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Backward transfer, the relationship between new learning and prior ways of reasoning, and action versus process views of linear functions

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Abstract

Backward transfer is defined as the influence that new learning has on individuals' prior ways of reasoning. In this article, we report on an explora-tory study that examined the influences that quadratic functions instruction in real classrooms had on students' prior ways of reasoning about linear functions. Two algebra classes and their teachers at two comprehensive high schools served as the participants. Both schools drew from low- socioeconomic urban populations. The study involved paper-and-pencil assessments about linear functions that were administered before and after a four- to five-week instructional unit on quadratic functions. The teachers were instructed to teach the quadratic functions unit using their regular approach. Qualitative analysis revealed three kinds of backward transfer influences and each influence was related to a shift in how the students reasoned about functions in terms of an action or process view of functions. Additionally, features of the instruction in each class provided plausible explanations for the similarities and differences in backward transfer effects across the two classrooms. These results offer insights into backward transfer, the relationship between prior knowledge and new learning, aspects of reasoning about linear functions, and instructional approaches to teaching functions.

Keywords

Transfer; functions; linear; quadratic; action view; process view

A well-established idea from mathematics education research is that "learning proceeds primarily from prior knowledge" (Roschelle, 1995, p. 37). In other words, learners' prior ways of reasoning are the foundation on which new knowledge is constructed. Also well-established is the idea that prior ways of reasoning continue to evolve over time (e.g., Moschkovich, 1998; Smith et al., 1994). Together, these ideas suggest a dynamic relationship exists in which learners construct new knowledge on a foundation of prior ways of reasoning that are themselves evolving (Roschelle, 1995).

The dynamics of this relationship is what we developed scientific curiosity about. Specifically, we sought to understand the influences that new learning can have on learners' still-developing prior ways of reasoning. These kinds of influences have, as yet, received relatively little attention from mathematics education researchers, especially in real classrooms, despite having potentially far-ranging implications for research and practice.

We conceptualize the influences described above as a type of transfer of learning called *backward transfer*. This type of transfer has been reported by several education researchers (e.g., Bagley et al., 2015; Gentner et al., 2004; Hohensee, 2014; Lima & Tall, 2008; Marton, 2006; Melhuish & Fagan, 2018; Moore, 2012). There also exists a body of research on backward transfer in the domain of second-language learning (e.g., Cook,2003).

We chose linear and quadratic functions as the context in which to study backward transfer and looked at how students' ways of reasoning about linear functions change when they learn about quadratic functions. Using Breidenbach et al.'s (1992) distinction between *action* and *process views* of functions as our lens, we addressed the following research questions: (a) In what ways are algebra students' prior ways of reasoning about linear functions with an action and/or a process view influenced by backward transfer, if at all, by their participation in an instructional unit on quadratic functions?; and (b) In what ways do particular instructional approaches to teaching quadratic functions offer plausible explanations for the changes in students' prior ways of reasoning about linear functions?

Theoretical orientation

In this section, we present our two-part theoretical orientation. First, we present our orientation toward backward transfer. Next, we present our orientation to functions.

Theoretical orientation toward backward transfer

Transfer of learning is the overarching category to which both *backward* and *forward transfer* belong (Gentner et al., 2004). Forward transfer, the more studied direction, is often referred to simply as *transfer* (e.g., Barnett & Ceci, 2002). Various perspectives have conceptualized forward transfer (e.g., Bransford & Schwartz, 1999; Lobato, 2012; Singley & Anderson, 1989; Thorndike & Woodworth, 1901). It was Lobato's (2012) *actor-oriented transfer* (AOT) perspective, which defined transfer as "the influence of a learner's prior activities on her activity in novel situations" (p. 223), that helped us conceptualize backward transfer.

According to AOT, forward transfer is when learners in novel situations are influenced by something they learned in the past, regardless of whether or not the influence led to correct application of knowledge. According to AOT, researchers should adopt the learner's (i.e., the actor's) point of view, rather than the observer's point of view, when deciding if forward transfer has occurred, hence the label *actor oriented*.

AOT was created to address critiques of the traditional conceptualization of transfer. Lave's (1988) critique was that the traditional conceptualization "underscores the static quality of transfer in experimental practice: it is treated as a process of taking a given item and applying it somewhere else" (p. 37), which is "a distorted representation of activity in everyday life" (p. 43). Additionally, although "transfer is necessarily a part of our moment-to-moment lives" (Beach, 1999, p. 101), Detterman's (1993) critique was that, for studies using the traditional conceptualization, there exists "little empirical evidence showing meaningful transfer to occur" (p. 21).

AOT addresses both critiques. By counting all influences that learning has on individuals' thinking in new contexts as evidence of forward transfer, not just correct knowledge application, AOT (a) is more consistent with what happens in moment-to-moment situations, and (b) allows for more instances of transfer to be realized (Lobato, 2012). Because we were interested in influences of all kinds, not just those that lead to correct performance or that involve static knowledge application from one context to the next, we used AOT as the foundation on which to build our orientation to backward transfer.

The definition guiding our study, which adapts the AOT definition of forward transfer, is that backward transfer is the influence that new learning has on prior ways of reasoning (Hohensee, 2014). Our definition of backward transfer is like Lobato's (2008) AOT definition of forward transfer, except that ours refers to influences backward *from* new learning *onto* prior ways of reasoning, instead of forward *from* prior activities *onto* new learning.

An illustrate example of backward transfer comes from Bagley et al.'s (2015) study of linear algebra students and their previously-established ways of reasoning about functions. The researchers wondered if learning about linear algebra "had a negative effect on their understanding of function . . . [or]

reinforces and enriches their understanding of function" (p. 36). After studying linear algebra, students were asked, "predict the result of composition of f with f^{-1}" (p. 44). Some students incorrectly concluded "it's just 1" (p. 45), instead of correctly concluding the result would be the identity function x. The researchers attributed this misconception to "backward transfer from the symbolism of linear algebra" (p. 45) (i.e., confusing the identity function with the identity matrix which is populated with ones along the main diagonal). In this example, the backward transfer influence went from the new learning about linear algebra onto students' prior ways of reasoning about functions.

Reasons we looked for backward transfer in real mathematics classrooms

One reason we looked for backward transfer in real mathematics classrooms was to identify backward transfer effects in real classrooms that undermine students' mathematical understanding. Macgregor and Stacey (1997) reported that some students in their study began reasoning incorrectly about algebraic symbols involving multiplication after learning about algebraic symbols involving exponents. Other studies have similarly found these undermining backward transfer influences (e.g., Bagley et al., 2015; Hohensee, 2014; Lima & Tall, 2008; Van Dooren et al., 2004). Identifying *undermining* backward transfer influences represents a first step toward developing instructional approaches that *minimize* those effects.

