

$$\left\{\begin{array}{lcl} \int_{\Omega_\epsilon}\underline{\underline{\sigma_{\mathbf{1}}}}(\mathbf{u}_\epsilon):\mathbf{E}(\mathbf{v})\,dV+\langle\mathbf{v}^-,\mathbf{W}\mathbf{u}_\epsilon^-+(-\frac{1}{2}\mathbf{I}+\mathbf{K}')\tau\rangle &=& \int_{\Omega_\epsilon}\mathbf{f}\cdot\mathbf{v}\,dV+\langle\mathbf{v}^-,\mathbf{W}\mathbf{p}_0+\mathbf{q}_0\rangle, \\ \langle(\frac{1}{2}\mathbf{I}-\mathbf{K})\mathbf{u}_\epsilon+\mathbf{V}\tau,\chi\rangle &=& \langle(\frac{1}{2}\mathbf{I}-\mathbf{K})\mathbf{p}_0,\chi\rangle. \end{array}\right. \quad (17)$$

for all $(\mathbf{v},\chi)\in\mathcal{H}_\epsilon$. We have established existence and uniqueness of solutions to this problem in the case of ϵ fixed, by using the variational equation

$$\mathcal{A}_\epsilon((\mathbf{u}_\epsilon,\tau),(\mathbf{v},\chi))=\mathcal{F}_\epsilon(\mathbf{v},\chi)\qquad\forall(\mathbf{v},\chi)\in\mathcal{H}_\epsilon.$$

At this stage, the most convenient coordinate system to be working in is a polar one – $\mathbf{x} = \mathbf{x}(r, \theta) \in \Omega_\epsilon$. We now need to scale the annular region to make the dependence on ϵ explicit. In what follows, we assume that the thickness $\epsilon > 0$ and the outer boundary is a fixed, smooth and rectifiable curve Γ_0. If we denote by Ω' the domain enclosed in Γ_0, then the thin domain and the inner boundary are defined precisely as

$$\Omega_\epsilon := \left\{ x \in \Omega' | \inf_{y \in \Gamma_0} \|x - y\| < \epsilon \right\}, \qquad \Gamma_\epsilon := \left\{ x \in \Omega' | \inf_{y \in \Gamma_0} \|x - y\| = \epsilon \right\}.$$

In particular, if we consider distances measured along the direction of the outer normal on Γ_0 as positive, we can scale the Euclidean distance $d(x, \Gamma_0)$ by

$$0 \leq t = \frac{-d(x, \Gamma_0) + \epsilon}{\epsilon} \leq 1.$$

Thus, when $d(x, \Gamma_0) = 0$, the thickness parameter $t = 1$ and $\mathbf{x} \in \Gamma_0$, whereas $d(\mathbf{x}, \Gamma_0) = \epsilon$, the thickness parameter is now $t = 0$, and the point under consideration is on Γ_ϵ. Provided ϵ is small, we can now use a parameterization of Ω_ϵ by the manifold $\Omega := [0, 2\pi] \times (0, 1)$ through the mapping

$$\left\{ \begin{array}{ccl} \Omega & \longrightarrow & \Omega_\epsilon, \\ (t, \theta) & \rightarrow & \mathbf{x} = \mathbf{X}(\theta) + \epsilon(t-1)\mathbf{n}(\theta) =: \mathbf{r}(\theta, t). \end{array} \right. \tag{18}$$

Here $\mathbf{n}(\theta)$ is the unit outward normal to Γ_0 with the polar angle $\theta, \mathbf{X}(\theta) \in \Gamma_0$ is the tail of $\mathbf{n}(\theta), t$ is the scaled length of the annular defined previously. In this way the points $\mathbf{x} = \mathbf{x}(t, \theta) = \mathbf{X}(\theta) + \epsilon(t-1)\mathbf{n}(\theta) \in \Omega_\epsilon$ are uniquely described. Note that $\mathbf{X}(\theta)$ is the projection of $\mathbf{x} \in \Omega_\epsilon$ onto Γ_0.

We assume that the mapping (18) is a diffeomorphism from Ω onto Ω_ϵ, that is, the Jacobian is

$$\begin{aligned} J &= det \begin{vmatrix} \frac{\partial x_1}{\partial t} & \frac{\partial x_1}{\partial \theta} \\ \frac{\partial x_2}{\partial t} & \frac{\partial x_2}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \epsilon n_1(\theta) & \dot{X}_1(\theta) + \epsilon(t-1)\dot{n}_1(\theta) \\ \epsilon n_2(\theta) & \dot{X}_2(\theta) + \epsilon(t-1)\dot{n}_2(\theta) \end{vmatrix} \\ &= \epsilon\, \mathbf{n}(\theta) \cdot \begin{pmatrix} \dot{X}_2(\theta) + \epsilon(t-1)\dot{n}_2 \\ -\dot{X}_1(\theta) - \epsilon(t-1)\dot{n}_1 \end{pmatrix} > 0. \end{aligned} \tag{19}$$

We have used the notion $\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2$.

