CAMERON-LIEBLER LINE CLASSES IN PG$(3, q)$

by

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ABSTRACT

This thesis concerns a research on Cameron-Liebler line classes in $\mathrm{PG}(3, q)$, which were introduced by Cameron and Liebler in their study of collineation groups of $\mathrm{PG}(3, q)$ having equally many orbits on points and lines. These line classes have appeared in different contexts under disguised names such as Boolean degree one functions, regular codes of covering radius one, and tight sets.

In this thesis we construct an infinite family of Cameron-Liebler line classes in $\mathrm{PG}(3, q)$ with new parameter $x = \frac{(q+1)^2}{3}$ for all prime powers q congruent to 2 modulo 3. The examples obtained when q is an odd power of two represent the first infinite family of Cameron-Liebler line classes in $\mathrm{PG}(3, q)$, q even. This result is joint work with Tao Feng, Koji Momihara, Morgan Rodgers and Qing Xiang.

Chapter 1

INTRODUCTION

1.1 Backgound

Cameron-Liebler line classes were first introduced by Cameron and Liebler in their study of collineation groups of a finite projective space $\mathrm{PG}(3, q)$ having the same number of orbits on points and lines of $\mathrm{PG}(3, q)$. It is well known that a collineation group of $\mathrm{PG}(n, q)$ has at least as many orbits on lines as on points. As the extreme situation, it is of great interest to discover when the equality happens. The problem is trivial for $n = 2$ as any collineation group has this property. For $n \geq 3$, Cameron and Liebler [20] made a progress on this problem in 1982. They conjectured that such a group is line transitive, or fixes a hyperplane H and is transitive on the lines of H, or fixes a point P and is transitive on the planes through P. At that time, they were able to prove the conjecture when the group is reducible. More precisely, they showed the following.

Proposition 1.1.1 ([20]). *Let G be a reducible collineation group of $\mathrm{PG}(n, q)$, $n \geq 3$, with equally many point and line orbits. Then either*

1. *G fixes a hyperplane H and is transitive on the lines of H; or*
2. *G fixes a point P and is transitive on the planes through P (or possibly both).*

Here a collineation group of $\mathrm{PG}(n, q)$, $n \geq 3$, is said to be *reducible* if it fixes a non-trivial subspace of $\mathrm{PG}(n, q)$.

Since then, the question remained open for irreducible groups for more than twenty years. The first complete proof of the conjecture did not come until 2008 when

Bamberg and Penttila [6] showed that the group, if irreducible, is transitive on lines, and therefore closed the gap.

The story went back to the paper by Cameron and Liebler. Besides the group-theoretic problem, they also studied its combinatorial analogue. Consider the point-line design of $\mathrm{PG}(n, q)$. A tactical decomposition of this design is a partition of the point set and a partition of the line set of $\mathrm{PG}(n, q)$ such that for each point class $\mathcal{P}$ and each line class $\mathcal{L}$, every point in $\mathcal{P}$ is incident with a fixed number of lines in $\mathcal{L}$, and every line in $\mathcal{L}$ contains a fixed number of points in $\mathcal{P}$, and these two constants only depend on $\mathcal{P}$ and $\mathcal{L}$, but not the representatives of them. By Block's lemma [9, Theorem 2.1], for any tactical decomposition, the number of line classes is greater than or equal to the number of point classes. This is a very similar situation, and in this case they intended to describe the tactical decompositions with equally many point and line classes, which they called symmetric.

A useful proposition to handle this problem is that such a decomposition induces a decomposition with the same property of any 3-dimensional subspace of $\mathrm{PG}(n, q)$. Therefore it is reasonable to focus on the case when $n = 3$, where they introduced certain "special" classes of lines of $\mathrm{PG}(3, q)$ with a number of interesting equivalent properties. These special line classes will be called Cameron-Liebler line classes and will be the subject of this thesis.

Definition 1.1.1. *A class of lines $\mathcal{L}$ in $\mathrm{PG}(3, q)$ is called* **special** *if every spread of $\mathrm{PG}(3, q)$ contains a fixed number of lines in $\mathcal{L}$.*

Cameron and Liebler showed that a line class in a symmetric tactical decomposition is special, but not conversely. Moreover, they also made two conjectures to classify symmetric tactical decompositions and special line classes in $\mathrm{PG}(3, q)$ respectively.

Conjecture 1.1.2 ([20])**.** *Any symmetric tactical decomposition of the point-line design of $\mathrm{PG}(n, q)$ (with more than one point class) either has a singleton point class $\{P\}$ and a line class consisting of all lines on P, or has just two point classes (a hyperplane*

H and its complement) and two line classes (the lines in H and those meeting H in a point).

Conjecture 1.1.3 ([20])**.** *The only special line classes in* $\mathrm{PG}(3,q)$ *are: the empty set, the set of all lines in a plane H, the set of all lines through a point P, the union of the classes of the previous two types for $P \notin H$, and the complements of them.*

Both conjectures are now known to be false. However, the study of these objects remains active, and the classification is far from complete.

Since Cameron-Liebler line classes were first introduced, many nice properties of Cameron-Liebler line classes have been discovered, and they are shown to have close connections to many geometric and combinatorial objects, such as projective two-weight codes, two-characters sets, strongly regular graphs, partial difference sets, tight sets in $\mathcal{Q}^+(5,q)$, complete regular codes with strength 1, equitable partitions, and Boolean degree 1 functions in the Grassmann graph $J_q(4,2)$ which is isomorphic to the collinearity graph of $Q^+(5,q)$. All these properties and connections make the study of Cameron-Liebler line classes a popular and active topic over the past few decades.

In this thesis, we construct a new infinite family of Cameron-Liebler line classes in $\mathrm{PG}(3,q)$ with a new parameter $x = \frac{(q+1)^2}{3}$ for all $q \equiv 2 \pmod 3$. It is worth noting that our construction provides the first infinite family of Cameron-Liebler line classes when q is even. Before this work, there are only a handful of sporadic examples of Cameron-Liebler line classes when q is even.

1.2 An outline of this thesis

In Chapter 2, we give a review of relevant objects of study and auxiliary results. Chapter 3 contains a thorough discussion of Cameron-Liebler line classes in $\mathrm{PG}(3,q)$, including the equivalent definitions, basic properties and facts, and give a survey of results. In particular, we collect all known constructions of infinite families and sporadic examples. The main result will be in Chapter 4, where we develop the tools we need, present our construction and determine the stabilizer of our example. Finally, we close the thesis with some remarks in Chapter 5.

Chapter 2

PRELIMINARIES

This chapter contains definitions and properties of some objects that will be used in later chapters. We give general references about these topics here. For finite fields, character theory, and Gauss sums, we refer readers to [7, 47, 55]. For topics in finite geometry, such as projective and polar spaces, and their groups, see for example [3, 8, 19, 43, 52, 59]. For strongly regular graphs and partial difference sets, details can be found in [14, 40, 48, 61].

2.1 Finite Fields

Let p be a prime and q a power of p. There is a unique field $\mathbb{F}_q$ of q elements with characteristic p. The multiplicative group $\mathbb{F}_q^*$ of $\mathbb{F}_q$ is cyclic, and a generator of $\mathbb{F}_q^*$ is called a *primitive element* of $\mathbb{F}_q$.

For $x \in \mathbb{F}_{q^n}$, the *relative trace* of x from $\mathbb{F}_{q^n}$ to $\mathbb{F}_q$ is defined by

$$\mathrm{Tr}_{q^n/q}(x) = x + x^q + x^{q^2} + \ldots + x^{q^{n-1}}.$$

If q is prime, then $\mathrm{Tr}_{q^n/q}(x)$ is called the *absolute trace* of x. The trace function has the following properties:

(i) $\mathrm{Tr}_{q^n/q}(x+y) = \mathrm{Tr}_{q^n/q}(x) + \mathrm{Tr}_{q^n/q}(y)$ for all $x, y \in \mathbb{F}_{q^n}$.

(ii) $\mathrm{Tr}_{q^n/q}(\lambda x) = \lambda \mathrm{Tr}_{q^n/q}(x)$ for all $\lambda \in \mathbb{F}_q, x \in \mathbb{F}_{q^n}$.

(iii) $\mathrm{Tr}_{q^n/q}$ is a linear transformation from $\mathbb{F}_{q^n}$ onto $\mathbb{F}_q$, where both $\mathbb{F}_{q^n}$ and $\mathbb{F}_q$ are viewed as vector spaces over $\mathbb{F}_q$.

(iv) $\mathrm{Tr}_{q^n/q}(\lambda) = n\lambda$ for all $\lambda \in \mathbb{F}_q$.

(v) $\mathrm{Tr}_{q^n/q}(x^q) = \mathrm{Tr}_{q^n/q}(x)$ for all $x \in \mathbb{F}_{q^n}$.

(vi) $\mathrm{Tr}_{q^m/q}(x) = \mathrm{Tr}_{q^n/q}(\mathrm{Tr}_{q^m/q^n}(x))$ for all $x \in \mathbb{F}_{q^m}$ where $n \mid m$.

For $x \in \mathbb{F}_{q^n}$, the *relative norm* of x from $\mathbb{F}_{q^n}$ to $\mathbb{F}_q$ is defined by

$$\mathrm{N}_{q^n/q}(x) = x \cdot x^q \cdot x^{q^2} \cdot \cdots \cdot x^{q^{n-1}}.$$

The norm function has the following properties:

(i) $\mathrm{N}_{q^n/q}(xy) = \mathrm{N}_{q^n/q}(x)\mathrm{N}_{q^n/q}(y)$ for all $x, y \in \mathbb{F}_{q^n}$.

(ii) $\mathrm{N}_{q^n/q}$ maps $\mathbb{F}_{q^n}$ onto $\mathbb{F}_q$, and $\mathbb{F}_{q^n}^*$ onto $\mathbb{F}_q^*$.

(iii) $\mathrm{N}_{q^n/q}(\lambda) = \lambda^n$ for all $\lambda \in \mathbb{F}_q$.

(iv) $\mathrm{N}_{q^n/q}(x^q) = \mathrm{N}_{q^n/q}(x)$ for all $x \in \mathbb{F}_{q^n}$.

(v) $\mathrm{N}_{q^m/q}(x) = \mathrm{N}_{q^n/q}(\mathrm{N}_{q^m/q^n}(x))$ for all $x \in \mathbb{F}_{q^m}$ where $n \mid m$.

2.2 Characters of Finite Abelian Groups

Let G be a finite abelian group of order n with identity element 1_G. A *character* χ of G is a homomorphism from G to the multiplicative group of the field of complex numbers $\mathbb{C}$, i.e.,

$$\chi(g_1 g_2) = \chi(g_1)\chi(g_2), \ \forall g_1, g_2 \in G.$$

The character χ_0 mapping every element of G to 1 is called the *trivial* character, or the *principal* character of G.

Proposition 2.2.1. *Let χ be a character of a finite abelian group G of order n.*

(i) $\chi(1_G) = 1$.

(ii) $\chi(g)$ *is an n-th root of unity in $\mathbb{C}$, for any $g \in G$.*

(iii) $\chi(g^{-1}) = \chi(g)^{-1} = \overline{\chi(g)}$.

It is well known that the set of all the characters of G with termwise multiplication forms a group, which is isomorphic to G. This group is called the *character group* of G, and denoted by $\widehat{G}$. The identity of $\widehat{G}$ is the trivial character χ_0. The results in the following theorem are called the orthogonality relations of characters.

Theorem 2.2.2. *Let χ be a character of a finite abelian group G of order n, then*

$$\sum_{g\in G}\chi(g)=\begin{cases} n, & \text{if } \chi=\chi_0, \\ 0, & \text{otherwise.} \end{cases}$$

If $g\in G$, then

$$\sum_{\chi\in\widehat{G}}\chi(g)=\begin{cases} n, & \text{if } g=1_G, \\ 0, & \text{otherwise.} \end{cases}$$

In a finite field $\mathbb{F}_q$, there are two natural and significant finite abelian groups, namely, the additive group and the multiplicative group of the field. We give the explicit formulas of the characters in both cases.

For the additive group of $\mathbb{F}_q$ with characteristic p, let $\zeta_p=e^{\frac{2\pi i}{p}}$ be a primitive p-th root of unity. Consider the function

$$\psi_{\mathbb{F}_q}(x):=\zeta_p^{\mathrm{Tr}_{q/p}(x)},\ \forall x\in\mathbb{F}_q,$$

where $\mathrm{Tr}_{q/p}(x)$ is the absolute trace of x. It is easy to check that $\psi_{\mathbb{F}_q}$ is a character, and it is called the *canonical additive character* of $\mathbb{F}_q$. Furthermore, every additive character of $\mathbb{F}_q$ is given by

$$\psi_a(x)=\psi_{\mathbb{F}_q}(ax)$$

for some $a\in\mathbb{F}_q$.

For the multiplicative group $\mathbb{F}_q^*$ of $\mathbb{F}_q$, let w be a primitive element of $\mathbb{F}_q$; then each multiplicative character is given by

$$\chi_j(w^k)=e^{\frac{2\pi ijk}{q-1}},\ \forall k=0,1,\ldots,q-2,$$

for some $j=0,1,\ldots,q-2$. Furthermore, the group of multiplicative characters of $\mathbb{F}_q$ is cyclic of order $q-1$ with identity χ_0.

2.3 Gauss Sums

Let $q = p^n$ with p a prime and $n \geq 1$, and let $\psi_{\mathbb{F}_q}$ be the canonical additive character of $\mathbb{F}_q$. For any multiplicative character χ, define the *Gauss sum* by

$$G_q(\chi) = \sum_{x \in \mathbb{F}_q^*} \psi_{\mathbb{F}_q}(x)\chi(x).$$

The following are some basic properties of Gauss sums:

(i) $G_q(\chi)\overline{G_q(\chi)} = q$ if χ is non-principal;

(ii) $G_q(\chi^{-1}) = \chi(-1)\overline{G_q(\chi)}$;

(iii) $G_q(\chi) = -1$ if χ is principal.

Gauss sums are instrumental in the transition from the additive to the multiplicative structure (or the other way around) of a finite field. This can be seen more precisely in the next theorem.

Theorem 2.3.1. *[47, (5.16) and (5.17)] The canonical additive character $\psi_{\mathbb{F}_q}$ of $\mathbb{F}_q$ can be expressed as a linear combination of the multiplicative characters:*

$$\psi_{\mathbb{F}_q}(x) = \frac{1}{q-1} \sum_{\chi \in \widehat{\mathbb{F}_q^*}} G_q(\chi^{-1})\chi(x), \ \forall x \in \mathbb{F}_q^*, \tag{2.3.1}$$

where $\widehat{\mathbb{F}_q^*}$ is the character group of $\mathbb{F}_q^*$. On the other hand, each nontrivial multiplicative character χ of $\mathbb{F}_q$ can also be expressed as a linear combination of the additive characters:

$$\chi(x) = \frac{1}{q} G_q(\chi) \sum_{a \in \mathbb{F}_q^*} \chi^{-1}(-a)\psi_{\mathbb{F}_q}(ax), \ \forall x \in \mathbb{F}_q^*.$$

Lemma 2.3.2. *Let C_0 be a multiplicative subgroup of $\mathbb{F}_q^*$ of index N, and let χ be a multiplicative character of $\mathbb{F}_q^*$ of order N. Then for any $x \in \mathbb{F}_q^*$ we have*

$$\frac{1}{N} \sum_{\ell=0}^{N-1} G_q(\chi^{-\ell})\chi^{\ell}(x) = \sum_{a \in C_0} \psi_{\mathbb{F}_q}(xa).$$

Proof. Let η be a multiplicative character of $\mathbb{F}_q^*$ of order $q-1$ such that $\chi = \eta^{(q-1)/N}$. By (2.3.1), we have

$$\sum_{a \in C_0} \psi_{\mathbb{F}_q}(xa) = \frac{1}{q-1} \sum_{i=0}^{q-1} G_q(\eta^{-i}) \eta^i(x) \sum_{a \in C_0} \eta^i(a).$$

The inner summation equals $(q-1)/N$ when $i \equiv 0 \pmod{(q-1)/N}$, and equals 0 otherwise, so it equals $\frac{1}{N} \sum_{j=0}^{N-1} G_q(\chi^{-j}) \chi^j(a)$ as desired. □

The following result on Eisenstein sums will be used in the proof of our main theorem.

Theorem 2.3.3 ([32, Theorem 2.1])**.** *Let L be a complete set of coset representatives of $\mathbb{F}_q^*$ in $\mathbb{F}_{q^3}^*$. Let*

$$\mathcal{S} = \{x \in L \mid \mathrm{Tr}_{q^3/q}(x) = 0\}.$$

If χ is a nontrivial character of $\mathbb{F}_{q^3}^*$ whose restriction on $\mathbb{F}_q^*$ is trivial, then

$$\chi(\mathcal{S}) = G_{q^3}(\chi)/q.$$

2.4 Finite Classical Projective Spaces

Let V be an $(n+1)$-dimensional vector space over the finite field $\mathbb{F}_q$. The n-dimensional finite classical projective space $\mathrm{PG}(n,q)$ is defined as follows.

The points of $\mathrm{PG}(n,q)$ are the 1-dim subspaces of V.

The lines of $\mathrm{PG}(n,q)$ are the 2-dim subspaces of V.

The planes of $\mathrm{PG}(n,q)$ are the 3-dim subspaces of V.

⋮

In general, the k-flats of $\mathrm{PG}(n,q)$ are the $(k+1)$-dim subspaces of V.

In particular, the n-dim subspaces of V (i.e., the $(n-1)$-dim subspaces of $\mathrm{PG}(n,q)$) are called the hyperplanes of $\mathrm{PG}(n,q)$.

Incidence of the flats of $\mathrm{PG}(n,q)$ is defined by the inclusion of the corresponding subspaces of V.

Proposition 2.4.1 (Counting subspaces)**.**

1. *The number of* k*-flats of* $\mathrm{PG}(n,q)$ *is*

$$\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_q := \frac{(q^{n+1}-1)(q^{n+1}-q)\cdots(q^{n+1}-q^k)}{(q^{k+1}-1)(q^{k+1}-q)\cdots(q^{k+1}-q^k)}.$$

2. *Assume* $\ell < k$*, the number of* k*-flats of* $\mathrm{PG}(n,q)$ *containing a fixed* ℓ*-flat is* $\begin{bmatrix} n-\ell \\ k-\ell \end{bmatrix}_q$.

An *isomorphism* between two projective spaces is a bijection between the point sets of the spaces which maps any subspace into a subspace.

A *collineation* is a bijection from a projective space to another (or to itself) which maps collinear points to collinear points. Hence a collineation of a projective space is an automorphism.

The set of all collineations of a projective space forms a group, called the *full collineation group*.

- For $\mathrm{PG}(1,q)$ which is a projective line, any bijection on the set of points is a collineation since all points lie on the unique line. So the collineation group of $\mathrm{PG}(1,q)$ is the full symmetric group of the points.
- For $n \geq 2$, there are two special types of collineations of $\mathrm{PG}(n,q)$: homographies and automorphic collineations.

The *general linear group* $\mathrm{GL}(n+1,q)$ is the group of all non-singular linear transformations of the vector space $V = \mathbb{F}_q^{n+1}$ over the field $\mathbb{F}_q$. It is isomorphic to the group of all $(n+1)\times(n+1)$ invertible matrices over $\mathbb{F}_q$.

Any element A of $\mathrm{GL}(n+1,q)$ maps subspaces of V into subspaces of the same dimension, and preserves inclusion; so it induces a collineation of $\mathrm{PG}(n,q)$:

$$\langle (x_0, x_1, \ldots, x_n) \rangle \mapsto \langle (x_0, x_1, \ldots, x_n)A \rangle.$$

Such a collineation is called a *homography.* The set of all the homographies is in fact a group (so a subgroup of the collineation group) called the *projective linear group*, denoted by $\mathrm{PGL}(n+1,q)$. Note that A and λA induces the same collineation for any $\lambda \in \mathbb{F}_q^*$. So we have

$$\mathrm{PGL}(n+1,q) \cong \mathrm{GL}(n+1,q)/Z(\mathrm{GL}(n+1,q)),$$

where $Z(\mathrm{GL}(n+1,q))$ is the group of non-singular scalar matrices.

Any automorphism σ of $\mathbb{F}_q$ also induces a collineation:

$$\langle(x_0,x_1,\ldots,x_n)\rangle \mapsto \langle(x_0^\sigma,x_1^\sigma,\ldots,x_n^\sigma)\rangle.$$

A collineation of this type is called an *automorphic collineation*. The set of automorphic collineations is a group which is isomorphic to $\mathrm{Aut}(\mathbb{F}_q)$.

The group of bijections of V generated by $\mathrm{GL}(n+1,q)$ and $\mathrm{Aut}(\mathbb{F}_q)$ (which is actually their semi-direct product) is denoted by $\Gamma\mathrm{L}(n+1,q)$, called the *general semilinear group*; its elements are called *semilinear transformations*.

The group of collineations of $\mathrm{PG}(n,q)$ induced by $\Gamma\mathrm{L}(n+1,q)$ is called the *projective semilinear group*, denoted by $\mathrm{P}\Gamma\mathrm{L}(n+1,q)$.

Proposition 2.4.2.

- $\Gamma\mathrm{L}(n+1,q) \cong \mathrm{GL}(n+1,q) \rtimes \mathrm{Aut}(\mathbb{F}_q)$.
- $\mathrm{P}\Gamma\mathrm{L}(n+1,q) \cong \Gamma\mathrm{L}(n+1,q)/Z(\Gamma\mathrm{L}(n+1,q)) \cong \mathrm{PGL}(n+1,q) \rtimes \mathrm{Aut}(\mathbb{F}_q)$.

Theorem 2.4.3 (**Fundamental Theorem of Projective Geometry**). *If $n \geq 2$, every collineation of $\mathrm{PG}(n,q)$ is the composition of an automorphic collineation and a homography.*

Corollary 2.4.4.

1. *For $n \geq 2$, the full collineation group of $\mathrm{PG}(n,q)$ is the group $\mathrm{P}\Gamma\mathrm{L}(n+1,q)$.*

2. *The kernel of the action of $\Gamma\mathrm{L}(n+1,\mathbb{F})$ on $PG(n,\mathbb{F})$ is the group of non-zero scalars (acting by left multiplication).*

Definition 2.4.1. *The set of subspaces of V of dimension k is called the* **Grassmannian** $G_k(V)$.

Proposition 2.4.5. *Let V be a vector space over a field $\mathbb{F}$ of dimension $n+1$. Then*

- *The group* $\mathrm{P\Gamma L}(n+1, \mathbb{F})$ *is faithful and transitive on* $G_k(V)$ *for* $k = 1, \ldots, n$.
- *The group* $\mathrm{P\Gamma L}(n+1, \mathbb{F})$ *is 2-transitive on* $G_1(V)$.

2.5 Dualities, Polarities, σ-sesquilinear Forms and Quadratic Forms

For an n-dimensional projective space S, there is an associated *dual* space S^*, whose points and hyperplanes are the hyperplanes and points of S, respectively. In general, the dual of a k-dimensional subspace of an n-dimensional projective space is an $(n-1-k)$-dimensional subspace. The dual space S^* is isomorphic to S.

A *duality* of $\mathrm{PG}(n, q)$ is an inclusion-reversing bijection among all the subspaces of $\mathrm{PG}(n, q)$. By definition, a k-dimensional subspace of $\mathrm{PG}(n, q)$ will be mapped to an $(n-1-k)$-dimensional subspace.

Note that the square of a duality is a collineation. The most important dualities are those whose square are the identity which will be called *polarities* because they give rise to polar spaces. Formally, a *polarity* of $\mathrm{PG}(n, q)$ is a duality $\perp$ such that $U^{\perp\perp} = U$ for all subspaces U of $\mathrm{PG}(n, q)$.

One effective way to study dualities of a projective space is by the correspondence between dualities and sesquilinear forms on the underlying vector space.