A second reason we looked for backward transfer in real mathematics classrooms was to identify backward transfer effects in real classrooms that enhance students' mathematical understanding. Arzi et al. (1985) reported that students in their study performed better on a retake of a seventh-grade science final exam after those students had taken eighth-grade science, despite eighth-grade science not revisiting seventh-grade science concepts. Identifying *enhancing* effects represents a first step toward developing instructional approaches that *maximize* these effects.

Although backward transfer effects have been identified in a number of studies, no systematic research efforts have, as yet, intentionally looked for backward transfer effects in real mathematics classrooms. Thus, our exploratory study represents a promising new direction for improving instruction.

Theoretical orientation toward reasoning about functions

Breidenbach et al. (1992) proposed the following two ways of reasoning about functions: as *actions*, and as *processes*.[1] Action view reasoning was defined as that which "emphasize[s] the act of substituting numbers for variables and calculating to get a number, but [does] not refer to any overall process of beginning with a value (numerical or otherwise) and doing something that resulted in a value" (p. 252). Weber (2002) illustrated action view reasoning in the context of $f(x) = b^x$: "repeatedly multiplying by b x times … students will not be able to do much with exponents besides compute these values and manipulate their formulas" (pp. 3–4). In this example, the focus was on computing values.

Process view reasoning was defined as that in which "the input, transformation, and output [are] present, integrated and fairly general" (Breidenbach et al., 1992, p. 252). Asiala et al. (1996) added that process view reasoning is about being able to "reflect on, describe, or even reverse the steps of the transformation without actually performing those steps" (p. 7). Weber (2002) illustrated process view reasoning in the context of $f(x) = 2^x$: "2^x will be a positive function since you start with the integer one and repeatedly multiply this by a positive number; it will be an increasing function since every time x increases by one, 2^x doubles" (p. 4). In this example, the focus was on how the function behaves, not on computing values.

It is critical for students to move from action- to process-view reasoning (Breidenbach et al., 1992; Sfard, 1991; Thompson, 1994), and the transition from action to process view has been labeled as *interiorization* (i.e., "an action … is relatively external to the thinking of the subject, whereas in a process it is more internal;" Breidenbach et al., p. 278). According to Breidenbach et al. and Sfard,

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students begin with an action view and interiorization occurs gradually. Furthermore, interiorization was characterized as a "struggle" that involves "reconstructing previous knowledge" (Breidenbach et al., p. 277). We were interested in whether backward transfer played any role in this process.

Reasons for looking for backward transfer in students' action/process view reasoning

We had three reasons for examining if and how backward transfer effects were connected to students' action- and process-view reasoning about functions. First, action- and process-view reasoning is, for any two kinds of functions, an aspect of mathematical reasoning they share. This is important because prior research has shown that when a particular aspect of reasoning applies to two different concepts, backward transfer can manifest itself in that aspect of reasoning about the earlier-encountered concept after an individual learns about the later-encountered concept (e.g., Hohensee et al., 2021; Lima & Tall, 2008). For example, Lima and Tall observed a student who learned about the quadratic formula (the later encountered concept), and then tried to apply it to reasoning about a linear equation (earlier encountered concept). Lima and Tall refer to this as a *met-after*, and we interpret met-afters as consistent with how we think of backward transfer. Similarly, because action- and process-view reasoning apply to reasoning about any two functions, we hypothesized that conceiving of new functions as actions or processes could transfer backward to reasoning about previously encountered functions.

Second, Breidenbach et al. (1992) claimed that interiorization is not unidirectional: "Many individuals will be in transition from action to process and, as with all cognitive transitions, the progress is never in a single direction" (p. 251). This suggests that students' action- or process-view reasoning might be susceptible to backward transfer influences. Note that backward transfer refers to influences that learning about a new concept has on individuals' prior ways of reasoning, not to the direction that action and process views are developing.

Third, Ed Dubinsky, a coauthor on Breidenbach et al. (1992), referenced backward transfer in the context of the curricular activities called *Trip Line*. A goal of Trip Line is to move students from action- to process-view reasoning. Dubinsky stated:

> When Bob Moses and I were writing the Trip Line, we talked a great deal about difficulties students had when their new knowledge did not seem to them consistent with their previous knowledge . . . many of the epistemological obstacles that permeate the literature are failures to make a backward transfer." (communication, March 11, 2017)

Together, the reasons outlined above motivated us to look at whether backward transfer is associated with action- and process-view reasoning.

Action- and process-view reasoning about linear and quadratic functions

The mathematics contexts in which we situated our study were linear and quadratic functions. Hines (2002) and Slavit (1997) were the lone articles we found that address action and process views of linear functions. Hines described interiorization for linear functions as going from reasoning that involves "repeatable actions, where although the procedures were consistent each time, the focus of . . . attention was not on the general consistency of the actions, but on individual input and output values" (p. 358), to reasoning that involves a "systematic *co-variation* [italics added] between two related variables in which functions are viewed as generalized processes" (p. 340). Covariational reasoning was a prominent part of our study, as will be explained later.

We also located just two articles addressing action and process views of quadratic functions, Childers and Vidakovic (2014), and Slavit (1997). Childers and Vidakovic described a student reasoning about quadratic functions with an action view as "she remembered the formula for finding the x-coordinate of the vertex, was able to find the y-coordinate by plugging into the formula and calculating it, and she knew that concavity is determined by the sign of the coefficient of x^2" (p. 13), and characterized a process view of quadratic functions as the "ability to transfer the idea of the vertex from the explicit problem to the real-world problem correctly . . . these actions have been interiorized and she has a process level of understanding of the vertex in the real-world problem" (p. 15). This is particularly relevant for our study, because the quadratic functions instruction we observed often focused on vertices.

Reasons for choosing linear and quadratic functions for a study on backward transfer

We chose linear and quadratic functions for our study for several reasons. First, linear functions are typically covered in curricula before quadratic functions (e.g., Discovering Algebra, Key Curriculum Press; CPM; CCSSM). With this order, ways of reasoning about linear functions would serve as the prior ways of reasoning that may *experience* a backward transfer influence and new learning about quadratic functions would serve as the source of the influence. If, in real classrooms, evidence of backward transfer for linear and quadratic functions was found, it would hold significance for the predominance of algebra curricula.