If ϕ is a scalar function in the original coordinate system $\mathbf{x} = \mathbf{x}(r, \theta)$, we have the following relationships between derivatives of $\hat{\phi}(t, \theta) := \phi(\mathbf{x})$ and those of ϕ:

$$\begin{aligned} \frac{\partial \hat{\phi}}{\partial t} &= \nabla \phi \cdot \frac{\partial \mathbf{x}}{\partial t} = \nabla \phi \cdot (\epsilon \mathbf{n}(\theta)), \\ \frac{\partial \hat{\phi}}{\partial \theta} &= \nabla \phi \cdot \frac{\partial \mathbf{x}}{\partial \theta} = \nabla \phi \cdot (\dot{\mathbf{X}(\theta)} + \epsilon(t-1)\dot{\mathbf{n}}(\theta)), \end{aligned}$$

where the dot means a derivative in the θ variable. These derivatives are well defined since the curve Γ_0 is smooth. The gradient is taken in the original coordinates,

$$\nabla \phi := \frac{\partial \phi}{\partial r} \hat{e}_r + \frac{1}{r} \frac{\partial \phi}{\partial \theta} \hat{e}_\theta.$$

Here $\hat{e}_r = \begin{pmatrix} cos\theta \\ sin\theta \end{pmatrix}$ and $\hat{e}_\theta = \begin{pmatrix} -sin\theta \\ cos\theta \end{pmatrix}$ are the orthogonal curvilinear basis vectors. We have $\hat{e}_r = \mathbf{n}(\theta), \hat{e}_\theta = \dot{\mathbf{n}}(\theta)$ and $\dot{\mathbf{X}}(\theta) \cdot (\mathbf{X} - \mathbf{x}(t,\theta)) = 0$. Therefore, $\dot{\mathbf{X}}(\theta) \perp \mathbf{n}(\theta)$ and $\dot{\mathbf{X}}(\theta) = C(\dot{\mathbf{X}})\hat{e}_\theta$, where $C(\dot{\mathbf{X}})$ is a scalar which characterizes the shape of the curve Γ_0. We obtain

$$\begin{aligned} \frac{\partial \hat{\phi}}{\partial t} &= \epsilon \frac{\partial \phi}{\partial r} = \epsilon \nabla \phi \cdot \mathbf{n}(\theta) && \text{(20a)} \\ \frac{\partial \hat{\phi}}{\partial \theta} &= [C(\dot{\mathbf{X}}) + \epsilon(t-1)] \frac{1}{r}\frac{\partial \phi}{\partial \theta} = \frac{m}{r}\frac{\partial \phi}{\partial \theta} = m \nabla \phi \cdot \dot{\mathbf{n}}(\theta) && \text{(20b)} \\ J &= \epsilon m, && \text{(20c)} \end{aligned}$$

where $m = C(\dot{\mathbf{X}}) + \epsilon(t-1)$. We assume m is bounded, that is $|m| < c$ for all $(t,\theta) \in \Omega$. We are now in a position to define function spaces on Ω which are independent of ϵ, and show that these have norms which can be related to those of analogous function spaces on Ω_ϵ. We introduce the function space

$$H_0^1(\Omega) := \left\{ \mathbf{u}(t,\theta) \in L^2(\Omega) | \ \frac{\partial \mathbf{u}}{\partial t}, \frac{\partial \mathbf{u}}{\partial \theta} \in L^2(\Omega), \mathbf{u}(t,\theta) = \mathbf{u}(t, \theta + 2\pi), \mathbf{u}|_{t=0} = \mathbf{0} \right\}$$

with norm

$$\|\mathbf{u}\|^2_{H_0^1(\Omega)} := \int_\Omega \left(|\mathbf{u}|^2 + \left| \frac{\partial \mathbf{u}}{\partial \theta} \right|^2 + \left| \frac{\partial \mathbf{u}}{\partial t} \right|^2 \right) d\Omega$$

and the associated product space

$$\mathcal{H}_0 := \left\{ (\mathbf{u}, \sigma) \, | \mathbf{u} \in H_0^1(\Omega), \sigma \in H_0^{-1/2}(\Gamma_0). \right\}.$$

Lemma 4.1 *The norms in $\mathcal{H}_0$ and $\mathcal{H}_\epsilon$ are equivalent in the sense that there is constant c independent of ϵ such that*

$$c\sqrt{\epsilon}||(\mathbf{u}, \sigma)||_{\mathcal{H}_\epsilon} \le ||(\mathbf{u}, \sigma||_{\mathcal{H}_0} \le \frac{c}{\sqrt{\epsilon}} ||(\mathbf{u}, \sigma)||_{\mathcal{H}_\epsilon}. \tag{21}$$

Proof. (a) Let $\hat{u} \in H_0^1(\Omega)$. Then from (20a), (20b), (20c), we obtain

$$\begin{aligned} ||\hat{u}||^2_{H_1(\Omega)} &= \int_\Omega (|\hat{u}|^2 + |\frac{\partial \hat{u}}{\partial t}|^2 + |\frac{\partial \hat{u}}{\partial \theta}|^2) dt d\theta \\ &= \int_{\Omega_\epsilon} (|u|^2 + \epsilon^2 |\nabla u \cdot \mathbf{n}(\theta)|^2 + m^2 |\nabla u \cdot \dot{\mathbf{n}}(\theta)|^2) \frac{1}{\epsilon m} dx_1 dx_2 \\ &\le \int_{\Omega_\epsilon} (|u|^2 + (\epsilon^2 + m^2)|\nabla u|^2) \frac{1}{\epsilon \, m} dx_1 dx_2 \\ &\le \frac{c}{\epsilon} \int_{\Omega_\epsilon} (|u|^2 + |\nabla u|^2) dx_1 dx_2 \end{aligned}$$

for $\epsilon < \epsilon_0$. The right-hand side inequality of (21) then follows, since $||\sigma||_{H^{-1/2}(\Gamma_0)}$ is independent of ϵ.

