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ON THE MEASUREMENT AND CALCULATION OF VERTICAL TRACK MODULUS +

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ABSTRACT

This paper presents the results of a series of tests and analyses directed towards the characterization of the track structure under vertical loads. It also presents and evaluates different analytical techniques for the calculation of the vertical track modulus.

In a series of tests at the Association of American Railroads' Track Structures Dynamic Test Facility, the response of the track was obtained by monitoring track deflection under increasing vertical loads. This load and deflection data was then used to calculate vertical track modulus, track stiffness and track compliance. Three widely used techniques were utilized to calculate the vertical modulus.

The results of the tests indicate that the modulus of the track is related to the level of loading; thus identical track can give different modulus values for different load levels. Of the three different techniques used to calculate track modulus, the beam-on-elastic-foundation technique was found to be the most applicable to field measurements since it requires a minimum number of track deflection values.

INTRODUCTION

Since the early days of the railroad industry, when track constructed with longitudinal steel rails and transverse wooden crossties was introduced, track engineers have desired a reliable method to quantify the response of the track structure to given loads. The ability to specify the load-carrying capacity of track, to determine the resulting rail stresses and accompanying track deformation, is considered to be essential to proper track design and maintenance.

Winkler (1) first proposed the use of an elastic beam theory to analyze rail stresses. His method assumed the rail to behave like a beam that was continuously supported on a uniform elastic foundation. He proposed the calculation of a fundamental parameter, called the track modulus, which was related to both the applied load and the resulting track deflection, measured at one location relative to the loading point. As more modern track structures evolved, using decreased tie spacings and heavier wheel loads, Winkler's original theory was shown to be justified.

Other investigators, including Gough (2), Czitary (3) and Wasiutynski (4), independently analyzed a track structure by two different methods, assuming: (1) a beam on discrete supports and (2) a beam on a distributed elastic (Winkler) foundation. Both methods produce similar results, although the Winkler method involves simpler calculations, and has gradually become accepted by the railroad industry for use in track design. More recent investigators using the method include Timoshenko (5) and the ASCE-AREA Special Committee on

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Stresses in Railroad Track under the leadership of Prof. A. N. Talbot(6). In the First Progress report of this committee, Talbot, employing the theory first proposed by Winkler, developed the notation which has been in common use by American railway engineers ever since.

The cornerstone of the beam on elastic foundation theory was the identification of a parameter which attempted to quantify in single term (u) the combined effects of cross ties, ballast and subgrade. This parameter, was referred to by Talbot as the modulus of elasticity of rail support. It is important to note that the effect of the rail itself enters into the beam on elastic foundation theory directly, and is consequently not represented by the modulus term.

After the validity of the Winkler method had been established, track moduli calculations became very important. In the original Winkler model (I), the foundation was assumed to behave like a continuous linear spring, and the calculated modulus was a measure of the spring's stiffness. This method, however, failed to account for interactions among soil particles in the foundation. In an attempt to correct this deficiency, many early investigators either modified the Winkler model, or tried to develop new models that could more accurately describe an actual track foundation's behavior under various applied loads. Reference (7) describes some of these alternate foundation models attributed to Filonenko-Borodich, Hetenyi, Pasternak, Vlasov and Reissner.

Although many mathematical track foundation models have been developed, little was done to determine track moduli from experimental data. The first attempt to do so was undertaken by the ASCE-AREA Special Committee on Stresses in Railroad Track (6). This second phase of the Talbot committee's investigation consisted of field measurement conducted on Illinois Central Railroad trackage near Champaign, Illinois. From the test results, Talbot determined the track modulus[*] for various combinations of rail size, tie size and spacing, and ballast depth and consolidation. Talbot's method assumed that the modulus was proportional to the applied load divided by the area under the track section's deflection curve. Since deflections were measured over the entire length of the depressed section caused by the load, both soil particle interactions and load distribution by beam action of the rails were taken into account. A major advantage of this method is the averaging effect acting over the entire length of the depressed area, which compensates for any track discontinuities that may be present.

This method, however, has three distinct disadvantages, namely (1) a large number of deflection measurements are needed on both sides of the applied loading point in order to accurately determine the shape of the deflection curve (2) since the foundation experiences compression only, any slack in the track is not taken into account, and (3) as described above, the effects of differing rail size are not taken into account.

To correct for the slack in the track, this method was modified, such that the modulus became equal to the difference between a light and heavy load, divided by the net area between the load deflection curves. While this eliminates the effects of free play, twice the number of deflections must be measured.

A third method for determining track modulus from experimental data is to use a modified version of the beam-on-an-elastic-foundation theory. This method, which accounts for differences in rail size, uses Winkler's equation to calculate the track modulus. The advantages of this method are: (1) measurements are required at only one deflection point, and (2) by taking rail stiffness into account, there is an averaging effect over the entire length of the depressed track section. This method for the determination of track modulus from

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[*] The committee described the parameter measured as the modulus of elasticity of rail support. However since the effect of rail stiffness was not considered, the terminology does not precisely correspond to the notation and terminology (u) previously defined by the committee.

measured data appears to be the easiest to use and has been used by a number of railroad investigators including Schoeneberg (8), Code, (9), and Way (10). However, it has not been directly compared with other techniques on the same or closely similar tracks.

In order to compare these three methods under identical track configuration and loading conditions, tests were conducted at the Association of American Railroads' Track Laboratory in Chicago, Illinois. This paper presents the test objective, instrumentation procedures and results. Theoretical track modulus values were calculated, for track loadings ranging from zero to 50 Kips. Other related variables, such as track stiffness and compliance, track deflections and rail bending stress were also obtained. This paper also discusses the three different methods, and compares the results with each other and with previously-published data.

VERTICAL MODULUS TESTS

A series of vertical modulus tests were conducted at the Association of American Railroads’ Track Laboratory in the fall of 1979. The test area (Figure 1) contained a 45 foot section of track constructed with 136 RE rail, hardwood cross-ties at *19-V2* inch spacing, 12 inches of AREA #4 limestone ballast, and 6 inches of limestone sub-ballast, all resting on the parent foundation of poorly graded sand.