Second, backward transfer between concepts too similar is trivial, and between two concepts too dissimilar is implausible (c.f. Barnett & Ceci's, 2002 discussion of near and far transfer). Linear and quadratic functions are sufficiently similar that backward transfer is plausibleand sufficiently dissimilar that backward transfer would be non-trivial. Moreover, studies in lab settings have already found evidence of backward transfer in the context of linear and quadratic functions (e.g., Hohensee, 2014; Lima & Tall, 2008), which adds to the plausibility that backward transfer involving linear and quadratic functions may be happening in real classroom settings. Finally, the relationship between students' ways of reasoning about linear and quadratic functions has been examined in the forward-transfer direction (e.g., Ellis & Grinstead, 2008; Zaslavsky, 1997). We extend the research described above by examining linear and quadratic functions in the backward transfer direction in real classrooms.

Prior research on teaching and learning about functions

Considerations pertaining more broadly to teaching and learning about functions were also relevant for our study. The first consideration was about strategies students use to find missing values of functions. Studies have shown that strategies are acquired in particular orders (Ayan et al., 2000; Hunt, 2015; Kaput & West, 1994). For instance, students solve missing-value problems using *buildup reasoning* before *abbreviated buildup reasoning* (Kaput & West, 1994).

Buildup reasoning involves finding a desired new coordinate pair for a linear function from a given coordinate pair by repeatedly adding. In our study, students applied the buildup strategy by adding the numerator of the unit rate of change for the function to the given dependent variable value and, simultaneously, repeatedly adding the denominator of the unit rate of change (i.e., 1 unit of independent variable) to the given independent variable value, until the desired new independent variable value was reached (Kaput & West, 1994).[2] In contrast, abbreviated buildup reasoning involves multiplying a constant unit rate of change by the desired independent variable value to find a desired new dependent variable value. Note, however, that when the linear function has a non-zero y-intercept, abbreviated buildup reasoning requires the additional step of adding the non-zero y-intercept to the product of the multiplication.

The second consideration was about whether students can find inverses of functions. Greer (2012) explained that "inverse functions are of great importance within algebra" (p. 432). However, research has shown that coming to understand inverted linear functions can be a challenge for students. For example, Cedillo (2001) found "only a few students were able to find a systematic way of inverting linear functions" (p. 243). Reasoning about inverting linear functions played a role in our study.

The third consideration was about generalizing. The type of generalizing relevant for our study, called *extending* (Ellis, 2007), is when "a student not only notices a pattern or a relationship of similarity, but then expands that pattern or relationship into a more general structure" (p. 241). In one example, a student realized that a line on a graph need not be restricted "to the first quadrant, but could extend it beyond the few points . . . [She] extended her reasoning by expanding the range of applicability" (p. 243). Extending showed up in our study.

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Given that little research exists about backward transfer in the context of functions in real classrooms, our exploratory study sought to generate hypotheses about backward transfer that could subsequently become the focus of study in future research. This aligns with Sloane's (2008) description of exploratory/basic research as "provid[ing] the intellectual fodder, in the form of hypotheses, for more rigorous inquiries" (p. 627).

Methods

The participants in our study came from two tenth-grade Integrated Mathematics 2 classes at different high schools located in the Mid-Atlantic region of the United States (*N*= 57). The majority of students came from a larger urban center and the schools were approximately 40% African American, 18% Hispanic, and 6% Asian. Additionally, according to the school demographics we were provided, approximately one-third of the student population at each school belonged to a family with sufficiently low socioeconomic status that they received breakfast and/or lunch during the school day at no cost or at a reduced cost. Although we were not provided with demographic information for individual students, both teachers reported that the demographic make-ups of the participating classes reflected the demographics of the school. The teachers in our study, Ms. H and Mr. A, had 8- and 17-years teaching experience, respectively. Ms. H taught 24 of our participants in 70 min. class periods. Mr. A taught 33 of our participants on a rotating schedule in which classes varied in length from 45 to 80 min.

Phases of data collection

This exploratory study had three data-collection phases: pre-instruction, instruction, and post-instruction. During the pre-instruction phase, we administered a 45-min. paper-and-pencil pre-assessment on linear functions. Additionally, we interviewed four randomly selected students per class, one-on-one, about their responses on the pre-assessment. During the instruction phase, students participated in a 12 lesson (Mr. A) or 17 lesson (Ms. H) quadratic functions unit. All lessons were observed by the research team. During the post-instruction phase, we administered a post-assessment on linear functions that mirrored the pre-assessment. We also interviewed the same four students per class about the post-assessment, using the same interview protocol.

Linear functions pre- and post-assessments

We designed two versions of the linear-functions assessment. Half the students were randomly assigned to take Version A as the pre-assessment and Version B as the post-assessment. The remaining students were assigned the assessments in reverse order.

The pre- and post-assessment problems focused on in this article were specifically designed to examine students' reasoning about linear functions in terms of action or process views.[3] Versions A and B had the same number of problems, the same order of problems, and the same underlying mathematics. The problem context for Version A was about a plant growing at a constant rate, whereas for Version B, the problem context was about a container filling with rainwater at a constant rate. Each version included a picture of the context (see Table 1 for the reasoning each problem was designed to examine).

Quadratic functions instruction

Both teachers primarily used a teacher-centered approach during the quadratic functions instruction phase. However, during practice time and whole-class discussions, the teachers often used a more student-centered approach. As for the classroom environments, students appeared comfortable with asking their teachers or each other for help, and with going in front of the class to show and explain their work. Both teachers effectively communicated to

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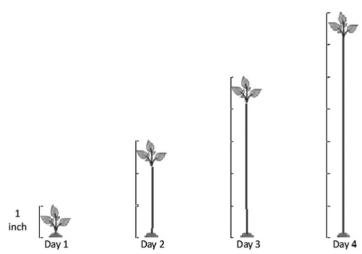
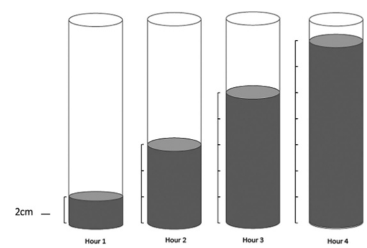
students that they were required to participate, regularly monitored students' progress, and held students accountable for their levels of participation.[4] In the results section, we present a deeper comparison of how quadratic functions were taught.

Data set

The data set consisted of written pre- and post-assessments, video-recorded one-on-one interviews, and observations of the quadratic functions instruction. In all, 114 assessments were completed (i.e., 2 assessments/student for 57 students); 16 one-on-one interviews were conducted (i.e., 4 interviews/assessment/class for 2 assessments and 2 classes); and 29 sets of observation field notes were created (i.e., 17 from Ms. H's class and 12 from Mr. A's class).