(b) Now let $u \in H_0^1(\Omega_\epsilon)$. Introducing the polar coordinates (r, θ) and using (20a), (20b), (20c), we see that

$$\begin{aligned}
\int_{\Omega_\epsilon} (|u|^2 + |\frac{\partial u}{\partial x_1}|^2 + |\frac{\partial u}{\partial x_2}|^2) dx_1 dx_2 &= \int_{\tilde{\Omega}_\epsilon} (|\tilde{u}|^2 + |\frac{\partial \tilde{u}}{\partial r}|^2 + \frac{1}{r^2}|\frac{\partial \tilde{u}}{\partial \theta}|^2) r dr d\theta \\
&= \int_{\Omega} (|\hat{u}|^2 + \frac{1}{\epsilon^2}|\frac{\partial \hat{u}}{\partial \theta}|^2 + \frac{1}{m^2}|\frac{\partial \hat{u}}{\partial \theta}|^2)\epsilon\, m dt d\theta \\
&= \int_{\Omega} (\epsilon\, m|\hat{u}|^2 + \frac{m}{\epsilon}|\frac{\partial \hat{u}}{\partial t}|^2 + \frac{\epsilon}{m}|\frac{\partial \hat{u}}{\partial \theta}|^2) dt d\theta \\
&\leq \frac{c}{\epsilon} \int_{\Omega} (|\hat{u}|^2 + |\frac{\partial \hat{u}}{\partial t}|^2 + |\frac{\partial \hat{u}}{\partial \theta}^2) dt d\theta.
\end{aligned}$$

This leads to the assertion (21).

Finally, if $\mathbf{u} \in H_0^1(\Omega_\epsilon)$, we associate with it the function $\hat{\mathbf{u}}(t, \theta) \in H_0^1(\Omega)$ through the change of variable

$$\hat{\mathbf{u}}(t, \theta) \to \mathbf{u}(x).$$

We drop the $\hat{\mathbf{u}}$ in what follows when there is no confusion, and use $\mathbf{u}$ instead, in the scaled domain.

In the original coordinates, if $\mathbf{u}(x) = \mathbf{u}(r, \theta) = u_r \hat{e}_r + u_\theta \hat{e}_\theta$ then

$$\mathbf{E}(\mathbf{u}) = \frac{1}{2} \begin{pmatrix} 2u_{r,r} & \frac{1}{r}u_{r,\theta} - \frac{u_\theta}{r} + u_{\theta,r} \\ \frac{1}{r}u_{r,\theta} - \frac{u_\theta}{r} + u_{\theta,r} & \frac{2}{r}u_{\theta,\theta} + \frac{u_r}{r} \end{pmatrix}.$$

Therefore, the strain tensor can be written in the new t, θ coordinates. After re-scaling the element of volume and collecting like powers of ϵ, the variational problem becomes (after some straightforward but tedious calculation): *Find* $(\mathbf{u}_\epsilon, \tau_\epsilon) \in \mathcal{H}_\epsilon$ *such that for all* $(\mathbf{v}, \chi) \in \mathcal{H}_\epsilon$,

$$\left\{ \begin{array}{rcl} \frac{1}{\epsilon}a_0(\mathbf{u}_\epsilon, \mathbf{v}) + a_1(\mathbf{u}_\epsilon, \mathbf{v}) + \langle \mathbf{v}^-, \mathbf{W}\mathbf{u}_\epsilon^- + (-\frac{1}{2}\mathbf{I} + \mathbf{K}')\tau_\epsilon \rangle & & \\ +\epsilon a_2(\mathbf{u}_\epsilon, \mathbf{v}) + ...\epsilon^{n-1}a_n(\mathbf{u}_\epsilon, \mathbf{v}) + \mathcal{Q}_n(\mathbf{u}_\epsilon, \mathbf{v}) & = & F_\epsilon(\mathbf{v}, \chi), \\ \langle (\frac{1}{2}\mathbf{I} - \mathbf{K})\mathbf{u}_\epsilon + \mathbf{V}\tau, \chi \rangle & = & G(\mathbf{v}, \chi), \end{array} \right. \tag{22}$$

with right - hand sides

$$F_\epsilon(\mathbf{v}, \chi) := \int_0^{2\pi} \int_0^1 \mathbf{f} \cdot \mathbf{v}\epsilon\, m\, dt d\theta + \langle \mathbf{v}^-, \mathbf{W}\mathbf{p}_0 + \mathbf{q}_0 \rangle, \qquad G(\mathbf{v}, \chi) := \langle (\frac{1}{2}\mathbf{I} - \mathbf{K})\mathbf{p}_0, \chi \rangle \tag{23}$$

for all $(\mathbf{v}, \chi) \in \mathcal{H}_\epsilon$. Note that since the boundary integral operators on Γ_0 are not affected by changing ϵ, the data term $G(\cdot, \cdot)$ is not subscripted by ϵ. In the expansion (22), the bilinear operators $a_0, a_1, ..., a_n$ are defined in terms of their action on vector fields $\mathbf{u} = u_t\hat{e}_r + u_\theta\hat{e}_\theta, \mathbf{v} =$