Vertical load was applied to the track structure through a specially designed loading bolster (Figure 2). A set of 50 Kip hydraulic jacks was used to apply the vertical load. Two jacks were used for the major portion of the test to represent single axle loading, and four jacks were used to simulate truck loading. The loading bolster was designed to approximate a conventional freight car track. Four 36 inch wheel segments were used to duplicate wheel- rail contact geometry.

The test .series consisted of three loading sequences in which simulated axle loads were applied through the loading bolster and measurements taken of track deflection and rail bending strain. In the first sequence, the load was applied in increasing increments from 0 to 50 Kips and data was recorded after each load increment. The second sequence was the unloading sequence and the data was taken after each decreasing increment of loading. At no time was the load returned to zero during the increasing or decreasing sequence. In the third sequence, the load was applied directly and then released to zero for each of the defined load levels.

Track deflections were measured at three locations using linear variable displacement transducers (LVDT) and at twenty-one locations using a surveyor's level. The deflections measured with LVDTs were read after each loading increment whereas the deflections measured with the level were read at only a limited set of load levels. All deflections measured were absolute, i.e. relative to a fixed zero point constant for all tests. To achieve this the L VDTs were mounted on a reference frame supported at the concrete walls of the test pit. (Figure 2) A triangular aluminum truss section was used for the reference frame. A cantilever beam extending from reference frame to rail provided the transducers support at each measurement station. The level readings were taken with the level outside the test pit and using a one-hundredth of an inch graduated scale held at the measurement point.

In addition to displacements, strains were monitored in the rails at five points. At two locations, strain gauge arrays were used to measure the applied load on each rail. This was done to provide a check on the load applied to the track. The other three arrays measured bending stresses in the rail at the load point, at 28.5 inches and 66.5 inches away from the load.

For all the loading sequences, data were recorded on both magnetic and paper tape, and reduced according to the techniques defined in Appendix A of Reference II.