Let V be a vector space over $\mathbb{F}_q$. Let σ be an automorphism of $\mathbb{F}_q$. A *σ-sesquilinear form* b is a function from $V \times V$ to $\mathbb{F}_q$ with the following properties:

- For fixed $v \in V$, $b(u, v)$ is linear;
- For fixed $u \in V$, $b(u, v_1 + v_2) = b(u, v_1) + b(u, v_2)$, $\forall v_1, v_2 \in V$;
- For fixed $u \in V$, $b(u, \lambda v) = \lambda^\sigma b(u, v)$, $\forall v \in V$.

In particular, if the automorphism σ is the identity, then b is bilinear. A σ-sesquilinear form b is *degenerate* if there is a non-zero vector $u \in V$ such that $b(u, v) = 0$ for all $v \in V$. An important property of a non-degenerate sesquilinear form is:

$$\dim(U) + \dim(U^\perp) = n,$$

for any subspace U of V, where

$$U^\perp := \{v \in V : b(u,v) = 0, \forall u \in U\}.$$

Given a duality π of S, consider a function θ from S to S^* mapping every subspace U to $\pi(U)$. It is a bijection mapping a k-dimensional subspace of S to a k-dimensional subspace of S^* and preserving inclusion. Therefore θ is a collineation.

By the Fundamental Theorem of Projective Geometry, the collineation θ can be represented by

$$\theta : \langle x \rangle \mapsto \langle x^\sigma A \rangle,\ \forall x \in V,$$

where $\sigma \in \mathrm{Aut}(\mathbb{F}_q)$ and $A \in \mathrm{GL}(n+1, q)$. Define a function:

$$\begin{aligned} b : V \times V &\to \mathbb{F}_q, \\ (u,v) &\mapsto v^\sigma A u^T. \end{aligned}$$

Then b is a non-degenerate σ-sesquilinear form.

Conversely, given a non-degenerate σ-sesquilinear form b of V, the map defined by

$$U \mapsto U^\perp$$

induces a duality of $\mathrm{PG}(n,q)$. Now the correspondence is established.

Since polarities are a special type of dualities, they will necessarily correspond to a special type of sesquilinear forms. A σ-sesquilinear form b is *reflexive* if $b(u,v) = 0$ implies $b(v,u) = 0$. Such forms have a nice property:

$$U \subseteq U^{\perp\perp},$$

for all subspaces U of V. If b is non-degenerate, then

$$U = U^{\perp\perp}.$$

This property together with the definition of polarity implies that a duality is a polarity if and only if the sesquilinear form defining it is reflexive and non-degenerate.

A classification result states that a non-degenerate reflexive σ-sesquilinear form b on V over the field $\mathbb{F}_q$, is, up to a scalar factor, one of the following types.

- alternating: $b(u, u) = 0$ for all $u \in V$.
- symmetric: $b(u, v) = b(v, u)$ for all $u, v \in V$, so f is bilinear.
- hermitian: $b(u, v) = b(v, u)^\sigma$ for all $u, v \in V$, with $\sigma^2 = id$ and $\sigma \neq id$.

A *quadratic form* on V is a map Q from V to $\mathbb{F}_q$ such that $Q(\lambda u) = \lambda^2 Q(u)$ for all $u \in V$. Given a quadratic form Q, define its polar form $b = b(u, v) := Q(u+v) - Q(u) - Q(v)$ for all $u, v \in V$. Then b is a bilinear form. Conversely, if the characteristic of $\mathbb{F}_q$ is not two, then $\frac{1}{2}b(u, u)$ is a quadratic form. Therefore when the characteristic of $\mathbb{F}_q$ is not two, the quadratic form Q and its polar form b determine each other.

A quadratic form is *singular* if there is a non-zero vector $u \in V$ such that $Q(u) = 0$ and $b(u, v) = 0$ for all $v \in V$. There are three types of non-singular quadratic forms: hyperbolic, parabolic and elliptic.

- hyperbolic: $Q(u) = u_1u_2 + u_3u_4 + \ldots + u_{2r-1}u_{2r}$.
- parabolic: $Q(u) = u_1u_2 + u_3u_4 + \ldots + u_{2r-1}u_{2r} + au_{2r+1}^2$,
  where $a = 1$ if q is even, and $a = 1$ or a chosen non-square if q is odd.
- elliptic: $Q(u) = u_1u_2 + u_3u_4 + \ldots + u_{2r-1}u_{2r} + u_{2r+1}^2 + au_{2r+1}u_{2r+2} + bu_{2r+2}^2$,
  where $b = 1$ and $\mathrm{Tr}_{q/2}(a^{-1}) = 1$ if q is even; and $a = 0$ and $-b$ is a chosen non-square if q is odd.

When the characteristic of $\mathbb{F}_q$ is not two, the quadratic form Q is non-singular if and only if its polar form b is non-degenerate.

2.6 Finite Classical Polar Spaces

Let V be a k-dimensional vector space over the finite field $\mathbb{F}_q$ equipped with a non-degenerate σ-sesquilinear form or a non-singular quadratic form. A finite classical polar space $\mathcal{S}$ constructed from V contains as subspaces the totally isotropic (totally singular) subspaces of V with respect to the sesquilinear form (quadratic form). A *totally isotropic* subspace with respect to a sesquilinear form b is a subspace U with

$b(u,v)=0$ for all $u,v\in U$, and a *totally singular* subspace with respect to a quadratic form Q is a subspace U with $Q(u)=0$ for all $u\in U$. The subspaces of maximal dimension are called the *generators* or the *maximals* of $\mathcal{S}$, and the dimension of a generator is called the *rank* of $\mathcal{S}$, or the *Witt index* of (V,b).

Based on the classification of σ-sesquilinear forms and quadratic forms, there are six types of finite classical polar spaces over $\mathbb{F}_q$:

Form	$k=\dim(V)$	Name	Notation	ϵ
Alternating	$2r$	Symplectic	$\mathcal{W}(2r-1,q)$	0
Hermitian	$2r$	Hermitian	$\mathcal{H}(2r-1,q)$	$-\frac{1}{2}$
Hermitian	$2r+1$	Hermitian	$\mathcal{H}(2r,q)$	$\frac{1}{2}$
Quadratic	$2r$	Hyperbolic	$\mathcal{Q}^{+}(2r-1,q)$	-1
Quadratic	$2r+1$	Parabolic	$\mathcal{Q}(2r,q)$	0
Quadratic	$2r+2$	Elliptic	$\mathcal{Q}^{-}(2r+1,q)$	1

Table 2.1: Finite classical polar spaces of rank r

The parameter ϵ in the above table is defined to take those values for different types of polar spaces so that we have uniform formulas for counting subspaces of a polar space of any type.

Proposition 2.6.1 (Counting subspaces)**.** *Let $\mathcal{S}$ be a finite classical polar space of rank r over $\mathbb{F}_q$.*

1. *The number of points of $\mathcal{S}$ is $(q^r-1)(q^{r+\epsilon}+1)/(q-1)$, of which $q^{2r-1+\epsilon}$ are not collinear with a given point.*

2. *The number of generators of $\mathcal{S}$ is $\prod_{i=1}^{r}(q^{i+\epsilon}+1)$.*

2.7 Orthogonal Groups

Let V be a vector space over $\mathbb{F}_q$ equipped with a non-singular quadratic form Q.

The *orthogonal group* associated with V and Q is

$$\mathrm{O}(V) := \{f \in \mathrm{GL}(V) \mid Q(f(v)) = Q(v) \text{ for all } v \in V\}.$$

The elements of $\mathrm{O}(V)$ are called *isometries* of (V, Q).

The *full orthogonal group* $\Gamma\mathrm{O}(V)$ consists of the σ-semilinear transformations f of V such that for some $a \in \mathbb{F}_q$

$$Q(f(v)) = a^\sigma Q(v), \forall v \in V.$$

The *general orthogonal group* is $\mathrm{GO}(V) = \Gamma\mathrm{O}(V) \cap \mathrm{GL}(V)$. The derived subgroup $\mathrm{O}(V)'$ of $\mathrm{O}(V)$ is denoted by $\Omega(V)$. In general, if $\mathrm{X}(V)$ denotes a group of transformations on V, the corresponding projective group is denoted by $\mathrm{PX}(V)$.

Theorem 2.7.1 ([59, Theorem 11.61])**.** *If V has Witt index m and dimension $2m$, then $\Omega(V)$ has two orbits on the set of maximals; two maximals U_1 and U_2 are in the same orbit if and only if $m - \dim(U_1 \cap U_2)$ is even.*

2.8 The Klein Correspondence

The Klein correspondence is a bijection between the set of lines of $\mathrm{PG}(3, q)$ and the set of points of $\mathcal{Q}^+(5, q)$. It is very useful in the study of Cameron-Liebler line classes of $\mathrm{PG}(3, q)$. The correspondence is set up with the help of the Plücker coordinates.

Let V be the 4-dimensional vector space underlying $\mathrm{PG}(3, q)$. For any two vectors $u = (u_1, u_2, u_3, u_4)$ and $v = (v_1, v_2, v_3, v_4)$, define

$$p_{ij} = \begin{vmatrix} u_i & u_j \\ v_i & v_j \end{vmatrix} = u_i v_j - u_j v_i, \ \forall i, j \in \{1, 2, 3, 4\}.$$

These numbers p_{ij} are called the *Plücker coordinates* of (u, v). Let τ be the map from the pairs of linearly independent vectors of V (which represent the lines of $\mathrm{PG}(3, q)$) to the points of $\mathrm{PG}(5, q)$ defined by

$$\tau(u, v) = \langle (p_{12}, p_{34}, p_{13}, p_{42}, p_{14}, p_{23}) \rangle.$$

Recall that $\mathcal{Q}^+(5, q)$ is defined by the quadratic form $Q(x) = x_1x_2 + x_3x_4 + x_5x_6$. By direct computing, one can see that

$$p_{12}p_{34} + p_{13}p_{42} + p_{14}p_{23} = 0.$$

Therefore $\langle (p_{12}, p_{34}, p_{13}, p_{42}, p_{14}, p_{23}) \rangle$ is a point of $\mathcal{Q}^+(5, q)$.

On the other hand, given a line $\ell = \langle u, v \rangle$, if we define $\theta(\ell) := \tau(u, v)$, one can easily check that $\theta(\ell)$ is independent of the choice of u and v. Moreover, we have the following.

Theorem 2.8.1. *The map τ induces a bijection θ between the lines of $\mathrm{PG}(3, q)$ and the points of $\mathcal{Q}^+(5, q)$.*

The bijection θ is called the Klein correspondence, and for this reason $\mathcal{Q}^+(5, q)$ is called the *Klein quadric*. By Theorem 2.7.1, the generators (planes) of $\mathcal{Q}^+(5, q)$ fall into two classes, called the *Latin* planes and the *Greek* planes. Two different planes are in the same class if and only if their intersection is a point.

Proposition 2.8.2 ([43, section 15.4], [27])**.**

- *The geometry with the Latin planes as "points", the points of $\mathcal{Q}^+(5, q)$ as "lines", and the Greek planes as "planes" is a 3-dimensional projective space $\mathrm{PG}(3, q)$.*
- *Every (partial) ovoid of $\mathcal{Q}^+(5, q)$ corresponds to a (partial) line spread of $\mathrm{PG}(3, q)$.*
- *An ovoid of $\mathcal{Q}^+(5, q)$ is an elliptic quadric $\mathcal{Q}^-(3, q)$ if and only if the corresponding line spread of $\mathrm{PG}(3, q)$ is regular.*
- *An ovoid of $\mathcal{Q}^+(5, q)$ is contained in a parabolic quadric $\mathcal{Q}(4, q)$ contained in $\mathcal{Q}^+(5, q)$ if and only if the corresponding line spread of $\mathrm{PG}(3, q)$ is also a spread of a symplectic geometry $W(3, q)$.*

2.9 Strongly Regular Graphs

A *strongly regular graph* $\mathrm{srg}(v,k,\lambda,\mu)$ is a simple and undirected graph, neither complete nor edgeless, that has the following properties:

(1) It is a regular graph of order v and valency k.

(2) For each pair of adjacent vertices x, y, there are exactly λ vertices adjacent to x and y.

(3) For each pair of nonadjacent vertices x, y, there are exactly μ vertices adjacent to x and y.

Let Γ be a (simple, undirected) graph. The adjacency matrix of Γ is the $(0,1)$-matrix A with both rows and columns indexed by the vertex set of Γ, where $A_{xy} = 1$ when there is an edge between x and y in Γ and $A_{xy} = 0$ otherwise. The *eigenvalues* of Γ are defined to be those of its adjacency matrix A. For convenience we call an eigenvalue of Γ *restricted* if it has an eigenvector which is not a multiple of the all-one vector $\mathbf{1}$. (For a k-regular connected graph, the restricted eigenvalues are simply the eigenvalues different from k.)

Theorem 2.9.1 ([14, Theorem 9.1.2]). *For a simple graph Γ of order v, neither complete nor edgeless, with adjacency matrix A, the following are equivalent:*

1. *Γ is strongly regular with parameters (v,k,λ,μ) for certain integers k,λ,μ,*

2. *$A^2 = (\lambda-\mu)A + (k-\mu)I + \mu J$ for certain real numbers k,λ,μ, where I, J are the identity matrix and the all-ones matrix, respectively,*

3. *A has precisely two distinct restricted eigenvalues α_1, α_2.*

The restricted eigenvalues and their multiplicities of a strongly regular graph can be computed explicitly by the parameters of the graph.

Theorem 2.9.2 ([61, Theorem 21.1]). *Let Γ be a strongly regular graph with parameters (v,k,λ,μ), whose restricted eigenvalues are r and s, where $r > s$. Then*

$$r = \frac{1}{2}\Big[\lambda-\mu+\sqrt{(\lambda-\mu)^2+4(k-\mu)}\Big],$$

and

$$s = \frac{1}{2}\Big[\lambda - \mu - \sqrt{(\lambda-\mu)^2 + 4(k-\mu)}\Big],$$

with multiplicities

$$f = \frac{1}{2}\Big[v - 1 + \frac{(v-1)(\mu-\lambda) - 2k}{\sqrt{(\mu-\lambda)^2 + 4(k-\mu)}}\Big],$$

and

$$g = \frac{1}{2}\Big[v - 1 - \frac{(v-1)(\mu-\lambda) - 2k}{\sqrt{(\mu-\lambda)^2 + 4(k-\mu)}}\Big],$$

respectively.

An effective method to construct strongly regular graphs is by the Cayley graph construction. Let G be an additively written abelian group of order v, and let D be a subset of G such that $0 \notin D$ and $-D = D$, where $-D = \{-d \mid d \in D\}$. The *Cayley graph on G with connection set D*, denoted by $\mathrm{Cay}(G, D)$, is the graph with the elements of G as vertices; two vertices are adjacent if and only if their difference belongs to D.

Let $\widehat{G}$ be the (complex) character group of G. All the eigenvalues of $\mathrm{Cay}(G, D)$ are given by $\psi(D) := \sum_{d \in D} \psi(d)$, $\psi \in \widehat{G}$. Note that $\psi_0(D) = |D|$, where ψ_0 is the principal character of G. By Theorem 2.9.1, the graph $\mathrm{Cay}(G, D)$ is strongly regular if and only if $\{\psi(D) : \psi \in \widehat{G} \setminus \{\psi_0\}\} = \{\alpha_1, \alpha_2\}$ with $\alpha_1 \neq \alpha_2$. If this is the case, then the connection set D is called a *partial difference set*, and the Delsarte *dual* of D is defined to be either of $\{\psi \in \widehat{G} : \psi(D) = \alpha_i\}$, $i = 1, 2$. Each of the Delsarte duals defines a strongly regular Cayley graph on $\widehat{G}$.

Let D be a partial difference set, and write $\widehat{G} = \{\psi_g : g \in G\}$ with ψ_0 be the trivial character. Let $D^* \subseteq G \setminus \{0\}$ be such that $\{\psi_g \in \widehat{G} : g \in D^*\}$ is one of the Delsarte duals. We say that D is *self-dual* if $D^* = D$. Self-dual partial difference sets (or self-dual strongly regular Cayley graphs) will be crucial in our construction of Cameron-Liebler line classes.

2.10 Intriguing Sets

Let $\mathcal{S}$ be a finite classical polar space of rank r over a finite field $\mathbb{F}_q$. We say that a set of points $\mathcal{I}$ in $\mathcal{S}$ is *intriguing* if for any point $P \in \mathcal{S}$,

$$|P^\perp \cap \mathcal{I}| = \begin{cases} h_1, & \text{if } P \in \mathcal{I}, \\ h_2, & \text{if } P \notin \mathcal{I}, \end{cases}$$

for some constants h_1 and h_2, which will be called the *intersection numbers* of $\mathcal{I}$.

There are two types of intriguing sets in a polar space. An intriguing set with intersection numbers

$$h_1 = (m-1)(1+q^{r-1+\epsilon}) + 1, \text{ and } h_2 = m(1+q^{r-1+\epsilon})$$

is called an *m-ovoid*. In particular, when $m = 1$, an m-ovoid is simply called an ovoid. An m-ovoid has $m(1+q^{r+\epsilon})$ points.

An intriguing set with intersection numbers

$$h_1 = i\frac{q^{r-1}-1}{q-1} + q^{r-1}, \text{ and } h_2 = i\frac{q^{r-1}-1}{q-1}$$

is called an *i-tight set*. An i-tight set contains $i \cdot \frac{q^r-1}{q-1}$ points.

Example 2.10.1.

- *The entire point set of $\mathcal{S}$ is a $\frac{q^r-1}{q-1}$-ovoid.*
- *Every generator of $\mathcal{S}$ is a 1-tight set.*

There are straightforward ways to construct intriguing sets. Let A and B be m-ovoid (i-tight set) and n-ovoid (j-tight set), respectively. If $A \subseteq B$, then $B \setminus A$ is an $(m-n)$-ovoid ($(j-i)$-tight set). If A and B are disjoint, then $A \cup B$ is an $(m+n)$-ovoid ($(i+j)$-tight set).

Properties about intriguing sets can be found in [5, 4]. Here we mention an important result which states that they give rise to strongly regular graphs.

Theorem 2.10.2 ([4, Theorem 11, 12]).

1. Let $\mathcal{S}$ be one of the polar spaces $H(2r, q^2), Q^-(2r+1, q)$, $W(2r-1, q)$ and let O be a proper m-ovoid that spans the ambient projective space. Then the set of points covered by O have two intersection numbers with respect to hyperplanes and so defines a strongly regular graph with parameters (v, k, λ, μ) as follows:

$\mathcal{S}$	$H(2r, q^2)$	$Q^-(2r+1, q)$	$W(2r-1, q)$
v	q^{4r+2}	q^{2r+2}	q^{2r}
k	$m(q^2-1)(q^{2r+1}+1)$	$m(q-1)(q^{r+1}+1)$	$m(q-1)(q^r+1)$
λ	$m(q^2-1)(3+m(q^2-1))-q^{2r+1}$	$m(3+m(q-1))(q-1)-q^r$	$m(3+m(q-1))(q-1)-q^r$
μ	$m(q^2-1)(m(q^2-1)+1)$	$m(q-1)(m(q-1)+1)$	$m(q-1)(m(q-1)+1)$

2. Let $\mathcal{S}$ be one of the polar spaces $H(2r-1, q^2)$, $Q^+(2r-1, q)$, $W(2r-1, q)$ and T be a proper i-tight set that spans the ambient projective space. Then the sets of points covered by T have two intersection numbers with respect to hyperplanes and so defines a strongly regular Cayley graph with parameters (v, k, λ, μ) as follows:

$\mathcal{S}$	$H(2r-1, q^2)$	$Q^+(2r-1, q)$	$W(2r-1, q)$
v	q^{4r}	q^{2r}	q^{2r}
k	$i(q^{2r}-1)$	$i(q^r-1)$	$i(q^r-1)$
λ	$i(i-3)+q^{2r}$	$i(i-3)+q^r$	$i(i-3)+q^r$
μ	$i(i-1)$	$i(i-1)$	$i(i-1)$

Note that this theorem has a partial converse: if D is a set of points of a polar space with the right size such that the corresponding Cayley graph is a self-dual strongly regular graph with the right parameters, then D is an m-ovoid or i-tight set (depending on the parameters). This partial converse gives a basic idea of our construction of Cameron-Liebler line classes. It will be discussed in detail in Chapter 4.

Chapter 3

CAMERON-LIEBLER LINE CLASSES IN PG(3, q)

In this chapter, we give a thorough discussion about Cameron-Liebler line classes in PG$(3, q)$, including definitions, properties, and a review of known results. Different approaches in the study of Cameron-Liebler line classes will be mentioned with examples. For details of some results, references will be listed.

3.1 Definitions and Properties

Let q be a prime power and let PG$(3, q)$ be the 3-dimensional projective space over the finite field $\mathbb{F}_q$ of order q. A *spread* in PG$(3, q)$ is a set of its lines which partitions its points. A counting argument implies that a spread contains $q^2 + 1$ skew lines. Let R be a set of skew lines of PG$(3, q)$. A line is called a *transversal* of R if it intersects every line in R. The set R is called a *regulus* if through each point of each line of R there is a transversal of R, and through each point of a transversal of R there is a line of R. It is easy to see that a regulus R contains $q + 1$ lines, and it has $q+1$ transversals which form another regulus, called the *opposite regulus* to R, denoted by R^{opp}. It has been proved that through any three skew lines of PG$(3, q)$, there is a unique regulus. A spread is *regular* if it contains all the reguli determined by any three of its lines. Note that there exist regulus-free spreads in PG$(3, q)$. More information of these objects can be found in [8, 2].

For a point P in PG$(3, q)$, let star(P) be the set of lines in PG$(3, q)$ passing through P. For a plane π in PG$(3, q)$, let line(π) be the set of lines in π. A *clique* is either a set star(P) or a set line(π). For an incident point-plane pair (P, π), let pencil(P, π) be set of lines in π passing through P.

Definition 3.1.1. *Let $\mathcal{L}$ be a set of lines of* $\mathrm{PG}(3,q)$*. We say that $\mathcal{L}$ is a Cameron-Liebler line class with parameter x if there exists a positive integer x such that $|\mathcal{L} \cap \mathcal{S}| = x$ for all spreads $\mathcal{S}$ of* $\mathrm{PG}(3,q)$*.*

Example 3.1.1.

- *The line set* $\mathrm{star}(P)$ *is a Cameron-Liebler line class with parameter 1 for any point P in* $\mathrm{PG}(3,q)$*.*
- *The line set* $\mathrm{line}(\pi)$ *of* $\mathrm{PG}(3,q)$ *is a Cameron-Liebler line class with parameter 1 for any plane π in* $\mathrm{PG}(3,q)$*.*
- *The line set* $\mathrm{star}(P) \cup \mathrm{line}(\pi)$ *is a Cameron-Liebler line class with parameter 2 for any non-incident point-plane pair (P,π) in* $\mathrm{PG}(3,q)$*.*

These examples of Cameron-Liebler line classes with parameter $x = 1$ or 2 are called *trivial*. Cameron and Liebler [20] proved that they are the only examples with parameters 1 or 2.

By definition and counting techniques, we get some properties of Cameron-Liebler line classes.

Proposition 3.1.2. *[20, Proposition 3.3] Let $\mathcal{L}$ be a Cameron-Liebler line class with parameter x in* $\mathrm{PG}(3,q)$*. Let $\chi_{\mathcal{L}}$ be the characteristic vector of $\mathcal{L}$ (which has length $\begin{bmatrix}4\\2\end{bmatrix}_q$), i.e., $\chi_{\mathcal{L}}(L) = 1$ if $L \in \mathcal{L}$ and $\chi_{\mathcal{L}}(L) = 0$ otherwise. Then*

1. *$|\mathcal{L}| = x(q^2+q+1)$.*
2. *For any line L, the number of lines m_1 of $\mathcal{L}$ skew to L is $(x - \chi_{\mathcal{L}}(L))q^2$.*
3. *For any two skew lines L_1, L_2, the number of lines m_2 of $\mathcal{L}$ skew to L_1 and L_2 is $(x - \chi_{\mathcal{L}}(L_1) - \chi_{\mathcal{L}}(L_2))q(q-1)$.*

Proof. Since $\mathrm{PGL}(4,q)$ acts transitively on the triples of pairwise skew lines, the number n_i of spreads containing i pairwise skew lines only depends on i for $i \le 3$.