$v_t\hat{e}_r + v_\theta\hat{e}_\theta$ via

$$\begin{aligned}
a_0(\mathbf{u},\mathbf{v}) &:= \int_0^{2\pi}\int_0^1 b_0(\mathbf{u},\mathbf{v})\, C(\dot{\mathbf{X}}) d\theta dt, \quad b_0(\mathbf{u},\mathbf{v}) := \left[(\lambda_1+2\mu_1)u_{t,t}v_{t,t} + \mu_1 u_{\theta,t}v_{\theta,t}\right],\\
a_1(\mathbf{u},\mathbf{v}) &:= \int_0^{2\pi}\int_0^1 (t-1)b_0(\mathbf{u},\mathbf{v}) + b_1(\mathbf{u},\mathbf{v})\, dt d\theta,\\
b_1(\mathbf{u},\mathbf{v}) &:= \lambda_1\left(u_{t,t}v_{\theta,\theta} + v_{t,t}u_{\theta,\theta} + u_{t,t}v_t + v_{t,t}u_t\right) + \mu_1\left(u_{\theta,t}v_{t,\theta} + v_{\theta,t}u_{t,\theta} - u_{\theta,t}v_\theta - v_{\theta,t}u_\theta\right),\\
a_2(\mathbf{u},\mathbf{v}) &:= \int_0^{2\pi}\int_0^1 \frac{1}{C(\dot{\mathbf{X}})} b_2(\mathbf{u},\mathbf{v})\, dt d\theta,\\
b_2(\mathbf{u},\mathbf{v}) &:= (\lambda_1+2\mu_1)(u_{\theta,\theta}+u_t)(v_{\theta,\theta}+v_t) + \mu_1(u_{t,\theta}-u_\theta)(v_{t,\theta}-v_\theta),\\
a_n(\mathbf{u},\mathbf{v}) &:= \int_0^{2\pi}\int_0^1 \frac{1}{C(\dot{\mathbf{X}})} b_2(\mathbf{u},\mathbf{v})(1-t)^{n-2}\left(\frac{1}{C(\dot{\mathbf{X}})}\right)^{n-2} d\theta dt.
\end{aligned}$$

The remainder term $\mathcal{Q}_n$ is defined via

$$\begin{aligned}
\mathcal{Q}_n(\mathbf{u},\mathbf{v}) &:= \int_0^{2\pi}\int_0^1 \epsilon b_2(\mathbf{u},\mathbf{v})\left(\frac{1}{C(\dot{\mathbf{X}})+\epsilon(t-1)} - \frac{1}{C(\dot{\mathbf{X}})}\sum_{j=0}^{n-2}\epsilon^j(1-t)^j\left(\frac{1}{C(\dot{\mathbf{X}})}\right)^j\right) d\theta dt\\
&= \int_0^{2\pi}\int_0^1 b_2(\mathbf{u},\mathbf{v})(1-t)^{n-1}\left(\frac{1}{C(\dot{\mathbf{X}})}\right)^{n-1}\frac{\epsilon^n}{C(\dot{\mathbf{X}})+\epsilon(t-1)}\, d\theta\, dt
\end{aligned}$$

For ϵ small enough and fixed $\mathbf{u},\mathbf{v} \in H_0^1(\Omega)$, it can be seen that $\mathcal{Q}_n(\mathbf{u},\mathbf{v}) = O(\epsilon^n)$. We thus write

$$\mathcal{Q}_n(\mathbf{u},\mathbf{v}) = \epsilon^n Q_n(\mathbf{u},\mathbf{v}).$$

At this stage we have developed the operators in the variational formulation in powers of ϵ. The boundary integral operators remain unchanged since the boundary Γ_0 does not move with changing ϵ. The formal asymptotic procedure will now consist of assuming a regular asymptotic series for the displacement $\mathbf{u}_\epsilon$ and traction τ_ϵ, inserting into the development above, and collecting like powers in ϵ, to arrive at a sequence of simpler problems for $(\mathbf{u}_j, \tau_j)$.

4.1 Formal asymptotic procedure

The illustrative example in Section (3) showed that $\mathbf{u}_\epsilon$ could be expressed in a regular asymptotic series in powers of ϵ. We shall therefore use this ansatz to develop a formal asymptotic procedure for the general transmission problem. To this end, let

$$\mathbf{u}_\epsilon = \sum_{j=0}^{n}\epsilon^j\mathbf{u}_j + \mathbf{R}_n, \qquad \tau_\epsilon = \sum_{j=0}^{n}\epsilon^j\tau_j + \mathbf{S}_n, \qquad n > 2, \tag{24}$$

where $(\mathbf{u}_j, \tau_j) \in \mathcal{H}_\epsilon$ for all $j = 0, 1, ...n$. When we substitute (24) into the variational problem (22), and collect like powers of ϵ, we get the following sequence of problems: At leading order, *find* $(\mathbf{u}_0, \tau_0) \in \mathcal{H}_0$ *such that for all* $(\mathbf{v}, \chi) \in \mathcal{H}_0$

$$\left\{\begin{aligned} a_0(\mathbf{u}_0,\mathbf{v}) &= 0,\\ \langle \mathbf{V}\tau_0, \chi\rangle &= -\langle(\tfrac{1}{2}\mathbf{I} - \mathbf{K})\mathbf{u}_0, \chi\rangle =: G_0(\chi). \end{aligned}\right. \tag{25a}$$