Table 1. Action vs process views problems on Version A and B of pre- and post-assessments.

	Growing Plant Problems Version A	Rain Water Problems Version B
Intro	The following diagram shows the height of a plant in inches each day that it is measured.	As part of a science project, you are measuring the amount of rain that falls during a storm. The following diagram shows the amount of rainwater you collected in the first four hours. Use the diagram to answer the questions below.
Picture		
Prob. (a)	Explain in words how to find the height of the plant on day 17.	Explain in words how to find the total amount of rainfall if the storm lasts for 11 hours.
Prob. (b)	Can you find the day the plant was measured if you were given the height? If yes, explain how. If no, explain why not.	Can you find the hour the rainwater was measured if given the height? If yes, explain how. If no, explain why not.
Prob. (c)	You have to leave the plant in your office over the weekend. You did not measure the plant for 2.5 days. The plant grows at the same rate the whole time. How much did the plant grow in the 2.5 days you were gone? Show any work that helped you decide.	You fall asleep while watching TV. You did not measure the rainwater for 3.5 hours. It rained the whole time at the same rate. How much rainwater was collected during the 3.5 hours that you were sleeping? Show any work that helped you decide.

Description of Reasoning each Problem was Designed to Examine (not shared with students)
Problem (a) was designed to examine how students reason about finding dep. variable values from indep. variable values of a linear function. Calculations-heavy explanations align with action-view reasoning.
Problem (b) was designed to examine how students reason about reversing a linear function (i.e., finding indep. variable values from dep. variable values) when not given specific values of the dep variable to work with. Reasoning that reverses a function without relying on specific values aligns with process-view reasoning.
Problem (c) was designed to examine how students reason to find the size of the interval of the dep. variable that corresponds to a given size of the indep. variable, when the interval of the indep. variable of a linear function has a given size but is not specific to a location on the function. Reasoning about corresponding intervals of a given size in ways that do not rely on a specific location on a function aligns with functions-as-processes reasoning.

Table 2. Description of codes for changes in student reasoning.

Code	Problem and Description of Code
Buildup vs. Abbreviated Buildup Process	Problem (a): To find the dep. variable from the indep. variable of a linear function, students changed from buildup process reasoning to abbreviated buildup process reasoning, or vice versa.
Not Reversing vs. Reversing the Function	Problem (b): To find the indep. variable from the dep. variable of a linear function, student changed from reasoning in ways that do not reverse the function to reasoning in ways that do reverse the function, or vice versa.
Intervals Tied to Specific Locations vs. Intervals not Tied to Specific Locations	Problem (c): To find the size of the interval of dep. variable that corresponds to a given interval size of the indep. variable, students changed from reasoning about intervals tied to specific locations on the function to reasoning about intervals in ways not tied to specific locations, or vice versa.

Data analysis

Data analysis for this study was conducted in three stages. During the first stage, we compared just the pre- and post-assessment and interview responses for the eight interviewed students. From these comparisons, we developed an initial set of codes that captured how reasoning changed from pre to post in terms of action and process views of functions. Codes were informed by a priori concepts contained in the action versus process view literature, and "other codes emerge[d] progressively during data collection" (Miles et al., 2014, p. 74). To each student's assessment response for each problem, and for each corresponding interview response, we assigned at most one change in reasoning code (i.e., each student's response to a particular problem was the unit of analysis). The initial codes and their supporting evidence were discussed and clarified among our research team until a shared understanding was achieved.

During the second stage of analysis, each research-team member was randomly assigned to code two-thirds of the non-interviewed students' assessment data, so that each student's responses were independently coded by two members of the research team. We started using the codes developed in the first stage of analysis. Codes were assigned when there was sufficient evidence to determine that a change in reasoning with respect to action and process views of functions had occurred from pre- to post-assessment. No codes were assigned to a problem when the student provided insufficient evidence to convince us a change in reasoning had occurred or when the change wasn't associated with action and process views of functions.

The research team also met in pairs during this stage of analysis to discuss assessments the pair both coded. When a pair could not reach consensus on a student's response to a particular problem, the third team member was brought in to help achieve consensus. Throughout the second stage of analysis, we engaged in *constant comparison* (Strauss & Corbin, 1994), meaning that codes were continuously revised and refined and new codes created when changes in reasoning that were identified were not adequately captured by codes developed in the first stage of analysis (see, Table 2 for the final codes). After reaching agreement on the codes, we recoded all assessments, including those from the interviewed students, to ensure coding consistency.

To conclude the second stage, we tabulated frequencies for each code and looked for patterns in the changes in reasoning. Note that, even though we think backward transfer could manifest itself differently for different students, for this study we operationalized backward transfer as when patterns of changes in reasoning were identified (i.e., a number of students exhibited a particular change in reasoning), and the changes closely associated with action and process views of functions. Examples of changes in reasoning we observed but did not include in our findings were when students changed from not using to using an equation and from not explicitly to explicitly accounting for a non-zero y-intercept.

During the third stage of analysis, we analyzed observation fieldnotes to identify mathematical foci for each quadratic functions lesson. Then, we compared the foci for each lesson with the observed patterns of changes in reasoning identified in the second stage. We did this to generate plausible

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explanations for the changes in reasoning, which was a goal of our exploratory study. This goal aligns with the approach to scientific explanation that Maxwell (2004) called *process theory*, which is "an analysis of the causal processes by which some events influence others" (p. 5). This goal is also consistent with the National Research Council's (2002) view that educational research "that explores students' and teachers' in-depth experiences, observes their actions, and documents the constraints that affect their day-to-day activities provides a key source of generating plausible causal hypotheses" (p. 109).

Results

To address our first research question, we present three kinds of changes in students' reasoning about linear functions that, according to our interpretation, involved shifts from an action to a process view or vice versa. The three kinds of changes involve: (a) finding linear function dependent variable values, (b) reversing the steps of a linear function, and (c) reasoning about intervals of a linear function. We will show that patterns of changes sometimes cut across classes and sometimes did not. To address RQ 2, we explain how similarities and dissimilarities between the instructional approaches to quadratic functions in the two classes helped us generate plausible explanations for the changes in students' reasoning about linear functions.

Changes in reasoning involving finding linear function dependent variable values

One pattern of changes in reasoning involved finding a value of the dependent variable of a linear function for a corresponding given value of the independent variable (see Prob(a) in Table 1). Several students, across both classes (i.e., 41%), changed from using *buildup* reasoning to using *abbreviated buildup* reasoning, or vice versa (Kaput & West, 1994). We illustrate this change in reasoning, present frequencies of students who exhibited this change, and provide an interpretation of the change in terms of action versus process views of linear functions.