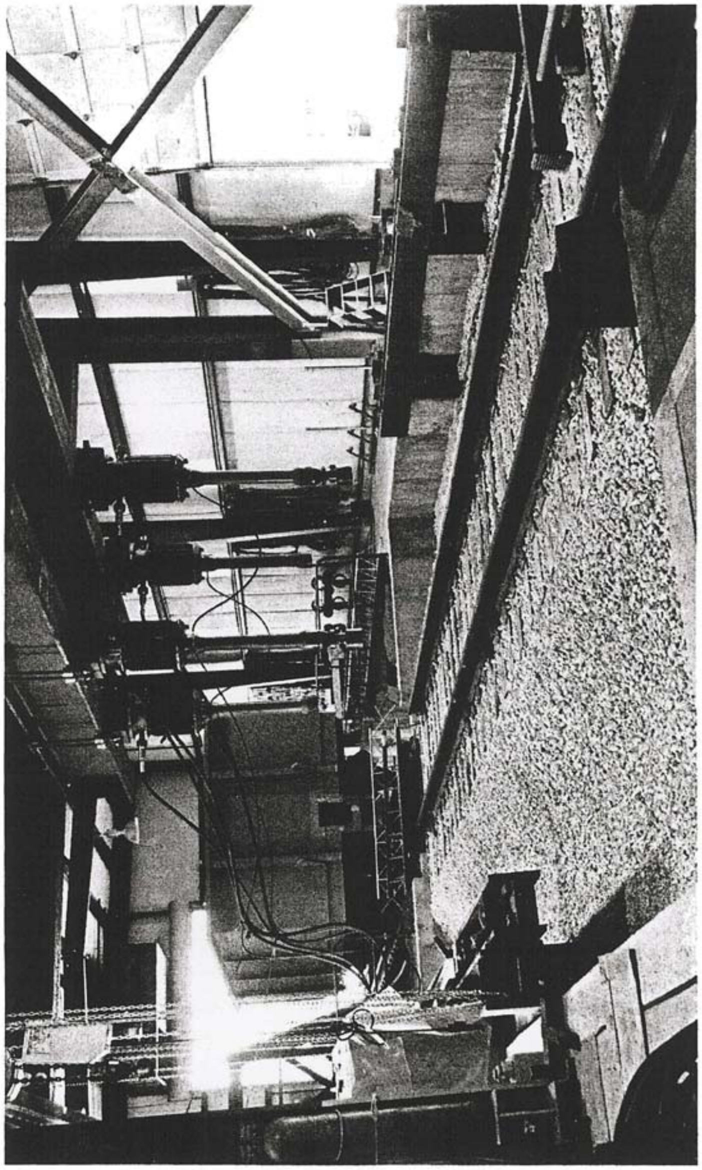

Figure 1. Test Track Used for Determining Vertical Track Modulus

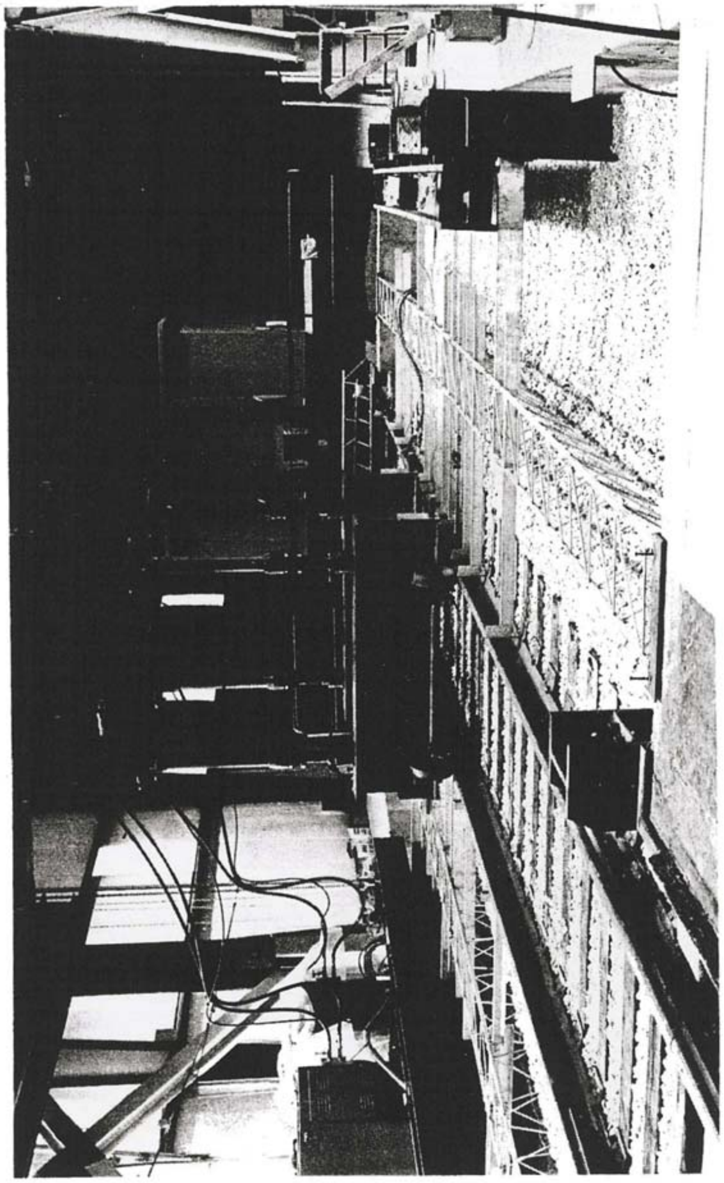

Figure 2. Test Set Up Showing Hydraulic Loading Jacks, Loading Bolster, Vertical Instrumentation, and Reference Fram.

TEST RESULTS

Three different analytical techniques were used to reduce the data with each method assuming a different definition of track modulus and also utilizing a separate procedure for calculating the modulus. The three methods are:

1. Deflection curve
2. Heavy-light wheel load deflection curve
3. Beam on elastic foundation

1. *Deflection Curve*

This method was used by the ASCE-AREA Special Committee (6) under the leadership of Talbot. The basic assumption of this method is that the applied wheel load divided by the area under the deflection curve is the track modulus, i.e.

$$u = \frac{P}{n} \quad \text{---(1)}$$
$$s\sum_{i}\left({}^{y_{-}y}{}_{1}\right)$$

Where u is the track modulus* (lb/in^2)

P is the applied wheel load (lb)

Y_i is the deflection of the ith tie (inches)

s is the tie spacing (inches)

n is the number of depressed ties.

Using this method, the modulus of the test track was found to be 4,712 lb/in^2 for a load of 39,566 lb and 4,796 lb/in^2 for a load of 50,327 lb. Figure 3 shows the deflection curve under the two loads.

2. *Heavy-Light Wheel Load Deflection Curve*

This method differs with the previous one in the way the applied load is taken into account. This method assumes that the track modulus is the difference of a heavy and a light wheel load divided by the net difference in area under these loads, i.e.

$$u = \frac{P - p}{n} \qquad \text{--(2)}$$
$$s\sum_{i}\left({}^{y_{-}y}{}_{1}\right)$$

Where u is the track modulus (lb/in^2)

P is heavy wheel load (lb)

p is light wheel load (lb)

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* u was defined to be the pressure per unit length of each rail necessary to depress the track one unit (modulus of elasticity of rail support)

s is tie spacing (inches)
y is the individual tie depression (inches) under P
y_i is the individual tie depression (inches) under p
n is the number of depressed ties

Using this method, the modulus of the track was found to be 5,016.5 lb/in^2 for a heavy load of 37,536 lb and a light load of 5,695 lb. Figure 4 shows the deflection curves under the heavy and light loads. It is evident from the data that it requires approximately 2,000 lb before the slack in the system is removed.

In methods one and two, the level measurements were used in determining the area under the load-deflection curve. The number of deflection points measured with the L VDT's was insufficient to establish a valid deflection curve.

3. *Beam On Elastic Foundation*

This method is based on the beam-on-an-elastic-foundation-theory, as defined by Winkler (1), and discussed by Talbot (6), which relates the deflection of the track, the applied load, and the track modulus. Solution of the beam on elastic foundation equation for the track modulus yields an equation for the modulus

$$u = \sqrt[3]{\frac{p^4}{64Ely^4}} \qquad \text{---}(3)$$

where u is the track modulus (lb/in^2)
EI is the stiffness of the rail (lb-in^2)

P is the applied wheel load (lb)

Y is the deflection under the load (inches)

This method differs from the previous two techniques in that the stiffness or bending rigidity of the rail is directly taken into account in the calculation of track modulus. Furthermore, only one deflection measurement, at the point of loading is required, rather than the entire deflection curve of the track.

Evaluation of the test data showed that the track modulus varied with the applied load. Figure 5 shows the track modulus *vs* the applied load for the loading and unloading sequence. It can be seen that for loads above 5,000 lb and up to 50,000 lb the modulus varies linearly with respect to the load for increasing loads. For decreasing loads it varies linearly from 50,000 lb to 10,000 lb. At loads less than 10,000 lb on the decreasing sequence, and less than 5,000 lb on the increasing sequence, the modulus variation was quite non-linear.

The difference in values for the track modulus shown for the loading and unloading sequence can be attributed to the permanent deformation in the track. The variability of the track modulus with load suggests that the track modulus should be measured as close as possible to the expected load environment of the given track. The difference between loading and unloading curves shown in Figure 5 would suggest that the values of the track modulus are dependent on the time duration of the load.