At the next order, *find* $(\mathbf{u}_1, \tau_1) \in \mathcal{H}_0$ *such that*

$$\left\{\begin{array}{rcl} a_0(\mathbf{u}_1, \mathbf{v}) & = & -a_1(\mathbf{u}_0, \mathbf{v}) - \langle \mathbf{v}^-, \mathbf{W}\mathbf{u}_0 + (-\frac{1}{2}\mathbf{I} + \mathbf{K}')\tau_0\rangle + \langle \mathbf{v}^-, \mathbf{W}\mathbf{p}_0 + \mathbf{q}_0\rangle \\ & =: & F_1(\mathbf{u}_0; \tau_0; \mathbf{v}), \\ & & \\ \langle \mathbf{V}\tau_1, \chi\rangle & = & -\langle(-\frac{1}{2}\mathbf{I} - \mathbf{K})\mathbf{u}_1, \chi\rangle =: G_1(\chi). \end{array}\right. \tag{25b}$$

The body force density $\mathbf{f}$ affects the asymptotic development beginning only at the next order: *find* $(\mathbf{u}_2, \tau_2) \in \mathcal{H}_0$ *such that for all* $(\mathbf{v}, \chi) \in \mathcal{H}_0$

$$\left\{\begin{array}{rcl} a_0(\mathbf{u}_2, \mathbf{v}) & = & -a_1(\mathbf{u}_1, \mathbf{v}) - \langle \mathbf{v}^-, \mathbf{W}\mathbf{u}_1 + (-\frac{1}{2}\mathbf{I} + \mathbf{K}')\tau_1\rangle + \int_0^{2\pi}\int_0^1 \mathbf{f}\cdot\mathbf{v}C(\dot{\mathbf{X}})\,dt\,d\theta \\ & =: & F_2(\mathbf{u}_0, \mathbf{u}_1; \tau_1; \mathbf{v}), \\ & & \\ \langle \mathbf{V}\tau_2, \chi\rangle & = & -\langle(\frac{1}{2}\mathbf{I} - \mathbf{K})\mathbf{u}_2, \chi\rangle = G_2(\chi). \end{array}\right. \tag{25c}$$

This means that we require at least two terms in the asymptotic expansion to capture the effects of the body force density.

Subsequently,

$$\left\{\begin{array}{rcl} a_0(\mathbf{u}_3, \mathbf{v}) & = & -a_1(\mathbf{u}_2, \mathbf{v}) - \langle \mathbf{v}^-, \mathbf{W}\mathbf{u}_2 + (-\frac{1}{2}\mathbf{I} + \mathbf{K}')\tau_2\rangle - a_2(\mathbf{u}_0, \mathbf{v}) \\ & & +\int_0^{2\pi}\int_0^1 \mathbf{f}\cdot\mathbf{v}(t-1)\kappa\,dt\,d\theta \\ & =: & F_3(\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2; \tau_2; \mathbf{v}), \\ & & \\ \langle \mathbf{V}\tau_3, \chi\rangle & = & -\langle(\frac{1}{2}\mathbf{I} - \mathbf{K})\mathbf{u}_3, \chi\rangle = G_3(\chi). \end{array}\right. \tag{25d}$$

The general term $(\mathbf{u}_j, \tau_j)$ for $j > 3$ in the asymptotic development of $(\mathbf{u}_\epsilon, \tau_\epsilon)$ solves the coupled system

$$\left\{\begin{array}{rcl} a_0(\mathbf{u}_j, \mathbf{v}) & = & -\sum_{k=1}^{j-1} a_k(\mathbf{u}_{j-k}, \mathbf{v}) - \langle \mathbf{v}^-, \mathbf{W}\mathbf{u}_{j-1} + (-\frac{1}{2}\mathbf{I} + \mathbf{K}')\tau_{j-1}\rangle \\ & =: & F_j(\mathbf{u}_0, \mathbf{u}_1, ...\mathbf{u}_{j-1}; \tau_{j-1}; (\mathbf{v})) \\ & & \\ \langle \mathbf{V}\tau_j, \chi\rangle & = & -\langle(\frac{1}{2}\mathbf{I} - \mathbf{K})\mathbf{u}_j, \chi\rangle =: G_j(\chi). \end{array}\right. \tag{25e}$$

At this stage it is worth drawing attention to some convenient facts. In this formal procedure each of the subproblems for $(\mathbf{u}_j, \tau_j)$ appears to be a coupled system. A closer look at any order reveals that one first obtains $\mathbf{u}_j$ by solving a problem of form

$$a_0(\mathbf{u}_j, \mathbf{v}) = F_j(\mathbf{u}_0, \mathbf{u}_1, ...\mathbf{u}_{j-1}; \tau_{j-1}, \mathbf{v})$$

and then solving the variational problem

$$\langle \mathbf{V}\tau_j, \chi\rangle = -\langle(\frac{1}{2}\mathbf{I} - \mathbf{K})\mathbf{u}_j, \chi\rangle = G_j(\chi)$$

for τ_j. We thus obtain a *staggered* asymptotic scheme, where the initially coupled system of equations for the pair $(\mathbf{u}_\epsilon, \tau_\epsilon)$ decouples for each pair $(\mathbf{u}_j, \tau_j)$ in the asymptotic development.