Illustrating this change in reasoning

To illustrate how students' reasoning changed, consider Prob(a) responses from Isaac (Ms. H's student). On the pre-assessment, Prob(a) asked "explain in words how to find the total amount of rainfall if the storm lasts 11 hours." Isaac correctly wrote down pairs of heights and hours, starting at 6 cm for hour 2 until he reached hour 11. This reasoning was consistent with buildup reasoning.

On Prob(a) of the post-assessment, Isaac's reasoning changed. Prob(a) asked "explain in words how to find the height of the plant on day 17." Isaac correctly wrote that the plant grew 2 inches/day and then multiplied the 2 by 17 (and added 1, presumably to account for the non-zero y-intercept). Thus, this reasoning involved multiplying the unit rate, which is consistent with abbreviated buildup reasoning (plus doing the extra step to deal with the y-intercept). Thus, Isaac exhibited a change from the buildup reasoning he had exhibited on the pre-assessment.

Also consider Prob(a) responses from Abby (Mr. A's student). On the pre-assessment, Abby responded to the plant problem by recording a set of heights and days, starting with 1 inch on day 1, and each day adding 2 inches to the height until she arrived at 33 inches for day 17. Abby then wrote, "day 17 will be 17 inches [sic] because for every day it ages it grows two inches." Although Abby incorrectly wrote "17," instead of 33 inches as her answer, her multi-step reasoning aligned with buildup process reasoning.

On the Prob(a) of the post-assessment, Abby's reasoning changed.[5] Specifically, on the plant problem Abby wrote the following: "The plant has a linear system of $2x - 1$; 2 is the m because it is the amount of inches grown per day. x is the # of day and the −1 is what the plants height was at day 0." Abby also drew arrows to x in $2x - 1$ and labeled it "the day" and to −1 and labeled it "if it's a two when it is at day 1, the plant is an inch below the ground." Finally, Abby provided the following two examples of her reasoning, "D1: $2(1) - 1 = 1$" and "D3: $2(3) - 1 = 5$." Even though Abby did not provide a final height for the plant, our interpretation was that Abby was no longer using buildup reasoning but multiplying the day number by 2 and subtracting 1, which is more consistent with abbreviated buildup reasoning.

Frequency of students exhibiting this change in reasoning

Twenty-three of Ms. H's students and 20 of Mr. A's students provided sufficient responses to Prob(a) of the pre- and post-assessments for us to determine if this change in reasoning had occurred. Of these 43 students, 9 students from each class exhibited this change in reasoning. In Ms. H's class, 6 of the 9 students changed *from* buildup *to* abbreviated buildup reasoning. In Mr. A's class, 7 of the 9 students exhibited the same change. The remaining 5 students, 3 from Ms. H's class and 2 from Mr. A's class, changed in the opposite direction.

Interpretation in terms of action and process views of functions

We interpreted the change from buildup to abbreviated buildup reasoning as a shift from action-view toward process-view reasoning. Action-view reasoning is when "the subject will tend to think about [a function] one step at a time" (Breidenbach et al., 1992, p. 251), and is "one of repeatable actions" (Hines, 2002, p. 358). Multi-step buildup reasoning aligns with these descriptions. In contrast, process-view reasoning is when "the input, transformation, and output were present, integrated and fairly general" (p. 252). To us, abbreviated buildup reasoning aligns with a process view because it requires the more general understanding that multi-step buildup reasoning can be collapsed into a single calculation. We note, however, that when using buildup reasoning, students in our study often found the correct answer, whereas when using abbreviated buildup reasoning, sometimes incorrectly accounted for the non-zero y-intercept. This indicates a complexity that needs to be navigated by students when forming a process view of functions.

Changes in reasoning about reversing the steps of a linear function

The second pattern of change involved how students reasoned about finding values of the independent variable of a linear function from given values of the dependent variable (see Prob(b) in Table 1). On one assessment, students *did not* reverse the steps of the function, either by deciding it *was not possible* or by reasoning with values of the *independent* variable to try to produce the given value of the *dependent* variable (i.e., by not reversing the steps). On the other assessment, these students *did* reverse the steps of the function to find a value of the independent variable for a given value of the dependent variable. Several students exhibited this change in reasoning about linear functions (i.e., 33%). We illustrate this change in reasoning, present frequencies of students who exhibited this change, and provide an interpretation of the change in terms of action versus process views of linear functions.

Illustrating this change in reasoning

To illustrate this change in reasoning, consider Prob(b) responses from Reece (Ms. H's student). On the pre-assessment, Reece wrote:

> No unless that start time is given other than that no because if just given the height we do not know how long it has been raining. If the equation is $y = 2x$, and we don't know any x the answer will be inaccurate. (?)

Here, Reece said it was not possible to reverse the steps and was unable to think of a way to find an x–value (indep. variable) for the equation $y = 2x$, if she knew the y value (dep. variable).

On Prob(b) of the post-assessment, Reece's reasoning changed. Reece wrote: "Yes you can find the day if just the height was given by simply solving the equation and finding x." Reece also illustrated reversing the steps by solving for x in the equation $33 = 2x - 1$. We interpreted her "yes" response and her illustration of reversing the steps as evidence of a change toward reasoning it was possible to reverse the steps.

Also consider Prob(b) responses from Arjun (Ms. H's student). On the pre-assessment, Arjun wrote:

> Yes you can because if you add one to the height and divide it by 2 you would get the day for example, if we were given the height of 9 inches and we added one in which we would get 10 and divide it by 2 we would get 5.50 on the 5th day we would have the height of 9 inches.

Despite the calculation error (i.e., $10 \div 2 \neq 5.50$), Arjun's pre-assessment response indicated he reversed the steps of the function for the plant, $y = 2x - 1$, by "added by 1 and divided by 2."

On Prob(b) of the post-assessment, Arjun's reasoning changed. He wrote: "No, because you can determine how much rain fell with within an hour but not the time unless there is a timer put with the collector." We interpreted Arjun's "no" as evidence he was no longer reversing the steps of the function. Thus, in contrast to Reece, Arjun's reasoning changed in the other direction, from it is possible to it is not possible to reverse the steps of a linear function.

Frequency of students exhibiting this change in reasoning

Fifteen of Ms. H's students and 18 of Mr. A's students provided sufficient responses on Prob(b) of the pre- and post-assessments for us to determine if this change in reasoning had occurred. Of these 33 students, 9 in Ms. H's class and 2 in Mr. A's class exhibited this change (i.e., 33%). Of the 11 who exhibited a change, 5 Ms. H's students and the 2 Mr. A's students changed from reasoning it is not to it is possible to reverse the steps. The other 4 Ms. H's students changed in the opposite direction.

