A STUDY OF FINITE PROJECTIVE PLANES: COORDINATISATION AND CONSTRUCTION

by

Kamal Joshi

A dissertation submitted to the Faculty of the University of Delaware in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics

Fall 2023

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ACKNOWLEDGEMENTS

My PhD journey has been possible on account of support, help, and kindness of many individuals. A few words in this page cannot do justice to my sense of gratitude and thankfulness to all of them. I hope the words carry the sentiments of my heart as I write them now.

My advisor, Dr. Robert S. Coulter, has been a guide and pillar of support both in the research and in my difficult times. I am truly fortunate and honored to have him as my PhD advisor. I would not have been able to complete this work if not for his constant encouragement, patience, and wise advice.

I would like to thank my dissertation committee Dr. Sebastian Cioabă, Dr. Alice Hui, and Dr. Chris Castillo for their valuable suggestions. Special thanks to Dr. Hui for her time and the intriguing questions at the time of writing the thesis.

The faculty and the staff in the department deserve my sincere thanks. Special regards to Dr. Mark Gockenbach who as the Chair has supported and advised me, Dr. Rakesh for his care, and Dr. Novi Bong for her friendly encouragements. Many thanks to David Caldwell for IT support and Pascale Larouche for making life easy in the department. I had received timely help from Deborah See and Pamela Irwin on numerous occasions whose delightful presence are missed by everyone.

My wife Shreeha has stood by me in all the ups and downs of the last five years. My sister Tara, brother Bishnu and their families, and my friend Avishek Mukherjee have been my allies.

Dedicated to the memories of my father.

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ABSTRACT

This dissertation consists of three parts. In the first part comprising of Chapters 2, 3, and 4, strategies for optimal coordinatisation of the finite projective planes are developed and illustrated with the coordinatisation of the planes of order 16. We have sought to establish the meaning or sense of optimality in the various coordinatisations. More than one aspect of optimality in different coordinatisations of the same plane are explored. Nice PTR polynomial representations of some finite projective planes are obtained as a consequence of optimal coordinatisations. We also present the PTR polynomials in forms that reveal some properties of the associated plane.

In the second part, comprising of Chapters 5, 6, and 7, we obtain some special planar ternary ring (PTR) polynomial representations of two individual planes and a class of planes. A brief description of each work will be given shortly.

In the third part, comprising of Chapter 8, a method of constructing some finite projective planes of Lenz-Barlotti (LB) types II.1 or above is given. The method constructs a PTR multiplication table of a plane with some given special subsets of the coordinatising set. The coordinatising set is taken to be the set of elements of a finite field and the special subsets correspond to various elation and homology groups admitted by the resulting plane. In this dissertation, the PTR addition table is always taken to be the finite field addition table which underlines our assumption that the resulting plane admits an elementary abelian transitive elation group. We have given examples of some planes of orders 16, 25, 32, 49, 64, and 81 constructed by our method.

Chapter 1 gives the preliminaries and background for the following chapters. In Chapter 2, we develop the theory of optimal coordinatisation with a special focus on the planes of LB types II.1 or above. Chapter 3 illustrates the theory developed in Chapter 2 with detailed descriptions and results of the coordinatisations of the planes of order 16 LB type II.1 or above. We continue the development of the theory of optimal coordinatisation in Chapter 4 by extending it to the planes of LB type I.1. Some coordinatisations of the planes of order 16 LB type I.1 are given. The next three chapters concern special PTR polynomial representations of some planes. In Chapter 5, a special coordinatisation and a PTR polynomial representation of the Figueroa plane of order 27 is given. A special PTR polynomial representation of the Hall planes from their definition as derived planes is obtained in Chapter 6. In Chapter 7, we obtain a PTR polynomial for the SEMI4 plane of order 16 by analysing the semifield multiplication given in [26]. Chapter 8 describes a method of constructing PTR multiplication tables of planes of LB type II.1 or above admitting an elementary abelian transitive elation group. Some planes of small orders constructed by this method are given. We give the planes in the form of PTR polynomials and supplement the data with the order of its full collineation group, p-rank, and a multiset giving the orders of central collineation groups having axes the lines of the plane.

Finally, some future directions are given in Chapter 9. Appendix A is the unpublished article [7]. The results in the article are used to develop the theory of optimal coordinatisation in this dissertation. Appendix B contains a polynomial which appears in the PTR polynomial of the Figueroa plane of order 27 given in Chapter 5.

Chapter 1

INTRODUCTION

The objective of this chapter is threefold: to give the definitions of the basic terms, to introduce the notations, and to state the preliminary results. The notation introduced in the definition of an object will be followed throughout unless specifically stated otherwise. Remarks are given where necessary with a focus on explaining or elaborating on a concept or a result to align with its application in the later chapters.

1.1 Incidence Structures

An incidence structure is a mathematical object that consists of a set of points and a set of lines, along with an incidence relation between the points and the lines. Let $\mathbf{p}$ be a point and $\mathcal{L}$ a line in an incident structure. Then the incidence (resp. non-incidence) between $\mathbf{p}$ and $\mathcal{L}$ is described variously using any of the following phrases and notations:

- $\mathbf{p}$ is incident to $\mathcal{L}$ (resp. $\mathbf{p}$ is not incident to $\mathcal{L}$).
- $\mathcal{L}$ contains $\mathbf{p}$ (resp. $\mathcal{L}$ does not contain $\mathbf{p}$).
- $\mathbf{p} \in \mathcal{L}$ (resp. $\mathbf{p} \notin \mathcal{L}$).
- $\mathbf{p}$ lies on $\mathcal{L}$ (resp. $\mathbf{p}$ does not lie on $\mathcal{L}$).
- $\mathcal{L}$ passes through $\mathbf{p}$ (resp. $\mathcal{L}$ does not pass through $\mathbf{p}$).

Two or more points in an incidence structure are said to be *collinear* if they are incident to a common line. Two or more lines are said to be *concurrent* if they contain a common point. A point-line pair in the incidence structure is called a *flag*. A flag $(\mathbf{p}, \mathcal{L})$ is an *incident flag* if $\mathbf{p} \in \mathcal{L}$ and a *non-incident flag* if $\mathbf{p} \notin \mathcal{L}$.

In this dissertation, we study special types of incidence structures called projective planes.

Definition 1.1.1. ***Projective Plane*** *A projective plane $\mathscr{P}$ is an incidence structure satisfying the following axioms:*

P1. ***Uniqueness of a Common Line:*** *Given any two distinct points $\mathbf{p}$ and $\mathbf{q}$, there exists a unique line $\mathcal{L}$ containing the two points. We denote $\mathcal{L} = \overline{\mathbf{p}\,\mathbf{q}}$.*

P2. ***Uniqueness of a Point of Intersection:*** *Given any two distinct lines $\mathcal{L}$ and $\mathcal{M}$, there exists a unique point $\mathbf{p}$ incident to both the lines. We denote $\mathbf{p} = \mathcal{L} \cap \mathcal{M}$.*

P3. ***(Non-degenerate case) Existence of a Quadrangle:*** *There exist at least four distinct points, no three of which are collinear (alternately, there exist at least four distinct lines, no three of which are concurrent).*

An incident structure satisfying the axioms $P1$ and $P2$ is called a *closed configuration*. Given a set of points and lines in a projective plane $\mathscr{P}$, the smallest closed configuration containing the set of points and lines is called the *closure* of the set in $\mathscr{P}$. A closed configuration $\mathscr{Q}$ is called a *subplane* of $\mathscr{P}$ if $\mathscr{Q}$ is a projective plane in its own right. A subplane $\mathscr{Q}$ is a *proper subplane* if $\mathscr{Q} \neq \mathscr{P}$.

Remark 1.1.2. *The axiomatic definition allows us to reason about the properties of projective planes without relying on a specific representation or model. Various models, such as the projective planes over finite fields (Definition 1.14.1), or projective* *planes over division rings, are used to instantiate the axioms and study the specific types of projective planes.* □

When the context is clear, we refer to a projective plane simply as a 'plane'.

1.1.1 Finite Projective Planes

A *finite projective plane* is a projective plane with a finite set of points (lines). A finite projective plane has the following combinatorial properties:

Theorem 1.1.3. *([17], Theorem 3.5) Let $\mathcal{P}$ be a finite projective plane. Then,*

(i) Every line $\mathcal{L}$ contains the same number of points.

(ii) There are the same number of lines incident to any point $\mathbf{p}$.

(iii) The total number of points and the total number of lines are the same.

A finite projective plane $\mathcal{P}$ is said to be of order n if a line in $\mathcal{P}$ contains $n+1$ points. A plane $\mathcal{P}$ of order n has $n+1$ points on every line, $n+1$ lines through every point, and totally n^2+n+1 number of points (lines).

Remark 1.1.4. *In this dissertation, we study the finite projective planes exclusively. Henceforth, a plane $\mathcal{P}$ refers to a finite projective plane of order n unless specified otherwise.* □

If $\mathcal{Q}$ is a proper subplane of order m of a plane $\mathcal{P}$ of order n, then either $n = m^2$ or $n \geq m^2 + m$ ([17], Theorem 3.7). The existence of subplanes of different orders plays an important role in the optimal coordinatisation of finite projective planes. This will become clear in Chapter 2.

1.2 Principle of Duality

Both in the axiomatic definition of a projective plane (Definition 1.1.1) and the combinatorial properties of a finite projective plane (Theorem 1.1.3), we see the 'points' and the 'lines' have an identical or symmetric relation. The observation is codified in the principle of duality.

Definition 1.2.1. ***Dual Plane*** *If $\mathcal{P}$ is a projective plane, then the incidence structure $\mathcal{P}^*$ obtained as follows is again a projective plane: the points of $\mathcal{P}^*$ are the lines of $\mathcal{P}$, the lines of $\mathcal{P}^*$ are the points of $\mathcal{P}$, and a point is incident to a line in $\mathcal{P}^*$ if the corresponding line and point are incident in $\mathcal{P}$. The plane $\mathcal{P}^*$ is called the dual plane of $\mathcal{P}$.*

Theorem 1.2.2. *([17], Theorem 3.2)* ***Principle of Duality*** *Let A be any theorem about projective planes. If A^* is the statement obtained by interchanging the words "points" and "lines", then A^* is a theorem about the dual planes. Hence, A^* is also a theorem about projective planes.*

1.3 Collineations of a Projective Plane

A *collineation* of a projective plane is a bijective map that maps points to points, lines to lines, and preserves the relation of incidence and collinearity between points and lines. In other words, a collineation is a transformation that preserves the set of collinear points and the set of concurrent lines in the plane. The collection of all collineations of $\mathcal{P}$ is a group under the composition of maps and called the *full collineation group* of $\mathcal{P}$, denoted as Aut $\mathcal{P}$.

A *central collineation* or *perspectivity* α is a collineation for which there is a unique line $\mathcal{L}$ called the *axis* and a unique point $\mathbf{p}$ called the *center* such that α fixes every point on $\mathcal{L}$ and every line through $\mathbf{p}$. We then say that α is a $(\mathbf{p}, \mathcal{L})$-*central* *collineation*. The identity map is considered a central collineation by convention. For a given flag $(\mathbf{p}, \mathcal{L})$, the set of all $(\mathbf{p}, \mathcal{L})$-central collineations denoted as $\Gamma(\mathbf{p}, \mathcal{L})$ is a subgroup of the full collineation group Aut $\mathcal{P}$. A $(\mathbf{p}, \mathcal{L})$-central collineation α is called an *elation* if the center lies on the axis i.e. $\mathbf{p} \in \mathcal{L}$ and a *homology* if the center does not lie on the axis i.e. $\mathbf{p} \notin \mathcal{L}$.

Whether or not the center is on the axis has a consequence on the orders of elations and homologies as elements of the group Aut $\mathcal{P}$.

Theorem 1.3.1. *([17], Lemma 4.10) Let $\mathcal{P}$ be a projective plane of order n with a central collineation α of order k. Then, either*

(i) $k|n$ *and* α *is an elation, or*

(ii) $k|(n-1)$ *and* α *is a homology.*

A projective plane $\mathcal{P}$ is said to be $(\mathbf{p}, \mathcal{L})$-transitive for point $\mathbf{p}$ and line $\mathcal{L}$ if for any two distinct points $\mathbf{a}$ and $\mathbf{b}$ on a line $\mathcal{M}$ through $\mathbf{p}$, $\mathcal{M} \neq \mathcal{L}$, there exists a $(\mathbf{p}, \mathcal{L})$-central collineation α of $\mathcal{P}$ such that $\mathbf{a}^\alpha = \mathbf{b}$. If $\mathcal{P}$ admits a $(\mathbf{p}, \mathcal{L})$-transitivity, then the group $\Gamma(\mathbf{p}, \mathcal{L})$ is called a transitive central collineation group. We use the terms '*a transitive elation group*' or '*a transitive homology group*' in the sense of their obvious meanings.

A $(\mathbf{p}, \mathcal{L})$-central collineation is completely determined by the image of any one point not on the axis and different from the center (see, for example, [17], Theorems 4.7 and 4.9). As a result, we make an observation on the orders of transitive central collineation groups in the form of a lemma.

Lemma 1.3.2. *([17], Lemma 4.18) Let $\mathcal{P}$ be a projective plane of order n with a transitive central collineation group $\Gamma = \Gamma(\mathbf{p}, \mathcal{L})$ for some flag $(\mathbf{p}, \mathcal{L})$. Then, either*

(i) $\mathbf{p} \in \mathcal{L}$ *i.e.* Γ *is a transitive elation group and* $|\Gamma| = n$*, or*

(ii) $\mathbf{p} \notin \mathcal{L}$ *i.e.* Γ *is a transitive homology group and* $|\Gamma| = n - 1$.

In both the cases, the orders of the groups are maximal. When the orders are not maximal, we can still make a statement resembling Theorem 1.3.1.

Theorem 1.3.3. *([17], Lemma 4.10) Let* $\mathcal{P}$ *be a projective plane of order* n *with a central collineation group* $\Gamma = \Gamma(\mathbf{p}, \mathcal{L})$ *for some flag* $(\mathbf{p}, \mathcal{L})$. *Let* $|\Gamma| = k$. *Then, either*

(i) $k|n$ *and* $\mathbf{p} \in \mathcal{L}$ *i.e.* Γ *is an elation group, or*

(ii) $k|(n-1)$ *and* $\mathbf{p} \notin \mathcal{L}$ *i.e.* Γ *is an homology group.*

The existence of various central collineation groups is crucial for the optimal coordinatisation of a plane (Chapter 2). Both the transitive groups and other smaller groups are utilized to obtain the optimal coordinatisations.

We close this section by giving some definitions for future use. For the definitions given below and for more definitions and results on the topic of central collineation groups, refer to [17], Chapter 4.

Definition 1.3.4. ***Translation Line*** *A line* $\mathcal{L}$ *of a plane* $\mathcal{P}$ *is said to be a translation line if the plane admits a transitive elation group* $\Gamma(\mathbf{p}, \mathcal{L})$ *for every* $\mathbf{p} \in \mathcal{L}$.

Definition 1.3.5. ***Translation Plane*** *A plane* $\mathcal{P}$ *with at least one translation line is called a translation plane.*

Definition 1.3.6. ***Central Collineation Group*** $\Gamma(\mathcal{L})$ *For any line* $\mathcal{L}$ *of a projective plane* $\mathcal{P}$, *the collection of all central collineations of* $\mathcal{P}$ *with axis* $\mathcal{L}$ *is a group denoted by* $\Gamma(\mathcal{L})$.

Lenz Type	$\mathcal{T}_e$
I	$\emptyset$
II	$\{(\mathbf{p}, \mathcal{L})\}$
III	$\{(\mathbf{p}, \overline{\mathbf{p}\,\mathbf{q}}) \ : \ \forall \ \mathbf{p} \in \mathcal{L}\}$ for some non-incident flag $(\mathbf{q}, \mathcal{L})$
IVa	$\{(\mathbf{p}, \mathcal{L}) \ : \ \forall \ \mathbf{p} \in \mathcal{L}\}$ for some line $\mathcal{L}$
IVb	$\{(\mathbf{q}, \mathcal{M}) \ : \ \forall \ \mathcal{M}$ through $\mathbf{q}\}$ for some point $\mathbf{q}$
V	$\{(\mathbf{q}, \mathcal{L}) \ : \ \forall \ \mathbf{q} \in \mathcal{L}\} \cup \{(\mathbf{p}, \mathcal{M}) \ : \ \forall \ \mathcal{M}$ through $\mathbf{p}\}$ for some incident flag $(\mathbf{p}, \mathcal{L})$
VII	$\{(\mathbf{p}, \mathcal{L}) \ : \ \forall \mathcal{L}$ and all $\mathbf{p} \in \mathcal{L}\}$

Table 1.1: The Lenz Classification.

1.4 The Lenz-Barlotti Classification

In our technique of optimal coordinatisation, the first step of the process is to identify what is called the Lenz-Barlotti class or Lenz-Barlotti type of the plane. The Lenz-Barlotti classification of projective planes was pioneered by Hanfried Lenz [27], [28] and refined by Adriano Barlotti [1] in the 1950s and 60s.

The *Lenz classification* identifies seven principal types to which any projective plane $\mathcal{P}$ belongs. It is based on the existence of transitive elation groups in $\mathcal{P}$. Define,

$$\mathcal{T}_e = \{(\mathbf{p}, \mathcal{L}) \ : \ \mathbf{p} \in \mathcal{L} \text{ and } \mathcal{P} \text{ is } (\mathbf{p}, \mathcal{L})\text{-transitive}\}.$$

The subscript e indicates the flags being considered correspond to the elation groups only. The classification is given in Table 1.1. Lenz type VI is omitted in the table. This because it has been shown no projective planes, finite or infinite, exist in that type ([9], Sec. 3.1, General Results 19, 20).

The *Lenz-Barlotti classification* refines the Lenz classification by introducing into the set of flags $\mathcal{T}_e$ the center-axis pairs corresponding to certain homology groups. We drop the condition $\mathbf{p} \in \mathcal{L}$ in the definition of $\mathcal{T}_e$ as well as the subscript e and define,

$$\mathcal{T} = \{(\mathbf{p}, \mathcal{L}) \ : \ \mathscr{P} \text{ is } (\mathbf{p}, \mathcal{L})\text{-transitive}\}.$$

As a result, each Lenz type is divided into sub-types, called the Lenz-Barlotti (LB) types. However, in the subsequent years some of the LB types have been proven to be empty. Specially in the case of finite planes, while the non-existence of planes has been proven for some LB types, neither a proof of non-existence nor any examples have yet been found for some others. See, for example, [9], Table 1 (p126) and [7]. Definitions of the LB types for which planes are known to exist or the existence is an open problem in the finite case are given in Table 1.2.

LB type IVb is omitted in Table 1.2 as the planes of type IVb are duals of the planes of type IVa. By virtue of the principle of duality (Section 1.2), it is sufficient to study one of the two types. Also, we have restricted the list to the LB types that are relevant to this dissertation.

1.5 Coordinatisation of a Projective Plane

One of the goals of this dissertation is to develop strategies to coordinatise projective planes in some optimal way. There are two concepts at play here: the 'coordinatisation' and the 'optimality of coordinatisation'. The optimality aspect is dealt with in Chapter 2. In this section, we introduce the notion of coordinatisation of a projective plane with occasional reference to optimality as a foresight.

Let $\mathcal{R}$ be a set of n symbols, two of which are special symbols 0 and 1. Define $\mathcal{R}^* = \mathcal{R} \setminus \{0\}$. Let ∞ be an additional symbol not in $\mathcal{R}$. Consider any three

LB Type	$\mathcal{T}$	**Finite Planes Exist?**
I.1	$\emptyset$	Yes
I.2	$\{(\mathbf{p},\mathcal{L})\}$ with $\mathbf{p}\notin\mathcal{L}$	Open
I.3	$\{(\mathbf{p},\mathcal{L}),(\mathbf{q},\mathcal{M})\}$ with $\mathbf{p}\in\mathcal{M}\setminus\mathcal{L},\ \mathbf{q}\in\mathcal{L}\setminus\mathcal{M}$	Open
I.4	$\{(\mathbf{p},\mathcal{L}),(\mathbf{q},\mathcal{M}),(\mathbf{r},\mathcal{N})\}$ where $\mathbf{p}\,\mathbf{q}\,\mathbf{r}$ is a triangle and $\mathcal{L}$ (resp. $\mathcal{M}$, $\mathcal{N}$) are opposite sides of $\mathbf{p}$ (resp. $\mathbf{q}$, $\mathbf{r}$)	Open
II.1	$\{(\mathbf{p},\mathcal{L})\}$ with $\mathbf{p}\in\mathcal{L}$	Yes
II.2	$\{(\mathbf{p},\mathcal{L}),\ (\mathbf{q},\mathcal{M})\}$ with $\mathbf{p}=\mathcal{L}\cap\mathcal{M},\ \mathbf{q}\in\mathcal{L}\setminus\mathcal{M}$	Open
IVa.1	$\{(\mathbf{p},\mathcal{L})\ :\ \forall\ \mathbf{p}\in\mathcal{L}\}$ for some line $\mathcal{L}$	Yes
IVa.2	$\{(\mathbf{r},\mathcal{L})\ :\ \forall\ \mathbf{r}\in\mathcal{L}\}\cup\{(\mathbf{p},\mathcal{M})\ :\ \forall\ \mathcal{M}$ through $\mathbf{q}\}\cup\{(\mathbf{q},\mathcal{N})\ :\ \forall\ \mathcal{N}$ through $\mathbf{q}\}$ with $\mathbf{p}\neq\mathbf{q}$ and $\mathbf{p},\mathbf{q}\in\mathcal{L}$	Yes
V.1	$\{(\mathbf{q},\mathcal{L})\ :\ \mathbf{q}\in\mathcal{L}\}\cup\{(\mathbf{p},\mathcal{M})\ :\ \mathcal{M}$ passes through $\mathbf{p}\}$ for some incident flag $(\mathbf{p},\mathcal{L})$	Yes
VII.2	$\{(\mathbf{p},\mathcal{L})\ :\ \forall\ \mathcal{L},\ \forall\ \mathbf{p}\}$	Yes

Table 1.2: The Lenz-Barlotti Classification.

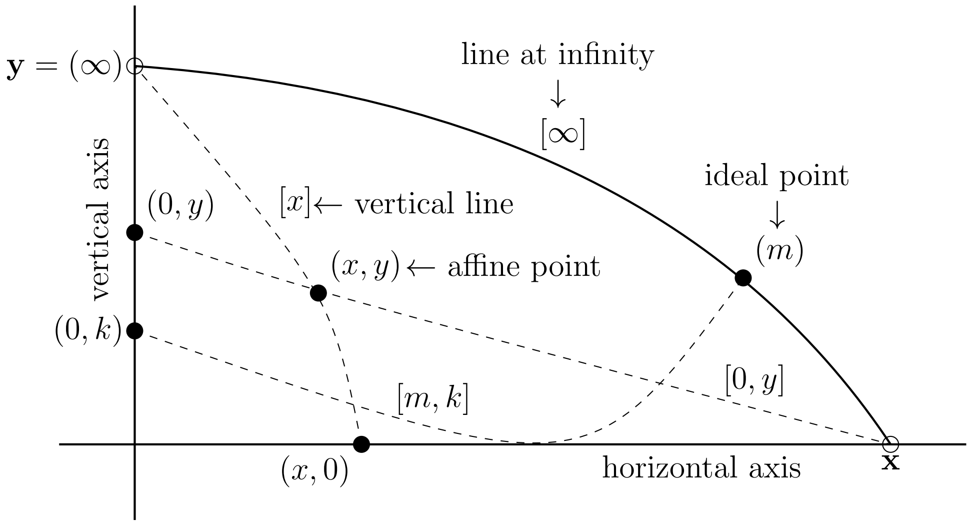

Figure 1.1: Notations for Points and Lines.

non-collinear points $\{\mathbf{y}, \mathbf{x}, \mathbf{O}\}$ in $\mathcal{P}$. We refer to this set of three points and the three lines determined by them as the *triangle* of coordinatisation. The line $\overline{\mathbf{x}\,\mathbf{y}}$ is designated the *line at infinity*. The point $\mathbf{y}$ is called the *point at infinity* and labeled $\mathbf{y} = (\infty)$. The points on the line at infinity other than (∞), called the *ideal points*, are labeled (m) for $m \in \mathcal{R}$. The remaining points of the plane, called the *affine points*, are labeled using the ordered pairs of $\mathcal{R} \times \mathcal{R}$. For the lines, let $\overline{\mathbf{x}\,\mathbf{y}} = [\infty]$. The other lines through (∞), called the *vertical lines*, are labeled $[x]$ for $x \in \mathcal{R}$. The remaining lines are labeled $[m, k]$ for $m, k \in \mathcal{R}$. Lines other than $[\infty]$ are called the *affine lines*. Also, the lines $\overline{\mathbf{x}\,\mathbf{y}}, \overline{\mathbf{O}\,\mathbf{y}}$, and $\overline{\mathbf{O}\,\mathbf{x}}$ are collectively referred to as the *axes*, and individually $\overline{\mathbf{O}\,\mathbf{y}}$ the *vertical axis*, $\overline{\mathbf{O}\,\mathbf{x}}$ the *horizontal axis*, and $\overline{\mathbf{x}\,\mathbf{y}}$ the *infinite axis*. Figure 1.1 is a pictorial representation of the points and lines along with their labeling.

Having established the notations and the terminology, we are now equipped to describe a method to obtain a complete labeling of points and lines of $\mathcal{P}$. First a remark,

Remark 1.5.1. *This is a note clarifying our use of the terms label, labeling, coordinates, coordinatising, and coordinatisation. For example, suppose a point denoted as* $\mathbf{O}$ *is labeled* $\mathbf{O} = (0, 0)$*. We then say the coordinates of* $\mathbf{O}$ *are* $(0, 0)$*. Generally speaking, we use the terms label or labeling during the coordinatisation process. Once the coordinatisation process is complete or when the labeling of a point or line is fixed, we use the term 'coordinates' to refer to the label. For example, the point denoted as* $\mathbf{y}$ *is usually labeled* $\mathbf{y} = (\infty)$ *early into the process of coordinatisation. The points are 'denoted', that is, named using bold face letters like* $\mathbf{x}, \mathbf{y}, \mathbf{O}, \mathbf{J}$ *etc and 'labeled' or 'assigned the coordinates' using symbols or ordered pair of symbols from the set* $\mathcal{R} \cup \{\infty\}$*. The term 'labeling' is used to describe the act of assigning labels to a point, a line, or a set of points and/or lines. The term 'coordinatising' is used in relation to the entire plane with all its points and lines. Coordinatising refers to the whole process from the initial labeling of individual points and lines to the final determination of the coordinates of all points and lines. Lastly, the term 'coordinatisation' is used to refer to both the process of coordinatising the plane and also as a noun to denote the coordinate system established after the completion of the process.* □

Procedure 1.5.1. The Labeling Procedure for a Coordinatisation of $\mathcal{P}$

Phase 1: the line at infinity

Step 1: Choose any point in $\mathcal{P}$ and label it $\mathbf{y} = (\infty)$.

Step 2: Choose any point in $\mathcal{P}$ other than $\mathbf{y}$ and label it $\mathbf{x} = (0)$.

Step 3: Label $\overline{\mathbf{x}\,\mathbf{y}} = [\infty]$, the line at infinity.

Step 4: Choose any point $\mathbf{J}$ on $\overline{\mathbf{x}\,\mathbf{y}}$—other than $\mathbf{x}, \mathbf{y}$—and label it $\mathbf{J} = (1)$.

Figure 1.2 is a pictorial representation of the points and lines labeled so far.

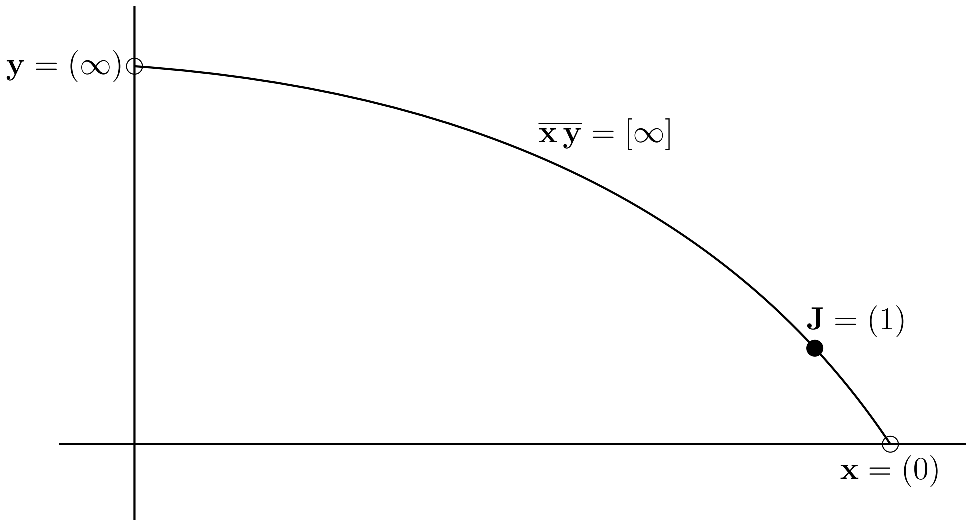

Figure 1.2: Labeling $\mathbf{y}, \overline{\mathbf{x}\,\mathbf{y}}, \mathbf{x}$, and $\mathbf{J}$.

Phase 2: the vertical axis

Step 5: Choose any affine point $\mathbf{O}$ i.e. any point not on the infinite axis $\overline{\mathbf{x}\,\mathbf{y}}$ and label it $\mathbf{O} = (0, 0)$.

Step 6: Label the line $\overline{\mathbf{O}\,\mathbf{y}} = [0]$, the vertical axis.

Step 7: Label the remaining points of $[0]$ by $(0, y)$ as y varies in $\mathcal{R}^*$.

Step 8: Identify the point $\mathbf{u} = (0, 1)$ on the vertical axis. (In a slightly different procedure explained in Section 1.5.1, a point not in any of the lines $\overline{\mathbf{O}\,\mathbf{x}}, \overline{\mathbf{O}\,\mathbf{y}}$, and $\overline{\mathbf{x}\,\mathbf{y}}$ is chosen and labeled as $\mathbf{I} = (1, 1)$ to obtain a *coordinatising quadrangle*.)

Phase 3: the rest of the plane

Step 9: Set $\overline{\mathbf{x}\,\mathbf{O}} = \overline{(0)(0,0)} = [0, 0]$.

Step 10: $\forall x \in \mathcal{R}$, let $\overline{(0, x)\,\mathbf{J}} \cap [0, 0] = (x, 0)$, and $\overline{(\infty)(x, 0)} = [x]$.

Step 11: In particular, $\overline{\mathbf{u}\,\mathbf{J}} \cap [0, 0] = (1, 0)$, and $\overline{(\infty)(1, 0)} = [1]$.

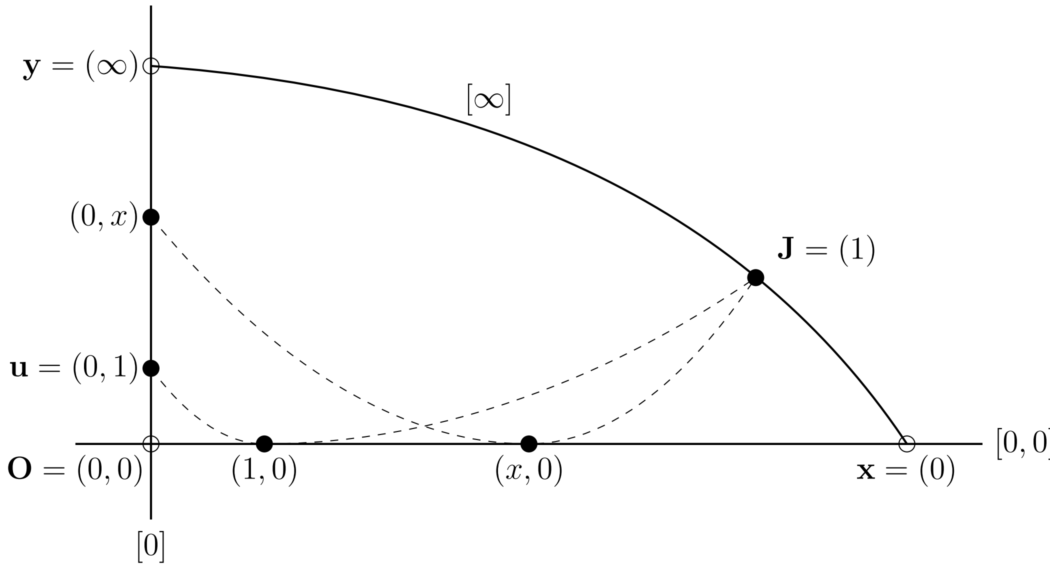

Figure 1.3: Labeling Points on the Vertical Axis $\overline{\mathbf{O}\,\mathbf{y}} = [0]$ and the Horizontal Axis $\overline{\mathbf{O}\,\mathbf{x}} = [0, 0]$.

Step 12: $\forall m, k \in \mathcal{R}$, let $\overline{(0, m)(1, 0)} \cap [\infty] = (m)$, and $\overline{(m)(0, k)} = [m, k]$.

Step 13: $\forall x, y \in \mathcal{R}$, define $(x, y) = [x] \cap [0, y]$. $\square$

The labeling of $\mathbf{O}$ and the remaining points on $\overline{\mathbf{O}\,\mathbf{y}} \setminus \{\mathbf{y}\}$, along with the points $\mathbf{x}$ and $\mathbf{J}$, determines the *coordinates* of all remaining points and lines in the plane. As we shall see later, some choices of labeling are better than others. In fact, one major component of our approach to an optimal coordinatisation of a plane is the labeling the points of $\overline{\mathbf{O}\,\mathbf{y}}$ suitably by letting $\mathcal{R}$ be the set of elements of a finite field (Definition 1.14.1). For the discussion here though, $\mathcal{R}$ is a set of arbitrary symbols and we assume an arbitrary labeling except for the origin $\mathbf{O}$.

Figure 1.3 shows the labeling of the points on the vertical axis and how the points on the horizontal axis labeled subsequently. Figure 1.4 shows the process of labeling the points on the line at infinity. Once all three axes and the points on the

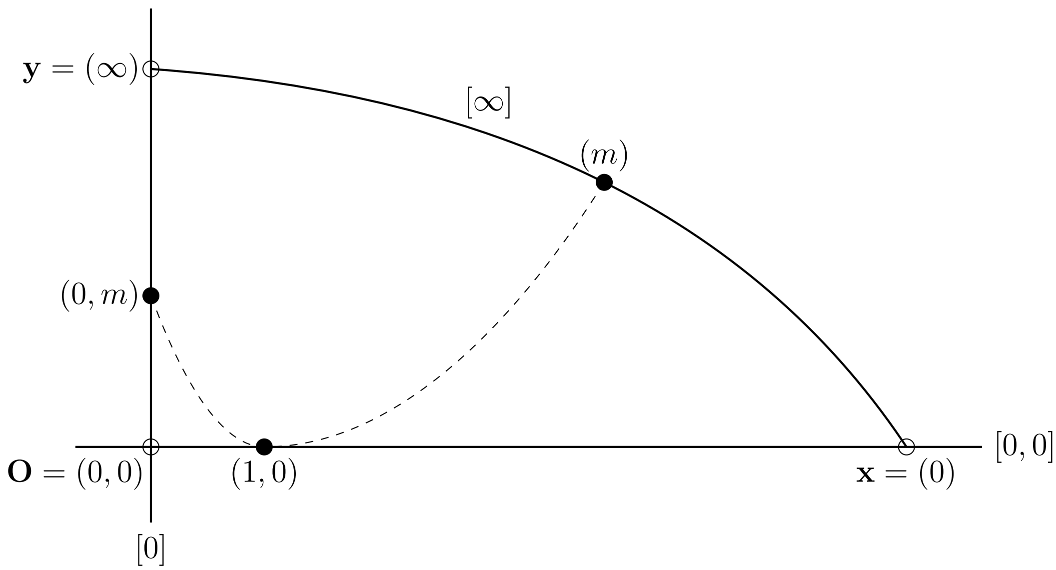

Figure 1.4: Labeling Points on the Infinite Axis $\overline{\mathbf{x}\,\mathbf{y}} = [\infty]$.

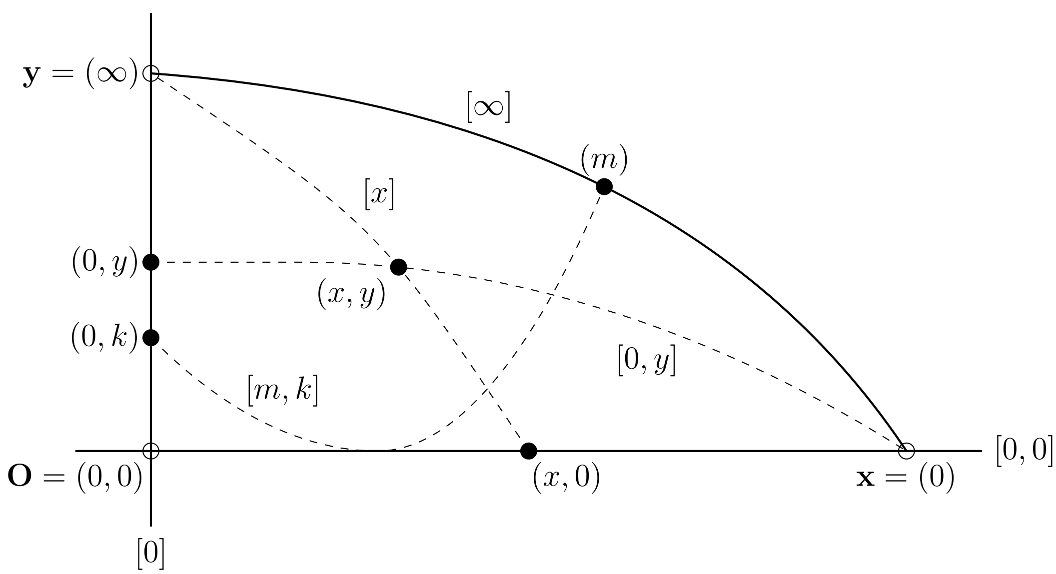

Figure 1.5: Labeling Affine Points and Affine Lines.

axes are labeled, we can determine the rest of the labels or coordinates from the above rules. Figure 1.5 shows the calculation of the coordinates of the remaining affine points and lines. $\square$

Remark 1.5.2. *We use the plural term 'coordinates' which suggests that the coordinates of any point should have more than one component. This notion is clearly compatible for the affine points whose coordinates are of the form* (x, y) *for some* $x, y \in \mathcal{R}$*. In the case of ideal points though, we need to interpret this carefully. For an ideal point, say* $\mathbf{q} = (m)$*, one component is the unique symbol* $m \in \mathcal{R}$ *and the other component is the information it lies on the line at infinity. In fact, we can interpret* (m) *to be the same as* (m, ∞)*. The point* (∞) *would then be interpreted as* (∞, ∞)*. We continue to use* (m) *and* (∞) *as per convention, but the use of the plural form 'coordinates' is now justified.* $\square$

Suppose a plane $\mathcal{P}$ is coordinatised with a coordinatising set $\mathcal{R}$. Denote the coordinates of a point $\mathbf{p}$ by $[\mathbf{p}]_c$. A *point-coordinates pair* is defined as the ordered pair $(\mathbf{p}, [\mathbf{p}]_c)$. A *line-coordinates* pair is defined similarly.

Definition 1.5.3. ***Coordinatisation of*** $\mathcal{P}$ *A coordinatisation of a plane* $\mathcal{P}$ *is the set of all point-coordinates pairs and line-coordinates pairs.*

1.5.1 The Quadrangle of Coordinatisation

Let $\mathbf{I} = [1] \cap \overline{\mathbf{x}\,\mathbf{u}} = [1] \cap [0, 1] = (1, 1)$. Then, the four points $\mathbf{O} = (0, 0)$, $\mathbf{I} = (1, 1)$, $\mathbf{x} = (0)$, and $\mathbf{y} = (\infty)$ constitute the *quadrangle of coordinatisation*. See Figure 1.6 for a marking and labeling of the quadrangle. Such quadrangle is generally labeled initially and used in the remainder of the labeling process. See, for example, [6], Section 2. In the Labeling Procedure 1.5.1, three of the four points of the

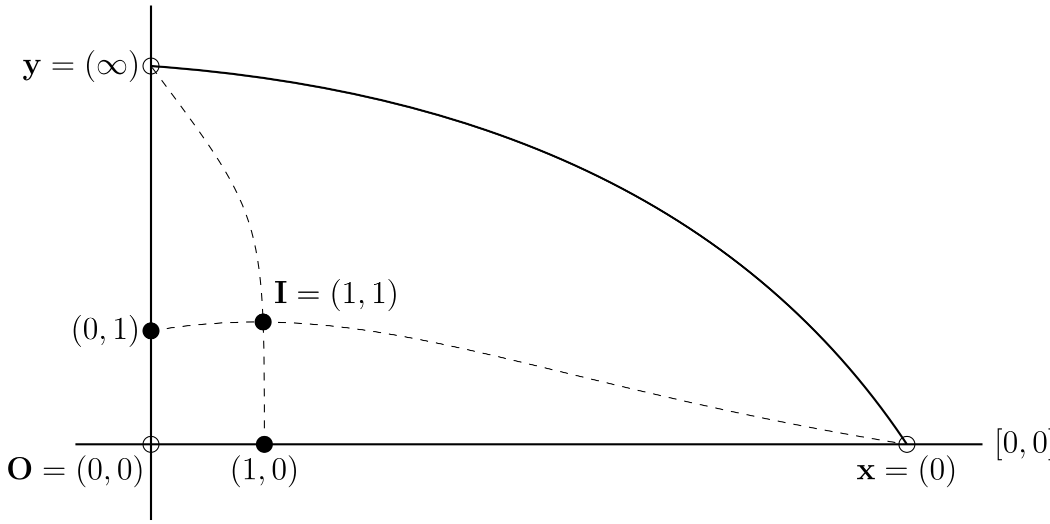

Figure 1.6: The Coordinatising Quadrangle $\{\mathbf{y}, \mathbf{x}, \mathbf{O}, \mathbf{I}\}$.

quadrangle are labeled initially. The points are $\mathbf{y}, \mathbf{x}$, and $\mathbf{O}$. The point $\mathbf{I}$ can be chosen from the affine points not on any of the three axes $[0]$, $[0,0]$, and $[\infty]$ to complete the quadrangle. If this method is chosen, the procedure will take a different path starting from Step 8 of the Labeling Procedure 1.5.1.

1.5.2 The Frame of Coordinatisation in this Dissertation

We use the term *frame of coordinatisation* or *frame* in short for the set of five points we have used in Labeling Procedure 1.5.1. See Figure 1.7 for a marking and labeling of the frame of coordinatisation. Formally,

Definition 1.5.4. ***The Frame of Coordinatisation*** *In a coordinatisation of the plane $\mathcal{P}$, the set of five points* $\{\mathbf{y}, \mathbf{x}, \mathbf{O}, \mathbf{u}, \mathbf{J}\}$ *labeled* $\mathbf{y} = (\infty), \mathbf{x} = (0), \mathbf{O} = (0,0), \mathbf{u} = (0,1)$*, and* $\mathbf{J} = (1)$ *is called the frame of coordinatisation.*

Remark 1.5.5. *We would like to make some clarifying remarks on the Labeling Procedure 1.5.1 and the resulting coordinate system:*

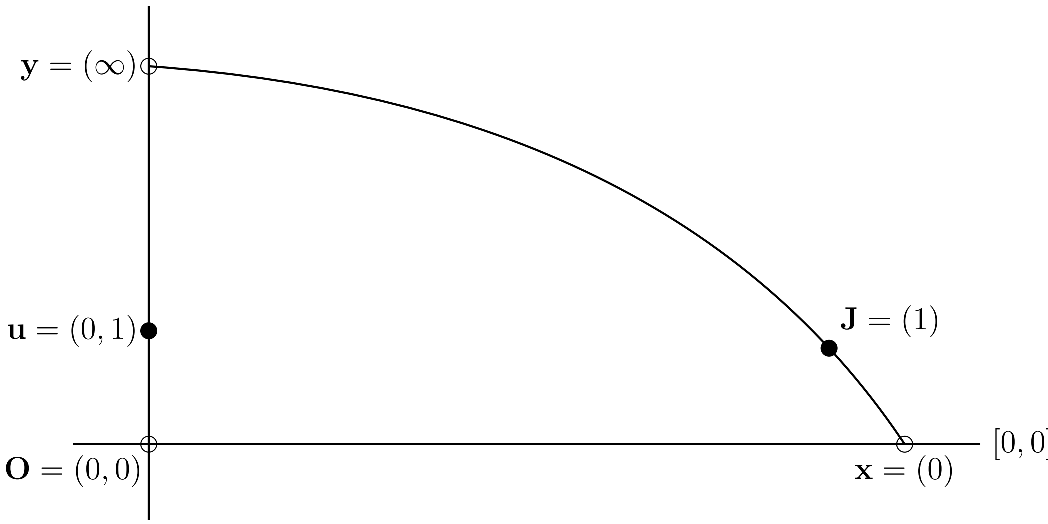

Figure 1.7: The Frame of Coordinatisation $\{\mathbf{y}, \mathbf{x}, \mathbf{O}, \mathbf{u}, \mathbf{J}\}$.

1. *Unlike in [6], Section 2, the method of labeling presented here does not begin by choosing the quadrangle* $\{\mathbf{y}, \mathbf{x}, \mathbf{O}, \mathbf{I}\}$*. In fact, the point* $\mathbf{I} = (1,1)$ *is determined by the choices of the five points* $\mathbf{y}, \mathbf{x}, \mathbf{O}, \mathbf{u}$*, and* $\mathbf{J}$ *which we defined as the frame. However, it should not be assumed that the two methods will result in different coordinatisations necessarily. The final coordinates of all the points and the lines of* $\mathcal{P}$ *depends on three crucial choices irrespective of the method or sequence of labeling—the choice of the triangle* $\{\mathbf{y}, \mathbf{x}, \mathbf{O}\}$*, the choice of the ideal point* $\mathbf{J}$ *on* $\overline{\mathbf{x}\,\mathbf{y}}$*, and the labeling of the points on* $\overline{\mathbf{O}\,\mathbf{y}}$*. This is evident from the labeling process described in Step* 2 *of Labeling Procedure 1.5.1.*

2. *The four points* $\mathbf{y}, \mathbf{x}, \mathbf{O}$*, and* $\mathbf{J}$ *do not form a quadrangle in the plane as* $\mathbf{y}, \mathbf{x}$*, and* $\mathbf{J}$ *are collinear. Similarly, the four points* $\mathbf{y}, \mathbf{x}, \mathbf{O}$*, and* $\mathbf{u}$ *also do not form a quadrangle. The four points* $\mathbf{x}, \mathbf{O}, \mathbf{u}$*, and* $\mathbf{J}$ *do form a quadrangle and* $\overline{\mathbf{x}\,\mathbf{J}} \cap \overline{\mathbf{O}\,\mathbf{u}} = (\infty)$*. But we do not use this relation to determine* (∞) *because either the line* $\overline{\mathbf{x}\,\mathbf{y}}$ *or the line* $\overline{\mathbf{O}\,\mathbf{y}}$ *with* $\mathbf{y} = (\infty)$ *in both cases is usually*

determined first in the process of coordinatisation.