For the size of $\mathcal{L}$, count line-spread pairs $(\ell, \mathcal{S})$ with $\ell \in \mathcal{L}$ and $\ell \in \mathcal{S}$. We have $|\mathcal{L}|n_1 = n_0 \cdot x$. On the other hand, by counting line-spread pairs $(\ell, \mathcal{S})$ with $\ell \in \mathcal{S}$ over all lines in $\mathrm{PG}(3,q)$, we have $(q^2+1)(q^2+q+1)n_1 = (q^2+1)n_0$ so $(q^2+q+1)n_1 = n_0$. Thus $|\mathcal{L}| = \frac{x \cdot n_0}{n_1} = x(q^2+q+1)$.

For the second statement, count line-spread pairs $(\ell, \mathcal{S})$ with $\ell \in \mathcal{L}$ skew to L and $\mathcal{S}$ containing ℓ, L. We have $m_1 \cdot n_2 = n_1(x - \chi_{\mathcal{L}}(L))$. Each point is on q^2+q+1 lines. So each line is incident with $(q+1)(q^2+q)$ lines, and is skew to $(q^2+1)(q^2+q+1) - (q+1)(q^2+q) - 1 = q^4$ lines. By counting $(\{\ell_1, \ell_2\}, \mathcal{S})$ with ℓ_1, ℓ_2 are skew lines and $\mathcal{S}$ a spread containing ℓ_1, ℓ_2, we have $\frac{(q^2+1)(q^2+q+1)q^4}{2} n_2 = n_0 \cdot \binom{q^2+1}{2} n_1 (q^2+q+1)\frac{(q^2+1)q^2}{2}$. Thus $m_1 = \frac{n_1}{n_2}(x - \chi_{\mathcal{L}}(L)) = q^2(x - \chi_{\mathcal{L}}(L))$.

For the last statement, count line-spread pairs $(\ell, \mathcal{S})$ with $\ell \in \mathcal{L}$ skew to L_1, L_2, and $\mathcal{S}$ containing ℓ, L_1, L_2. We have $m_2 \cdot n_3 = n_2(x - \chi_{\mathcal{L}}(L_1) - \chi_{\mathcal{L}}(L_2))$. The number of triples of pairwise skew lines is $\frac{(q^2+1)(q^2+q+1)q^4(q^2-1)(q^2-q)}{6}$. Counting $(\{\ell_1, \ell_2, \ell_3\}, \mathcal{S})$ over triples of pairwise skew lines, and spreads containing such triples yields $q(q-1)n_3 = n_2$. Therefore $m_2 = q(q-1)(x - \chi_{\mathcal{L}}(L_1) - \chi_{\mathcal{L}}(L_2))$.

□

The next theorem gives eight equivalent characterizations of Cameron-Liebler line classes. The first five statements were proved to be equivalent by Cameron and Liebler in [20]. Penttila [54] added the last three equivalent statements.

Theorem 3.1.3. *Let A be the point-line incidence matrix of* $\mathrm{PG}(3,q)$ *with respect to some ordering of the points and lines, and let $\mathcal{L}$ be a set of lines with characteristic vector $\chi_{\mathcal{L}}$. Then the following are equivalent:*

1. $\chi_{\mathcal{L}} \in row(A)$.

2. $\chi_{\mathcal{L}} \in (null(A^T))^{\perp}$.

3. *There exists a constant x such that $|\mathcal{L} \cap \mathcal{S}| = x$ for every spread $\mathcal{S}$.*

4. *There exists a constant x such that $|\mathcal{L} \cap \mathcal{S}| = x$ for every regular spread $\mathcal{S}$.*

5. $|\mathcal{L} \cap R| = |\mathcal{L} \cap R^{opp}|$ *for every regulus R and its opposite R^{opp}.*

6. *There exists a constant x such that*

$$|\text{star}(P) \cap \mathcal{L}| + |\text{line}(\pi) \cap \mathcal{L}| = x + (q+1)|\text{pencil}(P, \pi) \cap \mathcal{L}|,$$

for every incident point-plane pair (P, π).

7. *There exists a constant x such that*

$$|\{m \in \mathcal{L} : m \text{ meets } \ell, m \neq \ell\}| = (q+1)x + (q^2-1)\chi_{\mathcal{L}}(\ell),$$

for every line $\ell \in \mathrm{PG}(3, q)$.

8. *There exists a constant x such that*

$$|\{n \in \mathcal{L} : n \text{ is a transversal to } \ell \text{ and } m\}| = x + q(\chi_{\mathcal{L}}(\ell) + \chi_{\mathcal{L}}(m)),$$

for every pair of skew lines $\ell, m \in \mathrm{PG}(3, q)$.

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two Cameron-Liebler line classes in $\mathrm{PG}(3, q)$. We say that $\mathcal{L}_1$ and $\mathcal{L}_2$ are *projectively equivalent* if there exists a collineation of $\mathrm{PG}(3, q)$ that maps $\mathcal{L}_1$ to $\mathcal{L}_2$. Given two projectively equivalent line classes, they should have the same intersection numbers with cliques.

3.2 A Survey of Results

The main problem concerning Cameron-Liebler line classes in $\mathrm{PG}(3, q)$ is: for which values of x, do there exist Cameron-Liebler line classes with parameter x? There are two aspects of this problem: one can either show nonexistence or construct examples for certain values of x. When there exists a Cameron-Liebler line class with parameter x, one can further ask: how many inequivalent line classes with the same parameter x are there? In this section, we give a survey of results on these questions.

Note that the complement of a Cameron-Liebler line class with parameter x in the set of all lines of $\mathrm{PG}(3, q)$ is a Cameron-Liebler line class with parameter q^2+1-x. So without loss of generality we may assume that $x \leq \frac{q^2+1}{2}$ when discussing Cameron-Liebler line classes of parameter x in $\mathrm{PG}(3, q)$.

3.2.1 A conjecture and the first counterexamples

As for the first question, Cameron and Liebler [20] conjectured that the only special line classes are the ones in Example 3.1.1 and their complements, so the parameter can only be 1, 2, q^2-1, or q^2. They showed that the conjecture is true for $q=2$. Very shortly after the conjecture was made, Penttila [54] announced two supporting evidences.

Theorem 3.2.1.

1. *There are no Cameron-Liebler line classes with parameter 3 in* $\mathrm{PG}(3,q)$ *when* $q>2$.
2. *There are no Cameron-Liebler line classes with parameter 4 in* $\mathrm{PG}(3,q)$ *when* $q\geq 5$.

However this conjecture turns out to be false. The first counterexample was given by Drudge in 1999 [31] based on a connection between Cameron-Liebler line classes in $\mathrm{PG}(3,q)$ and blocking sets in $\mathrm{PG}(2,q)$.

A *blocking set* in $\mathrm{PG}(2,q)$ is a set $\mathcal{B}$ of points such that every line in $\mathrm{PG}(2,q)$ contains some point of $\mathcal{B}$ but $\mathcal{B}$ does not contain an entire line. Let $q>2$ and $\mathcal{L}$ a Cameron-Liebler line class with parameter x in $\mathrm{PG}(3,q)$. Assume that there exists no line class with parameter $x-1$. Drudge showed that for any clique $\mathcal{C}$ of $\mathrm{PG}(3,q)$, if $x<|\mathcal{C}\cap\mathcal{L}|\leq x+q$, then the lines of $\mathcal{C}\cap\mathcal{L}$ form a blocking set in $\mathcal{C}$, i.e., in the copy of $\mathrm{PG}(2,q)$ determined by $\mathcal{C}$. From this observation the next theorem follows.

Theorem 3.2.2. *[31, Theorem 2.1] Let s be the smallest integer such that there exists a blocking set of size s in* $\mathrm{PG}(2,q)$*. If $2<x<s-q$ then there exists no Cameron-Liebler line class of parameter x in* $\mathrm{PG}(3,q)$*.*

Since the size of blocking sets in $\mathrm{PG}(2,q)$ are well studied (see [10, 11, 17]), by using this theorem, he came up with a few results.

Theorem 3.2.3. *1. If p is an odd prime then there are no Cameron-Liebler line class of parameter $2 < x \leq \frac{p+1}{2}$ in* PG$(3, q)$.

2. If $q > 7$ is not a square and not equal to 27, then there are no Cameron-Liebler line class of parameter $2 < x < \sqrt{2q} + 1$ in PG$(3, q)$.

3. There are no Cameron-Liebler line class of parameter 4 in PG$(3, 3)$.

4. There exist a unique Cameron-Liebler line class of parameter 5 in PG$(3, 3)$.

A construction of Cameron-Liebler line class with parameter 5 is given. First note that the points of PG$(3, 3)$ satisfying the equation $x_0^2 + x_1^2 + x_2x_3 = 0$ form an elliptic quadric $\mathcal{E}$. Let $P = \langle(0, 0, 0, 1)\rangle \in \mathcal{E}$. Then the tangent plane to $\mathcal{E}$ at P is given by $x_2 = 0$. Let $\mathcal{L}_P$ be the two tangent lines $P \oplus \langle(1, 0, 0, 0)\rangle$ and $P \oplus \langle(0, 1, 0, 0)\rangle$. For each point $Q \in \mathcal{E} \setminus \{P\}$, define $\mathcal{L}_Q$ to be the two tangent lines to $\mathcal{E}$ at Q which intersect the two lines of $\mathcal{L}_P$. Now we have picked two tangent lines to $\mathcal{E}$ at each point of $\mathcal{E}$. Then these tangent lines together with all secant lines to $\mathcal{E}$ form a Cameron-Liebler line class of parameter 5 in PG$(3, 3)$.

So far it is still possible that the conjecture is true over fields with characteristic two as it is true for $q = 2$. Unfortunately it is again false for even q. In 2005, Govaerts and Penttila [41] gave the first counterexample when q is even, namely, a Cameron-Liebler line class with parameter 7 in PG$(3, 4)$. They provided a construction as follows.

Let (P, π) be a non-incident point-plane pair in PG$(3, 4)$ and O a hyperoval in π (a set of $q + 2$ points no three of which are collinear). Denote the cone with base O and vertex P by C. Then the set $\mathcal{L}$ consisting of all generators of C, all two-secants to C skew to O and all lines in π external to O is a Cameron-Liebler line class with parameter 7 in PG$(3, 4)$.

3.2.2 Complete classification/characterization for small q's

When $q = 3$, the classification of feasible parameters of Cameron-Liebler line class in PG$(3, 3)$ was complete in [31].

Theorem 3.2.4. *A Cameron-Liebler line class with parameter x exists in* $\mathrm{PG}(3,3)$ *if and only if* $x \in \{1,2,5\}$. *When* $x=5$, *there is a unique line class.*

As for $q=4$, Penttila's result (i.e. Theorem 3.2.1) shows that there are no Cameron-Liebler line classes with parameter 3 in $\mathrm{PG}(3,4)$. Govaerts and Penttila [41] proved that x can not be 4 or 5 by studying how the lines of the Cameron-Liebler line class are distributed among the cliques of $\mathrm{PG}(3,q)$ with the help of some results on multiple blocking sets in $\mathrm{PG}(2,4)$.

The classification of Cameron-Liebler line classes in $\mathrm{PG}(3,4)$ was complete in [39] where Gavrilyuk and Mogilnykh showed that there are no Cameron-Liebler line class with parameter 6 or 8 in $\mathrm{PG}(3,4)$ by studying the distribution of lines of a Cameron-Liebler line class over the set $\{\mathrm{pencil}(P,\pi) : P \in \ell, \ell \in \pi\}$ for a line ℓ. Furthermore they showed that any Cameron-Liebler line class with parameter 7 in $\mathrm{PG}(3,4)$ arises in the way that Govaerts and Penttila [41] constructed.

As a summary of the above results in $\mathrm{PG}(3,4)$, we have the following.

Theorem 3.2.5. *A Cameron-Liebler line class with parameter x exists in* $\mathrm{PG}(3,4)$ *if and only if* $x \in \{1,2,7\}$. *When* $x=7$, *there is a unique line class.*

For $q=5$, the upper bound for the parameter x is 13. Of course there are trivial examples with parameters 1 and 2. Parameters 3 and 4 are ruled out by the result of Penttila (i.e. Theorem 3.2.1), and $x=5$ is ruled out by a result of Govaerts and Storme [42]. There is an example $\mathcal{L}_R$ with parameter 12 found by Rodgers through computer search [57]. Since there exists a plane π that is disjoint from R, then $\mathcal{L}_{R^+} = \mathcal{L}_R \cup \mathrm{line}(\pi)$ is a Cameron-Liebler line class with parameter 13. Moreover, when $x=13$, there are two more examples: one example $\mathcal{L}_D$ in the infinite family constructed by Bruen and Drudge [16], and another one $\mathcal{L}_P$ in the infinite family constructed by Cossidente and Pavese [22] and by Gavrilyuk, Matkin and Penttila [37] independently. Finally Gavrilyuk and Metsch [38] proved that there are no Cameron-Liebler line classes with parameter 6, 7, 8, 9, or 11 in $\mathrm{PG}(3,5)$ and constructed an example $\mathcal{L}_G$ with parameter 10. The construction starts with a complete cap in $\mathrm{PG}(3,5)$.

A *k-cap* in $\mathrm{PG}(3,q)$ is a set of k points, no three of which are collinear. A k-cap is called *complete* if it is not contained in a $(k+1)$-cap. In [1] it was shown that $\mathrm{PG}(3,5)$ has a complete cap K_1 with 20 points which is missing exactly 16 planes, and this cap K_1 can be viewed as the union of five tetrahedra. Moreover, there are 20 planes that meet the cap in precisely three points and these are the 20 facets of the five tetrahedra. Let $\mathcal{L}$ be the set consisting of the following lines.

- the 120 intersection line of two planes missing K_1,
- the 160 lines that lie in a plane missing K_1 and two planes meeting K_1 in three points,
- the $5 \times 6 = 30$ lines that are edges of the tetrahedra, which are the secant lines to K_1 that lie in two planes meeting K_1 in three points.

Gavrilyuk and Metsch claimed that $\mathcal{L}$ is a Cameron-Liebler line class with parameter 10 in $\mathrm{PG}(3,5)$, and showed that all the Cameron-Liebler line classes with parameter 10 in $\mathrm{PG}(3,5)$ are projectively equivalent.

Theorem 3.2.6. *A Cameron-Liebler line class with parameter x exists in* $\mathrm{PG}(3,5)$ *if and only if $x \in \{1,2,10,12,13\}$. When $x = 10$, the line class is unique.*

The full characterization of Cameron-Liebler line classes in $\mathrm{PG}(3,5)$ was done by Gavrilyuk and Matkin [36]. They showed that the examples in $\mathrm{PG}(3,5)$ mentioned above are all the Cameron-Liebler line classes in $\mathrm{PG}(3,5)$. More precisely, they have the following theorem.

Theorem 3.2.7. *Up to a polarity of* $\mathrm{PG}(3,5)$ *or taking the complement, a nontrivial Cameron-Liebler line class in* $\mathrm{PG}(3,5)$ *is projectively equivalent to one of the following: $\mathcal{L}_G$, $\mathcal{L}_R$, $\mathcal{L}_{R^+}$, $\mathcal{L}_D$, and $\mathcal{L}_P$.*

3.2.3 Non-existence/Existence conditions in general

The first general non-existence result goes back to Bruen and Drudge's paper [15] where they showed that there are no Cameron-Liebler line classes with parameter $2 < x \leq \sqrt{q}$ in $\mathrm{PG}(3,q)$ by a counting argument. Very soon after that, Drudge

[31] observed the connection to blocking sets and proved non-existence for $2 < x < s - q$ where s is the size of the smallest blocking set in $\mathrm{PG}(2, q)$. Based on Drudge's observation, many non-existence results were obtained. One of the strongest results is by Govaerts and Storme [42] who excluded $2 < x \leq q$ when q is a prime. Then De Beule, Hallez and Storme [26] showed $2 < x \leq q/2$ is impossible for all prime powers q. Later Metsch [49] improved this result to $2 < x \leq q$ for all prime powers q. In 2014, Metsch [50] made a significant improvement of the upper bound by a different technique. He showed the following.

Theorem 3.2.8 (Theorem 1.1 in [50])**.** *If $\mathcal{L}$ is a Cameron-Liebler line class with parameter x of $\mathrm{PG}(3, q)$, then $x \leq 2$ or $x > q\sqrt[3]{q/2} - \frac{2}{3}q$.*

This means there exists a constant c such that no Cameron-Liebler line class exists with $2 < x < cq^{4/3}$.

In [39] Gavrilyuk and Mogilnykh introduced a new approach which allowed them to obtain a new existence condition.

Let $\mathcal{L}$ be a Cameron-Liebler line class in $\mathrm{PG}(3, q)$. Consider a line ℓ in $\mathrm{PG}(3, q)$. Then ℓ is on $q + 1$ planes, say $\pi_0, \pi_1, \ldots, \pi_q$. Let $P_0, P_1, \ldots, P_q$ be the $q + 1$ points on ℓ. Define a $(q + 1) \times (q + 1)$ matrix $\mathcal{T}(\ell)$ whose rows are labeled by $P_0, P_1, \ldots, P_q$ and columns are labeled by $\pi_0, \pi_1, \ldots, \pi_q$. For any $i, j \in \{0, 1, \ldots, q\}$, the (i, j)-entry of $\mathcal{T}(\ell)$ is:

$$t_{ij} := |(\mathrm{pencil}(P_i, \pi_j) \setminus \{\ell\}) \cap \mathcal{L}|.$$

The matrix $\mathcal{T}(\ell)$ will be called the *pattern* with respect to ℓ and we say a line ℓ has pattern $\mathcal{T}(\ell)$.

Lemma 3.2.9. *For every line ℓ in $\mathrm{PG}(3, q)$, the pattern $\mathcal{T}(\ell)$ satisfies the following properties.*

1. *$0 \leq t_{ij} \leq q$ for all $i, j \in \{0, 1, \ldots, q\}$.*
2. *$\sum_{i,j=0}^{q} t_{ij} = x(q + 1) + \chi_{\mathcal{L}}(\ell)(q^2 - 1)$.*

3. $\sum_{j=0}^{q} t_{sj} + \sum_{i=0}^{q} t_{it} + 2\chi_{\mathcal{L}}(\ell) = x + (q+1)(t_{st} + \chi_{\mathcal{L}}(\ell))$, *for all* $s, t \in \{0, 1, \ldots, q\}$.

4. $\sum_{i,j=0}^{q} t_{ij}^2 = (x - \chi_{\mathcal{L}}(\ell))^2 + q(x - \chi_{\mathcal{L}}(\ell)) + \chi_{\mathcal{L}}(\ell) q^2 (q+1)$.

By studying the pattern, Gavrilyuk and Metsch [38] got a very strong result.

Theorem 3.2.10. *Suppose* $\mathcal{L}$ *is a Cameron-Liebler line class with parameter* x *of* $\mathrm{PG}(3,q)$. *Then for every plane and every point of* $\mathrm{PG}(3,q)$

$$\binom{x}{2} + n(n-x) \equiv 0 \pmod{q+1} \tag{3.2.1}$$

where n *is the number of lines of* $\mathcal{L}$ *in the plane respectively through the point.*

Corollary 3.2.11. *Suppose* $\mathrm{PG}(3,q)$ *has a Cameron-Liebler line class with parameter* x. *Then equation (3.2.1) has a solution for* n *in the set* $\{0, 1, \ldots, q\}$.

The corollary above roughly eliminates at least half of all possible parameters x. This can be seen by analyzing the number of integers $x \in \{0, 1, \ldots, q^2+1\}$ for which the equation (3.2.1) has a solution n. It suffices to determine the number of integers x with $0 \leq x < 2(q+1)$ such that the equation $x(x-1) + 2n(n-x) \equiv 0 \pmod{2(q+1)}$ has a solution n. This number will be denoted by S.

Let $2(q+1) = p_1^{k_1} \cdots p_s^{k_s}$ be the prime factorization of $2(q+1)$. For a positive integer z, define

$$S(z) := \{x \in \{0, 1, \ldots, z-1\} \mid x(x-1) + 2n(n-x) \equiv 0 \pmod{z} \text{ for some integer } n\}.$$

The Chinese Remainder Theorem implies that

$$S = \prod_{i=1}^{s} |S(p_i^{k_i})|.$$

It can be shown that

$$|S(p^k)| = p^{k-1}s - \frac{p^k + (-1)^k p}{p+1} + \frac{3 + (-1)^k}{2}$$

where p is an odd prime, $s = \frac{p-1}{2}$ if $p \equiv 1 \pmod 4$, and $s = \frac{p+1}{2}$ if $p \equiv 3 \pmod 4$. For even prime powers, we have $|S(2)| = 2, |S(4)| = 3, |S(8)| = 4$ and

$$|S(2^k)| = 2^{k-2} + 3 + \frac{1}{3}(2^{k-3} + (-1)^k),$$

for $k \geq 4$.

More explicitly, the next few results show which values of x are eliminated for certain families of q.

Corollary 3.2.12. *If q is odd, then* $\mathrm{PG}(3,q)$ *does not have a Cameron-Liebler line class with parameter* $x \equiv 3 \pmod 4$.

Corollary 3.2.13. *If $q \equiv 4 \pmod 5$ odd, then* $\mathrm{PG}(3,q)$ *does not have a Cameron-Liebler line class with parameter* $x \equiv 3 \pmod 5$ *or* $x \equiv 4 \pmod 5$.

Corollary 3.2.14. *If $q \equiv 3 \pmod 4$, then* $\mathrm{PG}(3,q)$ *does not have a Cameron-Liebler line class with parameter x congruent to 3, 4, 6 or 7 modulo 8.*

3.2.4 Infinite families

3.2.4.1 The Bruen-Drudge construction

Bruen and Drudge [16] constructed the first infinite family of Cameron-Liebler line classes in $\mathrm{PG}(3,q)$ with parameter greater than 2. In particular, the characteristic of the underlying field in their example is odd, and the parameter of the line class is $\frac{q^2+1}{2}$. Their idea is as follows. Let $\mathcal{O}$ be an elliptic quadric in $\mathrm{PG}(3,q)$. For each point $P \in \mathcal{O}$ there are $q+1$ tangent lines to $\mathcal{O}$ at P. Take half of them, denoted by $\mathcal{L}_P$. Define a set of lines $\mathcal{L}$ by

$$\mathcal{L} = \{\text{all secants to } \mathcal{O}\} \cup \left(\bigcup_{P \in \mathcal{O}} \mathcal{L}_P \right). \tag{3.2.2}$$

It is easy to calculate the size of $\mathcal{L}$:

$$|\mathcal{L}| = \binom{q^2+1}{2} + (q^2+1)\frac{q+1}{2} = \frac{q^2+1}{2}(q^2+q+1).$$

If $\mathcal{L}$ is a Cameron-Liebler line class, the parameter is $\frac{q^2+1}{2}$. The question is: how to choose $\mathcal{L}_P$ for each point $P \in \mathcal{O}$ to make $\mathcal{L}$ a Cameron-Liebler line class?

Bruen and Drudge made use of an equivalence relation on circles in the inversive plane $IP(q)$ of order q, and a correspondence between the quadric $\mathcal{O}$ and $IP(q)$, i.e., by viewing plane sections of $\mathcal{O}$ as circles, $\mathcal{O}$ is isomorphic to $IP(q)$. In Orr's PhD thesis [51], he proved that the relation defined on circles of $IP(q)$ by $C_1 \sim C_2$ if there exists a circle tangent to both is an equivalence relation, and there are two equivalence classes, each containing $\frac{1}{2}q(q^2+1)$ circles. Now let $\mathcal{A}$ be either class and for each circle $C \in \mathcal{A}$ let $\mathcal{L}_C$ be the tangent lines of the plane section of $\mathcal{O}$ corresponding to C. Define $\mathcal{L}$ as

$$\mathcal{L} = \{\text{all secants to } \mathcal{O}\} \cup \left(\bigcup_{C \in \mathcal{A}} \mathcal{L}_C \right).$$

Bruen and Drudge prove that $\mathcal{L}$ is a Cameron-Liebler line class by showing it has a certain matching property presented in the following theorem.