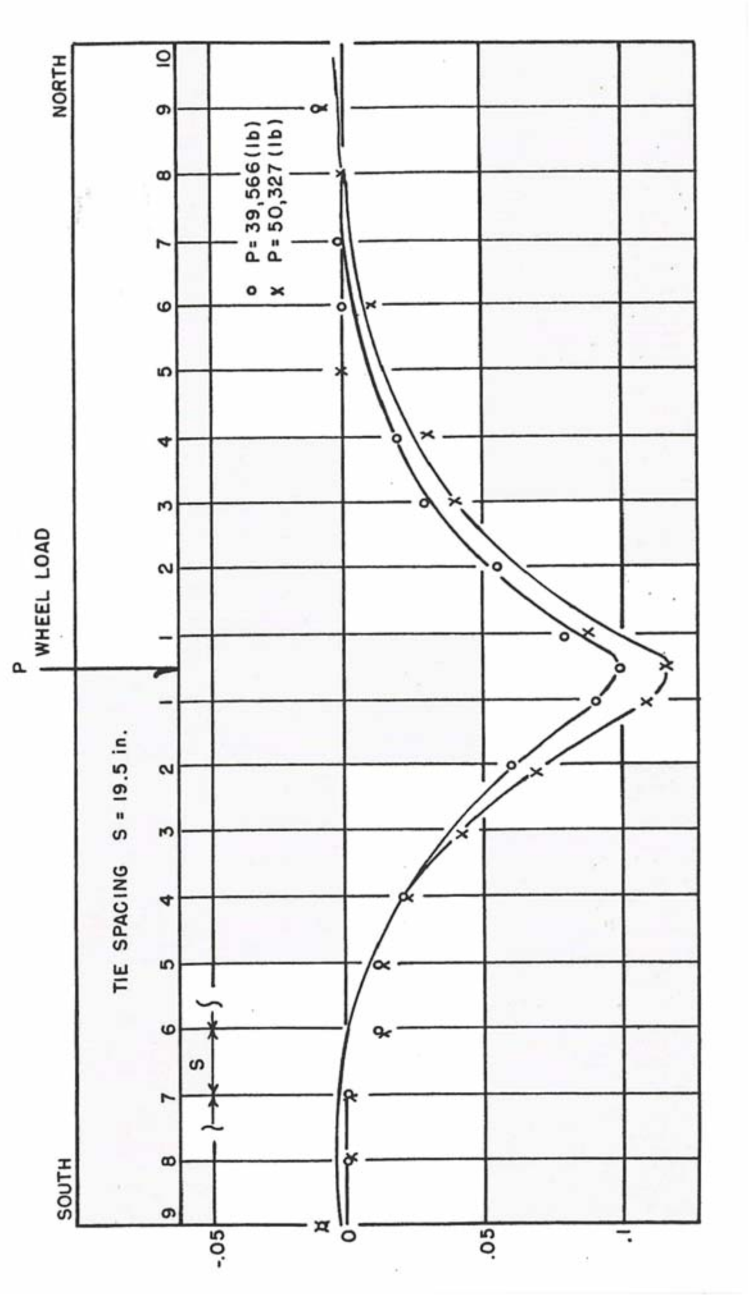

FIGURE 3. LOAD DEFLECTION CURVE FOR HEAVY WHEEL LOADING.

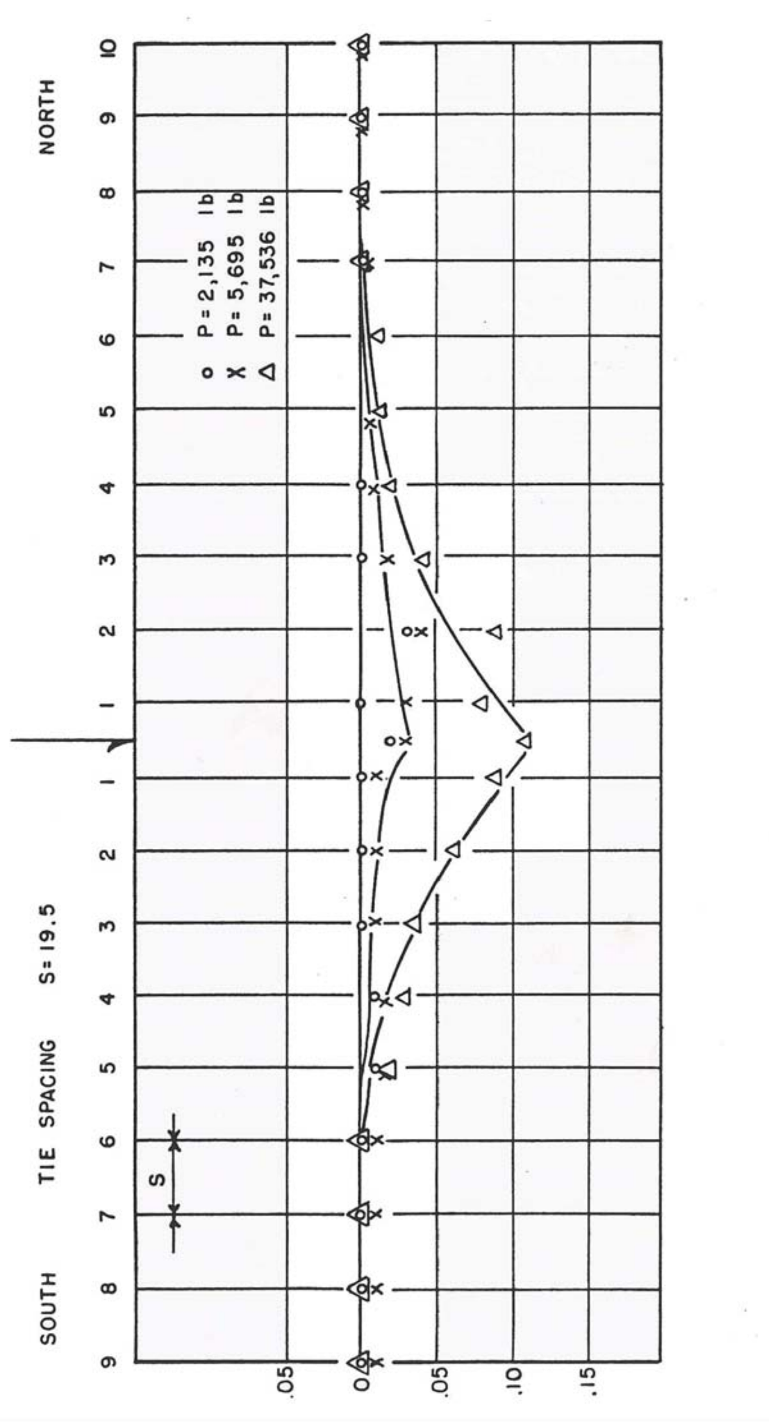

FIGURE 4. LOAD-DEFLECTION CURVE FOR HEAVY AND LIGHT WHEEL LOADS.