3. *The reason for identifying* $\overline{\mathbf{x}\,\mathbf{y}}$ *or* $\overline{\mathbf{O}\,\mathbf{y}}$ *and* $\mathbf{y}$ *to label first will become clear in Section 2.1. Also, our approach of using the frame* $\{\mathbf{y}, \mathbf{x}, \mathbf{O}, \mathbf{u}, \mathbf{J}\}$ *instead of the quadrangle* $\{\mathbf{y}, \mathbf{x}, \mathbf{O}, \mathbf{I}\}$ *is based on the sequence of steps we follow to obtain the optimal coordinatisations.*

4. *After the completion of the coordinatisation process, a point or a line is often referred to by its coordinates directly. This means, we write 'the point* (x, y)*' instead of 'the point whose coordinates are* (x, y)*'. Similarly with the lines.*

5. *At this stage, we have the point* (x, y) *is the intersection of the lines* $[x]$ *and* $[0, y]$*. But the point* (x, y) *also lies on* $n-1$ *other lines* $[m, k]$*. For those lines, an algebraic relation between* $x, y, m,$ *and* k *is not yet established. Similarly, the line* $[m, k]$ *contains the two points* (m) *and* $(0, k)$ *by definition. Therefore, it is the unique line containing those two points. But the line* $[m, k]$ *also contains* $n-1$ *other points* (x, y) *and no algebraic relation between* $x, y, m,$ *and* k *is yet established.*

6. *The method laid out in this section can be used to coordinatise a plane with any arbitrary set* $\mathcal{R}$ *of* n *symbols containing* 0 *and* 1*, and by making any valid choices for labeling the points and lines at every step of the process.*

7. *There are other methods for coordinatising a projective plane, for example, Pickert's method [38]. All methods lead to an algebraic representation of the plane which is the topic of the next section. The idea of coordinatisation of these types was originally introduced by Marshall Hall [15].* $\square$

1.6 Planar Ternary Ring (PTR) From Coordinatisation

The coordinatisation of $\mathcal{P}$ described in Section 1.5 lays the foundation for an algebraic representation of the plane. To obtain such representation, we define a tri-variate function $T : \mathcal{R}^3 \to \mathcal{R}$, called the *planar ternary ring* (PTR) such that $(x, y) \in [m, k] \iff T(m, x, y) = k$. The function T must satisfy certain properties which are described in a theorem by Hall. For a given coordinatisation of a plane $\mathcal{P}$, the function T serves as the most abstract relation between the affine points (x, y) and the affine lines $[m, k]$. This serves as the first step in an attempt to address the issue raised in Remark 1.5.5, (5). The following theorem establishes the fundamentals of the planar ternary rings:

Theorem 1.6.1. *([15], Theorem 5.4) Let $\mathcal{P}$ be a projective plane of order n and $\mathcal{R}$ be any set of cardinality n. Let $T : \mathcal{R}^3 \to \mathcal{R}$ be a PTR obtained from coordinatising $\mathcal{P}$. Then T must satisfy the following properties:*

(a) $T(a, 0, z) = T(0, b, z) = z$ *for all* $a, b, z \in \mathcal{R}$.

(b) $T(x, 1, 0) = x$ *and* $T(1, y, 0) = y$ *for all* $x, y \in \mathcal{R}$.

(c) If $a, b, c, d \in \mathcal{R}$ *with* $a \neq c$*, then there exists a unique* x *satisfying* $T(x, a, b) = T(x, c, d)$.

(d) If $a, b, c \in \mathcal{R}$*, then there is a unique* z *satisfying* $T(a, b, z) = c$.

(e) If $a, b, c, d \in \mathcal{R}$ *with* $a \neq c$*, then there is a unique pair* (y, z) *satisfying* $T(a, y, z) = b$ *and* $T(c, y, z) = d$.

Conversely, any tri-variate function T defined on $\mathcal{R}$ which satisfies Properties (c) through (e) can be used to define an affine plane $\mathcal{A}_T$ of order q as follows:

- *the points of $\mathcal{A}$ are (x, y), with $x, y \in \mathcal{R}$;*

- *the lines of $\mathcal{A}$ are the symbols $[m, a]$, with $m, a \in \mathcal{R}$, defined by*

$$[m, a] = \{(x, y) \in \mathcal{R} \times \mathcal{R} : a = T(m, x, y)\},$$

and the symbols $[c]$, with $c \in \mathcal{R}$, defined by

$$[c] = \{(c, y) : y \in \mathcal{R}\}.$$

When a coordinatizing set $\mathcal{R}$ is endowed with a function T as above, we also use the notation $(\mathcal{R}, T)$ to refer to the PTR thus defined. If the coordinatising set $\mathcal{R}$ is clear from the context, then we simply write the PTR T.

The converse part of Theorem 1.6.1 shows that we now have an algebraic representation of the plane $\mathcal{P}$, namely the PTR $(\mathcal{R}, T)$. However, upon using an arbitrary set of symbols $\mathcal{R}$ for coordinatisation as in Section 1.5, we obtain T as a tri-variate function on a set of n symbols of which we do not have any other representation yet. Compare this with, for example, any function defined on a finite field (Definition 1.14.1). A function defined on a finite field has a polynomial representation (Section 1.15.1). To develop a more descriptive representation of T, first we need to equip the set $\mathcal{R}$ with an algebraic structure and associate it with the PTR $(\mathcal{R}, T)$. Next two sections are devoted to this endeavor. Based on the structure thus defined, we later develop a theory of polynomial representation of the PTRs.

1.7 PTR Addition and PTR Multiplication

The PTR function T in Theorem 1.6.1 is used to define an addition $\oplus$ and a multiplication $\odot$ on the set $\mathcal{R}$.

Definition 1.7.1. ***PTR Addition and PTR Multiplication*** *For all $x, y \in \mathcal{R}$,*

$$x \oplus y = T(1, x, y), \text{ and}$$
$$x \odot y = T(x, y, 0).$$

We refer to $\oplus$ as the PTR addition, which distinguishes it from any other addition that $\mathcal{R}$ could be equipped with, e.g. the usual field addition $+$ when $\mathcal{R}$ is the set of elements of a field. Similarly, $\odot$ is the PTR multiplication.

There is a subtlety we need to be careful about when studying the PTRs and coordinatisation. Given the definitions of $\oplus$ and $\odot$, it is natural to ask if the relation $T(m, x, y) = (m \odot x) \oplus y$ holds for all $m, x, y \in \mathcal{R}$. The answer is it does not hold in general and when it does, the special case merits a definition.

Definition 1.7.2. ***Linear PTR*** *A planar ternary ring $(\mathcal{R}, T)$ is said to be linear if $T(m, x, y) = (m \odot x) \oplus y$ for all $m, x, y \in \mathcal{R}$.*

Remark 1.7.3. *This is a note on the precedence of the PTR multiplication over the PTR addition which has not been addressed in the literature to our best knowledge.*

1. *We used parenthesis on the right side of the equation $T(m, x, y) = (m \odot x) \oplus y$ in Definition 1.7.2 which means the product $m \odot x$ is evaluated first and then y is added to the result. Historically, the parenthesis has been omitted with an implicit assumption of the precedence of the PTR multiplication over the PTR addition (at least certainly in the case of a linear PTR). See, for example, [17], Theorem 6.1 which gives a necessary and sufficient condition for the linearity of the PTR in terms of the existence of some special little Desargues configurations. Refer to [17], Chapter IV, Section 7 for a definition of the Desargues (and little Desargues) configuration. In the proof of [17],*

Theorem 6.1, it is clear that $m \odot x \oplus y$ *is used to mean* $(m \odot x) \oplus y$ *(notations are different). In [17] or in any other work by any author, we have not found any justification or clarification provided for the assumption or convention.*

2. *The issue of precedence is critical when considering PTRs that are not linear. In the case of a non-linear PTR, the function* T *is simply a tri-variabled function on* $\mathcal{R}^3$*. We do not have a general representation of* T *in terms of* $\oplus$ *and* $\odot$ *as in the case of a linear PTR. Therefore, to assume a precedence between* $\oplus$ *and* $\odot$ *without specifying the context or providing any justification seems to be an oversight to us.*

3. *In the case of a linear PTR though, it is plausible why such precedence could be assumed, though not explicitly stated anywhere. As mentioned earlier, there is well-known geometric condition, the little Desargues configuration which is tied to the linearity by way of [17], Theorem* 6.1 *and the definition of linearity is* $T(m, x, y) = (m \odot x) \oplus y \ \forall m, x, y \in \mathcal{R}$*. We are not aware of any geometric property that is significant enough to result in some special properties of the corresponding plane and general enough that it exists in the planes of different classes like the different LB types, with the property being equivalent to* $T(m, x, y) = m \odot (x \oplus y) \ \forall m, x, y \in \mathcal{R}$*.*

4. *We have used parentheses explicitly throughout this dissertation.*

1.7.1 The Action of PTR Addition and PTR Multiplication on the Vertical Axis

From the Labeling Procedure 1.5.1, we know the elements of $\mathcal{R}$ are associated with the points on the vertical axis $\overline{\mathbf{O}\,\mathbf{y}} = [0]$, other than (∞), in a 1-1 correspondence. Therefore, it is useful to see the addition or multiplication by an element of

$\mathcal{R}$ vis-à-vis its action on the vertical axis.

Figures 1.8 and 1.9 describe the action of the PTR addition $\oplus$ on the vertical axis. The points on the axis $[0]$ that correspond to $x, y \in \mathcal{R}$ are $(0, x)$ and $(0, y)$ respectively. In a sentence, "The sum $x \oplus y$ on the vertical axis is the intersection of the line joining $\mathbf{J} = (1)$ and the point (x, y) with the vertical axis."

Given: $(0, x)$ and $(0, y)$ **To obtain:** $(0, x \oplus y)$

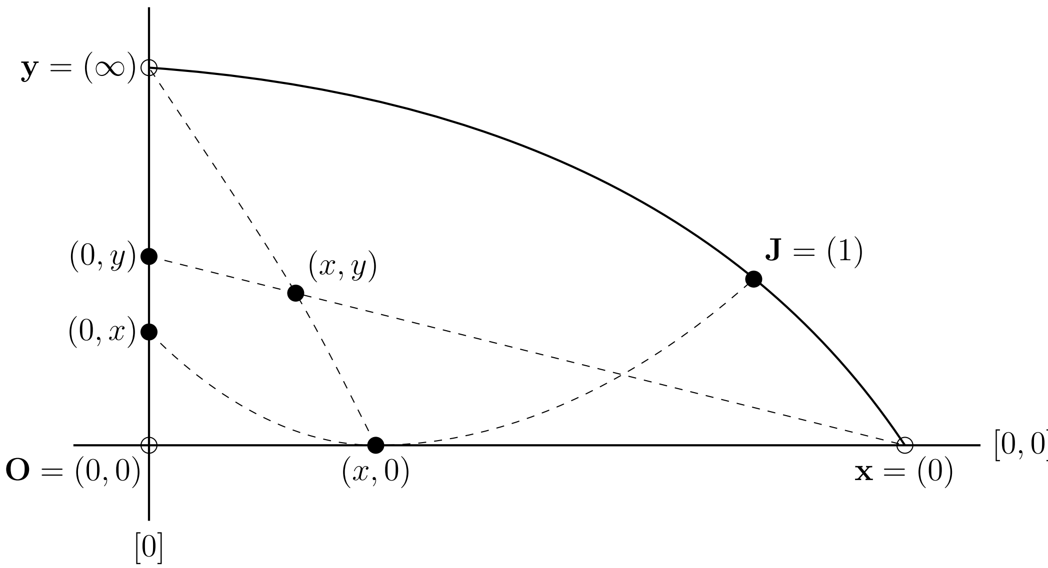

Figure 1.8: Step I - Locating the Point (x, y).

Sub-steps in Step I of the Action of $\oplus$ (Figure 1.8)

1. $(x, 0) = \overline{(0, x)\,\mathbf{J}} \cap [0, 0]$
2. $[x] = \overline{(x, 0)\,(\infty)}$
3. $[0, y] = \overline{(0)(0, y)}$
4. $(x, y) = [x] \cap [0, y]$

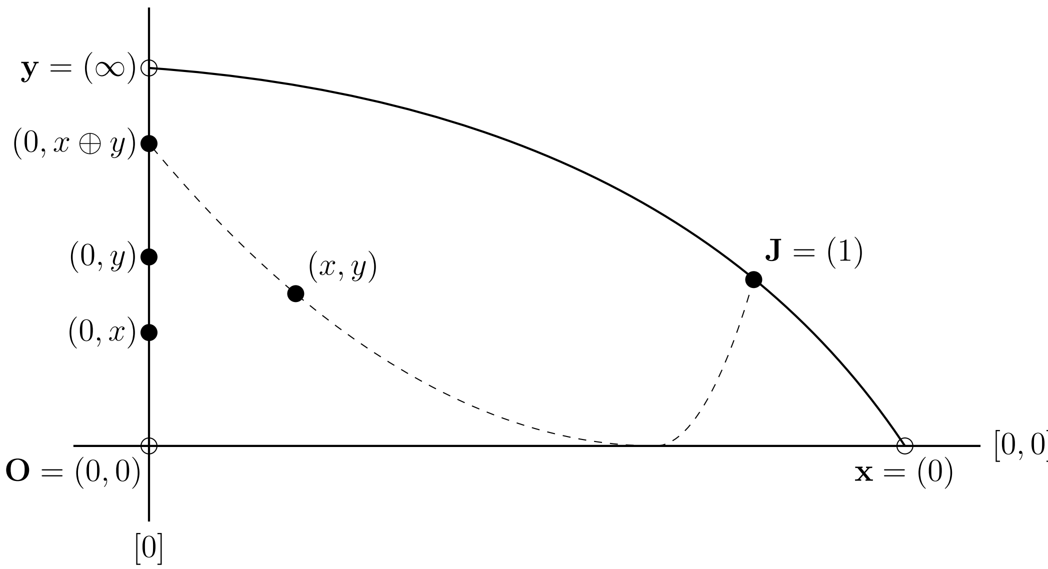

Figure 1.9: Step II - $(0, x \oplus y) = [0] \cap \overline{\mathbf{J}(x, y)}$.

Calculation in Step II of the Action of $\oplus$ (Figure 1.9)

$$
\begin{aligned}
&(x, y) \in [1, k] \\
\Longrightarrow\ &T(1, x, y) = k \\
\Longrightarrow\ &x \oplus y = k.
\end{aligned}
$$

Similarly, Figures 1.10 and 1.11 show the action of the PTR multiplication $\odot$ on the vertical axis. In a sentence, "The product $m \odot x$ on the vertical axis is the intersection of the line joining the ideal point (m) and the point $(x, 0)$ on the horizontal axis with the vertical axis."

Sub-steps in Step I of the Action of $\odot$ (Figure 1.10)

1. $(x, 0) = \overline{(0, x)\,\mathbf{J}} \cap [0, 0]$

2. $(1, 0) = \overline{\mathbf{u}\,\mathbf{J}} \cap [0, 0]$

Given: $(0, m)$ and $(0, x)$ **To obtain:** $(0, m \odot x)$

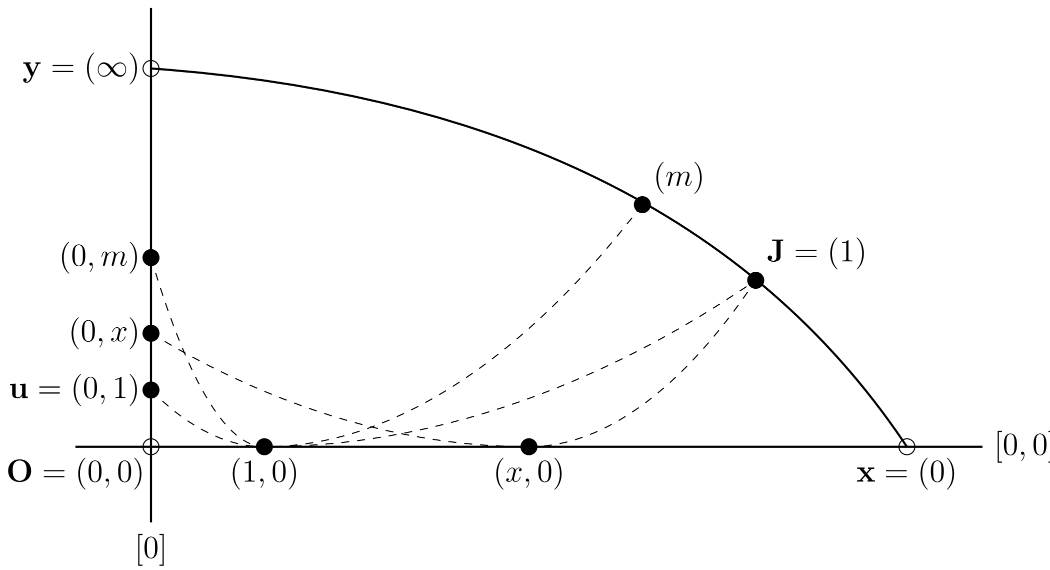

Figure 1.10: Step I - Locating the Points $(x, 0)$ and (m).

3. $(m) = \overline{(0, m)(1, 0)} \cap [\infty]$

Calculation in Step II of Action of $\odot$ (Figure 1.11)

$$
\begin{aligned}
& (x, 0) \in [m, k] \\
\Longrightarrow\ & T(1, x, y) = k \\
\Longrightarrow\ & x \oplus y = k.
\end{aligned}
$$

Remark 1.7.4. *It is clear from the actions of the PTR addition $\oplus$ and multiplication $\odot$ on the vertical axis that choosing and labeling the set of points $\{\mathbf{y}, \mathbf{x}, \mathbf{O}, \mathbf{u}, \mathbf{J}\}$ suffices to lay a geometric framework in which the operations $\oplus$ and $\odot$ can be executed on any pair $x, y \in \mathcal{R}$. We can also choose and label the quadrangle $\{\mathbf{y}, \mathbf{x}, \mathbf{O}, \mathbf{I}\}$ instead to achieve another framework which produces the same actions. Recall the*

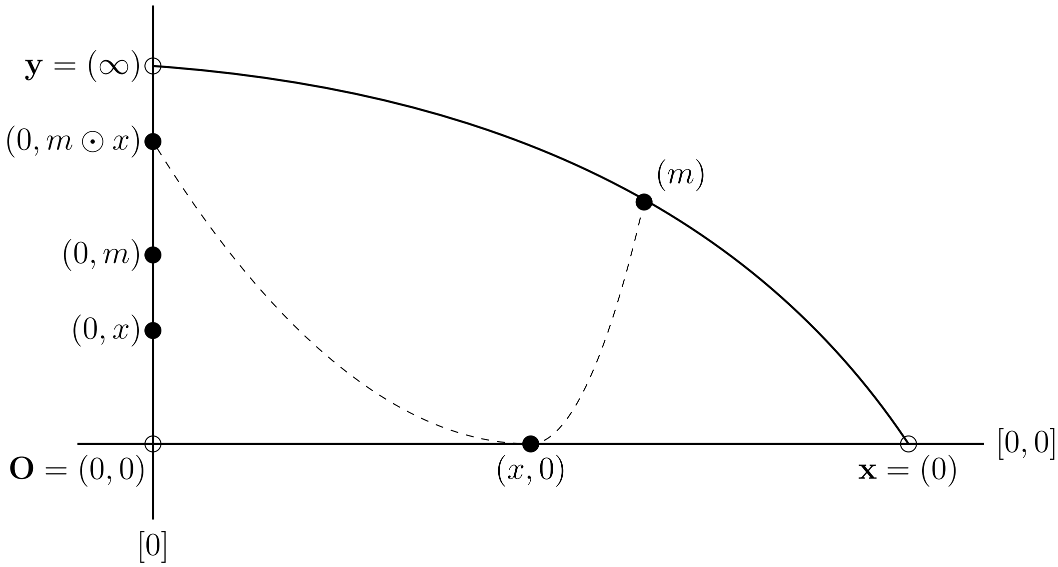

Figure 1.11: Step II - $(0, m \odot x) = \overline{(m)(x,0)} \cap [0]$.

discussion in Remark 1.5.5, (1). Also, observe $\mathbf{u} = \overline{\mathbf{x}\,\mathbf{I}} \cap \overline{\mathbf{O}\,\mathbf{y}}$, $(1,0) = \overline{\mathbf{y}\,\mathbf{I}} \cap \overline{\mathbf{O}\,\mathbf{x}}$, *and* $\mathbf{J} = \overline{\mathbf{u}(1,0)} \cap \overline{\mathbf{x}\,\mathbf{y}}$. □

1.8 The Algebraic Properties of $(\mathcal{R}, \oplus, \odot)$

The following properties of the PTR addition and multiplication follow from the defining properties of the PTR $(\mathcal{R}, T)$. See, for example, [17], Theorem 5.3.

1. $(\mathcal{R}, \oplus)$ is a loop with loop identity 0. This means, for any $a, b, c, d \in \mathcal{R}$, the equations

$$x \oplus a = b, \text{ and } c \oplus y = d$$

both have unique solutions x and y in $\mathcal{R}$. Also,

$$c \oplus 0 = 0 \oplus c = c \text{ for all } c \in \mathcal{R}.$$

The loop $(\mathcal{R}, \oplus)$ is called the *additive loop* of the PTR.

2. $(\mathcal{R}^*, \odot)$ is a loop with loop identity 1. This means, for any $a, c \in \mathcal{R}^*$ and $b, d \in \mathcal{R}$, the equations

$$x \odot a = b, \text{ and } c \odot y = d$$

both have unique solutions x and y in $\mathcal{R}$. Also,

$$c \odot 1 = 1 \odot c = c \text{ for all } c \in \mathcal{R}.$$

The loop $(\mathcal{R}^*, \odot)$ is called the *multiplicative loop* of the PTR.

3. $x \odot 0 = 0 \odot x = 0$ for all $x \in \mathcal{R}$.

No properties other than the loop properties of $(\mathcal{R}, \oplus)$ and $(\mathcal{R}^*, \odot)$, and the annihilating property of 0 (Property (3)) can be derived from the definitions of the PTR and the operations $\oplus$ and $\odot$. Therefore, in the case of an arbitrary PTR representation of a plane whose properties other than being a plane are not known (or not utilised during the coordinatisation), we have very little to infer from the PTR representation.

1.8.1 Additional Algebraic Properties of $(\mathcal{R}, \oplus, \odot)$ Based on the Geometric Properties of $\mathcal{P}$

In Section 1.4, we presented the Lenz classification and the Lenz-Barlotti classification of the projective planes. The classifications were based on the existence of two different types of central collineation groups, namely the elation groups and the homology groups, and the different combinations of them admitted by a plane $\mathcal{P}$. We use the term *geometric properties of* $\mathcal{P}$ to refer to the properties of the plane related to its various central collineation groups.

The geometric properties of $\mathcal{P}$, with suitable coordinatisation, yield additional properties for a PTR $(\mathcal{R}, T)$ representing the plane. As a result, the Lenz types and the LB types can be identified with special types of PTRs. Statements connecting the LB type of a plane with appropriate choices of some of the vertices and the axes of the triangle $\{\mathbf{y}, \mathbf{x}, \mathbf{O}\}$ for an optimal coordinatisation, and the properties of the resulting PTRs are given in [6], Lemma 15. We state the lemma in its entirety here because of its importance in our coordinatisation process.

Theorem 1.8.1. *([6], Lemma 15) The following statements hold.*

(i) A plane $\mathcal{P}$ which is only $((0), [0])$-transitive is necessarily LB type I.2. The plane $\mathcal{P}$ is $((0), [0])$-transitive if and only if it can be coordinatised by a linear PTR with associative multiplication $\odot$. In such cases, $\Gamma((0), [0])$ is isomorphic to the group described by $\odot$. Moreover, during coordinatisation, $\mathbf{x}$ is chosen to be the point (0).

(ii) A plane $\mathcal{P}$ which is only $((0), [0])$-transitive and $((\infty), [0, 0])$-transitive is necessarily LB type I.3. The plane $\mathcal{P}$ is $((0), [0])$-transitive and $((\infty), [0, 0])$ transitive if and only if it can be coordinatised by a linear PTR with associative multiplication $\odot$ and displaying a left distributive law.

(iii) A plane $\mathcal{P}$ which is $((\infty), [\infty])$-transitive is necessarily LB type at least II. The plane $\mathcal{P}$ is $((\infty), [\infty])$-transitive if and only if it can be coordinatised by a linear PTR with associative addition $\oplus$. In such cases, $\Gamma((\infty), [\infty])$ is isomorphic to the group described by $\oplus$. Moreover, during coordinatisation, $\mathbf{y}$ is chosen to be the point (∞).

(iv) A plane $\mathcal{P}$ which is a translation plane or dual translation plane is necessarily Lenz-Barlotti type at least IV. The plane $\mathcal{P}$ is a translation plane (resp. dual

translation plane) if and only if it can be coordinatised by a linear PTR with associative addition $\oplus$ and a left distributive law $x \odot (y \oplus z) = x \odot y + x \odot z$ (resp. a right distributive law $(x \oplus y) \odot z = x \odot z + y \odot z$). In such cases, the order of $\mathcal{P}$ must be a prime power q and the group described by $\oplus$ is elementary abelian. Moreover, during coordinatisation, $\overline{\mathbf{xy}}$ is the translation line (resp. $\mathbf{y}$ is the translation point).

(v) A plane $\mathcal{P}$ which is both a translation plane and a dual translation plane (so $[\infty]$ is a translation line and (∞) is a translation point) is necessarily Lenz-Barlotti type at least V. The plane $\mathcal{P}$ is Lenz-Barlotti type at least V if and only if it can be coordinatised by a linear PTR with associative addition $\oplus$ and both a left and right distributive law. In such cases, the order of $\mathcal{P}$ must be a prime power q and the group described by $\oplus$ is elementary abelian. Moreover, during coordinatisation, the $\overline{\mathbf{xy}}$ is the translation line and $\mathbf{y}$ is the translation point.

$\square$

1.9 Classification of the PTRs

1.9.1 PTRs Corresponding to the Lenz Types

Theorem 1.8.1 is an itemized summary of various results on the LB type of planes and the properties of the associated PTRs. In this subsection and the next, we discuss the classification of the PTRs based on their properties. We begin by stating some definitions:

Definition 1.9.1. ***Cartesian Group*** *A PTR $(\mathcal{R}, T)$ is said to be a cartesian group if the PTR addition $\oplus$ is associative.*

Definition 1.9.2. ***(Left) quasifield*** *A PTR $(\mathcal{R}, T)$ is said to be a left quasifield if it is a cartesian group and has a left distributive property.*

A right quasifield is defined similarly.

Definition 1.9.3. ***Semifield*** *A PTR $(\mathcal{R}, T)$ is said to be a semifield if it is a cartesian group and has both distributive properties.*

Definition 1.9.4. ***(Left) Nearfield*** *A PTR $(\mathcal{R}, T)$ is said to be a left nearfield if it is a left quasifield and the PTR multiplication is associative.*

The Lenz type of $\mathcal{P}$ depends on the transitive elation groups admitted by $\mathcal{P}$. The elation groups are related to the properties on the PTR addition $\oplus$. Theorem 1.8.1, (iii), for example, shows the existence of a transitive elation group can be utilised to impose the associativity of the PTR addition along with the linearity of the PTR. Existence of additional transitive elation groups results in additional properties of the PTR ([17], Section VI.3). Consequently, the Lenz type of $\mathcal{P}$ determines a broadly defined category of the PTR $(\mathcal{R}, T)$. See Table 1.3. To realise a PTR of the category given by Table 1.3, it is sufficient to choose the triangle of coordinatisation $\{\mathbf{y}, \mathbf{x}, \mathbf{O}\}$ appropriately.

Remark 1.9.5. *Given a Lenz type, the planes of types higher than the given type (i.e. higher roman numeral) all possess the properties of that type. For example, Table 1.4 shows a left quasifield is a Cartesian group which in turn is a linear PTR.*

Remark 1.9.6. *Any plane that is not of LB type I.1 is guaranteed to produce a linear PTR if coordinatised optimally. For type I.1 planes we have no such guarantees. Theorem 6.1 of [17] provides a necessary and sufficient condition for linearity of a PTR, but this falls short of showing that type I.1 planes cannot be coordinatised by a linear PTR. That said, at the time of writing, no LB type I.1 plane has ever been found that could be coordinatised by a linear PTR.*

However, within the Lenz type I, the planes of Lenz-Barlotti types I.2 or higher admit linear PTRs (recall Theorem 1.8.1, (i)). $\square$

Lenz Type	PTR Type	Properties of $(\mathcal{R}, \oplus, \odot)$
I	Ternary Ring	None
II	Cartesian Group	$(\mathcal{R}, \oplus)$ is a group
IVa	Left Quasifield	Cartesian Group with a left distributive property i.e. $\forall\ x, y, z \in \mathcal{R},\ x \odot (y \oplus z) = x \odot y + x \odot z$
V	Semifield	Cartesian Group with both left and right distributive properties i.e. $\forall\ x, y, z \in \mathcal{R},\ x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z)$, and $(y \oplus z) \odot x = (y \odot x) \oplus (z \odot x)$
VII	Field	All properties of a finite field

Table 1.3: Lenz Types and Corresponding PTRs.

1.9.2 PTRs Corresponding to the LB Types

While the elations are identified with additive properties of the PTR, the homologies are identified with its multiplicative properties ([17], Section VI.4). The Lenz-Barlotti classification, which takes into account the homologies of $\mathcal{P}$, produces a refinement of the PTR categories obtained from the Lenz types. In Table 1.5, we give the PTR types and the properties of the system $(\mathcal{R}, \oplus, \odot)$ corresponding to the different LB types. Note the LB type n.1 are the same as Lenz type n, where n can be any Roman numeral from I to VII. The additional types introduced by Barlotti's refinement are numbered n.2 or higher. Refer to Section 1.4 for the details. We repeat that in the finite case, many of the LB types are either empty or the existence of planes in those types are still open. The last column of Table 1.5 repeats this information from Table 1.2. See, for example, [9], Table 1 (p126) and [7]. We have omitted the LB types III.n and VII.1 which are known to be empty in the finite case.

Similar to the hierarchy of the PTR properties in the Lenz types, the higher

Lenz Type	**Linear PTR**	**Cartesian Group**	**Left Quasifield**	**Semifield**	**Field**
I	✓ if LB type I.2 or higher				
II	✓	✓			
IVa	✓	✓	✓		
V	✓	✓	✓	✓	
VII	✓	✓	✓	✓	✓

Table 1.4: Lenz Types and the PTR Hierarchy.

LB types within a Lenz type i.e. larger Arab numeral with the same Roman numeral, inherit the properties of the lower types. For example, a left nearfield (LB type IVa.2) is also a left quasifield (LB type IVa.1).

1.10 Substructures of $(\mathcal{R}, \oplus, \odot)$ - Further Discussion on the Algebra of the PTRs

In the previous section we discussed the broad classification of the PTRs based on the Lenz types and the LB types. Each PTR type implied certain algebraic properties for an associated system $(\mathcal{R}, \oplus, \odot)$. In this section, we discuss this connection between the geometric properties of $\mathscr{P}$ and the algebraic properties of a corresponding $(\mathcal{R}, \oplus, \odot)$ in more detail. Specifically, we will discuss the implications of the existence of non-transitive central collineation groups in $\mathscr{P}$. We begin by defining some subsets of $\mathcal{R}$ related to the algebraic properties of $(\mathcal{R}, \oplus, \odot)$.

LB Type	**PTR Type**	**Properties of** $(\mathcal{R}, \oplus, \odot)$	**Finite Plane Exists?**
I.1	Ternary Ring	None	Yes
I.2	Linear Ternary Ring	$(\mathcal{R}^*, \odot)$ is a group	Open
I.3	Linear Ternary Ring	$(\mathcal{R}^*, \odot)$ is a group and $(\mathcal{R}, \oplus, \odot)$ has a left distributive property	Open
I.4	Linear Ternary Ring	$(\mathcal{R}^*, \odot)$ is a group and $(\mathcal{R}, \oplus, \odot)$ has both left and right distributive properties	Open
II.1	Cartesian Group	$(\mathcal{R}, \oplus)$ is a group	Yes
II.2	Cartesian Group	Both $(\mathcal{R}^*, \odot)$ and $(\mathcal{R}, \oplus)$ are groups	Open
IVa.1	Left Quasifield	Cartesian group with a left distributive property	Yes
IVa.2	Left Nearfield	Left quasifield with associative multiplication i.e. both $(\mathcal{R}^*, \odot)$ and $(\mathcal{R}, \oplus)$ are groups and has a left distributive property	Yes
IVa.3	Exceptional Nearfield	Left nearfield with special properties	Yes (n=9 only)
V.1	Semifield	Cartesian group with both left and right distributive properties	Yes
VII.2	Field	All the properties of a finite field	Yes

Table 1.5: Lenz-Barlotti Types and Corresponding PTRs.

1.10.1 Some Special Subsets of $\mathcal{R}$

Definition 1.10.1. ***Measures of Associativity in*** $(\mathcal{R}^*, \odot)$

$$\mathcal{N}_l = \{a \in \mathcal{R} : a \odot (x \odot y) = (a \odot x) \odot y \textit{ for all } x, y \in \mathcal{R}\},$$
$$\mathcal{N}_m = \{a \in \mathcal{R} : x \odot (a \odot y) = (x \odot a) \odot y \textit{ for all } x, y \in \mathcal{R}\}, \textit{ and}$$
$$\mathcal{N}_r = \{a \in \mathcal{R} : x \odot (y \odot a) = (x \odot y) \odot a \textit{ for all } x, y \in \mathcal{R}\}.$$

The sets $\mathcal{N}_l$, $\mathcal{N}_r$, and $\mathcal{N}_m$ are called the *left, right,* and *middle nucleus* of the PTR, respectively. The relative orders of the nuclei with respect to the order of the coordinatising set $\mathcal{R}$ give a measure of the associativity in the multiplicative loop $(\mathcal{R}^*, \odot)$.

Remark 1.10.2. *In the interest of clarity, we note the element* $0 \in \mathcal{R}$ *is contained in each nucleus by definition. The annihilating property of* 0 *in the PTR multiplication operation (Section 1.8, Property 3) implies that it satisfies the associativity conditions trivially. Although,* $0 \notin \mathcal{R}^*$*. Under appropriate conditions which will become clear from Section 1.10.2, a nucleus excluding the* 0 *is a subgroup of* $(\mathcal{R}^*, \odot)$*. In the conventional notation, say for the middle nucleus, we would write that* $(\mathcal{N}_m^*, \odot)$ *is a subgroup of* $(\mathcal{R}^*, \odot)$*. In this dissertation, we will simply use the notation* $(\mathcal{N}_m, \odot)$ *for the subgroup noting that whether or not* 0 *is included in a particular discussion is usually clear from the context. Also, a trivial nucleus—left, middle, or right—is the set* $\{0, 1\} \subset \mathcal{R}$*.* □

Definition 1.10.3. ***Measures of Associativity in*** $(\mathcal{R}, \oplus)$

$$\mathcal{A}_l = \{a \in \mathcal{R} : a \oplus (x \oplus y) = (a \oplus x) \oplus y \textit{ for all } x, y \in \mathcal{R}\}, \textit{ and}$$
$$\mathcal{A}_r = \{a \in \mathcal{R} : x \oplus (y \oplus a) = (x \oplus y) \oplus a \textit{ for all } x, y \in \mathcal{R}\}.$$

We have not defined an additive analogue $\mathcal{A}_m$ of the middle nucleus $\mathcal{N}_m$ as we have not come across a result related to it. Also, we have not yet found an application of it in the optimal coordinatisation of planes. A *trivial* $\mathcal{A}_l$ or $\mathcal{A}_r$ is the set $\{0\} \subset \mathcal{R}$.

Definition 1.10.4. ***Measures of Distributivity in*** $(\mathcal{R}, \oplus, \odot)$

$$\begin{aligned}
\mathcal{D}_l &= \{a \in \mathcal{R} : a \odot (x \oplus y) = (a \odot x) \oplus (a \odot y) \textit{ for all } x, y \in \mathcal{R}\},\\
\mathcal{D}_r &= \{a \in \mathcal{R} : (x \oplus y) \odot a = (x \odot a) \oplus (y \odot a) \textit{ for all } x, y \in \mathcal{R}\},\\
\mathcal{D} &= \{a \in \mathcal{R} : x \odot (a \oplus y) = (x \odot a) \oplus (x \odot y) \textit{ for all } x, y \in \mathcal{R}\}, \textit{ and}\\
\mathcal{D}' &= \{a \in \mathcal{R} : (x \oplus a) \odot y = (x \odot y) \oplus (a \odot y) \textit{ for all } x, y \in \mathcal{R}\}.
\end{aligned}$$

Firstly, *distributive property* in $(\mathcal{R}, \oplus, \odot)$ has been defined here as the distribution of the PTR multiplication over the PTR addition. The sets $\mathcal{D}_l$ (resp. $\mathcal{D}_r$) consist of the elements of $\mathcal{R}$ that are left (resp. right) distributive on all of $\mathcal{R}$. The set $\mathcal{D}$ is known as the *distributor*. See, for example, [9], p. 134. It is related to the left distribution in $(\mathcal{R}, \oplus, \odot)$. We have introduced the right distribution analogue $\mathcal{D}'$ of the set $\mathcal{D}$. We use the terms *left middle distributor* for the set $\mathcal{D}$ and *right middle distributor* for the set $\mathcal{D}'$. The trivial values of $\mathcal{D}$ and $\mathcal{D}'$ are $\{0\} \subset \mathcal{R}$, and that of $\mathcal{D}_l$ and $\mathcal{D}_r$ are $\{0, 1\} \subset \mathcal{R}$.

Definition 1.10.5. ***Measure of Commutativity in*** $(\mathcal{R}^*, \odot)$

$$\mathcal{Z} = \{a \in \mathcal{R} : a \odot x = x \odot a \textit{ for all } x \in \mathcal{R}\}.$$

We call the set $\mathcal{Z}$ the *center* of the PTR multiplication. A *trivial center* is the set $\{0, 1\} \subset \mathcal{R}$.

Definition 1.10.6. ***Kernel*** *The kernel $\mathcal{K}$ is a subset of $\mathcal{R}$ defined for quasifield planes (LB type IV and above).*

$$\mathcal{K} = \mathcal{N}_l \cap \mathcal{D}_l.$$

The kernel of a quasifield is a field ([9], page 132).

1.10.2 Some Results Associating the Central Collineation Groups of $\mathcal{P}$ with Special Subsets of $\mathcal{R}$

In the unpublished paper [7], Coulter has taken the general approach to tie the existence of various elation and homology groups in $\mathcal{P}$, not necessarily transitive and therefore not necessarily relating to all of $\mathcal{R}$ (or $\mathcal{R}^*$), with various substructures of $(\mathcal{R}, \oplus, \odot)$. As noted in the paper, his results yield the results of [38] when the central collineation groups are transitive. Alternately, we can say the statements of the Theorem 1.8.1 are the special cases of the results in [7]. We state below some results from the work, to be used in optimal coordinatisation in later chapters, focusing on the special subsets of $\mathcal{R}$ and associated substructures of $(\mathcal{R}, \oplus, \odot)$.

Theorem 1.10.7. *([7] Theorem 7) Let $T : \mathcal{R}^3 \to \mathcal{R}$ be a planar ternary ring obtained from the projective plane $\mathcal{P}$ and let $\Gamma = \Gamma((0), [\infty])$. Set*

$$\mathcal{S} = \{s \in \mathcal{R} \,:\, (0,0)^\gamma = (s,0) \text{ for some } \gamma \in \Gamma\}.$$

Then, $\mathcal{S} \subseteq \mathcal{A}_l$ and $(\mathcal{S}, \oplus)$ is a group isomorphic to Γ. For all $m, x \in \mathcal{R}$, $s \in \mathcal{S}$, we have $T(m, s, m \odot x) = m \odot (s \oplus x)$. If T is linear, then $\mathcal{S} \subseteq \mathcal{A}_l \cap \mathcal{D}$. If T is a Cartesian group, then $\mathcal{S} = \mathcal{D}$.

Theorem 1.10.8. *([7], Theorem 10) Let $T : \mathcal{R}^3 \to \mathcal{R}$ be a planar ternary ring obtained from the projective plane $\mathcal{P}$ and let $\Gamma = \Gamma((0), [0])$. Set*

$$\mathcal{S} = \{s \in \mathcal{R} : (1,0)^\gamma = (s,0) \text{ for some } \gamma \in \Gamma\}.$$

Then, $\mathcal{S} \subseteq \mathcal{N}_m$ and $(\mathcal{S}, \odot)$ is a group isomorphic to Γ. For all $x, y \in \mathcal{R}$, $s \in \mathcal{S}$, we have $T(s,x,y) = (s \odot x) \oplus y$. If T is linear, then $\mathcal{S} = \mathcal{N}_m$.

Theorem 1.10.9. *([7], Theorem 12) Let $T : \mathcal{R}^3 \to \mathcal{R}$ be a planar ternary ring obtained from the projective plane $\mathcal{P}$ and let $\Gamma = \Gamma((\infty), [0,0])$. Set*

$$\mathcal{S} = \{s \in \mathcal{R} : (0,1)^\gamma = (0,s) \text{ for some } \gamma \in \Gamma\}.$$

Then, $\mathcal{S} \subseteq \mathcal{N}_l$ and $(\mathcal{S}, \odot)$ is a group isomorphic to Γ. For all $m, x, y \in \mathcal{R}, s \in \mathcal{S}$, we have $T(s \odot m, x, s \odot y) = s \odot T(m,x,y)$. If T is linear, then $\mathcal{S} = \mathcal{N}_l \cap \mathcal{D}_l$.

Theorem 1.10.10. *([7], Theorem 15) Let $T : \mathcal{R}^3 \to \mathcal{R}$ be a planar ternary ring obtained from the projective plane $\mathcal{P}$ and let $\Gamma = \Gamma((0,0), [\infty])$. Set*

$$\mathcal{S} = \{s \in \mathcal{R} : (0,1)^\gamma = (0,s) \text{ for some } \gamma \in \Gamma\}.$$

Then, $\mathcal{S} \subseteq \mathcal{N}_r \cap \mathcal{D}_r$ and $(\mathcal{S}, \odot)$ is a group isomorphic to Γ. For all $m, x, y \in \mathcal{R}$, $s \in \mathcal{S}$, we have $T(m, x \odot s, y \odot s) = T(m,x,y) \odot s$. If T is linear, then $\mathcal{S} = \mathcal{N}_r \cap \mathcal{D}_r$.

Theorem 1.10.11. *([7], Theorem 5) Let $T : \mathcal{R}^3 \to \mathcal{R}$ be a planar ternary ring obtained from the projective plane $\mathcal{P}$ and let $\Gamma = \Gamma((\infty), [\infty])$. Set*

$$\mathcal{S} = \{s \in \mathcal{R} : (0,0)^\gamma = (0,s) \text{ for some } \gamma \in \Gamma\}.$$

Then, $\mathcal{S} \subseteq \mathcal{A}_r$ and $(\mathcal{S}, \oplus)$ is a group isomorphic to Γ. For all $m, x \in \mathcal{R}$, $s \in \mathcal{S}$, we have $T(m, x, s) = (m \odot x) \oplus s$. If T is linear, then $\mathcal{S} = \mathcal{A}_r$.

The paper is included in appendix (Appendix A). We refer the reader to the paper for a detailed discussion and more results on the substructures of $(\mathcal{R}, \oplus, \odot)$. We have proved a new result, Theorem 2.0.1, related to the right middle distributor $\mathcal{D}'$ along similar lines.

1.11 $((\infty), [\infty])$-elations and the PTR Addition $\oplus$

In Section 1.7.1, Figures 1.8 and 1.9 we visualised the action of the PTR addition $\oplus$ on the vertical axis $\overline{\mathbf{O}\,\mathbf{y}} = [0]$. From Theorem 1.10.11, we know that there exists a subset $\mathcal{S}$ of $\mathcal{R}$ such that $(\mathcal{S}, \oplus)$ is isomorphic to the group $\Gamma = \Gamma((\infty), [\infty])$. This means, the action of an elation $\gamma \in \Gamma$ on a line other than the axis of the elation $[\infty]$ is identical to the action of some element of $\mathcal{S}$ on the vertical axis via the $\oplus$ operation. We want to elaborate on this connection between the $((\infty), [\infty])$-elations and the $\oplus$ operation. A pictorial illustration is also presented.

Let $\gamma_s \in \Gamma$ such that $(0, 0)^{\gamma_s} = (0, s)$. The point $\mathbf{x} = (0)$ is fixed because it lies on the axis of the elation. Therefore, $[0, 0]^{\gamma_s} = [0, s]$. Since $(x, s) \in [0, s]$ it follows that $(x, 0)^{\gamma_s} = (x, s)$. This means, $\overline{\mathbf{J}(x, 0)}^{\;\gamma_s} = \overline{\mathbf{J}(x, s)}$ because the point $\mathbf{J} = (1)$ is a fixed point too. But then, these lines meet the vertical axis $[0]$, which is fixed by the elation, at $(0, x)$ and $(0, x \oplus s)$. It follows, $(0, x)^{\gamma_s} = (0, x \oplus s)$.

By identifying the point $(0, s)$ with the element $s \in \mathcal{R}$, we say that the action of the elation $\gamma_s \in \Gamma((\infty), [\infty])$ on the vertical axis $[0]$ is equivalent to *right addition by s* in $(\mathcal{R}, \oplus)$. A pictorial view of the derivation is given in Figure 1.12.

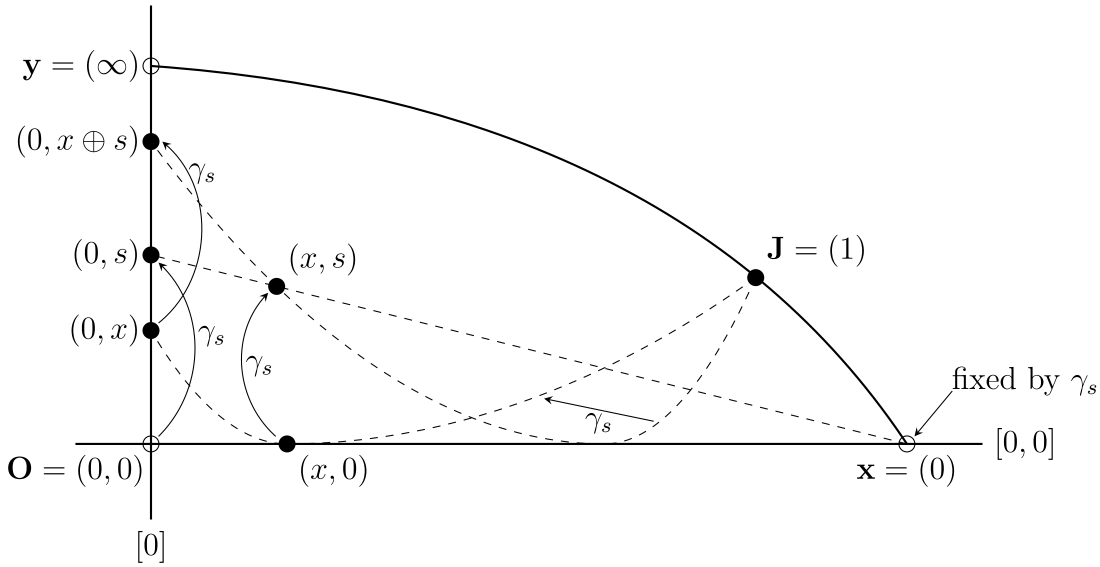

Figure 1.12: $((\infty), [\infty])$-elation γ_s and the Operation $x \oplus s$.

