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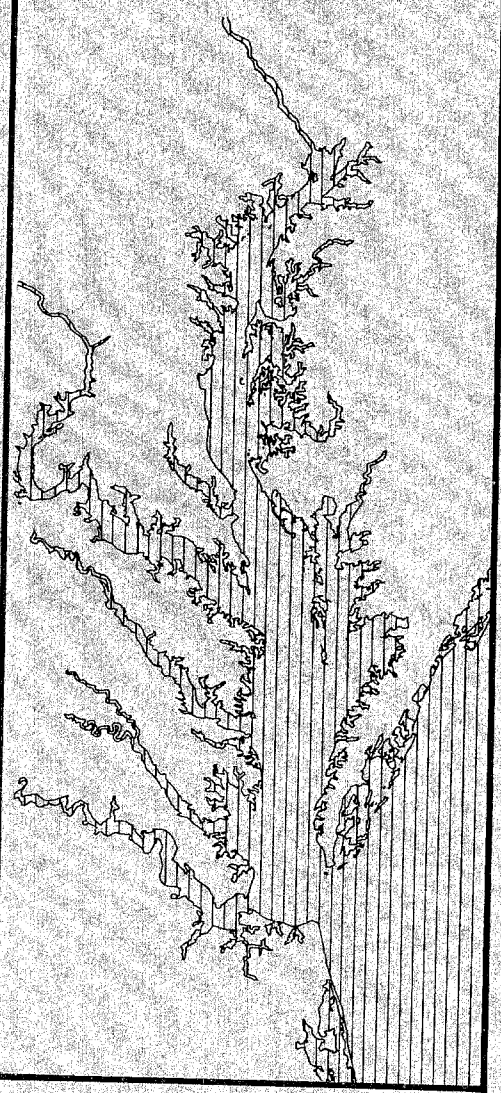

APPLICATION OF THE TELEGRAPH EQUATION TO OCEANIC DIFFUSION: ANOTHER MATHEMATICAL MODEL

Akira Okubo

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TECHNICAL REPORT 69

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by

Akira Okubo

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ABSTRACT

The solution of the conventional diffusion equation has an obvious shortcoming; that is, the substance concentration will rise instantaneously everywhere when substance is introduced at some point in the sea. Although such instantaneous propagation of substance makes a negligibly small contribution to the concentration at large distances from the source, it might cause serious error in predicting water pollution, micro-organism distributions, etc. A diffusion equation which overcomes this difficulty is the telegraph equation characterized by a finite propagation velocity. An *ad hoc* derivation of the telegraph equation from a set of hydromechanical equations identifies the parameters involved in the equation. Thus the propagation velocity is related to the correlation tensor of turbulent velocity. As a result, the one-particle dispersion law by Taylor and the relative diffusion law by Richardson can be deduced from the telegraph equation.

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1. *Introduction*

Many mathematical models of oceanic diffusion have thus far appeared (Okubo, 1962; Okubo, 1970). They are used for predicting the distribution of concentration from various sources as well as estimating the statistical moments, *e.g.*, the variance, of the distribution. Inherent in these models, is the characteristic that the rate of propagation of a diffusate is infinite, *i.e.*, the model predicts a finite amount of substance, though it is infinitesimally small near infinity after an infinitesimal period of time. Usually one can consider this difficulty as purely academic and ignore the infinitesimally small concentration at infinitely large distances so far as practical purposes are concerned.

However, when we are concerned with some biological problems, even apparently insignificant concentrations might be important. Animals' olfactories are so sensitive that threshold concentrations as low as 10^{-20} mol have been reported (Koch, Carr and Ehrenfeld, 1969). The mathematical models of oceanic diffusion so far developed are not designed for these biological problems but are concerned primarily with the pattern of diffusion of a patch at distances not much larger than the standard deviation of the distribution.

In this situation, it may be worthwhile to pay attention to models with finite propagation velocities. The so-called "telegraph equation" is known to be one of them. In this report, the author will explore the feasibility of the telegraph equation as a model for oceanic diffusion.

2. *Some preliminary remarks*

As an example of an oceanic diffusion model, consider the Joseph-Sendner

solution from an instantaneous line source (Joseph and Sendner, 1958),

$$S = \frac{MH^{-1}}{2\pi P^2 t^2} \exp\left(-\frac{r}{Pt}\right) ,$$

where S: concentration, M: total amount of substance released, H: the thickness of uniform layer, P: diffusion velocity, r: distance from the center of mass of the substance patch, t: diffusion time. After Joseph and Sendner, a typical value of P in the sea is 1 cm/sec.

As is seen from its form, the Joseph and Sendner solution predicts a finite amount of concentration near infinity even after a small period of time. We shall give an example of calculation of concentration. For $r = 10^9$cm (10,000 km) and $t = 4 \cdot 10^7$ sec ($\doteq$ 15 months), $S = M(2\pi H)^{-1} \cdot 10^{-26}/cm^2$. The above chosen time implies that the mean "traveling speed" of substance would be 25 cm/sec ($\doteq$ 1/2 knot). The diffusion time in question is much shorter than the time at which the substance patch grows into a size comparable to the distance $r = 10^9$cm. For this "short" time interval after release, the concentration at $r = 10^9$cm is negligibly small yet finite.

Thus, if we release 2 pounds of salt at the coast of California and we assume the mean thickness of the upper mixed layer to be 100 m, we then would find (statistically speaking) one molecule of salt in about 500 ℓ of water sampled at the coast of Japan after 15 months. Note that no allowance is made for mean currents in the calculation. The molecule which would arrive at the Japanese coast would have been driven with a mean (turbulent) speed of 1/2 knot. Although this may be possible in principle, we could hardly believe that it would happen in reality.

From a practical standpoint, the diffusion process with a finite

velocity of propagation is more appealing than that of infinite velocity. Actual field experiments of diffusion often show that a patch or plume of a passive contaminant introduced has a very distinct boundary beyond which apparently there is no contamination. In the diffusion of smoke in the atmosphere, Monin (1959) considered that the concentration beyond the visible smoke puff equals zero and led himself to the use of the telegraph equation.

3. *Features of the telegraph equation*

The telegraph equation (telegrapher's equation, telegraphist's equation) first appeared in a paper by Kirchhoff in 1857 and next in one by Heaviside in 1876. But it first attracted serious attention when treated by Poincaré in 1893. The original form of the telegraph equation reads

$$KL \frac{\partial^2 \varphi}{\partial t^2} + (RK + SL) \frac{\partial \varphi}{\partial t} + RS\varphi = \frac{\partial^2 \varphi}{\partial x^2} , \qquad (1)$$

where φ : electric potential or electric current, K: electrostatic capacity, L: self-inductance, R: resistance, S: leakage conductance, x: distance along the wire, t: time.

As its form suggests, the telegraph equation has features characteristic of both wave motion and diffusion. Thus, in the case of no resistance and leakage, it reduces to

$$\frac{\partial^2 \varphi}{\partial t^2} = \frac{1}{KL} \frac{\partial^2 \varphi}{\partial x^2} , \qquad (2)$$

which is the equation of wave motion with the phase speed of $(KL)^{-1/2}$. On the other hand, when the inductance is negligible compared with the

resistance and there is no leakage, we have

$$\frac{\partial \varphi}{\partial t} = \frac{1}{RK} \frac{\partial^2 \varphi}{\partial x^2} \quad , \tag{3}$$

which is the equation of diffusion with a diffusivity of $(RK)^{-1}$.

For the purpose of investigating briefly the properties of the telegraph equation in connection with the wave and diffusion equations, let us take the telegraph equation without leakage in the form:

$$\frac{\partial^2 \varphi}{\partial t^2} + \lambda \frac{\partial \varphi}{\partial t} = c^2 \frac{\partial^2 \varphi}{\partial x^2} \quad , \tag{4}$$

where λ $[T^{-1}]$ and C [LT] are positive constants.

Consider an infinite space, $-\infty < x < \infty$, and take the Fourier transform of φ, assuming it exists:

$$\tilde{\varphi}(t,k) = \int_{-\infty}^{\infty} \varphi(t,x)\, e^{ikx}\, dx$$

and

$$\varphi(t,x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{\varphi}(t,k)\, e^{-ikx} dk \quad .$$

Multiplication of (4) by e^{ikx} and integration over x result

$$\frac{d^2\tilde{\varphi}}{dt^2} + \lambda \frac{d\tilde{\varphi}}{dt} = - c^2 k^2 \tilde{\varphi} \quad . \tag{5}$$

Let φ initially be a Dirac function of x and $\partial\varphi/\partial t$ be zero. Then, $\tilde{\varphi} = 1$ and $d\tilde{\varphi}/dt = 0$ at $t = 0$. Solving (5) under these initial conditions, one obtains

$$\tilde{\varphi}(t,k) = e^{-\frac{\lambda}{2}t}\left[\cosh \frac{1}{2}\sqrt{\lambda^2 - 4c^2k^2}\ t + \frac{\lambda}{\sqrt{\lambda^2 - 4c^2k^2}} \sinh \frac{1}{2}\sqrt{\lambda^2 - 4c^2k^2}\ t\right] . \tag{6}$$

For values of $|k|$ smaller than $\lambda/2c$, $\tilde{\varphi}$ decreases monotonically with $|k|$ until reaching the value of $e^{-\frac{\lambda}{2}t}(1 + \lambda t)$ at $|k| = \lambda/2c$. Beyond this value of $|k|$, $\tilde{\varphi}$ begins to oscillate about the zero value (Fig. 1a). Asymptotically ($|k| \to \infty$), $\tilde{\varphi}$ behaves as $e^{-\frac{\lambda}{2}t}\cos ckt$. The oscillatory behavior of $\tilde{\varphi}$ for large values of $|k|$ is characteristic of wave motions in Fourier transformation space. As a matter of fact, the solution of the Fourier-transformed wave equation subject to the initial Dirac spike is given by $\cos ckt$ (Fig. 1b). The effect of the first-order term, $\lambda\, d\tilde{\varphi}/\partial t$, is to damp the amplitude of the Fourier mode by a factor of $e^{-\frac{\lambda}{2}t}$.