A second convenient observation concerns the solution of the variational problem

$$\begin{array}{rcl} a_0(\mathbf{u}_j, \mathbf{v}) & = & \displaystyle\int_0^{2\pi}\int_0^1 (\lambda_1 + 2\mu_1)u^j_{t,t}v_{t,t} + \mu_1 u^j_{\theta,t}v_{\theta,t}\,C(\dot{\mathbf{X}})\,dt\,d\theta \\ & = & F_j(\mathbf{u}_0, \mathbf{u}_1, ...\mathbf{u}_{j-1}; \tau_{j-1}, \mathbf{v}). \end{array}$$

This variational problem can be analytically solved, and we provide an explicit representation formula for it. This result is the main content of the next subsection.

4.2 Existence and uniqueness results

We recall that the variational formulation for the typical term $(\mathbf{u}_j, \tau_j)$ in the asymptotic sequence reads: *Find* $(\mathbf{u}_j, \tau_j) \in \mathcal{H}_0$ *such that for all* $(\mathbf{v}, \chi) \in \mathcal{H}_0$,

$$\left\{ \begin{array}{rcl} a_0(\mathbf{u}_j, \mathbf{v}) & = & F_j(\mathbf{u}_0, \mathbf{u}_1, ..., \mathbf{u}_{j-1}; \tau_{j-1}; \mathbf{v}) \\ \langle (\frac{1}{2}\mathbf{I} - \mathbf{K})\mathbf{u}_j + \mathbf{V}\tau_j, \chi \rangle & = & G_j(\chi). \end{array} \right. , \tag{26}$$

where F_j, G_j are defined as in the previous subsection. We need to establish existence and uniqueness of solutions for this system of uncoupled equations. We begin with the uniqueness result:

Theorem 4.2 *There is at most one solution* $(\mathbf{u}_j, \tau_j) \in \mathcal{H}_0$ *of* (26).

Proof. Suppose $(\mathbf{u}_j, \tau_j)$ and $(\mathbf{U}_j, \zeta_j)$ both satisfy (26). Then if $\mathbf{w} = \mathbf{u}_j - \mathbf{U}_j$, $\mathbf{w} = w_t \hat{e}_t + w_\theta \hat{e}_\theta$ solves

$$a_0(\mathbf{w}, \mathbf{v}) = \int_0^{2\pi} \int_0^1 (\lambda_1 + \mu_1) w_{t,t} v_{t,t} + \mu_1 w_{\theta,t} v_{\theta,t}) \, C(\dot{\mathbf{X}}) \, dt d\theta = 0.$$

This implies that $\frac{\partial w_t}{\partial t} = \frac{\partial w_\theta}{\partial t} = 0$ and hence w_t, w_θ are both functions of θ alone. However, since $\mathbf{w} \in H_0^1(\Omega)$, it has vanishing trace on the curve $t = 0$. Therefore, $w_t = w_\theta \equiv 0$. Moreover, τ_j, ζ_j satisfy

$$\langle (\frac{1}{2}\mathbf{I} - \mathbf{K})\mathbf{u}_j + \mathbf{V}\tau_j, \chi \rangle = \langle (\frac{1}{2}\mathbf{I} - \mathbf{K})\mathbf{u}_j + \mathbf{V}\zeta_j, \chi \rangle = G_j(\chi)$$

for all $\chi \in H_0^{-1/2}(\Gamma_0)$. Using the linearity of the single layer operator we get

$$\langle \mathbf{V}(\tau_j - \zeta_j), \chi \rangle = 0, \qquad \forall \chi \in H_0^{-1/2}(\Gamma_0).$$

However, $\mathbf{V}$ is $H_0^{-1/2}(\Gamma_0) - \textit{elliptic}$, and therefore $\zeta_j = \tau_j$, proving the assertion.

Note, here the strong use of the zero displacement condition on the inner boundary is necessary, without which the uniqueness assertion fails.

We demonstrate existence of solutions to the variational problem

$$a_0(\mathbf{u}_j, \mathbf{v}) = F_j(\mathbf{u}_0, \mathbf{u}_1, ..., \mathbf{u}_{j-1}; \tau_{j-1}, \mathbf{v})$$

by invoking the following result ([8])

Theorem 4.3 *Let* $F_1, F_2, P_1, P_2, Q_1, Q_2, R$ *be scalar functions such that for* $i = 1, 2$,

1. $F_i \in H^1(\Omega)$ *and* $F_i(\cdot, t), \frac{\partial F_i}{\partial \theta}, \frac{\partial^2 F_i}{\partial \theta^2} \in L^2(\Gamma_0)$ *for fixed* $t \in (0, 1)$. *That is,* F_i *are more regular in the* θ *variable than in the* t *variable.*
2. $P_i \in H^1(\Omega)$ *and* $\frac{\partial^2 P_i}{\partial t^2} \in L^2(0, 1)$ *for all fixed* θ.
3. $R \in H^{1/2}(\Gamma_0)$.
4. F_i, P_i, R *satisfy the periodicity requirement in the* θ *variable.*