1.12 $\Gamma((0), [0])$-homologies and the PTR Multiplication $\odot$

In Section 1.7.1, Figures 1.10 and 1.11 we visualised the action of the PTR multiplication $\odot$ on the vertical axis $\overline{\mathbf{O}\,\mathbf{y}} = [0]$. From Theorem 1.10.8, we know that there exists a subset $\mathcal{S}$ of $\mathcal{R}$ such that $(\mathcal{S}, \odot)$ is isomorphic to the group $\Gamma = \Gamma((0), [0])$. This means, the action of a homology $\gamma \in \Gamma$ on a line other than the axis of the homology $[0]$ is identical to the action of some element of $\mathcal{S}$ on the vertical axis via the $\odot$ operation. We want to elaborate on this connection between the $((0), [0])$-homologies and the $\odot$ operation. A pictorial illustration is also presented.

Let $\gamma_s \in \Gamma$ such that $(1, 0)^{\gamma_s} = (s, 0)$. The point $(0, s) \in [0]$ is fixed because it lies on the axis of the homology. Therefore, $\overline{(s)(0, s)}^{\,\gamma_s} = \overline{(t)(0, s)}$ for some $t \in \mathcal{R}$. Since $\overline{(s)(0, s)} \cap [0, 0] = (1, 0)$ and $(1, 0)^{\gamma_s} = (s, 0)$ it follows that $\overline{(t)(0, s)} \cap [0, 0] = (s, 0)$. Note that $[0, 0]$ is fixed by γ_s as it passes through the center (0) of the homology. Referring to the Labeling Procedure 1.5.1, it must be that $t = 1$ i.e. $(s)^{\gamma_s} = \mathbf{J} = (1)$.

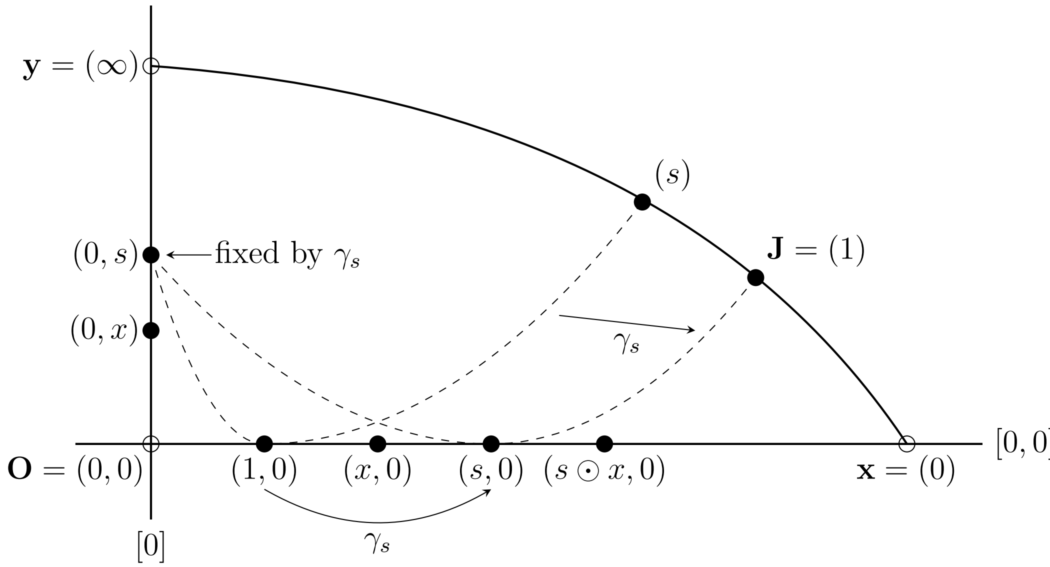

Figure 1.13: Action of the $((0), [0])$-homology γ_s on $\overline{(s)(1, 0)}$.

Consider the line $\overline{(s)(x, 0)}$. The line meets the axis of the homology $[0]$ at $(0, s \odot x)$. So, its image under the action of γ_s meets the axis $[0]$ at $(0, s \odot x)$ too. Now, $\overline{(s)(x, 0)}^{\,\gamma_s} = \overline{(s)^{\gamma_s}(x, 0)^{\gamma_s}} = \overline{\mathbf{J}\,(x, 0)^{\gamma_s}}$. Letting $(x, 0)^{\gamma_s} = (x', 0)$, we then have $\overline{\mathbf{J}(x', 0)} \cap [0] = (0, s \odot x)$. Clearly, $(x', 0) = (s \odot x, 0)$ i.e. $x' = s \odot x$.

By identifying the point $(s, 0)$ with the element $s \in \mathcal{R}$, we say that the action of the homology $\gamma_s \in \Gamma((0), [0])$ on the horizontal axis is equivalent to the *left multiplication by* s in $(\mathcal{R}^*, \odot)$. Finally, observe that the action of γ_s on the horizontal axis $[0, 0]$ can be projected on the vertical axis $[0]$ in a 1-1 correspondence because any point $(0, x) \in [0]$ is a projection of the point $(x, 0) \in [0, 0]$ via the ideal point $\mathbf{J}$, and vice-versa. A pictorial view of the derivation is given in Figures 1.13 and 1.14.

1.13 Principal Central Collineation Groups

From the discussions in Sections 1.8 and 1.10 on the connection of the central collineation groups admitted by a plane with the algebraic properties of a PTR

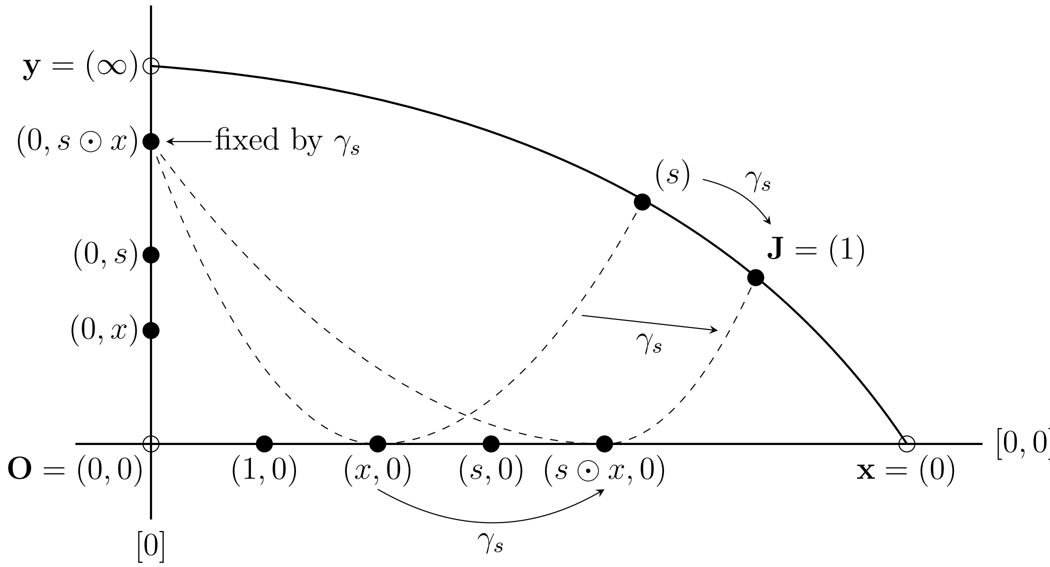

Figure 1.14: Action of γ_s and $\odot$.

obtained from its coordinatisation, we can infer the central collineation groups with centers the vertices and axes the sides of the triangle $\{\mathbf{y}, \mathbf{x}, \mathbf{O}\}$ play a special role. To highlight their importance and to make the reference easier, we refer to these central collineation groups collectively as the *principal central collineation groups*. The collection is denoted by $S(\Gamma)$.

Definition 1.13.1. ***Principal Central Collineation Groups***

$$S(\Gamma) = \{\Gamma((\infty), [\infty]), \Gamma((\infty), [0]), \Gamma((\infty), [0, 0]),$$
$$\Gamma((0), [\infty]), \Gamma((0), [0, 0]), \Gamma((0), [0]), \Gamma((0, 0), [\infty])\}.$$

The groups $\Gamma((0, 0), [0, 0])$ and $\Gamma((0, 0), [0])$ are not included as we are not aware of any results associating them with the algebraic properties of $(\mathcal{R}, \oplus, \odot)$. A subset of $S(\Gamma)$ consisting of all the elation groups will be called the set of *principal elation groups*, and same with the homologies.

1.14 Finite Fields

The discussion of the algebraic properties of a PTR in Section 1.8 and earlier remarks about the meaningful representations of $\mathscr{P}$ indicate that we are interested in associating the system $(\mathcal{R}, \oplus, \odot)$ resulting from a coordinatisation of $\mathscr{P}$ with some algebraic structure possessing stronger properties. In ring theory, it is well known that a field has the strongest of algebraic properties like the full left and the full right distribution, commutative addition and multiplication etc. We discuss the finite fields in this light.

Definition 1.14.1. ***Finite Field*** *A finite field $(\mathbb{F}_n, +, \cdot)$ is defined as a set $\mathbb{F}_n$ of n elements along with two binary operations, the addition $(+)$ and the multiplication $(\cdot)$, which satisfy the following axioms:*

1. *Closure: For any $a, b \in \mathbb{F}_n$, both $a + b$ and $a \cdot b$ belong to $\mathbb{F}_n$.*

2. *Commutativity: Addition and multiplication in $\mathbb{F}_n$ are commutative, i.e., $a + b = b + a$ and $a \cdot b = b \cdot a$ for all $a, b \in \mathbb{F}_n$.*

3. *Associativity: Addition and multiplication in $\mathbb{F}_n$ are associative, i.e., $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in \mathbb{F}_n$.*

4. *Existence of Additive and Multiplicative Identity: There exist unique elements 0 and 1 in $\mathbb{F}_n$ such that $a + 0 = a$ and $a \cdot 1 = a$ for all $a \in \mathbb{F}_n$.*

5. *Existence of Additive Inverse: For every $a \in \mathbb{F}_n$, there exists an element $b \in \mathbb{F}_n$ such that $a + b = 0$.*

6. *Existence of Multiplicative Inverse: For every nonzero element $a \in \mathbb{F}_n$, there exists an element $b \in \mathbb{F}_n$ such that $a \cdot b = 1$.*

7. *Distributivity: Multiplication distributes over addition, i.e.,* $a \cdot (b+c) = a \cdot b + a \cdot c$ *for all* $a, b, c \in \mathbb{F}_n$.

The multiplication $\cdot$ takes precedence over the addition $+$ in the evaluation of expressions over a field (for example, see Property 7 in the above definition). The well-established rule of precedence for evaluation of expressions over a field is not the same as the not-yet-settled but presumed rule of precedence in the system $(\mathcal{R}, \oplus, \odot)$ (recall Remark 1.7.3).

The finite fields have several interesting properties. We list some of them here for later use. For more details, refer to [29], Chapter 1.

1.14.2. *Some Properties of Finite Fields:*

1. *The number of elements of* $\mathbb{F}_n$ *is a prime power i.e.* $n = p^e$ *for some prime* p *and some positive integer* e.
2. *The characteristic of* $\mathbb{F}_n$ *is the smallest positive integer* m *such that* $1 + 1 + \cdots + 1$ *(m times)* $= 0$. *If* $n = p^e$, *then the characteristic equals* p.
3. *For every prime* p *and for every positive integer* e, *there exists a finite field of order* $n = p^e$.
4. *Any two finite fields of the same order are isomorphic to each other.*
5. $(\mathbb{F}_n, +)$ *is an elementary abelian* p*-group and* $(\mathbb{F}_n^*, \cdot)$ *is a cyclic group where* $\mathbb{F}_n^* = \mathbb{F}_n \setminus \{0\}$.

1.14.1 On the Generators of the Multiplicative Group of a Finite Field

Let $n = p^e$. Suppose, a be a generator of the (cyclic) multiplicative group $(\mathbb{F}_n^*, \cdot)$ of the finite field $(\mathbb{F}_n, +, \cdot)$. Then,

1. $a^{n-1} = 1$ and $\mathbb{F}_n = \{0, 1, a, a^2, \ldots, a^{n-2}\}$.

2. a^m is also a generator of $(\mathbb{F}_n^*, \cdot)$ for every $m \geq 1$, $\mathrm{Gcd}(m, n-1) = 1$.

3. For every integer $d \geq 1$, $d|e$, the restriction of the field addition and the multiplication to the set $\left\{0, 1, a^k, a^{2k}, \ldots, a^{\left(p^d-2\right)k}\right\}$ where $k = \dfrac{p^e - 1}{p^d - 1}$ forms a subfield of $(\mathbb{F}_n, +, \cdot)$. The subfield is isomorphic to $\left(\mathbb{F}_{p^d}, +, \cdot\right)$.

1.14.2 The Addition and Multiplication Tables of $(\mathbb{F}_{16}, +, \cdot)$

Since a major part of this dissertation involves coordinatising the planes of order 16 using the set of elements $\mathbb{F}_{16}$ as the coordinatising set and obtaining polynomial representations of the planes in $\mathbb{F}_{16}[X, Y, Z]$, it is useful to have the addition and the multiplication tables of the finite field $(\mathbb{F}_{16}, +, \cdot)$ handy. Let $\mathbb{F}_{16} = \{0, 1, a, a^2, \ldots, a^{14}\}$. The multiplication in $\mathbb{F}_{16}$ is defined via the rules:

- For all $x \in \mathbb{F}_{16}$, $0 \cdot x = x \cdot 0 = 0$.

- For all $1 \leq i \leq 15$, $a^i \cdot a^j = a^{i+j \text{ (mod 15)}}$ where $a^0 = 1$.

As for the addition, the rules are:

- For all $x \in \mathbb{F}_{16}$, $x + x = 0$. (The field is of characteristic 2)

- The $(x, 1+x)$ pairs are $1 + 1 = 0$, $1 + a^5 = a^{10}$, $1 + a = a^4$, $1 + a^2 = a^8$, $1 + a^3 = a^{14}$, $1 + a^{11} = a^{12}$, $1 + a^9 = a^7$, and $1 + a^6 = a^{13}$.

We also give the entire addition table for the convenience of lookup. The elements are ordered as the cosets of the subgroup $(\mathbb{F}_4, +)$ in the additive group $(\mathbb{F}_{16}, +)$. Considering the sets as ordered sets for the purpose of this discussion, let $\mathbb{F}_4 = \{0, 1, a^5, a^{10}\}$. Define, $\mathcal{G} = \mathbb{F}_4 \cup (\mathbb{F}_4 + a)$. Then, $\mathbb{F}_{16} = \mathcal{G} \cup (\mathcal{G} + a^3)$. We have chosen this ordering instead of listing the non-zero elements as ascending (or descending) powers of the generator a for a reason. In the coordinatisation of the planes of order 16, we often coordinatise a subplane of order 4 by the elements of $\mathbb{F}_4$ and use a method of labeling we call an additive labeling on the remaining points. See Section 2.3.1 for the details on additive labeling. In the Addition Table 1.6, the top left 4 by 4 block is the addition table of the PTR of a Desarguesian subplane of order 4 in a plane of order 16.

$+$	0	1	a^5	a^{10}	a	a^4	a^2	a^8	a^3	a^{14}	a^{11}	a^{12}	a^9	a^7	a^6	a^{13}
0	0	1	a^5	a^{10}	a	a^4	a^2	a^8	a^3	a^{14}	a^{11}	a^{12}	a^9	a^7	a^6	a^{13}
1	1	0	a^{10}	a^5	a^4	a	a^8	a^2	a^{14}	a^3	a^{12}	a^{11}	a^7	a^9	a^{13}	a^6
a^5	a^5	a^{10}	0	1	a^2	a^8	a	a^4	a^{11}	a^{12}	a^3	a^{14}	a^6	a^{13}	a^9	a^7
a^{10}	a^{10}	a^5	1	0	a^8	a^2	a^4	a	a^{12}	a^{11}	a^{14}	a^3	a^{13}	a^6	a^7	a^9
a	a	a^4	a^2	a^8	0	1	a^5	a^{10}	a^9	a^7	a^6	a^{13}	a^3	a^{14}	a^{11}	a^{12}
a^4	a^4	a	a^8	a^2	1	0	a^{10}	a^5	a^7	a^9	a^{13}	a^6	a^{14}	a^3	a^{12}	a^{11}
a^2	a^2	a^8	a	a^4	a^5	a^{10}	0	1	a^6	a^{13}	a^9	a^7	a^{11}	a^{12}	a^3	a^{14}
a^8	a^8	a^2	a^4	a	a^{10}	a^5	1	0	a^{13}	a^6	a^7	a^9	a^{12}	a^{11}	a^{14}	a^3
a^3	a^3	a^{14}	a^{11}	a^{12}	a^9	a^7	a^6	a^{13}	0	1	a^5	a^{10}	a	a^4	a^2	a^8
a^{14}	a^{14}	a^3	a^{12}	a^{11}	a^7	a^9	a^{13}	a^6	1	0	a^{10}	a^5	a^4	a	a^8	a^2
a^{11}	a^{11}	a^{12}	a^3	a^{14}	a^6	a^{13}	a^9	a^7	a^5	a^{10}	0	1	a^2	a^8	a	a^4
a^{12}	a^{12}	a^{11}	a^{14}	a^3	a^{13}	a^6	a^7	a^9	a^{10}	a^5	1	0	a^8	a^2	a^4	a
a^9	a^9	a^7	a^6	a^{13}	a^3	a^{14}	a^{11}	a^{12}	a	a^4	a^2	a^8	0	1	a^5	a^{10}
a^7	a^7	a^9	a^{13}	a^6	a^{14}	a^3	a^{12}	a^{11}	a^4	a	a^8	a^2	1	0	a^{10}	a^5
a^6	a^6	a^{13}	a^9	a^7	a^{11}	a^{12}	a^3	a^{14}	a^2	a^8	a	a^4	a^5	a^{10}	0	1
a^{13}	a^{13}	a^6	a^7	a^9	a^{12}	a^{11}	a^{14}	a^3	a^8	a^2	a^4	a	a^{10}	a^5	1	0

Table 1.6: Addition Table of $(\mathbb{F}_{16}, +)$.

1.15 Polynomials over Finite Fields

A *polynomial* over a finite field $(\mathbb{F}_n, +, \cdot)$ is an expression involving the elements of $\mathbb{F}_n$ and one or more indeterminates. A *term* of the polynomial is a product of an element of $\mathbb{F}_n$ and the indeterminate(s) raised to some non-negative exponent. The distinct terms of a polynomial are separated by a + sign. We say that the polynomial is the *sum* of its terms.

Definition 1.15.1. *A (univariate) polynomial $p(X)$ over $\mathbb{F}_n$ is an expression of the form*

$$p(X) = \sum_{i=0}^{r} c_i X^i,$$

where $r \geq 0$ is an integer and $c_i \in \mathbb{F}_n$ for $0 \leq i \leq r$.

Definition 1.15.2. *A (bivariate) polynomial $p(X, Y)$ over $\mathbb{F}_n$ is an expression of the form*

$$p(X, Y) = \sum_{i=0}^{r} \sum_{j=0}^{s} c_i d_j X^i Y^j,$$

where $r, s \geq 0$ are integers and $c_i, d_j \in \mathbb{F}_n$ for $0 \leq i \leq r$, $0 \leq j \leq s$.

The collection of all polynomials in a given set of indeterminates forms a *polynomial ring* over the field. A univariate polynomial ring over $\mathbb{F}_n$ is denoted as $\mathbb{F}_n[X]$, where X is the indeterminate. Similarly, a bivariate polynomial ring over $\mathbb{F}_n$ is denoted as $\mathbb{F}_n[X, Y]$ and so on. The *degree* of a polynomial is the largest sum of the indices of the indeterminates in any one term. For a univariate polynomial $p(X) \in \mathbb{F}_n$, an element $c \in \mathbb{F}_n$ is called a *zero* of the polynomial if $p(c) = 0$.

1.15.1 Interpolation Polynomials

A key result regarding the polynomials over the finite fields is the *polynomial interpolation theorem.* The theorem states that for any function defined over a finite field $(\mathbb{F}_n, +, \cdot)$, there exists a unique polynomial over $\mathbb{F}_n$ of degree at most $n - 1$ that interpolates the function at all n distinct elements of the field. Formally, let $f : \mathbb{F}_n \to \mathbb{F}_n$ be a function. Then, there exists a unique polynomial $p(X) \in \mathbb{F}_n[X]$

of degree at most $n-1$ such that $p(a) = f(a)$ for all $a \in \mathbb{F}_n$. In other words, if $\mathbb{F}_n = \{0, 1, a, a^2, \ldots, a^{n-2}\}$ and the function f takes the values $y_0 = f(0)$, $y_1 = f(a)$, $y_2 = f(a^2)$, $\ldots$, $y_{n-2} = f(a^{n-2})$, $y_{n-1} = f(1)$, then there exists a polynomial $p(X) \in \mathbb{F}_n[X]$ such that $p(0) = f(0) = y_0$ and $p\left(a^i\right) = y_i$ for $i = 1, 2, \ldots, (n-1)$. For the purposes of computing the PTR polynomials, other than using the software Magma we have used the following form of what is called the Lagrange interpolation:

$$p(X) = y_0 + \sum_{i=1}^{n-1} \left(\prod_{j=1, j \neq i}^{n-1} \frac{X - a^j}{a^i - a^j} \right) \frac{(y_i - y_0)X}{a^i}. \tag{1.15.1}$$

It is easy to see $p(0) = y_0$ and for every i_0, $1 \leq i_0 \leq (n-1)$, the inside product vanishes except when $i = i_0$. Consequently,

$$p\left(a^{i_0}\right) = y_0 + \frac{(y_{i_0} - y_0)a^{i_0}}{a^{i_0}} = y_{i_0}.$$

For multivariate functions, the univariate interpolation formula given by (1.15.1) can be applied repeatedly to obtain an interpolation polynomial of degree at most $n-1$ in each indeterminate.

An extensive study of the interpolation in finite fields is found in [4], Chapter 2. Special polynomials representing the different substructures of the finite field are presented.

1.15.2 Permutation Polynomials

An important class of polynomials over the finite fields are the permutation polynomials. A *permutation polynomial* or *PP* in short is a polynomial that maps every element of a finite field to a distinct element. In other words, given the finite field $(\mathbb{F}_n, +, \cdot)$, a polynomial $p(X) \in \mathbb{F}_n[X]$ is a PP if and only if $p(a) \neq p(b) \implies a \neq b$ in $\mathbb{F}_n$.

Permutation polynomials are used in the study of the PTRs in this dissertation. For instance, we discussed the additive loop $(\mathcal{R}, \oplus)$ and the multiplicative loop $(\mathcal{R}^*, \odot)$ in Section 1.8. As a consequence of the properties of a loop, for example, the mapping $x \to x \oplus c$ is a permutation of $\mathcal{R}$ for every fixed $c \in \mathcal{R}$. This means, if $\mathbb{F}_n$ is taken as the coordinatising set $\mathcal{R}$, then there exists a bivariate polynomial $p(X, Y) \in \mathbb{F}_n[X, Y]$ such that

(i) for all $x, y \in \mathcal{R}$, $p(x, y) = x \oplus y$, and

(ii) for any fixed $x \in \mathbb{F}_n$, the polynomial $p(x, Y)$ is a PP over $\mathbb{F}_n$.

1.15.3 Some Special Polynomials over Finite Fields

We define some special polynomials over finite fields to be used for a meaningful (and neater) presentation the PTR polynomials of the projective planes in later chapters. The PTR polynomials are introduced in Section 1.16. Let $n = p^e$ for some prime p and some positive integer e. Let $1 \leq d \leq e$ such that $d|e$ be an integer . The following polynomials are defined in $\mathbb{F}_n[X]$:

1. The polynomial $t_{p^d}(X)$ which vanishes on the subfield $\left(\mathbb{F}_{p^d}, +, \cdot\right)$ of $(\mathbb{F}_n, +, \cdot)$,

$$t_{p^d}(X) = X^{p^d} - X. \tag{1.15.2}$$

   The polynomials $t_{p^d}(X)$ for different values of d will typically be used to reflect the property of the PTRs in which a PTR behaves like a linear PTR on a subset of $\mathcal{R}^3$ and otherwise in the complement.

2. The trace $\mathrm{Tr}_{\mathbb{F}_{p^e}/\mathbb{F}_{p^d}}(X)$ which maps $\mathbb{F}_{p^e}$ into $\mathbb{F}_{p^d}$,

$$\mathrm{Tr}_{\mathbb{F}_{p^e}/\mathbb{F}_{p^d}}(X) = X + X^{p^d} + X^{p^{2d}} + \cdots + X^{p^{(k-1)d}}, \tag{1.15.3}$$

   where $k = \frac{e}{d}$. We will see the trace function in the PTR polynomials of some LB type I.1 planes.

3. The 'all ones' polynomial $h_k(X)$ as defined in [4], Section 1.6,

$$h_k(X) = 1 + X + X^2 + \cdots + X^k. \tag{1.15.4}$$

   We will see the polynomials $h_k(X)$ in some PTR polynomials of the planes we obtain in this dissertation. Examples can be found in the PTR polynomials of planes of order 16, a translation plane of order 49, and a special PTR polynomial representation of the Hall planes of any order (Chapter 6).

4. The linearised polynomials,

$$L(X) = \sum_{i=0}^{e-1} c_i X^{p^i}, \tag{1.15.5}$$

   where the coefficients c_i are in $\mathbb{F}_n$. The linearised polynomials act linearly over $\mathbb{F}_p$, that is, $L(rX + s) = rL(X) + L(s)$ for all $r \in \mathbb{F}_p$ and $s \in \mathbb{F}_n$. In particular, $L(x + y) = L(x) + L(y)$ for all $x, y \in \mathbb{F}_n$. Therefore, we will see the linearised polynomials in the PTR polynomials of the left or the right distributive PTRs.

1.16 PTR Polynomials

Given a PTR $(\mathcal{R}, T)$ obtained from a coordinatisation of a plane $\mathcal{P}$, a corresponding *PTR polynomial* is a polynomial interpolation of the tri-variate PTR function $T : \mathcal{R}^3 \to \mathcal{R}$.

Remark 1.16.1. *The PTR polynomials in this dissertation are defined over a finite field. This implies the planes are assumed to be of prime power order, the prime power conjecture notwithstanding. The conjecture states that the order of any finite projective plane must be a prime power. The set $\mathcal{R}$ of n symbols used in the co-ordinatisation is taken to be the set of elements $\mathbb{F}_n$ of the finite field $(\mathbb{F}_n, +, \cdot)$ i.e. $\mathcal{R} = \mathbb{F}_n$. We use the notations $\mathcal{R}$ or $\mathbb{F}_n$ interchangeably to denote the coordinatising set.* $\square$

A formal definition of the PTR polynomial is given next.

Definition 1.16.2. *([6], Definition 5)* ***PTR Polynomial*** *A PTR polynomial $T(X, Y, Z)$ over $(\mathbb{F}_n, +, \cdot)$ is any three variabled polynomial in $\mathbb{F}_n[X, Y, Z]$ resulting from the co-ordinatisation of a plane $\mathcal{P}$ of order n through labelling the points of $\mathcal{P}$ using elements of $\mathbb{F}_n$ and where we label $\mathbf{O} = (0, 0)$ and $\mathbf{I} = (1, 1)$.*

From Section 1.15.2, we know a PTR polynomial $T(X, Y, Z)$ of degree at most $n - 1$ each indeterminate can be obtained by polynomial interpolation. However, we are interested in obtaining simpler or more optimal forms of the PTR polynomials. To obtain such optimal polynomials is one aspect of the optimal coordinatisation of projective planes. In [6], Theorems 6 through 18, various restrictions on the form of the PTR polynomials for the planes of different LB types are derived. Based on the discussions and results in [6] and with some additional remarks, we state the following theorem:

Theorem 1.16.3. ***Restrictions on the Form of PTR Polynomials*** *([6]) Let $\mathcal{P}$ be a projective plane of order n where $n = p^e$ is a prime power. Let $(\mathcal{R}, \oplus, \odot)$ be a planar ternary ring obtained from a coordinatisation of $\mathcal{P}$. Let $(\mathcal{R}, T)$ be a PTR* *representation of $\mathcal{P}$ where $\mathcal{R}$ is taken as the set of elements $\mathbb{F}_n$ of the finite field $(\mathbb{F}_n, +, \cdot)$. Let b_{ijk}, c_{ij}, d_{ij} be elements of $\mathbb{F}_n$ which denote the coefficients of the polynomials. The coefficients denoted similarly, say c_{ij} each time, in two different statements do not necessarily represent the same element or equal value in $\mathbb{F}_n$.*

The following statements hold.

(i) Most generally,

$$T(X,Y,Z) = XYZ\sum_{i=0}^{n-3}\sum_{j=0}^{n-3}\sum_{k=0}^{n-3} b_{ijk}X^iY^jZ^k + \sum_{i=1}^{n-2}\sum_{j=1}^{n-2} c_{ij}X^iY^j + Z, \quad (1.16.1)$$

where the double sum in the middle represents the PTR multiplication i.e. $x \odot y = \sum_{i=1}^{n-2}\sum_{j=1}^{n-2} c_{ij}x^iy^j$ for all $x, y \in \mathcal{R}$.

(ii) If $\mathcal{P}$ is of LB type II.1 or above and admits an elementary abelian transitive elation group, then by coordinatising $\mathcal{P}$ such that the PTR addition is the field addition,

$$T(X,Y,Z) = \sum_{i=1}^{n-2}\sum_{j=1}^{n-2} c_{ij}X^iY^j + Z. \quad (1.16.2)$$

(iii) In (ii), if $\mathcal{P}$ is of LB type at least IVa.1 i.e. a left quasifield plane, then

$$T(X,Y,Z) = \sum_{i=0}^{e-1}\sum_{j=1}^{n-1} c_{ij}X^{p^i}Y^j + Z. \quad (1.16.3)$$

(iv) In (iii), if $\mathcal{P}$ is of LB type at least V.1 i.e. a semifield plane, then

$$T(X,Y,Z) = \sum_{i=0}^{e-1}\sum_{j=0}^{e-1} c_{ij}X^{p^i}Y^{p^j} + Z. \quad (1.16.4)$$

$\square$

Some remarks on Theorem 1.16.3:

Remark 1.16.4. *A plane of LB type II.1 or above admits a transitive elation group by definition. For planes of LB type IV or above, any transitive elation group centered on a translation line is elementary abelian (see, for example, [17], Theorem 4.14). For all the known planes of LB type II.1, the unique transitive elation groups in the planes are elementary abelian as well.*

Remark 1.16.5. *The sum on the right hand side of Equation 1.16.2 represents the PTR multiplication i.e.* $\forall x, y \in \mathcal{R},\ x \odot y = \sum_{i=1}^{n-2} \sum_{j=1}^{n-2} c_{ij} x^i y^j$.

Remark 1.16.6. *Equation (1.16.3) shows the PTR polynomial is linearized in X. Defined over a finite field of characteristic p as the PTR polynomial is, the linearization of the PTR polynomial is a consequence of the left distributive property of the PTR exhibited by any translation plane (LB type IVa.1 or higher) when coordinatised optimally. For all $x_1, x_2, y \in \mathcal{R}$,*

$$T(x_1 + x_2, y, 0) = T(x_1, y, 0) + T(x_2, y, 0).$$

We used $+$ in place of $\oplus$ on both sides since the PTR addition is the field addition by hypothesis.

Remark 1.16.7. *A statement similar to (iii) can be made for a dual plane of LB type at least IVb, with the PTR multiplication being linearized in Y.*

Remark 1.16.8. *Equation 1.16.4 shows that the PTR polynomial is linearized in both X and Y which aligns with the property that the PTR is both left and right distributive.* $\square$

In the same paper, [6], Theorem 16 gives the restrictions on the form of PTRs when the PTR multiplication $\odot$ can be identified with the field multiplication. This is possible in the presence of a cyclic transitive homology group chosen to be $\Gamma((0),[0])$. We encounter such homology groups in the Desarguesian planes. The Desarguesian plane of order 16 is coordinatised in Chapter 3. We have used a transitive homology group for labeling purpose in the coordinatisation.

Definition 1.16.9. ***Length of a Polynomial*** *Given a (reduced) polynomial over a finite field, the length of the polynomial is is the number of nonzero terms in the polynomial in expanded form.*

1.17 Background Topics for Chapter 6

In this section, we give the definitions and some historical notes on objects that we will refer to in Chapter 6 in which we obtain a generalised polynomial representation of the Hall planes.

Derivation is a method of constructing a new projective plane from a given plane if the plane has a special set of points on a line satisfying a subplane condition. The technique which evolved to become the derivation process was first developed by Ostrom [34] in 1960. The class of planes obtained from the process in [34] included, among others, the Hall planes. Since the starting point of the process was a finite field, we can say the work in the paper was a derivation of the Desarguesian plane. In the following decade, the process was generalised to other classes (or, LB types). As a result, a number of new classes of planes were obtained as also the process crystallized to its current form. Additionally, many known classes of planes were proved obtainable as derived planes. See, for example, [36], [35], [18] etc. To understand and apply the derivation on a plane, we need the definitions of a Baer subplane and a derivation set.

Definition 1.17.1. ***Baer Subplane*** *A subplane $\mathcal{Q}$ of a projective plane $\mathcal{P}$ is called a Baer subplane if every point of $\mathcal{P}$ is incident to a line of $\mathcal{Q}$ and every line of $\mathcal{P}$ is incident to a point of $\mathcal{Q}$.*

In the finite case, $\mathcal{P}$ must be of square order $n = m^2$ and any Baer subplane $\mathcal{Q}$ of $\mathcal{P}$ is of order m. See [17], Theorem 3.8.

$\square$

Definition 1.17.2. ***Derivation Set*** *Let $\mathcal{P}$ be a finite projective plane of order $n = m^2$. Let $[\infty]$ be a special line of $\mathcal{P}$ having a property we describe shortly. Obtain the affine plane $\mathcal{A} = \mathcal{P}^{[\infty]}$ by removing $[\infty]$ and all its points from $\mathcal{P}$. Suppose there is a set $\mathcal{S}$ of $m+1$ points of $[\infty]$ such that for any two points $\mathbf{a}$ and $\mathbf{b}$ of $\mathcal{A}$, there exists a Baer subplane $\mathcal{Q}$ of $\mathcal{P}$ containing $\mathbf{a}, \mathbf{b}$, and $\mathcal{S}$. Then, the set $\mathcal{S}$ is called a derivation set.*

Definition 1.17.3. ***Derivable Plane*** *A plane which contains at least one derivation set is a derivable plane.*

$\square$

1.17.1 The Derivation Process

Let $\mathcal{P}$ be a derivable projective plane. Let $[\infty]$ be a chosen line at infinity of $\mathcal{P}$ containing a derivation set $\mathcal{S}$. Let $\mathcal{A} = \mathcal{P}^{[\infty]}$ be the corresponding affine plane. At the end of the process we are describing now, a new plane will be obtained which will be called a derived plane of $\mathcal{P}$. The apostrophe $'$ is added to the symbol of an object like a point, a line, a function, or a subplane in the original plane $\mathcal{P}$ or related to the original plane to denote the corresponding object in the derived plane or related to the derived plane.

For every line $\mathcal{M}$ of the affine plane $\mathscr{A}$ which meets the line at infinity $[\infty]$ at a point in the derivation set $\mathcal{S}$, consider the points $\mathbf{a} = \mathcal{M} \cap [0, 0]$ and $\mathbf{b} = \mathcal{M} \cap [0]$. Let $\mathscr{B}$ be the Baer subplane (Definition 1.17.1) of $\mathscr{P}$ containing the points $\mathbf{a}, \mathbf{b}$, and the set $\mathcal{S}$. Then, the set of points of the affine subplane $\mathscr{B}^{[\infty]}$ is a line $\mathcal{M}'$ in the derived affine plane $\mathscr{A}'$ (replacing $\mathcal{M}$). All the affine lines of $\mathscr{A}$ meeting the line $[\infty]$ at a point not in the derivation set $\mathcal{S}$ are retained in the derived plane. The new set of lines consisting of the replaced lines and the retained lines forms the line set of a new affine plane $\mathscr{A}'$ whose closure is the derived plane $\mathscr{P}'$.

$\square$

The next definition and subsequent discussion are adapted from [32]. We will refer to the paper later again in the discussion of projective planes coordinatised by an isotopic pair of a quasifield and a pre-quasifield.

Definition 1.17.4. ***Isotope of a PTR*** *A PTR $(\mathcal{R}', T')$ is said to be an isotope of or isotopic to a given PTR $(\mathcal{R}, T)$ if there exist permutations α, β, and γ of $\mathcal{R}$ such that $T'(x, y, z) = T(x^{\alpha}, y^{\beta}, z^{\gamma})$ for all $x, y, z \in \mathcal{R}$.*

The triplet of permutations (α, β, γ) is called an isotopism. An isotopism in which $\alpha = \beta = \gamma$ is an isomorphism. The main result on isotopism for us is that isotopic PTRs coordinatise isomorphic projective planes.

Theorem 1.17.5. *([32], Theorem 11) If two PTR are isotopic, then the projective planes which they induce are isomorphic.*

$\square$

Definition 1.17.6. ***Pre-quasifield*** *A left pre-quasifield or simply a pre-quasifield is a non-empty set $\mathcal{R}$ equipped with two operations $\oplus$ and $\odot$ such that,*

(i) *$(\mathcal{R}, \oplus)$ is a group with an identity element 0.*

(ii) For all $b, c, d \in \mathcal{R},\ b \neq 0$, the equations $b \odot x = c$ and $y \odot b = d$ have unique solutions x and y in $\mathcal{R}$. In other words, $(\mathcal{R}^*, \odot)$ is a quasiloop or a loop without an identity element.

(iii) For all $b, c, d \in \mathcal{R},\ b \neq c$, the equation $b \odot x = (c \odot x) \oplus d$ has a unique solution x in $\mathcal{R}$.

(iv) For all $b, c, d \in \mathcal{R}$, we have $b \odot (c \oplus d) = (b \odot c) \oplus (b \odot d)$ i.e. $(\mathcal{R}, \oplus, \odot)$ has a left distributive property.

Definition 1.17.7. ***Quasifield*** *A quasifield is a pre-quasifield with a multiplicative unit i.e. there is a special element $1 \in \mathcal{R}$ such that $1 \odot b = b \odot 1 = b$ for all $b \in \mathcal{R}$.*

Let $(\mathcal{R}, \oplus, \odot)$ be a pre-quasifield. From Properties (i) and (iv) of a pre-quasifield, $b \odot 0 = 0$ for all $b \in \mathcal{R}$ follows easily. We say the pre-quasifield has a *right zero 0*. As a ternary system with a right zero in the sense of [32], the pre-quasifield $(\mathcal{R}, \oplus, \odot)$ has a left zero ([32], Theorem 12) and therefore a *zero* ([32], Theorem 13). Consequently, the pre-quasifield $(\mathcal{R}, \oplus, \odot)$ is isotopic to some quasifield $(\mathcal{R}, \oplus', \odot')$ by [32], Proposition 3. The two systems viz. the pre-quasifield and its isotopic quasifield, coordinatise isomorphic planes ([32], Theorem 11).

1.18 Background on the Planes of Order 16

The coordinatisation of the known planes of order 16 is presented in Chapters 3 and 4. In this section, we give a brief history of the discovery, the classification, and the interrelation of the planes of order 16. There are 22 known planes of order 16. In the descending order of LB types (skipping the dual planes and up to isomorphism), the planes are:

1. **LB type VII.2** Clearly, there is only one plane of this LB type, *the Desarguesian plane* $PG(2, 16)$ which is the classical plane of order 16 coordinatised by $(\mathbb{F}_{16}, +, \cdot)$.

2. **LB type V.1** *The SEMI4 plane* and *the SEMI2 plane* are the two semifield planes of order 16. The algebraic systems which are semifields were first studied by [13] in 1906. [25] enumerated the 23 non-isomorphic semifields of order 16 in 1960 and showed the semifields coordinatise two non-isomorphic planes. The computations were done by a combination of calculations-by-hand and the use of the SWAC (Standards Western Automatic Computer) computer system. The term semifield itself was coined by [26] in 1965 . He gave the algebraic construction of the semifields of order 16 via the field addition and two specially defined multiplications in a vector space of dimension 2 over $(\mathbb{F}_4, +, \cdot)$ ([26], Section 2).

   The Desarguesian and the semifield planes are self-dual planes.

3. **LB type IVa.1** There are 5 planes of this LB type:

   (a) *The Hall Plane of order 16* is a left quasifield plane (see also Definition 1.17.7 of a left quasifield). The Hall planes are coordinatised by the Hall quasifields which were defined by Marshall Hall [15] in 1943 for orders q^2 where q is a prime power. In the article by Hall, the quasifields are called left Veblen-Wedderburn systems following [40]. In the 1907 paper, Oswald Veblen and Joseph H. Maclagan-Wedderburn first introduced the left distributive ternary systems that came to be known as the Veblen-Wedderburn (VW) systems. The term 'quasifield' seems to have originated in the 1950s and 60s and become popular by the early 70s.

For example, Gunter Pickert uses the term '*Quasikörpen*' which translates to 'quasibodies' to describe the VW systems in his 1955 book [38]. The German word '*körper*' is typically rendered as 'field' in mathematical translations. See also [20].

The Hall planes are also derived planes obtained from the derivation of the Desarguesian plane. See Definition 1.17.2 and Chapter 6 for a description of the derivation process.

The dual Hall plane is a right quasifield plane of LB type IVb.1. The derivation of the dual Hall plane gives two non-translation planes BBH1 and BBH2 which will be described below.

(b) *The Johnson-Walker Plane* was constructed independently by Norman L. Johnson in 1971 [20] and Michael Walker in 1975 [41]. The construction of the plane as well as two other planes is implicit in [20] and Johnson determines the three non-isomorphic planes implied in the paper in a separate note published in 1978 [21].

(c) *The Lorimer-Rahilly Plane* Alan Rahilly described a translation plane of order 16 in his PhD thesis [39] at the University of Sydney in 1973. A translation plane of order 16 was constructed by Peter Lorimer in 1972 [30] (published 1974). In 1977, [24] showed that the two planes are the same and they named it the Lorimer-Rahilly plane. In an article dedicated to Henry G. Forder on his 90th birthday, [31] states that the plane was also discovered by Norman L. Johnson in 1971. Indeed, [20] obtains a list of five left quasifield multiplications for the order 16 as a result of the derivation of the SEMI4 plane. He further proves the quasifields coordinatise planes of LB type IVa.1 (see [20], Theorems 3.4 and 4.1). The paper does not classify the quasifields or explore further properties of the planes coordinatised by them. Therefore, as noted by [41], the construction of the Lorimer-Rahilly plane is implicit in the 1971 work of Johnson.

(d) *The Derived Semifield Plane* We mentioned above the derivations of the SEMI4 plane outlined in [20] yield translation planes of LB type IVa.1. and the planes were classified into three non-isomorphic classes by Johnson again in 1978 [21]. By the definition, all three derived planes in [21] are 'derived semifield planes'. Two of the planes were named the Lorimer-Rahilly plane and the Johnson-Walker plane. The remaining plane came to be called the Derived Semifield Plane, although not being 'the plane' of the type.

    Of the total five translation planes of order 16 LB type IVa.1, the other two planes, namely the Hall plane and the Dempwolff plane can also be obtained as derived planes. The original discoveries of both the planes was not through the process of derivation though.

(e) *The Dempwolff Plane* The plane was first constructed by Dempwolff [10] (unpublished) in the early 1980s. In 1983, [12] obtained a classification of the translation planes of order 16. The classification was obtained through a computer-aided exhaustive search made feasible by the use of a special equivalence class defined on what the authors called the coordinate sets of the translation planes of order 16. See [12], Section 2 for details. The method yielded all known translation planes of order 16 at the time, including the Dempwolff plane.

    Around the same time, [22] showed the Dempwolff plane can also be obtained from the derivation of the SEMI2 plane.

4. **LB type II.1** Chronologically the last plane of order 16 discovered, the *Mathon Plane* was obtained by Rudi Mathon around 1990. According to Resmini [8], Mathon presented the plane along with a complete classification of the projective planes of order 16 in a conference in 1992. At the time of writing, we have not been able to obtain the presentation of Mathon at the conference in print form (if one ever existed).

   The Mathon plane is the only known plane of order 16 LB type II.1. It is the smallest known plane in this LB type. Also, it is the only non-translation plane of order 16 not obtainable from the derivation of a translation plane (see, for example, [23]).

5. **LB type I.1** There is one self-dual and three dual pairs of planes of order 16 LB type I.1. The planes are examples of semi-translation planes in the sense of [36]. The semi-translation planes obtained from a derivation of the dual Hall planes were shown to be LB type I.1 in [18]. About a year later, Johnson extended his own result to obtain two non-isomorphic semi-translations planes from the derivation of any right quasifield plane of order 16 with a kernel $(\mathbb{F}_4, +, \cdot)$. Since the SEMI4 plane (self-dual) also satisfies this property, we obtain four non-isomorphic semi-translation planes of order 16; two each from the derivations of the SEMI4 plane and the dual Hall plane of order 16. The planes are:

   (a) *The BBH1 plane,* the only self-dual non-translation plane of order 16. It can be obtained from a derivation of the dual Hall plane of order 16.

   (b) *The BBH2 plane*, also obtained from a derivation of the dual Hall plane of order 16.

(c) *The Johnson plane and the BBS4 plane*, obtained from the derivations of the SEMI4 plane.

Remark 1.18.1. *The LB type I.1 planes of order 16 are all semi-translation planes. However, not all semi-translation planes are of LB type I.1. In fact, by the definition in [36] any translation plane is a semi-translation plane. There exist semi-translation planes of LB type II.1 as well. See, for example, [19].* □

1.19 Miscellaneous Definitions

Definition 1.19.1. ***(0,1)-Incidence Matrix*** *The (0,1)-incidence matrix of a finite projective plane is a matrix whose rows represent the lines and columns represent the points of the plane, or vice versa, such that an entry in the matrix is 1 if there is an incidence relation between the corresponding pair of line and point, and it is 0 otherwise.*

Definition 1.19.2. ***p-rank*** *The p-rank of a projective plane is the rank of its (0,1)-incidence matrix over a field of characteristic p.*

1.20 Use of Magma Software

We have used the Magma software package [3] exclusively for all computations in this dissertation. All codes for the coordinatisation of planes, and the construction of PTR multiplication tables—two major components of our research—are written and executed in Magma. Among the available Magma procedures and functions, we have mainly utilised routines related to projective and affine planes, finite fields, and some general purpose functions. The online handbook [2] was immensely useful for quick searches and examples.

Chapter 2

INTRODUCTION TO OPTIMAL COORDINATISATION OF PROJECTIVE PLANES

In this chapter, we develop general strategies for the optimal coordinatisation of projective planes. To define the sense of optimality and to integrate it in the coordinatisation process, we use results that associate the geometric properties of the plane $\mathscr{P}$ with the algebraic properties of $(\mathcal{R}, \oplus, \odot)$. Most of these results are stated and discussed in the Sections 1.8 and 1.10.