On the other hand, the solution of the Fourier-transformed diffusion equation subject to the Dirac spike is given by $e^{-\frac{c^2}{\lambda}k^2 t}$ (Fig. 1c). It can be seen that the $\tilde{\varphi}$-curve (Fig. 1a) behaves as $e^{-\frac{c^2}{\lambda}k^2 t}$ for the values of $|k| \ll \lambda/c$. For large t, the $\tilde{\varphi}$-curve has a significant value only in the vicinity of $k = 0$. This implies that a solution of the telegraph equation tends to that of diffusion for large values of t. In fact, the first term on the left-hand side of (4) becomes small in comparison with the second term as t becomes much larger than λ^{-1}.

To gain further insight into features of the telegraph equation, we shall first consider a wave-like solution to (4) modified by the presence of the λ-term. Let $f(x-ct)g(t)$ be such a solution. Substitution of this form into (4) yields

$$c^2 f''g - cf'g' + fg'' - c\lambda f'g + \lambda fg' = c^2 f''g \quad ,$$

which is satisfied by $g = e^{-\lambda t}$. In other words, the telegraph equation represents a damped wave motion: $\varphi \propto e^{-\lambda t} f(x - ct)$, the form of f being determined by the initial condition (Fig. 2).

Next we will investigate the possible existence of a diffusion-like solution for the telegraph equation. Consider a release of substance at the origin, x = 0. Assume that some residual effects remain behind the damped waves traveling with a speed of c. Let $\varphi = e^{-\sigma t} H\{\beta(c^2t^2-x^2)^{1/2}\}$ describe the effect, where β and σ are constants to be determined, and the proper range of the argument of H is $ct \geq |x|$. Substitution of this form into (4) yields

$$H'' + \{\frac{1}{\beta(c^2t^2- x^2)^{1/2}} - \frac{t}{\beta(c^2t^2-x^2)^{1/2}} (2\sigma - \lambda)\} H'$$

$$+ \frac{1}{\beta^2c^2} \sigma(\sigma - \lambda) H = 0 \quad , \tag{7}$$

where $H' \equiv dH(\eta)/d\eta$ and $H'' \equiv d^2H(\eta)/d\eta^2$ with $\eta \equiv \beta(c^2t^2- x^2)^{1/2}$. In order for (7) to be satisfied with the similarity form for H, it is necessary for $2\sigma - \lambda$ to vanish and for $\beta^{-2}c^{-2}\sigma(\sigma - \lambda)$ to be a numerical constant (say, one). Hence,

$$\sigma = \lambda/2$$

$$\beta = \lambda/2c \quad .$$

Therefore,

$$\varphi = e^{-\frac{\lambda}{2}t} H\{\frac{\lambda}{2c} (c^2t^2- x^2)^{1/2}\} \quad , \tag{8}$$

and

H satisfies the following differential equation,

$$H'' + \eta^{-1} H - H = 0 \quad . \tag{9}$$

Equation (9) has solutions of $I_o(\eta)$ and $K_o(\eta)$, but only $I_o(\eta)$ is physically

admissible so that

$$\varphi = e^{-\frac{\lambda}{2}t} I_o\{\frac{\lambda}{2c}(c^2t^2 - x^2)^{1/2}\} \quad . \tag{10}$$

The corresponding initial conditions are a unity of concentration and a finite rate of change of concentration at the origin. Fig. 3 describes how the solution (10) looks at some time t after release. Note that no substance is found beyond $| x | > ct$.

The situation of nondispersal of substance beyond $| x | > ct$ can be seen directly from the telegraph equation (4) without solving it for a particular problem. To see this, we first put

$$\varphi = e^{-\frac{\lambda}{2}t} \psi \quad , \tag{11}$$

and substitute it into (4). The result is

$$\frac{\partial^2\psi}{\partial t^2} = c^2 \frac{\partial^2\psi}{\partial x^2} + \frac{\lambda^2}{4} \psi \tag{12}$$

or changing to the normal form by putting

$$\begin{aligned} \alpha &\equiv \frac{\lambda}{2c}(x + ct) \quad , \\ \beta &\equiv \frac{\lambda}{2c}(x - ct) \quad , \end{aligned} \tag{13}$$

we have

$$\frac{\partial^2\psi}{\partial\alpha\partial\beta} + \frac{\psi}{4} = 0 \quad . \tag{14}$$

Let the initial conditions be $\psi(o,x) = f(\frac{\lambda}{2c}x)$ for $| x | \leq x_1$ and 0 for $| x | > x_1$, $(\partial\psi/\partial t)(o,x) = \frac{\lambda}{2} g(\frac{\lambda}{2c}x)$ for $| x | \leq x_1$ and 0 for $| x | > x_1$. In terms of α and β, the initial conditions read

$$\text{for } \alpha = \beta,\ \psi = f(\alpha) = f(\beta) = \begin{cases} \neq 0 \text{ for } |\alpha|, |\beta| \leqq \dfrac{\lambda}{2c} x_1 \equiv \alpha_1 \equiv \beta_1 \\ = 0 \text{ for } |\alpha|, |\beta| > \alpha_1 = \beta_1 \end{cases}$$

$$\frac{\partial \psi}{\partial \alpha} - \frac{\partial \psi}{\partial \beta} = g(\alpha) = g(\beta) = \begin{cases} \neq 0 \text{ for } |\alpha|, |\beta| \leqq \dfrac{\lambda}{2c} x_1 \equiv \alpha_1 \equiv \beta_1 \\ = 0 \text{ for } |\alpha|, |\beta| > \alpha_1 = \beta_1 \ . \end{cases} \tag{15}$$

Since $t \geqq 0$, $\alpha \geqq \beta$, the domain of interest excludes the area to the left of the line $\alpha = \beta$ (Fig. 4). We define three sub-domains that

I. $\alpha > \beta > \alpha_1 = \beta_1$ or $x > ct + x_1$.

II. $\alpha \geqq \alpha_1$, $\beta \leqq \beta_1$ or $ct + x_1 \geqq x \geqq - ct + x_1$.

III. $-\beta > - \alpha > \alpha_1 = \beta_1$ or $- ct + x_1 > x$.

D and D' in Fig. 4 are points (α_1, β_1) and $(-\alpha_1, -\beta_1)$, respectively.

Let χ be Riemann's function satisfying the following conditions

$$\frac{\partial^2 \chi}{\partial \alpha \partial \beta} + \frac{\chi}{4} = 0 \qquad \text{for } \alpha \geqq \beta \ ,$$

$$\chi = 1 \quad \text{for } \alpha = \alpha_1 \text{ and } \alpha \geqq \beta \ , \tag{16}$$

$$\chi = 1 \quad \text{for } \beta = \beta_1 \text{ and } \alpha \geqq \beta \ .$$

Construct new functions $Q(\alpha,\beta)$ and $R(\alpha,\beta)$ from ψ and χ as

$$Q \equiv \chi \frac{\partial \psi}{\partial \alpha} - \psi \frac{\partial \chi}{\partial \alpha} \ ,$$

$$R \equiv \chi \frac{\partial \psi}{\partial \beta} - \psi \frac{\partial \chi}{\partial \beta} \ . \tag{17}$$

It is then seen that $\partial Q/\partial\beta + \partial R/\partial\alpha = 0$. Hence,

$$\iint \left(\frac{\partial Q}{\partial \beta} + \frac{\partial R}{\partial \alpha}\right) d\alpha d\beta = \oint_c (Q d\alpha - R d\beta) = 0 \quad . \tag{18}$$

The line integral is carried out along a curve lying in the domain where $\alpha \geq \beta$.

Choose a point $P(\alpha,\beta)$ in Subdomain I and consider the path, APB, shown in Fig. 4. Applying the line integral (18) to the path, we obtain

$$\int_A^P Q d\alpha - \int_P^B R d\beta + \int_B^A (Q d\alpha - R d\beta) = 0 \quad . \tag{19}$$

The last integral on the line segment BA vanishes from (15) and (17), and hence (19) is reduced to

$$\int_A^P \left(\chi \frac{\partial \psi}{\partial \alpha} - \psi \frac{\partial \chi}{\partial \alpha}\right) d\alpha - \int_P^B \left(\chi \frac{\partial \psi}{\partial \beta} - \psi \frac{\partial \chi}{\partial \beta}\right) d\beta = 0 \quad ,$$

or

$$(\chi\psi)_P = \frac{1}{2}(\chi\psi)_B - \frac{1}{2}(\chi\psi)_A - \int_A^P \psi \frac{\partial \chi}{\partial \alpha} d\alpha + \int_P^B \psi \frac{\partial \chi}{\partial \beta} d\beta \quad .$$

Let $\chi = 1$ along AP and PB, so that

$$(\psi)_P = \frac{1}{2}(\psi)_B - \frac{1}{2}(\psi)_A$$

Since A and B are located outside the line segment, DD', $(\psi)_A = (\psi)_B = 0$. Therefore, $(\psi)_P \equiv 0$ for P in Subdomain I.