Theorem 3.2.15 (Theorem 2.1 in [16]). *Fix an elliptic quadric $\mathcal{O}$ in $\mathrm{PG}(3,q)$ and let $T_P(\mathcal{O})$ be the set of tangent lines to $\mathcal{O}$ at P. Let $\mathcal{L}$ be a set of lines of $\mathrm{PG}(3,q)$ defined in (3.2.2). Then the following conditions on $\mathcal{L}$ are equivalent:*

1. *$\mathcal{L}$ is a Cameron-Liebler line class of parameter $x = (q^2+1)/2$.*
2. *$\mathcal{L}$ matches along $T_{P_1}(\mathcal{O}) \cap T_{P_2}(\mathcal{O})$ for all points $P_1, P_2 \in \mathcal{O}$, i.e., for each point $Q \in T_{P_1}(\mathcal{O}) \cap T_{P_2}(\mathcal{O})$, the line $\overline{P_1Q} \in \mathcal{L}$ if and only if the line $\overline{P_2Q} \in \mathcal{L}$.*
3. *For any $P \in \mathcal{O}$, $T_P(\mathcal{O}) \cap \mathcal{L} = \mathcal{L}_P$, and if π is not a tangent plane to $\mathcal{O}$ then $line(\pi) \cap \mathcal{L}$ is all secants to the nonsingular conic $\pi \cap \mathcal{O}$ plus either all or none of its tangent lines.*
4. *For any $P \in \mathcal{O}$, $star(P) \cap \mathcal{L}$ is $\mathcal{L}_P$ together with all secants to $\mathcal{O}$ on P; and if $P \notin \mathcal{O}$ then $star(P) \cap \mathcal{L}$ is all secants to $\mathcal{O}$ on P together with either all or none of the tangent lines to $\mathcal{O}$ on P.*

In fact, one can change part of the lines in $\mathcal{L}$ to get another example. More precisely, the set of all tangent lines to $\mathcal{O}$ not in $\mathcal{L}$ together with all secant lines is also a Cameron-Liebler line class $\mathcal{M}$, projectively equivalent to $\mathcal{L}$. The set of all tangents lines to $\mathcal{O}$ in $\mathcal{L}$ together with all external lines to $\mathcal{O}$ is again a Cameron-Liebler line class, which is projectively distinct from $\mathcal{L}$, but can be mapped to either $\mathcal{L}$ or $\mathcal{M}$ under the polarity of $\mathrm{PG}(3, q)$ which induces $\mathcal{O}$.

They also mentioned that these sets of lines can be defined in other ways. For example, such a set is a union of two line orbits of $PSL(2, q^2) \times (\mathbb{Z}/2\mathbb{Z})$ which can be viewed as a subgroup of the collineations of $\mathrm{PG}(3, q)$ fixing $\mathcal{O}$.

The intersection numbers of $\mathcal{L}$ with respect to planes and points of $\mathrm{PG}(3, q)$ are given below.

- A plane π of $\mathrm{PG}(3, q)$ contains $\frac{q+1}{2}$, $\frac{q(q+1)}{2}$, or $\frac{(q+1)(q+2)}{2}$ lines of $\mathcal{L}$.
- A point P of $\mathrm{PG}(3, q)$ is on $q^2 + \frac{q+1}{2}$, $\frac{q(q-1)}{2}$, or $\frac{q(q+1)}{2} + 1$ lines of $\mathcal{L}$.

3.2.4.2 Constructions derived from Bruen-Drudge

Cossidente-Pavese Construction 1 [22].

Cossidente and Pavese first gave an alternative description of the Bruen-Drudge construction. Let q be odd and $\mathcal{Q}^-(3, q)$ a fixed quadric in $\mathrm{PG}(3, q)$ with quadratic form Q. Let $G = \mathrm{P}\Omega^-(4, q)$ be the derived subgroup of the full stabilizer of $\mathcal{Q}^-(3, q)$ in $\mathrm{PGL}(4, q)$. The group G has three orbits on the points of $\mathrm{PG}(3, q)$, namely the points on $\mathcal{Q}^-(3, q)$ and two other orbits: $O_s = \{x : Q(x) \in \Box_q\}$ and $O_n = \{x : Q(x) \notin \Box_q\}$. On the set of lines of $\mathrm{PG}(3, q)$, the group G has four orbits: $\mathcal{L}_1, \mathcal{L}_2$, both of size $(q+1)(q^2+1)/2$ consisting of tangent lines to $\mathcal{Q}^-(3, q)$, and $\mathcal{L}_3, \mathcal{L}_4$, both of size $q^2(q^2+1)/2$ consisting of secant and external lines to $\mathcal{Q}^-(3, q)$, respectively.

Let

$$\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_3. \tag{3.2.3}$$

Then $\mathcal{L}$ is a Cameron-Liebler line class constructed by Bruen and Drudge. Starting form here, Cossidente and Pavese gave their new construction.

Let $P \in \mathcal{Q}^-(3,q)$ and let π be the tangent plane to $\mathcal{Q}^-(3,q)$ at P. The lines of $\mathrm{PG}(3,q)$ can be further partitioned into eight sets:

$$t_1 : (q+1)/2 \text{ lines in } \mathcal{L}_1 \text{ through } P.$$

$$t_2 : (q+1)/2 \text{ lines in } \mathcal{L}_2 \text{ through } P.$$

$$s_1 : q^2 \text{ secant lines on } P.$$

$$s_2 : q^2(q^2-1)/2 \text{ secant lines not on } P.$$

$$u_1 : q^2(q+1)/2 \text{ lines in } \mathcal{L}_1 \setminus t_1.$$

$$u_2 : q^2(q+1)/2 \text{ lines in } \mathcal{L}_2 \setminus t_2.$$

$$e_1 : q^2 \text{ external lines lying in } \pi.$$

$$e_2 : q^2(q^2-1)/2 \text{ external lines not in } \pi.$$

Let $\mathcal{L}' = t_1 \cup s_1 \cup u_1 \cup e_2$. By counting they show that $\mathcal{L}'$ satisfies statement 7 in Theorem 3.1.3. Therefore $\mathcal{L}'$ is a Cameron-Liebler line class with intersection numbers respect to planes and points of $\mathrm{PG}(3,q)$ as follows.

- A plane of $\mathrm{PG}(3,q)$ has $\frac{q+1}{2}$, $\frac{q^2-q-2}{2}$, $\frac{q^2+q}{2}$, $\frac{q^2+3q+2}{2}$, or $q^2 + \frac{q-1}{2}$ lines of $\mathcal{L}'$.
- A point of $\mathrm{PG}(3,q)$ is on $\frac{q+3}{2}$, $\frac{q^2-q}{2}$, $\frac{q^2+q+2}{2}$, $\frac{q^2+3q+4}{2}$, or $q^2 + \frac{q+1}{2}$ lines of $\mathcal{L}'$.

Therefore unless $q = 3$, $\mathcal{L}'$ is not equivalent to $\mathcal{L}$ under the action of $\mathrm{P\Gamma L}(4,q)$ and the polarity inducing $\mathcal{Q}^-(3,q)$. When $q = 3$, Drudge's result [31] which states that there is a unique Cameron-Liebler line class in $\mathrm{PG}(3,3)$ with parameter 5 implies that $\mathcal{L}'$ and $\mathcal{L}$ are the same.

Gavrilyuk-Matkin-Penttila Construction [37].

These authors used a switching technique as shown in the following lemma to construct a new example from the Bruen-Drudge example.

Lemma 3.2.16 (Lemma 1 and 2 in [37])**.** *Let $\mathcal{L}$ be a Cameron-Liebler line class such that there exists an incident point-plane pair (P, π) satisfying the following conditions:*

1. $(\text{line}(\pi) \setminus \text{star}(P)) \cap \mathcal{L} = \emptyset$,

2. $\text{star}(P) \setminus \text{line}(\pi) \subseteq \mathcal{L}$.

Then

$$\mathcal{L}'' := \mathcal{L} \cup (\text{line}(\pi) \setminus \text{star}(P)) \setminus (\text{star}(P) \setminus \text{line}(\pi))$$

is a Cameron-Liebler line class with the same parameter.

Furthermore, if the conditions are satisfied, then the parameter x of $\mathcal{L}$ is equal to q^2 or $\frac{q^2+1}{2}$.

Now let $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_3$ as defined in (3.2.3). For a point P of $\mathcal{Q}^-(3, q)$ and its tangent plane π, it is easy to see that

$$\text{line}(\pi) \setminus \text{star}(P) \subseteq \mathcal{L}_4,\ \text{star}(P) \setminus \text{line}(\pi) \subseteq \mathcal{L}_3 \subseteq \mathcal{L},$$

so the pair (P, π) satisfies the conditions in Lemma 3.2.16. Therefore $\mathcal{L}''$ as defined in the lemma is a Cameron-Liebler line class with parameter $\frac{q^2+1}{2}$. This line class $\mathcal{L}''$ is not equivalent to $\mathcal{L}$ under the action of $\text{P}\Gamma\text{L}(4, q)$ or a duality since they have different intersection numbers with respect to planes and points of $\text{PG}(3, q)$.

Note that if we define $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_4$, then $\mathcal{L}'$ and $\mathcal{L}''$ are the same line class. Furthermore, $\mathcal{L}''$ admits a group K as the stabilizer of the point P in G.

Cossidente-Pavese Construction 2 [24].

The idea is similar to their first construction. They start with a line class $\mathcal{L}$ consisting of half of the tangent lines and all the external lines to a quadric. Then switch some external lines in $\mathcal{L}$ and some secant lines. The substitution depends on two more quadrics. The precise description is given below.

Let w be a non-square of $\mathbb{F}_q$. Let $\mathcal{E}$ be the elliptic quadric of $\text{PG}(3, q)$ with equation $x_1^2 - wx_2^2 + x_3x_4 = 0$. Consider the group $G = \text{P}\Omega^-(4, q)$. The point orbits O_s, O_n and the line orbits $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4$ are as defined in the previous example. Then $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_4$ is a Cameron-Liebler line class. Let π be the plane given by the equation

$x_4 = 0$, and define $\pi_0 = O_s \cap \pi$ and $\pi_1 = O_n \cap \pi$. For an element $\lambda \in \mathbb{F}_q$, let $\mathcal{E}_\lambda$ be the elliptic quadric with equation $x_1^2 - wx_2^2 + \lambda x_4^2 + x_3x_4 = 0$. Let $\lambda_1 \neq 0$ be a fixed square in $\mathbb{F}_q$ and let λ_2 be a fixed non-square in $\mathbb{F}_q$. Define the following sets of lines:

$$T_{10} = \{r \in \mathcal{L}_3 : |r \cap \mathcal{E}_{\lambda_1}| = 1, |r \cap \pi_0| = 1\},$$

$$T_{11} = \{r \in \mathcal{L}_4 : |r \cap \mathcal{E}_{\lambda_1}| = 1, |r \cap \pi_1| = 1\},$$

$$T_{20} = \{r \in \mathcal{L}_4 : |r \cap \mathcal{E}_{\lambda_2}| = 1, |r \cap \pi_0| = 1\},$$

$$T_{21} = \{r \in \mathcal{L}_3 : |r \cap \mathcal{E}_{\lambda_2}| = 1, |r \cap \pi_1| = 1\},$$

Let $\mathcal{A} = T_{11} \cup T_{20}$ and $\mathcal{B} = T_{10} \cup T_{21}$. They prove that $\mathcal{L}''' = (\mathcal{L} \setminus \mathcal{A}) \cup \mathcal{B}$ is a Cameron-Liebler line class admitting a subgroup of K of order $q^2(q+1)$. It has the following intersection numbers.

- A plane of $\mathrm{PG}(3,q)$ contains $q^2 + \frac{q+1}{2}, q^2 - \frac{3(q+1)}{2}, \frac{q(q-1)}{2} + 3(q+1), \frac{q(q-1)}{2} + 2(q+1), \frac{q(q-1)}{2} + q + 1, \frac{q(q-1)}{2}, \frac{q(q-1)}{2} - (q+1)$, or $\frac{q(q+1)}{2} - 2(q+1)$ lines of $\mathcal{L}'''$.
- A point of $\mathrm{PG}(3,q)$ is on $\frac{q+1}{2}, \frac{5(q+1)}{2}, \frac{q(q+1)}{2} - 2(q+1), \frac{q(q+1)}{2} - (q+1), \frac{q(q+1)}{2}, \frac{q(q+1)}{2} + q + 1, \frac{q(q+1)}{2} + 2(q+1)$, or $\frac{q(q+1)}{2} + 3(q+1)$ lines of $\mathcal{L}'''$.

Cossidente-Pavese Construction 3 [23].

This construction is again by a switching technique which is stated in the following theorem. Let $\mathcal{A}, \mathcal{B}$ be two line sets of $\mathrm{PG}(3,q)$. For a line ℓ of $\mathrm{PG}(3,q)$ consider the following sets.

$$\mathcal{A}_\ell = \{r \in \mathcal{A} : |r \cap \ell| \geq 1\}, \mathcal{B}_\ell = \{r \in \mathcal{B} : |r \cap \ell| \geq 1\}.$$

Theorem 3.2.17 (Theorem 3.11 in [23]). *Let $\mathcal{L}$ be a Cameron-Liebler line class with parameter $(q^2+1)/2$ and let $\mathcal{A}, \mathcal{B}$ be line sets of equal size of $\mathrm{PG}(3,q)$ such that*

1. *$\mathcal{A} \subseteq \mathcal{L}$ and $|\mathcal{B} \cap \mathcal{L}| = 0$,*
2. *if $\ell \notin \mathcal{A} \cup \mathcal{B}$, then $|\mathcal{A}_\ell| = |\mathcal{B}_\ell|$,*
3. *if $\ell \in \mathcal{A}$, then $|\mathcal{A}_\ell| - |\mathcal{B}_\ell| = q^2$,*

4. if $\ell \in \mathcal{B}$, then $|\mathcal{B}_\ell| - |\mathcal{A}_\ell| = q^2$.

Then the set $\bar{\mathcal{L}} = (\mathcal{L} \setminus \mathcal{A}) \cup \mathcal{B}$ is a Cameron-Liebler line class with parameter $(q^2+1)/2$.

They managed to find such line sets $\mathcal{A}$ and $\mathcal{B}$ when $q \equiv 1 \pmod 4$ by considering the line orbits of a group that is isomorphic to $\mathrm{PGL}(2,q)$. By the theorem, the new line set $\bar{\mathcal{L}}$ is a Cameron-Liebler line class. The intersection numbers are the following.

- A plane of $\mathrm{PG}(3,q)$ contains $\frac{3q+5}{2}$, $\frac{q^2+q}{2} - q$, $\frac{q^2+q}{2} + 1$, $\frac{q^2+q}{2} + q + 2$, $\frac{q^2+q}{2} + 2q + 3$, or $q^2 + \frac{q+1}{2}$ lines in $\bar{\mathcal{L}}$.
- A point of $\mathrm{PG}(3,q)$ is on $\frac{q+1}{2}$, $\frac{q^2+q}{2} - 2(q+1)$, $\frac{q^2+q}{2} - (q+1)$, $\frac{q^2+q}{2}$, $\frac{q^2+q}{2} + q + 1$, or $q^2 - \frac{q+3}{2}$ lines in $\bar{\mathcal{L}}$.

Therefore when $q \geq 9$, the line class $\bar{\mathcal{L}}$ is not equivalent to any of the known examples.

3.2.4.3 Feng-Momihara-Xiang Construction with parameter $\frac{q^2-1}{2}$

This an algebraic construction. The first step is prescribing an automorphism group for the Cameron-Liebler line class that one intends to construct. Then the Cameron-Liebler line class will be a union of orbits of the prescribed automorphism group on the set of lines of $\mathrm{PG}(3,q)$. In Rodger's PhD thesis [56] he found examples in this way for small q's by computer search. Feng, Momihara and Xiang [34] successfully generalized these examples to an infinite family of Cameron-Liebler line classes with parameter $\frac{q^2-1}{2}$ for $q \equiv 5$ or $9 \pmod{12}$. A line class in the family has an automorphism group isomorphic to $(\mathbb{Z}_{q^2+q+1} \times \mathbb{Z}_{q-1}) \rtimes \mathbb{Z}_3$.

By the construction, for any line class $\mathcal{L}$ in the family there is a plane π and a point $P \notin \pi$ such that $\mathcal{L}$ does not have any line in the plane π or any line through the point P. Therefore $\mathcal{L}$ together with all the lines in π or all the lines through P is a Cameron-Liebler line class of parameter $\frac{q^2+1}{2}$.

It is worth noting that almost at the same time De Beule, Demeyer, Metsch and Rodger [25] obtained the same result by a more geometric approach.

3.2.5 Sporadic examples

We collect sporadic examples of Cameron-Liebler line classes in $\mathrm{PG}(3,q)$ which are not in any of the known infinite families.

- One example in $\mathrm{PG}(3,4)$ with $x = 7$. [41]
- One example in $\mathrm{PG}(3,5)$ with $x = 10$. [38]
- One example in $\mathrm{PG}(3,27)$ with $x = 336$. [56]
- Two examples in $\mathrm{PG}(3,32)$ with $x = 495$. [56]

Chapter 4

MAIN RESULT: A NEW INFINITE FAMILY

We give an overview of our construction here. The initial step is to prescribe an automorphism group for the Cameron-Liebler line classes that we intend to construct; once this is done, the Cameron-Liebler line classes we want to construct will be unions of orbits of lines under the action of the prescribed automorphism group. For the choices of automorphism groups, we follow the idea in [56]; that is, we will choose a cyclic group of order $q^2 + q + 1$ as the prescribed automorphism group. Examples of Cameron-Liebler classes with parameter $(q+1)^2/3$ have been found in this way by using a computer for all $q < 150$ with $q \equiv 2 \bmod 3$ (see [56]). The difficulty lies in how to come up with a choice of orbits for general q which will always give a Cameron-Liebler line class in $\mathrm{PG}(3, q)$ with parameter $(q + 1)^2/3$. The examples in [56] provided vital clues for a general choice; also the computations of additive character sums (needed to prove that the union of the chosen orbits is a Cameron-Liebler line class) gave us hints for making correct choices of orbits. In Section 3, we come up with an explicit choice of orbits that will give a Cameron-Liebler line classes with parameter $x = (q + 1)^2/3$ for all q congruent to 2 modulo 3.

4.1 Cameron-Liebler line classes and self-dual partial difference sets

In this section, we describe a connection between Cameron-Liebler line classes of $\mathrm{PG}(3, q)$ to certain partial difference sets, which will be used to prove a line set of $\mathrm{PG}(3, q)$ is a Cameron-Liebler line class.

We first translate Cameron-Liebler line classes in $\mathrm{PG}(3, q)$ into the setting of the 5-dimensional hyperbolic quadric $\mathcal{Q}^+(5, q)$ using the Klein correspondence. Let $\mathcal{L}$ be a set of lines of $\mathrm{PG}(3, q)$ with $|\mathcal{L}| = x(q^2 + q + 1)$, x a positive integer, and let

$\mathcal{M}$ be the image of $\mathcal{L}$ under the Klein correspondence. Then it is known that $\mathcal{L}$ is a Cameron-Liebler line class with parameter x in $\mathrm{PG}(3, q)$ if and only if $\mathcal{M}$ is an x-tight set of $\mathcal{Q}^+(5, q)$. Moreover, if $\mathcal{L}$ is a Cameron-Liebler line class with parameter x, by [50, Theorem 2.1(b)] it holds that $|P^\perp \cap \mathcal{M}| = x(q+1) + q^2$ for any point $P \in \mathcal{M}$ and $|P^\perp \cap \mathcal{M}| = x(q+1)$ for any point $P \notin \mathcal{M}$ (here P can be in the exterior of $\mathcal{Q}^+(5, q)$); consequently $\mathcal{M}$ is a projective two-intersection set in $\mathrm{PG}(5, q)$ with intersection sizes $h_1 = x(q+1) + q^2$ and $h_2 = x(q+1)$. We summarize these known facts as follows.

Result 4.1.1. *Let $\mathcal{L}$ be a set of $x(q^2+q+1)$ lines in $\mathrm{PG}(3, q)$ with $1 \leq x \leq (q^2+1)/2$, and let $\mathcal{M}$ be the image of $\mathcal{L}$ under the Klein correspondence. Then $\mathcal{L}$ is a Cameron-Liebler line class with parameter x if and only if $\mathcal{M}$ is an x-tight set in $\mathcal{Q}^+(5, q)$; moreover, in the case when $\mathcal{L}$ is a Cameron-Liebler line class, we have*

$$|P^\perp \cap \mathcal{M}| = \begin{cases} x(q+1) + q^2, & \text{if } P \in \mathcal{M}, \\ x(q+1), & \text{otherwise.} \end{cases}$$

It was pointed out in [4, Theorem 12] that if $\mathcal{M}$ is a proper x-tight set of $\mathcal{Q}^+(5, q)$ that spans the ambient projective space, then $\{\lambda v : \lambda \in \mathbb{F}_q^*, \langle v \rangle \in \mathcal{M}\}$ is a partial difference set in $(V, +)$ with parameters $(q^6, x(q^3-1), x(x-3) + q^3, x(x-1))$, where V is the 6-dimensional vector space underlying $\mathcal{Q}^+(5, q)$. In fact, a partial converse of this statement is also true, which we present in the next theorem.

Theorem 4.1.2. *Let $\mathcal{L}$ be a set of $x(q^2+q+1)$ lines in $\mathrm{PG}(3, q)$ with $1 \leq x \leq \frac{q^2+1}{2}$, and let $\mathcal{M}$ be the image of $\mathcal{L}$ under the Klein correspondence. Define*

$$D = \{\lambda v : \lambda \in \mathbb{F}_q^*, \langle v \rangle \in \mathcal{M}\} \subset (V, +),$$

where V is the 6-dimensional vector space underlying $\mathcal{Q}^+(5, q)$. Then $\mathcal{L}$ is a Cameron-Liebler line class with parameter x if and only if D is a self-dual partial difference set in $(V, +)$ with parameters

$$(q^6, x(q^3-1), x(x-3) + q^3, x(x-1)).$$

Proof. Let $\mathcal{Q}^+(5,q)$ be defined by the quadratic form Q with its polar form f. Then each additive character of $(V,+)$ is of form ψ_w for some $w \in V$, where

$$\psi_w(v) = \psi_{\mathbb{F}_q}(f(w,v)), \quad \forall v \in V.$$

Here $\psi_{\mathbb{F}_q}$ is the canonical additive character of $\mathbb{F}_q$. It is clear that ψ_w is trivial on the hyperplane $\langle w\rangle^\perp$.

Assume D is a self-dual partial difference set with the above parameters. Then $\mathrm{Cay}(V,D)$ is a strongly regular graph and by Theorem 2.9.2 its restricted eigenvalues are $r = q^3 - x$ and $s = -x$. Since D is self-dual, then $\psi_w(D) = r$ for all $w \in D$, and $\psi_w(D) = s$ for all $w \in V \setminus (\{0\} \cup D)$. On the other hand, for any $w \in V \setminus \{0\}$, we have

$$\begin{aligned}\psi_w(D) &= \sum_{\langle v\rangle \in \mathcal{M}} \sum_{\lambda \in \mathbb{F}_q^*} \psi_w(\lambda v) = \sum_{\langle v\rangle \in \mathcal{M}} \sum_{\lambda \in \mathbb{F}_q^*} \psi_{\mathbb{F}_q}(\lambda f(w,v)) \\ &= \sum_{\langle v\rangle \in \mathcal{M}} (q\mathbb{1}_{\langle w\rangle^\perp}(\langle v\rangle) - 1) = -|\mathcal{M}| + q|\langle w\rangle^\perp \cap \mathcal{M}|,\end{aligned}$$

where $\mathbb{1}_{\langle w\rangle^\perp}(\langle v\rangle)$ is the characteristic function taking value 1 if $\langle v\rangle \in \langle w\rangle^\perp$, and 0 otherwise. Thus

$$|\langle w\rangle^\perp \cap \mathcal{M}| = \frac{\psi_w(D) + |\mathcal{M}|}{q} = \begin{cases} q^2 + x(q+1), & \text{if } \langle w\rangle \in \mathcal{M}, \\ x(q+1), & \text{otherwise.} \end{cases}$$

Therefore $\mathcal{L}$ is a Cameron-Liebler line class with parameter x by Result 4.1.1.