Knowing that $(\mathcal{R}, \oplus, \odot)$ is obtained from a coordinatisation of $\mathscr{P}$, the general approach to optimal coordinatisation is to analyse the behavior induced by the various central collineation groups of $\mathscr{P}$ on the various subsets of $\mathcal{R}$ with respect to the PTR addition $\oplus$, or the PTR multiplication $\odot$, or both. The analysis also includes the comparison of the properties of $(\mathcal{R}, \oplus, \odot)$ resulting from different choices made during the coordinatisation process (Section 1.5). The subsets could either be proper subsets of $\mathcal{R}$ or the entire $\mathcal{R}$ itself (or $\mathcal{R}^*$). A restricted approach would be to consider only the transitive central collineation groups that relate to all of $\mathcal{R}$ (or $\mathcal{R}^*$). Historically, the latter approach has been adopted with significant outcomes (see, for example, [38]). We take the general approach i.e. both the transitive and non-transitive central collineation groups are taken into consideration simultaneously.

In our method, the LB type of a plane, based on the existence of various transitive central collineation groups, dictates a rough outline of its coordinatisation. The existence of other smaller (non-transitive) central collineation groups is used to embed substructures like a subgroup in either $(\mathcal{R}, \oplus)$ or $(\mathcal{R}^*, \odot)$, or a subfield in $(\mathcal{R}, \oplus, \odot)$ and to optimally label the points on the vertical axis $\overline{\mathbf{O}\,\mathbf{y}}$. The results stated in Section 1.10.2 are utilised to achieve this. At the end of Section 1.10.2, we mentioned a new result along similar lines concerning the right middle distributor. We begin by proving the result.

Theorem 2.0.1. *Let $\mathcal{P}$ be a projective plane of LB type at least II and $T : \mathcal{R}^3 \to \mathcal{R}$ a Cartesian group obtained from a coordinatisation $\mathcal{P}$. Let $\Gamma = \Gamma((\infty), [0])$. Define,*

$$\mathcal{S} = \{s \in \mathcal{R} : (0)^\gamma = (s) \text{ for some } \gamma \in \Gamma\}\,, \text{ and}$$
$$\mathcal{D}' = \{a \in \mathcal{R} : (x \oplus a) \odot y = (x \odot y) \oplus (a \odot y) \text{ for all } x, y \in \mathcal{R}\}.$$

Then, $\mathcal{S} = \mathcal{D}'$.

Proof. An elation in Γ is uniquely determined by the image of any one point not on the axis $[0]$ of the elation. Let $\gamma_s : (0)^\gamma = (s)$ be the unique elation corresponding to some $s \in \mathcal{S}$. Since γ_s fixes every line through the center (∞), it induces a permutation σ_s on the points of $[\infty] \setminus \{(\infty)\}$, and permutations σ_{s_x} on the affine points of the vertical line $[x]$ for every $x \in \mathcal{R}$. Therefore, the action of γ_s on the points and lines of $\mathcal{P}$ is given by the mappings:

$$(m) \mapsto (m^{\sigma_s})$$
$$(x, y) \mapsto (x, y^{\sigma_{s_x}})$$
$$[x] \mapsto [x]$$
$$[m, k] \mapsto [m^{\sigma_s}, k].$$

Since $(\mathcal{R}, T)$ is a Cartesian group, the PTR is linear. So, $(x, y) \in [m, k] \Longleftrightarrow (m \odot x) \oplus y = k$. Therefore,

$$(x, y^{\sigma_{s_x}}) \in [m^{\sigma_s}, k] \Longleftrightarrow (m^{\sigma_s} \odot x) \oplus y^{\sigma_{s_x}} = k$$
$$\text{i.e. } (m^{\sigma_s} \odot x) \oplus y^{\sigma_{s_x}} = (m \odot x) \oplus y. \tag{2.0.1}$$

The image $y^{\sigma_{s_x}}$ of y depends on $y, x,$ and s but is independent of m. For $x = 0$, the permutation σ_{s_x} is the identity map since the vertical line $[0]$ is the axis of the elation γ_s. Consequently, $y^{\sigma_{s_0}} = y$. By setting $m = 0$ in (2.0.1), we get

$$(s \odot x) \oplus y^{\sigma_{s_x}} = y. \tag{2.0.2}$$

Using (2.0.2) and the associativity of addition in $(\mathcal{R}, T)$,

$$(m^{\sigma_s} \odot x) \oplus y^{\sigma_{s_x}} = (m \odot x) \oplus y$$
$$\Longleftrightarrow (m^{\sigma_s} \odot x) \oplus y^{\sigma_{s_x}} = (m \odot x) \oplus ((s \odot x) \oplus y^{\sigma_{s_x}})$$
$$\Longleftrightarrow (m^{\sigma_s} \odot x) \oplus y^{\sigma_{s_x}} = ((m \odot x) \oplus (s \odot x)) \oplus y^{\sigma_{s_x}} \tag{2.0.3}$$
$$\Longleftrightarrow m^{\sigma_s} \odot x = (m \odot x) \oplus (s \odot x)\,. \tag{2.0.4}$$

Equation (2.0.4) follows from (2.0.3) by cancellation laws since $(\mathcal{R}, T)$ is a Cartesian group i.e. $(\mathcal{R}, \oplus)$ is a group. Setting $x = 1$ gives $m^{\sigma_s} = m \oplus s$. Substituting in (2.0.4), we have

$$(m \oplus s) \odot x = (m \odot x) \oplus (s \odot x), \tag{2.0.5}$$

proving $s \in \mathcal{D}'$.

Conversely, suppose $s \in \mathcal{D}'$. That is, (2.0.5) holds for all $m, x \in \mathcal{R}$. Let γ_s be a mapping on the set of points and lines of $\mathcal{P}$ defined by,

$$\begin{aligned}
(\infty) &\mapsto (\infty) \\
(m) &\mapsto (m \oplus s) \\
(x, y) &\mapsto (x, (-(s \odot x)) \oplus y) \\
[x] &\mapsto [x] \\
[m, k] &\mapsto [m \oplus s, k]
\end{aligned}$$

for all $x.y, m, k \in \mathcal{R}$.

The sign '$-$' is used to denote the additive inverse of an element in the group $(\mathcal{R}, \oplus)$. We will show the mapping γ_s is an elation in $\Gamma = \Gamma((\infty), [0])$. Using the linearity of $(\mathcal{R}, T)$ and the group properties $(\mathcal{R}, \oplus)$, we get

$$\begin{aligned}
&(x, y) \in [m, k] \\
\Longleftrightarrow &(m \odot x) \oplus y = k \\
\Longleftrightarrow &((m \odot x) \oplus ((s \odot x) - (s \odot x))) \oplus y = k \\
\Longleftrightarrow &((m \odot x) \oplus (s \odot x)) \oplus ((-(s \odot x)) \oplus y) = k.
\end{aligned} \tag{2.0.6}$$

Now use (2.0.5) to make a substitution on the left side of (2.0.6),

$$\begin{aligned}
&((m \oplus s) \odot x) \oplus ((-(s \odot x)) \oplus y) = k \\
\Longleftrightarrow &(x, (-(s \odot x)) \oplus y) \in [m \oplus s, k] \\
\Longleftrightarrow &(x, y)^{\gamma_s} \in [m, k]^{\gamma_s}.
\end{aligned}$$

We have shown $(x, y) \in [m, k] \Longleftrightarrow (x, y)^{\gamma_s} \in [m, k]^{\gamma_s}$ i.e. the mapping γ_s is a collineation. The collineation γ_s is central since it fixes all the vertical lines along with the line at infinity. Since $(0, y) \mapsto (0, y)$ for all $y \in \mathcal{R}$, the axis of γ_s is the line $[0]$. The center (∞) through which all the fixed lines pass is on the axis $[0]$. Thus, γ_s is an elation. Since $(0)^{\gamma_s} = (0 \oplus s) = (s)$, we get $s \in \mathcal{S}$ and the theorem is proved. $\square$

We are now ready to build a guideline for the optimal coordinatisation of the projective planes.

2.1 On Using the Lenz-Barlotti Classification to Initiate an Optimal Coordinatisation

In the introduction to coordinatisation (Section 1.5, Remarks 1.5.5), we noted the coordinates of all points and lines of $\mathcal{P}$ are determined by the choices made in the following three steps:

(i) In selecting the triangle $\{\mathbf{y}, \mathbf{x}, \mathbf{O}\}$,

(ii) In choosing the point $\mathbf{J} = (1)$ on $[\infty]$, and

(iii) In labeling the points of $\overline{\mathbf{O}\,\mathbf{y}}$ by the elements of $\mathcal{R}$.

Let us consider in detail the implications of the choice made in the first step. In the triangle $\{\mathbf{y}, \mathbf{x}, \mathbf{O}\}$, the side $\overline{\mathbf{x}\,\mathbf{y}}$ is designated the line at infinity $[\infty]$, while the side $\overline{\mathbf{O}\,\mathbf{y}}$ becomes the vertical axis $[0]$. The point $\mathbf{y}$ is taken as the point at infinity (∞). In Section 1.7.1, we studied the actions of the PTR addition and multiplication on the vertical axis $[0]$ (Figures 1.9 and 1.11). For both the actions, the labeling of the points on the infinite axis $\overline{\mathbf{x}\,\mathbf{y}} = [\infty]$ plays an important role in determining the result of the operation. In fact, $\mathbf{J} = (1)$ is the projection of $\mathbf{u} = (0, 1)$ on the infinite axis $[\infty]$ via the point $(1, 0)$ on the horizontal axis $[0, 0]$. Similarly, the ideal point (m) is the projection of the point $(0, m)$. Since the point (∞) is common to both the axes $[0]$ and $[\infty]$, we conclude the very first step in the coordinatisation process should be to select a point of the projective plane to be the point at infinity (∞) and a line of the plane to be either the vertical axis $[0]$ or the infinite axis $[\infty]$. The criterion for selection is the point and the line so chosen should facilitate an imposition of some stronger algebraic structure on the system $(\mathcal{R}, \oplus, \odot)$. This is achieved in conjunction with suitable choices of the remaining points $\mathbf{O}$, $\mathbf{x}$, and $\mathbf{J}$ of the frame of coordinatisation, and the labeling on the vertical axis $\overline{\mathbf{O}\,\mathbf{y}} = [0]$. In this context we invoke the LB classification of projective planes to pick a point $\mathbf{y}$ and a line $\overline{\mathbf{x}\,\mathbf{y}}$ or $\overline{\mathbf{O}\,\mathbf{y}}$. The imposition of an algebraic structure on $(\mathcal{R}, \oplus, \odot)$ is then a consequence of the isomorphism between certain central collineation groups and the subgroups of either $(\mathcal{R}, \oplus)$ or $(\mathcal{R}^*, \odot)$.

Since the initiation of the coordinatisation process is dictated by the LB type of the plane, we present the methods for different LB types separately. Some steps of the process are identical for the planes of LB types equal to or higher than a certain LB type. So, for every step or strategy, we also specify the LB types to which the strategy can be applied. Finally, in the case of the planes of LB type IV, we give methods for the translation planes only (LB type IVa), noting the principle of duality can be applied to obtain the methods for the dual translation planes (LB type IVb).

2.1.1 Choices of (∞) and $[\infty]$

A projective plane of LB type at least II.1 admits at least one transitive elation group. We utilize one such group in the spirit of Theorem 1.8.1, (iii). This means, an incident flag $(\mathbf{q}, \mathcal{M})$ is identified such that $\mathscr{P}$ is $(\mathbf{q}, \mathcal{M})$-transitive. Then we label $\mathbf{q} = \mathbf{y} = (\infty)$, $\mathcal{M} = \overline{\mathbf{x}\,\mathbf{y}} = [\infty]$. The exact details of how the step is completed depends on the exact LB type of the plane.

(i) **LB Type VII.2 -** The plane $\mathscr{P}$ is the desarguesian plane of the given order which is a prime power. All flags in $\mathscr{P}$, incident and non-incident, are transitive. As such, any incident flag can be chosen to be $((\infty), [\infty])$.

(ii) **LB Type V.1 -** The plane $\mathscr{P}$ is a semifield plane. There is exactly one incident flag $(\mathbf{q}, \mathcal{M})$ such that $\mathscr{P}$ is $(\mathbf{p}, \mathcal{M})$-transitive for every point $\mathbf{p} \in \mathcal{M}$, and $(\mathbf{q}, \mathcal{L})$-transitive for every line $\mathcal{L}$ through $\mathbf{q}$. Therefore, choosing any of the flags $(\mathbf{p}, \mathcal{M})$ or $(\mathbf{q}, \mathcal{L})$ to be $((\infty), [\infty])$ forces the resulting $(\mathcal{R}, \oplus)$ to be a group (in fact, an elementary abelian group as a consequence of [17], Theorem 4.14). However, choosing the unique flag $(\mathbf{q}, \mathcal{M})$ to be $((\infty), [\infty])$ yields an additional structure on $(\mathcal{R}, \oplus, \odot)$ in the form of distributivity (Theorem 1.8.1, (v)).

(iii) **LB Type IVa.n -** The plane $\mathscr{P}$ is a translation plane. There exists a translation line $\mathcal{M}$ such that $\mathscr{P}$ is $(\mathbf{p}, \mathcal{M})$-transitive for every $\mathbf{p} \in \mathcal{M}$. Thus, $\mathcal{M} = [\infty]$ and any point on the line $\mathcal{M}$ is a potential (∞). For any such choice of $((\infty), [\infty])$, the resulting PTR has an elementary abelian addition and a left distributive property (Theorem 1.8.1, (iv)).

Remark 2.1.1. *For the planes of LB types V.1 and IVa.n, the translation line $\mathcal{M}$ is chosen to be the axis $[\infty]$. By definition, the point (∞) must be chosen from the points on the line $[\infty]$. The choice of a specific point is affected by the existence of other central collineation groups, transitive and non-transitive, and the properties we seek to equip $(\mathcal{R}, \oplus, \odot)$ with. For the planes of LB type V.1, choosing the translation point $\mathbf{q}$ to be (∞) yields a right distributive property in addition to the left distributive property obtained by choosing the translation line $\mathcal{M}$ to be $[\infty]$. A different choice means a loss of right distribution but it could yield a different nice property on some substructure of $(\mathcal{R}, \oplus, \odot)$. Similarly, for the planes of LB type IVa.n, different choices of the point (∞) from* *among the points of the translation line $\mathcal{M}$ can result in different amounts of associativity and commutativity in $(\mathcal{R}^*, \odot)$, or distributivity in $(\mathcal{R}, \oplus, \odot)$. We will see various cases in the coordinatisation of the planes of order 16 LB type II.1 or above in Chapter 3.* □

1. **LB Type II.1 -** The plane $\mathcal{P}$ admits exactly one incident transitive flag $(\mathbf{q}, \mathcal{M})$. Coordinatising the plane with $\mathbf{q} = \mathbf{y} = (\infty)$ and $\mathcal{M} = \overline{\mathbf{x}\,\mathbf{y}} = [\infty]$ results in a Cartesian group $(\mathcal{R}, \oplus)$ (Theorem 1.8.1, (iii)). Essentially, there is only one good choice of $((\infty), [\infty])$. We also make the following remarks in this regard, some of which are clarifying restatements of the above lemmas.

Remark 2.1.2. ***On $((\infty), [\infty])$-transitivity, the group structure of $(\mathcal{R}, \oplus)$, and the form of the PTR polynomial***

(a) *From Theorem 1.8.1, (iii), if $(\mathcal{R}, T)$ is linear, then the PTR addition $\oplus$ is not associative in $\mathcal{R}$ if the flag $((\infty), [\infty])$ is not a transitive flag. In other words, under the condition that a PTR be linear, a group structure can be imposed on all of $\mathcal{R}$ with the PTR addition $\oplus$ as the binary group operation only if there exists at least one incident transitive flag and only if one such flag is chosen to be $((\infty), [\infty])$. Hence, the plane must be of LB type at least II.1 for $(\mathcal{R}, \oplus)$ to be a group in a linear $(\mathcal{R}, T)$. We make this choice, that is, we have a transitive flag as $((\infty), [\infty])$ in all our coordinatisations of the planes of LB type II.1 or above.*

(b) *As mentioned in Remark 1.16.4, all known examples of the planes of LB type II.1 have an elementary abelian $\Gamma((\infty), [\infty])$. Consequently, $(\mathcal{R}, \oplus)$ is an elementary abelian group in our coordinatisations. Since the known planes are of prime power order, $(\mathcal{R}, \oplus)$ is identified with the additive* *group* $(\mathbb{F}_n, +)$ *of the finite field* $(\mathbb{F}_n, +, \cdot)$*. This means, we achieve* $x \oplus y = x + y \; \forall x, y \in \mathcal{R}$ *by a suitable labeling (see Section 2.3.1 on additive labeling with a fully optimised PTR addition).*

(c) *Hence in our coordinatisations of the known planes of LB type II.1 or above, the field addition* $+$ *is equivalent to the PTR addition* $\oplus$*. Also taking into account the linearity of* $(\mathcal{R}, T)$ *that comes with the* $((\infty), [\infty])$*-transitivity, this means, a general PTR polynomial* $T(X, Y, Z) \in \mathbb{F}_n[X, Y, Z]$ *of the planes of LB type II.1 or above has a form*

$$T(X, Y, Z) = T(X, Y, 0) + Z = M(X, Y) + Z, \tag{2.1.1}$$

for some bivariate polynomial $M(X, Y) \in \mathbb{F}_n[X, Y]$*. Note that* $M(x, y) = x \odot y$ *for all* $x, y \in \mathcal{R}$*.*

(d) *Combining the discussion in the parts (a), (b), and (c), we can formulate the idea of an aspect of 'optimality' in our optimal coordinatisations. The aspect is related to the form of the PTR polynomial resulting from a coordinatisation. With a suitable choice of* $((\infty), [\infty])$ *and careful labeling of the axis* $[0]$ *in the planes of LB type II.1 or above, we can achieve two things—separation of the PTR addition component from the PTR multiplication component in the PTR polynomial i.e. linearity, and (at least) in the case of all known planes, a replacement of* $\oplus$ *of the PTR by* $+$ *of the finite field. Both achievements are implicit in (2.1.1).*

$\square$

2.1.2 Choices of (∞) and $[\infty]$ in the Planes of LB Type I

For the planes of LB types I.2, I.3, and I.4, an optimal coordinatisation process would be initiated by choosing a non-incident transitive flag to serve as the centre-axis pair of the homology group $\Gamma((0),[0])$. To see the reason for this, recall Theorem 1.8.1, (i) and (ii). At the time of writing, no planes of these LB types are known in the finite case (see Table 1.5).

We will discuss in more detail the optimal choices of the point (∞) and the line $[\infty]$ for the planes of LB type I.1 in Chapter 4.

2.2 Choice of the Origin

After selecting the line at infinity $\overline{\mathbf{x}\,\mathbf{y}} = [\infty]$ and the point at infinity $\mathbf{y} = (\infty)$, we aim to complete the triangle $\{\mathbf{y}, \mathbf{x}, \mathbf{O}\}$ by identifying an ideal point to be $\mathbf{x} = (0)$ and an affine point to be the origin $\mathbf{O} = (0,0)$. To devise a strategy for this, we will first study the role of the triangle (the vertices and the axes) in determining the properties of $(\mathcal{R}, \oplus, \odot)$ obtained eventually. We have already discussed the role of the point $\mathbf{y}$ and the line $\overline{\mathbf{x}\,\mathbf{y}}$. The following results are related to the points $\mathbf{O}$ and $\mathbf{x}$, and the lines $\overline{\mathbf{O}\,\mathbf{y}}$ and $\overline{\mathbf{O}\,\mathbf{x}}$.

2.2.1 Role of the Principal Homology Groups

The principal homology groups correspond to the multiplicative properties in the loop $(\mathcal{R}^*, \odot)$, and to two measures of the distributive properties in $(\mathcal{R}, \oplus, \odot)$, namely the set $\mathcal{D}_l$ of left distributive elements and the set $\mathcal{D}_r$ of right distributive elements.

1. Consider Theorem 1.10.8: The set $\mathcal{S}$ in the theorem is a subset of the points on the horizontal axis $\overline{\mathbf{O}\,\mathbf{x}}$. But, according to Labeling Procedure 1.5.1, we use a labeling of points on the vertical axis $\overline{\mathbf{O}\,\mathbf{y}}$ to determine the labeling of

the remaining points and lines. This is quickly resolved since, given a point $(y, 0)$ on the horizontal axis $\overline{\mathbf{O}\,\mathbf{x}} = [0, 0]$, we can obtain its projection onto the vertical axis $\overline{\mathbf{O}\,\mathbf{y}} = [0]$ by $(0, y) = \overline{\mathbf{J}(y, 0)} \cap [0]$.

The central collineation group Γ in Theorem 1.10.8 is a homology group. Hence, the implication of the theorem is: if there exists a non-trivial homology group centered at some $\mathbf{x} = (0)$, $\mathbf{x} \neq \mathbf{y}$ on the axis $[\infty]$, and having its axis a line $\overline{\mathbf{O}\,\mathbf{y}} = [0]$ through $\mathbf{y}$, then the resulting $(\mathcal{R}, \oplus, \odot)$ has a non-trivial subgroup $(\mathcal{S}, \odot)$ in its multiplicative loop $(\mathcal{R}^*, \odot)$. Thus, Theorem 1.10.8 provides a criterion to select a point $\mathbf{x}$ and a line $\overline{\mathbf{O}\,\mathbf{y}}$. By having the order of $\Gamma((0), [0])$ as large as possible, we obtain a largest possible subgroup $(\mathcal{S}, \odot)$ of $(\mathcal{R}^*, \odot)$; assuming other restrictions are identical. We reiterate the theorem provides one of the many criteria for the selection of the points $\mathbf{x}$ and $\mathbf{O}$.

Since the set $\mathcal{S} \subseteq \mathcal{N}_m$, the middle nucleus, the subset $\mathcal{S}$ of $\mathcal{R}$ thus obtained has the following additional property related to the associativity of the PTR multiplication (Definition 1.10.1):

$$\forall s \in \mathcal{S},\ \forall x, y \in \mathcal{R}, \quad x \odot (s \odot y) = (x \odot s) \odot y. \tag{2.2.1}$$

In other words, with the orders of other principal central collineation groups (Definition 1.13.1) fixed, a larger $|\Gamma((0), [0])|$ corresponds to higher associativity in the multiplicative loop $(\mathcal{R}^*, \odot)$.

For the planes of LB type I.2 or above, the optimal PTRs are linear (Theorem 1.8.1). By Theorem 1.10.8, we have $\mathcal{S} = \mathcal{N}_m$. Then, the middle nucleus $(\mathcal{N}_m, \odot)$ is a subgroup of $(\mathcal{R}^*, \odot)$.

2. Consider Theorem 1.10.9: The theorem concerns a homology group again,

$\Gamma = \Gamma((\infty), [0,0])$. The effect similarly is to obtain a subgroup $(\mathcal{S}, \odot)$ of the multiplicative loop $(\mathcal{R}^*, \odot)$. We also get two additional properties for the elements of the set $\mathcal{S}$:

(a) Since $\mathcal{S} \subseteq \mathcal{N}_l$,

$$\forall s \in \mathcal{S},\ \forall x, y \in \mathcal{R}, \quad s \odot (x \odot y) = (s \odot x) \odot y. \tag{2.2.2}$$

(b) For the planes of LB type I.2 or above, since an optimal PTR is linear, $\mathcal{S} \subseteq \mathcal{D}_l$ and we get

$$\forall s \in \mathcal{S},\ \forall x, y \in \mathcal{R}, \quad s \odot (x \oplus y) = (s \odot x) \oplus (s \odot y). \tag{2.2.3}$$

Since the axis of the central collineations in $\Gamma((\infty), [0,0])$ is the horizontal axis $\overline{\mathbf{O}\,\mathbf{x}} = [0,0]$, Theorem 1.10.9 is applicable towards a selection of the points $\mathbf{x} = (0)$ and $\mathbf{O} = (0,0)$. Implications of (2.2.2) are similar to that of (2.2.1). Equation (2.2.3) is of particular interest in the planes of LB type less than IVa.1 and higher than I.1. To see this, recall Section 2.1.1. The planes of LB type IVa.1 or above have a left distributive PTR so long as a point on a translation line is chosen as $\mathbf{y} = (\infty)$. Whereas, (2.2.3) implies a partial left distributive property in $(\mathcal{R}, \oplus, \odot)$ is possible for the planes without a translation line. For this, a non-trivial $\Gamma((\infty), [0,0])$ must be chosen in the coordinatisation process and a linear PTR must be obtained. Larger the order of $\Gamma((\infty), [0,0])$, higher is the amount of left distributive property in the resulting $(\mathcal{R}, \oplus, \odot)$.

For the planes of LB type IVa.1 or higher, we have $\mathcal{D}_l = \mathcal{R}$ giving $\mathcal{S} = \mathcal{N}_l$ in Theorem 1.10.9. Then, the left nucleus $(\mathcal{N}_l, \odot)$ is a subgroup of $(\mathcal{R}^*, \odot)$.

3. Consider Theorem 1.10.10: The theorem provides yet another criterion to make an optimal selection of the points $\mathbf{O} = (0,0)$ and $\mathbf{x} = (0)$. A subgroup $(\mathcal{S}, \odot)$ of $(\mathcal{R}^*, \odot)$ is obtained from $\Gamma((0,0), [\infty])$. The elements of the set $\mathcal{S}$ also have a right distributive property and right associativity of multiplication:

   (a) $\mathcal{S} \subseteq \mathcal{N}_r$. So,

$$\forall s \in \mathcal{S},\ \forall x, y \in \mathcal{R}, \quad (x \odot y) \odot s = x \odot (y \odot s). \tag{2.2.4}$$

   (b) $\mathcal{S} \subseteq \mathcal{D}_r$. So,

$$\forall s \in \mathcal{S},\ \forall x, y \in \mathcal{R}, \quad (x \oplus y) \odot s = (x \odot s) \oplus (y \odot s). \tag{2.2.5}$$

   Unlike the implication of Theorem 1.10.9 for the left distributivity in $(\mathcal{R}, \oplus, \odot)$, Theorem 1.10.10 guarantees right a distributive property in the form of (2.2.5) for the planes of any LB type. For the dual translation planes of LB types IVb.n and the planes of LB type V.1 or higher, we have $\mathcal{D}_r = \mathcal{R}$ in an optimal PTR. As a result, $\mathcal{S} = \mathcal{N}_r$ i.e. the right nucleus $(\mathcal{N}_r, \odot)$ is a subgroup of $(\mathcal{R}^*, \odot)$ in the optimal PTRs of those planes. □

The choice of the origin $\mathbf{O}$ is discussed again in Section 2.4.

2.2.2 Role of the Principal Elation Groups

In this section, we discuss the influence of the principal elation groups. The elation groups correspond to the additive properties in the loop $(\mathcal{R}, \oplus)$, and to two other measures of distributive properties in $(\mathcal{R}, \oplus, \odot)$, namely the left middle distributor $\mathcal{D}$ and the right middle distributor $\mathcal{D}'$ (Definition 1.10.4). Like before, we will recall the appropriate theorems and discuss their implications.

1. Consider Theorem 1.10.7: The theorem concerns the elation group $\Gamma = \Gamma((0), [\infty])$. A subgroup $(\mathcal{S}, \oplus)$ of the additive loop is guaranteed where $\mathcal{S} = (0,0)^{\Gamma} = \{(0,0)^{\gamma} : \gamma \in \Gamma\}$. Also, the elements of the set $\mathcal{S}$ are left associative in $(\mathcal{R}, \oplus)$. These two implications of the theorem are of significance in the planes of LB type I.n where the additive loop $(\mathcal{R}, \oplus)$ is not guaranteed to be a group. They are redundant when considering the planes of LB type II.2 or above.

   Consider the planes of LB type I.2 or above for the remainder of the discussion (so the optimal PTRs are linear). By the theorem, $\mathcal{S} \subseteq \mathcal{D}$, the left middle distributor. When considering the planes of LB type IVa.n (the translation planes) or the planes of LB type V.1 or higher, this implies $\mathcal{S} = \mathcal{D} = \mathcal{R}$ as the system $(\mathcal{R}, \oplus, \odot)$ is left distributive in the optimal coordinatisations of those planes. See also [7], Corollary 8. Now, consider the planes of LB types I.2, I.3, I.4, II.1, II.2, and IVb.n. For the planes of these LB types, $(\mathcal{R}, \oplus, \odot)$ is not guaranteed to be fully left distributive. However, a partial left distributivity can be imposed on $(\mathcal{R}, \oplus, \odot)$ if it is possible to choose the flag $((0), [\infty])$ such that $\Gamma((0), [\infty])$ is non-trivial. For an optimal amount of left distributivity, we must choose a $\Gamma((0), [\infty])$ of largest possible order. Finally, for the planes of LB type II.1 or above, the left middle distributor $(\mathcal{D}, \oplus)$ is a subgroup of the additive group $(\mathcal{R}, \oplus)$ of the PTR.

2. Theorem 2.0.1: Similar to the implications of Theorem 1.10.7, a larger right middle distributor $\mathcal{D}'$ in $(\mathcal{R}, \oplus, \odot)$ is obtained by choosing a larger $\Gamma((\infty), [0])$ in the planes of LB types II.n and IVa.n. Also, the theorem is redundant for the dual translation planes of LB type IVb.n and the planes of LB types V.1 or above. Recall Theorem 1.8.1, (iv), (v). For these planes, $(\mathcal{R}, \oplus, \odot)$ is fully right distributive if a translation point is chosen to be the point at infinity (∞).

In our optimal coordinatisations, we always make this choice.

3. Theorem 1.10.11: Firstly, for the planes of LB type II.1 or above, the theorem specializes to the the case of a transitive $\Gamma((\infty),[\infty])$ and $(\mathcal{R},\oplus)$ is a group. So, the theorem is especially useful when coordinatising the planes of LB type I.n. By choosing a larger $\Gamma((\infty),[\infty])$, we obtain a larger subgroup $(\mathcal{S},\oplus)$ of the additive loop $(\mathcal{R},\oplus)$. Since $\mathcal{S}\subseteq\mathcal{A}_r$, the elements of the set $\mathcal{S}$ also contribute to the amount of right associativity in all of $(\mathcal{R},\oplus)$.

Remark 2.2.1. *The results in [7] we have used for discussions on the role of the principal central collineation groups also contain statements showing 'nice' behavior of the PTR function T on some special subsets of $\mathcal{R}^3$. See, for example, [7], Theorem 7 (iii). The subsets of $\mathcal{R}^3$ depend on the particular set of points $\mathcal{S}$ in the respective theorems. We have saved the discussion on these statements for Chapter 4 where we coordinatise the planes of order 16 LB type I.1. See also the note in Section 2.1.2 in this regard.* □

2.3 Choices of J on $\overline{\mathbf{x}\,\mathbf{y}}$, u on $\overline{\mathbf{O}\,\mathbf{y}}$, and Labeling the Remaining Points on $\overline{\mathbf{O}\,\mathbf{y}}$ Optimally

In Sections 2.1 and 2.2, we used the existence of non-trivial principal central collineation groups for choosing and labeling the triangle $\{\mathbf{y},\mathbf{x},\mathbf{O}\}$ optimally. It remains to identify a point $\mathbf{J}=(1)$ on the infinite axis $[\infty]$ and to label the points in $\overline{\mathbf{O}\,\mathbf{y}}\setminus\{\mathbf{y},\mathbf{O}\}$. This is the second phase of coordinatisation and concerns finding a 1-1 correspondence between the elements of $\mathbb{F}_n$ and the affine points of $[0]$ so that as many substructures of $(\mathcal{R},\oplus,\odot)$ as possible are isomorphic to the substructures of $(\mathbb{F}_n,+,\cdot)$. Like the previous steps of the coordinatisation process, the choices for obtaining this 1-1 correspondence are influenced, in part, by the orders of the principal central collineation groups. Apart from that, factors like the existence of subplanes of certain orders are taken into consideration. For example, if a plane $\mathcal{P}$ of order n is known to have Desarguesian subplanes of order m, then we aim to coordinatise the plane $\mathcal{P}$ such that for some subplane $\mathcal{Q}$ of order m,

- the frame of coordinatisation $\{\mathbf{y}, \mathbf{x}, \mathbf{O}, \mathbf{u}, \mathbf{J}\}$ of $\mathcal{P}$ is contained in $\mathcal{Q}$,
- the frame serves as the frame of coordinatisation for $\mathcal{Q}$ as well,
- the affine points on the vertical axis of $\mathcal{Q}$ i.e. the points on the vertical axis [0] of $\mathcal{P}$ are identified with the elements of the subfield $(\mathbb{F}_m, +, \cdot)$ of $(\mathbb{F}_n, +, \cdot)$, and
- the PTR addition $\oplus$ when restricted to the elements of $\mathbb{F}_m$ behaves like the field addition $+$ and likewise with the PTR multiplication $\odot$. This means, for all $x, y \in \mathbb{F}_m \subset \mathbb{F}_n$, we have $x \oplus y = x + y$ and $x \odot y = xy$.

If the above conditions can be satisfied, then a PTR polynomial $T(X, Y, Z) \in \mathbb{F}_n[X, Y, Z]$ representing $\mathcal{P}$ will take the form $T(X, Y, Z) = XY + Z$ when restricted to the subplane $\mathcal{Q}$. Here, restricting to the subplane $\mathcal{Q}$ means the indeterminates X, Y, and Z take values in the subset $\mathbb{F}_m$ of $\mathbb{F}_n$ which is the coordinatising set of $\mathcal{Q}$.

2.3.1 Optimising $\oplus$ or $\odot$ in Labeling

Consider the additive loop $(\mathcal{R}, \oplus)$ of a PTR $(\mathcal{R}, T)$ obtained from a coordinatisation of $\mathcal{P}$ where the coordinatising set $\mathcal{R}$ is taken as the set $\mathbb{F}_n$ of the elements of the finite field $(\mathbb{F}_n, +, \cdot)$.

Definition 2.3.1. *For a given labeling of the points of* $\overline{\mathbf{O}\,\mathbf{y}} \setminus \{\mathbf{y}\}$, *let* $\mathcal{S}_1$ *be the subset of* $\mathcal{R}$ *such that* $x \oplus y = x + y \ \forall x, y \in \mathcal{S}_1$ *where* $\oplus$ *is the PTR addition as given by Definition 1.7.1 and* $+$ *is the usual field addition. Let* $\mathcal{U} \subseteq \mathcal{S}_1$ *be a largest subset* *of $\mathcal{S}_1$ such that $(\mathcal{U}, \oplus)$ is isomorphic to a subgroup of $(\mathbb{F}_n, +)$. We say a labeling of the points of $\overline{\mathbf{O\,y}} \setminus \{\mathbf{y}\}$ is optimised with respect to the PTR addition $\oplus$ if $|\mathcal{U}|$ is the largest possible for any coordinatisation of $\mathcal{P}$ using the set of elements $\mathbb{F}_n$ as the coordinatising set. The $\oplus$ and $+$ coincide when restricted to the set $\mathcal{U}$.*

Definition 2.3.2. *We say a labeling of the points of $\overline{\mathbf{O\,y}} \setminus \{\mathbf{y}\}$ is optimised with respect to the PTR addition $\oplus$ if $|\mathcal{U}|$ is the largest possible for any coordinatisation of $\mathcal{P}$ using the set of elements $\mathbb{F}_n$ as the coordinatising set. The $\oplus$ and $+$ coincide when restricted to the set $\mathcal{U}$.*

Definition 2.3.3. *We say the PTR addition is fully optimised if $(\mathcal{R}, \oplus) \cong (\mathbb{F}_n, +)$ and $x \oplus y = x + y \ \forall x, y \in \mathcal{R}$.*

If $(\mathcal{R}, T)$ is linear and the PTR addition is fully optimised, we usually replace the $\oplus$ by $+$ in the tri-variate form $(x \odot y) \oplus z$ of the PTR function T i.e. $T(x, y, z) = (x \odot y) + z \ \forall x, y, z \in \mathcal{R}$. As a PTR polynomial, we write $T(X, Y, Z) = M(X, Y) + Z$ where $M(x, y) = x \odot y \ \forall x, y \in \mathcal{R}$.

Remark 2.3.4. *There is clearly a connection between optimising the labeling of the points of $\overline{\mathbf{O\,y}} \setminus \{\mathbf{y}\}$ with respect to the PTR addition and the having a largest elation group to be $\Gamma((\infty), [\infty])$ in accordance with the strategy of choosing the point (∞) and the line $[\infty]$ in Section 2.1.1. It is easy to see the strategy leads to a fully optimised PTR addition in the planes of LB types II.1 or above. However, the two actions are not the same. While the strategy of choosing the point (∞) and the line $[\infty]$ searches for a single largest elation group, the optimisation with respect to the PTR addition is achieved by a combination of suitably chosen principal central collineation groups $S(\Gamma)$ and a systematic labeling of the points of $\overline{\mathbf{O\,y}} \setminus \{\mathbf{y}\}$. The distinction is seen clearly in the coordinatisation of the planes of order 16 LB type I.1 where the order* *of largest additive groups under the PTR addition $\oplus$ are larger than the order of the largest elation groups in the planes. In fact, the PTR additions are fully optimised when there are no transitive elation groups admitted by the planes. For details, refer to Chapter 4, especially the 'Measures of the Algebraic Properties of $(\mathcal{R}, \oplus, \odot)$' for each coordinatisation of the planes of order 16, LB type I.1.* $\square$

Definition 2.3.5. *For a given labeling of the points of $\overline{\mathbf{O}\,\mathbf{y}} \setminus \{\mathbf{y}\}$, suppose $\mathcal{S}_2$ is a subset of $\mathcal{R}$ such that $x \odot y = xy\ \forall x, y \in \mathcal{S}_2$. Let $\mathcal{V} \subseteq \mathcal{S}_2$ be a largest subset of $\mathcal{S}_2$ such that $(\mathcal{V}^*, \odot)$ is isomorphic to a subgroup of $(\mathbb{F}_n^*, \cdot)$. We say a labeling of the points on $\overline{\mathbf{O}\,\mathbf{y}} \setminus \{\mathbf{y}\}$ is optimised with respect to the PTR multiplication $\odot$ if $|\mathcal{V}|$ is the largest possible for any coordinatisation of $\mathcal{P}$.*

Definition 2.3.6. *We say the PTR multiplication is fully optimised if $(\mathcal{R}^*, \odot) \cong (\mathbb{F}_n^*, \cdot)$ and $x \odot y = xy\ \forall x, y \in \mathcal{R}$.*

If the PTR $(\mathcal{R}, T)$ is linear and the PTR multiplication is fully optimised, we usually replace the $\odot$ by juxtaposition in the tri-variate form $(x \odot y) \oplus z$ of the PTR function T i.e. $T(x, y, z) = (xy) \oplus z\ \forall x, y, z \in \mathcal{R}$. As a PTR polynomial, we write $T(X, Y, Z) = T(1, XY, Z)$.

Remark 2.3.7. *While we focus on choosing larger $\Gamma((0), [0]), \Gamma((\infty), [0, 0])$, and $\Gamma((0, 0), [\infty])$ to obtain a larger subgroup of the multiplicative loop $(\mathcal{R}^*, \odot)$, the goal of optimising the PTR multiplication is achieved by a combination of suitable choices of the vertices $\{\mathbf{y}, \mathbf{x}, \mathbf{O}, \mathbf{u}, \mathbf{J}\}$ of the frame of coordinatisation and a systematic labeling of the points of $\overline{\mathbf{O}\,\mathbf{y}} \setminus \{\mathbf{y}\}$. The order of the largest subgroup $(\mathcal{S}, \odot)$ of $(\mathcal{R}^*, \odot)$ need not be equal to the largest order of a homology group in $\mathcal{P}$.* $\square$

Except for the Desarguesian planes, it is not possible to fully optimize both $\oplus$ and $\odot$ and obtain a linear PTR. Therefore, we make a choice and prioritise one of the two PTR operations for an optimal labeling. For example, if a plane admits a transitive elation group which is also elementary abelian but does not admit any transitive homology group, then we ensure the PTR addition is fully optimised while also obtaining a largest possible multiplicative subgroup of $(\mathcal{R}^*, \odot)$.

Next, we discuss the steps taken to optimise the labeling with respect to a particular operation. We will give generalised examples to illustrate the concept. Suppose the order of $\mathcal{P}$ is $n = p^e$ for some prime p and a positive integer e. The coordinatising set $\mathcal{R}$ is taken to be the set of elements $\mathbb{F}_n$ of the field $(\mathbb{F}_n, +, \cdot)$. Assume that the frame $\{\mathbf{y}, \mathbf{x}, \mathbf{O}, \mathbf{u}, \mathbf{J}\}$ has been chosen and labeled. Assume $(\mathcal{R}, \oplus)$ is elementary abelian and consider the PTR addition $\oplus$. Suppose we label a point of $\overline{\mathbf{O}\,\mathbf{y}} \setminus \{\mathbf{y}, \mathbf{O}, \mathbf{u}\}$ by $(0, x)$ for some $x \in \mathcal{R} \setminus \{0, 1\}$. Based on the labeling of this one point, we can label $2p - 3$ other points as described next.

Procedure 2.3.1. Additive Labeling Recall that $\mathcal{R} = \mathbb{F}_{p^e}$, the set of elements of the finite field of order p^e. Let $\mathbb{F}_{p^e} = \left\{0, 1, a, \ldots, a^{p^e-2}\right\}$ where a is the generator of $\left(\mathbb{F}^*_{p^e}, \cdot\right)$. Suppose $(\mathcal{R}, \oplus)$ is an elementary abelian group i.e. $(\mathcal{R}, \oplus) \cong (\mathbb{F}_{p^e}, +)$. Consider the labeling on the vertical axis [0] using the following substitutions:

- $(0, x \oplus x) = (0, a + a), \ldots\ldots, (0, x \oplus x \cdots \oplus x) = (0, a + a \cdots + a)$, by which $p - 2$ new points are labeled i.e. $(0, 2a), \ldots, (0, (p-1)a)$.

- $(0, 1 \oplus x) = (0, 1 + a), \ldots\ldots, (0, 1 \oplus x \oplus x \cdots \oplus x) = (0, 1 + a + a \cdots + a)$, by which $p - 1$ new points are labeled i.e. $(0, 1 + a), \ldots, (0, 1 + (p-1)a)$.

No parentheses are needed for the sums on either side because both $(\mathcal{R}, \oplus)$ and $(\mathbb{F}_n, +)$ are elementary abelian groups. We will refer to any labeling procedure similar to Procedure 2.3.1, that is, based on labeling the PTR sum $x \oplus y$ of any two elements by the sum $x + y$ in $(\mathbb{F}_n, +, \cdot)$ as *additive labeling on [0]*. It is easy to see the procedure can be generalised to non-optimal additive structures like a proper subgroup $(\mathcal{S}, \oplus)$ of the loop $(\mathcal{R}, \oplus)$ or a group $(\mathcal{R}, \oplus)$ that is not elementary abelian. The algebraic structure whose elements are used in the labeling, like the additive group $(\mathbb{F}_n, +)$ in the above example, must be isomorphic to the loop $(\mathcal{R}, \oplus)$ or $(\mathcal{S}, \oplus)$ as the case may be.

A multiplicative labeling on the vertical axis $[0]$ is similarly defined based on the finite field multiplication. For the planes we have coordinatised, the multiplicative loop is not optimal in most of them. This means, $(\mathcal{R}^*, \odot)$ is not a cyclic group (often, not a group). The Desarguesian planes are exceptions. However, there exist subgroups $(\mathcal{S}, \odot)$ of $(\mathcal{R}^*, \odot)$ as given by Theorems 1.10.8 etc. (recall Section 2.2.1). We can adopt a multiplicative labeling on a subset of $\mathcal{R}^*$ in such cases.

2.4 Homology Group $\Gamma((0,0), [\infty])$ in the Planes of LB Type IV or Above

We prove a lemma in connection with the optimal choice of the origin $\mathbf{O} = (0,0)$ based on Theorem 1.10.10.

Lemma 2.4.1. *Let $\mathcal{P}$ be a projective plane of LB type IVa or above. Suppose, in a coordinatisation of $\mathcal{P}$, a translation line $\mathcal{L}$ is chosen to be the line at infinity $[\infty]$. Let $\mathbf{p}$ and $\mathbf{q}$ be any two distinct affine points. Then $\Gamma(\mathbf{p}, [\infty]) \cong \Gamma(\mathbf{q}, [\infty])$.*

Proof. The plane $\mathcal{P}$ is a translation plane and the line $\mathcal{L} = [\infty]$ is a translation line of $\mathcal{P}$. Therefore, given two distinct affine points $\mathbf{p}$ and $\mathbf{q}$, there exists a central collineation α having its axis the line $[\infty]$ and satisfying $\mathbf{p}^\alpha = \mathbf{q}$. Consider any homology $\gamma \in \Gamma(\mathbf{p}, [\infty])$. By Corollary 2 of [17], Lemma 4.11, $\alpha^{-1}\Gamma(\mathbf{p}, [\infty])\alpha = \Gamma(\mathbf{p}^\alpha, [\infty]^\alpha) = \Gamma(\mathbf{q}, [\infty])$. Since both central collineation groups are subgroups of Aut $\mathcal{P}$, we have proved $\Gamma(\mathbf{p}, [\infty]) \cong \Gamma(\mathbf{q}, [\infty])$.

□

An implication of Lemma 2.4.1 is that the search for an optimal choice of the origin $\mathbf{O} = (0,0)$ based on the order of $\Gamma((0,0),[\infty])$ is meaningful only in the planes of LB types less than IVa (and for the dual translation planes of LB type IVb). We would like to note the lemma does not imply all possible choices of the point $\mathbf{O}$ are equally optimal. The order of $\Gamma((0,0),[\infty])$ is only one of the many aspects of an optimal coordinatisation.

2.5 Conclusion

We have discussed the general theory and developed broad strategies for the optimal coordinatisation of the projective planes. We would like to illustrate them with concrete examples. Therefore, Chapter 3 is devoted to the coordinatisation of some planes of order 16.

Chapter 3

SOME OPTIMAL COORDINATISATIONS OF PROJECTIVE PLANES OF ORDER 16 LB TYPE II.1 OR ABOVE

Throughout this chapter and the next, $\mathscr{P}$ is a plane of order 16 and coordinatised with the set of elements $\mathbb{F}_{16}$ of the field $(\mathbb{F}_{16}, +, \cdot)$ of order 16. Refer to Section 1.18 for a note on the known planes of order 16. Following the notations in Section 1.14.1, we have $\mathcal{R} = \left\{0, 1, a, a^2, \cdots, a^{14}\right\}$ where a is a generator of the multiplicative group $(\mathbb{F}_{16}^{*}, \cdot)$. Let $(\mathbb{F}_4, +, \cdot)$ be the subfield of order 4 of $(\mathbb{F}_{16}, +, \cdot)$. Then, $\mathbb{F}_4 = \{0, 1, a^5, a^{10}\}$.