Interpretation in terms of action and process views of functions

We interpreted the change from not reversing to reversing the steps of a linear function as a shift from action- to process-view reasoning. We based this interpretation on Asiala et al.'s (1996) description that "an individual who has a process conception . . . [can] reverse the steps of the transformation" (p. 7), and their conclusion that individuals unable to reverse steps would not yet have attained process-view reasoning. Thus, a significant number of Ms. H's students and a small number of Mr. A's students changed in terms of this aspect of reasoning about linear functions.

Changes in reasoning about intervals on a linear function

The third pattern of change involved reasoning about intervals on the dependent variable of a linear function that correspond to given intervals of the independent variable (see Prob(c) in Table 1). There were two main ways of reasoning about interval sizes, and several students, across both classes, changed their reasoning from pre- to post-assessment (i.e., 64%). However, the trends differed for the two classes.

One way students reasoned about intervals was to incorrectly assume that a given interval size of the independent variable was tied to a specific location on a linear function (i.e., starting at a specific day or hour) and that the corresponding interval of the dependent variable was tied to a specific location on the function as well (i.e., at a specific height of the plant or rainwater). In contrast, the other way students reasoned was to correctly assume that a given interval size of the independent variable was not tied to a specific location on the function and that the corresponding interval of the dependent variable was not tied to a specific location either.

Illustrating this change in reasoning

To illustrate this change in reasoning, consider the Prob(c) response from Kira (Mr. A's student). On the pre-assessment, Kira wrote the following: "day 4: plant is 7 in + 2 in" and wrote the total as "9 in." Then, Kira added "+1 in" with an arrow indicating "most likely grow 1 inch in half a day." Finally, Kira wrote " = 10 inches" and circled this phrase. According to our interpretation, Kira was attempting to find the specific height on a specific day (i.e., day 6.5), by adding onto the specific height of 7 inches for the specific day, day 4, rather than finding the change in height over a 2.5-day interval (note she also should have added 4 inches rather than 2 inches for 2 days).

On Prob(c) of the post-assessment, Kira wrote "14 cm of water was collected . . . 4 cm per hour, 3.5 hours." According to our interpretation, Kira reasoned that, at 4 cm per hour for 3.5 hours, 14 cm would collect. This response no longer referenced specific days or heights. We concluded Kira was now reasoning about a 14 cm interval over an unspecified 3.5-hour interval (i.e., she changed to reasoning as if the interval was not tied to a specific location).

Also consider the Prob(c) responses for Kelly (Ms. H's student). On the pre-assessment, Kelly wrote the following: "It grew 5 inches" and "1 day = 2 in," "1 day = 2 in," "half day = 1 in," and " = 5." According to our interpretation, Kelly was reasoning with height and time intervals that were not tied to specific days (i.e., 5 inches over a general 2.5-day interval).

On Prob(c) of the post-assessment, Kelly wrote the specific hours, 4, 5, 6, 7 and 7.5. Then, under each corresponding hour, she wrote the respective specific heights 14, 16, 18, 20 or 21. She also noted that, for each hour, the height increased by 2 cm, whereas from 7 to 7.5 hours, the height increased by 1 cm. Finally, Kelly wrote down and circled 21 cm as the final answer. Thus, Kelly had changed from reasoning about a general interval to reasoning about specific heights (i.e., 14, 16, 18, 20, and 21 cm) at specific hours (i.e., hour 4, 5, 6, 7, and 7.5).

Frequency of students exhibiting this change in reasoning

Of the 34 students that provided a sufficient response to Prob(c) of the pre- and post-assessments for us to determine if this change had occurred, 9 of 13 Ms. H's students and 13 of 21 Mr. A's students exhibited this change reasoning. Six of the 9 Ms. H's students, and 5 of the 13 Mr. A's students changed from reasoning about intervals of linear functions as if they were tied to specific locations to reasoning about general intervals. The remaining students changed in the other direction.

Interpretation in terms of action and process views of functions

According to our interpretation, reasoning with intervals of the independent variable of a linear function in ways that are tied to a specific location on the function is more consistent with an action view. As Slavit (1997) described, "An action conception is concerned with the computation of a single quantity for a single numeric value" (p. 261). Students who are thinking about functions as computations of single outputs for single inputs, would likely think about intervals for inputs and outputs as tied to the specific location on a function on which they were calculated.

In contrast, when students reasoned about independent variable intervals for a linear function as general intervals, we interpreted that as more consistent with a process view. Our interpretation is based on descriptions that process-view reasoning is when learners understand "it is not necessary to perform the operations [of the function], but to only think about them being performed" (Asiala et al., 1996, p. 8). Being able to think about a function without operating on inputs to find outputs is necessary for reasoning about general intervals of linear functions (i.e., those not tied to specific locations). This aligns with Hines (2002) description that a process view involves reasoning about linear functions as generalized processes.

Features of the instructional approaches that may account for changes in reasoning

Our second research question asked, In what ways do particular instructional approaches to teaching quadratic functions offer plausible explanations for the changes in students' prior ways of reason about linear functions? To address this question, we examined observation fieldnotes from the quadratic functions units to identify plausible explanations—as per Maxwell's process theory (Maxwell, 2004) and Sloane's (2008) view of the purpose of exploratory/basic research—for why students' reasoning changed. We focused our analysis for the second question on how the instructional approaches during the quadratic functions units addressed (a) finding values of the dependent variable of a quadratic function, (b) reversing steps of a quadratic function, and (c) reasoning about quadratic functions in ways tied or not tied to specific locations. Our analysis of the fieldnotes revealed plausible explanations for the changes we observed in students' prior ways of reasoning about linear functions.

Finding values of the dependent variable of a quadratic function

Recall that to find values of the dependent variable of a linear function, several students in both classes on one assessment used buildup process reasoning and, on the other assessment, used abbreviated buildup process reasoning. When we examined our fieldnotes from the quadratic functions units, we found that both teachers generated dependent variable values exclusively by substituting values of the independent variable into equations. This approach was used by Mr. A in Lesson 1 (L1) and L2 and by Ms. H in L3, L4, and L5. In neither class was buildup process reasoning exhibited. The following excerpt from Mr. A illustrates how both teachers generated dependent variable values for quadratic functions:

> How would I graph that? . . . Somebody said, put a zero in, well where am I putting a zero in? . . . For your *x*'s. Sort of like what [S] just told us, you plugged them in for the *x*'s to get your *y* value.

We interpreted this approach as more consistent with abbreviated buildup process reasoning because dependent variable values were generated without building up on previous values.