Conversely, if $\mathcal{L}$ is a Cameron-Liebler line class with parameter x, we know by [4, Theorem 12] that D is a partial difference set with the required parameters and we only need to show it is self-dual. But it just follows from the computation above that $D^* = \{w \in V : \psi_w(D) = r\} = \{w \in V : \langle w\rangle \in \mathcal{M}\} = D$. This means D is indeed self-dual and the proof is complete. □

As a corollary of this theorem, we can characterize Cameron-Liebler line classes by the character values on the corresponding sets in V. More precisely, we have the following result.

Corollary 4.1.3. *Let $\mathcal{L}$ be a set of $x(q^2+q+1)$ lines in $\mathrm{PG}(3,q)$ with $1 \le x \le \frac{q^2+1}{2}$, and let $\mathcal{M}$ be the image of $\mathcal{L}$ under the Klein correspondence. Define*

$$D = \{\lambda v : \lambda \in \mathbb{F}_q^*, \langle v \rangle \in \mathcal{M}\} \subset (V, +).$$

Then $\mathcal{L}$ is a Cameron-Liebler line class with parameter x if and only if $|D| = (q^3-1)x$ and for any $P \in \mathrm{PG}(5,q)$,

$$\psi(D) = \begin{cases} -x + q^3, & \text{if } P \in \mathcal{M}, \\ -x, & \text{otherwise,} \end{cases}$$

where ψ is any non-principal character of V that is principal on the hyperplane $P^\perp$.

4.2 Cubic polynomials over $\mathbb{F}_q$

Our construction of new Cameron-Liebler line classes is based on the image sets of certain cubic polynomials. This idea was previously used in [30] for constructing new difference sets with Singer parameters.

Let $q = p^n$ be a prime power, where $p \neq 3$ is a prime. Let $f(X) = X^3 + cX + d$ be a cubic polynomial over $\mathbb{F}_q$, and let $\gamma_1, \gamma_2, \gamma_3$ be its roots in some extension field of $\mathbb{F}_q$. The discriminant of f is

$$\Delta(f) := (\gamma_1 - \gamma_2)^2(\gamma_2 - \gamma_3)^2(\gamma_3 - \gamma_1)^2,$$

which equals $-4c^3 - 27d^2$ for all q. Hence f has no repeated roots if and only if $\Delta(f) \neq 0$. In particular, when q is even, we have $\Delta(f) = d^2$. We shall need the following theorem giving the number of roots of f in $\mathbb{F}_q$ in various situations.

Theorem 4.2.1. *[29, 62] Let $p \neq 3$ be a prime and $q = p^n$. Suppose that $f(X) = X^3 + cX + d$ is a polynomial over $\mathbb{F}_q$ with discriminant $\Delta(f) \neq 0$.*

(i) If q is odd, f has exactly one root in $\mathbb{F}_q$ if $\Delta(f)$ is a nonsquare in $\mathbb{F}_q$ and 0 or 3 roots in $\mathbb{F}_q$ otherwise.

(ii) If q is even, f has exactly one root in $\mathbb{F}_q$ if $\mathrm{Tr}_{q/2}(c^3d^{-2}) \neq \mathrm{Tr}_{q/2}(1)$ and 0 or 3 roots in $\mathbb{F}_q$ otherwise.

4.3 The construction

Our construction will be presented in this section. The set E will correspond to a choice of orbits and the set $\mathcal{M}$ will be the union of the chosen orbits.

4.3.1 The set E

Throughout the rest of this thesis, we always assume that q is a prime power such that $q \equiv 2 \pmod 3$. We define

$$T_0 = \{x \in \mathbb{F}_{q^3}^* : \mathrm{Tr}_{q^3/q}(x) = 0\},$$
$$L_0 = \{x \in T_0 : \mathrm{N}_{q^3/q}(x) = 1\},$$

where $\mathrm{Tr}_{q^3/q}$ and $\mathrm{N}_{q^3/q}$ are the relative trace and norm from $\mathbb{F}_{q^3}$ to $\mathbb{F}_q$, respectively. Then $|T_0| = q^2 - 1$, $|L_0| = q + 1$ and $L_0 \cdot \mathbb{F}_q^* = T_0$. Since $\gcd(q-1, q^2+q+1) = 1$, we have $C_0 \cdot \mathbb{F}_q^* = \mathbb{F}_{q^3}^*$, where C_0 is the subgroup of $\mathbb{F}_{q^3}^*$ of order $q^2 + q + 1$.

Lemma 4.3.1. *If $q \equiv 2 \pmod 3$ with q odd, then -3 is a nonsquare in $\mathbb{F}_q$.*

Proof. Write $q = p^n$ with p an odd prime. Then $p \equiv 2 \pmod 3$ and n is odd. It suffices to show that -3 is a nonsquare in $\mathbb{F}_p$. By the law of quadratic reciprocity we have

$$\left(\frac{-3}{p}\right) = \left(\frac{-1}{p}\right) \cdot \left(\frac{3}{p}\right) = (-1)^{(p-1)/2} \cdot (-1)^{(p-1)/2 \cdot (3-1)/2} \cdot \left(\frac{p}{3}\right) = -1.$$

Here, $(\frac{\cdot}{p})$ is the Legendre symbol. The proof is complete. □

Lemma 4.3.2. *If z is an element of $\mathbb{F}_{q^3}^*$ such that $\mathrm{Tr}_{q^3/q}(z) = 0$, then $\mathrm{Tr}_{q^3/q}(z^{1+q}) \neq 0$.*

Proof. We have $z \notin \mathbb{F}_q$, since otherwise $3z = \mathrm{Tr}_{q^3/q}(z) = 0$. If $\mathrm{Tr}_{q^3/q}(z^{1+q}) = 0$, then the minimal polynomial of z over $\mathbb{F}_q$ is $X^3 - c$, where $c = \mathrm{N}_{q^3/q}(z)$. Since $q \equiv 2 \pmod 3$, we have $\gcd(q-1, 3) = 1$, so $X^3 = c$ has exactly one root in $\mathbb{F}_q$: a contradiction to the irreducibility of $X^3 - c$. This completes the proof. □

Since $q \equiv 2 \pmod 3$, we have $\gcd(q-1,3) = 1$ and so the map $y \mapsto y^3$ is a permutation of $\mathbb{F}_q$. We write $y \mapsto y^{1/3}$ for its inverse map. We define two multisets as follows:

$$D_1 = [x\mathrm{N}_{q^3/q}(\lambda + x^q - x^{q^2})^{1/3} : x \in L_0, \lambda \in \mathbb{F}_q],$$
$$D_2 = [\beta^{-1}x\mathrm{N}_{q^3/q}(\lambda + x^q - x^{q^2})^{-1/3} : x \in L_0, \lambda \in \mathbb{F}_q],$$

where $\beta = -3^{-1} \in \mathbb{F}_q$. Set $\gamma := \beta^{-3} = -27$.

Lemma 4.3.3. *Let $x \in L_0$ and set $z := x^q - x^{q^2}$. For each $\alpha \in \mathbb{F}_q^*$, set*

$$c_\alpha := |\{\lambda \in \mathbb{F}_q : \mathrm{N}_{q^3/q}(\lambda + z) = \alpha\}| + |\{\lambda \in \mathbb{F}_q : \gamma\mathrm{N}_{q^3/q}(\lambda + z)^{-1} = \alpha\}|.$$

Then $c_\alpha = 1$ or 4.

Proof. Write $a := \mathrm{Tr}_{q^3/q}(z^{1+q})$, $b := \mathrm{N}_{q^3/q}(z)$, and set $u := -\frac{1}{3}a$. It is clear that $\mathrm{Tr}_{q^3/q}(z) = 0$, so $a \neq 0$ by Lemma 4.3.2. The minimal polynomial of x over $\mathbb{F}_q$ is $g(X) := X^3 + eX - 1$ with $e = \mathrm{Tr}_{q^3/q}(x^{1+q})$. The discriminant of g is $z^{2+2q+2q^2} = b^2$ by definition, and it equals $-4e^3 - 27$. So $b^2 = -4e^3 + \gamma$. On the other hand, we have

$$a = \mathrm{Tr}_{q^3/q}((x^q - x^{q^2})^{1+q}) = \mathrm{Tr}_{q^3/q}(x^{q+1} - x^2)$$
$$= \mathrm{Tr}_{q^3/q}(x^{q+1} + x(x^q + x^{q^2})) = 3e.$$

We thus have $b^2 - 4u^3 = \gamma$, which is a nonsquare in $\mathbb{F}_q$ when q is odd by Lemma 4.3.1.

For any element $\alpha \in \mathbb{F}_q^*$, let $f_1(X) := X^3 + aX + b - \alpha$, $f_2(X) := X^3 + aX + b - \beta^{-3}\alpha^{-1}$. Upon expansion we deduce that

$$c_\alpha = |\{\lambda \in \mathbb{F}_q : f_1(\lambda) = 0\}| + |\{\lambda \in \mathbb{F}_q : f_2(\lambda) = 0\}|.$$

The discriminants of the two cubic polynomials f_1 and f_2 are

$$\Delta_1 = -4a^3 - 27(b - \alpha)^2 = \gamma((\alpha - b)^2 - 4u^3),$$
$$\Delta_2 = -4a^3 - 27(b - \beta^{-3}\alpha^{-1})^2 = \gamma((\gamma\alpha^{-1} - b)^2 - 4u^3),$$

respectively. We claim that either none or both of Δ_1, Δ_2 are zero, and if the latter occurs then $\alpha^2 - 2b\alpha + \gamma = 0$. We compute that

$$\begin{aligned}\Delta_1\Delta_2 =& \gamma^2((\alpha - b)^2 - 4u^3)((\gamma\alpha^{-1} - b)^2 - 4u^3)\\ =& \gamma^2(\alpha^2 - 2\alpha b + \gamma)(\gamma^2\alpha^{-2} - 2\gamma\alpha^{-1}b + \gamma)\\ =& \gamma^2(\alpha^2\gamma + \gamma^3\alpha^{-2} - 4b\gamma\alpha - 4b\gamma^2\alpha^{-1} + 2\gamma^2 + 4b^2\gamma)\\ =& \gamma^3(\alpha + \gamma\alpha^{-1})^2 + 4b\gamma^3(b - \alpha - \gamma\alpha^{-1})\\ =& \gamma^3(\alpha + \gamma\alpha^{-1} - 2b)^2.\end{aligned}$$

If $\Delta_1 = 0$, then $\Delta_1\Delta_2 = 0$ and so $\alpha - b = b - \gamma\alpha^{-1}$, which in turn implies that $\Delta_2 = 0$. The converse is also true. The claim is now established.

We first consider the case where $\Delta_1 = \Delta_2 = 0$, so that $\alpha^2 - 2b\alpha + \gamma = 0$. In this case, $f_1(X)$ has a repeated root η. If η is not in $\mathbb{F}_q$, then η^q would also be a repeated root of $f_1(X)$, contradicting the fact that $\deg(f_1) = 3$. Hence all the roots of $f_1(x) = 0$ lie in $\mathbb{F}_q$. The three roots of f_1 can not be all equal: if $f_1(X) = (X - \eta)^3$, then $\eta = 0$ by comparing the coefficients of X^2, and so $a = 0$: a contradiction. To conclude, $f_1(X)$ has two distinct roots in $\mathbb{F}_q$. The same is true for f_2 by the same argument. We thus deduce that $c_\alpha = 4$ in this case.

We next consider the case $\Delta_1\Delta_2 \neq 0$. In the case where q is odd, $\Delta_1\Delta_2$ is a nonsquare in $\mathbb{F}_q$ by Lemma 4.3.1. That is, exactly one of Δ_1 and Δ_2 is a nonsquare of $\mathbb{F}_q$. By (i) of Theorem 4.2.1, we conclude that one of f_1 and f_2 has exactly one zero in $\mathbb{F}_q$ and the other has 0 or 3 zeros in $\mathbb{F}_q$. It follows that $c_\alpha = 1$ or 4 as desired. In the case where q is even, we have $\gamma = \beta = 1$ and $z = x$. It follows that $a = \mathrm{Tr}_{q^3/q}(x^{1+q}) = \mathrm{Tr}_{q^3/q}(x^{-q^2})$ and $b = \mathrm{N}_{q^3/q}(x) = 1$ by the fact $x \in L_0$. The two cubic polynomials take the form $X^3 + aX + 1 + \alpha$, $X^3 + aX + 1 + \alpha^{-1}$, respectively.

Since $x \in L_0$, we have $\mathrm{Tr}_{q^3/q}(x) = 0$ and $x^{1+q+q^2} = 1$. We compute that

$$\begin{aligned}
&\mathrm{Tr}_{q/2}\left(\frac{a^3}{1+\alpha^2}\right) + \mathrm{Tr}_{q/2}\left(\frac{a^3}{1+\alpha^{-2}}\right) = \mathrm{Tr}_{q/2}(a^3)\\
=&\mathrm{Tr}_{q/2}\Big((x^{-1}+x^{-q}+x^{-q^2})^3\Big) = \mathrm{Tr}_{q^3/2}(x^{-3}+x^{q-1}+x^{q^2-1})\\
=&\mathrm{Tr}_{q^3/2}(x^{q^2+q+1-3}+1) = \mathrm{Tr}_{q^3/2}(x^{q-1}\cdot x^{q^2-1})+1\\
=&\mathrm{Tr}_{q^3/2}(x^{q-1}x^{-1}(x+x^q))+1 = \mathrm{Tr}_{q^3/2}(x^{2(q-1)}+x^{q-1})+1\\
=&1.
\end{aligned}$$

As in the case where q is odd, we deduce that $c_\alpha = 1$ or 4 by using (ii) of Theorem 4.2.1. □

Proposition 4.3.4. *There is a subset $E \subseteq T_0 \subseteq \mathbb{F}_{q^3}^*$ of size $\frac{(q+1)^2}{3}$ such that*

$$D_1 + D_2 = 3E + T_0$$

in the group ring $\mathbb{Z}[\mathbb{F}_{q^3}^*]$.

Proof. For $x \in L_0$, set $z := x^q - x^{q^2}$, and define a multiset

$$W_x := [\mathrm{N}_{q^3/q}(\lambda+z) : \lambda \in \mathbb{F}_q] \cup [\gamma \mathrm{N}_{q^3/q}(\lambda+z)^{-1} : \lambda \in \mathbb{F}_q]. \tag{4.3.1}$$

For any element $\alpha \in \mathbb{F}_q^*$, its multiplicity in W_x equals c_α, where c_α is as defined in Lemma 4.3.3. We have $c_\alpha \in \{1, 4\}$ by the same lemma. Therefore, there is a subset L_x of $\mathbb{F}_q^*$ such that $W_x = \mathbb{F}_q^* + 3L_x^{(3)}$, where $L_x^{(3)} = \sum_{z \in L_x} z^3$. Moreover, it is clear that $|L_x| = \frac{2q - |\mathbb{F}_q^*|}{3} = \frac{q+1}{3}$.

It is routine to check that $D_1 + D_2 = \sum_{x \in L_0} x W_x^{(1/3)}$. Set $E := \sum_{x \in L_0} x L_x$, which is a subset of $\mathbb{F}_{q^3}^*$ of size $(q+1)^2/3$. Then the claim in the proposition follows from the fact $W_x = \mathbb{F}_q^* + 3L_x^{(3)} \in \mathbb{Z}[\mathbb{F}_{q^3}^*]$ for $x \in L_0$, and $L_0 \cdot \mathbb{F}_q^* = T_0$. The proof is now complete. □

4.3.2 The set $\mathcal{M}$

Let $V = \mathbb{F}_{q^3} \times \mathbb{F}_{q^3}$, which is viewed as a 6-dimensional vector space over $\mathbb{F}_q$. Define a map $Q : V \to \mathbb{F}_q$ by

$$Q((x,y)) = \mathrm{Tr}_{q^3/q}(xy),\ \forall (x,y) \in V.$$

It is easy to check that Q is a non-degenerate hyperbolic quadratic form on V. The quadric defined by Q will be our model for $\mathcal{Q}^+(5,q)$ whose points can be expressed as $\{\langle (x,y)\rangle : Q((x,y)) = 0\}$. The polar form $f : V \times V \to \mathbb{F}_q$ of Q is given by

$$f((x,y),(a,b)) = \mathrm{Tr}_{q^3/q}(bx + ay).$$

For a point $P = \langle (x_0, y_0)\rangle$, its polar hyperplane $P^\perp$ is given by

$$P^\perp = \{\langle (x,y)\rangle : \mathrm{Tr}_{q^3/q}(xy_0 + x_0 y) = 0\}.$$

Let w be a primitive element of $\mathbb{F}_{q^3}$. Let C_0 be the subgroup of $\mathbb{F}_{q^3}^*$ of order q^2+q+1. For any $\mu \in C_0$, consider the following map on $\mathcal{Q}^+(5,q)$ defined by

$$\langle (x,y)\rangle \mapsto \langle (\mu x, \mu^{-1} y)\rangle.$$

Then, C_0 is embedded as a subgroup $i(C_0)$ of $\mathrm{PGO}^+(6,q)$, and it acts semi-regularly on the points of $\mathcal{Q}^+(5,q)$. So the points of $\mathcal{Q}^+(5,q)$ are partitioned into orbits of this action; each orbit has length q^2+q+1, and the number of orbits is q^2+1. We denote the orbit containing the point $\langle (a,b)\rangle$ by $O_{(a,b)}$. Then, the orbits are $O_{(0,1)}$ and $O_{(1,z)}$, $z \in \mathbb{F}_{q^3}$ with $\mathrm{Tr}_{q^3/q}(z) = 0$. The following is our main theorem.

Theorem 4.3.5. *Let $q \equiv 2 \pmod 3$ be a prime power. Let $\mathcal{M} = \bigcup_{z\in E} O_{(1,z)}$, where E is defined in Proposition 4.3.4. Then, the line set $\mathcal{L}$ in $\mathrm{PG}(3,q)$ corresponding to $\mathcal{M}$ under the Klein correspondence forms a Cameron-Liebler line class with parameter $x = \frac{(q+1)^2}{3}$.*

Let $\mathcal{M}$ be defined as in Theorem 4.3.5. Clearly $|\mathcal{M}| = x(q^2+q+1)$ with $x = \frac{(q+1)^2}{3}$. Set $D = \{\lambda v : \lambda \in \mathbb{F}_q^*, \langle v\rangle \in \mathcal{M}\}$. Then

$$|D| = (q-1)|\mathcal{M}| = (q-1)|E|(q^2+q+1) = (q^3-1)\frac{(q+1)^2}{3}.$$

To prove Theorem 4.3.5, we need to show that D has the correct character values as specified in Corollary 4.1.3. Each additive character of $(V,+)$ is of the form $\psi_{a,b}$ for some $(a,b) \in V$, where

$$\psi_{a,b}((x,y)) = \psi_{\mathbb{F}_q}(f((a,b),(x,y))) = \psi_{\mathbb{F}_{q^3}}(bx + ay),\ \forall (x,y) \in V$$

Here $\psi_{\mathbb{F}_q}$ and $\psi_{\mathbb{F}_{q^3}}$ are the canonical additive characters of $\mathbb{F}_q$ and $\mathbb{F}_{q^3}$, respectively. It is clear that $\psi_{a,b}$ is trivial on the hyperplane $\langle(a,b)\rangle^\perp$. By Corollary 4.1.3, in order to prove Theorem 4.3.5 it suffices to prove the following claim: for any nonzero element (a,b) of V, we have

$$\psi_{a,b}(D) = \begin{cases} -\frac{(q+1)^2}{3} + q^3, & \text{if } (a,b) \in D, \\ -\frac{(q+1)^2}{3}, & \text{otherwise.} \end{cases} \tag{4.3.2}$$

In other words, we want to prove that D is a self-dual partial difference set in V. This will be accomplished in the next section.

4.4 Proof of the main theorem

Let $\mathcal{M}$ be defined as in Theorem 4.3.5, and let D be the corresponding subset of nonzero vectors in $V = \mathbb{F}_{q^3} \times \mathbb{F}_{q^3}$. Take the same notation as introduced in Subsection 4.3.2, and set $N := q^2 + q + 1$. To simplify notation, we write $G(\chi)$ for the Gauss sum $G_{q^3}(\chi)$, where χ is a multiplicative character of $\mathbb{F}_{q^3}^*$. We need to evaluate the character sum

$$\psi_{a,b}(D) = \sum_{z\in E} \psi_{a,b}(\mathbb{F}_q^* O_{(1,z)})$$

for nonzero $(a,b) \in V$, where $\mathbb{F}_q^* O_{(1,z)} = \{(y\mu, y\mu^{-1}z) : y \in \mathbb{F}_q^*, \mu \in C_0\}$. For $z \in E$, we have

$$\psi_{a,b}(\mathbb{F}_q^* O_{(1,z)}) = \sum_{y\in\mathbb{F}_q^*} \sum_{\mu\in C_0} \psi_{\mathbb{F}_{q^3}}(by\mu + ay\mu^{-1}z). \tag{4.4.1}$$

The inner sum in the right hand side of (4.4.1) is an incomplete Kloosterman sum.

Definition 4.4.2. *Let ψ be a non-trivial additive character of $\mathbb{F}_q$ and let $a, b \in \mathbb{F}_q$. Then the sum*

$$K(\psi; a, b) = \sum_{c\in\mathbb{F}_q^*} \psi(ac + bc^{-1})$$

is called a Kloosterman sum.

It is in general very difficult to evaluate incomplete Kloosterman sums exactly. Here we are dealing with certain sums $\psi_{a,b}(D)$ of incomplete Kloosterman sums; and for these sums we can evaluate them exactly.