Likewise one can prove that

$$(\psi)_P \equiv 0 \text{ for P in Subdomain III.}$$

On the other hand, if we take the path, EPF, for a point P in Sub-

domain II and if we let $\chi = 1$ along EP and PF, we obtain after similar calculation,

$$(\chi)_P = \int_D^{D'} (Q d\alpha - R d\beta) \quad .$$

The integral along DD' does not vanish so that χ has a value only in Subdomain II.

4. Random-walk considerations of the telegraph equation

Goldstein (1951) first derived the telegraph equation from a generalized random-walk model. Bourret (1960) achieved a significant generalization of the Goldstein equation. Here we shall study the stochastic model with emphasis upon the differences among the equations of telegraph, wave and diffusion.

Consider a random walk in one dimension (Fig. 5). At time $t = 0$, a particle starts from the origin $x = 0$ and moves with a constant velocity v either to the right (in the positive direction) or to the left (in the negative direction) with equal probability. We call v the "path velocity." At the end of a time interval τ, the particle starts a new movement with the same velocity v and in a direction, either positive or negative x. Let $\eta = v\tau$ denote the distance covered by the particle during τ. Unlike the simple random walk in which half move to the right and half to the left after each interval of time τ, we now assume that the direction taken by any one time interval be correlated with that in the interval immediately following (*i.e.*, partially-conscious-drunkard-walk).

Our problem is to obtain the equation for the probability $P(x,t)$ that the position x is occupied by the particle at time t. Let $P^+(x,t)$ and $P^-(x,t)$ be the probabilities that the position x is occupied, respec-

tively, by the rightward-moving particle and by the leftward-moving particle at time t. For the moment, x and t are assumed to be discrete, *i.e.*, $|x|/d$ and t/τ are integers.

$$P(x,t) = P^{+}(x,t) + P^{-}(x,t) \quad . \tag{20}$$

The rightward-moving particle has come from either the right or the left. If a is the probability for the particle to move in the same direction as in the previous time interval and b is the probability for the particle to move in the opposite direction in the previous time interval, it can be shown that

$$P^{+}(x,t) = a\, P^{+}(x-\eta,\, t-\tau) + b\, P^{-}(x-\eta,\, t-\tau) \quad . \tag{21}$$

$$P^{-}(x,t) = a\, P^{-}(x+\eta,\, t-\tau) + b\, P^{+}(x+\eta,\, t-\tau) \quad . \tag{22}$$

From now on we regard x, η, t and τ as continuous variables.

For a random walk with no directional correlation (completely-unconscious-drunkard-walk) between any two consective movements, $a = b = 1/2$. Hence, from (20), (21), and (22)

$$P(x,t) = \frac{1}{2} P(x+\eta,\, t-\tau) + \frac{1}{2} P(x-\eta,\, t-\tau) \quad .$$

Expansion of the right-hand side in the powers of η and τ yields

$$P(x,t) = \frac{1}{2}\Big[P(x,t) + \eta \frac{\partial P}{\partial x} - \tau \frac{\partial P}{\partial t} + \frac{1}{2}\eta^2 \frac{\partial^2 P}{\partial x^2} - \eta\tau \frac{\partial^2 P}{\partial x \partial t} + \frac{1}{2}\tau^2 \frac{\partial^2 P}{\partial t^2} +$$

$$P(x,t) - \eta \frac{\partial P}{\partial x} - \tau \frac{\partial P}{\partial t} + \frac{1}{2}\eta^2 \frac{\partial^2 P}{\partial x^2} + \eta\tau \frac{\partial^2 P}{\partial x \partial t} + \frac{1}{2}\tau^2 \frac{\partial^2 P}{\partial t^2}\Big] + O\{(\eta+\tau)^3\},$$

or

$$\frac{\partial P}{\partial t} = \frac{1}{2}\frac{\eta^2}{\tau} \frac{\partial^2 P}{\partial x^2} + \frac{1}{2}\tau \frac{\partial^2 P}{\partial t^2} + O\{ \frac{(\eta+\tau)^3}{\tau} \} \quad .$$

Assuming $\eta = O(\tau^{1/2})$ and taking the limit $\tau \to 0$, we obtain

$$\frac{\partial P}{\partial t} = D \frac{\partial^2 P}{\partial x^2} \quad , \tag{23}$$

where $D \equiv \eta^2/2\tau$: diffusivity. Of course, (23) is the Fickian diffusion equation for P.

On the other hand, if the directional correlation is perfect (completely-conscious-teetotaler-walk), $a = 1$, $b = 0$. Then, (21) and (22) become

$$P^+(x,t) = P^+(x-\eta, t-\tau) \quad , \tag{24}$$

$$P^-(x,t) = P^-(x+\eta, t-\tau) \quad . \tag{25}$$

Expansion of (24) in the powers of η and τ yields

$$P^+(x,t) = P^+(x,t) - \eta \frac{\partial P^+}{\partial x} - \tau \frac{\partial P^+}{\partial t} + O\{(\eta+\tau)^2\} \quad ,$$

or

$$\eta \frac{\partial P^+}{\partial x} + \tau \frac{\partial P^+}{\partial t} = O\{(\eta+\tau)^2\} \quad . \tag{26}$$

Similarily from (25)

$$-\eta \frac{\partial P^-}{\partial x} + \tau \frac{\partial P^-}{\partial t} = O\{(\eta+\tau)^2\} \quad . \tag{27}$$

Assuming $\eta = O(\tau)$ and taking the limit $\tau \to 0$, we obtain from (26) and (27),

$$\frac{\partial P^+}{\partial t} + v \frac{\partial P^+}{\partial x} = 0 \quad , \tag{28}$$

$$\frac{\partial P^-}{\partial t} - v \frac{\partial P^-}{\partial x} = 0 \quad , \tag{29}$$

where $v = \eta/\tau$: path velocity. Eliminating P^+ and P^- from (20), (28), and

(29), we have

$$\frac{\partial^2 P}{\partial t^2} = v^2 \frac{\partial^2 P}{\partial x^2} \quad . \tag{30}$$

Equation (30) is the wave equation for P with v being the phase speed.

What will happen if there is a partial correlation between any two consecutive directions of motion? In this case, $1 > a > b > 0$. Let $a - b = c$ and $P^+ - P^- = Q$. Since $a + b = 1$ and $P^+ + P^- = P$,

$$\begin{aligned} a &= \frac{1}{2}(1 + c) \quad , \\ b &= \frac{1}{2}(1 - c) \quad , \\ P^+ &= \frac{1}{2}(P + Q) \quad , \\ P^- &= \frac{1}{2}(P - Q) \quad . \end{aligned} \tag{31}$$

Expanding (21) in the powers of η and τ and using (31), we obtain

$$(1 - c)\, Q = -\eta\frac{\partial P}{\partial x} - c\eta\frac{\partial Q}{\partial x} - \tau\frac{\partial P}{\partial t} - c\tau\frac{\partial Q}{\partial t} + O\{(\eta+\tau)^2\} \quad . \tag{32}$$

Likewise from (22) we obtain

$$(1 - c)\, Q = -\eta\frac{\partial P}{\partial x} + c\eta\frac{\partial Q}{\partial x} + \tau\frac{\partial P}{\partial t} - c\tau\frac{\partial Q}{\partial t} + O\{(\eta+\tau)^2\} \quad . \tag{33}$$

Elimination of Q between (32) and (33) results in

$$\frac{\partial^2 P}{\partial t^2} + \left(\frac{1-c}{c\tau}\right)\frac{\partial P}{\partial t} = \frac{\eta^2}{\tau^2}\frac{\partial^2 P}{\partial x^2} + O\left\{\frac{(\eta+\tau)^3}{\tau^2}\right\} \quad . \tag{34}$$

Assuming $\underline{\dfrac{1 - c}{c} = O(\tau)}$ and $\underline{\eta = O(\tau)}$ and taking the limit $\tau \to 0$, we obtain

$$\frac{\partial^2 P}{\partial t^2} + n\frac{\partial P}{\partial t} = v^2 \frac{\partial^2 P}{\partial x^2} \quad , \tag{35}$$

where $n \equiv \frac{1 - c}{c\tau}$ and $v \equiv \eta/\tau$ (path velocity). Having the dimensions of time, n^{-1} represents a characteristic time scale for the correlation of particle movements. Equation (35) can also be expressed as

$$\frac{\partial P}{\partial t} + n^{-1} \frac{\partial^2 P}{\partial t^2} = A \frac{\partial^2 P}{\partial x^2} \quad , \tag{36}$$

where $A \equiv v^2 n^{-1}$ [L^2T^{-1}] : diffusivity.

Note that there is a difference in the physical meaning between D and A, both being defined as diffusivity. For D, $\eta = O(\tau^{1/2})$ so that the path velocity $v = \eta/\tau = O(\tau^{-1/2})$. Hence, in the limit of $\tau \to 0$, the path velocity becomes infinite. This must be the reason why the ordinary diffusion represents an infinitely rapid propagation of substance. For A, on the other hand, the path velocity as well as n remains finite. Substance spreads with a finite "diffusion velocity." At no time, is substance found beyond certain spatial limits.

Bourret (1960) argued that, because of the persistent effect of the successive paths, the process involved in the telegraph equation is of the non-Markovian character and the diffusive flux of substance depends on the history of the concentration gradient, not simply on the concentration gradient alone. To simplify, we shall first express the diffusion equation for P in the form

$$\frac{\partial P}{\partial t} + \frac{\partial J}{\partial x} = 0 \quad , \tag{37}$$

where J represents the diffusive flux. From (36) and (37), we have

$$\frac{\partial}{\partial x} \left(J + n^{-1} \frac{\partial J}{\partial t} + A \frac{\partial P}{\partial x} \right) = 0 \quad .$$

Therefore, integration results in

$$\frac{\partial J}{\partial t} + nJ = - An \frac{\partial P}{\partial x} + H(t) \quad , \qquad (38)^1$$

where H(t) is an arbitrary function of time only.