Instead of this approach, the teachers could have used buildup process reasoning that built each successive value of the dependent variable of a quadratic function on the previous value (see the Appendix for an illustration). We hypothesize that the lack of buildup process reasoning was a potential reason why more students changed from using buildup process reasoning on the pre-assessment to using the abbreviate buildup process reasoning on the post-assessment. In other words, we hypothesize that how teachers generate dependent variable values for quadratic functions may influence students as to how they generate dependent variable values when they subsequently reason about linear functions (i.e., a backward transfer influence).

Reversing the steps of quadratic functions

Recall that several students, especially Ms. H's students, reversed the steps of a linear function to (i.e., they solved for the independent variable) on one but not both assessments. When we examined the instructional approaches during the quadratic function units, we found dissimilarities in the ways the teachers reversed steps of quadratic functions. These dissimilarities in the quadratic function instruction may help explain differences in changes in reasoning about linear functions across the two classes.

During the quadratic functions units, the teachers promoted four strategies for reversing the steps of quadratic equations: (a) using the *square root operation* when the side of the equation containing the independent variable is a perfect square (e.g., $(x+4)^2 = 10$); (b) *completing the square* when the side containing the independent variable is not a perfect square (e.g., $x^2 + 6\underline{x} + 8 = 7$); (c) *factoring* and then using the *zero-product property* when the equation is equal to zero and factorable (e.g., $x^2 - 6x + 8 = 0$); and (d) using the *quadratic formula* for any and all quadratic equations.

There were significant dissimilarities between classes with respect to using these strategies. Ms. H used the zero-product property in part of L6 and L7, and the quadratic formula in part of L8, L15, and L16. In contrast, Mr. A used the square root operation in part of L3 and L5, and all of L4; factoring and the zero-product property in part of L5 and L9, and all of L8; completing the square in part of L9 and all of L10; and the quadratic formula in all of L11 and part of L12. Thus, Mr. A offered more experiences reversing the steps of quadratic functions.

More focus on reversing the steps of quadratic equations in Mr. A's class may help explain why Mr. A's students only changed from not reversing the steps of a linear function on the pre-assessment to reversing the steps on the post-assessment. In contrast, the more limited focus on reversing the steps of quadratic equations in Ms. H's class may help explain why similar numbers of her students changed their reasoning about reversing the steps of linear functions from pre- to post-assessment in either direction. In other words, we hypothesize that different levels of focus on reversing the steps in quadratic contexts may lead to different backward transfer influences on how students subsequently reverse the steps in linear contexts.

Reasoning about quadratic functions in ways tied or not tied to specific locations

Recall that several students in both classes reasoned on one assessment about intervals of a linear function in ways tied to specific locations on the function, and on the other assessment in ways not tied to specific locations. When we examined the instructional approaches during the quadratic function units, we found dissimilarities that may help explain the differences in changes in reasoning across the two classes. Specifically, we found dissimilarities that have to do with identifying and reasoning about landmark features of quadratic functions. By *landmark features*, we mean features of a quadratic function that help define the entire function.

In Ms. H's quadratic functions unit, during part of L1, nearly all of L2 and L3, and part of L4, she focused on finding and reasoning about the following four landmark features: the axis of symmetry, the vertex, the maximum or minimum, and the y-intercept. In contrast, in Mr. A's unit, only brief parts of L2 and L3 focused on landmark features. Instead, his unit focused more on calculating non-landmark points of quadratic functions.

According to our interpretation, finding and reasoning about landmark features of quadratic functions is more consistent with finding and reasoning about intervals of linear functions in ways that are not tied to specific locations on the functions. Our reasoning is that landmark features of a function have relevance for the entire function, not just for a specific location on the function. For example, the axis of symmetry has relevance for an entire quadratic function. Ms. H's greater emphasis on landmark features for quadratic functions may help explain why more of Ms. H's students went from reasoning on the pre-assessment about a specific interval on a linear function to reasoning on the post-assessment about a general interval on a linear function. In other words, we hypothesize that emphasizing landmark features of quadratic functions during instruction may have backward transfer influences on how students reason about intervals on linear functions that are not tied to specific locations.

We also note an additional consideration we made regarding intervals at specific locations. Specifically, we considered the property of quadratic functions that for a particular given size of interval of the independent variable, the specific location of that interval will determine the size of interval of the corresponding dependent variable, and we initially wondered if this property would influence students to focus more on specific intervals for linear functions. However, we ended up not considering this a likely influence from either quadratic functions unit because neither teacher focused on intervals during their unit.

Discussion

This study demonstrated that, despite the myriad of reported findings about how prior ways of reasoning influence new learnings, discoveries remain to be made about how new learnings influence prior ways of reasoning (i.e., discoveries about backward transfer). Our study adds to the growing evidence of the relationship between prior ways of reasoning and new learning in the less-researched backward direction (e.g., Bagley et al., 2015; Gentner et al., 2004; Macgregor & Stacey, 1997; Melhuish & Fagan, 2018). Although evidence of backward transfer from quadratic functions instruction to linear functions reasoning already exists (e.g., Hohensee, 2014; Lima & Tall, 2008), previously there has not been a systematic examination of students' linear function reasoning while they learn about quadratic functions in real classrooms. Our findings offer a window into what actually happens. Our exploratory/basic research has also generated hypotheses that need to be tested in future research. This aligns with Sloane's (2008) description of exploratory/basic research:

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> The researcher's or practitioner's intuition leads him or her to some conclusion based on limited data, with myriad alternative hypothesis and with great opportunities to be wrong. The point, however, is that the researcher also has the possibility to be right. (p. 627)

Tests of the hypotheses we generated have the potential to help shape the field's thinking about how knowledge of linear and quadratic functions develops, and perhaps also about how knowledge of mathematics more broadly develops.

We make two additional comments. First, our teachers did not intentionally attend to influences their quadratic functions instruction had on their students' reasoning about linear functions, and thus may have unintentionally produced inconsistent influences. Thus, we find it understandable that students in the same class may have been influenced in different ways. Second, in our experience, reasoning does not just change on a whim, but is usually due to some influence. Thus, we think it more likely that the changes in reasoning found in our study were due to some influence, rather than that they were arbitrary or capricious. Next, we discuss insights our exploratory study generated.

Insights into backward transfer in real algebra classrooms

Our study led us to new insights into ways backward transfer can manifest itself in real classrooms. First, our study suggests backward transfer can manifest itself as students changing strategies to solve problems (e.g., changing from using a buildup process to an abbreviated buildup process). Second, backward transfer can manifest itself as students reversing their conclusions (e.g., reversing from concluding it is not possible to find independent variable values from dependent variable values to concluding it is possible). Third, backward transfer can manifest itself as students changing their interpretation of a problem (e.g., changing how an interval for a function is interpreted). Our findings also have implications for teaching and learning about functions.