Lemma 4.4.1. *If $ab = 0$ but $(a, b) \neq (0, 0)$, then $\psi_{a,b}(D) = -\frac{(q+1)^2}{3}$.*

Proof. Recall that $\mathbb{F}_{q^3}^* = C_0 \cdot \mathbb{F}_q^*$. When $a = 0$ and $b \neq 0$,

$$\psi_{0,b}(\mathbb{F}_q^* O_{(1,z)}) = \sum_{\theta \in \mathbb{F}_q^*} \sum_{\mu \in C_0} \psi_{\mathbb{F}_{q^3}}(b\theta\mu) = \sum_{x \in \mathbb{F}_{q^3}^*} \psi_{\mathbb{F}_{q^3}}(bx) = -1.$$

The computations in the case when $a \neq 0$ and $b = 0$ are similar. This completes the proof. □

From now on, we assume that $ab \neq 0$. Let χ be a generator of the multiplicative character group of $\mathbb{F}_{q^3}^*$, and set

$$\chi_1 := \chi^{q-1}, \quad \chi_2 := \chi^N.$$

The orders of χ_1 and χ_2 are N and $q-1$, respectively. For a subset Y of $\mathbb{F}_{q^3}^*$ (possibly a multiset) and a multiplicative character χ^i, we write $\chi^i(Y) := \sum_{x \in Y} \chi^i(x)$. It is well known that

$$\chi^k(C_0) = \begin{cases} N, & \text{if } k \equiv 0 \pmod{N}, \\ 0, & \text{otherwise}, \end{cases} \tag{4.4.3}$$

and

$$\chi^k(\mathbb{F}_q^*) = \begin{cases} q-1, & \text{if } k \equiv 0 \pmod{q-1}, \\ 0, & \text{otherwise}. \end{cases} \tag{4.4.4}$$

We introduce two auxiliary exponential sums:

$$S_1 = \frac{1}{q^3 - 1} \sum_{\ell=0}^{N-1} G(\chi_1^{-\ell})^2 \chi_1^\ell(ab) \chi_1^\ell(E)$$

and

$$S_2 = \frac{1}{q^3 - 1} \sum_{i=1}^{q-2} \sum_{\ell=0}^{N-1} G(\chi_2^i \chi_1^{-\ell}) G(\chi_2^{-i} \chi_1^{-\ell}) \chi_1^\ell(ab) \chi_2^i(ab^{-1}) \chi_2^i \chi_1^\ell(E).$$

Lemma 4.4.2. *It holds that $\psi_{a,b}(D) = S_1 + S_2$.*

Proof. By Theorem 2.3.1, we have

$$\begin{aligned}
\psi_{a,b}(\mathbb{F}_q^* O_{(1,z)}) &= \sum_{\theta\in\mathbb{F}_q^*}\sum_{\mu\in C_0}\psi_{\mathbb{F}_{q^3}}(b\theta\mu + a\theta\mu^{-1}z)\\
&= \frac{1}{(q^3-1)^2}\sum_{i=0}^{q^3-2}\sum_{j=0}^{q^3-2}\sum_{\theta\in\mathbb{F}_q^*}\sum_{\mu\in C_0} G(\chi^{-i})\chi^i(b\theta\mu)G(\chi^{-j})\chi^j(a\theta\mu^{-1}z)\\
&= \frac{1}{(q^3-1)^2}\sum_{i=0}^{q^3-2}\sum_{j=0}^{q^3-2}\sum_{\theta\in\mathbb{F}_q^*} G(\chi^{-i})G(\chi^{-j})\chi^i(b\theta)\chi^j(a\theta z)\sum_{\mu\in C_0}\chi^{i-j}(\mu)\\
&\overset{(4.4.3)}{=} \frac{N}{(q^3-1)^2}\sum_{j=0}^{q^3-2}\sum_{h=0}^{q-2}\sum_{\theta\in\mathbb{F}_q^*} G(\chi^{-j-Nh})G(\chi^{-j})\chi^{j+Nh}(b\theta)\chi^j(a\theta z)\\
&= \frac{N}{(q^3-1)^2}\sum_{j=0}^{q^3-2}\sum_{h=0}^{q-2} G(\chi^{-j-Nh})G(\chi^{-j})\chi^{j+Nh}(b)\chi^j(az)\chi^{2j+Nh}(\mathbb{F}_q^*).
\end{aligned}$$

By (4.4.4), $\chi^{2j+Nh}(\mathbb{F}_q^*) = q-1$ if and only if $2j+Nh \equiv 0 \pmod{q-1}$, i.e., $h \equiv \frac{2j(q-2)}{3} \pmod{q-1}$. We thus have

$$\psi_{a,b}(\mathbb{F}_q^* O_{(1,z)}) = \frac{1}{q^3-1}\sum_{j=0}^{q^3-2} G(\chi^{-j-\frac{2j(q-2)N}{3}})G(\chi^{-j})\chi^{j+\frac{2j(q-2)N}{3}}(b)\chi^j(az).$$

Since $N = q^2+q+1$ and $\gcd(N, q-1) = 1$, any integer $j \in [0, q^3-2]$ can be uniquely written as $j = Ni + (q-1)\ell$ for some integers $0 \le i \le q-2$ and $0 \le \ell \le q^2+q$ by the Chinese Remainder Theorem. Moreover, with $j = Ni + (q-1)\ell$, we have

$$j + \frac{2j(q-2)N}{3} \equiv -Ni + (q-1)\ell \pmod{q^3-1}$$

by the fact $\frac{(q-2)}{3}N \equiv -1 \pmod{q-1}$. By rewriting j as $Ni + (q-1)\ell$ we deduce that

$$\begin{aligned}
\psi_{a,b}(\mathbb{F}_q^* O_{(1,z)}) &= \frac{1}{q^3-1}\sum_{i=0}^{q-2}\sum_{\ell=0}^{N-1} G(\chi_2^i\chi_1^{-\ell})G(\chi_2^{-i}\chi_1^{-\ell})\chi_2^{-i}\chi_1^{\ell}(b)\chi_2^i\chi_1^{\ell}(az)\\
&= \frac{1}{q^3-1}\sum_{\ell=0}^{N-1} G(\chi_1^{-\ell})^2\chi_1^{\ell}(b)\chi_1^{\ell}(az)\\
&\quad + \frac{1}{q^3-1}\sum_{i=1}^{q-2}\sum_{\ell=0}^{N-1} G(\chi_2^i\chi_1^{-\ell})G(\chi_2^{-i}\chi_1^{-\ell})\chi_2^{-i}\chi_1^{\ell}(b)\chi_2^i\chi_1^{\ell}(az).
\end{aligned}$$

Taking summation over $z \in E$, we get the desired equality $\psi_{a,b}(D) = S_1 + S_2$. □

We now explicitly evaluate S_1 and S_2. Write $ab = w^{Ns_0+(q-1)t_0}$ for some $s_0 \in \{0, 1, \dots, q-2\}$ and $t_0 \in \{0, 1, \dots, q^2+q\}$.

Lemma 4.4.3. *It holds that*

$$S_1 = \begin{cases} \dfrac{(q+1)(q^3-q^2+1)}{3(q-1)}, & \text{if } \mathrm{Tr}_{q^3/q}(w^{(q-1)t_0}) = 0, \\ -\dfrac{(q+1)^2}{3}, & \text{otherwise.} \end{cases}$$

Proof. From the proof of Proposition 4.3.4, we see that $E = \bigcup_{x \in L_0} xL_x$, where each L_x is a subset of $\mathbb{F}_q^*$ of size $\frac{q+1}{3}$. By Theorem 2.3.3, for any $1 \le \ell \le N-1$ we have

$$\chi_1^\ell(E) = \sum_{x \in L_0} \sum_{y \in L_x} \chi_1^\ell(xy) = \frac{q+1}{3} \sum_{x \in L_0} \chi_1^\ell(x) = \frac{(q+1)}{3q} G(\chi_1^\ell).$$

Together with the fact $G(\chi_1^{-\ell})G(\chi_1^\ell) = q^3$ for $1 \le \ell \le N-1$, we have

$$\begin{aligned} S_1 =& \frac{1}{q^3-1} \sum_{\ell=0}^{N-1} G(\chi_1^{-\ell})^2 \chi_1^\ell(ab)\chi_1^\ell(E) \\ =& \frac{|E|}{q^3-1} + \frac{1}{q^3-1} \sum_{\ell=1}^{N-1} G(\chi_1^{-\ell})^2 \chi_1^\ell(ab)\chi_1^\ell(E) \\ =& \frac{(q+1)^2}{3(q^3-1)} + \frac{q+1}{3q(q^3-1)} \sum_{\ell=1}^{N-1} G(\chi_1^{-\ell})G(\chi_1^{-\ell})G(\chi_1^\ell)\chi_1^\ell(ab) \\ =& \frac{(q+1)^2}{3(q^3-1)} + \frac{(q+1)q^2}{3(q^3-1)} \sum_{\ell=1}^{N-1} G(\chi_1^{-\ell})\chi_1^\ell(ab). \end{aligned}$$

By Lemma 2.3.2 and the fact that $G(\chi_1^0) = -1$, we have

$$S_1 = \frac{(q+1)^2}{3(q^3-1)} + \frac{(q+1)q^2}{3(q-1)} \left(\psi_{\mathbb{F}_{q^3}}(ab\mathbb{F}_q^*) + \frac{1}{N} \right).$$

We compute

$$\psi_{\mathbb{F}_{q^3}}(ab\mathbb{F}_q^*) = \psi_{\mathbb{F}_q}(w^{Ns_0}\mathrm{Tr}_{q^3/q}(w^{(q-1)t_0})\mathbb{F}_q^*) = \begin{cases} q-1, & \text{if } \mathrm{Tr}_{q^3/q}(w^{(q-1)t_0}) = 0, \\ -1, & \text{otherwise.} \end{cases} \quad (4.4.5)$$

Hence we finally obtain

$$S_1 = \begin{cases} \frac{(q+1)(q^3-q^2+1)}{3(q-1)}, & \text{if } \mathrm{Tr}_{q^3/q}(w^{(q-1)t_0}) = 0, \\ -\frac{(q+1)^2}{3}, & \text{otherwise.} \end{cases}$$

This completes the proof of the lemma. ∎

Next we evaluate S_2. By the definition of E, we have

$$\chi_2^i\chi_1^\ell(E) = \frac{1}{3}\left(\chi_2^i\chi_1^\ell(D_1) + \chi_2^i\chi_1^\ell(D_2) - \chi_2^i\chi_1^\ell(T_0)\right).$$

Therefore $S_2 = \Sigma_1 + \Sigma_2 + \Sigma_3$, where

$$\Sigma_1 = \frac{1}{3(q^3-1)}\sum_{i=1}^{q-2}\sum_{\ell=0}^{N-1} G(\chi_2^i\chi_1^{-\ell})G(\chi_2^{-i}\chi_1^{-\ell})\chi_1^\ell(ab)\chi_2^i(ab^{-1})\chi_2^i\chi_1^\ell(D_1),$$
$$\Sigma_2 = \frac{1}{3(q^3-1)}\sum_{i=1}^{q-2}\sum_{\ell=0}^{N-1} G(\chi_2^i\chi_1^{-\ell})G(\chi_2^{-i}\chi_1^{-\ell})\chi_1^\ell(ab)\chi_2^i(ab^{-1})\chi_2^i\chi_1^\ell(D_2),$$
$$\Sigma_3 = \frac{-1}{3(q^3-1)}\sum_{i=1}^{q-2}\sum_{\ell=0}^{N-1} G(\chi_2^i\chi_1^{-\ell})G(\chi_2^{-i}\chi_1^{-\ell})\chi_1^\ell(ab)\chi_2^i(ab^{-1})\chi_2^i\chi_1^\ell(T_0).$$

Write $ab = w^{Ns_0+(q-1)t_0}$, $ab^{-1} = w^{Nu_0+(q-1)v_0}$, and set $z_0 = w^{Nu_0+(q-1)t_0}$ and $z_1 = w^{-Nu_0+(q-1)t_0}$. The next proposition gives the values of Σ_1, Σ_2 and Σ_3.

Theorem 4.4.4. *For each $z \in T_0$, define*

$$\mu_z := |\{(y,\lambda) \in C_0 \times \mathbb{F}_q : y - (z^q - z^{q^2}) + \lambda = 0\}|,$$
$$\mu'_z := |\{(y,\lambda) \in C_0 \times \mathbb{F}_q : y - \beta(z^q - z^{q^2}) + \lambda = 0\}|,$$

where $\beta = -3^{-1} \in \mathbb{F}_q$. Then it holds that

$$(\Sigma_1, \Sigma_2, \Sigma_3) = \begin{cases} (\frac{q^3}{3}\mu_{z_0} - \frac{q^4}{3(q-1)},\ \frac{q^3}{3}\mu'_{z_1} - \frac{q^4}{3(q-1)},\ 0), & \text{if } w^{(q-1)t_0} \in T_0, \\ (0,\ 0,\ 0), & \text{otherwise.} \end{cases}$$

The proof of Theorem 4.4.4 involves very complicated computations of exponential sums. To streamline the presentation, we delay the proof to the end of this chapter.

Proof of Theorem 4.3.5. As mentioned in Subsection 4.3.2, it suffices to establish (4.3.2) for nonzero element (a,b) of V, i.e.,

$$\psi_{a,b}(D) = \begin{cases} -\frac{(q+1)^2}{3} + q^3, & \text{if } (a,b) \in D, \\ -\frac{(q+1)^2}{3}, & \text{otherwise.} \end{cases} \tag{4.4.6}$$

If either $a = 0$ or $b = 0$, then $(a, b) \notin D$ and the claim has been established in Lemma 4.4.1.

We next treat the case when $ab \neq 0$. Write $ab = w^{Ns_0+(q-1)t_0}$, $ab^{-1} = w^{Nu_0+(q-1)v_0}$, and set $x_0 = w^{(q-1)t_0}$, $\theta_0 = w^{Nu_0}$. It follows that $a^2 = w^{N(u_0+s_0)+(q-1)(t_0+v_0)}$. In the rest of this proof, we use the notation introduced in the statement of Theorem 4.4.4. In particular, $z_0 = x_0\theta_0$ and $z_1 = x_0\theta_0^{-1}$.

We claim that $(a, b) \in D$ if and only if $z_1 \in E$. We write $a = x\mu$ for some $x \in \mathbb{F}_q^*$ and $\mu \in C_0$. From $a^2 = w^{N(u_0+s_0)+(q-1)(t_0+v_0)}$ we deduce that $x^2 = w^{N(u_0+s_0)}$. It is straightforward to check that $ab = x^2 z_1$, so $b = (x\mu)^{-1}x^2 z_1 = x\mu^{-1}z$. That is, $(a, b) = (x\mu, x\mu^{-1}z_1)$. The claim now follows from the definition of D.

By Lemma 4.4.3 and Theorem 4.4.4, we have

$$\psi_{a,b}(D) = \begin{cases} \frac{(q+1)(q^3-q^2+1)}{3(q-1)} + \frac{q^3}{3}(\mu_{z_0} + \mu'_{z_1}) - \frac{2q^4}{3(q-1)}, & \text{if } x_0 \in T_0, \\ -\frac{(q+1)^2}{3}, & \text{otherwise.} \end{cases}$$

Hence, we need to compute $\mu_{z_0} + \mu'_{z_1}$ under the assumption that $x_0 \in T_0$. Set $\tilde{z}_0 := x_0^q - x_0^{q^2}$. We now have

$$\begin{aligned}
&\mu_{z_0} + \mu'_{z_1} \\
=&|\{(y, \lambda) \in C_0 \times \mathbb{F}_q : y = (z_0^q - z_0^{q^2}) - \lambda\}| + |\{(y, \lambda) \in C_0 \times \mathbb{F}_q : y = \beta(z_1^q - z_1^{q^2}) - \lambda\}| \\
=&|\{\lambda \in \mathbb{F}_q : \mathrm{N}_{q^3/q}(\lambda + \theta_0\tilde{z}_0) = 1\}| + |\{\lambda \in \mathbb{F}_q : \mathrm{N}_{q^3/q}(\lambda + \theta_0^{-1}\beta\tilde{z}_0) = 1\}| \\
=&|\{\lambda \in \mathbb{F}_q : \mathrm{N}_{q^3/q}(\lambda + \tilde{z}_0) = \theta_0^{-3}\}| + |\{\lambda \in \mathbb{F}_q : \beta^3\mathrm{N}_{q^3/q}(\lambda + \tilde{z}_0)^{-1} = \theta_0^{-3}\}|,
\end{aligned}$$

which is the multiplicity of θ_0^{-3} in the multiset W_{x_0}. By Proposition 4.3.4, this multiplicity is equal to 1 or 4, and correspondingly $\psi_{a,b}(D) = -\frac{(q+1)^2}{3}$ or $q^3 - \frac{(q+1)^2}{3}$. Moreover $\mu_{z_0} + \mu'_{z_1} = 4$ if and only if $\theta_0^{-1} \in L_{x_0}$, i.e., $z_1 = x_0\theta_0^{-1} \in E$. To sum up, we have shown that

$$\psi_{a,b}(D) = \begin{cases} q^3 - \frac{(q+1)^2}{3}, & \text{if } z_1 \in E, \\ -\frac{(q+1)^2}{3}, & \text{otherwise.} \end{cases}$$

Also we have shown that $(a, b) \in D$ if and only if $z_1 \in E$. The proof is now complete.

□

4.5 The stabilizer of $\mathcal{M}$ in $\mathrm{P}\Omega(V)$

It should be pointed out that the results in this section are mainly attributed to Tao Feng in our joint work [33].

Recall that $V = \mathbb{F}_{q^3} \times \mathbb{F}_{q^3}$, and $Q : V \to \mathbb{F}_q$ defined by $Q((x, y)) = \mathrm{Tr}_{q^3/q}(xy)$, $\forall (x, y) \in V$, is a nonsingular hyperbolic quadratic form on V. Let $\Omega(V)$ be the derived subgroup of the isometry group of the quadratic space (V, Q), and let $\mathrm{P}\Omega(V)$ be the quotient group modulo its center. In this section, we determine the stabilizer of $\mathcal{M}$ in $\mathrm{P}\Omega(V)$.

Set $W := \{x \in \mathbb{F}_{q^3} :\ \mathrm{Tr}_{q^3/q}(x) = 0\}$, $U_1 := \mathbb{F}_{q^3} \times \{0\}$, and $U_2 := \{0\} \times \mathbb{F}_{q^3}$. Let E be the subset of $\mathbb{F}^*_{q^3}$ as in Proposition 4.3.4. For each $x \in L_0$, there exists a subset L_x of $\mathbb{F}^*_q$ of size $\frac{q+1}{3}$ such that $E = \cup_{x\in L_0} xL_x$ by the proof of Proposition 4.3.4.

Let $\square$ be the set of squares of $\mathbb{F}^*_q$ (so in the case when q is even, $\square = \mathbb{F}^*_q$). For two subsets A, B of $\mathbb{F}^*_{q^3}$, define $A \cdot B := \{ab :\ a \in A,\ b \in B\}$. In particular, if $A = \{a\}$, we write aB for $A \cdot B$.

Lemma 4.5.1. *As $\mathbb{F}_q$-vector spaces, $\mathbb{F}_{q^3} = W \oplus \mathbb{F}_q$. Furthermore $\mathbb{F}^*_q \cdot E = W \setminus \{0\}$.*

Proof. Since $\gcd(3, q-1) = 1$, W and $\mathbb{F}_q$ intersect trivially, and the first claim follows. The second is clear from the above description of E. □

For each $u \in L_0$, we define $B_u := \{y^{q^2}u^q - y^q u^{q^2} :\ y \in L_0 \setminus \{u\}\}$. It is routine to check that the elements of B_u lie in $\mathbb{F}^*_q$ by the fact $\mathrm{Tr}_{q^3/q}(y) = 0$ for $y \in L_0$.

Lemma 4.5.2. *For $u \in L_0$, we have $|B_u| = \frac{2}{3}(q+1) - 1$ and $B_u \cdot L_u = \mathbb{F}^*_q$.*

Proof. We have shown that W and $\mathbb{F}_q$ intersect trivially, so $u \notin \mathbb{F}_q$ and $W = \langle u,\ u^q\rangle_{\mathbb{F}_q}$. Moreover, $u^{q-1} \notin \mathbb{F}^*_q$ by the fact that $\gcd(q-1, q^2+q+1) = 1$. We thus have $L_0 = \left\{\frac{u^q+\lambda u}{\mathrm{N}_{q^3/q}(u^q+\lambda u)^{1/3}} :\ \lambda \in \mathbb{F}_q\right\} \cup \{u\}$. It follows that $B_u = \left\{\frac{u^{q+1}-u^{2q^2}}{\mathrm{N}_{q^3/q}(u^q+\lambda u)^{1/3}} :\ \lambda \in \mathbb{F}_q\right\}$, and its size equals that of $\{\mathrm{N}_{q^3/q}(u^{q-1}+\lambda) :\ \lambda \in \mathbb{F}_q\}$. Write $x := \lambda + \frac{1}{3}\mathrm{Tr}_{q^3/q}(u^{q-1}) \in \mathbb{F}_q$. Then

$$\mathrm{N}_{q^3/q}(u^{q-1} + \lambda) = x^3 + bx + c$$

for some $b, c \in \mathbb{F}_q$. The polynomial $X^3 + bX + c \in \mathbb{F}_q[X]$ has no roots in $\mathbb{F}_q$, so it is irreducible over $\mathbb{F}_q$. It follows that $bc \neq 0$. The polynomial $X^3 + bX$ is a Dickson polynomial of degree 3 [47], and its value set over $\mathbb{F}_q$ has size $\frac{q-1}{2\cdot\gcd(3,q-1)} + \frac{q+1}{2\cdot\gcd(3,q+1)} = \frac{2}{3}(q+1) - 1$ by Theorem 10 and Theorem 10' of [21]. For any $a \in \mathbb{F}_q^*$, we have

$$|\{ay^{-1} :\, y \in L_u\} \cap B_u| \geq |L_u| + |B_u| - |\mathbb{F}_q^*| \geq 1.$$

This show that $a \in B_u \cdot L_u$. Hence $B_u \cdot L_u = \mathbb{F}_q^*$. The proof of the lemma is now complete. □

The generators of $(V,\, Q)$ fall into two equivalence classes; two generators U and U' are equivalent if and only if $U \cap U'$ has dimension 1, cf. [44, Theorem 1.39]. The group $\Omega(V)$ stabilizes each equivalence class, cf. [46, p. 30]. The two subspaces U_1 and U_2 are both generators of the quadratic space $(V,\, Q)$, and they are in different equivalence classes.

Lemma 4.5.3. *The only generators of (V, Q) that are disjoint from $\mathcal{M}$ are U_1 and U_2.*

Proof. It is clear that U_1 and U_2 are disjoint from $\mathcal{M}$. Suppose that U is a generator other than U_1, U_2. We will show that U intersects $\mathcal{M}$ nontrivially.

We first consider the case where U and U_2 are equivalent. In this case, $U \cap U_2$ is a projective point P. By applying the action of some element in $i(C_0) \leq PGO^+(6, q)$ if necessary, we may assume without loss of generality that $P = \langle (0, 1) \rangle$. It is clear that $P^\perp = W \times \mathbb{F}_{q^3}$. Since $\mathbb{F}_{q^3} = W \oplus \mathbb{F}_q$, we identify $W \times W$ with $P^\perp/P$ naturally. In this way, $W \times W$ becomes a quadratic space $\mathcal{Q}^+(3, q)$ whose inherited quadratic form is the same as the restriction of Q to $W \times W$. We have $\mathcal{M} \cap P^\perp = \{\langle (y,\, y^{-1}x) \rangle :\, x \in E,\, y \in L_0\}$, and the corresponding set in $W \times W$ is $\mathcal{M}_P := \{\langle (y,\, \tau_y(x)) \rangle :\, x \in E,\, y \in L_0\}$, where $\tau_y(x) := y^{-1}x - \frac{1}{3}\mathrm{Tr}_{q^3/q}(y^{-1}x)$. It is straightforward to check that

$$\ker(\tau_y) = \mathbb{F}_q \cdot y,\ \mathrm{Im}(\tau_y) \leq W,\ \tau_y(W) \leq \{z \in W :\, \mathrm{Tr}_{q^3/q}(yz) = 0\} = \langle y^q - y^{q^2} \rangle_{\mathbb{F}_q}.$$

We thus have $\tau_y(W) = \langle y^q - y^{q^2} \rangle_{\mathbb{F}_q}$ by comparing dimensions. Let U' be the totally singular line of $W \times W$ corresponding to U. To show that U intersects $\mathcal{M}$ nontrivially, it suffices to show that U' intersects $\mathcal{M}_P$ nontrivially. There are $2(q+1)$ totally singular lines of $\mathcal{Q}^+(3,q)$, these are $\ell_y = \langle (y,0), (0, y^q - y^{q^2}) \rangle$ with $y \in L_0$, $\ell'_a = \{\langle (x, ax^q - ax^{q^2}) \rangle : x \in W\}$ with $a \in \mathbb{F}_q$ and $\ell'_\infty = \{0\} \times W$. The last line ℓ'_∞ corresponds to the generator U_2, so $U' \neq \ell'_\infty$.

(1) If $U' = \ell_y$ for some $y \in L_0$, then the point $\langle (y, \tau_y(x)) \rangle$ with $x \in E$ is in U' if there exists $\lambda \in \mathbb{F}_q^*$ such that $\tau_y(\lambda x) = y^q - y^{q^2}$. By Lemma 4.5.1, we have $\mathbb{F}_q^* \cdot E = W \setminus \{0\}$. The existence of such $x \in E$ and $\lambda \in \mathbb{F}_q^*$ now follows from the fact that $\tau_y(W) = \langle y^q - y^{q^2} \rangle_{\mathbb{F}_q}$.

(2) If $U' = \ell'_a$ for some fixed $a \in \mathbb{F}_q$, then $U' \cap \mathcal{M}_P \neq \emptyset$ if there exists $y \in L_0$, $u \in L_0$ and $c \in L_u$ such that $\tau_y(uc) = ay^q - ay^{q^2}$. The left hand side equals $z^q - z^{q^2}$ with $z = -\frac{1}{3}(y^{-1}u)^q c + \frac{1}{3}(y^{-1}u)^{q^2} c$, so $ay - z \in \mathbb{F}_q$. By taking the relative trace, we see that

$$\begin{aligned} ay - z &= \mathrm{Tr}_{q^3/q}(ay - z) = a\mathrm{Tr}_{q^3/q}(y) - \mathrm{Tr}_{q^3/q}(z) \\ &= \frac{1}{3} c\mathrm{Tr}_{q^3/q}(y^{-1}u) - \frac{1}{3} c\mathrm{Tr}_{q^3/q}(y^{-1}u) = 0. \end{aligned}$$

By the fact that $\mathrm{N}_{q^3/q}(y) = 1$ for $y \in L_0$, we deduce that

$$\begin{aligned} a &= a\mathrm{N}_{q^3/q}(y) = ay^{1+q+q^2} = y^{q+q^2} z \\ &= y^{q+q^2}\left(-\frac{1}{3} y^{-q} u^q c + \frac{1}{3} y^{-q^2} u^{q^2} c\right) \\ &= -\frac{1}{3}(y^{q^2} u^q - y^q u^{q^2})c. \end{aligned}$$

When $a = 0$, we can simply take $y = u \in L_0$ and $c \in L_u$ arbitrarily. When $a \neq 0$, we take u to be any element of L_0 and the existence of the desired (y, c) pair follows from Lemma 4.5.2.