The general solution of (38) may be written as

$$J(x,t) = \{J(x,o) + \int_o^t H(t') \, e^{nt'} dt'\} \, e^{-nt} - An \int_o^t e^{n(t'-t)} \frac{\partial P(t',x)}{\partial x} \, dt' \quad ,$$

or

$$J(x,t) = \{J(x,o) + \int_o^t H(t') \, e^{nt'} dt'\} \, e^{-nt} - An \int_o^t e^{-nt'} \frac{\partial P(x,t-t')}{\partial x} \, dt' \quad . \qquad (39)$$

The convolution integral (2nd term) on the right-hand side of (39) should be noticed. The diffusive flux depends upon the history of the concentration gradient. This process might well be called "memory diffusion" (after Dr. A.D. Kirwan, personal communication). The duration of the memory persistence is of the order of n^{-1}. For $n^{-1} \to 0$, the convolution integral becomes $\frac{\partial}{\partial x} P(x,t)$, thus reducing to ordinary diffusive flux.

5. *An ad hoc derivation of the telegraph equation from a basic set of hydromechanical equations*

The random motion discussed in the previous section is still essentially a discontinuous one. Since the motion of a turbulent fluid is continuous and capable of diffusing fluid properties such as a contaminant, the derivation of the telegraph equation based on the concept of a continuum

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[1] Monin (1955) approached the problem from a reverse direction. He developed the same differential equation as (38) for the flux J from a more or less independent concept. Monin (1959) later found the telegraph equation useful in the discussion of smoke plume diffusion in the atmosphere.

would be more desirable. We shall derive the telegraph equation from a basic set of hydromechanical equations.

For simplicity consider an incompressible Newtonian flow in a non-rotating system. We thus have

$$\frac{\partial u_i}{\partial x_i} = 0 \quad , \tag{40}$$

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_j}(u_i u_j) = -\rho^{-1}\frac{\partial p}{\partial x_i} + \nu\nabla^2 u_i + g_i \quad , \tag{41}$$

where $u_i (i = 1,2,3)$ is an instantaneous velocity component, x_i a space coordinate, t the time, ρ the density, ν the kinematic viscosity, p the instantaneous pressure, and g_i a component of the body force.

An instantaneous equation for conservative concentration, S, is written as

$$\frac{\partial S}{\partial t} + \frac{\partial}{\partial x_i}(u_i S) = D\nabla^2 S \quad , \tag{42}$$

where D is the molecular diffusivity.

Denoting an ensemble average by an overbar and a fluctuation by a prime, we can express the instantaneous velocity, pressure and concentration as

$$\begin{aligned} u_i &= \overline{u}_i + u'_i \quad , \\ p &= \overline{p} + p' \quad , \\ g_i &= \overline{g}_i + g'_i \quad , \\ S &= \overline{S} + S' \quad . \end{aligned} \tag{43}$$

For simplicity we ignore the density fluctuation. Substituting (43) into

(40), (41) and (42) and averaging, we obtain

$$\frac{\partial \bar{u}_i}{\partial x_i} = 0 \quad . \tag{44}$$

$$\frac{\partial \bar{u}_i}{\partial t} + \frac{\partial}{\partial x_j}(\bar{u}_i \bar{u}_j) + \frac{\partial}{\partial x_j}\overline{u'_i u'_j} = -\rho^{-1}\frac{\partial \bar{p}}{\partial x_i} + \nu\nabla^2 \bar{u}_i + \bar{g}_i \quad . \tag{45}$$

$$\frac{\partial \bar{S}}{\partial t} + \frac{\partial}{\partial x_j}(\bar{u}_j \bar{S}) + \frac{\partial}{\partial x_j}\overline{u'_j S'} = D\nabla^2 \bar{S} \quad . \tag{46}$$

Subtraction of the averaged equations from the corresponding instantaneous equations leads to the equations for the fluctuating quantities:

$$\frac{\partial u'_i}{\partial x_i} = 0 \quad . \tag{47}$$

$$\frac{\partial u'_i}{\partial t} + \frac{\partial}{\partial x_j}(\bar{u}_i u'_j) + \frac{\partial}{\partial x_j}(u'_i \bar{u}_j) + \frac{\partial}{\partial x_j}(u'_i u'_j) - \frac{\partial}{\partial x_j}\overline{u'_i u'_j}$$

$$= -\rho^{-1}\frac{\partial p'}{\partial x_i} + \nu\nabla^2 u'_i + g'_i \quad . \tag{48}$$

$$\frac{\partial S'}{\partial t} + \frac{\partial}{\partial x_j}(\bar{u}_j S') + \frac{\partial}{\partial x_j}(u'_j \bar{S}) + \frac{\partial}{\partial x_j}(u'_j S') - \frac{\partial}{\partial x_j}\overline{u'_j S'}$$

$$= D\nabla^2 S' \quad . \tag{49}$$

We wish to close equation (46) in order to obtain a diffusion equation for $\bar{S}$. There, unfortunately, is the well-known trouble. The turbulent flux $\overline{u'_j S'}$ is unknown. Its determination is therefore the central problem of our investigation. For this purpose let us derive the equation for $\overline{u'_j S'}$ from (48) and (49).

Multiplication of equation (48) by S' and of equation (49) by u'_i, adding the resultant equations and averaging, gives the result

$$\frac{\partial}{\partial t}\overline{u'_i S'} + \bar{u}_j \frac{\partial}{\partial x_j}\overline{u'_i S'} + \overline{u'_j S'}\frac{\partial \bar{u}_i}{\partial x_j} + \overline{u'_i u'_j}\frac{\partial \bar{S}}{\partial x_j} + \frac{\partial}{\partial x_j}\overline{u'_j u'_i S'}$$

$$= -\rho^{-1}\overline{S'\frac{\partial p'}{\partial x_i}} + \nu\overline{S'\nabla^2 u'_i} + D\overline{u'_i \nabla^2 S'} + \overline{S' g'_i} \quad , \tag{50}$$

where the incompressibility conditions, (44) and (47), are used.

By cross differentiating (46) and (50) with respect to t and x_i, respectively, and by subtracting the resulting equations, one obtains

$$\frac{\partial^2}{\partial t^2}\bar{S} + \frac{\partial^2}{\partial t \partial x_i}(\bar{u}_i \bar{S}) - \frac{\partial}{\partial x_i}\{\bar{u}_j \frac{\partial}{\partial x_j}(\overline{u'_i S'})\} - \frac{\partial}{\partial x_i}(\overline{u'_j S'}\frac{\partial \bar{u}_i}{\partial x_j})$$

$$- \frac{\partial}{\partial x_i}(\overline{u'_i u'_j}\frac{\partial \bar{S}}{\partial x_j}) - \frac{\partial^2}{\partial x_i \partial x_j}(\overline{u'_i u'_j S'})$$

$$= D\frac{\partial}{\partial t}(\nabla^2 \bar{S}) + \rho^{-1}\frac{\partial}{\partial x_i}(\overline{S'\frac{\partial p'}{\partial x_i}}) - \nu\frac{\partial}{\partial x_i}(\overline{S'\nabla^2 u'_i})$$

$$- D\frac{\partial}{\partial x_i}(\overline{u'_i \nabla^2 S'}) - \frac{\partial}{\partial x_i}(\overline{S' g'_i}) \quad . \tag{51}$$

We now assume:

i) No mean flow, $\bar{u}_i \equiv 0$; in a case of constant mean flow, $\bar{u}_i = u_o \delta_{i1}$ in x_1 direction, one can eliminate the flow taking the moving coordinate system in such a way that $\xi_i = x_i - u_o \delta_{i1} t$.

ii) All molecular processes are ignored.

iii) No correlation between the fluctuating external force and concentration: $\overline{S' g'_i} = 0$.

iv) No correlation between the fluctuating pressure force and concentration: $\overline{S'\frac{\partial}{\partial x_i}(p'/\rho)} = 0$.

Under these assumptions, equation (51) is reduced to

$$\frac{\partial^2}{\partial t^2}\overline{S} - \frac{\partial^2}{\partial x_i \partial x_j}\left(\overline{u'_j u'_i S'}\right) = \frac{\partial}{\partial x_i}\left(\overline{u'_i u'_j}\frac{\partial \overline{S}}{\partial x_j}\right) . \tag{52}$$

And equation (46) is reduced to

$$\frac{\partial \overline{S}}{\partial t} + \frac{\partial}{\partial x_i}\left(\overline{u'_i S'}\right) = 0 . \tag{53}$$

We shall further hypothesize;

$$\frac{\partial}{\partial x_j}\overline{u'_i u'_j S'} = \sigma_{jj}\,\overline{u'_i S'} \equiv \tau^{-1}\,\overline{u'_i S'} \quad (\tau \geq 0) , \tag{54}$$

where τ is a time scale. By the use of (54), equation (52) becomes

$$\frac{\partial^2 \overline{S}}{\partial t^2} - \tau^{-1}\frac{\partial}{\partial x_i}\left(\overline{u'_i S'}\right) = \frac{\partial}{\partial x_i}\left(\overline{u'_i u'_j}\frac{\partial \overline{S}}{\partial x_j}\right) . \tag{55}$$

Elimination of $\overline{u'_i S'}$ between (53) and (55) yields

$$\frac{\partial^2 \overline{S}}{\partial t^2} + \tau^{-1}\frac{\partial \overline{S}}{\partial t} = \frac{\partial}{\partial x_i}\left(\overline{u'_i u'_j}\frac{\partial \overline{S}}{\partial x_j}\right) . \tag{56}$$

For stationary and isotropic turbulence,

$$\overline{u'_i u'_j} = \overline{u'^2}\,\delta_{ij} \equiv w^2\,\delta_{ij}$$

Therefore,

$$\frac{\partial^2}{\partial t^2}\overline{S} + \tau^{-1}\frac{\partial \overline{S}}{\partial t} = w^2 \nabla^2 \overline{S} . \tag{57}$$

This is the telegraph equation in three-dimensional space.