We coordinatise the planes of order 16 LB types II.1 or above in this chapter. The theory developed in Chapter 2 is used to make 'optimal' choices. Often, there is more than one aspect of the optimality. In such cases, the sense of optimality is described. For each plane,

- a description of all collineation groups admitted by $\mathscr{P}$ is given,
- the sizes of the principal central collineation groups $S(\Gamma)$ are tabulated, and
- a description of the coordinatisation process or general strategies for coordinatisation if there are more than one type of optimal coordinatisations is given along with the reasons behind the choices made.

For each coordinatisation,

- the measures of some algebraic properties of $(\mathcal{R}, \oplus, \odot)$ are given (refer to Section 1.10.1 for the definitions of the measures), and

- the PTR polynomial(s) is (are) obtained. The form and the complexity of the polynomials and the information readily presented by them is a measure of the optimality of coordinatisation.

The descriptions of the central collineation groups of all the planes coordinatised in this chapter and in later chapters are obtained with the help of magma. After computing all non-trivial central collineation groups using magma, we analysed the centers and axes of the groups to obtain the descriptions given here.

Remark 3.0.1. ***On Linearity of the PTRs and the PTR Addition*** *By Theorem 1.8.1, (iii), and since we make the optimal choices based on the theory developed in Chapter 2, all PTRs obtained in this chapter are linear. Additionally, the PTR additions* $\oplus$ *in all algebraic systems* $(\mathcal{R}, \oplus, \odot)$ *are made to coincide with the field addition* $+$ *in* $(\mathbb{F}_{16}, +, \cdot)$. *This is based on Remark 2.1.2, (a)-(c) and the use of additive labeling to obtain a fully optimised PTR addition. Refer to Section 2.3.1 for a description of the additive labeling. The details of the techniques applied to attain a fully optimised PTR addition are given in the description of the coordinatisation strategy of each plane.* $\square$

The points and lines data of the planes are downloaded from the website Moorehouse [33]. Each plane data consists of 273 linesets containing 17 points each.

Remark 3.0.2. ***A note on uses of*** $\oplus$ ***and*** $\odot$ ***as binary operations****: By Definition 1.7.1, the operations* $\oplus$ *and* $\odot$ *are defined for the elements of the coordinatising set* $\mathcal{R}$. *On the other hand, from the actions of* $\oplus$ *and* $\odot$ *on the vertical axis as described in Section 1.7.1, we have interpretations of the 'addition' and 'multiplication'* *directly on the points of the vertical axis. Based on these two meanings of the operations and the one-one relation between the points of the vertical axis* $[0]$ *and the elements of* $\mathcal{R}$*, we will sometimes use the notations of the type* $\mathbf{a} \oplus \mathbf{b}$ *to refer to the point obtained as a result of the action of* $\oplus$ *on the points* $\mathbf{a}$ *and* $\mathbf{b}$ *of the axis* $[0]$*. We will also sometimes stretch the interpretation to add two points on the horizontal axis if from the context it is clear that the plane has been coordinatised and the corresponding points on the vertical axis can be obtained via projection.*

3.1 The Desarguesian Plane

The Desarguesian plane $\mathscr{P}$ of order 16 is the classical plane $PG(2, 16)$ of LB type VII.2.

3.1.1 Collineation Groups of $\mathscr{P}$

Every flag $(\mathbf{p}, \mathcal{L})$ of $\mathscr{P}$, incident or non-incident, admits a transitive central collineation group.

3.1.2 Coordinatisation Strategy

As every flag is optimal in terms of its central collineation group, the triangle of coordinatisation $\{\mathbf{y}, \mathbf{x}, \mathbf{O}\}$ is chosen and labeled arbitrarily. The orders of the principal central collineation groups $S(\Gamma)$ are given in Table 3.1.

(V, l)	$((\infty), [\infty])$	$((0), [0])$	$((\infty), [0, 0])$	$((0), [\infty])$	$((0, 0), [\infty])$	$((\infty), [0])$
$\lvert\Gamma(V, l)\rvert$	16	15	15	16	15	16

Table 3.1: The Desarguesian Plane - Orders of $\Gamma \in S(\Gamma)$.

Following the notations in the Labeling Procedure 1.5.1, we have $\mathbf{x} = (0)$, $\mathbf{O} = (0, 0)$, $\mathbf{y} = (\infty)$, and $\overline{\mathbf{O}\,\mathbf{y}} = [0]$. Next, any point $\mathbf{J} = (1)$ on $\overline{\mathbf{x}\,\mathbf{y}} \setminus \{\mathbf{x}, \mathbf{y}\}$ and any point $\mathbf{u} = (0,1)$ on $\overline{\mathbf{O}\,\mathbf{y}} \setminus \{\mathbf{O}, \mathbf{y}\}$ are chosen. In summary, any valid frame of coordinatisation $\{\mathbf{y}, \mathbf{x}, \mathbf{O}, \mathbf{u}, \mathbf{J}\}$ can be chosen in $\mathcal{P}$.

The order of the homology group $\Gamma((0),[0])$ is 15 and the group is cyclic. This means, $(\mathcal{R}^*, \odot)$ is isomorphic to the multiplicative group $(\mathbb{F}_{16}^*, \cdot)$. In fact, this is true for all finite Desarguesian planes of any order. A generator of the homology group is chosen, say γ. We identify the homology γ with a generator of $(\mathcal{R}^*, \odot)$, say $\mathcal{S}$. There are 8 distinct generators in a cyclic group of order 15. We will discuss the choice of the generator in detail when obtaining the PTR polynomial for the coordinatisation. Recall from Section 1.12 that the action of the homology γ is equivalent to a left multiplication by $\mathcal{S}$ in $(\mathcal{R}^*, \odot)$. Accordingly, the points on $\overline{\mathbf{O}\,\mathbf{y}} \setminus \{\mathbf{y}, \mathbf{O}\}$ are labeled using the rule $\mathbf{s} = (0, a^i)$ if $\overline{\mathbf{s}\,\mathbf{J}} \cap [0,0] = (1,0)^{\gamma^i}$ for all $1 \leq i \leq 15$.

3.1.3 Measures of the Algebraic Properties of $(\mathcal{R}, \oplus, \odot)$

Measures of associativity of addition: Since $(\mathcal{R}, \oplus)$ is a group,

$$\mathcal{A}_l = \mathcal{A}_m = \mathcal{A}_r = \mathcal{R}.$$

Measures of associativity of multiplication: Since $(\mathcal{R}^*, \odot)$ is a group,

$$\mathcal{N}_l = \mathcal{N}_m = \mathcal{N}_r = \mathcal{R}.$$

Measures of distributive property: $(\mathcal{R}, \oplus, \odot)$ is both left and right distributive. Consequently, the left and right middle distributors are also maximal.

$$\mathcal{D}_l = \mathcal{D}_r = \mathcal{D} = \mathcal{D}' = \mathcal{R}.$$

Measure of commutativity of multiplication: Since $(\mathcal{R}^*, \odot) \cong (\mathbb{F}_{16}^*, \cdot)$,

$$\mathcal{Z} = \mathcal{R}.$$

3.1.4 The PTR Polynomial

In the coordinatisation process, we optimised the behavior of the PTR multiplication $\odot$ by choosing a generator of the cyclic homology group $\Gamma((0), [0])$ and using its action repeatedly to label the affine points on the vertical axis (except the origin). In short, we have a fully optimised PTR multiplication. We can replace the PTR multiplication $\odot$ by the field multiplication to obtain the following form of the PTR polynomial

$$T(X, Y, Z) = (XY) \oplus Z. \tag{3.1.1}$$

The additive group $(\mathcal{R}, \oplus)$ isomorphic to $(\mathbb{F}_{16}, +)$. But it is not enough to replace the $\oplus$ by $+$ in the PTR polynomial. For that, the labeling on $[0]$ must be fully optimized with respect to $\oplus$ as well. In this context we take a closer look at the choice of the generator of $(\mathcal{R}^*, \odot)$ in the coordinatisation process.

The generators of $(\mathbb{F}_{16}^*, \cdot)$ are $\{a, a^2, a^4, a^8, a^7, a^{11}, a^{13}, a^{14}\}$. The set of generators can be divided into two subsets; $\mathcal{G} = \{a, a^2, a^4, a^8\}$ and $\mathcal{H} = \{a^7, a^{11}, a^{13}, a^{14}\}$. For any $g \in \mathcal{G}$, we have $1 + g = g^4$ and for any $h \in \mathcal{H}$, we have $1 + h = h^{12}$. So, if the chosen generator γ of the homology group $\Gamma((0), [0])$ satisfies $\mathbf{u} \oplus \mathbf{s} = \mathbf{r}$ where $\mathbf{s}$ is the projection of $(1, 0)^\gamma$ and $\mathbf{r}$ the projection of $(1, 0)^{\gamma^4}$ onto the vertical axis $[0]$, then we identify γ with any generator g in $\mathcal{G}$. Without loss of generality, we chose $g = a$ in the coordinatisation above. The choice to identify γ with a generator in $\mathcal{H}$ can be made in a similar way. Note the set $\{0, 1, g, g^4\}$ is a subgroup of $(\mathbb{F}_{16}, +)$ having index 4 for every $g \in \mathcal{G}$ and similarly the set $\{0, 1, h, h^{12}\}$ for every $h \in \mathcal{H}$.

The labeling done this way is fully optimised with respect to the PTR addition $\oplus$ as well. Replacing $\oplus$ by $+$ in (3.1.1),

$$T(X, Y, Z) = XY + Z. \tag{3.1.2}$$

We dropped the parenthesis in (3.1.2) because the expression on the right side is now a polynomial in $\mathbb{F}_{16}[X, Y, Z]$.

We end the discussion on coordinatising the Desarguesian plane optimally with yet another note on the optimal labeling of points on the vertical axis $[0]$. There are many different ways to approach this part of the coordinatisation process. For example, we could start by choosing a homology $\gamma \in \Gamma((0), [0])$ of order 5. Then γ can be identified with an element of order 5 in $(\mathcal{R}^*, \odot)$, say a^3. Using this, we can label the points $(0, x)$ for $x \in \{a^3, a^6, a^9, a^{12}, a^{15}\}$ where $a^{15} = 1$ must correspond to $\mathbf{u} = (0, 1)$. Or, we could first coordinatise a subplane of order 4 sharing the same frame of coordinatisation as $\mathcal{P}$ and its affine points on the vertical axis corresponding to the elements in the set $\mathbb{F}_4 = \{0, 1, a^5, a^{10}\}$. This is usually achieved by using an appropriate homology γ of order 3. Each of these approaches requires additional steps to be taken to ensure a labeling that is fully optimised in both addition $\oplus$ and multiplication $\odot$, but the idea is clear by now.

3.2 The SEMI4 Plane

The SEMI4 plane $\mathcal{P}$ is the semifield plane of order 16 with kernel $GF(4)$. The semifield planes are of LB type V.1.

3.2.1 Collineation Groups of $\mathscr{P}$

There is a translation line $\mathcal{M}$ and a translation point $\mathbf{q}$ in $\mathscr{P}$. Besides, there are a total of 768 non-incident flags $(\mathbf{p}, \mathcal{L})$ with $|\Gamma(\mathbf{p}, \mathcal{L})| = 3$. They can be grouped into three sets of 256 flags each. We call the corresponding $\Gamma(\mathbf{p}, \mathcal{L})$ the *homology groups of type* A, B, or C.

A. $\mathbf{p} \notin \mathcal{M}$, $\mathcal{L} = \mathcal{M}$.

B. $\mathbf{p} = \mathbf{q}$, $\mathbf{q} \notin \mathcal{L}$.

C. $\mathbf{q} \neq \mathbf{p} \in \mathcal{M}$, $\mathbf{q} \in \mathcal{L} \neq \mathcal{M}$.

3.2.2 Coordinatisation Strategy

From Theorem 1.8.1, (v), the choice of the translation point as $\mathbf{y} = (\infty)$ and the translation line as $\overline{\mathbf{x}\,\mathbf{y}} = [\infty]$ is clear. This ensures $(\mathcal{R}, \oplus, \odot)$ is both left and right distributive. Also, the orders of all $\Gamma \in S(\Gamma)$ are determined by this choice. The orders are given in Table 3.2. The reason why they are fixed is discussed next.

(V, l)	$((\infty), [\infty])$	$((0), [0])$	$((\infty), [0, 0])$	$((0), [\infty])$	$((0, 0), [\infty])$	$((\infty), [0])$
$\lvert\Gamma(V, l)\rvert$	16	3	3	16	3	16

Table 3.2: The SEMI4 Plane - Orders of $\Gamma \in S(\Gamma)$.

All three principal elation groups are transitive i.e. of order 16 each since they are centered on $\mathbf{y}$ and $\mathbf{x}$, and $\overline{\mathbf{x}\,\mathbf{y}}$ is a translation line and $\mathbf{y}$ is a translation point. For the principal homology groups, consider the three types of homology groups A, B, and C. In the set of flags for each type, the existence of one flag with a homology group Γ automatically determines that every other flag in the set admits a homology group isomorphic to Γ. This follows from the application of [17], Lemma

4.11 and its corollaries and consequences. For example, recall Lemma 2.4.1 which we proved based on Corollary 2 of [17], Lemma 4.11. Lemma 2.4.1 yields the set for the homology groups of type A once it is known that a randomly chosen flag in the set admits a homology group of order 3. Now, from the description of the homology groups of type C, we see the order of $\Gamma((0),[0])$ is 3 for any choice of the vertex $\mathbf{x}=(0)$ on the axis $[\infty]$ and any choice of the axis $\overline{\mathbf{O}\,\mathbf{y}}=[0]$ through the vertex (∞). Similarly from the descriptions of the homology groups of the other two types, the orders of $\Gamma((0,0),[\infty])$ and $\Gamma((\infty),[0,0])$ are both 3 for any possible choice of flags.

As a result, our coordinatisation strategy for this part focuses on optimising the substructures of $(\mathcal{R},\oplus,\odot)$ resulting from the choice of the $\{\mathbf{y},\mathbf{x},\mathbf{O}\}$. This means, we vary the points $\mathbf{x}$ and $\mathbf{O}$ in the respective sets of possible points and analyse the subsets of $\mathcal{R}$ associated with the groups in $S(\Gamma)$. We also compute and analyse the PTR polynomials arising from each choice. Another noticeable aspect of the coordinatisation strategy for the SEMI4 plane is that we do not make random choices of the points $\mathbf{u}=(0,1)$ on the axis $[0]$ and $\mathbf{J}=(1)$ on the axis $[\infty]$. The reason for this is self-explained as we develop the strategy.

Since the PTR is linear, in applying Theorem 1.10.8 to the PTR we find the set $\mathcal{S}=\mathcal{N}_m$, the middle nucleus. Therefore, $(\mathcal{N}_m,\odot)$ is a group and $|\mathcal{N}_m|=|\mathcal{S}|=|\,\Gamma((0),[0])\,|=3$. Additionally, the PTR is left distributive i.e. $\mathcal{D}_l=\mathcal{R}$. So, in applying Theorem 1.10.9 we have the set $\mathcal{S}=\mathcal{N}_l$, the left nucleus. Therefore, $(\mathcal{N}_l,\odot)$ is a group and $|\mathcal{N}_l|=|\mathcal{S}|=|\,\Gamma((\infty),[0,0])\,|=3$. Again, since the PTR is right distributive as well i.e. $\mathcal{D}_r=\mathcal{R}$, in applying Theorem 1.10.10 we have the set $\mathcal{S}=\mathcal{N}_r$, the right nucleus. Therefore, $(\mathcal{N}_r,\odot)$ is a group and $|\mathcal{N}_r|=|\,\Gamma((0,0),[\infty])\,|=3$. In summary, the three theorems guarantee that $(\mathcal{N}_m,\odot)\,,(\mathcal{N}_l,\odot)$, and $(\mathcal{N}_r,\odot)$ are all subgroups of order 3 of the multiplicative loop $(\mathcal{R}^*,\odot)$. Clearly, they are isomorphic to each other. Whether or not any two of them or all of them are also equal as sets depends on the choices made in choosing the frame $\{\mathbf{y}, \mathbf{x}, \mathbf{O}, \mathbf{u}, \mathbf{J}\}$. For some choices of $\{\mathbf{y}, \mathbf{x}, \mathbf{O}, \mathbf{u}, \mathbf{J}\}$, the same set of three points on the vertical axis exhibit the left, middle, and right associativity of multiplication on all of $\mathcal{R}$. For some other choices, there are 2 or 3 distinct sets of points, each exhibiting one or two of the associative properties. Connect this with the earlier statement that $\mathbf{u}$ and $\mathbf{J}$ are not chosen randomly. We have obtained coordinatisations admitting each of the possibilities.

Before developing the coordinatisation strategy further, let us take a closer look at the multiplicative structure in the loop $(\mathcal{R}^*, \odot)$ induced by the principal homologies. Let γ be a non-trivial homology in any of the principal homology groups i.e. $\gamma \neq 1$ and $\gamma^3 = 1$. Let $\mathcal{L}^*$ be the set of 15 points on a line $\mathcal{L}$ through the center of the homology γ excluding the center itself and the intersection of the line $\mathcal{L}$ with the axis of γ. Consider the permutation induced by γ on the set $\mathcal{L}^*$. By the definition of a homology, the permutation has no fixed points in $\mathcal{L}^*$. Permutations like this induced on the different lines all have the same cycle structure, that is, 5 distinct orbits of lengths 3 each. This is because all non-trivial homologies are of order 3 and all non-axial lines contain 15 points not fixed by the corresponding homology. For instance, let the homology be $1 \neq \gamma \in \Gamma((0), [0])$ and let the line be $\overline{\mathbf{O}\,\mathbf{x}} = [0, 0]$. So, $\mathcal{L}^* = [0, 0] \setminus \{(0), (0, 0)\}$, that is, the 15 non-fixed points of the horizontal axis $[0, 0]$ under the action of γ are $\mathcal{L}^* = \{(x, 0) : x \in \mathcal{R}^*\}$. An orbit in the permutation of $\mathcal{L}^*$ induced by γ could be, for example, $((1, 0), (a^6, 0), (a^{13}, 0))$.

Recall from Section 1.12 that the action of γ on the points $(x, 0)$, $x \in \mathcal{R}$ is equivalent to the left multiplication in $(\mathcal{R}^*, \odot)$ by some element $s \in \mathcal{R}$ where $(1, 0)^\gamma = (s, 0)$. This means, assuming that a frame of coordinatisation $\{\mathbf{y}, \mathbf{x}, \mathbf{O}, \mathbf{u}, \mathbf{J}\}$ has been labeled and letting $\mathbf{s} = (1, 0)$, we have $\Gamma((0), [0]) \cong \left(\left\{\mathbf{s}, \mathbf{s}^\gamma, \mathbf{s}^{\gamma^2}\right\}, \odot\right)$. At the same time, $\Gamma((0), [0]) \cong (\mathcal{N}_m, \odot)$ by Theorem 1.10.8. Therefore, by an appropriate choice of $\mathbf{s}$, we can have $\mathcal{N}_m$ correspond to the set of points $\left\{\mathbf{s}, \mathbf{s}^{\gamma}, \mathbf{s}^{\gamma^2}\right\}$. Note the set $\left\{\mathbf{s}, \mathbf{s}^{\gamma}, \mathbf{s}^{\gamma^2}\right\}$ can also be obtained as the image of $(1,0)$ under the action of the homology group $\Gamma((0),[0])$. As $\gamma^3 = 1$, an obvious choice of labeling is $\mathbf{s}^{\gamma} = (1,0)^{\gamma} = (a^5,0)$ and $\mathbf{s}^{\gamma^2} = \left(a^5,0\right)^{\gamma} = \left(a^{10},0\right)$. Also, the corresponding points $(0,a^5)$ and $(0,a^{10})$ on the axis $[0]$ can be labeled by projection as discussed at the end of Section 1.12.

The identification of the middle nucleus $\mathcal{N}_m$ with the set $\{1, a^5, a^{10}\} \subset \mathcal{R}$ is one aspect of the optimal labeling. The labeling contributes to optimising the PTR with respect to the PTR multiplication $\odot$. But we would like to optimize the labeling in terms of the PTR addition $\oplus$ too. By trial and error, we select a $\mathbf{J} = (1)$ on $[\infty]$ such that $\mathbf{s}^{\gamma} \oplus \mathbf{s}^{\gamma^2} = \mathbf{s}$. The actual PTR addition is carried out on the vertical axis though. The relation $\mathbf{s}^{\gamma} \oplus \mathbf{s}^{\gamma^2} = \mathbf{s}$ is translated as $a^5 + a^{10} = 1$ in $(\mathbb{F}_{16}, +)$. In fact, we have identified and coordinatised a subplane of order 4 in this process. The subplane is the closure of the set of points $\{\mathbf{y}, \mathbf{x}, \mathbf{O}, \mathbf{u}, \mathbf{J}\} \cup \left\{\mathbf{s}^{\gamma}, \mathbf{s}^{\gamma^2}\right\}$.

Remark 3.2.1. *We can also use $\Gamma((0,0),[\infty])$ instead to optimally label a subgroup $(\mathcal{N}_r, \odot)$ of $(\mathcal{R}^*, \odot)$, or use $\Gamma((\infty),[0,0])$ to label $(\mathcal{N}_l, \odot)$. In the former case, the set $\mathcal{N}_r$ corresponds to the affine points on the vertical axis of a subplane of order 4 called the kernel subplane. See [17], Lemma 7.9 for details.* □

In the next steps, we choose any point $(0,a)$ from the remaining points on the axis $[0]$. The point is used for an additive labeling with the previously labeled points $\{(0,0),(0,1),(0,a^5),(0,a^{10})\}$. The additive labeling ensures the PTR addition $\oplus$ is optimised. As a result, 4 new points are labeled, namely $\{(0,a),(0,a^4),(0,a^2),(0,a^8)\}$ where, $a = a \oplus 0,\ a^4 = a \oplus 1,\ a^2 = a \oplus a^5,$ and $a^8 = a \oplus a^{10}$ (see Table 1.6). Finally, we choose any point $(0,a^3)$ from the remaining 8 points on the axis $[0]$, which then is used to complete the labeling on the vertical axis via additive labeling with the 8 points previously labeled.

3.2.3 Measures of the Algebraic Properties of $(\mathcal{R}, \oplus, \odot)$

Measures of associativity of addition: Since $(\mathcal{R}, \oplus)$ is a group,

$$\mathcal{A}_l = \mathcal{A}_m = \mathcal{A}_r = \mathcal{R}.$$

Measures of associativity of multiplication: As mentioned in the coordinatisation process, all the nuclei are of the same order. Including 0, we have

$$|\mathcal{N}_l| = |\mathcal{N}_m| = |\mathcal{N}_r| = 4.$$

The exact set of elements of the different nuclei vary with the choice of the frame of coordinatisation. Therefore, they are given separately for each optimal coordinatisation along with the corresponding PTR polynomial.

Measures of distributive property: The system $(\mathcal{R}, \oplus, \odot)$ is both left and right distributive. Consequently, the left and the right middle distributors are maximal as well.

$$\mathcal{D}_l = \mathcal{D}_r = \mathcal{D} = \mathcal{D}' = \mathcal{R}.$$

Measure of commutativity of multiplication: The center of this semifield is trivial.

$$\mathcal{Z} = \{0, 1\}.$$

3.2.4 The PTR Polynomials

Firstly, the PTRs are linear with a fully optimised PTR addition. So, the PTR polynomials have a general form,

$$T(X,Y,Z) = (X \odot Y) + Z. \tag{3.2.1}$$

The PTR multiplication is optimized to the extent of the nuclei subgroups in $(\mathcal{R}^*, \odot)$. We ensure at least one of the nuclei is labeled by, including 0, the set $\mathbb{F}_4 = \{0, 1, a^5, a^{10}\}$. We also ensure the labeling of the nucleus is optimised for the PTR addition. As a result, a subplane $\mathcal{Q}$ of order 4 is coordinatised by the subfield $(\mathbb{F}_4, +, \cdot)$ of $(\mathbb{F}_{16}, +, \cdot)$. This means, $T(x,y,z) = xy + z$ when restricted to the coordinates of $\mathcal{Q}$. Now, using notations from Section 1.15.3,

$$\boxed{t_4(X) = \prod_{x \in \mathbb{F}_4} X - x = X^4 - X.}$$

Therefore, we can update the PTR polynomial form (3.2.1) to

$$T(X,Y,Z) = XY + p(X,Y)t_4(X) + q(X,Y)t_4(Y) + Z,$$

for some polynomials $p(X,Y), q(X,Y) \in \mathbb{F}_{16}[X,Y]$.

Theorem 3.2.2. *The PTR polynomial form*

$$T(X,Y,Z) = XY + p(X,Y)t_4(X) + q(X,Y)t_4(Y) + Z, \tag{3.2.2}$$

for some polynomials $p(X,Y), q(X,Y) \in \mathbb{F}_{16}[X,Y]$ is obtainable for all planes of order 16 LB type II.1 or above.

$\square$

Remark 3.2.3. *There are two reasons behind the validity of the statement in Theorem 3.2.2:*

(i) The PTR addition can be fully optimized. See Remark 3.0.1.

(ii) Every known plane of order 16 LB type II.1 or above has subplanes of order 4. It turns out we can always find at least one (numerous actually) subplane that can be coordinatised with a frame of coordinatisation that also fully optimises the PTR addition in $\mathcal{P}$.

$\square$

The first two PTR polynomials we have listed below are more optimal than the form (3.2.2) in the sense that either $q(X, Y)$ or $p(X, Y)$ is vanished. The reason for this is not explained by our coordinatisation. We will see an explanation in the form of an alternative method to obtain a PTR polynomial representation of the SEMI4 plane. The result is presented in Chapter 7.

We have listed three of the many PTR polynomials obtained by different coordinatisations of the SEMI4 plane. The nuclei in each case are given following the polynomial. Observe that each nucleus is an additive subgroup of $(\mathbb{F}_{16}, +)$.

(i) $T(X, Y, Z) = XY + a^{11}\left(Y^8 + a^{10}Y^2 + a^5Y\right)t_4(X) + Z,$

with nuclei $\mathcal{N}_l = \mathcal{N}_m = \mathcal{N}_r = \mathbb{F}_4$.

(ii) $T(X, Y, Z) = XY + a^2\left(X^8 + a^8X^2 + a^2X\right)t_4(Y) + Z,$

with nuclei $\mathcal{N}_l = \{0, 1, a^7, a^9\}$, and $\mathcal{N}_m = \mathcal{N}_r = \mathbb{F}_4$.

(iii) $T(X, Y, Z) = XY + a^4 t_4(Y)^2 t_4(X)^2 + p(Y)t_4(X) + q(X)t_4(Y) + Z,$

where $p(Y) = a^{10}\big(Y^8 + a^9Y^2 + a^7Y\big)$ and $q(X) = a^8\left(X^8 + a^{11}X^4 + a^{12}X^2\right)$.

The nuclei are $\mathcal{N}_l = \{0, 1, a^7, a^9\}$, $\mathcal{N}_m = \mathbb{F}_4$, and $\mathcal{N}_r = \{0, 1, a^6, a^{13}\}$.

3.2.4.1 On the Special Properties of Some PTR Multiplications

In all SEMI4 PTRs given in this section, The PTR multiplications given by $x \odot y = T(x, y, 0)$ for all $x, y \in \mathbb{F}_4$ reduce to $x \odot y = xy$ i.e. coincide with the field multiplication for all $x, y \in \mathbb{F}_4$ since the optimal coordinatisations involve coordinatising a Desarguesian subplane of order 4 by $(\mathbb{F}_4, +, \cdot)$. However, some PTR polynomials of the SEMI4 plane show a PTR multiplication that coincides with the field multiplication on a larger subset of $\mathcal{R}\times\mathcal{R}$. For example, consider the polynomial

$$T(X, Y, Z) = XY + a^{11}\left(Y^8 + a^{10}Y^2 + a^5Y\right)t_4(X) + Z.$$

From the polynomial, we obtain

$$x \odot y = xy \text{ if } x \in \mathbb{F}_4 \text{ or } y \in \{0, 1, a^2, a^8\}. \tag{3.2.3}$$

The coinciding is not surprising for $x \in \mathbb{F}_4$ since $\mathbb{F}_4$ is a left nucleus of $(\mathcal{R}, \oplus, \odot)$ and it is well known that the algebraic system $(\mathcal{R}, \oplus, \odot)$ must then be a left vector space of dimension 2 over $(\mathbb{F}_4, +, \cdot)$. See, for example, [37], page 123. In fact, the same set $\mathbb{F}_4$ is also the right nucleus of the PTR so that $(\mathcal{R}, \oplus, \odot)$ is also a right vector over $\mathbb{F}_4$. But the vector multiplication rule is not a linear transformation over $\mathbb{F}_4$ as in the case of left nucleus. We find $x \odot y = xy + at_4(x)t_2(y)$ for all $x \in \mathcal{R}$ if $y \in \mathbb{F}_4$. See also [26], page 212 for a note on this behavior.

On the other hand, $(\{0, 1, a^2, a^8\}, \oplus)$ is a subgroup of $(\mathcal{R}, \oplus)$ but the structure $(\{0, 1, a^2, a^8\}, \oplus, \odot)$ is not a subfield of $(\mathcal{R}, \oplus, \odot)$ as $a^2 \odot a^8 = a^2 \cdot a^8 = a^{10}$. This is interesting in the light of the multiplication rule above. Once again, we refer the reader to Chapter 7.

3.3 The SEMI2 Plane

The SEMI2 plane $\mathscr{P}$ is the semifield plane of order 16 with kernel $GF(2)$ and LB type V.1.

3.3.1 Collineation Groups of $\mathscr{P}$

There is a translation line $\mathcal{M}$ and a translation point $\mathbf{q}$ in $\mathscr{P}$. The plane does not admit any non-trivial central collineations other than the ones in its 33 transitive elation groups. The transitive elation groups themselves can be grouped into two categories as follows. Suppose $(\mathbf{p}, \mathcal{L})$ is an incident transitive flag in $\mathscr{P}$. Then, either

A. $\mathbf{p} \in \mathcal{M}$ and $\mathcal{L} = \mathcal{M}$, or

B. $\mathbf{p} = \mathbf{q}$ and $\mathcal{L} \neq \mathcal{M}$.

3.3.2 Coordinatisation Strategy

Similar to the SEMI4 plane, the choice of the translation point $\mathbf{q}$ as the vertex $\mathbf{y} = (\infty)$ and the choice of translation line $\mathcal{M}$ as the axis $\overline{\mathbf{x}\,\mathbf{y}} = [\infty]$ is obvious.

(V, l)	$((\infty), [\infty])$	$((0), [0])$	$((\infty), [0, 0])$	$((0), [\infty])$	$((0, 0), [\infty])$	$((\infty), [0])$
$\lvert\Gamma(V, l)\rvert$	16	1	1	16	1	16

Table 3.3: The SEMI2 Plane - Orders of $\Gamma \in S(\Gamma)$.

Again, by similar reasons as for the SEMI4 plane, all principal elation groups are transitive and all principal homology groups are trivial as seen in Table 3.3.

Clearly, no strategies related to optimising the multiplicative structure of $(\mathcal{R}^*, \odot)$ by using a homology group are possible. As such, we vary the choices of all four remaining points of the frame i.e. $\mathbf{x}$ in $\overline{\mathbf{x}\,\mathbf{y}}\setminus\{\mathbf{y}\}$, $\mathbf{O}$ in all affine points, $\mathbf{u}$ in $\overline{\mathbf{O}\,\mathbf{y}}\setminus\{\mathbf{y}, \mathbf{O}\}$, and $\mathbf{J}$ in $\overline{\mathbf{x}\,\mathbf{y}}\setminus\{\mathbf{x}, \mathbf{y}\}$ and compare the resulting PTR polynomials by their complexity. The meaning of complexity of a polynomial implied here is difficult to put in a formal definition. To get a sense of it, for example, consider the form (3.2.2) guaranteed by Theorem 3.2.2. We compare two PTR polynomials by comparing the respective polynomial coefficients $p(X, Y)$ and $q(X, Y)$ in the form (3.2.2). However, we would like to underline that the complexity of a PTR polynomial is not necessarily about the degree and the number of terms in the polynomial. It is also related to the information relayed by the polynomial. For example, a PTR polynomial for a plane of order 16 with coefficients in $\mathbb{F}_2$ alone is in some sense more optimal than a different PTR polynomial for the same plane but with coefficients in $\mathbb{F}_4$, regardless of the number of terms or the degree of the polynomials.

We label the remaining points on the axis $[0] = \overline{\mathbf{O}\,\mathbf{y}}$ using the method of additive labeling by successively choosing any point from the remaining points to be labeled $(0, a^5), (0, a)$, and $(0, a^3)$ in that order.

The general outline of the additive labeling method is similar to the one explained in the coordinatisation strategy for the SEMI4 plane in Section 3.2. However, we note the PTR of the SEMI2 plane does not possess any nuclei of order 4 which could be readily identified with the affine points of a subplane of order 4. Therefore, in theory, we fix a frame $\{\mathbf{y}, \mathbf{x}, \mathbf{O}, \mathbf{u}, \mathbf{J}\}$ and then conduct a brute force search for a subplane of order 4 sharing the same frame of coordinatisation. The actual search is relatively simpler though. We look for two points $\mathbf{r}$ and $\mathbf{s}$ on $\overline{\mathbf{O}\,\mathbf{y}}\setminus\{\mathbf{y}, \mathbf{O}, \mathbf{u}\}$ such that,

(i) $\mathbf{r} \oplus \mathbf{s} = \mathbf{u}$ [$\because$ akin to $a^5 + a^{10} = 1$], and

(ii) $\mathbf{s} \odot \mathbf{r} = \mathbf{r} \odot \mathbf{s} = \mathbf{u}$ [$\because$ akin to $a^5 \cdot a^{10} = a^{10} \cdot a^5 = 1$].

Note that we do not need to test the commutativity for the PTR addition $\oplus$ in (i) but we need to for the multiplication $\odot$ in (ii). This is because $(\mathcal{R}, \oplus)$ is an elementary abelian group while $(\mathcal{R}^*, \odot)$ is simply a loop with an identity. The commutativity of $\odot$ in the set $\{\mathbf{O}, \mathbf{u}, \mathbf{r}, \mathbf{s}\}$ is essential in order to coordinatise a subplane of order 4 which is necessarily Desarguesian. Since the point $\mathbf{O}$ corresponds to the element $0 \in \mathcal{R}$ and the point $\mathbf{u}$ corresponds to the element $1 \in \mathcal{R}$, the test in (ii) suffices to obtain a subplane $\mathcal{Q}$ of order 4. The subplane $\mathcal{Q}$ is the closure of the configuration $\{\mathbf{y}, \mathbf{x}, \mathbf{O}, \mathbf{u}, \mathbf{J}\} \cup \{\mathbf{r}, \mathbf{s}\}$ in $\mathcal{P}$.

The subplane of the SEMI2 plane of order 4 obtained in our coordinatization is however not the kernel of the plane. In particular, the elements a^5 and a^{10} in the coordinatising set of the subplane do not possess the left associativity of multiplication over $\mathcal{R}$. In other words, the left nucleus $\mathcal{N}_l$ is trivial instead of $\mathbb{F}_4$ as in the case of SEMI4 plane.

3.3.3 Measures of the Algebraic Properties of $(\mathcal{R}, \oplus, \odot)$

Measures of associativity of addition: Since $(\mathcal{R}, \oplus)$ is a group,

$$\mathcal{A}_l = \mathcal{A}_m = \mathcal{A}_r = \mathcal{R}.$$

Measures of associativity of multiplication: Since there are no non-trivial homologies and the PTR is linear, the middle nucleus is necessarily trivial by

Theorem 1.10.8. Since the PTR is left and right distributive as well, the left and the right nuclei are trivial too by Theorems 1.10.9 and 1.10.10 respectively. Hence,

$$\mathcal{N}_l = \mathcal{N}_m = \mathcal{N}_r = \{0, 1\}.$$

Remark 3.3.1. *The reasoning given here for the nuclei of the PTR to be necessarily trivial does not apply to a non-linear PTR. For example, consider Theorem 1.10.8. In a linear PTR, the set $\mathcal{S}$ in the theorem equals the middle nuclei $\mathcal{N}_m$. So, a non-trivial middle nuclei becomes equivalent to a non-trivial $\Gamma((0), [0])$. For the left and the right nuclei though, the conclusion does not follow from linearity alone. For instance, in Theorem 1.10.9, the set $\mathcal{S} = \mathcal{N}_l$ if the PTR is left distributive so that $\mathcal{D}_l = \mathcal{R}$. It should be pointed out the left distributivity is a sufficient condition in this case. A similar argument can be made for the right nucleus as well.*

Whereas in a non-linear PTR, the middle nuclei could potentially contain the set $\mathcal{S}$ properly so that the equivalence is no longer implied. We will see examples of this in Chapter 4 where we coordinatise the planes of order 16 LB type I.1. □

Measures of distributive property: The system $(\mathcal{R}, \oplus, \odot)$ is both left and right distributive. Consequently, the left and the right middle distributors are maximal as well.

$$\mathcal{D}_l = \mathcal{D}_r = \mathcal{D} = \mathcal{D}' = \mathcal{R}.$$

Measure of commutativity of multiplication: We obtained different PTRs with centers of different orders. The centers are given with the respective PTR polynomials.

3.3.4 The PTR Polynomials

From the outset, we can expect a PTR polynomial of the form give by (3.2.2). Unlike the SEMI4 PTR polynomials however, we can expect the polynomial coefficients $p(X,Y)$ of $t_4(X) = X^4 - X$ and $q(X,Y)$ of $t_4(Y) = Y^4 - Y$ to have more complex forms. This can be owed to the lack of multiplicative substructures in SEMI2 PTRs. Listed below are some of the PTR polynomials representing the SEMI2 plane followed by the centers of the corresponding system $(\mathcal{R}, \oplus, \odot)$.

(i) $T(X,Y,Z) = XY + a^9 t_4(X)^2 t_4(Y)^2 + a^{11}\left(X^8 + a^2X^2 + a^8X\right)t_4(Y) + Z,$

with center $\mathcal{Z} = \{0,1\}$.

(ii) $T(X,Y,Z) = XY + p(Y)t_4(X)^2 + q(Y)t_4(X) + r(X)t_4(Y)^2 + s(X)t_4(Y) + Z,$

where $p(Y) = a^{10}(Y^2 + Y),$

$$q(Y) = Y^2 + Y,$$

$$r(X) = a^5X^8 + X^4 + a^{10}X^2, \text{ and}$$

$$s(X) = a^5X^8 + a^{10}X^4 + X^2,$$

with center $\mathcal{Z} = \{0, 1, a^2, a^8\}$.

(iii) $T(X,Y,Z) = XY + p(Y)t_4(X) + q(X)t_4(Y)^2 + r(X)t_4(Y) + Z,$

where $p(Y) = Y^2 + Y,$

$$q(X) = X^4 + a^5X^2 + a^{10}X, \text{ and}$$

$$r(X) = X^4 + X^2,$$

with center $\mathcal{Z} = \{0,1\}$.

3.4 The Hall Plane

The Hall plane $\mathcal{P}$ is a translation plane of LB type IVa.1. The plane is a case study for comparing and contrasting the different choices made in the process of coordinatising a plane optimally. This is the first example of a plane with a relatively rich collection of non-transitive elations and homologies on the descending order of LB types in which we have chosen to coordinatise the planes of order 16.

3.4.1 Collineation Groups of $\mathscr{P}$

There is a translation line $\mathcal{M}$. Besides the transitive elation groups centered on each point of the translation line, the plane $\mathscr{P}$ admits a variety of smaller elation and homology groups. The non-transitive central collineation groups of the plane can be grouped into three types as follows:

A. Elation group $\Gamma(\mathbf{p}, \mathcal{L})$ with $\mathbf{p} \in \mathcal{M}$, $\mathcal{L} \neq \mathcal{M}$, $|\Gamma| = 2$.
   There are 80 elation groups of this type with centers on 5 distinct points of $\mathcal{M}$ with 16 groups on each center.

B. Homology group $\Gamma(\mathbf{p}, \mathcal{L})$ with $\mathbf{p} \in \mathcal{M}$, $|\Gamma| = 5$.
   There are 192 homology groups of this type with centers on 12 distinct points of $\mathcal{M}$ with 16 groups on each center.

C. Homology group $\Gamma(\mathbf{p}, \mathcal{M})$ with center on an affine point $\mathbf{p} \notin \mathcal{M}$, $|\Gamma| = 3$.
   As discussed in the case of SEMI4 plane, there are 256 homology groups of this type with each affine point a center.

Note: The 5 centers of the elation groups of type A and the 12 centers of the homology groups of type B do not overlap. Together they constitute the 17 points on the translation line $\mathcal{M}$.

3.4.2 General Coordinatisation Strategies

By Theorem 1.8.1, (iv), the translation line $\mathcal{M}$ is chosen as the infinite axis $\overline{\mathbf{x}\mathbf{y}} = [\infty]$. Any point on the line $\overline{\mathbf{x}\mathbf{y}}$ is a potential $\mathbf{y} = (\infty)$. The resulting PTR will be a left quasifield for any choice of the point $\mathbf{y}$ on the line $\overline{\mathbf{x}\,\mathbf{y}} = [\infty]$. We obtain a Cartesian group with a left distributive property (see Table 1.5). Therefore, along with fully optimising the additive group $(\mathcal{R}, \oplus)$, we will focus on optimising two other aspects: the multiplicative subgroups in $(\mathcal{R}^*, \odot)$ and the right distributive property in $(\mathcal{R}, \oplus, \odot)$.

(i) *The multiplicative subgroups in* $(\mathcal{R}^*, \odot)$*(Coordinatisation I)*: For this we will optimize the orders of a principal homology groups. From the descriptions of the central collineation groups of types A, B, and C, we deduce the type B homologies can be used to optimise two of the three principal homology groups. To be precise, choose the points $\mathbf{y}$ and $\mathbf{x}$ on the line $[\infty]$ such that $|\Gamma(\mathbf{y}, \mathcal{L}_1)| = 5$ for any $\mathcal{L}_1 \neq \mathcal{M}$ through $\mathbf{x}$ and symmetrically, $|\Gamma(\mathbf{x}, \mathcal{L}_2)| = 5$ for any $\mathcal{L}_2 \neq \mathcal{M}$ through $\mathbf{y}$. By fixing and letting $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively be the axes $[0, 0]$ and $[0]$, we achieve $|\,\Gamma((\infty), [0, 0])\,| = |\,\Gamma((0), [0])\,| = 5$. From Theorems 1.10.9 and 1.10.8, we obtain $(\mathcal{N}_l, \odot)$ and $(\mathcal{N}_m, \odot)$ as subgroups of order 5 of the multiplicative loop $(\mathcal{R}^*, \odot)$. Note the origin $\mathbf{O} = \mathcal{L}_1 \cap \mathcal{L}_2$.

Similarly, the homology groups of type C can be identified with $\Gamma((0, 0), [\infty])$ and be used to optimally label the points associated with the right nucleus $\mathcal{N}_r$. We save the discussion on these homology groups for part (ii) where the right distributive property of the PTR is analysed since the same homology groups are involved there too.

Remark 3.4.1. *Optimising any one of* $\Gamma((\infty), [0, 0])$ *or* $\Gamma((0), [0])$ *suffices to obtain a multiplicative subgroup of order 5 in* $(\mathcal{R}^*, \odot)$*. However, we chose to optimise the orders of both groups so as to optimise both the left and middle associativity of multiplication in the form of* $\mathcal{N}_l$ *and* $\mathcal{N}_m$ *respectively.* □

Remark 3.4.2. *As in the case of SEMI4 plane, the two nuclei* $\mathcal{N}_l$ *and* $\mathcal{N}_m$ *may or may not be equal as sets. By trial and error, we found that the PTR polynomial has a simpler form when the two sets are equal. This is likely because the same set of elements are equipped with both the left and the middle associativity of multiplication but we have not studied this sufficiently to make a definitive statement.*

We have also obtained another form of optimal coordinatisation where the left nucleus $\mathcal{N}_l$ *is trivial. Both coordinatisations are presented below.* □

(ii) *The right distributive property in* $(\mathcal{R}, \oplus, \odot)$ *(Coordinatisation II)*: There are two results associated with the right distributivity in $(\mathcal{R}, \oplus, \odot)$ viz. Theorem 1.10.10 and 2.0.1. For the Hall plane, we take a closer look at both the results.

- Based on Theorem 1.10.10, if possible, we should choose a larger $\Gamma((0,0),[\infty])$ to obtain a larger $\mathcal{D}_r$ which is the set of elements right distributive on all of $\mathcal{R}$. Note the set $\mathcal{S}$ in Theorem 1.10.10 is a subset of both $\mathcal{N}_r$ and $\mathcal{D}_r$. Unlike in the semifield planes, the set $\mathcal{S}$ is not necessarily equal to the right nucleus $\mathcal{N}_r$ since we do not know the order and the exact elements of the set $\mathcal{D}_r$. The order of the set $\mathcal{S}$ is the same as the order of $\Gamma((0,0),[\infty])$. Since $\Gamma((0,0),[\infty])$ is a homology group of type C, $|\mathcal{S}|$ is fixed and equal to 3 (Lemma 2.4.1). As a result, $|\mathcal{D}_r| \geq 4$ and $|\mathcal{N}_r| \geq 4$ with 0 included.
- Based on Theorem 2.0.1, we have to choose $\Gamma((\infty),[0])$ as large as possible to optimise the order of the right middle distributor $\mathcal{D}'$. The elation groups of type A are the only possible non-trivial elation groups for this choice because the translation line $\mathcal{M}$ is fixed as the infinite axis $[\infty]$. However,

making this choice means we lose on the optimal choice for $\Gamma((0),[0])$. Recall the centers of the elation groups of type A and the homology groups of type B do not overlap. This means, we have to make a choice between the two aspects of optimality. We have coordinatised the plane $\mathcal{P}$ separately based on each choice. In fact, we explored similar options on the vertex $\mathbf{x} = (0)$ where we could choose to have $|\Gamma((0),[0,0])| = 2$ instead of $|\,\Gamma((\infty),[0,0])\,| = 5$ as chosen earlier to optimise the order of the left nucleus $\mathcal{N}_l$. One of the resulting PTRs with this choice is sufficiently 'nice' as can be seen in its polynomial representation. However, we do not have a theoretical result showing the effect of the $\Gamma((0),[0,0])$ on the structure of $(\mathcal{R},\oplus,\odot)$.

3.4.3 Coordinatisation I

First we present coordinatisations that optimise the orders of the left and the middle nuclei. Table 3.4 gives the orders of the principal central collineation groups $S(\Gamma)$ for this case.

(V,l)	$((\infty),[\infty])$	$((0),[0])$	$((\infty),[0,0])$	$((0),[\infty])$	$((0,0),[\infty])$	$((\infty),[0])$
$\lvert\Gamma(V,l)\rvert$	16	5	5	16	3	1

Table 3.4: The Hall Plane - Orders of $\Gamma \in S(\Gamma)$.

Labeling Procedure: The triangle $\{\mathbf{y},\mathbf{x},\mathbf{O}\}$ has been labeled. Choose any point $\mathbf{s} = (1,0)$ on $[0,0]\setminus\{(0),(0,0)\}$. Choose any $1 \neq \gamma \in \Gamma((0),[0])$. Note that $\gamma^5 = 1$. Define $\mathbf{s}^\gamma = (1,0)^\gamma = (a^3,0)$. Three more points on $[0,0]$ can now be labeled using the multiplicative labeling (Section 2.3.1). They are, $(a^6,0),(a^9,0)$, and $(a^{12},0)$. The set of points labeled in this multiplicative labeling correspond to the subgroup $(\mathcal{N}_m,\odot)$ of the multiplicative loop $(\mathcal{R}^*,\odot)$. The remaining points on the horizontal axis $[0,0]$ can be labeled by an additive labeling. The vertical axis $[0]$ is labeled by using the projection $(0,y) = \overline{\mathbf{J}(y,0)} \cap [0]$.