In both cases, we have shown that U' intersects $\mathcal{M}_P$ nontrivially. This establishes the claim in the case U is in the same equivalence class as U_2.

We next consider the case where U and U_1 are equivalent. Observe that $O_{(1,xa)} = O_{(xa^{-1},1)}$ for $x \in C_0$ and $a \in \mathbb{F}_q^*$, so $\mathcal{M} = \cup_{x \in E'} O_{(x,1)}$, where $E' = \cup_{x \in L_0} x L'_x$ with $L'_x = \{a^{-1} : a \in L_x\}$. The argument is exactly the same as in the previous case. □

Let K be the stabilizer of U_1 and U_2 in $\Omega(V)$, i.e., $K = \{\alpha \in \Omega(V) : \alpha(U_1) = U_1 \text{ and } \alpha(U_2) = U_2\}$. By [46, Lemma 4.1.9], K consists of

$$\kappa(h, h^*) : V \to V, \ (x, y) \mapsto (h(x), h^*(y)),$$

where both h and h^* are bijective $\mathbb{F}_q$-linear transformations of $\mathbb{F}_{q^3}$ such that $\det(h)$, $\det(h^*) \in \square$ and $Q((x, y)) = Q((h(x), h^*(y)))$ for all $x, y \in \mathbb{F}_{q^3}$. Here, $\det(h)$ is the determinant of h with respect to any $\mathbb{F}_q$-basis of $\mathbb{F}_{q^3}$. For each bijective $\mathbb{F}_q$-linear transformation h of $\mathbb{F}_{q^3}$ with $\det(h) \in \square$, there is a unique h^* such that $\kappa(h, h^*) \in K$, and vice versa.

We now describe some special elements of K. For $a \in \mathbb{F}_{q^3}^*$, define

$$h_a : \mathbb{F}_{q^3} \to \mathbb{F}_{q^3}, \quad x \mapsto ax,$$

and set $\kappa_a := \kappa(h_a, h_{a^{-1}})$. An element $z \in C_0$ (which we identify with the corresponding element in $i(C_0)$) acts on V in exactly the same way as κ_z.

Lemma 4.5.4. *For $a \in \mathbb{F}_{q^3}^*$, κ_a is in K if and only if a is a square in $\mathbb{F}_{q^3}^*$.*

Proof. The linear transformation κ_a clearly has determinant 1 and stabilizes the generators U_1 and U_2, so it suffices to show that $\det(h_a) \in \square$ if and only if a is a square in $\mathbb{F}_{q^3}^*$. For $a \in C_0$, we have $h_a^{q^2+q+1} = \mathrm{id}_{\mathbb{F}_{q^3}}$, so $\det(h_a)^{q^2+q+1} = 1$. It follows that $\det(h_a) = 1$ from the fact $\gcd(q^2 + q + 1, q - 1) = 1$. For $a \in \mathbb{F}_q^*$, we have $\det(h_a) = a^3$, which is a square if and only if a is. The claim then follows readily from the fact that $\mathbb{F}_{q^3}^* = C_0 \cdot \mathbb{F}_q^*$. $\square$

We define $\iota : K \to \mathrm{PGL}(3, q)$ such that $\iota(g)$ is the quotient image of $g|_{U_1}$ in $\mathrm{PGL}(3, q)$, where $g|_{U_1}$ is the restriction of g to U_1. Since $\gcd(3, q - 1) = 1$, we have $\mathrm{PGL}(3, q) = \mathrm{PSL}(3, q)$. The homomorphism ι is surjective by the above description of K.

Lemma 4.5.5. *We have* $\ker(\iota) = \kappa_\square$, *where* $\kappa_\square = \{\kappa_a : a \in \square\}$.

Proof. If $\kappa = \kappa(h, h^*) \in \ker(\iota)$, then $h = h_a$ for some $a \in \mathbb{F}_q^*$ and correspondingly $\kappa = \kappa_a$. The claim is now an easy consequence of Lemma 4.5.4. $\square$

Let σ be the $\mathbb{F}_q$-linear transformation of V such that $\sigma((x, y)) = (x^q, y^q)$. It has order 3 and stabilizes both U_1 and U_2.

Lemma 4.5.6. *We have $\sigma \in K$, and $\sigma(\mathcal{M}) = \mathcal{M}$.*

Proof. The first claim follows by the same argument as in the proof of Lemma 4.5.4. The second claim is equivalent to $\sigma(E) = E$, or equivalently, $L_{x^q} = \{a^q : a \in L_x\}$ for each $x \in L_0$. This is clear from the definition of L_x in the proof of Proposition 4.3.4. □

Let G be the stabilizer of $\mathcal{M}$ in $\Omega(V)$. Let $\alpha \in G$. From $U_1 \cap \mathcal{M} = \emptyset$ and $U_2 \cap \mathcal{M} = \emptyset$, we obtain $\alpha(U_1) \cap \mathcal{M} = \emptyset$ and $\alpha(U_2) \cap \mathcal{M} = \emptyset$. By Lemma 4.5.3 and the fact that U_1 and U_2 are in different equivalence classes, we deduce that $\alpha(U_1) = U_1$ and $\alpha(U_2) = U_2$, and so $\alpha \in K$. We have shown that $G \leq K$. Moreover, G contains the subgroup H generated by σ and $i(C_0)$.

Lemma 4.5.7. *The group $\iota(G)$ has order $3(q^2 + q + 1)$ when $q > 2$.*

Proof. The group $\iota(H)$ has order $3(q^2+q+1)$ and is a maximal subgroup of $\mathrm{PSL}(3, q)$ by [13, Table 8.3]. Hence either $\iota(G) = \mathrm{PSL}(3, q)$ or $\iota(G) = \iota(H)$.

Suppose that $\iota(G) = \mathrm{PSL}(3, q)$. Fix an element $u \in L_0$, and take λ to be a primitive element of $\mathbb{F}_q^*$. Let $g = \kappa(h, h^*)$ be the element of K such that $h^*(1) = 1$, $h^*(u) = \lambda u$ and $h^*(u^q) = \lambda^{-1}u^q$. We deduce that $h(1) = 1$ from the property $Q((1, x)) = Q((h(1), h^*(x)))$. By our assumption there exists $a \in \square$ such that $\kappa_a g \in G$, i.e., $\kappa_a g$ stabilizes $\mathcal{M}$. The image of $\{\langle(1, x)\rangle : x \in E\} \subseteq \mathcal{M}$ under $\kappa_a g$ is $\{\langle(1, a^{-2}h^*(x))\rangle : x \in E\}$, so we have $E = a^{-2}h^*(E)$. Comparing both sides, we deduce that $uL_u = a^{-2}\lambda u L_u$, $u^q L_{u^q} = a^{-2}\lambda^{-1}u^q L_{u^q}$. Taking the product over the set on each side, we get $(a^{-2}\lambda)^{(q+1)/3} = 1$, $(a^{-2}\lambda^{-1})^{(q+1)/3} = 1$. It follows that $\lambda^{2(q+1)/3} = 1$. If $q > 5$, then $\frac{2(q+1)}{3} < q - 1$, and the equality $\lambda^{2(q+1)/3} = 1$ contradicts the assumption that λ is primitive. If $q = 5$, we have $a^4 = 1$ and this leads to $\lambda^2 = 1$, again contradicting the assumption that λ is primitive. The proof is now complete. □

Lemma 4.5.8. *If q is odd, then $|L_x \cap \square| = \frac{q+1}{6}$ for each $x \in L_0$.*

Proof. From the proof of Proposition 4.3.4, we know that $W_x = \mathbb{F}_q^* + 3L_x^{(3)}$ in the group ring $\mathbb{Z}[\mathbb{F}_q^*]$, where W_x is the same as in (4.3.1). Let ρ be the quadratic character of $\mathbb{F}_q^*$, which maps squares to 1 and nonsquares to -1. Then $\rho(\mathbb{F}_q^*) = 0$ and $\rho(W_x) = 3\rho(L_x)$. Since $\gamma = -27$ is a nonsquare in $\mathbb{F}_q^*$, we deduce that $\rho(W_x) = 0$. It follows that $\rho(L_x) = 0$, i.e., L_x has the same number of squares as nonsquares. This completes the proof. $\square$

Theorem 4.5.9. *The group G has order $3(q^2+q+1)s$, where $s = 1$ or $s = \gcd(2, \frac{q-1}{2})$ according as q is even or odd.*

Proof. The case $q = 2$ is verified by Magma [12]; so from now on we assume that $q > 2$. By Lemma 4.5.7, G lies in the group $H \times \kappa_\square$, where $\kappa_\square$ is as in Lemma 4.5.5. We have shown that $H \leq G$, so $G = H \times (G \cap \kappa_\square)$. It now suffices to determine the stabilizer of $\mathcal{M}$ in $\kappa_\square$.

Suppose that κ_a stabilizes $\mathcal{M}$, where a is a square of $\mathbb{F}_q^*$. The condition $\kappa_a(\mathcal{M}) = \mathcal{M}$ is equivalent to $a^2 E = E$, i.e., $a^2 L_x = L_x$ for each $x \in L_0$. Taking the product over the set on each side, we deduce that $a^{2(q+1)/3} = 1$. If q is even, then the order of a divides $\gcd(2(q+1)/3, q-1) = 1$, implying $a = 1$. If $q \equiv 3 \pmod 4$, then $\gcd(\frac{q-1}{2}, \frac{2(q+1)}{3}) = 1$ and we also get $a = 1$. If $q \equiv 1 \pmod 4$, then from $a^2 L_x = L_x$ we deduce that $a^2(L_x \cap \square) = L_x \cap \square$. By Lemma 4.5.8 we have $|L_x \cap \square| = \frac{q+1}{6}$. By the same argument we get $a^{(q+1)/3} = 1$. In this case, we have $\gcd(\frac{q+1}{3}, \frac{q-1}{2}) = 2$, so $a^2 = 1$, i.e., $a = \pm 1$. Since -1 is in $\square$, we see that indeed κ_{-1} is in G. This completes the proof. $\square$

As a corollary, the stabilizer of $\mathcal{M}$ in $\mathrm{P}\Omega(V)$ has order $3(q^2+q+1)$. By the isomorphism $\mathrm{P}\Omega^+(6,q) \cong \mathrm{PSL}(4,q)$, we see that the stabilizer of the corresponding Cameron-Liebler line class in $\mathrm{PSL}(4,q)$ has order $3(q^2+q+1)$.

4.6 Proof of Theorem 4.4.4

In this section, we will prove Theorem 4.4.4. Recall that $N = q^2 + q + 1$. We start with an observation on Gauss sums. Let S be any subset of $\mathbb{F}_{q^3}^*$, and set

$T_S := \{(s,t) : 0 \le i \le N-1, 0 \le t \le q-2, w^{s(q-1)+tN} \in S\}$. By the definition of Gauss sums, for any integers i and ℓ and $\epsilon, \delta \in \{1,-1\}$ we have

$$G(\chi_2^{\epsilon i}\chi_1^{\delta\ell})\chi_2^i\chi_1^\ell(S) = \sum_{y\in\mathbb{F}_{q^3}^*}\sum_{z\in S}\chi_2^{\epsilon i}\chi_1^{\delta\ell}(y)\chi_2^i\chi_1^\ell(z)\psi_{\mathbb{F}_{q^3}}(y)$$
$$= \sum_{y\in\mathbb{F}_{q^3}^*}\sum_{(s,t)\in T_S}\chi_2^{\epsilon i}(yw^{\epsilon s(q-1)+\epsilon tN})\chi_1^{\delta\ell}(yw^{\delta s(q-1)+\delta tN})\psi_{\mathbb{F}_{q^3}}(y). \quad (4.6.1)$$

Since $\chi_2(w^{s(q-1)}) = 1$ and $\chi_1(w^{tN}) = 1$, continuing from (4.6.1), we have

$$G(\chi_2^{\epsilon i}\chi_1^{\delta\ell})\chi_2^i\chi_1^\ell(S) = \sum_{y\in\mathbb{F}_{q^3}^*}\sum_{(s,t)\in T_S}\chi_2^{\epsilon i}(yw^{\delta s(q-1)+\epsilon tN})\chi_1^{\delta\ell}(yw^{\delta s(q-1)+\epsilon tN})\psi_{\mathbb{F}_{q^3}}(y)$$
$$= \sum_{z\in\mathbb{F}_{q^3}^*}\sum_{(s,t)\in T_S}\chi_2^{\epsilon i}\chi_1^{\delta\ell}(z)\psi_{\mathbb{F}_{q^3}}(zw^{-\delta s(q-1)-\epsilon tN}). \quad (4.6.2)$$

This identity will be used in the rest of the proof.

Let $D_3 = \beta D_2 = [x\mathrm{N}_{q^3/q}(\lambda + x^q - x^{q^2})^{-\frac{1}{3}} : x \in L_0, \lambda \in \mathbb{F}_q]$.

To evaluate Σ_1, we need the following observation: By (4.6.2) we have

$$G(\chi_2^i\chi_1^{-\ell})\chi_2^i\chi_1^\ell(D_1) = \sum_{z\in\mathbb{F}_{q^3}^*}\chi_2^i\chi_1^{-\ell}(z)\psi_{\mathbb{F}_{q^3}}(zD_3) \quad (4.6.3)$$

for any $1 \le i \le q-2$ and $0 \le \ell \le q^2+q$. Next we compute $\psi_{\mathbb{F}_{q^3}}(zD_3)$.

Lemma 4.6.1. *Let $R = \{\lambda + (h^{q^2} - h^q) : \lambda \in \mathbb{F}_q, h \in L_0\}$. For $z \in \mathbb{F}_{q^3}^*$, it holds that*

$$\psi_{\mathbb{F}_{q^3}}(zD_3) = \begin{cases} q^2+q, & \text{if } z \in \mathbb{F}_q^*, \\ -1 + \psi_{\mathbb{F}_{q^3}}(eC_0), & \text{if } z \in eR \text{ for some } e \in \mathbb{F}_q^*. \end{cases}$$

Proof. We first note that R is a system of coset representatives for $(\mathbb{F}_{q^3}^* \setminus \mathbb{F}_q^*)/\mathbb{F}_q^*$. Assume that there are $\lambda_1, \lambda_2 \in \mathbb{F}_q$, $d \in \mathbb{F}_q^*$ and $h_1, h_2 \in L_0$ such that $\lambda_1 + (h_1^{q^2} - h_1^q) = d\lambda_2 + d(h_2^{q^2} - h_2^q)$. Then by taking trace of both sides, we have $\lambda_1 = d\lambda_2$. Note that $h_1^{q^2} - h_1^q = d(h_2^{q^2} - h_2^q)$ implies that $h_1^{q^2} - dh_2^{q^2} = h_1^q - dh_2^q$, i.e., $h_1 - dh_2 \in \mathbb{F}_q$. Hence, we have

$$0 = \mathrm{Tr}_{q^3/q}(h_1) - d\mathrm{Tr}_{q^3/q}(h_2) = \mathrm{Tr}_{q^3/q}(h_1 - dh_2) = 3(h_1 - dh_2),$$

which implies that $h_1 = dh_2$. By taking norm of both sides, we have $d = 1$, $\lambda_1 = \lambda_2$ and $h_1 = h_2$. It is clear that none of the elements of R is in $\mathbb{F}_q^*$. Hence R is a system of coset representatives of $(\mathbb{F}_{q^3}^* \setminus \mathbb{F}_q^*)/\mathbb{F}_q^*$.

Next we evaluate $\psi_{\mathbb{F}_{q^3}}(zD_3)$. Let η_{q-1} be a fixed multiplicative character of order $q-1$ of $\mathbb{F}_q$. Then, we have

$$\begin{aligned}
\psi_{\mathbb{F}_{q^3}}(zD_3) =& \sum_{\lambda \in \mathbb{F}_q} \sum_{x \in L_0} \psi_{\mathbb{F}_{q^3}}(zx \mathrm{N}_{q^3/q}(\lambda + x^q - x^{q^2})^{-1/3}) \\
=& \frac{1}{q-1} \sum_{c \in \mathbb{F}_q^*} \sum_{\lambda \in \mathbb{F}_q} \sum_{x \in L_0} \sum_{i=0}^{q-2} \psi_{\mathbb{F}_{q^3}}(zxc^{-1}) \eta_{q-1}^i((\lambda + x^q - x^{q^2})^N) \eta_{q-1}^{-3i}(c) \\
=& \frac{1}{q-1} \sum_{c \in \mathbb{F}_q^*} \sum_{\lambda \in \mathbb{F}_q} \sum_{x \in L_0} \sum_{i=1}^{q-2} \psi_{\mathbb{F}_{q^3}}(zxc^{-1}) \eta_{q-1}^i((\lambda + x^q - x^{q^2})^N) \eta_{q-1}^{-3i}(c) \quad (4.6.4) \\
&+ \frac{1}{q-1} \sum_{c \in \mathbb{F}_q^*} \sum_{\lambda \in \mathbb{F}_q} \sum_{x \in L_0} \psi_{\mathbb{F}_{q^3}}(zxc^{-1}). \quad (4.6.5)
\end{aligned}$$

Denote the summands in (4.6.4) and (4.6.5) by W_1 and W_2, respectively. Then, $\psi_{\mathbb{F}_{q^3}}(zD_3) = W_1 + W_2$. Here, it is easy to see that

$$\begin{aligned}
W_2 =& \frac{1}{q-1} \sum_{c \in \mathbb{F}_q^*} \sum_{d \in \mathbb{F}_q} \sum_{x \in C_0} \psi_{\mathbb{F}_{q^3}}(zxc^{-1} + xd) \\
=& \frac{1}{q-1} \sum_{d \in \mathbb{F}_q} \sum_{x \in \mathbb{F}_{q^3}^*} \psi_{\mathbb{F}_{q^3}}(x(z+d)) = \frac{1}{q-1} \begin{cases} q^3 - q, & \text{if } z \in \mathbb{F}_q, \\ -q, & \text{if } z \notin \mathbb{F}_q. \end{cases}
\end{aligned}$$

We next evaluate W_1. Let ρ_{q-1} be the lift of η_{q-1} to $\mathbb{F}_{q^3}^*$, i.e., $\rho_{q-1}(x) = \eta_{q-1}(x^N)$. We note that for any $s \in \mathbb{F}_q^*$, $\rho_{q-1}(s) = \eta_{q-1}(s^N) = \eta_{q-1}(s^3)$. Then

$$W_1 = \frac{1}{q-1} \sum_{c \in \mathbb{F}_q^*} \sum_{\lambda \in \mathbb{F}_q} \sum_{x \in L_0} \sum_{i=1}^{q-2} \psi_{\mathbb{F}_{q^3}}(zxc^{-1}) \rho_{q-1}^i(\lambda + x^q - x^{q^2}) \rho_{q-1}^{-i}(c).$$

By Theorem 2.3.1, we have

$$\rho_{q-1}^i(\lambda + x^q - x^{q^2}) = \frac{G(\rho_{q-1}^i)}{q^3} \sum_{b \in \mathbb{F}_{q^3}^*} \psi_{\mathbb{F}_{q^3}}(b(\lambda + x^q - x^{q^2})) \rho_{q-1}^{-i}(-b), \quad 1 \le i \le q-2.$$

Substituting $\rho_{q-1}^{i}(\lambda+x^q-x^{q^2})$ in the expression for W_1 by the right-hand-side expression of the above equation, we have

$$\begin{aligned}
W_1 =& \frac{1}{q^3(q-1)} \sum_{c\in\mathbb{F}_q^*} \sum_{\lambda\in\mathbb{F}_q} \sum_{x\in L_0} \sum_{i=1}^{q-2} \psi_{\mathbb{F}_{q^3}}(zxc^{-1}) G(\rho_{q-1}^{i}) \sum_{b\in\mathbb{F}_{q^3}^*} \psi_{\mathbb{F}_{q^3}}(b(\lambda+x^q-x^{q^2}))\rho_{q-1}^{-i}(-bc) \\
=& \frac{1}{q^3(q-1)} \sum_{c,d\in\mathbb{F}_q^*} \sum_{\lambda\in\mathbb{F}_q} \sum_{x\in L_0} \sum_{i=1}^{q-2} \sum_{h\in C_0} G(\rho_{q-1}^{i}) \psi_{\mathbb{F}_{q^3}}(zxc^{-1}+hd(\lambda+x^q-x^{q^2}))\rho_{q-1}^{-i}(-dhc) \\
=& \frac{1}{q^3(q-1)} \sum_{c,d\in\mathbb{F}_q^*} \sum_{\lambda\in\mathbb{F}_q} \sum_{x\in L_0} \sum_{i=0}^{q-2} \sum_{h\in C_0} G(\rho_{q-1}^{i}) \psi_{\mathbb{F}_{q^3}}(zxc^{-1}+hd(\lambda+x^q-x^{q^2}))\rho_{q-1}^{-i}(-dc)
\end{aligned} \tag{4.6.6}$$

$$- \frac{G(\rho_{q-1}^{0})}{q^3(q-1)} \sum_{c,d\in\mathbb{F}_q^*} \sum_{\lambda\in\mathbb{F}_q} \sum_{x\in L_0} \sum_{h\in C_0} \psi_{\mathbb{F}_{q^3}}(zxc^{-1}+h\lambda+hd(x^q-x^{q^2})). \tag{4.6.7}$$

Denote the summands in (4.6.6) and (4.6.7) by W_3 and W_4, respectively. Then $W_1 = W_3 + W_4$. Here, W_4 is reformulated as

$$\begin{aligned}
W_4 =& \frac{1}{q^3(q-1)} \sum_{c,d\in\mathbb{F}_q^*} \sum_{x\in L_0} \sum_{h\in L_0} \psi_{\mathbb{F}_{q^3}}(zxc^{-1}+hd(x^q-x^{q^2})) \\
=& \frac{1}{q^3(q-1)} \sum_{c,d\in\mathbb{F}_q^*} \sum_{\lambda'\in\mathbb{F}_q} \sum_{x\in C_0} \sum_{h\in L_0} \psi_{\mathbb{F}_{q^3}}(zxc^{-1}+x\lambda'+hd(x^q-x^{q^2})) \\
=& \frac{1}{q^3(q-1)} \sum_{c,d\in\mathbb{F}_q^*} \sum_{\lambda'\in\mathbb{F}_q} \sum_{x\in C_0} \sum_{h\in L_0} \psi_{\mathbb{F}_{q^3}}(zxc^{-1}+x\lambda')\psi_{\mathbb{F}_q}(d\mathrm{Tr}_{q^3/q}(hx^q-hx^{q^2})) \\
=& \frac{1}{q^3(q-1)} \sum_{c,d\in\mathbb{F}_q^*} \sum_{\lambda'\in\mathbb{F}_q} \sum_{x\in C_0} \sum_{h\in L_0} \psi_{\mathbb{F}_{q^3}}(zxc^{-1}+x\lambda')\psi_{\mathbb{F}_q}(d\mathrm{Tr}_{q^3/q}(h^{q^2}x-h^qx)) \\
=& \frac{1}{q^3(q-1)} \sum_{c,d\in\mathbb{F}_q^*} \sum_{\lambda'\in\mathbb{F}_q} \sum_{x\in C_0} \sum_{h\in L_0} \psi_{\mathbb{F}_{q^3}}(zxc^{-1}+x(\lambda'+d(h^{q^2}-h^q))).
\end{aligned}$$