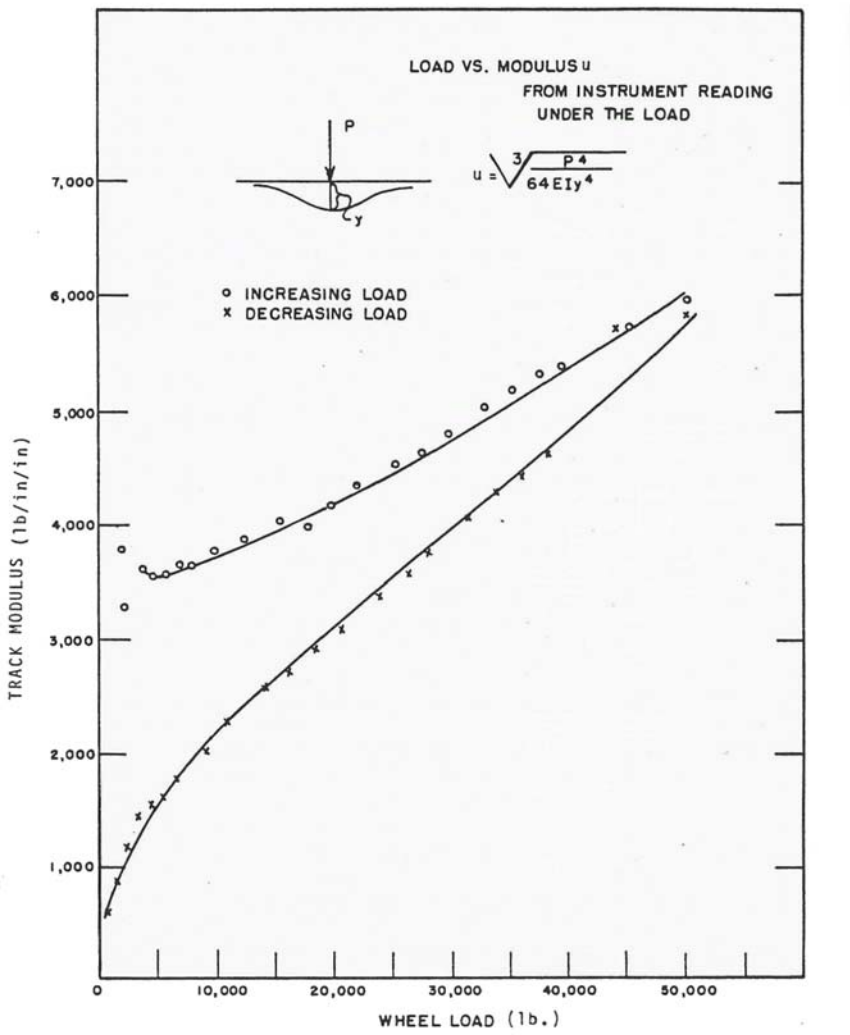

FIGURE 5. TRACK MODULUS VS. WHEEL LOAD FOR THE INCREASING AND DECREASIN LOAD SEQUENCE.

Table 1 gives the calculated modulus values for the increasing and decreasing load sequences. Table 2 gives the modulus value calculated for the third loading sequence.

Table 3 presents a summary comparison of the track modulus values calculated for each of these three methods for three sets of applied wheel loads. Note that in all cases shown, the three methods all yielded modulus values within a 20% band.

In order to calculate track modulus under multiple wheel loading using the deflection curve methods, it is necessary only to sum the applied loads and divide this applied load value by the area of the deflection curve, as appropriate. However in order to utilize beam on elastic foundation theory, an iterative calculation is required to determine the modulus.

Noting the equation for deflection of the track

$$y(x) = \frac{P\beta}{2} \qquad \eta(x) \text{ ---(4)}$$

where y(x) is the track deflection at x (inches)

x is the distance from the load (inches)

P is the applied load (lb)

u is the track modulus (lb/in^2)

$$\eta(x) = e^{Bx} \qquad (\cos \beta\chi - \sin \beta\chi) \text{ ---(5)}$$

and $$\beta = \frac{u}{4EI} \; 1/4 \qquad \text{---(6)}$$

where EI is the stiffness of the rail and using superposition theory for multiple loads, the track modulus equation becomes:

$$u = \frac{\beta}{2y} \sum_{i=1}^{n} P_i \eta_i \qquad \text{---(7)}$$

where n is the number of loads

i is the ith load

Rewriting equation (7) as

$$\left| u - \frac{\beta}{2\gamma} \sum_{i=1}^{\eta} P_i \eta_i \right| \leq \in \qquad \text{---(8)}$$

a solution can be obtained using an iterative approach by choosing a value of u and systematically changing it until equation (8) is satisfied, to a preassigned accuracy, $\in$.