Remark 3.4.3. *We also require the multiplicative labeling in the above procedure to be synchronized with a fully optimised PTR addition $\oplus$ in order to replace the $\oplus$ by $+$. That is, $x \oplus y = x + y$ must hold for all $x, y \in \mathcal{R}$. This is achieved by trial and error.* □

Remark 3.4.4. *We can label the points $\{(1,0),(a^3,0),(a^6,0),(a^9,0)\}$ alone via the multiplicative labeling and use the additive labeling on the rest because the set $\{1,a^3,a^6,a^9\}$ is a basis of $(\mathbb{F}_{16},+)$ as a vector space over $(\mathbb{F}_2,+)$.* □

3.4.3.1 Measures of the Algebraic Properties of $(\mathcal{R},\oplus,\odot)$

Measures of associativity of addition: Since $(\mathcal{R},\oplus)$ is a group,

$$\mathcal{A}_l = \mathcal{A}_m = \mathcal{A}_r = \mathcal{R}.$$

Measures of associativity of multiplication: As mentioned in the coordinatisation strategy, the orders of the nuclei $\mathcal{N}_m$ and $\mathcal{N}_l$ are 5 each excluding 0. The sets $\mathcal{N}_m$ and $\mathcal{N}_l$ may or may not be equal. We give here the sets for the PTRs given later, and include 0. We also discussed $|\mathcal{N}_r| \geq 4$ including 0. The set $\mathcal{N}_r$ we give here is from the coordinatisations whose PTR polynomials we give later.

$$\mathcal{N}_l = \mathcal{N}_m = \{0,1,a^3,a^6,a^9,a^{12}\}, \text{ and } \mathcal{N}_r = \mathbb{F}_4.$$

Measures of distributive property: The system $(\mathcal{R},\oplus,\odot)$ has a left distributive property. Consequently, the left middle distributor is maximal. The order of the set $\mathcal{D}_r$ of right distributive elements is at least 4. By Theorem 2.0.1, the right middle distributor $\mathcal{D}'$ is trivial.

$$\mathcal{D}_l = \mathcal{D} = \mathcal{R},\ \mathcal{D}_r = \mathbb{F}_4,\ \text{and } \mathcal{D}' = \{0\}$$

We obtained three distinct optimal coordinatisations of the Hall plane with this strategy of coordinatisation. The measures of the algebraic properties listed above are identical for all three coordinatisations. They differ in the center $\mathcal{Z}$ which is given separately for each PTR polynomial.

3.4.3.2 The PTR Polynomials

(i) $T(X,Y,Z) = XY + a^{10}X\left(X^5-1\right)\left(X^5-a^5\right)t_4(Y) + Z$,
with center $\mathcal{Z} = \{0, 1, a^{10}\}$.

(ii) $T(X,Y,Z) = XY + a^{5}X\left(X^5-1\right)\left(X^5-a^{10}\right)t_4(Y) + Z$,
with center $\mathcal{Z} = \{0, 1, a^{5}\}$.

(iii) $T(X,Y,Z) = XY + X^6\left(X^5-1\right)t_4(Y) + Z$,
with center $\mathcal{Z} = \{0, 1\}$.

3.4.4 Coordinatisation II

In this coordinatisation we optimise the right middle distributor $\mathcal{D}'$. The orders of the principal central collineation groups $S(\Gamma)$ are given in Table 3.5. Note the order of $\mathcal{D}'$ is equal to $|\,\Gamma((\infty),[0])\,| = 2$ in the optimal case while $|\,\Gamma((\infty),[0,0])\,|$ can take values either 1 or 5. In the coordinatisation presented here, the value is 1.

Labeling Procedure: The triangle $\{\mathbf{y}, \mathbf{x}, \mathbf{O}\}$ has been labeled. We want to identify a point $\mathbf{u} = (0,1)$ and a point $\mathbf{J} = (1)$ such that the resulting $(\mathcal{R}, \oplus, \odot)$ has the following two sets contain the elements as stated:

(V,l)	$((\infty),[\infty])$	$((0),[0])$	$((\infty),[0,0])$	$((0),[\infty])$	$((0,0),[\infty])$	$((\infty),[0])$
$\lvert\Gamma(V,l)\rvert$	16	1	1	16	3	2

Table 3.5: The Hall Plane - Orders of $\Gamma \in S(\Gamma)$

(i) The right middle distributor $\mathcal{D}' = \{0,1\}$. Recall $|\mathcal{D}'| = |\,\Gamma((\infty),[0])\,| = 2$.

(ii) Let $1 \neq \gamma \in \Gamma((0,0),[\infty])$ be chosen and fixed. Then, $\{1, a^5, a^{10}\} \subset \mathcal{N}_r$ where $(0,a^5) = (0,1)^\gamma$ and $(0,a^{10}) = (0,1)^{\gamma^2}$. Additionally, $a^5 \oplus a^{10} = 1$ must hold.

Remark 3.4.5. *The idea behind requiring $\mathcal{N}_r$ to contain the points corresponding to the elements of $\mathbb{F}_4$ as given above is to optimise the multiplicative structure of the PTR while also satisfying the primary objective of this coordinatisation which is to optimise the amount of right middle distribution. The only non-zero element in $\mathcal{D}'$ can be any element of $\mathcal{R}^*$ so long as the optimality is measured by the order of $\mathcal{D}'$. However, requiring it to be* 1 *produces a PTR polynomial which we found to be nicer in comparison. The observation is in line with other general observations that a PTR polynomial is more optimal when the various special subsets of $\mathcal{R}$ defined in Section 1.10.1 contain elements of either a subfield of $(\mathbb{F}_n, +, \cdot)$ or a subgroup of either $(\mathbb{F}_n, +)$ or $(\mathbb{F}_n^*, \cdot)$.* □

Once a pair of points $\mathbf{u} = (0,1)$ and $\mathbf{J} = (1)$ satisfying the above two conditions is found, we choose any remaining point on the vertical axis $[0]$ to label as $(0,a)$. By additive labeling, we determine 3 more points corresponding to the elements in $\{a^4, a^2, a^8\}$. Next, we choose any $(0,a^3)$ and complete the labeling of the axis $[0]$ by additive labeling. The choices of $(0,a)$ and $(0,a^3)$ are varied to compare the resulting PTR polynomials.

3.4.4.1 Measures of the Algebraic Properties of $(\mathcal{R}, \oplus, \odot)$

Measures of associativity of addition: Since $(\mathcal{R}, \oplus)$ is a group,

$$\mathcal{A}_l = \mathcal{A}_m = \mathcal{A}_r = \mathcal{R}.$$

Measures of associativity of multiplication: The two nuclei $\mathcal{N}_m$ and $\mathcal{N}_l$ are trivial as are $\Gamma((0), [0])$ and $\Gamma((\infty), [0, 0])$. This follows from Theorems 1.10.8 and 1.10.9 respectively. Also recall Remark 3.3.1. The order of the right nucleus $|\mathcal{N}_r| \geq 4$ as $|\,\Gamma((0, 0), [\infty])\,| = 3$. The set $\mathcal{N}_r$ given below is obtained from the coordinatisation whose corresponding PTR polynomial is given later. As would be expected from the discussions until now, the right nucleus is the set $\mathbb{F}_4$ in all coordinatisations with optimal PTR polynomials.

$$\mathcal{N}_l = \mathcal{N}_m = \{0, 1\}, \text{ and } \mathcal{N}_r = \mathbb{F}_4.$$

Measures of distributive property: The system $(\mathcal{R}, \oplus, \odot)$ has a left distributive property. Consequently, the left middle distributor is maximal. The orders of the set $\mathcal{D}_r$ of right distributive elements is at least 4. The right middle distributor $\mathcal{D}'$ is of order 2 and we have forced its exact elements in the coordinatisation. In the optimal coordinatisation,

$$\mathcal{D}_l = \mathcal{D} = \mathcal{R}, \; \mathcal{D}_r = \mathbb{F}_4, \text{ and } \mathcal{D}' = \{0, 1\}.$$

Measure of commutativity of multiplication: Interestingly, the PTR obtained from this coordinatisation has a larger center compared to the previously given PTRs. The center coincides with the right nucleus $\mathcal{N}_r$ and the set $\mathcal{D}_r$ of right distributive elements.

$$\mathcal{Z} = \mathbb{F}_4.$$

3.4.4.2 The PTR Polynomial

One of the PTR polynomials obtained with this optimal coordinatisation is given. The polynomial exhibits an even stronger property regarding the center. The PTR product $x \odot y$ coincides with the field product xy when either x or y is in the center. Note the center is also the coordinatising set of a subplane of order 4.

$$T(X,Y,Z) = XY + \left(X^2 + X + a^{10}\right) t_4(X)^2 t_4(Y) + Z.$$

3.5 Dempwolff Plane

3.5.1 Collineation Groups of $\mathscr{P}$

The Dempwolff plane $\mathscr{P}$ is a translation plane of LB type IVa.1. There is a translation line $\mathcal{M}$. Besides, there are 16 homology groups of order 3 centered on a point $\mathbf{p} \in \mathcal{M}$ and all their axes intersecting the line $\mathcal{M}$ at a point $\mathbf{q}$.

3.5.2 Coordinatisation Strategy

Translation line $\mathcal{M}$ is chosen as the infinite axis $\overline{\mathbf{x}\,\mathbf{y}} = [\infty]$. By Theorem 1.8.1, (iv) and Remark 3.0.1, a PTR $(\mathcal{R}, T)$ with a fully optimised PTR addition and a left distributive property can be constructed based on this choice alone. We want to optimise the multiplicative structure of $(\mathcal{R}^*, \odot)$ along with those two properties.

There are two optimal choices for choosing the pair of points $\mathbf{y} = (\infty)$ and $\mathbf{x} = (0)$ on the line at infinity. We consider the choices equally optimal as far as the algebraic properties induced on $(\mathcal{R}, \oplus, \odot)$ are concerned. This ranking of optimality will be explained shortly. We also obtained PTR polynomials for coordinatisations based on each of the two optimal choices. When comparing the polynomials though, one choice is more optimal than the other. The choices are:

(i) Let $\mathbf{y} = \mathbf{q}$ and $\mathbf{x} = \mathbf{p}$ i.e. the homology groups of order 3 are centered at $\mathbf{x} = (0)$ and any vertical line through $\mathbf{y} = (\infty)$ is a potential vertical axis $\overline{\mathbf{O}\,\mathbf{y}} = [0]$. By Theorem 1.10.8, the middle nuclei $(\mathcal{N}_m, \odot)$ is of order 3 and a subgroup of $(\mathcal{R}^*, \odot)$.

(ii) Let $\mathbf{y} = \mathbf{p}$ and $\mathbf{x} = \mathbf{q}$ i.e. the homology groups of order 3 are centered at $\mathbf{y} = (\infty)$ and any horizontal line through $\mathbf{x} = (0)$ is a potential horizontal axis $\overline{\mathbf{O}\,\mathbf{x}} = [0, 0]$. By Theorem 1.10.9, the left nuclei $(\mathcal{N}_l, \odot)$ is of order 3 and a subgroup of $(\mathcal{R}^*, \odot)$. Note the PTR is linear and left distributive i.e. $\mathcal{D}_l = \mathcal{R}$. As a result, $\mathcal{S} = \mathcal{N}_l$ where the set $\mathcal{S}$ is as defined in the theorem.

With either choice, we obtain a subgroup of order 3 in $(\mathcal{R}^*, \odot)$. So, the measures of associativity in the resulting systems $(\mathcal{R}, \oplus, \odot)$ are of the same order too. In an identical situation in terms of the orders of the homology groups and linearity of the PTR but where the left distributivity is not already given, the second choice (ii) would be considered more optimal. This is because a partial left distributive property in the form of a non-trivial set $\mathcal{D}_l$ is considered more optimal than no such property resulting from first choice (i).

3.5.3 Coordinatisation I

The coordinatisation is based on the first choice above. The orders of the principal central collineation groups for this choice are given in Table 3.6.

Labeling Procedure: Choose a line through $\mathbf{y} = (\infty)$ other than the infinite axis $[\infty]$ to be the vertical axis $\overline{\mathbf{O}\,\mathbf{y}} = [0]$. Choose any two distinct points $\mathbf{O} = (0, 0)$ and $\mathbf{u} = (0, 1)$ on the axis $[0]$, other than the point (∞). Take any one of the two

(V, l)	$((\infty), [\infty])$	$((0), [0])$	$((\infty), [0, 0])$	$((0), [\infty])$	$((0, 0), [\infty])$	$((\infty), [0])$
$\lvert\Gamma(V, l)\rvert$	16	3	1	16	1	1

Table 3.6: The Dempwolff Plane - Orders of $\Gamma \in S(\Gamma)$ in Coordinatisation I.

non-trivial homologies in $\Gamma((0), [0])$, say γ. So, $\gamma \neq 1,\ \gamma^3 = 1$. Obtain a point $\mathbf{s} \in [\infty]$ such that the closure of the configuration $\{\mathbf{y}, \mathbf{x}, \mathbf{O}, \mathbf{u}\} \cup \left\{\mathbf{s}, \mathbf{s}^{\gamma}, \mathbf{s}^{\gamma^2}\right\}$ in $\mathcal{P}$ is a subplane of order 4. Label $\mathbf{s} = \mathbf{J} = (1)$, $\mathbf{s}^{\gamma} = (a^5)$, and $\mathbf{s}^{\gamma^2} = (a^{10})$. Obtain $(1, 0) = \overline{\mathbf{J}\,\mathbf{u}} \cap [0, 0]$ which in turn gives $(0, a^5) = \overline{(a^5)(1, 0)} \cap [0]$ and $(0, a^{10}) = \overline{(a^{10})(1, 0)} \cap [0]$. Follow this by additive labeling on the vertical axis $[0]$ as described in previous coordinatisations so that the resulting PTR has a fully optimised PTR addition.

Remark 3.5.1. *From a purely procedural view point, we just described a different approach to labeling than the ones used previously. However, the approaches are not different from a conceptual view point. As discussed earlier, a homology induces identical permutations on different lines through its center. The labels on any one axis can be translated to that on any other axis by projection as soon as the coordinatising frame is labeled. The central collineations of a plane induce central collineations in its subplanes. Specifically in the case of homologies of order 3 in the planes of order 16, this means the subplane concerned is of order 4. By forcing the same frame of coordinatisation on the subplane as that of $\mathcal{P}$, we ensure that that the subplane is coordinatised by a set of order 4 which must be of the form $\{0, 1, x, y\}$ for some $x, y \in \mathcal{R}$ and satisfy the axioms of a finite field of order 4. The obvious choice of labeling then becomes $\{0, 1, a^5, a^{10}\}$. In summary, we are using many variations of the same general theory in all labeling procedures.* $\square$

3.5.3.1 Measures of the Algebraic Properties of $(\mathcal{R}, \oplus, \odot)$

Measures of associativity of addition: Since $(\mathcal{R}, \oplus)$ is a group,

$$\mathcal{A}_l = \mathcal{A}_m = \mathcal{A}_r = \mathcal{R}.$$

Measures of associativity of multiplication: The middle nucleus has an order at least 4 (including 0) since $|\,\Gamma((0),[0])\,| = 3$. The left nucleus must be trivial as the PTR is left distributive with a trivial $\Gamma((\infty),[0,0])$. In the optimal PTR we obtain, the right nucleus is trivial as well.

$$\mathcal{N}_m = \mathbb{F}_4, \text{ and } \mathcal{N}_l = \mathcal{N}_r = \{0,1\}.$$

Measures of distributive property: The system $(\mathcal{R}, \oplus, \odot)$ is left distributive. Consequently, the left middle distributor is maximal. The right middle distributor $\mathcal{D}'$ is necessarily trivial since $\Gamma((\infty),[0])$ is trivial. For the coordinatisation we obtained, the set $\mathcal{D}_r$ of right distributive elements is also trivial.

$$\mathcal{D}_l = \mathcal{D} = \mathcal{R},\ \mathcal{D}' = \{0\}, \text{ and } \mathcal{D}_r = \{0,1\}.$$

Measure of commutativity of multiplication: Trivial.

$$\mathcal{Z} = \{0,1\}.$$

3.5.3.2 The PTR Polynomial

We have organised the terms of the polynomial for a better presentation as well as for comparing with the polynomial obtained in Coordinatisation II. It is easily seen the polynomial can be reorganised in the form of (3.2.2) which gives a general form of the PTR polynomials of the planes of order 16 LB type II.1 or above. Note the coefficients of the polynomial are in $\mathbb{F}_4$.

$$T(X,Y,Z) = XY + \left(X^3+1\right)t_4\left(X\right)t_4\left(Y\right)^2 + a^{10}Xt_4\left(X\right)t_4\left(Y\right) + Xt_4\left(X\right)^3 t_2\left(Y\right) + Z. \tag{3.5.1}$$

We can rewrite the PTR polynomial in the following form

$$T(X,Y,Z) = XY + Z + p(X,Y)t_4\left(X\right), \tag{3.5.2}$$

where $p(X,Y) = \left(X^3+1\right)t_4\left(Y\right)^2 + a^{10}Xt_4\left(Y\right) + Xt_4\left(X\right)^2 t_2\left(Y\right)$.

The PTR polynomial form (3.5.2) highlights that the PTR T coincides with the function $f(x,y,z) = xy + z$ whenever $x \in \mathbb{F}_4$. Keeping in perspective this is a linear PTR with a fully optimised addition, we have $x \odot y = xy$ for all $x \in \mathbb{F}_4$ and $y \in \mathcal{R}$. This shows $(\mathcal{R}, \oplus, \odot)$ is a vector space of dimension 2 over $(\mathbb{F}_4, +, \cdot)$. Also, from (3.5.1) we get $x \odot y = xy + xt_4(x)^3 t_2(y)$ for all $x \in \mathcal{R}$ and $y \in \mathbb{F}_4$. Since $t_4(x) = x^4 - x = x^4 + x \in \mathbb{F}_4^*$ for $x \notin \mathbb{F}_4$, we have $t_4(x)^3 = 1$ for $x \notin \mathbb{F}_4$. Thus, we can further simplify the previous expressions and give the following rule:

$$x \odot y = \begin{cases} xy & \text{if } x \in \mathbb{F}_4, \\ xy^2 & \text{if } x \notin \mathbb{F}_4,\ y \in \mathbb{F}_4, \text{ and} \\ xy + p(x,y)t_4(x) & \text{if } x \notin \mathbb{F}_4,\ y \notin \mathbb{F}_4. \end{cases} \tag{3.5.3}$$

We will compare this multiplication with the multiplication in the optimal

PTR of the derived semifield plane (DSFP) which we coordinatise in Section 3.6.

3.5.4 Coordinatisation II

The coordinatisation is based on the second optimal choice above. The orders of the principal central collineation groups for this choice are given in Table 3.7.

(V,l)	$((\infty),[\infty])$	$((0),[0])$	$((\infty),[0,0])$	$((0),[\infty])$	$((0,0),[\infty])$	$((\infty),[0])$
$\lvert\Gamma(V,l)\rvert$	16	1	3	16	1	1

Table 3.7: The Dempwolff Plane - Orders of $\Gamma \in S(\Gamma)$ in Coordinatisation II.

Labeling Procedure: Choose a line through $\mathbf{x} = (0)$ other than the infinite axis $[\infty]$ to be the horizontal axis $\overline{\mathbf{O}\,\mathbf{x}} = [0,0]$. Choose any point $\mathbf{O} = (0,0)$ other than (0) on the axis $[0,0]$ and any point $\mathbf{J} = (1)$ other than (0) on the axis $[\infty]$. Take any of the two non-trivial homologies in $\Gamma((\infty),[0,0])$, say γ. So, $\gamma \neq 1,\ \gamma^3 = 1$. Obtain a point $\mathbf{u} \in [0]$ such that the closure of the configuration $\left\{\mathbf{y},\mathbf{x},\mathbf{O},\mathbf{J},\mathbf{u},\mathbf{u}^{\gamma},\mathbf{u}^{\gamma^2}\right\}$ in $\mathscr{P}$ is a subplane of order 4. Label $\mathbf{u} = (0,1)$, $\mathbf{u}^{\gamma} = (0,a^5)$, and $\mathbf{u}^{\gamma^2} = (0,a^{10})$. Follow this by additive labeling on the vertical axis $[0]$ as described in previous coordinatisations.

3.5.4.1 Measures of the Algebraic Properties of $(\mathcal{R},\oplus,\odot)$

All measures are the same as in Coordinatisation I except for the swapping of the left and the middle nuclei as a result of optimising $\Gamma((\infty),[0,0])$ instead of $\Gamma((0),[0])$. Thus,

$$\mathcal{N}_l = \mathbb{F}_4,\ \text{and } \mathcal{N}_m = \mathcal{N}_r = \{0,1\}.$$

3.5.4.2 The PTR Polynomial

In this coordinatisation we chose to prioritize the order of $\Gamma((\infty),[0,0])$ over $\Gamma((0),[0])$. We obtain a PTR polynomial which can be written in the same general form (3.2.2) but has coefficients in $\mathbb{F}_{16}$. In this sense, we consider the PTR polynomial obtained in Coordinatisation I to be more optimal.

$$T(X,Y,Z) = XY + \big(p(X)t_4\,(Y)^2 + q(X)t_4\,(Y) + r(X)t_2\,(Y)\big)\,t_4\,(X) + Z,$$

where $p(X) = a^{11}X^9 + a^9X^6 + aX^3 + a^5$, $q(X) = a^{10}X^9 + a^{14}X^6 + a^2X^3 + a^{14}$, and $r(X) = a^7X^9 + aX^6 + a^6X^3 + a^2$.

The new PTR T also coincides with the function $f(x,y,z) = xy + z$ whenever $x \in \mathbb{F}_4$.

3.6 Derived Semifield Plane

3.6.1 Collineation Groups of $\mathscr{P}$

The derived semifield plane (DSFP) $\mathscr{P}$ of order 16 is a translation plane of LB type IVa.1. There is a translation line $\mathcal{M}$. Besides, there are 32 homology groups of order 3 centered on two distinct points $\mathbf{p}, \mathbf{q} \in \mathcal{M}$ such that the axes of the homologies centered at $\mathbf{p}$ intersect the line $\mathcal{M}$ at the point $\mathbf{q}$ and vice-versa.

3.6.2 Coordinatisation Strategy

The translation line $\mathcal{M}$ is chosen as the infinite axis $\overline{\mathbf{x}\,\mathbf{y}} = [\infty]$. As in the case of Dempwolff plane, a PTR $(\mathcal{R}, T)$ with a fully optimised PTR addition and a left distributive property can be constructed based on this choice. Similarly again, we want to optimise the multiplicative structure of $(\mathcal{R}^*, \odot)$ along with those two properties.

Since there is a symmetry between the points $\mathbf{p}$ and $\mathbf{q}$ in the sense that both points are incident on the translation line and the plane admits two isomorphic central collineation groups centered on a point each, the two choices in choosing $\mathbf{x}$

and $\mathbf{y}$ are equally optimal according to the theory of optimal coordinatisation we have developed. Compare this with the choice between the same two points on the translation line of the Dempwolff plane in Section 3.5. In the Dempwolff plane, there was an asymmetry in the sense of the structures of the central collineation groups centered on the two points. As a result, we had to choose between optimising either the middle or the left nucleus.

Without loss of generality, let $\mathbf{p} = \mathbf{y} = (\infty)$ and $\mathbf{q} = \mathbf{x} = (0)$. With the choice, $\Gamma((0),[0])$ and $\Gamma((\infty),[0,0])$ are of order 3 each. The orders of the principal central collineation groups are given in Table 3.8.

(V,l)	$((\infty),[\infty])$	$((0),[0])$	$((\infty),[0,0])$	$((0),[\infty])$	$((0,0),[\infty])$	$((\infty),[0])$
$\lvert\Gamma(V,l)\rvert$	16	3	3	16	1	1

Table 3.8: The Derived Semifield Plane - Orders of $\Gamma \in S(\Gamma)$.

Labeling Procedure: Choose a line through (∞) other than the infinite axis $[\infty]$ to be the vertical axis $\overline{\mathbf{O}\,\mathbf{y}} = [0]$. Choose any two distinct points $\mathbf{O} = (0,0)$ and $\mathbf{u} = (0,1)$ on the axis $[0]$, other than (∞). Take any one of the two non-trivial homologies in $\Gamma((0),[0])$, say γ. So, $\gamma \neq 1$, $\gamma^3 = 1$. Obtain a point $\mathbf{s} \in [0,0]$ such that the closure of the configuration $\{\mathbf{y},\mathbf{x},\mathbf{O},\mathbf{u}\} \cup \left\{\mathbf{s},\mathbf{s}^{\gamma},\mathbf{s}^{\gamma^2}\right\}$ in $\mathscr{P}$ is a subplane of order 4. Label $\mathbf{s} = (1,0)$, $\mathbf{s}^{\gamma} = (a^5,0)$, and $\mathbf{s}^{\gamma^2} = (a^{10},0)$. Obtain $\mathbf{J} = \overline{\mathbf{u}\,\mathbf{s}} \cap [\infty]$ which completes the coordinatising frame $\{\mathbf{y},\mathbf{x},\mathbf{O},\mathbf{u},\mathbf{J}\}$. Follow this by additive labeling on the vertical axis $[0]$ as described in previous coordinatisations so that the resulting PTR has a fully optimised PTR addition.

3.6.3 Measures of the Algebraic Properties of $(\mathcal{R}, \oplus, \odot)$

Measures of associativity of addition: Since $(\mathcal{R}, \oplus)$ is a group,

$$\mathcal{A}_l = \mathcal{A}_m = \mathcal{A}_r = \mathcal{R}.$$

Measures of associativity of multiplication: Left left and the middle nuclei are of order 4 each since $\Gamma((\infty), [0,0])$ and $\Gamma((0), [0])$ are both of order 3. As must be expected at this point, the two nuclei coincide with $\mathbb{F}_4$ in our optimal coordinatisation. The right nucleus is trivial as is $\Gamma((0,0), [\infty])$ in the PTR we have given.

$$\mathcal{N}_l = \mathcal{N}_m = \mathbb{F}_4, \text{ and } \mathcal{N}_r = \{0,1\}.$$

Measures of distributive property: The system $(\mathcal{R}, \oplus, \odot)$ is left distributive. Consequently, the left middle distributor is maximal. The right middle distributor $\mathcal{D}'$ is necessarily trivial since $\Gamma((\infty), [0])$ is trivial. In the optimal coordinatisation we obtained, the set $\mathcal{D}_r$ of right distributive elements is also trivial.

$$\mathcal{D}_l = \mathcal{D} = \mathcal{R}, \ \mathcal{D}' = \{0\}, \text{ and } \mathcal{D}_r = \{0,1\}.$$

Measure of commutativity of multiplication: Trivial.

$$\mathcal{Z} = \{0,1\}.$$

3.6.4 The PTR Polynomial

Similar to the optimal PTR polynomials of the Dempwolff plane, we have organised the terms of the optimal PTR polynomial obtained for the DSFP plane to reflect the property of the PTR $(\mathcal{R}, T)$ that the product $x \odot y$ coincides with the product xy in $(\mathbb{F}_{16}, +, \cdot)$ whenever $x \in \mathbb{F}_4$. The subfield $(\mathbb{F}_4, +, \cdot)$ coordinatises a (Desarguesian) subplane of order 4 sharing the same frame of coordinatisation.

$$T(X, Y, Z) = XY + \left((X^3 + a^{10})t_4(Y)^2 + Xt_4(X)^2 t_2(Y)\right) t_4(X) + Z.$$

The coefficients of the polynomial are in $\mathbb{F}_4$, which is similar to the PTR Polynomial (3.5.1) of the Dempwolff plane. We also note the PTR multiplication is given by $x \odot y = xy + xt_4(x)^3 t_2(y)$ for all $x \in \mathcal{R}$ and $y \in \mathbb{F}_4$. This is again the same as in the PTR Polynomial (3.5.1) of the Dempwolff plane. Finally, we can give a multiplication rule for the optimal PTR obtained for the DSFP plane by

$$x \odot y = \begin{cases} xy & \text{if } x \in \mathbb{F}_4, \\ xy^2 & \text{if } x \notin \mathbb{F}_4,\ y \in \mathbb{F}_4, \text{ and} \\ xy^2 + (x^3 + a^{10})t_4(x)t_4(y)^2 & \text{if } x \notin \mathbb{F}_4,\ y \notin \mathbb{F}_4. \end{cases} \tag{3.6.1}$$

3.7 The Johnson-Walker Plane

3.7.1 Collineation Groups of $\mathcal{P}$

The Johnson-Walker (JOWK) plane $\mathcal{P}$ of order 16 is a translation plane of LB type IVa.1. There is a translation line $\mathcal{M}$. Besides, there are 48 elation groups of order 2 centered on three distinct points $\mathbf{p}, \mathbf{q}, \mathbf{r} \in \mathcal{M}$ with 16 groups on each center.

3.7.2 Coordinatisation Strategy

The translation line $\mathcal{M}$ is chosen as the infinite axis $\overline{\mathbf{x}\mathbf{y}} = [\infty]$. A PTR $(\mathcal{R}, T)$ with a fully optimised PTR addition and a left distributive property can be constructed based on this choice. Since there are no homologies, we do not have a strategy to optimise the PTR multiplication via subgroups of $(\mathcal{R}^*, \odot)$. The non-trivial elations centered on the points $\mathbf{p}, \mathbf{q}$, and $\mathbf{r}$ of the line $[\infty]$ provide a way to optimise the algebraic structure of the PTR by increasing the amount of the distributive property in $(\mathcal{R}, \oplus, \odot)$.

Firstly, consider the symmetry in the geometric structure of $\mathscr{P}$ in terms of the collineation groups centered on the three points $\mathbf{p}, \mathbf{q}$, and $\mathbf{r}$. Observing the symmetry, we make the choices of $\mathbf{y}, \mathbf{x}$, and $\mathbf{J}$ of the frame of coordinatisation from among the three points in a random way. Say, $\mathbf{p} = \mathbf{y} = (\infty)$, $\mathbf{q} = \mathbf{x} = (0)$, and $\mathbf{r} = \mathbf{J} = (1)$. The orders of the principal central collineation groups $S(\Gamma)$ are determined with this choice. The orders are given in Table 3.9. Not seen in the table is the order of $\Gamma((0), [0, 0])$ which is 2.

(V, l)	$((\infty), [\infty])$	$((0), [0])$	$((\infty), [0, 0])$	$((0), [\infty])$	$((0, 0), [\infty])$	$((\infty), [0])$
$\lvert\Gamma(V, l)\rvert$	16	1	1	16	1	2

Table 3.9: The Johnson-Walker Plane - Orders of $\Gamma \in S(\Gamma)$.

We make an observation on the relation among the three points of the frame thus chosen. Let any vertical line i.e. a line through (∞) other than $[\infty]$ be chosen as the vertical axis $[0]$. Let $1 \neq \gamma \in \Gamma((\infty), [0])$ be the non-trivial elation in the elation group which is of order 2. Then, $(0)^\gamma = (1)$ i.e. $\mathbf{x}^\gamma = \mathbf{J}$. The relation follows again from [17], Lemma 4.11. For if α is a $((0), \mathcal{L})$-elation for some line $\mathcal{L} \neq \mathcal{M}$ through (0), then $\gamma^{-1}\alpha\gamma$ is a $((0)^\gamma, \mathcal{L}^\gamma)$-elation. If α is non-trivial then so is $\gamma^{-1}\alpha\gamma$. Finally, $(0)^\gamma \in [\infty]$ but $(0)^\gamma \neq (\infty)$ as (∞) is the center of the elation γ. The only other center of a non-trivial elation with axis $[\infty]$ is the point (1).

By Theorem 2.0.1, the choice of $\mathbf{J}$ has the effect of including $1 \in \mathcal{R}$ in the right middle distributor $\mathcal{D}'$. We consider this inclusion optimal in the following sense.

Since the order of the set $\mathcal{D}'$ equals $|\,\Gamma((\infty),[0])\,| = 2$ in any coordinatisation given the choice of (∞), we would prefer the set to consist of 0 and 1, with 0 being included trivially. When this is true, the following relation holds for all $x, y \in \mathcal{R}$ provided the PTR addition is fully optimised:

$$(x+1) \odot y = (x \odot y) + y \tag{3.7.1}$$

We obtained some PTR polynomials with coefficients entirely in $\mathbb{F}_2$ when this is the case with right middle distribution. No such polynomials are obtained in any other case. Also, we observed the set $\mathcal{D}_r$ of right distributive elements of the PTR always equals the union of $\mathbb{F}_4$ and its coset $\mathbb{F}_4 + a$ in $(\mathbb{F}_{16}, +)$ when the PTR polynomial has all coefficients in $\mathbb{F}_2$.

With this foresight, choose any line through $\mathbf{y} = (\infty)$ other than the infinite axis $[\infty]$ to be the vertical axis $\overline{\mathbf{O}\,\mathbf{y}} = [0]$. Choose any two points $\mathbf{O} = (0,0)$ and $\mathbf{u} = (0,1)$ on the line $[0]$, other than the point (∞). The frame of coordinatisation $\{\mathbf{y}, \mathbf{x}, \mathbf{O}, \mathbf{u}, \mathbf{J}\}$ thus labeled has the property that any point $\mathbf{s}$ on $[0] \setminus \{(0,0),(0,1),(\infty)\}$ together with the frame generates a subplane of order 4 as its closure in $\mathcal{P}$. It is easy to see the point $\mathbf{s}$ could be chosen on any other axis too so long as its projection on the vertical $[0]$ via $(1,0)$ or $\mathbf{J}$ as needed is not in $\{\mathbf{O}, \mathbf{u}, \mathbf{y}\}$. This is interesting because the JOWK plane does not admit any homologies of order 3 which we have used in other planes to obtain subplanes of order 4 based on a single point on a suitable axis (the point must not be one of the vertices of the frame). The property of the configuration consisting of the frame and one additional point closing to a subplane of order 4 does not hold if the three points $\mathbf{y}, \mathbf{x}$, and $\mathbf{J}$ are not chosen to be the three centers of the non-trivial elation groups.

Hence, three points $(0,a^5),(0,a)$, and $(0,a^3)$ on the axis [0] are chosen randomly from the available set of points in that order, each choice followed by an additive labeling on the vertical axis [0]. An optimal coordinatisation is then determined by analysing the resulting PTR polynomial.

3.7.3 Measures of the Algebraic Properties of $(\mathcal{R},\oplus,\odot)$

Measures of associativity of addition: Since $(\mathcal{R},\oplus)$ is a group,

$$\mathcal{A}_l = \mathcal{A}_m = \mathcal{A}_r = \mathcal{R}.$$

Measures of associativity of multiplication: Since there are no non-trivial homologies and the PTR is linear, by Theorems 1.10.8 and 1.10.9, the middle and the left nuclei respectively are necessarily trivial. In the optimal PTR we have given, the right nucleus is trivial too. Hence,

$$\mathcal{N}_m = \mathcal{N}_l = \mathcal{N}_r = \{0,1\}.$$

Measures of distributive property: The system $(\mathcal{R},\oplus,\odot)$ is left distributive. Consequently, the left middle distributor is maximal. The elements in the right middle distributor $\mathcal{D}'$ are forced as explained in the coordinatisation process. The set $\mathcal{D}_r$ of right distributive elements is of order 8 in every coordinatisation we obtained. We have given here the $\mathcal{D}_r$ corresponding to the PTR polynomial given later. Recall the discussion in the coordinatisation strategy above for a relation between the exact elements included in $\mathcal{D}_r$ and the coefficients of the corresponding PTR polynomial.

$$\mathcal{D}_l = \mathcal{D} = \mathcal{R},\ \mathcal{D}' = \{0,1\},\ \text{and } \mathcal{D}_r = \{0,1,a^5,a^{10},a,a^4,a^2,a^8\}.$$

Measure of commutativity of multiplication: Trivial.

$$\mathcal{Z} = \{0, 1\}.$$

3.7.4 The PTR Polynomial

The terms of the PTR polynomial have been organised to show the PTR product $x \odot y$ coincides with the product xy in $(\mathbb{F}_{16}, +, \cdot)$ if and only if $x, y \in \mathbb{F}_4$. The subfield $(\mathbb{F}_4, +, \cdot)$ is the coordinatising set of a subplane $\mathcal{Q}$ of order 4 sharing the frame of coordinatisation $\{\mathbf{y}, \mathbf{x}, \mathbf{O}, \mathbf{u}, \mathbf{J}\}$ of $\mathcal{P}$. The coefficients of the PTR polynomial are in $\mathbb{F}_2$.

$$\begin{aligned} T(X, Y, Z) = XY &+ t_4\,(X)^2 t_2\,(X)^2\ \mathrm{Tr}_{\mathbb{F}_{16}/\mathbb{F}_2}(Y) \\ &+ t_2\,(Y)^4\ \mathrm{Tr}_{\mathbb{F}_{16}/\mathbb{F}_2}(X) + t_2\,(X)^2 t_4\,(Y)^2 + Z. \end{aligned}$$

3.8 The Lorimer-Rahilly Plane

3.8.1 Collineation Groups of $\mathcal{P}$

The Lorimer-Rahilly (LMRH) plane $\mathcal{P}$ of order 16 is a translation plane of LB type IVa.1. There is a translation line $\mathcal{M}$. Besides, there are 48 elation groups of order 2 centered on three distinct points $\mathbf{p}, \mathbf{q}, \mathbf{r} \in \mathcal{M}$ with 16 groups on each center. We note the description is identical to the description of the collineation groups of the JOWK plane.

3.8.2 Coordinatisation Strategy

The coordinatisation strategy for the LMRH plane $\mathcal{P}$ is identical to that of the JOWK plane described in Section 3.7.2. The orders of the principal central collineation groups are given in Table 3.10. The abundance of subplanes of order 4 in our choice of frame is identical too. That is, having fixed the frame $\{\mathbf{y}, \mathbf{x}, \mathbf{O}, \mathbf{u}, \mathbf{J}\}$ as in Section 3.7.2, choose any point $\mathbf{s}$ on the vertical axis $[0]$ other than the points $(0,0)$, $(0,1)$, and (∞). The closure of the configuration $\{\mathbf{y}, \mathbf{x}, \mathbf{O}, \mathbf{u}, \mathbf{J}\} \cup \{\mathbf{s}\}$ is a subplane of $\mathcal{P}$ of order 4. Thus, label $\mathbf{s} = (0, a^5)$, and follow up with additive labeling to successively label the points on the axis $[0]$ by the subspaces of $(\mathbb{F}_{16}, +)$ until the axis points are all labeled.

(V, l)	$((\infty), [\infty])$	$((0), [0])$	$((\infty), [0,0])$	$((0), [\infty])$	$((0,0), [\infty])$	$((\infty), [0])$
$\lvert\Gamma(V, l)\rvert$	16	1	1	16	1	2

Table 3.10: The Lorimer-Rahilly Plane - Orders of $\Gamma \in S(\Gamma)$.

3.8.3 Measures of the Algebraic Properties of $(\mathcal{R}, \oplus, \odot)$

Measures of associativity of addition: Since $(\mathcal{R}, \oplus)$ is a group,

$$\mathcal{A}_l = \mathcal{A}_m = \mathcal{A}_r = \mathcal{R}.$$

Measures of associativity of multiplication: Following similar arguments as in the JOWK plane,

$$\mathcal{N}_m = \mathcal{N}_l = \mathcal{N}_r = \{0, 1\}.$$

Measures of distributive property: The system $(\mathcal{R}, \oplus, \odot)$ is left distributive. Consequently, the left middle distributor is maximal. The elements in the right middle distributor $\mathcal{D}'$ are forced as explained in the coordinatisation process. In contrast to the JOWK plane, the set $\mathcal{D}_r$ of right distributive elements is trivial in the LMRH plane.

$$\mathcal{D}_l = \mathcal{D} = \mathcal{R},\ \mathcal{D}' = \{0, 1\},\ \text{and } \mathcal{D}_r = \{0, 1\}.$$

Measure of commutativity of multiplication: Trivial.

$$\mathscr{Z} = \{0, 1\}.$$

3.8.4 The PTR Polynomial

We have organised the terms of the PTR polynomial similarly to the PTR polynomial obtained for the JOWK plane. The coefficients of this PTR polynomial are also entirely in $\mathbb{F}_2$.

$$\begin{aligned} T(X,Y,Z) = XY &+ t_4(X)^3 t_2(Y)^2 + t_4(X)^2 t_2(Y)^2 + t_2(X)\, t_4(Y)^2 \\ &+ t_2(X)\, t_4(X)\, t_4(Y)^2 + t_2(X)^2 t_4(X)^2 t_4(Y) + Z. \end{aligned}$$

3.9 The Mathon Plane

3.9.1 Collineation Groups of $\mathscr{P}$

The Mathon plane $\mathscr{P}$ is the only known plane of order 16 LB type II.1. There is one incident transitive flag $(\mathbf{q}, \mathcal{M})$. Besides, there are 32 elation groups of order 2 which are all centered on the points of $\mathcal{M}$. No homologies are admitted by $\mathscr{P}$. We divide the 32 elation groups into two collections of 16 elation groups called type A and type B elation groups as follows:

A. $\Gamma(\mathbf{q}, \mathcal{L})$ for all $\mathcal{L} \neq \mathcal{M}$ through $\mathbf{q}$, and

B. $\Gamma(\mathbf{p}, \mathcal{M})$ for all $\mathbf{p} \in \mathcal{M}$, $\mathbf{p} \neq \mathbf{q}$.

3.9.2 Coordinatisation Strategy

By Theorem 1.8.1,(iii), we choose $\mathbf{q}$ to be the point at infinity $\mathbf{y} = (\infty)$ and $\mathcal{M}$ to be the line at infinity $\overline{\mathbf{x}\,\mathbf{y}} = [\infty]$. This guarantees we can obtain a fully optimised PTR addition. From the description of the collineation groups of $\mathscr{P}$, any choice of the vertical axis $\overline{\mathbf{O}\,\mathbf{y}} = [0]$ is the axis of an elation group of type A. Similarly, any choice of the ideal point $\mathbf{x} = (0)$ is the center of an elation group of type B. Therefore, choosing the $((\infty), [\infty])$ flag in the above manner fixes the orders of all principal central collineation groups as given in Table 3.11.

(V, l)	$((\infty), [\infty])$	$((0), [0])$	$((\infty), [0, 0])$	$((0), [\infty])$	$((0, 0), [\infty])$	$((\infty), [0])$
$\lvert\Gamma(V, l)\rvert$	16	1	1	2	1	2

Table 3.11: The Mathon Plane - Orders of $\Gamma \in S(\Gamma)$.

The choice of the triangle $\{\mathbf{y}, \mathbf{x}, \mathbf{O}\}$ cannot be used to optimise the multiplicative structure of $(\mathcal{R}^*, \odot)$ since there are no homologies of $\mathcal{P}$. So, we focus on the measures of distributive properties. Even there, we can achieve neither a left nor a right distributive property on all of $\mathcal{R}$ since there is no translation line or translation point in $\mathcal{P}$. However, the orders of $\Gamma((0), [\infty])$ and $\Gamma((\infty), [0])$ are 2 each so we can obtain a left middle distributor $\mathcal{D}$ and a right middle distributor $\mathcal{D}'$ containing two elements each. Similar to the discussion in the coordinatisation strategy of the JOWK plane in Section 3.7.2, we want the non-trivial element in both $\mathcal{D}$ and $\mathcal{D}'$ to be $1 \in \mathcal{R}$. When this is achieved, the PTR multiplication will have the following property for all $x, y \in \mathcal{R}$ provided the PTR addition is fully optimised:

$$(1 + x) \odot (1 + y) = 1 + x + y + (x \odot y) \tag{3.9.1}$$

We used the commutative property $x + 1 = 1 + x$ in (3.9.1) since the fully optimised PTR addition coincides with the finite field addition.

Theorem 1.10.7 is applied to achieve $1 \in \mathcal{D}$. Suppose the triangle of coordinatisation $\{\mathbf{y}, \mathbf{x}, \mathbf{O}\}$ has been labeled. We will discuss the optimal choice of the origin $\mathbf{O} = (0, 0)$ in the next step of coordinatisation. For the purpose of this discussion, consider any affine point $\mathbf{O}$. Take the elation $1 \neq \gamma \in \Gamma((0), [\infty])$. The elation is unique as $|\,\Gamma((0), [\infty])\,| = 2$. Use the elation γ to label $(0,0)^{\gamma} = (1,0)$. Similarly, we apply Theorem 2.0.1 to achieve $1 \in \mathcal{D}'$. Consider the elation $1 \neq \alpha \in \Gamma((\infty), [0])$. Use the elation α to label $(0)^{\alpha} = \mathbf{J} = (1)$. The frame of coordinatisation is completed by labeling the point $\mathbf{u} = (0,1) = \overline{\mathbf{J}(1,0)} \cap [0]$. Any PTR obtained from this frame must have $\mathcal{D} = \mathcal{D}' = \{0,1\}$ by Theorems 1.10.7 and 2.0.1.

We have utilised the existence of the different central collineation groups in $\mathcal{P}$ to the fullest for optimising the coordinatisation. The next step in optimisation is to ensure the subfield $(\mathbb{F}_4, +, \cdot)$ coordinatises, with the same frame $\{\mathbf{y}, \mathbf{x}, \mathbf{O}, \mathbf{u}, \mathbf{J}\}$ of coordinatisation as that of $\mathcal{P}$, a subplane $\mathcal{Q}$ of order 4. Recall we have so far chosen and labeled three points on the line at infinity viz. $\mathbf{y}, \mathbf{x}$, and $\mathbf{J}$. We also have a strategy in place to determine the $\mathbf{u}$ once the origin $\mathbf{O} = (0,0)$ is chosen. This discussion is aimed at an optimal choice of the point $\mathbf{O}$. Now, inspired by the cases of the JOWK and the LMRH planes, we would like the frame $\{\mathbf{y}, \mathbf{x}, \mathbf{O}, \mathbf{u}, \mathbf{J}\}$ be such that we can choose any point $\mathbf{s}$ on $\overline{\mathbf{O}\,\mathbf{y}} \setminus \{\mathbf{y}, \mathbf{u}, \mathbf{O}\}$ and have the closure of the configuration $\{\mathbf{y}, \mathbf{x}, \mathbf{O}, \mathbf{u}, \mathbf{J}\} \cup \{\mathbf{s}\}$ in $\mathcal{P}$ a subplane of order 4. In the cases of the JOWK and the LMRH planes, the optimal choices made in choosing $\mathbf{y}, \mathbf{x}$, and $\mathbf{J}$ sufficed for the purpose. The choice of the point $\mathbf{O}$ was random to the extent of available choices for the horizontal axis $\overline{\mathbf{O}\,\mathbf{x}}$ through $\mathbf{x}$ and the vertical axis $\overline{\mathbf{O}\,\mathbf{y}}$ through $\mathbf{y}$. Coming to the Mathon plane, the choice of the vertical axis can still be random i.e. any line through the point $\mathbf{y}$ other than the infinite axis $[\infty]$ already chosen. But in choosing the horizontal axis $\overline{\mathbf{O}\,\mathbf{x}}$, or alternately in choosing a point $\mathbf{O}$ on a chosen vertical axis $\overline{\mathbf{O}\,\mathbf{y}}$, we have to go by trial and error. This is an observation based on our coordinatisations. We state a few more observations in this regard, enumerated for clarity.