Interpretation of τ

$$\frac{\partial}{\partial x_j} \overline{u'_i u'_j S'} \equiv \tau^{-1} \overline{u'_i S'} \quad ,$$

[2]

or

$$\operatorname{div} (\overline{q \underset{\sim}{u}'}) = \tau^{-1} \overline{q} \; (q \equiv u'_i S') \quad . \tag{58}$$

Integrating (58) over an infinitesimally small volume ΔV (Fig. 6)

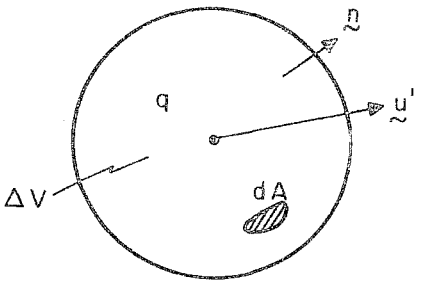

Figure 6. Interpretation of τ.

$$(\overline{q \underset{\sim}{u}'})_n \Delta A = \tau^{-1} \overline{q} \Delta V \quad ,$$

namely, the flux of q through ΔA is proportional to the average amount of q in ΔV, or

$$\tau = \overline{q} \Delta V / (\overline{q \underset{\sim}{u}'})_n \Delta A = (\text{total amount of } \overline{q} \text{ in volume } \Delta V) / (\text{total flux of q through } \Delta A).$$

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[2]Concerning this hypothesis, Dr. Stanley Corrsin (personal communication) has pointed out an obvious counter-example. For the case of homogeneous transport possibly nonstationary, $\frac{\partial}{\partial x_j} \overline{u'_j u'_i S'} = 0$, whereas $\tau^{-1} \overline{u'_i S'} \neq 0$. This is a warning against speculative attempts to estimate triple moments in terms of double moments. Double and triple moments may represent very different aspects of random processes.

Hence τ is characteristic of the mean removal (or renewal) time of q due to turbulence. To put it another way

$$\tau \sim L/v' = (\text{integral length-scale of } \overline{q\,\underset{\sim}{u}'} \text{ variation})/(\text{intensity of turbulence}).$$

Still another interpretation of τ will be seen later.

6. *Properties of the telegraph equation and the implications to oceanic diffusion*

We shall now investigate some features of the telegraph equation which are pertinent to oceanic diffusion. First of all one must remember that the telegraph equation thus far discussed in this report refers to a fixed coordinate system. In this sense, the telegraph equation should be applied to the turbulent diffusion of matter released continuously from a fixed source placed in a turbulent flow ("absolute diffusion"). We will first consider absolute diffusion.

Assume that the turbulent field is homogeneous and stationary so that the parameters τ and w are constant. If we take a one-dimensional diffusion, the telegraph equation reads

$$\frac{\partial^2 \overline{S}}{\partial t^2} + \tau^{-1} \frac{\partial \overline{S}}{\partial t} = w^2 \frac{\partial^2 \overline{S}}{\partial x^2} \quad . \tag{59}$$

Proper initial conditions for "absolute diffusion" from a plane source at $x = 0$ are

$$\overline{S}\,(o,x) = \delta(x)$$

$$\frac{\partial \overline{S}}{\partial t}\,(o,x) = 0 \quad . \tag{60}$$

Obviously

$$\int_{-\infty}^{\infty} \bar{S}\, d\,x = 1 \ ,$$

$$\int_{-\infty}^{\infty} x\, \bar{S}\, d\,x = 0 \ .$$

The equation for the variance, $\sigma^2 \equiv \int_{-\infty}^{\infty} x^2\, \bar{S}\, d\,x$, becomes

$$\frac{d^2\sigma^2}{dt^2} + \tau^{-1}\, \frac{d\sigma^2}{dt} = 2\, w^2 \ . \tag{61}$$

The initial conditions for σ^2 read. At

$$t = 0,\ \sigma^2 = 0,\ d\sigma^2/dt = 0 \ . \tag{62}$$

The solution of (61) subject to (62) is given by

$$\sigma^2 = 2w^2\, \tau^2\, \{\, \tau^{-1} - 1 + e^{-\tau^{-1}t} \,\} = 2w^2 \int_0^t dt' \int_0^{t'} e^{-\tau^{-1}t''} dt'' \ . \tag{63}$$

Limiting behaviors for σ^2 are:

$$\begin{aligned} &\text{for} \quad t \ll \tau \ , \quad \sigma^2 \sim w^2 t^2 \ . \\ &\text{for} \quad t \gg \tau \ , \quad \sigma^2 \sim 2w^2 \tau t \ . \end{aligned} \tag{64}$$

The relation (64) is the same as Taylor's (Taylor, 1921) for one-particle diffusion (Fig. 7). From (63) and Taylor's formula for the variance, the Lagrangian correlation coefficient for the telegraph equation is given by $R(\xi) = e^{-\tau^{-1}\xi}$, and τ is identified with the integral time scale (Corrsin, 1962) since $\int_0^\infty R(\xi)d\xi = \int_0^\infty e^{-\tau^{-1}\xi} d\xi = \tau$. In other words, τ is a measure of the largest time during which, on the average, a particle persists its motion.

We shall now discuss the concentration distribution described by the telegraph equation (59). For a general purpose, we use, instead of (60),

the following initial conditions; at

$$t = 0 \quad , \quad \bar{S} = f(x) \quad , \quad \frac{\partial \bar{S}}{\partial t} = g(x) \quad . \tag{65}$$

A procedure similar to that in Section 3, is used to write a formal solution for (59) subject to (65) by the use of the Fourier transformation,

$$\bar{S}(t,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\frac{-\tau^{-1}t}{2}} \Bigg[F(k) \cosh\left\{\frac{\sqrt{\tau^{-2}-4w^2k^2}}{2}\, t\right\}$$

$$+ \{2G(k) + \tau^{-1}F(k)\} \sinh\left\{\frac{\sqrt{\tau^{-2}-4w^2k^2}}{2}\, t\right\} / \sqrt{\tau^{-2}-4w^2k^2} \Bigg] e^{-ikx} dk \quad ,$$

or

$$\bar{S}(t,x) = \frac{e^{-t/2\tau}}{2\pi} \iint_{-\infty}^{\infty} f(x') \cosh \tfrac{1}{2}(\tau^{-2}-4w^2k^2)^{1/2} t \cdot e^{ik(x'-x)} dx'dk$$

$$+ \frac{e^{-t/2\tau}}{2\pi} \iint_{-\infty}^{\infty} \{2g(x')+\tau^{-1}f(x')\} \frac{\sinh \frac{1}{2}(\tau^{-2}-4w^2k^2)^{1/2} t}{(\tau^{-2}-4w^2k^2)^{1/2}} e^{ik(x'-x)} dx'dk \, . \tag{66}$$

Note that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\sinh\, a(\lambda^2-\nu^2)^{1/2}}{(\lambda^2-\nu^2)^{1/2}} e^{-ib\nu}\, d\nu = \begin{cases} \frac{1}{2} I_o\{\lambda(a^2-b^2)^{1/2}\} & |a| \geq |b| \\ 0 & |a| < |b| \end{cases} \tag{67}$$

where I_o is the zero-order modified Bessel function of the first kind.

Hence the second integral of (66) becomes

$$\frac{\pi}{w}\int_{x-wt}^{x+wt} g(x')\, I_o\left[\frac{1}{2\tau w}\{w^2t^2-(x-x')^2\}^{1/2}\right] dx' \quad +$$

$$\frac{\pi}{2w\tau}\int_{x-wt}^{x+wt} f(x')\, I_o\left[\frac{1}{2\tau w}\{w^2t^2-(x-x')^2\}^{1/2}\right] dx' \quad . \qquad (68)$$

Since $\frac{d}{db}\left(\frac{\sinh bA}{A}\right) = \cosh bA$, the first integral of (66) can easily be evaluated from the formula (67), so that

$$\int_{-\infty}^{\infty}\int f(x')\cosh\tfrac{1}{2}(\tau^{-2}-4w^2k^2)^{1/2}t \cdot e^{ik(x'-x)}dx'dk$$

$$= \frac{\pi}{w}\frac{\partial}{\partial t}\int_{x-wt}^{x+wt} f(x')\, I_o\left[\frac{1}{2\tau w}\{w^2t^2-(x-x')^2\}^{1/2}\right] dx'$$

$$= \pi\{f(x+wt)+f(x-wt)\}$$

$$+\frac{\pi t}{2\tau}\int_{x-wt}^{x+wt} f(x')\,\frac{I_1\left[\frac{1}{2\tau w}\{w^2t^2-(x-x')^2\}^{1/2}\right]}{\{w^2t^2-(x-x')^2\}^{1/2}}\, dx' \quad ,$$

where I_1 is the first-order modified Bessel function of the first kind.