Track stiffness and track compliance values calculated from the test data are also given in Tables 1 and 2. The track stiffness was determined by dividing the applied load by the deflection of the track under the load, i.e.

$$K = \frac{P}{y} \qquad \text{---(9)}$$

TABLE 1: SUMMARY OF RESULTS FOR THE LOADING AND UNLOADING SEQUENCE

LOAD (LB)	DEFLECTION (IN)	TRACK MODULUS (LB/IN x IN)	TRACK STIFFNESS (LB/IN)	TRACK COMPLIANCE (IN/LB)
		INCREASING LOADS		
641.9	0013	6967.50	0.49379E+06	0.20252E-05
1867.4	0060	3765.39	0.31124E+06	0.32130E-05
2135.9	0076	3286.26	0.28104E+06	0.35583E-05
3583.1	0119	3602.80	0.30110E+06	0.33211E-05
4586.9	0154	3550.97	0.29785E+06	0.33574E-05
5695.6	0191	3556.59	0.29820E+06	0.33534E-05
6804.4	0224	3645.40	0.30377E+06	0.32920E-05
7808.2	0257	3646.21	0.30382E+06	0.32914E-05
9990.7	0321	3765.39	0.31124E+06	0.32130E-05
12185.0	0384	3863.76	0.31732E+06	0.31514E-05
15476.3	0472	4036.33	0.32789E+06	0.30498E-05
17588.8	0541	3990.92	0.32512E+06	0.30758E-05
19771.4	0588	4174.13	0.33625E+06	0.29740E-05
21930.6	0636	4316.62	0.34482E+06	0.29001E-05
25151.9	0704	4525.68	0.35727E+06	0.27990E-05
27486.2	0755	4640.63	0.36406E+06	0.27468E-05
29738.8	0797	4795.56	0.37313E+06	0.26800E-05
32936.7	0854	5011.69	0.38568E+06	0.25929E-05
35247.7	0894	5161.12	0.39427E+06	0.25363E-05
37535.3	0933	5301.89	0.40231E+06	0.24857E-05
39566.1	0975	5363.46	0.40581E+06	0.24642E-05
45285.1	1063	5722.46	0.42601E+06	0.23474E-05
50268.8	1144	5963.71	0.43941E+06	0.22758E-05
		DECREASING LOADS		
50222.1	.1166	5806.95	0.43072E+06	0.23217E-05
44246.3	.1111	5230.83	0.39826E+06	0.25109E-05
38188.9	.1048	4546.44	0.36440E+05	0.27443E-05
35983.0	.1019	4455.71	0.35312E+06	0.28319E-05
33835.4	.0987	4283.11	0.34281E+06	0.29171E-05
31524.5	.0954	4078.37	0.33045E+06	0.30262E-05
28128.1	.0899	3791.94	0.31288E+06	0.31961E-05
26097.3	.0868	3595.73	0.30066E+06	0.33260E-05
23751.3	.0825	3393.64	0.28789E+06	0.34735E-05
20553.4	.0759	3127.58	0.27080E+06	0.36928E-05
18382.5	.0714	2923.89	0.25745E+06	0.38841E-05
16153.2	.0659	2738.54	0.24512E+06	0.40797E-05
13932.3	.0597	2577.28	0.23421E+06	0.42697E-05
10807.7	.0507	2273.29	0.21317E+06	0.46911E-05
8496.8	.0435	2023.21	0.19533E+06	0.51196E-05
6501.0	.0366	1782.45	0.17762E+06	0.56299E-05
5333.8	.0320	1637.59	0.16668E+06	0.59994E-05
4423.5	.0283	1503.10	0.15631E+06	0.63977E-05
3443.1	.0224	1469.88	0.15371E+06	0.65058E-05
2205.9	.0170	1172.74	0.12976E+06	0.770566-05
1283.9	.0117	937.84	0.10973E+05	0.91132E-05
543.6	.0072	576.60	0.76188E+05	0.13125E-04
175.1	.0047	222.08	0.37249E+05	0.26846E-04

TABLE 2: SUMMARY OF RESULTS FOR THE THIRD LOADING SEQUENCE

LOAD (LB)	DEFLECTION (IN)	TRACK MODULUS (LB/IN x IN)	TRACK STIFFNESS (LB/IN)	TRACK COMPLIANCE (IN/lb
		INCREMENTED LOADS		
5940.7	.0230	2936.55	0.25829E+06	0.38716E-05
10037.4	.0359	3263:80	0.27959E+06	0.35766E-05
19794.7	.0595	4106.05	0.33213E+06	0.30109E-05
29937.2	.0795	4854.50	0.37657E+06	0.26556E-05
39659.5	.0963	5469.92	0.41183E+06	0.24282E-05
50420.5	.1142	6001.71	0.44151E+06	0.22650E-05

TABLE 3: SUMMARY OF MODULUS VALUES

Load (lb)	Modulus (lb/in^2)		
	Method 1	Method 2	Method 3
37,536	4,465	5,167	4,257
39,566	4,712	5,658	4,667
50,327	4,769	5,529	5,604

Where K is the track stiffness (lb/in)

P is the applied wheel load (lb)

y is the deflection under the load (inches)

The track compliance is defined as the inverse of the stiffness:

$$C = \frac{1}{K} \qquad \text{--(10)}$$

Where K is the track stiffness (lb/in)

C is the track compliance (in/lb)

The relationship between track modulus and track stiffness is shown in Figure 6.

Finally, utilizing data from the bending strain gauge arrays, the bending stresses in the rail were obtained and compared with the moment influence line predicted by beam on elastic foundation theory (12). Note the excellent agreement between the test data and the theory. (Figure 7)

For a more detailed discussion of the test data and test results, the reader is referred to reference (13).

CONCLUSIONS

In comparing the methods of determining track modulus, one should concentrate not only on the results that best represent the track response but also on the technique that is easiest to use.

Thus, 'before any conclusion can be made as to the "best" method for calculating track modulus, consideration should be given to the practical problem of collecting data. As noted previously, method 1, the deflection curve technique, appears to be the most accurate because it takes into account a large portion of the track, thus eliminating local effects. However, the use of this method in the field could be cumbersome, since along with the load information, at least six absolute deflection measurements have to be taken on each side of the load. This can be done either manually with a level, thus creating a time delay in the readings, or mechanically with displacement transducers, which requires significant instrumentation. Time delay especially, in soft track, could give erroneous readings due to creeping under load. These disadvantages tend to outweigh the accuracy obtained by using this method. Method 2, the heavy-light load deflection curve, has the same disadvantages as method 1. In fact, they are even more severe since two deflection curves have to be obtained, thus doubling the number of measurements and consequently the test time. Furthermore, it is expected that this method would always give a higher track modulus than that "seen" by a vehicle in service since the initial slack in the track structure has been eliminated.