(i) The structure of the group of central collineations of the Mathon plane lacks two aspects found in the group of central collineations of the JOWK plane (or LMRH plane for that matter), especially in the context of the groups centered on the vertices of the triangle of coordinatisation $\{\mathbf{y}, \mathbf{x}, \mathbf{O}\}$. Firstly, there is no translation line. Secondly, $\Gamma((0), [0, 0])$ is trivial in the Mathon plane while it is of order 2 in the JOWK plane.

(ii) Even in the Mathon plane, some choices of $\overline{\mathbf{O}\,\mathbf{y}}$ result in a frame that closes to a subplane of order 4 for any point in $\overline{\mathbf{O}\,\mathbf{y}} \setminus \{\mathbf{y}, \mathbf{u}, \mathbf{O}\}$. Of course, the remaining choices for $\overline{\mathbf{O}\,\mathbf{y}}$ also contain at least one point each such that it generates a subplane of order 4 in conjunction with the frame.

(iii) The observations show that a 'decreasing' structure of the group of central collineations in the planes corresponds to lesser options for an optimal coordinatisation in the sense we have defined.

(iv) Regarding the observation in (ii), we have computed identical PTR polynomials under both types of the vertical axes. We have listed two such PTR polynomials below. Therefore, the only difference for the purposes of optimal coordinatisation is in the ease of optimising a subplane of order 4 along the way.

After coordinatising the subplane $\mathcal{Q}$ of order 4, the rest of the labeling on the vertical axis [0] is completed by additive labeling as usual.

We make a final note regarding the coordinatisation of the Mathon plane based on our computations. We found that the set $\mathcal{D}_l$ of elements left distributive on all of $\mathcal{R}$ is of order 8 in every coordinatisation in which the point $\mathbf{J}$ is chosen by the above method. This is largest among all coordinatisations we have obtained. We clarify the set $\mathcal{D}_l$ is of order 8 in some coordinatisations obtained by choosing the point $\mathbf{J}$ randomly as well. But when we choose $\mathbf{J}$ by the above method, some of the PTR polynomials obtained from the coordinatisation have coefficients entirely in $\mathbb{F}_2$. No other choice of the point $\mathbf{J}$ resulted in a PTR polynomial with its coefficients entirely in $\mathbb{F}_2$. Moreover, the set $\mathcal{D}_l$ is precisely the union of elements of $\mathbb{F}_4$ and its coset $\mathbb{F}_4 + a$ in $(\mathbb{F}_{16}, +)$ in every coordinatisation whose corresponding PTR polynomial has all its coefficients in $\mathbb{F}_2$.

3.9.3 Measures of the Algebraic Properties of $(\mathcal{R}, \oplus, \odot)$

Measures of associativity of addition: Since $(\mathcal{R}, \oplus)$ is a group,

$$\mathcal{A}_l = \mathcal{A}_m = \mathcal{A}_r = \mathcal{R}.$$

Measures of associativity of multiplication: We can only claim a non-trivial middle nucleus in this case. Recall Theorem 1.10.8. We computed the left and the right nuclei as being trivial in the optimal coordinatisations we present below.

$$\mathcal{N}_m = \mathcal{N}_l = \mathcal{N}_r = \{0, 1\}.$$

Measures of distributive property: Putting together the discussions made in the coordinatisation strategy section,

$$\mathcal{D}_l = \{0, 1, a^5, a^{10}, a, a^4, a^2, a^8\}, \text{ and } \mathcal{D} = \mathcal{D}' = \mathcal{D}_r = \{0, 1\}.$$

Measure of commutativity of multiplication: Trivial.

$$\mathcal{Z} = \{0, 1\}.$$

3.9.4 The PTR Polynomials

We give two PTR polynomials for the Mathon plane. We have organised the terms of the polynomials in the form (3.2.2). The coefficient polynomials $p(X,Y)$ and $q(X,Y)$ have been distributed for a better appearance. Note all the coefficients of the PTR polynomials are in $\mathbb{F}_2$. The trace $\mathrm{Tr}_{\mathbb{F}_{16}/\mathbb{F}_2}(x) = 0$ for $x \in \mathbb{F}_4 \cup (\mathbb{F}_4 + a)$ which is equal to the set $\mathcal{D}_l$ in each PTR. The PTR polynomials reduce to linearised polynomials in that case showing the left distributive property of the PTR in $\mathcal{D}_l \times \mathcal{R}$.

(i) $T(X,Y,Z) = XY + p(X,Y)\mathrm{Tr}_{\mathbb{F}_{16}/\mathbb{F}_2}(X) + t_4(X)\,t_4(Y)^2 + t_2(X)\,t_4(Y) + Z$, where $p(X,Y) = t_2(X)^2\,t_4(Y)^2 + t_4(X)\,t_2(Y)^2 + \mathrm{Tr}_{\mathbb{F}_{16}/\mathbb{F}_2}(Y) + Y^4(1 + h_4(Y))^2$.

(ii) $T(X,Y,Z) = XY + p(X,Y)\mathrm{Tr}_{\mathbb{F}_{16}/\mathbb{F}_2}(X) + t_2(X)\,\mathrm{Tr}_{\mathbb{F}_{16}/\mathbb{F}_2}(Y) + t_4(X)\,t_2(Y)^2 + Z$, where $p(X,Y) = t_4(X)\,t_2(Y)^2 + t_2(X)\,t_4(Y)^2 + (Y^2+1)\mathrm{Tr}_{\mathbb{F}_{16}/\mathbb{F}_2}(Y) + t_4(Y^3)$.

Chapter 4

COORDINATISATION OF PROJECTIVE PLANES OF ORDER 16 LB TYPE I.1

In this chapter, we develop the notion of optimal coordinatisation for the planes of LB type I.1. We obtain some optimal coordinatisations of the planes of order 16 LB type I.1. The planes are the BBH1, the BBH2, the Johnson and the BBS4 planes. Refer to Section 1.18 for a brief note on the discovery of the planes and their relation to the other planes of order 16.

4.1 Meaning of Optimal Coordinatisation for LB Type I.1 Planes

In the LB classification, the LB type I.1 planes are the lowest in the hierarchy of planes. By definition, the planes do not admit any transitive flag (Table 1.2). As a result, the following two properties do not co-exist in any PTR $(\mathcal{R}, T)$ obtained from a coordinatisation of a LB type I.1 plane:

(i) linearity (Definition 1.7.2), and

(ii) either $(\mathcal{R}, \oplus)$ or $(\mathcal{R}^*, \odot)$ is a group.

We clarify further the non-existence of the two properties simultaneously in a PTR. By Theorem 1.8.1, (iii), the existence of an incident transitive flag implies and is implied by the simultaneous occurrence of '$(\mathcal{R}, \oplus)$ is a group' and '$(\mathcal{R}, T)$ is linear' (when appropriately coordinatised). Again by Theorem 1.8.1, (i), the existence of a non-incident transitive flag implies and is implied by the simultaneous occurrence of '$(\mathcal{R}^*, \odot)$ is a group' and '$(\mathcal{R}, T)$ is linear'. Emphasis is on the two statements not ruling out the possibility of obtaining a linear PTR $(\mathcal{R}, T)$ or of realising the corresponding loops $(\mathcal{R}, \oplus)$ and/or $(\mathcal{R}^*, \odot)$ as groups - just that they cannot both occur in any one PTR obtained by coordinatising a LB type I.1 plane.

The discussion in the previous paragraph has a direct impact on our efforts to develop the meaning of optimal coordinatisation of the planes of LB type I.1. Recalling Remark 1.9.6, while we are not aware of any results obtaining a linear PTR representation of a plane of LB type I.1, it is shown in [5] every finite derivable plane admits a PTR whose additive loop $(\mathcal{R}, \oplus)$ is an elementary abelian group. In a non-linear PTR, the ability to impose a structure as regular as an elementary abelian group on an object defined via the PTR i.e. the additive loop $(\mathcal{R}, \oplus)$ in this case, can be immensely useful in obtaining a meaningful representation of the PTR function T such as a polynomial over a finite field. Therefore, we still consider the group structure within the additive loop $(\mathcal{R}, \oplus)$ or the multiplicative loop $(\mathcal{R}^*, \odot)$ a measure of optimality of coordinatisation of the LB type I.1 planes even though the PTR is not expressible in terms of $\oplus$ and $\odot$ directly. We do not completely ignore the linearity aspect of the PTR either. Recall the coordinatisation the planes of order 16 LB type II.1 or above in Chapter 3. In the coordinatisation of every plane there, we had a Desarguesian subplane of order 4 coordinatised by the subfield $(\mathbb{F}_4, +, \cdot)$ of $(\mathcal{R}, \oplus, \odot)$. By way of converting this observation to intuition, we will look for optimising the 'parts' of the domain $\mathcal{R}^3$ of the PTR function T where the linearity holds. So, a measure of linearity of the resulting PTR will be a measure of optimality of the coordinatisation.

We introduce some definitions to formalize the discussion of the measure of linearity in the PTRs of the LB type I.1 planes.

Definition 4.1.1. *Given a PTR* $(\mathcal{R}, T)$ *and a tuple* $(m, x, y) \in \mathcal{R}^3$*, we say that* T *has a linear property with respect to* (m, x, y) *if* $T(m, x, y) = (m \odot x) \oplus y$ *where* $\oplus$ *and* $\odot$ *are defined as in Definition 1.7.1.*

Clearly, a linear PTR has a linear property with respect to every tuple (m, x, y) in $\mathcal{R}^3$. Note the cartesian product $\mathcal{R}^3$ is the domain of the PTR function T. Given this perspective, we define,

Definition 4.1.2. ***Domain of Linearity of a PTR*** *Let* $(\mathcal{R}, T)$ *be a planar ternary ring obtained from a coordinatisation of a projective plane* $\mathcal{P}$*, with the PTR addition* $\oplus$ *and multiplication* $\odot$*. The domain of linearity of the PTR* $(\mathcal{R}, T)$ *is defined as the following subset of* $\mathcal{R}^3$*:*

$$\{(m, x, y) \in \mathcal{R}^3 : T(m, x, y) = (m \odot x) \oplus y\}.$$

We use Definition 4.1.2 as a tool to measure the optimality of the PTRs presented in this chapter. While the entirety of $\mathcal{R}^3$ is trivially the domain of linearity for any PTR obtained from an optimal coordinatisation of a plane of LB type other than I.1, the relative order of the domain of linearity is not small in the PTRs we obtain for the planes of LB type I.1 (of order 16 in this chapter). In the finite cases, as we have shown for the PTRs we obtain, it is possible to state this measure of linearity as a fraction of the order of $\mathcal{R}^3$.

4.2 General Coordinatisation Strategies for LB Type I.1 Planes

At the outset, the strategies for optimal coordinatisation of LB type I.1 planes are informed by the strategies for the planes of LB type I.2 or above discussed in Chapter 2. The planes of LB type I.1 do not admit any transitive central collineation groups. This means, we cannot obtain a group structure on all of $(\mathcal{R}, \oplus)$ or $(\mathcal{R}^*, \odot)$

by associating the points of an axis of the frame of coordinatisation with the actions of the central collineations in a central collineation group. So, we look to maximize the group structures in either $(\mathcal{R}, \oplus)$ or $(\mathcal{R}^*, \odot)$ by a combination of associating some points of an axis with a central collineation group and using additive or multiplicative labeling. A suitable frame of coordinatisation will prove to be crucial to obtaining maximal group structures.

The subsets of $(\mathcal{R}^*, \odot)$ are associated with the principal homology groups. Refer to Section 2.2.1 for the discussion on the optimal choices guided by the homology groups. Similarly, the subsets of $(\mathcal{R}, \oplus)$ are associated with the principal elation groups. Section 2.2.2 discusses the optimal choices based on the principal elation groups.

As an example, we identify a largest central collineation group to be either $\Gamma((\infty), [\infty])$ or $\Gamma((0), [0])$ depending on, respectively, whether it is an elation group or a homology group. Other optimising strategies like increasing the amount of associativity of either the PTR addition or the multiplication, increasing the amount of the distributive properties, or forcing certain elements into some special subset of $\mathcal{R}$ are similarly motivated. Unlike in the higher LB types where linearity is guaranteed in the optimal coordinatisations, increasing the amount of linearity will also be a goal of the optimal coordinatisation of LB type I.1 planes.

The coordinatisation process is described in the same way as in Chapter 3. We will draw analogies to the strategies used for the planes of LB types II.1 or above in specific cases. The PTR polynomial part however contains more discussions. We discuss the relation of the PTR polynomial with the PTR addition and the PTR multiplication. We also discuss the inferences drawn from the PTR polynomial representation of the plane.

4.3 General Form of the PTR Polynomials of LB Type I.1 Planes of Order 16

Similar to the coordinatisations of the planes of order 16 LB type II.1 or higher in Chapter 3, the coordinatisation we obtain for a plane $\mathscr{P}$ of order 16 LB type I.1 also coordinatises a subplane $\mathscr{Q}$ of order 4. Moreover, the frame $\{\mathbf{y}, \mathbf{x}, \mathbf{O}, \mathbf{u}, \mathbf{J}\}$ of coordinatisation of $\mathscr{P}$ is inherited by the the subplane $\mathscr{Q}$ and the induced coordinatisation of the subplane is optimal too. This means, the subplane $\mathscr{Q}$ is coordinatised by the subfield $(\mathbb{F}_4, +, \cdot)$ of $(\mathcal{R}, \oplus, \odot)$ which in turn implies the PTR function T is given by $T(x, y, z) = xy + z$ for $(x, y, z) \in \mathbb{F}_4^3$. The product xy in this case is also the product in the PTR multiplication i.e. $x, y \in \mathbb{F}_4 \Longrightarrow x \odot y = xy$. Similarly, the sum $xy + z$ is also the sum in the PTR addition i.e. $xy, z \in \mathbb{F}_4 \Longrightarrow (xy) \oplus z = xy + z$. By combining the results, we get $T(x, y, z) = (x \odot y) \oplus z$ for all $x, y, z \in \mathbb{F}_4$. Hence, the domain of linearity contains all $(x, y, z) \in \mathbb{F}_4^3$. Combining this with the statement on the general form a PTR polynomial given by Theorem 1.16.3, (i), we deduce the following statement:

Definition 4.3.1. *A general form of the PTR polynomials of the planes of order 16 LB type I.1 obtained in this chapter is*

$$T(X, Y, Z) = (X \odot Y) + Z + p(X, Y, Z)t_4(X) + q(X, Y, Z)t_4(Y), \qquad (4.3.1)$$

for some polynomials $p(X, Y, Z)$ and $q(X, Y, Z)$ in $\mathbb{F}_{16}[X, Y, Z]$.

Recall from Section 1.15.3 the polynomials $t_4(X)$ and $t_4(Y)$ vanish on the subfield $(\mathbb{F}_4, +, \cdot)$ of $(\mathcal{R}, \oplus, \odot)$.

4.4 The BBH1 Plane

The BBH1 plane $\mathscr{P}$ of order 16 is a self-dual plane of LB type I.1.

4.4.1 Central Collineation Groups of $\mathscr{P}$

There is symmetry between points and lines in terms of the number and orders of the central collineation groups with a given center and the groups with a given axis. The largest central collineation group is a unique elation group of order 8. Call it the $(\mathbf{q}, \mathcal{M})$-elation group. Every central collineation admitted by $\mathscr{P}$ is centered on $\mathcal{M}$. By duality, the axis of every central collineation of $\mathscr{P}$ is incident to $\mathbf{q}$. We divide the central collineation groups of $\mathscr{P}$ based on the center-axis flags $(\mathbf{p}, \mathcal{L})$ as follows:

A. The unique incident flag $(\mathbf{q}, \mathcal{M})$ with $|\Gamma(\mathbf{q}, \mathcal{M})| = 8$.

B. $\mathbf{p} = \mathbf{q}$, $\mathbf{p} \in \mathcal{L} \neq \mathcal{M}$, $|\Gamma(\mathbf{p}, \mathcal{L})| = 4$. There are 4 elation groups of this type.

C. $\mathbf{p} = \mathbf{q}$, $\mathbf{p} \in \mathcal{L} \neq \mathcal{M}$, $|\Gamma(\mathbf{p}, \mathcal{L})| = 2$. There are 12 elation groups of this type.

D. $\mathbf{q} \neq \mathbf{p} \in \mathcal{M}$, $\mathcal{L} = \mathcal{M}$, $|\Gamma(\mathbf{p}, \mathcal{L})| = 4$. There are 4 elation groups of this type.

E. $\mathbf{q} \neq \mathbf{p} \in \mathcal{M}$, $\mathcal{L} = \mathcal{M}$, $|\Gamma(\mathbf{p}, \mathcal{L})| = 2$. There are 12 elation groups of this type.

F. $\mathbf{q} \neq \mathbf{p} \in \mathcal{M}$, $\mathbf{q} \in \mathcal{L} \neq \mathcal{M}$, $|\Gamma(\mathbf{p}, \mathcal{L})| = 3$. There are 16 homology groups of this type on 4 distinct centers and sharing 4 distinct axes. Each center admits 4 homology groups with 4 distinct axes through $\mathbf{q}$ and each axis admits 4 homology groups with 4 distinct centers on $\mathcal{M}$. The 4 centers are the same as the centers of the elation groups of type D. The axes are the same as the axes of the elation groups of type B.

4.4.2 Coordinatisation Strategy

We draw analogy with the semifield planes for choosing the $((\infty), [\infty])$ flag since the BBH1 plane is also self-dual. Therefore, $\mathbf{y} = (\infty) = \mathbf{q}$ and $\overline{\mathbf{x}\,\mathbf{y}} = [\infty] = \mathcal{M}$. Next, the choice of $\mathbf{x} = (0)$ is made easy by the descriptions of the elation and homology groups above. Choosing any of the 4 centers of type F (which are also the centers of type D) as $\mathbf{x} = (0)$ has the following optimisation benefits:

(i) We have $\Gamma((0),[0])$ a homology group of type F. By Theorem 1.10.8, the PTR $(\mathcal{R}, T)$ has a subgroup $(\mathcal{S}_1, \odot)$ of order 3 in its multiplicative loop $(\mathcal{R}^*, \odot)$. Additionally, the elements of $\mathcal{S}_1$ have a middle associativity of multiplication and the set $\{(s, x, y) : s \in \mathcal{S},\ x, y \in \mathcal{R}\}$ is a subset of the domain of linearity.

(ii) We have $\Gamma((0),[\infty])$ an elation group of type D. By Theorem 1.10.7, the PTR $(\mathcal{R}, T)$ has a subgroup $(\mathcal{S}_2, \oplus)$ of order 4 in the additive loop $(\mathcal{R}, \oplus)$. Additionally, the elements of $\mathcal{S}_2$ have a left associativity of addition on all of $\mathcal{R}$.

In part (ii), we cannot make a statement on the left middle distributor $\mathcal{D}$ since $(\mathcal{R}, T)$ is not a Cartesian group. Consequently, we cannot extract a relation among s, m, and x as in Theorem 1.10.7 to increase the amount of linearity in the PTR.

The elation group $\Gamma((\infty),[0])$ is of type B. However again, since the linearity of the resulting PTR is not guaranteed we cannot invoke Theorem 2.0.1 to claim the resulting PTR must have a right middle distributor $\mathcal{D}'$ of order 4. This does not mean the orders of the distributors are necessarily trivial or less than the orders of the corresponding central collineation groups. For example, in Theorem 1.10.7, the PTR being a Cartesian group is a sufficient but not necessary condition to obtain $\mathcal{S} = \mathcal{D}$ (note $|\mathcal{S}| = |\,\Gamma((0),[\infty])\,| = 4$ in this case). Similarly for the set $\mathcal{D}'$ in Theorem 2.0.1. Interestingly, the optimal PTR we present below achieves $\mathcal{D} = \mathcal{S}_2$ while $\mathcal{D}'$ is trivial. The corresponding PTR polynomial is also the most optimal among all PTR polynomials obtained for the BBH1 plane.

With the choice of $\mathbf{x} = (0)$, the orders of the principal central collineation groups are determined. See Table 4.1.

(V,l)	$((\infty),[\infty])$	$((0),[0])$	$((\infty),[0,0])$	$((0),[\infty])$	$((0,0),[\infty])$	$((\infty),[0])$
$\lvert\Gamma(V,l)\rvert$	8	3	1	4	1	4

Table 4.1: The BBH1 Plane - Orders of $\Gamma \in S(\Gamma)$.

Let the centers in type F other than $\mathbf{x} = (0)$ be denoted $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$. Let $\Gamma((0),[0]) = \{1, \gamma, \gamma^2\}$ and $1 \neq \alpha \in \Gamma(\mathbf{a},[0])$ be one of the two non-trivial $(\mathbf{a},[0])$-homologies. Define $\beta = \gamma^{-1}\alpha\gamma$. By [17], Lemma 4.11, β is a $(\mathbf{a}^\gamma, [0]^\gamma)$-central collineation. But $[0]^\gamma = [0]$. We have $\beta \neq 1$ for otherwise $\gamma = \alpha\gamma \Rightarrow \alpha = 1$, a contradiction. Also, $\beta^3 = \gamma^{-1}\alpha^3\gamma = 1$. In short, β is a non-trivial homology in a homology group of type F. Now, $\mathbf{a}^\gamma \neq (0)$ which is the center of the homology γ and $\mathbf{a}^\gamma \neq \mathbf{a}$ since $\gamma \neq 1$. Thus, $\mathbf{a}^\gamma \in \{\mathbf{b}, \mathbf{c}\}$. We see $1 \neq \gamma \in \Gamma((0),[0])$ permutes $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ cyclically. Based on the observation, one of the remaining three centers of the homology groups of type F is chosen as the point $\mathbf{J} = (1)$ on the axis $[\infty]$ followed by $(1)^\gamma = (a^5)$, and $(1)^{\gamma^2} = (a^{10})$. Recall a similar discussion in the coordinatisation of the SEMI4 plane (Section 3.2).

As a result of this labeling of the four points on the infinite axis $[\infty]$ by $\{(0), (1), (a^5), (a^{10})\}$, the PTR $(\mathcal{R}, T)$ has a subgroup $(\mathcal{S}, \odot) = (\{1, a^5, a^{10}\}, \odot)$ in the multiplicative loop $(\mathcal{R}^*, \odot)$. By Theorem 1.10.8, $T(s,x,y) = (s \odot x) \oplus y \ \forall x, y \in \mathcal{R},\ s \in \mathcal{S}$. That is, the PTR is linear with respect to any point (s,x,y) with $s \in \mathbb{F}_4$. Consequently, we can improve on the general PTR polynomial form (4.3.1) for this coordinatisation of the BBH1 plane. We get

$$T(X,Y,Z) = (X \odot Y) + Z + p(X,Y,Z)t_4(X)\,, \tag{4.4.1}$$

for some polynomial $p(X,Y,Z)$ in $\mathbb{F}_{16}[X,Y,Z]$.

Our choice of the point $\mathbf{J}$ and the labeling of its images under $\Gamma((0),[0])$ have an additional optimal feature. The remaining two points of the frame i.e. $\mathbf{O} = (0, 0)$ and $\mathbf{u} = (0, 1)$ can be chosen to be any two distinct points other than (∞) on the vertical axis [0] to obtain a subplane of order 4. The subplane is obtained as a closure of the configuration $\{\mathbf{y}, \mathbf{x}, \mathbf{O}, \mathbf{u}, \mathbf{J}\} \cup \{(a^5)\}$. The vertical axis [0] itself can be any of the four axes of the homology groups of type F. However, we optimise the choice of the points $\mathbf{O}$ and $\mathbf{u}$ by varying the axes and the point $\mathbf{u}$ on the chosen axis [0] such that the set of points $\{(0, 0), (0, 1), (0, a^5), (0, a^{10})\}$ on [0] is a subset of the images of the point $(0, 0)$ under the action of $\Gamma((\infty), [\infty])$. Note the elation group is of order 8 and is elementary abelian.

After obtaining the points $(0, a^5)$ and $(0, a^{10})$ by projection, the remaining points on the axis [0] are labeled by additive labeling as in the case of most planes of LB type II.1 or above. All possible choices in the additive labeling process achieve a fully optimised PTR addition. This is remarkable from two perspectives. Firstly, the elementary abelian $(\mathcal{R}, \oplus)$ guaranteed by [5] (refer to Section 4.4) is achieved, that too with numerous possible frames and labeling. Every point of the frame other than (∞) is can be chosen from a set containing more than one point, and so with axes other than $[\infty]$. Secondly, the PTR addition coincides with $(\mathbb{F}_{16}, +)$ in all cases. We see the two aspects as additional proofs of optimality of our coordinatisation strategy.

In the additive labeling of the axis [0] we choose any unlabeled point on the line to be $(0, a)$, perform the additive labeling, and follow it by choosing any of the remaining points to be $(0, a^3)$ and finally labeling all remaining points, also additively. For the BBH1 plane, since $\Gamma((\infty), [\infty])$ is of order 8, we can optimise the choice of the first point $(0, a)$ to improve the form of the PTR polynomial of the resulting PTR. By Theorem 1.10.11, if $\mathcal{S}$ is the subset of $\mathcal{R}$ corresponding to the points labeled $(0, s)$ which are obtained as images of the origin $(0, 0)$ under the action of $\Gamma((\infty), [\infty])$, then the PTR is linear with respect to the points (m, x, s) $\forall x, y \in \mathcal{R},\ s \in \mathcal{S}$. So, by choosing $(0,0)^\alpha = (0, a)$ for some $\alpha \in \Gamma((0), [\infty])$ and ensuring this is distinct from $(0,0), (0,1), (0, a^5)$, and $(0, a^{10})$ labeled previously, we obtain a PTR which satisfies $T(m, x, s) = (m \odot x) \oplus s$ $\forall x, y \in \mathcal{R},\ s \in \mathcal{S} = \{0, 1, a^5, a^{10}, a, a^4, a^2, a^8\}$. Recall from the Addition Table 1.6 of $(\mathbb{F}_{16}, +)$ that $1 + a = a^4$ etc. As a result, we can improve the form of the the PTR polynomial (4.4.1) to

$$T(X, Y, Z) = (X \odot Y) + Z + q(X, Y, Z)t_4(X)\left(\prod_{z \in \mathcal{S}} Z - z\right), \tag{4.4.2}$$

where the set $\mathcal{S}$ is as above and $q(X, Y, Z)$ is some polynomial in $\mathbb{F}_{16}[X, Y, Z]$. For the set $\mathcal{S} = \{0, 1, a^5, a^{10}, a, a^4, a^2, a^8\}$, we have

$$\prod_{z \in \mathcal{S}} Z - z = Z^8 + Z^4 + Z^2 + Z = \mathrm{Tr}_{\mathbb{F}_{16}/\mathbb{F}_2}(Z).$$

By trial and error we are able to further optimise the labeling of the points on the vertical axis such that the image of the point $(0,0)$ under the action of $\Gamma((0), [\infty])$ is as good as it can be i.e. $\{(0,0), (1,0), (a^5, 0), (a^{10}, 0)\}$. Also, the the left middle distributor $\mathcal{D}$ of the PTR we obtained is $\mathbb{F}_4$. By Theorem 1.10.7 the distributor $\mathcal{D}$ would be guaranteed in a linear PTR. We obtained it for a non-linear PTR of a LB type I.1 plane by making careful choices in the coordinatisation process followed by trial and error.

Obtaining a nice left middle distributor $\mathcal{D}$ has a further benefit. Again by Theorem 1.10.7, we have $T(x, y, x \odot w) = x \odot (y \oplus w)$ for all $x, w \in \mathcal{R}, y \in \mathbb{F}_4$. Since $\mathcal{D} = \mathbb{F}_4$, we get $T(x, y, x \odot w) = (x \odot y) \oplus (x \odot w)$ for all $x, w \in \mathcal{R}, y \in \mathbb{F}_4$. For a given x, the product $x \odot w$ varies in $\mathcal{R}$ as w varies. Thus, we have obtained the property: 'linear with respect to (x, y, z) for all $x, z \in \mathcal{R}, y \in \mathbb{F}_4$'. Consequently, a factor $t_4(Y)$ of the polynomial $q(X,Y,Z)$ is introduced in (4.4.2). Recall that $\prod_{y\in\mathbb{F}_4} Y-y = Y^4-Y = t_4(Y)$. The updated PTR polynomial form is

$$T(X,Y,Z) = (X\odot Y)+Z+r(X,Y,Z)t_4(X)\,t_4(Y)\,\mathrm{Tr}_{\mathbb{F}_{16}/\mathbb{F}_2}(Z), \tag{4.4.3}$$

for some polynomial $r(X,Y,Z)$ in $\mathbb{F}_{16}[X,Y,Z]$.

4.4.3 Measures of the Algebraic Properties of $(\mathcal{R},\oplus,\odot)$

Measures of associativity of addition: Since $(\mathcal{R},\oplus)$ is a group,

$$\mathcal{A}_l = \mathcal{A}_m = \mathcal{A}_r = \mathcal{R}.$$

Measures of associativity of multiplication: The groups $\Gamma((\infty),[0,0])$ and $\Gamma((0,0),[\infty])$ are trivial and so are the left and the right nuclei. From the discussion in the coordinatisation process, $\{1,a^5,a^{10}\}\subseteq\mathcal{N}_m$. Including 0, we have

$$\mathcal{N}_l = \mathcal{N}_r = \{0,1\},\ \text{and}\ \mathcal{N}_m = \mathbb{F}_4.$$

Remark 4.4.1. ***Two Aspects of Optimality:*** *We know a larger nucleus corresponds to a larger amount of associativity in the multiplicative loop $(\mathcal{R}^*,\odot)$ of a PTR. In this sense, a non-trivial nucleus is more optimal than a trivial one. This is meaningful especially in the PTRs of the planes of LB type I.1 which are not linear and the orders of the nuclei do not necessarily correspond to the orders of the principal homology groups. On the other hand, if a principal homology group is trivial, then a trivial corresponding nucleus is optimal in the sense of the behavior of the PTR. We clarify this with the above example of the BBH1 plane PTR. By Theorem 1.10.9, a trivial $\Gamma((\infty),[0,0])$ implies a trivial left nucleus provided the PTR is linear* *and left distributive. Similarly, by Theorem 1.10.10, a trivial* $\Gamma((0,0),[\infty])$ *implies a trivial right nucleus provided the PTR is linear and right distributive. This means, the PTR we obtained for the BBH1 plane has a property similar to a linear PTR with both a left and a right distributive property as far as the orders on the left and the right nuclei are concerned. We see this as an aspect of optimality of the PTR and therefore of the coordinatisation. Recall that we drew analogy with the semifield plane at the beginning of the coordinatisation process.* □

Measures of distributive property: The left and right middle distributors were discussed in the coordinatisation process. The sets $\mathcal{D}_l$ and $\mathcal{D}_r$ of the left and the right distributive elements respectively cannot be claimed to contain non-trivial elements by using Theorems 1.10.11 and 1.10.7 since the concerned principal homology groups are trivial. However, they both equal $\mathbb{F}_4$ in the optimal PTR we obtained.

$$\mathcal{D}_l = \mathcal{D}_r = \mathcal{D} = \mathbb{F}_4, \text{ and } \mathcal{D}' = \{0\}.$$

Measure of commutativity of multiplication: The middle nucleus is also the center in the optimal PTR.

$$\mathcal{Z} = \mathbb{F}_4.$$

4.4.4 The PTR Polynomial

We give three polynomials - the PTR addition polynomial, the PTR multiplication polynomial, and the PTR polynomial - as generated by the polynomial interpolation of the functions $T(1,y,z)$, $T(x,y,0)$, and $T(x,y,z)$ respectively. For the convenience of reading, we have replaced $\mathrm{Tr}_{\mathbb{F}_{16}/\mathbb{F}_2}$ by Tr in this chapter.

(i) $Y \oplus Z = Y + Z$.

(ii) $X \odot Y = XY + X^8\ t_4(X)^2\ t_4(Y) + X^4\ t_4(X)^2\ \operatorname{Tr}\left(Y^3\right)$
$$+\, t_4(X)\ \operatorname{Tr}\left(Y^3\right) + \operatorname{Tr}\left(X^3\right)\ t_4(Y) + X^4\ \operatorname{Tr}(Y)$$
$$+\, Y^4\ \operatorname{Tr}(X) + \operatorname{Tr}(XY) + a^{10}\ t_4(X)^2\ t_4(Y)^2.$$

(iii) $T(X,Y,Z) = (X \odot Y) + Z + t_4(X)^2\ t_4(Y)^2\ \operatorname{Tr}(Z)$.

The equation in (i) shows the PTR is fully optimised with respect to addition. Unlike the PTR polynomials of the planes of LB type II.1 or above, as seen in Chapter 3, we will find terms of the form $X^iY^jZ^j$ in this PTR polynomial (when expanded). Recall Theorem 1.16.3, (i). The PTR polynomial is of the form (1.16.1). The PTR polynomial is in the form (4.4.3) with $r(X,Y,Z) = t_4(X)\,t_4(Y)$.

4.4.4.1 Interpreting the PTR polynomial

In this section we obtain some deductions from the PTR polynomial of the BBH1 plane. The first two deductions are known facts about the coordinatisation and the corresponding PTR. The third deduction gives a measure of the linearity in the PTR. The fourth deduction simply shows the PTR multiplication is simpler in specific coordinates. The exercise is intended to show the efficacy of the PTR polynomial representation of a projective plane.

(i) Combining $T(X,Y,Z) = (X \odot Y) + Z + t_4(X)^2\ t_4(Y)^2\ \operatorname{Tr}(Z)$ with $Y \oplus Z = T(1,Y,Z) = Y + Z$, we get $T(X,Y,Z) \neq (X \odot Y) \oplus Z$ identically on $\mathcal{R}^3$. Thus, the PTR $(\mathcal{R}, T)$ is not linear.

(ii) We have $x \odot y = xy$ in $\mathbb{F}_4^2$ since every term other than xy on the right side in (ii) contains a $t_4(x)$ and/or $t_4(y)$. Note that $\operatorname{Tr}(x) = t_4(x)^2 + t_4(x)$, $\operatorname{Tr}(x^3) = t_4(x)^3$, and $\operatorname{Tr}(xy) = t_2\left(t_4(x)t_4(y) + xt_4(y) + yt_4(x)\right)$ as the field is of characteristic 2. Since the PTR addition coincides with the field addition on all of $\mathcal{R}$, this shows a Desarguesian subplane of order 4 exists and has been optimally coordinatised by the subfield $(\mathbb{F}_4, +, \cdot)$ of $(\mathcal{R}, \oplus, \odot)$.

(iii) The PTR is linear with respect to (x, y, z) where $t_4(x)^2\ t_4(y)^2\ \operatorname{Tr}(z) = 0$. The term $t_4(x)^2\ t_4(y)^2\ \operatorname{Tr}(z)$ vanishes on $\mathcal{S} \times \mathbb{F}_{16} \times \mathbb{F}_{16}$, $\mathbb{F}_{16} \times \mathbb{F}_4 \times \mathbb{F}_{16}$, and $\mathbb{F}_{16} \times \mathbb{F}_{16} \times \mathbb{F}_4$ where $\mathcal{S} = \mathbb{F}_4 \cup (\mathbb{F}_4 + a)$. Hence, the domain of linearity of the PTR is

$$(\mathcal{S} \times \mathbb{F}_{16} \times \mathbb{F}_{16}) \cup (\mathbb{F}_{16} \times \mathbb{F}_4 \times \mathbb{F}_{16}) \cup (\mathbb{F}_{16} \times \mathbb{F}_{16} \times \mathbb{F}_4)\,.$$

A simple calculation shows the order of the domain of linearity of the PTR is a $\frac{23}{32}$th of the order of $\mathcal{R}^3$.

(iv) The PTR multiplication satisfies

$$x \odot y = \begin{cases} xy & \text{if } x \in \mathbb{F}_4,\ y \in \mathbb{F}_4, \\ xy + t_4(x)^2 & \text{if } x \notin \mathbb{F}_4,\ y \in \mathbb{F}_4 \setminus \mathbb{F}_2, \text{ and} \\ xy + t_4(y)^2 & \text{if } x \in \mathbb{F}_4 \setminus \mathbb{F}_2,\ y \notin \mathbb{F}_4. \end{cases}$$

4.4.4.2 On the Presentations of a PTR Polynomial

Clearly, the deductions from the PTR polynomial of the BBH1 plane are facilitated by the form in which we have written the polynomial. On one hand, theory helps in predicting an expected form of a PTR polynomial in a coordinatisation of a plane. On the other hand, a PTR polynomial obtained from the computations reveals the properties of the plane when written appropriately. In our work, we have gone back and forth between the two while always going forward in the understanding of the subject.

In many cases we have considered different ways to look at the same PTR polynomial. The forms presented in this work are the ones we find most revealing (and neater to see). In this context, there are some connections with other well known polynomials we think might lead to further insights but have not explored at the time of writing. For instance, the terms of the type $\mathrm{Tr}\,(x^3)$ that appear in the given PTR polynomial of the BBH1 plane can also be written as $\mathrm{Tr}\,(x^3) = -1 + h_4(x^3)$ in any characteristic apart from $\mathrm{Tr}\,(x^3) = t_4(x)^3$ in characteristic 2. The h_k polynomials have been applied to construct polynomial indicator functions and for polynomial representation of groups in the PhD thesis of [4].

4.5 The BBS4 Plane

The BBS4 plane $\mathcal{P}$ of order 16 is a semi-translation plane of LB type I.1.

4.5.1 Central Collineation Groups of $\mathcal{P}$

All central collineation groups admitted by the BBS4 plane of order 16 are centred on a unique line $\mathcal{M}$. There are 5 distinct centers of the groups which we denote as $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{q}$, and $\mathbf{r}$. Let $(\mathbf{p}, \mathcal{L})$ denote the center-axis flag of a central collineation group. The groups can be classified into the following three types:

A. $\mathbf{p} \in \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{q}, \mathbf{r}\}$, $\mathcal{L} = \mathcal{M}$, $|\Gamma(\mathbf{p}, \mathcal{L})| = 4$. There are are 5 elation groups of this type.

B. $\mathbf{p} = \mathbf{q}$, $\mathbf{r} \in \mathcal{L} \neq \mathcal{M}$, $|\Gamma(\mathbf{p}, \mathcal{L})| = 3$. There are 4 homology groups of this type.

C. $\mathbf{p} = \mathbf{r}$, $\mathbf{q} \in \mathcal{L} \neq \mathcal{M}$, $|\Gamma(\mathbf{p}, \mathcal{L})| = 3$. There are 4 homology groups of this type.

4.5.2 Coordinatisation Strategy

The unique line $\mathcal{M}$ is the obvious choice for the line at infinity $\overline{\mathbf{x}\,\mathbf{y}} = [\infty]$. While all 5 centers of the elation groups are equally optimal choices to maximize the group structure in $(\mathcal{R}, \oplus)$, thanks to the homology groups of types B and C, the centers $\mathbf{q}$ and $\mathbf{r}$ can contribute to the optimisation of the group structure in $(\mathcal{R}^*, \odot)$ as well. Considering the symmetry in the group of central collineations of $\mathcal{P}$ around the two points $\mathbf{q}$ and $\mathbf{r}$, we let $\mathbf{q} = \mathbf{y} = (\infty)$ and $\mathbf{r} = \mathbf{x} = (0)$ without loss of generality. Thus, by choosing the lines $\overline{\mathbf{O}\,\mathbf{x}} = [0,0]$ and $\overline{\mathbf{O}\,\mathbf{y}} = [0]$ from the pool of the axes of the homology groups of types B and C respectively, we get $|\,\Gamma((\infty),[0,0])\,| = |\,\Gamma((0),[0])\,| = 3$. By Theorems 1.10.9 and 1.10.8, we obtain subgroups $(\mathcal{S}_1, \odot)$ and $(\mathcal{S}_2, \odot)$ of $(\mathcal{R}^*, \odot)$ respectively, each of order 3. Based on prior experience, we will attempt to force the two sets to be equal i.e. $\mathcal{S}_1 = \mathcal{S}_2$ in the process of coordinatisation. This will ensure at least two non-trivial elements (elements other than 0 and 1) have both a left and a middle associativity in $(\mathcal{R}^*, \odot)$. To see this, consider for example, the set $\mathcal{S}_2$. Since $\mathcal{S}_2 \subseteq \mathcal{N}_l$ by Theorem 1.10.9, the elements of the set $\mathcal{S}_2$ are left associative on all of $(\mathcal{R}^*, \odot)$.

The orders of all principal central collineation groups are decided by the above choices. The orders are given in Table 4.2.

(V,l)	$((\infty),[\infty])$	$((0),[0])$	$((\infty),[0,0])$	$((0),[\infty])$	$((0,0),[\infty])$	$((\infty),[0])$
$\lvert\Gamma(V,l)\rvert$	4	3	3	4	1	1

Table 4.2: The BBS4 Plane - Orders of $\Gamma \in S(\Gamma)$.

Next we want to make an optimal choice of the point $\mathbf{J} = (1)$. Recall the discussion in the coordinatisation strategy of the BBH1 plane in Section 4.4.2 for choosing a point $\mathbf{J}$ optimally. Skipping an analogous argument showing the set of points $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ is the image of any of the points in the set, say point $\mathbf{a}$, under the action of any of $\Gamma((0), [0])$ or $\Gamma((\infty), [0, 0])$, let $\mathbf{J} = (1) = \mathbf{a}$, $(1)^{\gamma} = (a^5) = \mathbf{b}$, and $(1)^{\gamma^2} = (a^{10}) = \mathbf{c}$ for some $1 \neq \gamma \in \Gamma((0), [0])$. Now choose any point $\mathbf{u} = (0, 1)$ on the vertical axis and label the following points: $\mathbf{s} = (1, 0) = \overline{\mathbf{J}\,\mathbf{u}} \cap [0, 0]$, $(0, a^5) = \overline{(a^5)\,\mathbf{s}} \cap [0]$, and $(0, a^{10}) = \overline{(a^{10})\,\mathbf{s}} \cap [0]$. Therefore, $(\{1, a^5, a^{10}\}, \odot)$ is a cyclic subgroup of $(\mathcal{R}^*, \odot)$. This means, the PTR multiplication coincides with the field multiplication on, at least, $\{0, 1, a^5, a^{10}\}$. Since the set of points $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\} = \{(1), (a^5), (a^{10})\}$ is an orbit in the permutations of the points of $[\infty] \setminus \{(\infty), (0)\}$ under either $\Gamma((0), [0])$ or $\Gamma((\infty), [0, 0])$, the choice of the point $\mathbf{J}$ and its images from the set $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ achieves $\mathcal{S}_1 = \mathcal{S}_2$ as well.

Some more properties for $(\mathcal{R}, \oplus, \odot)$ are achieved by this labeling. Let $\mathcal{S}_1 = \mathcal{S}_2 = \mathcal{S}$. By Theorem 1.10.8, $\forall\, x, y \in \mathcal{R}$, $s \in \mathcal{S}$, we have $T(s, x, y) = (s \odot x) \oplus y$. In other words, the PTR is linear with respect to (s, x, y) for any $x, y \in \mathcal{R}$ so long as $s \in \mathcal{S}$. Including $s = 0$ which satisfies the equation trivially, we have $\mathbb{F}_4 \times \mathbb{F}_{16} \times \mathbb{F}_{16}$ as a subset of the domain of linearity. By Theorem 1.10.9, $\forall\, m, x, y \in \mathcal{R}, s \in \mathcal{S}$, we have $T(s \odot m, x, s \odot y) = s \odot T(m, x, y)$. Let $m = 1$. Then, $T(s, x, s \odot y) = s \odot T(1, x, y)$. Combining with the previous statement on linearity, we get $(s \odot x) \oplus (s \odot y) = s \odot (x \oplus y)$. The same is stated in [7], Lemma 14. Hence, the elements of $\mathcal{S}$ have left distributivity for all of $\mathcal{R}$. Note the left distributivity for the elements of $\mathcal{S}$ does not follow from Theorem 1.10.9 alone since the PTR is not linear. Also note $\mathcal{S}_1 = \mathcal{S}_2 = \mathcal{S}$ is essential to achieve this.

The optimal PTR we obtain has an even stronger property: the PTR is fully left distributive. We do not, however, have a theoretical reason for this.

We move to the optimal choices of the points $\mathbf{O} = (0, 0)$ and $\mathbf{u} = (0, 1)$ next. For this part, we consider $\Gamma((\infty), [\infty])$ and $\Gamma((0), [\infty])$ both of which are of order 4. To understand the role of the points $\mathbf{O}$ and $\mathbf{u}$ in optimising the PTR, we will assume arbitrary choices first and discuss the special subsets of $\mathcal{R}$ associated to the elation groups. Choose any of the axes of the homology groups of type B as the horizontal axis $[0,0]$ and any of the axes of the homology groups of type C as the vertical axis $[0]$. The point $\mathbf{O} = (0,0)$ is determined as $(0,0) = [0,0] \cap [0]$. Then choose any point $(0,1)$ on the line $[0]$. At this point, the frame of coordinatisation $\{\mathbf{y}, \mathbf{x}, \mathbf{O}, \mathbf{u}, \mathbf{J}\}$ has been labeled. Updating the labeling on the vertical axis $[0]$ via the projection of the points labeled on the infinite axis $[\infty]$, we obtain the points $(0, a^5)$ and $(0, a^{10})$. This means, the points $(1,0), (a^5, 0)$, and $(a^{10}, 0)$ on the horizontal axis $[0,0]$ have been determined too. The role of the points $\mathbf{O}$ and $\mathbf{u}$ in optimising the PTR is based on this. Clearly, we would like the points labeled so far (and the lines based on the labeling of points) to constitute a subplane of order 4. In this context we consider the subsets of $\mathcal{R}$ associated with $\Gamma((\infty), [\infty])$ and $\Gamma((0), [\infty])$. In Theorem 1.10.11, the image of the point $\mathbf{O} = (0,0)$ under the action of $\Gamma((\infty), [\infty])$ is the set $\mathcal{S}$ which forms a subgroup $(\mathcal{S}, \oplus)$ of $(\mathcal{R}, \oplus)$ and has a right associativity of addition. Obviously, we want the choice of the points $\mathbf{O} = (0,0)$ and $\mathbf{u} = (0,1)$ to be such that $\mathcal{S} = \{0, 1, a^5, a^{10}\}$. To achieve this, we vary the choices of the axes and the point $\mathbf{u}$ as described earlier and look for the following to be satisfied: $\mathbf{u} = \mathbf{O}^{\alpha}$ for some $\alpha \in \Gamma((\infty), [\infty])$ and the resulting labeling satisfies $(0, a^5) = \mathbf{O}^{\beta}$, $(0, a^{10}) = \mathbf{O}^{\omega}$ for the other two non-trivial elations $\beta, \omega \in \Gamma((\infty), [\infty])$. We have achieved this in our coordinatisation. Since $\Gamma((\infty), [\infty])$ is elementary abelian, the labeling so far has coordinatised a (Desarguesian) subplane of order 4. In addition, $\mathbb{F}_{16} \times \mathbb{F}_{16} \times \mathbb{F}_4$ is now a subset of the domain of linearity (refer to Theorem 1.10.11).