Insights into the teaching and learning of functions

Several insights emerged from our study that pertain to the teaching and learning of functions in general. First, our study suggests potential ways students' action versus process views of functions may change and evolve. Breidenbach et al. (1992) pointed out that the process of moving from an action to a process view, also referred to as the process of interiorization, involves "reconstructing previous knowledge to deal with new situations" (p. 277). Our study suggests that interiorization may sometimes involve setbacks, at least temporarily, toward more of an action view.

Second, this study suggests that students' conceptions of inverses of previously encountered functions may be influenced by the learning of new functions. This is important because inverses of functions are of central importance (Greer, 2012), but can be difficult to learn about (Cedillo, 2001). Backward transfer offers a potential mechanism for how students' reasoning about inverses of previously encountered functions can be productively influenced.

Third, this study suggests that instruction about new functions may change how students think about extending patterns or relationships for functions previously encountered. Recall that Ellis (2007) called it *extending* when students increase the range to which a mathematical relationship applies. Students in our study demonstrated extending when they went from interpreting an interval as applying to a specific point on a function to interpreting the interval as general (i.e., applying to the entire function). As per Sloane (2008), these insights from our exploratory/basic research should become the subject of more rigorous inquiry in the future.

Insights into the teaching of quadratic functions

Our study of backward transfer also provides potential insights into teaching quadratic functions specifically. First, as stated above, our study suggests it may sometimes be useful for quadratic functions instruction to engage students in reasoning about quadratic functions that resembles buildup process reasoning with linear functions, so as not to create a backward transfer influence that discourages buildup process reasoning in linear functions contexts (see the Appendix for an illustration). This insight emerged from our observation that students in our study were more often correct on linear function problems with non-zero y-intercepts when using buildup process reasoning. Teachers may be advised to use quadratic functions instruction to encourage some buildup process reasoning, thereby supporting students who have not yet learned how to correctly use abbreviated buildup reasoning in non-zero y-intercept linear function contexts.

Second, our study suggests that teachers should consider engaging students in significant reasoning about landmark features of quadratic functions. This is because of the potential that exists for backward transfer influences to be produced that focus students on general features of linear functions instead of just on specific values (i.e., that promote *extending*). Focusing on general features over specific values is also more consistent with a process view of functions (Breidenbach et al., 1992). These insights about quadratic functions instruction, if implemented together, could create synergistic productive backward transfer effects on students' ways of reasoning about linear functions, and thus need to be subsequently investigated with more rigorous research, as per Sloane (2008).

Insights into the learning of linear functions

Finally, the results from this backward transfer study also provide new insights into how students learn about linear functions. Specifically, our study shows that aspects of reasoning about linear functions that have been reported in the mathematics education literature may be involved in backward transfer effects. For example, Hines's (2002) finding that students either reason about linear functions as "individual input and output values" (p. 358) or as "generalized processes" (p. 340) may be associated with backward transfer because some students in our study moved from one to the other. This suggests that the transition from thinking about linear functions as inputs and outputs to thinking about them as generalized processes may be influenced by quadratic functions instruction.

Second, our study suggests that the transition from using buildup process reasoning to using abbreviated buildup process reasoning about linear functions may also be initiated when students learn about quadratic functions. However, as stated above, this transition may not be something teachers should overemphasize while students are not yet able to appropriately engage in abbreviated buildup process reasoning in non-zero y-intercept linear function contexts. Future research is needed to unpack how and when to implement these instructional principles.

Conclusion

This exploratory study set out to look for backward transfer effects from quadratic functions instruction on students' prior ways of reasoning about linear functions in two real mathematics classrooms. We showed that three categories of changes in prior ways of reasoning about linear functions were produced, that those changes involved action versus process views of functions, and that the teachers' approaches to quadratic functions helped to account for similarities and differences in changes in reasoning across the two classrooms. Significantly, our findings represent the first reported evidence from real classrooms of the relationship between prior ways of reasoning and new learning in the backwards direction (i.e., from new learning to prior ways of reasoning). Thus, our study takes an important first look at backward transfer effects in real classrooms, and reveals various ways backward transfer effects on prior ways of reasoning about linear functions may manifest themselves.

Notes

1. Breidenbach et al. (1992) also proposed two other categories, namely reasoning about functions as *objects* and as *schemas*. However, in our study we did not see evidence of these ways of reasoning about functions. Also, our focus on just action and process views is consistent with the focus of Breidenbach et al.
2. An abbreviated buildup process could be used apart from a linear equation or as part of using a linear equation.
3. Other problems on the assessment were designed to look at correspondence vs covariational reasoning about functions (Confrey & Smith, 1995), and at levels of covariational reasoning (Carlson et al., 2002). Since this was an exploratory study, we used an array of types of problems to capture as many influences on students' reasoning as we could. We reported findings from the other problems elsewhere (Hohensee et al., 2021).
4. The following link provides an overview of quadratic function topics covered in each lesson: (https://drive.google.com/file/d/1IVukvuOni8WfNvC-iEgT_GwHjxGA0faX/view?usp=sharing).
5. Note that Abby completed the same version of the assessment pre and post.

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Appendix

To explain what we mean by engaging students in reasoning about quadratic functions that *resembles* buildup process reasoning, consider the following description. A teacher could help students see that for $y = x^2$ (i.e., the parent quadratic function), as x starts at 0 and builds up by 1, y starts at 0 and builds up by 1, by 3, by 5, by 7, etc. Then, for quadratic functions with a constant term, the teacher could help students see that a similar buildup of y occurs as for the parent function, except that as x starts at 0 and builds up by 1, y starts at the constant term and the buildup for y is adjusted by adding the constant term (e.g., for $y = x^2 + -2$, as x starts at 0 and builds up by 1, y starts at −2, and rather than building up by 1, by 3, by 5, by 7, etc., builds up by 1 + -2, by 3 + -2, by 5 + -2, by 7 + -2, etc.). For quadratic functions with a linear term, the teacher could help students see that a similar buildup of y occurs as for the parent function, except that as x starts at 0 and builds up by 1, the buildup for y is adjusted by adding the value of the linear term (e.g., for $y = x^2 + x$, as x starts at 0 and builds up by 1, y starts at 0, but rather than building by 1, by 3, by 5, by 7, etc., builds up by 1 + 1, by 3 + 2, by 5 + 3, by 7 + 4, etc.).