Then the variational equation

$$\begin{aligned}\int_0^{2\pi}\int_0^1 \big((\lambda+2\mu)\tfrac{\partial u_t}{\partial t}\tfrac{\partial \bar{v_1}}{\partial t} & + \mu\tfrac{\partial u_\theta}{\partial t}\tfrac{\partial \bar{v_2}}{\partial t}\big)\, dtd\theta \\ & = -\sum_{i=1}^{2}\int_0^{2\pi}\int_0^1 \left[\tfrac{\partial F_i}{\partial \theta}\tfrac{\bar{v_i}}{\partial \theta} + (t-1)\tfrac{\partial P_i}{\partial t}\tfrac{\partial \bar{v_i}}{\partial t} + Q_i v_i\right] dtd\theta + \int_0^{2\pi} Rv_2 d\theta,\end{aligned} \tag{27}$$

for all $\mathbf{v} = v_1\hat{e_t} + v_2\hat{e_\theta} \in \mathcal{H}_0$ *determines a unique* $\mathbf{u} = u_t\hat{e_t} + u_\theta\hat{e_\theta} \in \mathcal{H}_0$*, given explicitly by*

$$u_t(\theta,t) = \int_0^t\int_\tau^1 \frac{\partial^2 F_1}{\partial \theta^2} - Q_1(\theta,y)dyd\tau - \int_0^1 (\tau-1)\frac{\partial P_1}{\partial \tau}(\theta,\tau)d\tau + tR(\theta), \tag{28a}$$

and

$$u_\theta(\theta,t) = \int_0^t\int_\tau^1 \frac{\partial^2 F_2}{\partial \theta^2} - Q_2(\theta,y)dyd\tau - \int_0^1 (\tau-1)\frac{\partial P_2}{\partial \tau}(\theta,\tau)d\tau. \tag{28b}$$

The proof is straightforward, and follows by examining solutions of a suitable ordinary differential equation in the t variable. A similar result was proved in [8].

Once $\mathbf{u}_j$ is obtained, the existence of solutions τ_j in (26) is guaranteed by the Lax-Milgram lemma, provided G_k is a continuous linear functional on $H_0^{-1/2}(\Gamma_0)$. This is due to the fact that the boundary integral operator $\mathbf{V}$ satisfies the coercivity property

$$\langle \chi, \mathbf{V}\chi\rangle \geq \alpha\|\chi\|^2_{H^{-1/2}(\Gamma_0)}, \qquad \forall\chi \in H_0^{-1/2}(\Gamma_0).$$

The continuity of G_k will be fulfilled, following the properties of the solution $\mathbf{u}_j$ from Theorem 4.2. However, in general, the asymptotic analysis requires σ_j to be more than merely $H^{-1/2}(\Gamma_0)$ regular. Thus one requires that the given data $\mathbf{p}_0, \mathbf{q}_0$ and $\mathbf{f}$ be smooth enough to ensure that $-(\frac{1}{2}\mathbf{I} - \mathbf{K})\mathbf{u}_j \in H^{s+1}(\Gamma_0), s > -\frac{1}{2}$. With this regularity in $\mathbf{u}_j$, the solution τ_j of the integral equation

$$\mathbf{V}\tau_j = -(\frac{1}{2}\mathbf{I} - \mathbf{K})\mathbf{u}_j,$$

will be in $H^s(\Gamma_0)$.

We note again the convenient staggered sequence process just developed. In order to obtain the term $(\mathbf{u}_j, \tau_j)$ in the asymptotic sequence, we can use the explicit representation formula given by (28) to compute $\mathbf{u}_j$ in terms of previously computed $(\mathbf{u}_i, \tau_i), i = 0, 1, ...j-1)$. We then use the boundary element formulation in (26) to compute τ_j. Indeed, for $j > 0$, we are always solving a problem of the form

$$\langle \mathbf{V}\tau_j, \chi\rangle = -\langle(\frac{1}{2}\mathbf{I} - \mathbf{K})\mathbf{u}_j, \chi\rangle \qquad \forall\chi \in H_0^{-1/2}(\Gamma_0).$$

At the implementation level, this means that once we form the matrices for discretilized version of this boundary element problem, we simply need to solve the same linear system for different right hand sides.

5 Justification of formal asymptotic procedure

In this section, we shall justify the use of the asymptotic expansion developed in the previous section. We begin by determining the variational problem satisfied by the remainder terms $(\mathbf{R}_n, \mathbf{S}_n)$ in (24), and then use the estimates derived in Theorem 2.2 to show that

$$\|(\mathbf{R}_n, \mathbf{S}_n)\| = O(\epsilon^m)$$

in some appropriate norm, where m increases monotonically with n, and $m > 0$ for all $n > N_0$. For this problem, the scaled differential operator contains leading order terms of $O(\epsilon^{-3/2})$, and therefore $m = n - 3/2,\ n \geq 2$. This means that we need two terms in the asymptotic expansion of $\mathbf{u}_\epsilon, \tau_\epsilon$ to capture the angular behavior of the solutions. This "lag' in the asymptotics does not present any problems since we already have a precise representation formula for the $\mathbf{u}_j$.