Thus, of the three methods, method 3, the beam on elastic foundation, appears to be the most suitable for general use. This method is substantially easier to collect data for, since it requires only one deflection value together with the applied load. Its accuracy, as compared to method 1, which is considered by many to be the "correct" method, is quite good. Therefore, it is recommended by the authors that the beam on elastic foundation theory be used where track modulus is required. Furthermore, it should be determined using a load level corresponding to the level of traffic experienced by the track. Thus for track that sees 100 ton car traffic, a wheel load of approximately 33,000 lb should be used to calculate the track modulus. For track that sees lighter traffic, an appropriate lower load should be used. If a comparison of

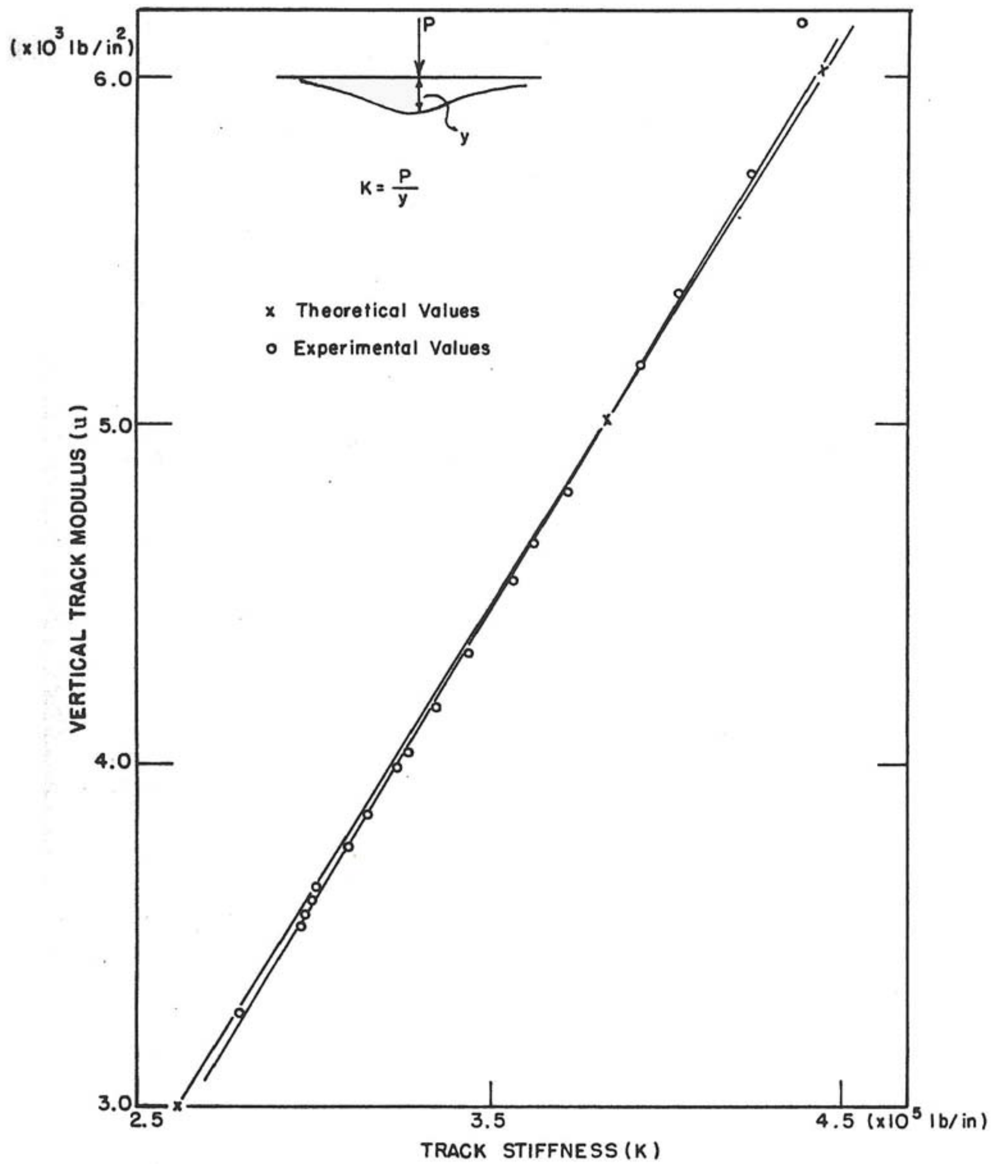

FIGURE 6. TRACK MODULUS VS. TRACK STIFFNESS FROM RAIL DEFLECTION UNDER THE LOAD.