One finally obtains the solution,

$$\bar{S} = \frac{1}{2} e^{-t/2\tau} \{ f(x + wt) + f(x = wt) \}$$

$$+ \frac{t}{4\tau} e^{-t/2\tau} \int_{x-wt}^{x+wt} f(x') \frac{I_1\{\frac{1}{2\tau w}(w^2t^2-(x-x')^2)^{1/2}\}}{(w^2t^2-(x-x')^2)^{1/2}} dx'$$

$$+ \frac{1}{4w} e^{-t/2\tau} \int_{x-wt}^{x+wt} \{2g(x') + \tau^{-1}f(x')\} I_o\{\frac{1}{2\tau w}(w^2t^2-(x-x')^2)^{1/2}\}dx' . \quad (69)$$

For an instantaneous plane source placed at the origin,

$$f(x') = \delta(x') \quad \text{and} \quad g(x') = 0 ,$$

so that

$$\bar{S} = \begin{cases} = \frac{1}{4\tau w} e^{-t/2\tau} I_o\{\frac{1}{2\tau w}(w^2t^2-x^2)^{1/2}\} + \\ \qquad \frac{t}{4\tau} e^{-t/2\tau} \frac{I_1\{\frac{1}{2\tau w}(w^2t^2-x^2)^{1/2}\}}{(w^2t^2-x^2)^{1/2}} & \text{for } |x| < wt , \\ = \frac{1}{2} e^{-t/2\tau} \{ \delta(x-wt) + \delta(x+wt) \} & \text{for } |x| = wt , \\ = 0 & \text{for } |x| > wt . \end{cases} \quad (70)$$

The concentration distribution is characterized by two leading edges ("fronts") located at $x = wt$ and $x = -wt$. The fronts are indicative of wave propagation with the phase speed w. The wave is damped exponentially with time, the characteristic time of decay being τ. The telegraph equation resembles the wave equation in this respect. There is, however, an

entirely new phenomenon shown in the telegraph equation. As the solution (70) reveals, a diffusion-like pattern of the concentration appears in the region inside the fronts: the concentration shows a maximum at the origin and decreases monotonically towards the fronts at which a heap in concentration is evident. It can be shown that the concentration distribution in the region tends to Gaussian for large values of t. To this end, we make use of the relation,

$$I_{\nu}\{\frac{1}{2\tau w}(w^2t^2-x^2)^{1/2}\} \sim \pi^{1/2}\tau^{-1/2}t^{1/2}(1-x^2/w^2t^2)^{1/2} \exp\{\frac{t}{2\tau}(1-x^2/w^2t^2)^{1/2}\}$$

as $t \to \infty$. We thus have

$$\lim_{t\to\infty} \bar{S}\,(\,|x| < wt\,) = \frac{1}{2\sqrt{\pi At}}\, e^{-\frac{x^2}{4At}} ,$$

where $A = w^2\tau$. Fig. 8 shows the concentration versus distance relationship for $t = 0.2\tau$, 2τ, 4τ, 10τ, 20τ.

Some modification is necessary when one attempts to use the telegraph equation in relative-diffusion problems. Thus, if one deals with the turbulent diffusion of a patch of substance, one should pay attention primarily to the spread of substance around the center of mass rather than to the spread around a fixed point. This means that the coordinate system pertinent to the problem is taken relative to the moving center of mass. This modification leads to the generalized telegraph equation which is formally the same as equation (56) with the understanding that the velocity correlation tensors $\overline{u'_i u'_j}$ stand for the velocity components relative to the center of mass.

Consider a two-dimensional radially symmetrical diffusion. The telegraph equation can then be written as

$$\frac{\partial^2 \bar{S}}{\partial t^2} + \tau^{-1} \frac{\partial \bar{S}}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left(r \, \overline{u'_r u'_r} \, \frac{\partial \bar{S}}{\partial r} \right) \quad , \tag{71}$$

when $\overline{u'_r u'_r}$ is the auto-correlation function of the relative velocity in the radial direction. We can not determine $\overline{u'_r u'_r}$ from first principles, however. We will use a conventional approach to the correlation function, *i.e.* specify $\overline{u'_r u'_r}$ as a function of t and r in general.

For simplicity, assume

$$\overline{u'_r u'_r} = \text{fn(t only)} \equiv \frac{\omega^2 t}{2\tau} \quad , \tag{72}$$

when ω has dimensions of velocity. The chosen form (72) is consistent with a model of oceanic diffusion developed by Okubo and Pritchard (Okubo, 1962). Substitution of (72) into (71) yields

$$\tau \frac{\partial^2 \bar{S}}{\partial t^2} + \frac{\partial \bar{S}}{\partial t} = \frac{\omega^2 t}{2r} \frac{\partial}{\partial r} \left(r \frac{\partial \bar{S}}{\partial r} \right) \quad . \tag{73}$$

Initial conditions for a plane-source release are

$$\begin{aligned} \bar{S}\,(o,r) &= \frac{1}{\pi r} \delta(r) \quad , \\ \frac{\partial \bar{S}}{\partial t}\,(o,r) &= 0 \quad . \end{aligned} \tag{74}$$

We shall compute the variance, $\sigma^2 = \int_o^\infty r^2 \bar{S} \, 2\pi r dr$, from (73). Multiplying $2\pi r^3$ by (73) and integrating over r from 0 to ∞, one obtains

$$\tau \frac{d^2\sigma^2}{dt^2} + \frac{d\sigma^2}{dt} = 2\omega^2 t \quad . \tag{75}$$

From (74),

$$\sigma^2(o) = \frac{d}{dt}\sigma^2(o) = 0 \quad . \tag{76}$$

The solution of (75) subject to (76) is found to be

$$\sigma^2 = \omega^2 t^2 - 2\omega^2 \tau t + 2\omega^2 \tau^2 \left(1 - e^{-t/\tau}\right) \quad . \tag{77}$$

$$\text{For} \quad t << \tau \ , \quad \sigma^2 \sim \frac{1}{3}\omega^2 \tau^{-1} t^3 \quad , \tag{78}$$

$$\text{and} \quad t >> \tau \ , \quad \sigma^2 \sim \omega^2 t^2 \quad . \tag{79}$$

The result for a time period much smaller than the integral time scale is impressive since (78) agrees with the law of the variance in relative diffusion that is derivable from the similarity theory of turbulence (Batchelor, 1950). After a sufficiently long time, the variance increases as the second power of time which is characteristic of the diffusion model by Okubo and Pritchard.

7. *Discussion*

The telegraph equation sheds light on the study of oceanic diffusion. Mainly, the finite propagation velocity characteristic of the telegraph equation is a promising feature. The behavior of the variance in both absolute and relative diffusion agrees with most other models. On the other hand, the presence of an advancing front where the substance is heaped, looks unrealistic for oceanic diffusion, although the heaping effect results partially from the purely mathematical initial condition of a Dirac spike. So far, we have not observed this heaped effect in diffusion experiments. Aside from the presence of this singular front,

the telegraph equation can produce realistic patterns of the concentration distribution almost equal to that of other diffusion models.

Further extensions of the telegraph equation are possible. One can include the Coriolis forces in the hydrodynamical equations and develop the telegraph equation in a rotating system, using the same *ad hoc* hypothesis as in Section 5. The effect of Coriolis forces on oceanic diffusion is not fully understood though some studies have been made on the matter (Welander, 1966). It seems that the inclusion of the Coriolis effect may lead to a telegraph equation which describes an asymmetric pattern of diffusion. At present, the complete derivation of the telegraph equation with the inclusion of the Coriolis forces is not possible.

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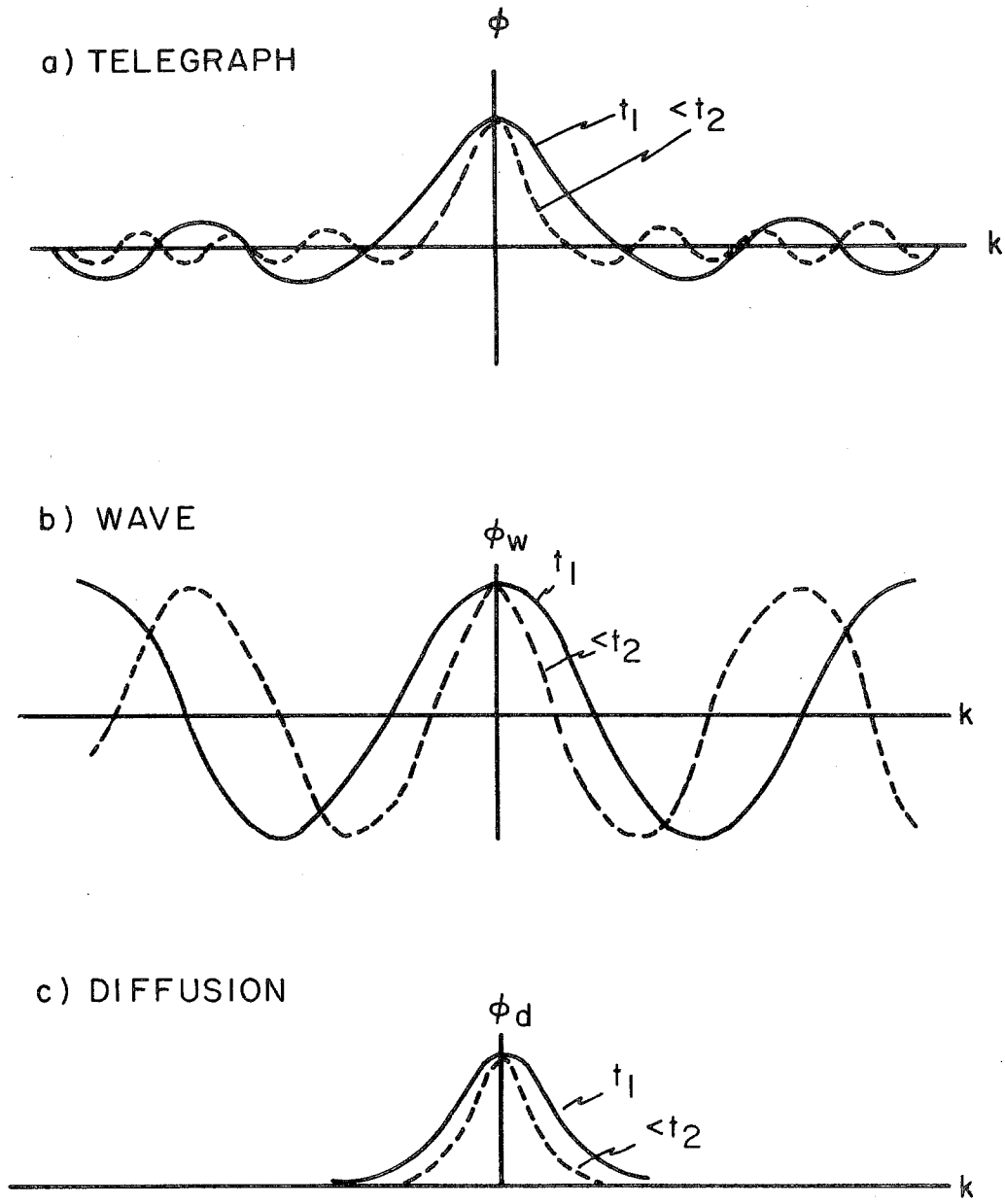