5.1 Variational formulation for remainder terms

Using the weak formulation $(\mathbf{P}_\epsilon)$ for the non-local boundary value problem (7), and the scaled problems developed in the previous section, we are able to derive a system for the remainder terms $(\mathbf{R}_n, \mathbf{S}_n)$: We have the equation for $\mathbf{R}_n$,

$$\begin{aligned}
&\int_{\Omega_\epsilon} \underline{\underline{\sigma_1}}(\mathbf{R}_n) : \mathbf{E}(\mathbf{v})\, d\Omega_\epsilon + \langle \mathbf{v}^-, \mathbf{W}(\mathbf{R}_n) + (-\frac{1}{2}\mathbf{I} + \mathbf{K}')\mathbf{S}_n \rangle \\
= &\int_{\Omega_\epsilon} \underline{\underline{\sigma_1}} \left(\mathbf{u}_\epsilon - \sum_{j=0}^{n} \epsilon^j \mathbf{u}_j \right) : \mathbf{E}(\mathbf{v}) + \left\langle \mathbf{v}^-, \mathbf{W}(\mathbf{u}_\epsilon - \sum_{j=0}^{n} \epsilon^j \mathbf{u}_j) \right\rangle + \left\langle \mathbf{v}^-, (-\frac{1}{2}\mathbf{I} + \mathbf{K}')(\tau - \sum_{j=0}^{n} \epsilon^j \tau_j) \right\rangle \\
= & -\epsilon^n \left[\sum_{j=1}^{n} a_j(\mathbf{u}_{n+1-j}, \mathbf{v}) + \epsilon \sum_{j=2}^{n} a_j(\mathbf{u}_{n+2-j}, \mathbf{v}) + ... + \epsilon^{n-1} a_n(\mathbf{u}_n, \mathbf{v}) \right] \\
& -\epsilon^n Q_n(\sum_{j=0}^{n} \epsilon^j \mathbf{u}_j, \mathbf{v}) - \epsilon^n \langle \mathbf{v}, (\mathbf{W}\mathbf{u}_n + (-\frac{1}{2}\mathbf{I} + \mathbf{K}')\tau_n \rangle \\
=: & \ \mathcal{F}_n(\mathbf{v})
\end{aligned}$$

where the linear functional $\mathcal{F}_n(\cdot)$ depends on the terms $(\mathbf{u}_1, \dots, \mathbf{u}_n, \tau_n)$ in the asymptotic sequence. The equation for $\mathbf{S}_n$ is much simpler:

$$\langle (\frac{1}{2}\mathbf{I} - \mathbf{K})\mathbf{R}_n + \mathbf{V}\mathbf{S}_n, \chi \rangle = 0.$$

Combining these results, we see that the remainder $(\mathbf{R}_n, \mathbf{S}_n)$ satisfies the variational equation

$$\mathcal{A}_\epsilon\left((\mathbf{R}_n, \mathbf{S}_n), (\mathbf{v}, \chi)\right) \ = \ \mathcal{F}_n(\mathbf{v}) + < \mathbf{0}, \chi > =: \ell_n(\mathbf{v}, \chi)$$

where $\mathcal{A}_\epsilon$ is the bilinear form defined in Section 2.4. It is not difficult to see that $|\ell_n(\mathbf{R}_n, \mathbf{S}_n)| = O(\epsilon^n)$, or more precisely we have

$$|\ell_n((\mathbf{R}_n, \mathbf{S}_n))| \leq c_n \epsilon^n ||(\mathbf{R}_n, \mathbf{S}_n)||_{\mathcal{H}_\epsilon} \tag{29}$$

while Theorem 2.2 implies that

$$\alpha\|(\mathbf{R}_n, \mathbf{S}_n)\|_{\mathcal{H}_\epsilon}^2 \leq |\mathcal{A}_\epsilon\left((\mathbf{R}_n, \mathbf{S}_n), (\mathbf{R}_n, \mathbf{S}_n)\right)|.$$

Then using the norm relation between $\mathcal{H}_0$ and $\mathcal{H}_\epsilon$ from Lemma 4.1, we see that

$$\|(\mathbf{R}_n, \mathbf{S}_n)\|_{\mathcal{H}_0}^2 \leq (c^2/\epsilon)\alpha|(\mathcal{A}_\epsilon\left((\mathbf{R}_n, \mathbf{S}_n), (\mathbf{R}_n, \mathbf{S}_n)\right)| = (c^2/\epsilon)\alpha|\ell_n(\mathbf{R}_n, \mathbf{S}_n)| \leq C_n'\epsilon^{n-3/2}\|(\mathbf{R}_n, \mathbf{S}_n)\|_{\mathcal{H}_0}$$

where the constant C_n' depends on $(\mathbf{u}_0, \mathbf{u}_1, \cdots, \mathbf{u}_{n-1})$ or on $(\mathbf{f}, \mathbf{p}_0.\mathbf{q}_0)$ implicitly. We have thus proved

Theorem 5.1 *Under the hypotheses of theorem* (2.2)*, and for small enough* $\epsilon > 0$*, the following estimates hold:*

$$\left\|(\mathbf{u}_\epsilon, \tau_\epsilon) - \left(\sum_{j=0}^{n} \epsilon^j \mathbf{u}_j, \sum_{j=0}^{n} \epsilon^j \tau_j\right)\right\|_{\mathcal{H}_0} = O(\epsilon^{n-3/2}), \qquad \text{as } \epsilon \to 0^+ \tag{30}$$

where the terms $\{(\mathbf{u}_j, \tau_j)\}$ *are constructed by the formal asymptotic procedure.*

We note that at least two terms in the asymptotic sequence are required; this agrees with our observation that the body force density $\mathbf{f}$ enters into the scheme only at second order. Thus, to obtain a solution which is asymptotically accurate to $O(\epsilon^n)$, one needs to compute $n + 2$ terms in the asymptotic sequence. Since the solution of the finite-element formulation is given via an explicit representation formula, this *lag* does not adversely impact the asymptotic strategy.

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