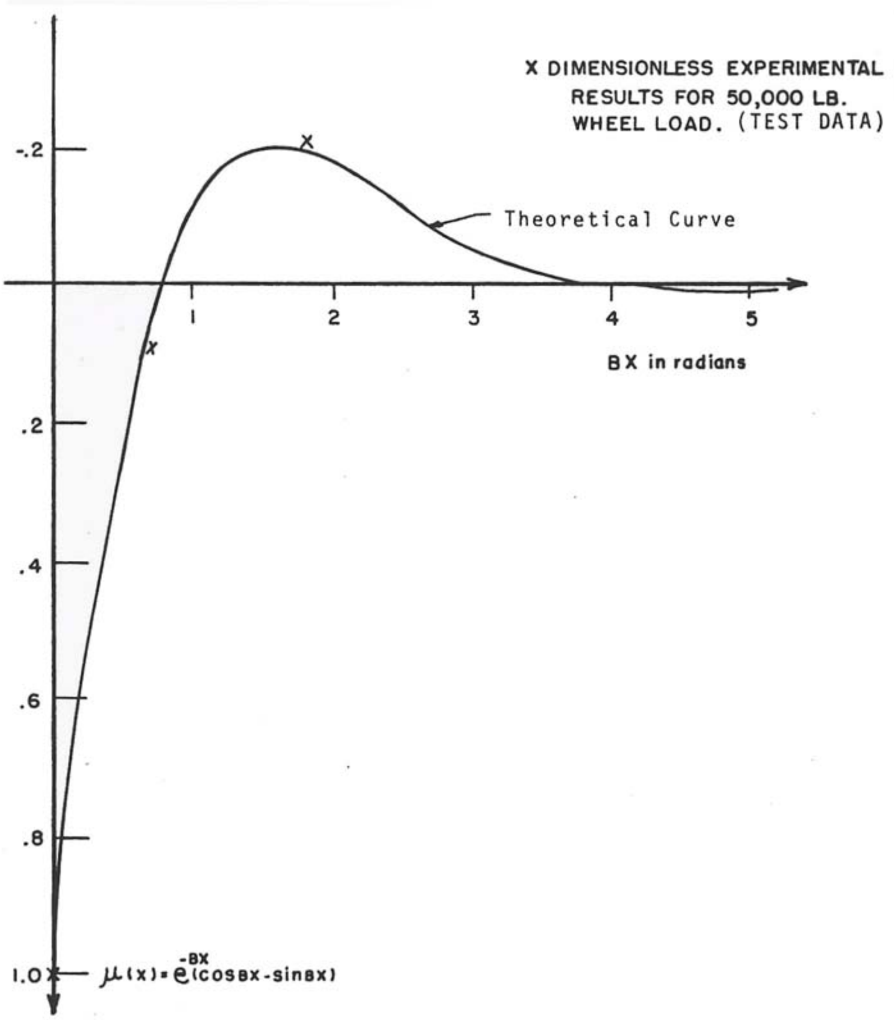

FIGURE 7

DIMENSIONLESS INFLUENCE LINE FOR THEORETICAL MOMENT different track with varying support conditions is desired, a lighter wheel load, possibly 27,5000 lb (70 ton car), should be used.

Once the track modulus is known, the stiffness and the track compliance can be readily determined from Figure 8, which gives the relation between track modulus and track stiffness for a range of rail sizes. With this information, the track engineer is then in a better position to evaluate the condition of his track, and it ability to support service traffic.

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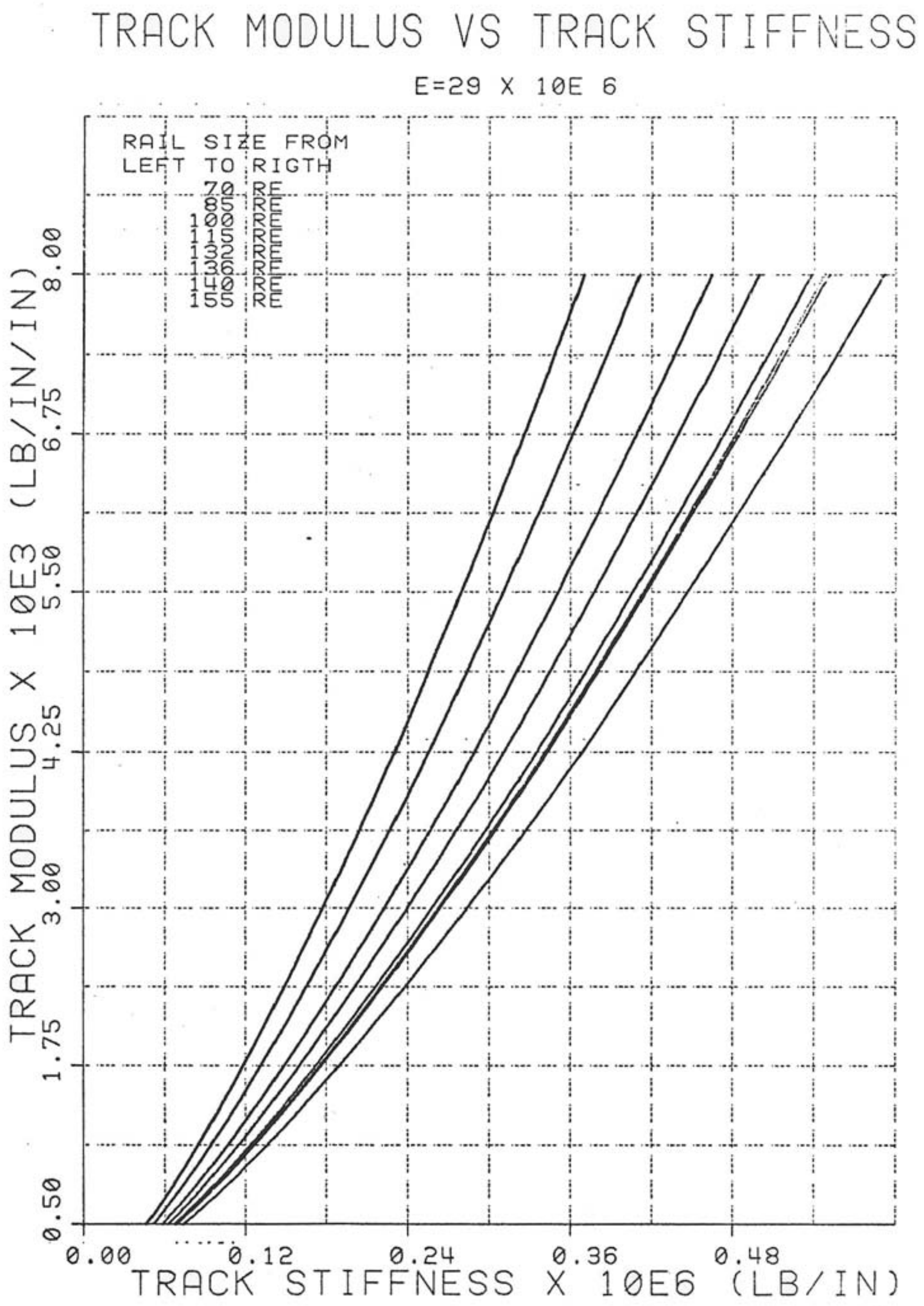

FIGURE 8. TRACK MODULUS VS TRACK STIFFNESS,
THEORETICAL RESULTS FOR VARIOUS SIZE OF RAILS.