Figure 1. Behaviors in wavenumber space
a) Telegraph equation
b) Wave equation
c) Diffusion equation

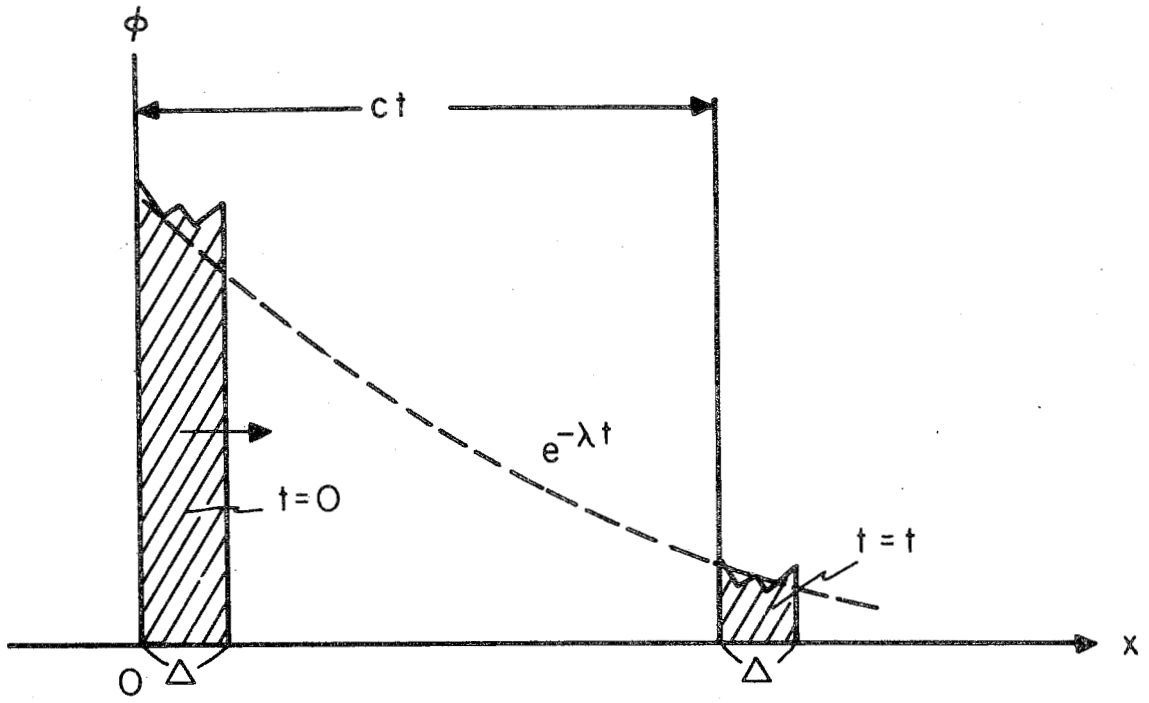

Figure 2. A damped wave motion described by the telegraph equation.

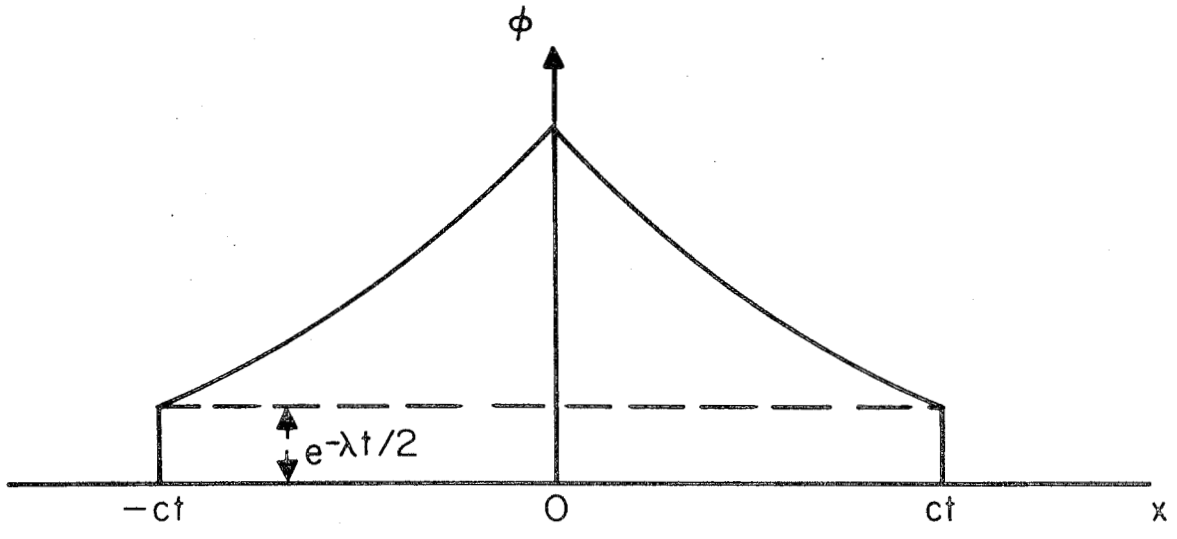

Figure 3. A diffusion-like pattern described by the telegraph equation.

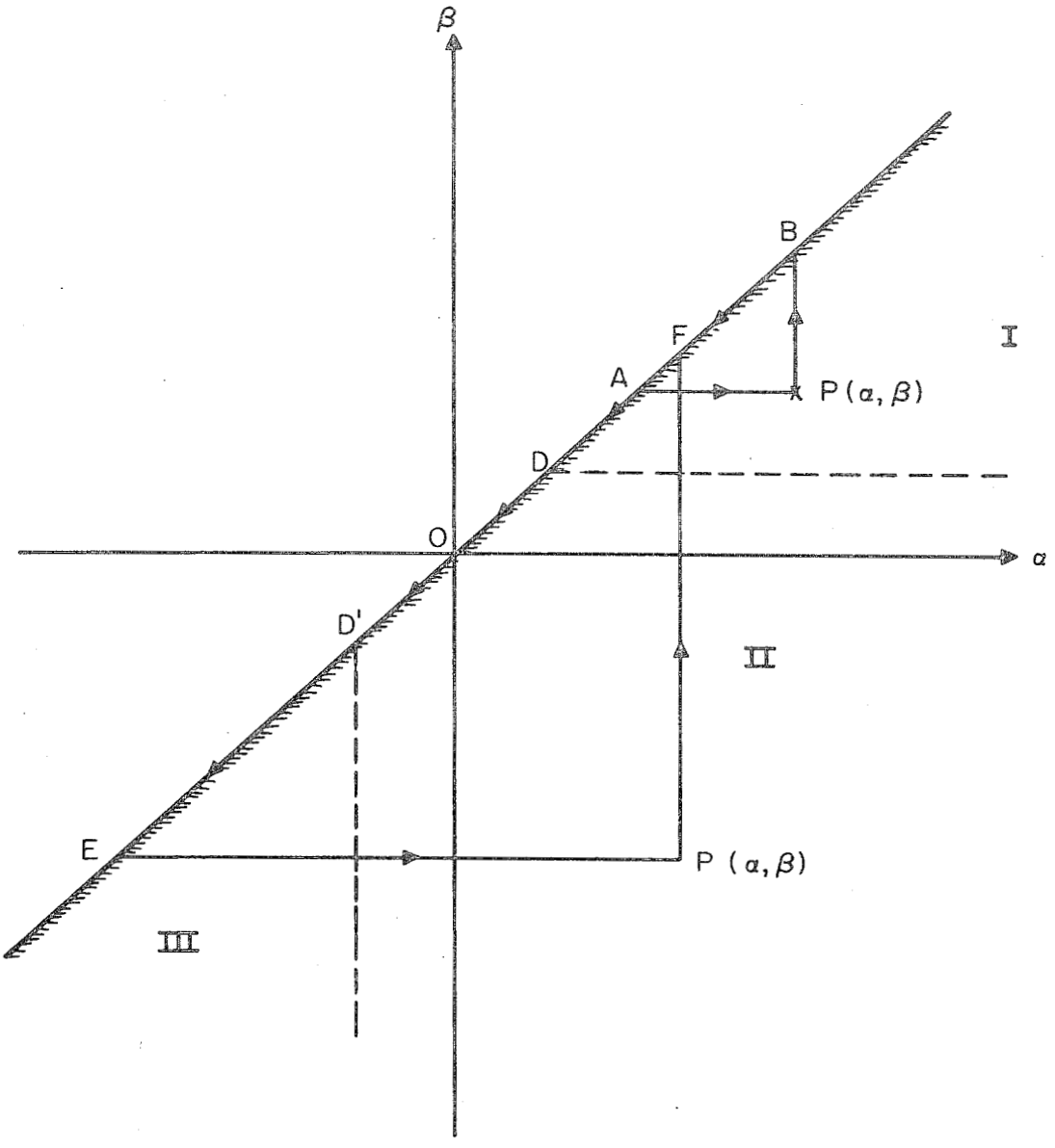

Figure 4. α - β domain.