Since R is a system of representatives of $(\mathbb{F}_{q^3}^* \setminus \mathbb{F}_q^*)/\mathbb{F}_q^*$, we have

$$\begin{aligned}
&\{\lambda'+d(h^{q^2}-h^q) : \lambda'\in\mathbb{F}_q, d\in\mathbb{F}_q^*, h\in L_0\} \\
=&\{d(\lambda'+h^{q^2}-h^q) : \lambda'\in\mathbb{F}_q, d\in\mathbb{F}_q^*, h\in L_0\} = \mathbb{F}_{q^3}\setminus\mathbb{F}_q.
\end{aligned}$$

Then,

$$\begin{aligned}
W_4 =& \frac{1}{q^3(q-1)} \sum_{c\in\mathbb{F}_q^*} \sum_{x\in C_0} \sum_{y\in\mathbb{F}_{q^3}} \psi_{\mathbb{F}_{q^3}}(zxc^{-1}+xy) - \frac{1}{q^3(q-1)} \sum_{c\in\mathbb{F}_q^*} \sum_{x\in C_0} \sum_{y\in\mathbb{F}_q} \psi_{\mathbb{F}_{q^3}}(zxc^{-1}+xy) \\
=& -\frac{1}{q^3(q-1)} \sum_{c\in\mathbb{F}_q^*} \sum_{x\in C_0} \sum_{y\in\mathbb{F}_q} \psi_{\mathbb{F}_{q^3}}(xc(z+y)) \\
=& -\frac{1}{q^3(q-1)} \begin{cases} q^3-q, & \text{if } z\in\mathbb{F}_q, \\ -q, & \text{if } z\notin\mathbb{F}_q. \end{cases}
\end{aligned}$$

To evaluate W_3, we use

$$\sum_{i=0}^{q-2} G(\rho_{q-1}^i)\rho_{q-1}^{-i}(-dc) = (q-1)\sum_{y\in C_0} \psi_{\mathbb{F}_{q^3}}(-dcy),$$

which follows from Lemma 2.3.2. Then

$$\begin{aligned}
W_3 =& \frac{1}{q^3} \sum_{c,d\in\mathbb{F}_q^*} \sum_{\lambda\in\mathbb{F}_q} \sum_{x\in L_0} \sum_{h,y\in C_0} \psi_{\mathbb{F}_{q^3}}(zxc^{-1} + hd(\lambda + x^q - x^{q^2}) - dcy) \\
=& \frac{1}{q^3} \sum_{c,d\in\mathbb{F}_q^*} \sum_{\lambda'\in\mathbb{F}_q} \sum_{h\in L_0} \sum_{x,y\in C_0} \psi_{\mathbb{F}_{q^3}}(zxc^{-1} + x\lambda' + xd(h^{q^2} - h^q))\psi_{\mathbb{F}_{q^3}}(-dcy).
\end{aligned}$$

Since R is a system of coset representatives for $(\mathbb{F}_{q^3}^*\setminus\mathbb{F}_q^*)/\mathbb{F}_q^*$, we have

$$\begin{aligned}
W_3 =& \frac{1}{q^3} \sum_{c,d\in\mathbb{F}_q^*} \sum_{z'\in R} \sum_{x,y\in C_0} \psi_{\mathbb{F}_{q^3}}(zxc^{-1} + xdz')\psi_{\mathbb{F}_{q^3}}(-dcy) \\
=& \frac{1}{q^3} \sum_{c,e\in\mathbb{F}_q^*} \sum_{z'\in R} \sum_{x,y\in C_0} \psi_{\mathbb{F}_{q^3}}(zxc^{-1} + xc^{-1}ez')\psi_{\mathbb{F}_{q^3}}(-ey) \\
=& \frac{1}{q^3} \sum_{e\in\mathbb{F}_q^*} \sum_{z'\in R} \sum_{y\in C_0} \sum_{u\in\mathbb{F}_{q^3}^*} \psi_{\mathbb{F}_{q^3}}(u(z-ez'))\psi_{\mathbb{F}_{q^3}}(ey).
\end{aligned}$$

Here, if $z\in\mathbb{F}_q$,

$$W_3 = -(q^2+q)\sum_{e\in\mathbb{F}_q^*}\sum_{y\in C_0} \psi_{\mathbb{F}_{q^3}}(ey) = q^2+q.$$

If $z \notin \mathbb{F}_q$, there is a unique $e \in \mathbb{F}_q^*$ such that $e^{-1} \in z^{-1}R \cap \mathbb{F}_q^*$. For this e, we have

$$\begin{aligned} W_3 =& -|R| \sum_{e' \in \mathbb{F}_q^* \setminus \{e\}} \sum_{y \in C_0} \psi_{\mathbb{F}_{q^3}}(e'y) + (q^3-1) \sum_{y \in C_0} \psi_{\mathbb{F}_{q^3}}(ey) - (|R|-1) \sum_{y \in C_0} \psi_{\mathbb{F}_{q^3}}(ey) \\ =& -|R| \sum_{e' \in \mathbb{F}_q^*} \sum_{y \in C_0} \psi_{\mathbb{F}_{q^3}}(e'y) + q^3 \psi_{\mathbb{F}_{q^3}}(eC_0) \\ =& -|R| \sum_{u \in \mathbb{F}_{q^3}^*} \psi_{\mathbb{F}_{q^3}}(u) + q^3 \psi_{\mathbb{F}_{q^3}}(eC_0) \\ =& q^2 + q + q^3 \psi_{\mathbb{F}_{q^3}}(eC_0). \end{aligned}$$

Summing up, we have

$$\psi_{\mathbb{F}_{q^3}}(zD_3) = W_2 + W_3 + W_4 = \begin{cases} q^2+q, & \text{if } z \in \mathbb{F}_q, \\ -1 + \psi_{\mathbb{F}_{q^3}}(eC_0), & \text{if } z \in eR. \end{cases}$$

This completes the proof of the lemma. □

Lemma 4.6.2. *For $e \in \mathbb{F}_q^*$, let $R'_e = \{x^{-1}y : x \in C_0, y \in \mathbb{F}_q^*, xy \in eR\}$. Then, $R'_e = -eR$.*

Proof. It is clear that $R'_e = eR'_1$. Therefore we only need to show that $R'_1 = -R$. Take $z = xy \in R$ with $x \in C_0$ and $y \in \mathbb{F}_q^*$. There exists $(\lambda, h) \in \mathbb{F}_q \times L_0$ such that

$$xy = \lambda + h^{q^2} - h^q.$$

Let $\lambda_2 = \frac{-y\mathrm{Tr}_{q^3/q}(x^{-1})}{3}$ and $h_2 = \frac{y(x^{-q^2}-x^{-q})}{3}$. A direct computation shows that

$$x^{-1}y = -(\lambda_2 + h_2^{q^2} - h_2^q),$$

It is clear that $\lambda_2 \in \mathbb{F}_q$. To complete the proof, it suffices to show that $h_2 \in L_0$.

Using $\lambda = xy - (h^{q^2} - h^q)$ and $\lambda^q = \lambda$, we obtain

$$x^q y - h + h^{q^2} = xy - h^{q^2} + h^q,$$

which implies that $(x^q - x)y = -3h^{q^2}$, i.e., $h = \frac{y(x^{q^2}-x^q)}{-3}$. Note that

$$h_2 = \frac{y(x^{q+1} - x^{q^2+1})}{3} = \frac{xy(x^{q^2} - x^q)}{-3} = xh.$$

Hence, $\mathrm{N}_{q^3/q}(h_2) = 1$. It is clear that $\mathrm{Tr}_{q^3/q}(h_2) = 0$. Therefore, $h_2 \in L_0$. □

We now give the promised proof of Theorem 4.4.4.

Proof of Theorem 4.4.4. By (4.6.3), we have

$$\Sigma_1 = \frac{1}{3(q^3-1)} \sum_{i=1}^{q-2} \sum_{\ell=0}^{N-1} \sum_{z \in \mathbb{F}_{q^3}^*} G(\chi_2^{-i}\chi_1^{-\ell})\chi_2^i\chi_1^{-\ell}(z)\chi_1^\ell(ab)\chi_2^i(ab^{-1})\psi_{\mathbb{F}_{q^3}}(zD_3). \tag{4.6.8}$$

By Lemma 4.6.1, continuing from (4.6.8), we have

$$\begin{aligned}\Sigma_1 =& \frac{N-1}{3(q^3-1)} \sum_{i=1}^{q-2} \sum_{\ell=0}^{N-1} \sum_{z \in \mathbb{F}_q^*} G(\chi_2^{-i}\chi_1^{-\ell})\chi_2^i\chi_1^{-\ell}(z)\chi_1^\ell(ab)\chi_2^i(ab^{-1}) \\ &+ \frac{1}{3(q^3-1)} \sum_{e \in \mathbb{F}_q^*} \sum_{z \in eR} \sum_{i=1}^{q-2} \sum_{\ell=0}^{N-1} (-1+\psi_{\mathbb{F}_{q^3}}(eC_0))G(\chi_2^{-i}\chi_1^{-\ell})\chi_2^i\chi_1^{-\ell}(z)\chi_1^\ell(ab)\chi_2^i(ab^{-1}).\end{aligned} \tag{4.6.9}$$

Here, by (4.6.2) and Lemma 4.6.2, continuing from (4.6.9), we have

$$\Sigma_1 = \frac{N-1}{3(q^3-1)} \sum_{i=1}^{q-2} \sum_{\ell=0}^{N-1} \sum_{z \in \mathbb{F}_{q^3}^*} \chi_2^{-i}\chi_1^{-\ell}(z)\chi_1^\ell(ab)\chi_2^i(ab^{-1})\psi_{\mathbb{F}_{q^3}}(z\mathbb{F}_q^*) \tag{4.6.10}$$

$$- \frac{1}{3(q^3-1)} \sum_{e \in \mathbb{F}_q^*} \sum_{z \in \mathbb{F}_{q^3}^*} \sum_{i=1}^{q-2} \sum_{\ell=0}^{N-1} \chi_2^{-i}\chi_1^{-\ell}(z)\chi_1^\ell(ab)\chi_2^i(ab^{-1})\psi_{\mathbb{F}_{q^3}}(-zeR) \tag{4.6.11}$$

$$+ \frac{1}{3(q^3-1)} \sum_{e \in \mathbb{F}_q^*} \sum_{z \in \mathbb{F}_{q^3}^*} \sum_{i=1}^{q-2} \sum_{\ell=0}^{N-1} \psi_{\mathbb{F}_{q^3}}(eC_0)\chi_2^{-i}\chi_1^{-\ell}(z)\chi_1^\ell(ab)\chi_2^i(ab^{-1})\psi_{\mathbb{F}_{q^3}}(-zeR). \tag{4.6.12}$$

We denote the summands in (4.6.10), (4.6.11) and (4.6.12) by Ω_1, Ω_2 and Ω_3, respectively. Then $\Sigma_1 = \Omega_1 + \Omega_2 + \Omega_3$.

Since $\psi_{\mathbb{F}_{q^3}}(z\mathbb{F}_q^*) = \sum_{y\in\mathbb{F}_q^*} \psi_{\mathbb{F}_q}(\mathrm{Tr}_{q^3/q}(z)y)$, we have

$$\begin{aligned}
\Omega_1 =& \frac{N-1}{3(q^3-1)} \sum_{i=1}^{q-2} \sum_{\ell=0}^{N-1} \sum_{z\in\mathbb{F}_{q^3}^*} \chi_2^{-i}\chi_1^{-\ell}(z)\chi_1^{\ell}(ab)\chi_2^{i}(ab^{-1}) \sum_{y\in\mathbb{F}_q^*} \psi_{\mathbb{F}_q}(\mathrm{Tr}_{q^3/q}(z)y) \\
=& \frac{(N-1)(q-1)}{3(q^3-1)} \sum_{i=1}^{q-2} \sum_{\ell=0}^{N-1} \sum_{z\in T_0} \chi_2^{-i}\chi_1^{-\ell}(z)\chi_1^{\ell}(ab)\chi_2^{i}(ab^{-1}) \\
&- \frac{N-1}{3(q^3-1)} \sum_{i=1}^{q-2} \sum_{\ell=0}^{N-1} \sum_{z\in\mathbb{F}_{q^3}^*\setminus T_0} \chi_2^{-i}\chi_1^{-\ell}(z)\chi_1^{\ell}(ab)\chi_2^{i}(ab^{-1}) \\
=& \frac{q(N-1)}{3(q^3-1)} \sum_{i=1}^{q-2} \sum_{\ell=0}^{N-1} \sum_{z\in T_0} \chi_2^{-i}\chi_1^{-\ell}(z)\chi_1^{\ell}(ab)\chi_2^{i}(ab^{-1}).
\end{aligned}$$

We next evaluate Ω_2. Note that

$$\begin{aligned}
\sum_{e\in\mathbb{F}_q^*} \psi_{\mathbb{F}_{q^3}}(-zeR) =& \sum_{e\in\mathbb{F}_q^*} \sum_{\lambda\in\mathbb{F}_q} \sum_{h\in L_0} \psi_{\mathbb{F}_{q^3}}(-ze(\lambda + h^{q^2} - h^q)) \\
=& \sum_{e\in\mathbb{F}_q^*} \sum_{\lambda\in\mathbb{F}_q} \sum_{h\in L_0} \psi_{\mathbb{F}_q}(-e\lambda \mathrm{Tr}_{q^3/q}(z))\psi_{\mathbb{F}_{q^3}}(-ze(h^{q^2} - h^q)).
\end{aligned}$$

If $\mathrm{Tr}_{q^3/q}(z) \neq 0$, we have $\sum_{e\in\mathbb{F}_q^*} \psi_{\mathbb{F}_{q^3}}(-zeR) = 0$. If $\mathrm{Tr}_{q^3/q}(z) = 0$, we have

$$\begin{aligned}
\sum_{e\in\mathbb{F}_q^*} \psi_{\mathbb{F}_{q^3}}(-zeR) =& \sum_{e\in\mathbb{F}_q^*} \sum_{\lambda\in\mathbb{F}_q} \sum_{h\in L_0} \psi_{\mathbb{F}_{q^3}}(-ze(h^{q^2} - h^q)) \\
=& \sum_{e\in\mathbb{F}_q^*} \sum_{\lambda'\in\mathbb{F}_q} \sum_{h\in C_0} \psi_{\mathbb{F}_{q^3}}(-eh(z^q - z^{q^2}) + h\lambda') \\
=& \sum_{x\in\mathbb{F}_{q^3}^*} \sum_{\lambda'\in\mathbb{F}_q} \psi_{\mathbb{F}_{q^3}}(x(\lambda' + z^{q^2} - z^q)) = -q,
\end{aligned}$$

where the last equality follows from $z^{q^2} - z^q \notin \mathbb{F}_q$. Therefore, we have

$$\Omega_2 = \frac{q}{3(q^3-1)} \sum_{z\in T_0} \sum_{i=1}^{q-2} \sum_{\ell=0}^{N-1} \chi_2^{-i}\chi_1^{-\ell}(z)\chi_1^{\ell}(ab)\chi_2^{i}(ab^{-1}).$$

We next evaluate Ω_3. Note that

$$\begin{aligned}
\sum_{e\in\mathbb{F}_q^*} \psi_{\mathbb{F}_{q^3}}(eC_0)\psi_{\mathbb{F}_{q^3}}(-zeR) =& \sum_{e\in\mathbb{F}_q^*} \sum_{y\in C_0} \sum_{\lambda\in\mathbb{F}_q} \sum_{h\in L_0} \psi_{\mathbb{F}_{q^3}}(ey - ze(\lambda + h^{q^2} - h^q)) \\
=& \sum_{e\in\mathbb{F}_q^*} \sum_{y\in C_0} \sum_{\lambda\in\mathbb{F}_q} \sum_{h\in L_0} \psi_{\mathbb{F}_q}(-e\lambda \mathrm{Tr}_{q^3/q}(z))\psi_{\mathbb{F}_{q^3}}(ey - ze(h^{q^2} - h^q)).
\end{aligned}$$

If $\mathrm{Tr}_{q^3/q}(z) \neq 0$, we have $\sum_{e\in\mathbb{F}_q^*} \psi_{\mathbb{F}_{q^3}}(eC_0)\psi_{\mathbb{F}_{q^3}}(-zeR) = 0$. If $\mathrm{Tr}_{q^3/q}(z) = 0$, we have

$$\begin{aligned}\sum_{e\in\mathbb{F}_q^*} \psi_{\mathbb{F}_{q^3}}(eC_0)\psi_{\mathbb{F}_{q^3}}(-zeR) =& \sum_{e\in\mathbb{F}_q^*}\sum_{y\in C_0}\sum_{\lambda\in\mathbb{F}_q}\sum_{h\in L_0} \psi_{\mathbb{F}_{q^3}}(ey - ze(h^{q^2} - h^q)) \\ =& \sum_{e\in\mathbb{F}_q^*}\sum_{\lambda'\in\mathbb{F}_q}\sum_{h,y\in C_0} \psi_{\mathbb{F}_{q^3}}(ey - eh(z^q - z^{q^2}) + h\lambda') \\ =& \sum_{e\in\mathbb{F}_q^*}\sum_{\lambda'\in\mathbb{F}_q}\sum_{h,y\in C_0} \psi_{\mathbb{F}_{q^3}}(ey(1 - y^{-1}h(z^q - z^{q^2}) + y^{-1}he^{-1}\lambda')) \\ =& \sum_{x\in\mathbb{F}_{q^3}^*}\sum_{\lambda\in\mathbb{F}_q}\sum_{h\in C_0} \psi_{\mathbb{F}_{q^3}}(x(1 - h(z^q - z^{q^2}) + h\lambda)) \\ =& q^3\mu_z - qN.\end{aligned}$$

Hence we have

$$\Omega_3 = \frac{1}{3(q^3-1)}\sum_{z\in T_0}\sum_{i=1}^{q-2}\sum_{\ell=0}^{N-1}(q^3\mu_z - qN)\chi_2^{-i}\chi_1^{-\ell}(z)\chi_1^{\ell}(ab)\chi_2^{i}(ab^{-1}).$$

Summing up Ω_i, $i = 1, 2, 3$, above, we obtain

$$\begin{aligned}\Sigma_1 =& \frac{q^3}{3(q^3-1)}\sum_{z\in T_0}\sum_{i=1}^{q-2}\sum_{\ell=0}^{N-1}\mu_z\chi_1^{\ell}(abz^{-1})\chi_2^{i}(ab^{-1}z^{-1}) \\ =& \frac{q^3}{3(q^3-1)}\sum_{w^{Nk+(q-1)j}\in T_0}\sum_{i=0}^{q-2}\sum_{\ell=0}^{N-1}\mu_z\chi_1^{\ell}(w^{(q-1)(t_0-j)})\chi_2^{i}(w^{N(u_0-k)}) \\ &- \frac{q^3}{3(q^3-1)}\sum_{w^{Nk+(q-1)j}\in T_0}\sum_{\ell=0}^{N-1}\mu_z\chi_1^{\ell}(w^{(q-1)(t_0-j)}) \\ =& \begin{cases} \frac{q^3}{3}\mu_{z_0} - \frac{q^3}{3(q-1)}\sum_{k=0}^{q-2}\mu_{w^{Nk+(q-1)t_0}}, & \text{if } w^{(q-1)t_0}\in T_0, \\ 0, & \text{if } w^{(q-1)t_0}\notin T_0,\end{cases}\end{aligned}$$

Finally, we need to compute $\sum_{k=0}^{q-2}\mu_{w^{Nk+(q-1)t_0}}$ under the assumption that $w^{(q-1)t_0}\in T_0$. Let $x_0 = w^{(q-1)t_0}$. Since $\mathrm{N}_{q^3/q}(\lambda + x_0^q - x_0^{q^2}) \neq 0$ for any $\lambda\in\mathbb{F}_q$, we have

$$\begin{aligned}\sum_{\theta\in\mathbb{F}_q^*}\mu_{x_0\theta} =& \sum_{\theta\in\mathbb{F}_q^*}\#\{(y,\lambda)\in C_0\times\mathbb{F}_q : y = \theta(x_0^q - x_0^{q^2}) - \lambda\} \\ =& \sum_{\theta\in\mathbb{F}_q^*}\#\{\lambda\in\mathbb{F}_q : \mathrm{N}_{q^3/q}(\lambda + x_0^q - x_0^{q^2}) = \theta^{-3}\} \\ =& \#\{\lambda\in\mathbb{F}_q : \mathrm{N}_{q^3/q}(\lambda + x_0^q - x_0^{q^2})\in\mathbb{F}_q^*\} = q.\end{aligned}$$

Therefore,

$$\Sigma_1 = \begin{cases} \frac{q^3}{3}\mu_{z_0} - \frac{q^4}{3(q-1)}, & \text{if } w^{(q-1)t_0} \in T_0, \\ 0, & \text{if } w^{(q-1)t_0} \notin T_0. \end{cases}$$

Similarly, by noting that

$$\begin{aligned} G(\chi_2^{-i}\chi_1^{-\ell}) \sum_{x \in D_2} \chi_2^i \chi_1^\ell(x) &= \sum_{z \in \mathbb{F}_{q^3}^*} \chi_2^{-i}\chi_1^{-\ell}(z)\psi_{\mathbb{F}_{q^3}}(zD_2) \\ &= \sum_{z \in \mathbb{F}_{q^3}^*} \chi_2^{-i}\chi_1^{-\ell}(z)\psi_{\mathbb{F}_{q^3}}(z\beta^{-1}D_3), \end{aligned}$$

we have

$$\Sigma_2 = \begin{cases} \frac{q^3}{3}\mu'_{z_1} - \frac{q^4}{3(q-1)}, & \text{if } w^{(q-1)t_0} \in T_0, \\ 0, & \text{if } w^{(q-1)t_0} \notin T_0. \end{cases}$$

Finally, we evaluate Σ_3. Recall that

$$\Sigma_3 = -\frac{1}{3(q^3-1)} \sum_{i=1}^{q-2} \sum_{\ell=0}^{N-1} G(\chi_2^i\chi_1^{-\ell})G(\chi_2^{-i}\chi_1^{-\ell})\chi_1^\ell(ab)\chi_2^i(ab^{-1}) \sum_{z \in T_0} \chi_2^i\chi_1^\ell(z).$$

Since for $i \neq 0$

$$\begin{aligned} \sum_{z \in T_0} \chi_2^i\chi_1^\ell(z) &= \sum_{x \in L_0} \sum_{y \in \mathbb{F}_q^*} \chi_2^i(xy)\chi_1^\ell(xy) \\ &= \left(\sum_{x \in L_0} \chi_1^\ell(x) \right) \left(\sum_{y \in \mathbb{F}_q^*} \chi_2^i(y) \right) = 0, \end{aligned}$$

we have $\Sigma_3 = 0$. This completes the proof of the proposition. □

Chapter 5

CONCLUDING REMARKS

In this work, we have constructed Cameron-Liebler line classes in $\mathrm{PG}(3,q)$ with parameter $x=(q+1)^2/3$ for all prime powers q congruent to 2 modulo 3. This is a contribution to the study of the central problem about Cameron-Liebler line classes in $\mathrm{PG}(3,q)$. Besides the trivial examples with $x=1,2$, all known infinite families of Cameron-Liebler line classes prior to our work have parameters $x=(q^2-1)/2$ or $x=(q^2+1)/2$, up to complement.

Most notably, we have constructed the first infinite family of nontrivial Cameron-Liebler line classes in $\mathrm{PG}(3,q)$ with q even. In contrast, the first nontrivial infinite family of Cameron-Liebler line classes in $\mathrm{PG}(3,q)$ for odd q was constructed by Bruen and Drudge [16] twenty years ago. The major obstacle in the characteristic two case seems to be that such line classes, if they exist, tend not to be highly symmetric. In our construction, the Cameron-Liebler line classes have automorphism groups of medium sizes. This fact makes it difficult to give a neat geometric description of the objects we have constructed. Our proof is very algebraic, due to the nature of our construction.

In Section 4.5, we have determined the stabilizers of our Cameron-Liebler line classes in $\mathrm{PSL}(4,q)$. The size of the stabilizer is $3(q^2+q+1)$. It will be of particular interest to find infinite families of Cameron-Liebler line classes whose stabilizers in $\mathrm{PSL}(4,q)$ do not grow as q increases.

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