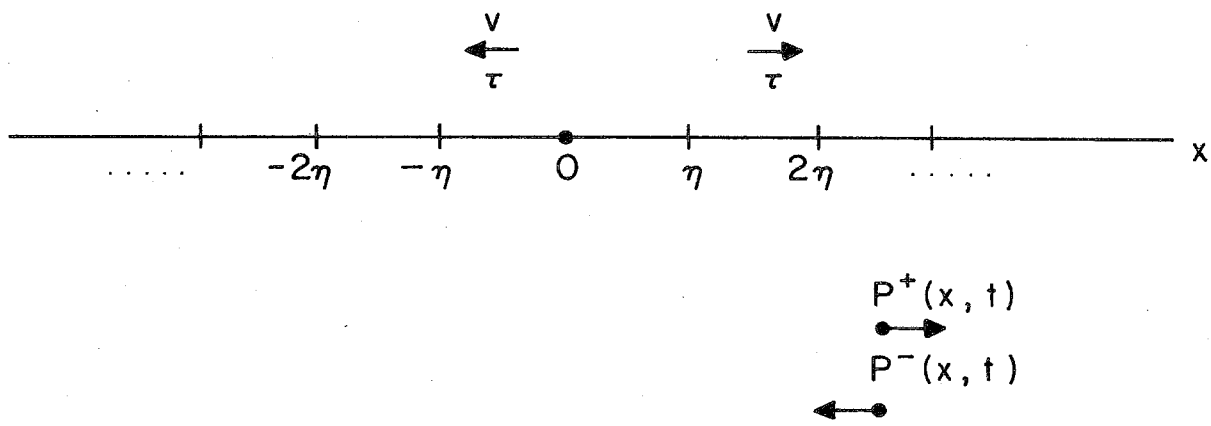

Figure 5. Random walk in one-dimensional space.

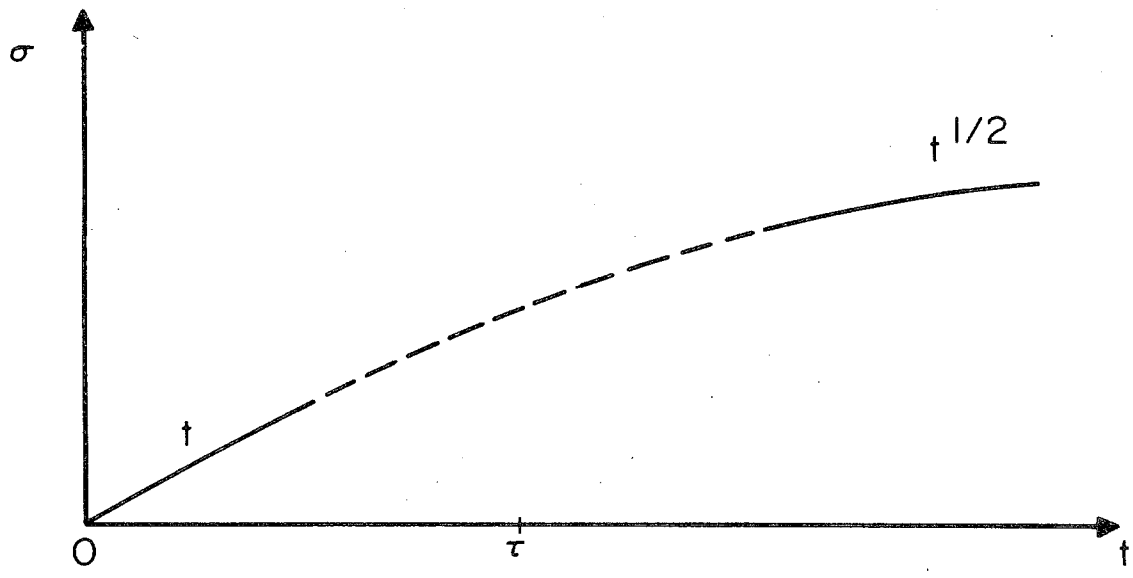

Figure 7. Time behavior of the standard deviation of particle displacement obtained from the telegraph equation.

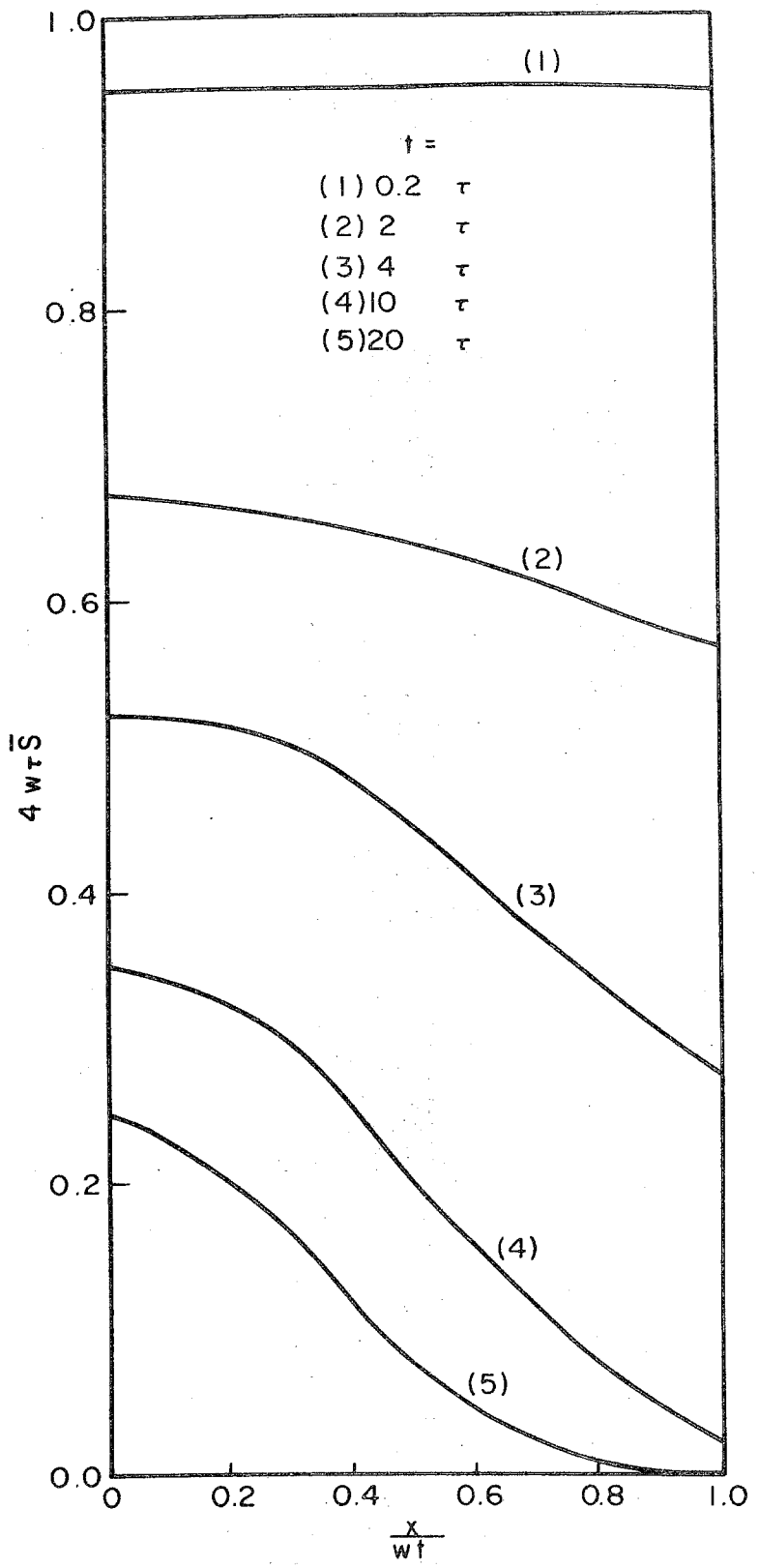

Figure 8. Spatial distribution of concentration obtained from the telegraph equation.

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13. ABSTRACT

The solution of the conventional diffusion equation has an obvious shortcoming; that is, the substance concentration will rise instantaneously everywhere when substance is introduced at some point in the sea. Although such instantaneous propagation of substance makes a negligibly small contribution to the concentration at large distances from the source, it might cause serious error in predicting water pollution, micro-organism distributions, etc. A diffusion equation which overcomes this difficulty is the telegraph equation characterized by a finite propagation velocity. An *ad hoc* derivation of the telegraph equation from a set of hydromechanical equations identifies the parameters involved in the equation. Thus the propagation velocity is related to the correlation tensor of turbulent velocity. As a result, the one-particle dispersion law by Taylor and the relative diffusion law by Richardson can be deduced from the telegraph equation.

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mathematical model of oceanic diffusion						
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telegraph equation						
velocity correlation tensor						