Regarding $\Gamma((0), [\infty])$, also of order 4, and the associated Theorem 1.10.7, we found our choice of $\mathbf{J}$ to always result in $\mathcal{S}_1 = \mathcal{S}_2$ where the sets $\mathcal{S}_1$ and $\mathcal{S}_2$ are subsets of $\mathcal{R}$ obtained from the image sets of $\mathbf{O} = (0,0)$ under the actions of the two elation groups centered on (0) and (∞), provided the labeling on $[0]$ is optimised for the PTR addition as we do. At the time of writing we do not have a proof for this. However, the implication of this equality is that $\mathcal{S} = \{0, 1, a^5, a^{10}\}$ has a left associativity of addition on all of $\mathcal{R}$ and the following distributive property: $m \odot (s \oplus x) = (m \odot s) \oplus (m \odot x)$ for all $s \in \mathcal{S}$ provided $m \odot x \in \mathcal{S}$. This follows from application of Theorem 1.10.7. In fact, [7], Lemma 9 states this. We suspect the combination of the partial left middle distributive property of $\mathcal{S}$ obtained in the equation and the left distributive property of $\mathcal{S}$ we proved earlier together contribute to the full left distributivity obtained for the PTR but again, we do not have a proof at the time of writing.

The remaining points on the axis [0] are labeled additively. Similar to the coordinatisation of BBH1 plane, we are able to fully optimise the PTR addition for any choice of $(0, a)$ from the unlabeled points on the axis [0] (and similarly any $(0, a^3)$). For an optimal labeling though, we ensure the above distributive properties are satisfied too and then compare the resulting PTR polynomials for their complexity.

4.5.3 Measures of the Algebraic Properties of $(\mathcal{R}, \oplus, \odot)$

Measures of associativity of addition: Since $(\mathcal{R}, \oplus)$ is a group,

$$\mathcal{A}_l = \mathcal{A}_m = \mathcal{A}_r = \mathcal{R}.$$

Measures of associativity of multiplication: The left and middle associativity were discussed in the coordinatisation. The respective theorems do not imply the left and middle nuclei are necessarily equal to the set $\mathcal{S}$ since the PTR is not linear though they both equal $\mathcal{S}$ in the optimal PTR we have obtained. The right nucleus is trivial as is $\Gamma((0,0),[\infty])$.

$$\mathcal{N}_l = \mathcal{N}_m = \mathbb{F}_4, \text{ and } \mathcal{N}_r = \{0,1\}.$$

Measures of distributive property: While the PTR we obtained is fully left distributive as stated in the coordinatisation, both the measures related to the distribution on the right are trivial. Note the two associated principal central collineation groups, $\Gamma((\infty),[0])$ and $\Gamma((0,0),[\infty])$ are both trivial.

$$\mathcal{D}_l = \mathcal{D} = \mathcal{R},\ \mathcal{D}_r = \{0,1\}, \text{ and } \mathcal{D}' = \{0\}.$$

Measure of commutativity of multiplication: Trivial.

$$\mathcal{Z} = \{0,1\}.$$

4.5.4 The PTR Polynomial

We give three polynomials—the PTR addition polynomial, the PTR multiplication addition polynomial, and the PTR polynomial—as generated by the polynomial interpolation of the respective functions.

(i) $Y \oplus Z = Y + Z$.

(ii) $X \odot Y = XY + Xt_4(X)^3 t_2(Y) + \left(X^3 + a^5\right) t_4(X) t_4(Y)^2$.

(iii) $T(X,Y,Z) = (X \odot Y) + Z + p(X,Y,Z)t_4(X) t_4(Y)^2 t_4(Z)$,

where $p(X,Y,Z) = a^5 X t_4(X) t_4(Y) + a^{10} t_4(Z)^2$.

4.5.4.1 Interpreting the PTR polynomial

Full optimisation of the addition, non-linearity of the PTR, and optimal co-ordinatisation of a subplane of order 4 can be inferred from the PTR polynomial similarly as for the BBH1 plane.

Besides, we see the PTR multiplication is linearized in Y showing the left distributivity of the PTR. As for the domain of linearity, the PTR is linear with respect to (x, y, z) if and only if any of x, y, or z is in $\mathbb{F}_4$. We will prove this. The sufficiency part is obvious. For the necessary part, assume to the contrary $x, y, z \notin \mathbb{F}_4$ so that $t_4(x), t_4(y), t_4(z) \neq 0$. Then, $T(x, y, z) = (x \odot y) + z \iff p(x, y, z) = 0$. Since $t_4(y) \neq 0$, we have $xt_4(x) + a^5 t_4(z)^2 = 0$ which is impossible as $a^5, t_4(x), t_4(z) \in \mathbb{F}_4$ but $x \notin \mathbb{F}_4$ by hypothesis. The fact that $t_4(x) = x^4 - x$ is the same as the trace $\mathrm{Tr}_{\mathbb{F}_{16}/\mathbb{F}_4}(x) = x^4 + x$ over $(\mathbb{F}_{16}, +, \cdot)$ is crucial for this. The order of the domain of linearity is therefore a $\frac{37}{64}$th of the order $\mathcal{R}^3$. Finally, the PTR multiplication $x \odot y$ is simpler when one of the coordinates is in $\mathbb{F}_4$. Indeed,

$$x \odot y = \begin{cases} xy & \text{if } x \in \mathbb{F}_4, \\ xy^2 & \text{if } x \notin \mathbb{F}_4,\ y \in \mathbb{F}_4,\ \text{and} \\ xy^2 + (x^3 + a^5)\, t_4(x) t_4(y)^2 & \text{if } x \notin \mathbb{F}_4,\ y \notin \mathbb{F}_4. \end{cases}$$

We have used $t_4(x)^3 = 1$ for $x \notin \mathbb{F}_4$ to arrive at the given multiplication rule.

4.6 The Johnson Plane

The Johnson plane $\mathcal{P}$ of order 16 is a semi-translation plane of LB type I.1.

4.6.1 Central Collineation Groups of $\mathcal{P}$

The Johnson plane of order 16 does not admit any homologies. The elations are all centered on a unique line $\mathcal{M}$. There are 5 distinct centers of the elation groups which we denote as $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{q}$, and $\mathbf{r}$. Let $(\mathbf{p}, \mathcal{L})$ denote the center-axis flag of an elation group. The groups can be classified into the following two types:

A. $\mathbf{p} \in \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{q}, \mathbf{r}\}$, $\mathcal{L} = \mathcal{M}$, $|\Gamma(\mathbf{p}, \mathcal{L})| = 4$. There are are 5 elation groups of this type.

B. $\mathbf{p} \in \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$, $\mathbf{p} \in \mathcal{L} \neq \mathcal{M}$, $|\Gamma(\mathbf{p}, \mathcal{L})| = 2$. There are 12 elation groups of this type, 4 on each center.

4.6.2 General Coordinatisation Strategies

We present three distinct optimal coordinatisations of the Johnson plane from three different perspectives. The first coordinatisation is optimal in the sense of a subplane generating property of the frame of coordinatisation. The second coordinatisation yields the highest amount of distributivity in $(\mathcal{R}, \oplus, \odot)$ among all coordinatisations. The last coordinatisation we present has a mix of an optimal distributive property and a relatively simple PTR polynomial.

The unique line $\mathcal{M}$ is the only suitable choice for the line at infinity $\overline{\mathbf{x}\,\mathbf{y}} = [\infty]$. The choices of three points $\mathbf{y} = (\infty), \mathbf{x} = (0)$, and $\mathbf{J} = (1)$ on the infinite axis $[\infty]$ are crucial in determining the properties of the resulting PTR. Regarding these choices, we now describe the observations from our coordinatisations of the Johnson plane backed by theoretical reasoning where we have one. The observations have been verified exhaustively.

(i) A fully optimised PTR addition can be achieved only if the three points $\mathbf{y}, \mathbf{x}$, and $\mathbf{J}$ are selected from among the centers of the elation groups. By Theorem

1.10.11 (resp. Theorem 1.10.7), choosing any of the 5 centers as $\mathbf{y}$ (resp. $\mathbf{x}$) results in $(\mathcal{R}, \oplus)$ having a subgroup $(\mathcal{S}, \oplus)$ of order 4. Our observation indicates that in the absence of a transitive central collineation group, maximizing the orders of all central collineation groups centered on the vertices of the frame of coordinatisation can help to obtain more group structure in the PTR than guaranteed by the theorems in Section 1.10.2.

(ii) With the line $\mathcal{M}$ fixed for the infinite axis $[\infty]$, choosing any of the centers $\{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{q}, \mathbf{r}\}$ as the point at infinity (∞) and labeling $\mathbf{u} = \mathbf{O}^{\alpha}$ i.e. $(0,1) = (0,0)^{\alpha}$ for some non-trivial elation $\alpha \in \Gamma((\infty), [\infty])$ guarantees a Fano subplane is coordinatised by $\mathbb{F}_2$. To see this, note the non-trivial elations in $\mathscr{P}$ are all of order 2 by [17], Theorem 4.14. Using the equivalence of the PTR addition and the actions of $((\infty), [\infty])$-elations from Section 1.11, we have $1 \oplus 1 = 0$. This is sufficient to coordinatise a Fano subplane containing the frame $\{\mathbf{y}, \mathbf{x}, \mathbf{O}, \mathbf{u}, \mathbf{J}\}$ of coordinatisation by the subfield $(\mathbb{F}_2, +, \cdot)$ of $(\mathcal{R}, \oplus, \odot)$.

(iii) By [36], Lemma 1, the five centers $\{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{q}, \mathbf{r}\}$ of elation groups on the infinite axis $[\infty]$ constitute a derivation set. Hence, choosing the points $\mathbf{y} = (\infty), \mathbf{x} = (0)$, and $\mathbf{J} = (1)$ from the set ensures that $0, 1 \in \mathcal{R}$ are included in the coordinatising set of a Baer subplane of order 4 associated with the derivation set. This is similar to the cases of the BBH1 or the BBS4 planes.

(iv) Unlike in the cases of the BBH1 or the BBS4 planes, however, we do not have a homology of order 3 in the Johnson plane to readily identify two other points on the axis $[\infty]$ to label by (a^5) and (a^{10}). We instead use the elations centered on the points (∞) and (0), and a test with the PTR multiplication $\odot$ to obtain the points $(0, a^5)$ and $(0, a^{10})$. Recall a similar strategy in the coordinatisation of the SEMI2 plane in Section 3.3. Since the Johnson plane does not have a transitive elation group like the SEMI2 plane, the choice of points are limited as we explain next.

(v) Since $|\,\Gamma((\infty),[\infty])\,| = |\,\Gamma((0),[\infty])\,| = 4$ by choosing $\mathbf{y}$ and $\mathbf{x}$ from the set $\{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{q},\mathbf{r}\}$, the subgroups $(\mathcal{S}_1,\oplus)$ and $(\mathcal{S}_2,\oplus)$ of $(\mathcal{R},\oplus)$ given by Theorems 1.10.11 and 1.10.7 respectively are of orders 4 each. As before, we want to obtain $\mathcal{S}_1 = \mathcal{S}_2 = \mathbb{F}_4$ where the points corresponding to the sets in the respective theorems are points in a Baer subplane of $\mathcal{P}$. To achieve this, we vary the axes $[0]$ and $[0,0]$ such that the images of the origin $\mathbf{O} = [0] \cap [0,0]$ under the actions of $\Gamma((\infty),[\infty])$ and $\Gamma((0),[\infty])$ are contained in the same Baer subplane of $\mathcal{P}$.

(vi) The choice of the points $\mathbf{y},\mathbf{x}$, and $\mathbf{J}$ as in (iii) along with additive labeling on the vertical axis $[0]$ is sufficient to yield a fully optimised PTR addition on the resulting PTR (irrespective of the choices of $(0,0),(0,1),(0,a^5),(0,a)$, and $(0,a^3)$ from the available set of points at each step on the axis $[0]$).

For the next step in coordinatisation, a point $\mathbf{O} = (0,0)$ and a point $\mathbf{u} = (0,1)$ need to be chosen to complete the frame of coordinatisation. The choices of the points $\mathbf{O}$ and $\mathbf{u}$ provide us some options. Firstly, if the points $\mathbf{y}$ and $\mathbf{x}$ are chosen from the set $\{\mathbf{a},\mathbf{b},\mathbf{c}\}$, then the choice of the point $\mathbf{O} = (0,0)$ determines the orders of the elation groups $\Gamma((\infty),[0])$ and $\Gamma((0),[0,0])$. Each elation group can be either trivial or of order 2 (see the description of the central collineation groups of $\mathcal{P}$). The choice of the point $\mathbf{u} = (0,1)$, from our analysis of the different coordinatisations, affects the form of the resulting PTR polynomial by determining whether the coefficients are in $\mathbb{F}_2$, $\mathbb{F}_4$, or $\mathbb{F}_{16}$. By varying the choice of the vertical axis $[0]$ among the vertical lines and choosing $\mathbf{u} = (0,1)$ based on the elations, we have obtained PTR polynomials with coefficients in $\mathbb{F}_2$ in all three types of coordinatisations we have presented below.

4.6.3 Coordinatisation I

In this coordinatisation, we maximise the orders of all elation groups centered on the three points $\mathbf{y}, \mathbf{x}$, and $\mathbf{J}$ of the frame of coordinatisation lying on the line $\overline{\mathbf{x}\,\mathbf{y}} = [\infty]$. The orders of the principal central collineation groups are given in Table 4.3. Not included in the table are $\Gamma((0), [0, 0])$ of order 2, $\Gamma((1), [\infty])$ of order 4, and $\Gamma((1), \mathcal{L})$ of order 2 where $\mathcal{L} \neq \mathcal{M}$ is one of the 4 lines admitting an elation group of type B centered on $\mathbf{J} = (1)$.

(V, l)	$((\infty), [\infty])$	$((0), [0])$	$((\infty), [0, 0])$	$((0), [\infty])$	$((0, 0), [\infty])$	$((\infty), [0])$
$\lvert\Gamma(V, l)\rvert$	4	1	1	4	1	2

Table 4.3: The Johnson Plane - Orders of $\Gamma \in S(\Gamma)$: Coordinatisation I.

Our observations about the coordinatisations obtained by starting with the above choices are as follows:

(i) We can show that $\mathbf{J} = (1) = (0)^{\alpha}$ for the non-trivial elation $1 \neq \alpha \in \Gamma((\infty), [0])$ by using [17], Lemma 4.11. So, if the PTR were linear, then by Theorem 2.0.1 we would get $1 \in \mathcal{D}'$, the right middle distributor. This also requires assuming the additive loop $(\mathcal{R}, \oplus)$ is a group which is true in the PTR we obtain. But the PTR is not linear. We find $1 \notin \mathcal{D}'$. In other words, optimising the order of $\Gamma((\infty), [0])$ does not necessarily increase the amount of right distribution in a non-linear PTR.

(ii) The coordinatisation is optimised for identifying some subplanes of order 4 of the Johnson plane but the PTR polynomial is not the most optimal we obtained. This is in spite of optimising the orders of all possible central collineation groups centered on the points of the coordinatising frame. We note this as a deviation from the usual behaviour observed in the PTRs of planes of higher LB types.

4.6.3.1 Measures of the Algebraic Properties of $(\mathcal{R}, \oplus, \odot)$

Measures of associativity of addition: Since $(\mathcal{R}, \oplus)$ is a group,

$$\mathcal{A}_l = \mathcal{A}_m = \mathcal{A}_r = \mathcal{R}.$$

Measures of associativity of multiplication: The nuclei of the PTR are all trivial. However, this is not a consequence guaranteed by the absence of homologies in the plane which is the case with linear PTRs having appropriate distributive properties. See Theorems 1.10.8 and 1.10.9.

$$\mathcal{N}_l = \mathcal{N}_m = \mathcal{N}_r = \{0, 1\}.$$

Measures of distributive property: Since there are no non-trivial homologies in $\mathcal{P}$, both $\Gamma((\infty), [0, 0])$ and $\Gamma((0, 0), [\infty])$ are trivial. So, we cannot claim a non-trivial set $\mathcal{D}_l$ of left distributive elements or set $\mathcal{D}_r$ of right distributive elements based on Theorems 1.10.9 and 1.10.10 respectively. But, the PTR we obtained has each set equal to $\mathbb{F}_4$. On the other hand, we ensure in our coordinatisation the origin $(0, 0)$ has the image $\{(0, 0), (0, 1)\}$ under $\Gamma((\infty), [0])$ and $\{(0, 0), (1, 0), (a^5, 0), (a^{10}, 0)\}$ under $\Gamma((0), [\infty])$. Yet, unlike in the planes of LB type II.1 or above we do not obtain non-trivial middle distributors $\mathcal{D}$ and $\mathcal{D}'$.

$$\mathcal{D}_l = \mathcal{D}_r = \mathbb{F}_4, \text{ and } \mathcal{D} = \mathcal{D}' = \{0\}.$$

Measure of commutativity of multiplication: Trivial.

$$\mathcal{Z} = \{0, 1\}.$$

4.6.3.2 The PTR Polynomial

The nicest PTR polynomial obtained for this coordinatisation has length 270 with a PTR multiplication polynomial of length 77. The coefficients are in $\mathbb{F}_2$. We give three polynomials—the PTR addition polynomial, the PTR multiplication polynomial, and the PTR polynomial by regrouping the terms.

(i) $Y \oplus Z = Y + Z$

(ii) $$\begin{aligned} X \odot Y = XY &+ X^{11}\ t_4(Y) + X^{10}\ t_4(X)\ t_4(Y)^2 + X^8\ p(Y)\ t_4(X) \\ &+ q(Y)\ t_4(X)^2 + XY^5\ t_4(X)\ t_4(Y) + X^7\ \mathrm{Tr}\left(Y^3\right) + X^5\ t_4\left(Y^3\right) \\ &+ X^2\ r(Y)\ t_4(X) + s(Y)\ t_4(X) + X^4\ \mathrm{Tr}(Y) + Y^8\ t_4(X), \end{aligned}$$

where $p(Y) = Y^{13} + Y^{12} + Y^{10} + Y^8 + Y^7 + Y^3 + Y^2 + Y,$

$q(Y) = Y^{12} + Y^8 + Y^6 + Y^3 + Y^2 + Y,$

$r(Y) = Y^{13} + Y^{10} + Y^9 + Y^8 + Y^7 + Y^6 + Y^2 + Y,$ and

$s(Y) = Y^{14} + Y^{11} + Y^9 + Y^6 + Y^5.$

(iii) $$\begin{aligned} T(X, Y, Z) = (X \odot Y) + Z &+ u(X, Y)\ t_4(Z) \\ &+ v(X, Y)\ \mathrm{Tr}(Z) + w(X, Y)\ \mathrm{Tr}\left(Z^3\right), \end{aligned}$$

where $u(X, Y) = X^2\ t_4(X)\ t_4(Y)^2 + Xt_4(Y),$

$v(X, Y) = t_4(X)^2\ t_4(Y)^2,$ and

$w(X, Y) = X^2 t_4(X)^2\ t_4(Y).$

From the PTR polynomial we obtain that the PTR multiplication coincides with the field multiplication in $\mathbb{F}_4$. Also, the domain of linearity includes $\mathbb{F}_{16} \times \mathbb{F}_{16} \times \mathbb{F}_4$ and $\mathbb{F}_4 \times \mathbb{F}_4 \times \mathbb{F}_{16}$.

4.6.4 Coordinatisation II

In this coordinatisation, we choose the points $\mathbf{y}, \mathbf{x}$, and $\mathbf{J}$ as in Coordinatisation I. We alter the strategy for choosing the point $\mathbf{O} = (0,0)$ such that the resulting $\Gamma((\infty),[0])$ and $\Gamma((0),[0,0])$ are trivial. The orders of the principal central collineation groups are given in Table 4.4. Besides, $|\Gamma((0),[0,0])| = 1$ and $|\Gamma((1),[\infty])| = 4$. We noticed in Coordinatisation I that choosing larger $\Gamma((\infty),[0])$ did not yield a larger right middle distributor $\mathcal{D}'$ as would be the case in LB type II.1 or higher planes. In this coordinatisation, the elation group is trivial but we get $\mathcal{D}'$ of order 2. Also, the PTR is fully left distributive and the set $\mathcal{D}_r$ of right distributive elements is of order 8 (half of the order of $\mathcal{R}$). The amount of distributive property in the PTR is surprisingly large.

(V,l)	$((\infty),[\infty])$	$((0),[0])$	$((\infty),[0,0])$	$((0),[\infty])$	$((0,0),[\infty])$	$((\infty),[0])$
$\lvert\Gamma(V,l)\rvert$	4	1	1	4	1	1

Table 4.4: The Johnson Plane - Orders of $\Gamma \in S(\Gamma)$: Coordinatisation II.

4.6.4.1 Measures of the Algebraic Properties of $(\mathcal{R}, \oplus, \odot)$

Measures of associativity of addition: Since $(\mathcal{R}, \oplus)$ is a group,

$$\mathcal{A}_l = \mathcal{A}_m = \mathcal{A}_r = \mathcal{R}.$$

Measures of associativity of multiplication: Trivial.

$$\mathcal{N}_l = \mathcal{N}_m = \mathcal{N}_r = \{0, 1\}.$$

Measures of distributive property: Since $(\mathcal{R}, \oplus, \odot)$ is left distributive, the left middle distributor $\mathcal{D}$ is maximal too. The set $\mathcal{D}_r$ of right distributive elements which is of order 8 is particularly nice in the optimal PTR. It is the set of elements with trace $\mathrm{Tr}_{\mathbb{F}_{16}/\mathbb{F}_2}$ equal to 0.

$$\mathcal{D}_l = \mathcal{D} = \mathcal{R},\ \mathcal{D}_r = \mathbb{F}_4 \cup (\mathbb{F}_4 + a)\,,\ \text{and } \mathcal{D}' = \{0, 1\}.$$

Measure of commutativity of multiplication: Trivial.

$$\mathcal{Z} = \{0, 1\}.$$

4.6.4.2 The PTR Polynomial

The nicest PTR polynomial obtained for this coordinatisation has length 94 with a PTR multiplication polynomial of length 21. The coefficients are in $\mathbb{F}_2$. We give three polynomials—the PTR addition polynomial, the PTR multiplication polynomial, and the PTR polynomial by regrouping the terms.

(i) $Y \oplus Z = Y + Z$.

(ii) $X \odot Y = XY + t_4(X)^2 t_4(Y)^2 + t_2(X)^2\, t_4(X)^2\, \mathrm{Tr}(Y) + t_4(X)\, t_2(Y)^4\,.$

(iii) $$T(X, Y, Z) = (X \odot Y) + Z + r(X, Y)\ t_4(Z) + s(X, Y)\ t_4(Z)^2 + u(X, Y)\ \mathrm{Tr}(Z) + v(X, Y)\ \mathrm{Tr}\left(Z^3\right),$$

where $r(X,Y) = X^4\ t_4\,(X)^2\ t_4\,(Y) + t_4\,(X)^2\ t_4\,(Y)^3\,,$

$$s(X,Y) = X^4\ t_4\,(X)^2\ t_4\,(Y)^3\,,$$
$$u(X,Y) = X^4\ t_4\,(X)^2\ t_4\,(Y)^2\,,\ \text{and}$$
$$v(X,Y) = t_4\,(X)^2\ t_4\,(Y)^2\,.$$

The PTR multiplication given by (ii) is linearized in Y showing the left distributive property of $(\mathcal{R}, \oplus, \odot)$. From the PTR polynomial in (iii), the domain of linearity includes $\mathbb{F}_4 \times \mathbb{F}_{16} \times \mathbb{F}_{16}$, $\mathbb{F}_{16} \times \mathbb{F}_4 \times \mathbb{F}_{16}$, and $\mathbb{F}_{16} \times \mathbb{F}_{16} \times \mathbb{F}_4$. With some calculations we can show the union of these three sets is all of the domain of linearity. The order is $\frac{37}{64}$th of the order of $\mathcal{R}^3$. Also, we can obtain the following rule for the PTR multiplication from (ii):

$$x \odot y = \begin{cases} xy & \text{if } x \in \mathbb{F}_4 \text{ or } y \in \mathbb{F}_2,\ \text{and} \\ xy + t_4(x) & \text{if } x \notin \mathbb{F}_4,\ y \in \mathbb{F}_4 \setminus \mathbb{F}_2. \end{cases}$$

4.6.5 Coordinatisation III

In this coordinatisation, we choose the points $\mathbf{y} = (\infty)$ and $\mathbf{x} = (0)$ to be the points $\mathbf{q}$ and $\mathbf{r}$ (in any order). Recall the points $\mathbf{q}$ and $\mathbf{r}$ are the two centers on the infinite axis $[\infty]$ which do not admit non-trivial elations other than the ones with axis $[\infty]$. The orders of the principal central collineation groups are given in Table 4.5. We see the orders are identical with the ones in Coordinatisation II. The difference is in this coordinatisation $\Gamma((\infty), [0])$ and $\Gamma((0), [0, 0])$ are necessarily trivial while the same was by choice in Coordinatisation II. Refer to the description of the central collineation groups of $\mathscr{P}$ to see this. Also, the point $\mathbf{J} = (1)$ is in the set $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ in the optimal PTR we obtained. As expected, the five centers of the non-trivial elation groups on $[\infty]$ are the points of the infinite axis of the Baer subplane of $\mathscr{P}$ coordinatised by the subfield $(\mathbb{F}_4, +, \cdot)$ of $(\mathcal{R}, \oplus, \odot)$.

The PTR is fully optimised for addition and is left distributive while every other measure of the algebraic properties is trivial.

(V,l)	$((\infty),[\infty])$	$((0),[0])$	$((\infty),[0,0])$	$((0),[\infty])$	$((0,0),[\infty])$	$((\infty),[0])$
$\lvert\Gamma(V,l)\rvert$	4	1	1	4	1	1

Table 4.5: The Johnson Plane - Orders of $\Gamma \in S(\Gamma)$: Coordinatisation III.

4.6.5.1 Measures of the Algebraic Properties of $(\mathcal{R}, \oplus, \odot)$

Measures of associativity of addition: Since $(\mathcal{R}, \oplus)$ is a group,

$$\mathcal{A}_l = \mathcal{A}_m = \mathcal{A}_r = \mathcal{R}.$$

Measures of associativity of multiplication:

$$\mathcal{N}_l = \mathcal{N}_m = \mathcal{N}_r = \{0, 1\}.$$

Measures of distributive property:

$$\mathcal{D}_l = \mathcal{D} = \mathcal{R},\ \mathcal{D}_r = \{0, 1\},\ \text{and}\ \mathcal{D}' = \{0\}.$$

Measure of commutativity of multiplication:

$$\mathcal{Z} = \{0, 1\}.$$

4.6.5.2 The PTR Polynomial

The PTR polynomial of length 64 is the shortest in all three coordinatisations and has an equally nice form when compared to Coordinatisation II. The coefficients are in $\mathbb{F}_2$. We give three polynomials—the PTR addition polynomial, the PTR multiplication polynomial, and the PTR polynomial by regrouping the terms.

(i) $Y \oplus Z = Y + Z$.

(ii) $X \odot Y = XY + Xt_2(Y)\, t_4(X)^3 + \left(X^2+1\right)\left(X^3+1\right) t_4(X)\, t_4(Y)^2$.

(iii) $T(X,Y,Z) = (X \odot Y) + Z + u(X,Y)t_4(Z) + v(X,Y)t_4(Z)^2 + w(X,Y)t_4(Z)^3$,

where $u(X,Y) = X\ t_4(X)^2\ t_4(Y)^3$,

$v(X,Y) = X\ t_4(X)^2\ t_4(Y) + t_4(X)\ \ t_4(Y)$, and

$w(X,Y) = t_4(X)\ t_4(Y)^2$.

The left distributivity and the domain of linearity in this coordinatisation is the same as in Coordinatisation II. The PTR multiplication, however, can be defined by simpler rules and on all of $\mathcal{R} \times \mathcal{R}$:

$$x \odot y = \begin{cases} xy & \text{if } x \in \mathbb{F}_4 \text{ or } y \in \mathbb{F}_2, \\ (x+1)y & \text{if } x \notin \mathbb{F}_4,\ y \in \mathbb{F}_4 \setminus \mathbb{F}_2, \text{ and} \\ xy^2 + (x+x^{-1})t_4(x)^2 t_4(y)^2 & \text{if } x \notin \mathbb{F}_4,\ y \notin \mathbb{F}_4. \end{cases}$$

4.7 The BBH2 Plane

The BBH2 plane $\mathcal{P}$ of order 16 is a semi-translation plane of LB type I.1.

4.7.1 Central Collineation Groups of $\mathscr{P}$

The plane $\mathscr{P}$ does not admit any homologies. The elations are all centered on a unique line $\mathcal{M}$. There are 5 distinct centers of the elation groups which we denote as $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{q}$, and $\mathbf{r}$. Let $(\mathbf{p}, \mathcal{L})$ denote the center-axis flag of an elation group. The groups can be classified into the following two types:

A. $\mathbf{p} \in \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{q}, \mathbf{r}\}$, $\mathcal{L} = \mathcal{M}$, $|\Gamma(\mathbf{p}, \mathcal{L})| = 4$. There are are 5 elation groups of this type.

B. $\mathbf{p} \in \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{q}, \mathbf{r}\}$, $\mathbf{p} \in \mathcal{L} \neq \mathcal{M}$, $|\Gamma(\mathbf{p}, \mathcal{L})| = 2$. There are 20 elation groups of this type, 4 on each center.

4.7.2 Coordinatisation Strategy

Coordinatisation of the BBH2 plane is similar to the Johnson plane except there are not as many options in choosing the frame of coordinatisation. The five centers $\{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{q}, \mathbf{r}\}$ in BBH2 plane are symmetrical in terms of the elation groups centered on them. Unlike the Johnson plane, however, we did not obtain a PTR polynomial over $\mathbb{F}_2$. On the other hand, the optimal PTR we obtain has a non-trivial right nucleus $\mathcal{N}_r$ and a non-trivial center $\mathcal{Z}$, apart from being left distributive and having a set $\mathcal{D}_r$ of right distributive elements of order 4. This is intriguing given there are no homologies in the group of central collineations of $\mathscr{P}$. In summary, the algebraic structure of the PTRs of the planes of LB type I.1 can vary widely for two planes with a similar structure of the group of central collineations as well as for different coordinatisations of the same plane.

The orders of the principal central collineation groups are given in Table 4.6. Also, $|\Gamma((1), [\infty])| = 4$ in the optimal PTR.

(V,l)	$((\infty),[\infty])$	$((0),[0])$	$((\infty),[0,0])$	$((0),[\infty])$	$((0,0),[\infty])$	$((\infty),[0])$
$\lvert\Gamma(V,l)\rvert$	4	1	1	4	1	1

Table 4.6: The BBH2 Plane - Orders of $\Gamma \in S(\Gamma)$.

4.7.2.1 Measures of the Algebraic Properties of $(\mathcal{R}, \oplus, \odot)$

Measures of associativity of addition: Since $(\mathcal{R}, \oplus)$ is a group,

$$\mathcal{A}_l = \mathcal{A}_m = \mathcal{A}_r = \mathcal{R}.$$

Measures of associativity of multiplication:

$$\mathcal{N}_l = \mathcal{N}_m = \{0,1\}, \text{ and } \mathcal{N}_r = \mathbb{F}_4.$$

Measures of distributive property:

$$\mathcal{D}_l = \mathcal{D} = \mathcal{R},\ \mathcal{D}_r = \{0,1\}, \text{ and } \mathcal{D}' = \mathbb{F}_4.$$

Measure of commutativity of multiplication:

$$\mathcal{Z} = \mathbb{F}_4.$$

4.7.2.2 The PTR Polynomial

The PTR polynomial has length 94 with a PTR multiplication polynomial of length 13. The coefficients are in $\mathbb{F}_4$. We give three polynomials—the PTR addition polynomial, the PTR multiplication polynomial, and the PTR polynomial by regrouping the terms.

(i) $Y \oplus Z = Y + Z$.

(ii) $X \odot Y = XY + \left(X^2 + X + a^5\right)\ t_4\left(X\right)^2\ t_4\left(Y\right).$

(iii) $T(X,Y,Z) = (X \odot Y) + Z + p(X,Y)\,\mathrm{Tr}\left(Z\right)$

$$+\, q(X,Y)t_4\left(Z\right) + r(X,Y)t_4\left(Z\right)^2 + s(X,Y)t_4\left(Z\right)^3,$$

where $p(X,Y) = Xt_4\left(X\right)^2\ t_4\left(Y\right)^2,$

$$q(X,Y) = \left(X + a^{10}\right)t_4\left(X\right)^2\ t_4\left(Y\right),$$

$$r(X,Y) = \left(Xt_4(Y^3) + a^{10}\ t_4\left(Y\right)^2 + t_4\left(Y\right)\right)t_4\left(X\right)^2,\ \text{and}$$

$$s(X,Y) = t_4\left(X\right)^2\ t_4\left(Y\right)^2.$$

The domain of linearity is $(\mathbb{F}_4 \times \mathbb{F}_{16} \times \mathbb{F}_{16}) \cup (\mathbb{F}_{16} \times \mathbb{F}_4 \times \mathbb{F}_{16}) \cup (\mathbb{F}_{16} \times \mathbb{F}_{16} \times \mathbb{F}_4)$ which has an order $\frac{37}{64}$th of the order of $\mathcal{R}^3$. The PTR multiplication is linearized in Y. We also obtain the following rules for the multiplication on given subsets:

$$x \odot y = \begin{cases} xy & \text{if } x \in \mathbb{F}_4 \text{ or } y \in \mathbb{F}_4,\ \text{and} \\ xy + x^2 + x + a^5 & \text{if } x, y \notin \mathbb{F}_4, t_4(x) = t_4(y). \end{cases}$$

Chapter 5

COORDINATISATION OF THE FIGUEROA PLANE OF ORDER 27

In this chapter, we present a special coordinatisation method for the Figueroa plane $\mathscr{P}$ of order 27. The Figueroa planes are a class of LB type I.1 planes defined originally by Figueroa [14] in 1982 for prime powers of order q^3, $q > 2, q \neq 1 \pmod 3$. Subsequently, [16] extended the construction to all prime powers and [11] extended it to include infinite orders.

The speciality of the coordinatisation we obtain is that the ternary system $(\mathcal{R}, \oplus, \odot)$ obtained from the corresponding PTR $(\mathcal{R}, T)$ is the finite field $(\mathbb{F}_{27}, +, \cdot)$. That is, we take the coordinatising set $\mathcal{R}$ to be the set of elements $\mathbb{F}_{27}$ and the PTR addition $\oplus$ and multiplication $\odot$ coincide with the addition and the multiplication in $(\mathbb{F}_{27}, +, \cdot)$. Note this does not mean $T(m, x, y) = mx + y \ \forall x, y, z \in \mathcal{R}$ as the PTR $(\mathcal{R}, T)$ is non-linear (the plane is a LB type I.1 plane). We believe the result is new for any Figueroa plane or indeed, any non-derived plane.

Like before we denote the generator of the multiplicative group $(\mathbb{F}_{27}^*, \cdot)$ of the field by a. Hence, $\mathbb{F}_{27} = \{0, 1, a, \ldots, a^{25}\}$. Note $a^{26} = 1$ and $a^{13} = 2$.

5.1 Central Collineation Groups of $\mathscr{P}$

The plane $\mathscr{P}$ is self-dual. Therefore, there is a symmetric relation between the centers and the axes of the central collineation groups admitted by $\mathscr{P}$. There are 13 distinct centers $\mathbf{p}$ and 13 distinct axes $\mathcal{L}$ based on which are defined 169 center-axis flags $(\mathbf{p}, \mathcal{L})$ admitting all non-trivial $(\mathbf{p}, \mathcal{L})$-central collineation groups of $\mathscr{P}$. Of the

169 groups, 52 are elation and 117 are homology groups. Let $(\mathbf{q}, \mathcal{M})$ be the incident center-axis flag of an elation group. Then, there are 4 distinct centers $\{\mathbf{q}, \mathbf{a}, \mathbf{b}, \mathbf{c}\}$ on $\mathcal{M}$ and 4 distinct axes $\{\mathcal{M}, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3\}$ through $\mathbf{q}$ such that the following 25 $(\mathbf{p}, \mathcal{L})$-central collineation groups are admitted by $\mathcal{P}$.

A. $\mathbf{p} \in \{\mathbf{q}, \mathbf{a}, \mathbf{b}, \mathbf{c}\}$, $\mathcal{L} = \mathcal{M}$, $|\Gamma(\mathbf{p}, \mathcal{L})| = 3$. There are 4 elation groups of this type.

B. $\mathbf{p} = \mathbf{q}$, $\mathcal{L} \in \{\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3\}$, $|\Gamma(\mathbf{p}, \mathcal{L})| = 3$. There are 3 elation groups of this type.

C. $\mathbf{p} \in \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$, $\mathcal{L} \in \{\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3\}$, $|\Gamma(\mathbf{p}, \mathcal{L})| = 2$. There are 9 homology groups of this type, 3 on each center (and 3 with each axis).

D. $\mathbf{p} = \mathbf{q}$, $\mathbf{r} \in \mathcal{L} \neq \mathcal{M}$ for some $\mathbf{r} \in \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$, $|\Gamma(\mathbf{p}, \mathcal{L})| = 2$. There are 9 homology groups of this type, with 3 axes each through each point in $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$.

We did not classify all 169 $(\mathbf{p}, \mathcal{L})$-central collineation groups of $\mathcal{P}$ in the above description for two reasons. Firstly, every $(\mathbf{p}, \mathcal{L})$-central collineation group of $\mathcal{P}$ belongs to one of the types A, B, C, or D for multiple incident flags of the type $(\mathbf{q}, \mathcal{M})$ and 25 such central collineation groups correspond to every incident flag of type $(\mathbf{q}, \mathcal{M})$. Secondly, the listing of 25 central collineation groups corresponding to a chosen incident center-axis flag $(\mathbf{q}, \mathcal{M})$ is intended to facilitate a description of the coordinatisation process.

5.2 The Coordinatisation

We give a step-wise description our coordinatisation process for the Figueroa plane $\mathcal{P}$ of order 27 supplemented by additional notes:

1. Choose any of the 52 incident center-axis flags $(\mathbf{q}, \mathcal{M})$ corresponding to the elation groups and label $\mathbf{q} = \mathbf{y} = (\infty)$ and $\mathcal{M} = \overline{\mathbf{x}\,\mathbf{y}} = [\infty]$.

2. Choose any axis of the elation groups of type B as the vertical axis $\overline{\mathbf{O}\,\mathbf{y}} = [0]$. Note the axis is also an axis of a homology group of type C.

3. Choose any center in $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$, say $\mathbf{a}$, and label it $\mathbf{a} = \mathbf{x} = (0)$.

4. Choose any axis of the homology groups of type D passing through $\mathbf{a}$ as the horizontal axis $\overline{\mathbf{O}\,\mathbf{x}} = [0, 0]$. Note the axis is also the axis of one of the four elation groups centered on the point $\mathbf{a}$.

5. Obtain the point $\mathbf{O} = (0, 0)$ as $(0, 0) = [0] \cap [0, 0]$. The point $\mathbf{O}$ thus determined is also one of the 13 centers of the non-trivial central collineation groups of $\mathcal{P}$.

   With the triangle $\{\mathbf{y}, \mathbf{x}, \mathbf{O}\}$ of coordinatisation labeled so far, we have the following orders of the principal central collineation groups:

   - $|\,\Gamma((\infty), [\infty])\,| = |\,\Gamma((0), [\infty])\,| = |\,\Gamma((\infty), [0])\,| = 3.$
   - $|\,\Gamma((0), [0])\,| = |\,\Gamma((\infty), [0, 0])\,| = |\,\Gamma((0, 0), [\infty])\,| = 2.$

   Besides, $|\Gamma((0), [0, 0])| = 3$ and $|\Gamma((0, 0), [0, 0])| = |\Gamma((0, 0), [0])| = 3$ as well. In short, every central collineation group centered on a vertex of the triangle of coordinatisation with its axis being an axis of the coordinate system is of maximum possible order in the plane.

6. Choose any of the two remaining centers $\{\mathbf{b}, \mathbf{c}\}$ on $\mathcal{M}$, say $\mathbf{b}$, and label it $\mathbf{b} = \mathbf{J} = (1)$.

7. Let $1 \neq \gamma \in \Gamma((0), [0])$ be the non-trivial homology in the group. Define $(2) = (1)^{\gamma}$.

   To explain the rationale behind the choices of the ideal points (1) and (2) on the infinite axis $[\infty]$ we recall similar choices made in the coordinatisations in Chapters 3 and 4 to label three points on the infinite axis of a plane of order 16 as $(1), (a^5)$, and (a^{10}) whenever the plane admitted either a $\Gamma((0),[0])$ or a $\Gamma((\infty),[0,0])$ of order 3. Similarly, in this case the centers $\mathbf{b}$ and $\mathbf{c}$ on the infinite axis $[\infty] = \mathcal{M}$ constitute a point-orbit of the cyclic (involutory) homology group $\Gamma((0),[0])$ of order 2. Thus, it happens $\mathbf{c} = (2)$ under the above labeling process.

Also note we could use a non-trivial homology γ in $\Gamma((\infty),[0,0])$ instead to define (2) as the image of (1). We could also have labeled any point on the vertical axis [0] other than (∞) and $(0,0)$ as $\mathbf{u} = (0,1)$ and labeled $(0,2)$ as its image under the non-trivial homology in $\Gamma((0,0),[\infty])$. All of the choices work the same way as far as obtaining an optimal coordinatisation is concerned because of the symmetrical structure of the central collineation groups around the vertices of the triangle $\{\mathbf{y},\mathbf{x},\mathbf{O}\}$ in our technique of coordinatisation.

8. Choose any $\mathbf{u} = (0,1)$ on $[0] \setminus \{(\infty),(0,0)\}$. With this, a frame of coordinatisation $\{\mathbf{y},\mathbf{x},\mathbf{O},\mathbf{u},\mathbf{J}\}$ is determined as well as a subplane $\mathcal{Q}$ of order 3 sharing the same frame of coordinatisation. The affine points of the subplane $\mathcal{Q}$ are the points of $\mathcal{P}$ determined by $\mathbb{F}_3^2$.

   We will discuss the properties of the coordinatising frame thus obtained in the subsequent section.

   Labeling of the frame of coordinatisation also means the PTR addition $\oplus$ and multiplication $\odot$ can now be defined on the coordinatising set $\mathcal{R}$ via their actions on the vertical axis as described in Section 1.7.1.

9. A labeling of the remaining points on the vertical axis [0] is obtained by following an additive labeling. Recall the discussion on choosing a suitable generator

of $(\mathbb{F}_{16}^*, \cdot)$ in the coordinatisation of the Desarguesian plane of order 16 (Section 3.1). In a similar way, here we choose a point $\mathbf{s}$ on the vertical axis [0] to label as $\mathbf{s} = (0, a)$ such that the following relation holds: $\mathbf{u} \oplus (\mathbf{s} \odot \mathbf{s} \odot \mathbf{s}) = \mathbf{s}$ (akin to $1 + a^3 = a$ in $\mathbb{F}_{27}$). In our search, we were able to always find a point $\mathbf{s}$ admitting this property. A point $\mathbf{s}$ chosen this way fully optimises the PTR addition as well as the multiplication. The reason to not use parenthesis in the triple product $\mathbf{s} \odot \mathbf{s} \odot \mathbf{s}$ will be explained in the next section. Once a point $\mathbf{s} = (0, a)$ is chosen, repeated additive labeling with a and a^2 or any other result of the first round of additive labeling produces a labeling on all the remaining points of the axis [0]. Here are few examples:

- Suppose $\mathbf{u} = (0, 1)$ and $\mathbf{s} = (0, a)$. Then $\mathbf{u} \oplus \mathbf{s} = (0, 1 + a) = (0, a^9)$.
- Continuing from above, $\mathbf{u} \oplus \mathbf{u} = (0, 2) = \mathbf{r}$ (say). Then, $\mathbf{r} \oplus \mathbf{s} = (0, 2 + a) = (0, a^3) = \mathbf{r}'$ (say). From this, $\mathbf{s} \oplus \mathbf{r}' = (0, a + a^3) = (0, a^{22})$ and so on.

With the frame of coordinatisation and the vertical axis labeled, the coordinates of all points and lines of $\mathcal{P}$ can be determined.

5.3 Some Observations on the Coordinatisation Method

We list a few observations regarding the coordinatisation described in the previous section. The observations are justifications for some parts of the labeling procedure as well. We do not have a theoretical proof of the observations. We have verified the claims made in the observations by exhaustive computation. Aspects of the observations not verified by exhaustive computation are noted as such.

(i) With any triangle of coordinatisation $\{\mathbf{y}, \mathbf{x}, \mathbf{O}\}$ chosen by following the Coordinatisation Steps 1 through 5, we can obtain a coordinatisation whose associated PTR is fully optimised for the PTR addition i.e. $x \oplus y = x + y \ \forall x, y \in \mathcal{R}$.

(ii) Obtaining a PTR fully optimised in both the PTR addition and multiplication is possible after choosing the point $\mathbf{J} = (1)$ and $(1)^\gamma = (2)$ as described in the Coordinatisation Step 7. With the choice, any valid choice of $\mathbf{u} = (0, 1)$ in the Coordinatisation Step 8 can be completed to a labeling of the points on the vertical axis $[0]$ that produces a PTR fully optimised in both the PTR addition and the PTR multiplication.

At the time of writing, we have not obtained any other process of coordinatisation that resulted in such a doubly optimised PTR.

(iii) With the PTR addition $\oplus$ and multiplication $\odot$ on $\mathcal{R}$ obtained from a labeling of the frame of coordinatisation as given by Coordinatisation Steps 1 through 8, the resulting system $(\mathcal{R}, \oplus, \odot)$ is always isomorphic to the finite field $(\mathbb{F}_{27}, +, \cdot)$. This means,

- the additive loop $(\mathcal{R}, \oplus)$ is an elementary abelian group, and
- the multiplicative loop $(\mathcal{R}^*, \odot)$ is a cyclic group.

Group structure of the multiplicative loop is the reason we ignored the parenthesis in the PTR multiplication in the Coordinatisation Step 9. Moreover, the additive labeling in the same step is defined unambiguously since the additive loop is a group.

Referring to the isomorphism of $(\mathcal{R}, \oplus, \odot)$ and $(\mathbb{F}_{27}, +, \cdot)$, we would like note the plane defined by the PTR $(\mathcal{R}, T)$ is the Figueroa plane which is not isomorphic to the Desarguesian plane of order 27. The PTR $(\mathcal{R}, T)$ is not linear and hence the relation of incidence between points and lines in the Figueroa plane is *not given* by the relation $T(x, y, z) = (x \odot y) \oplus z$.

(iv) Every frame of coordinatisation obtained by following the Coordinatisation Steps 1 through 8 can be completed to a labeling of the points on the vertical axis [0] that is fully optimised in both the PTR addition and multiplication. Note the $\oplus$ and $\odot$ are defined by the frame of coordinatisation as actions on the vertical axis (Section 1.7.1). Moreover, such optimised labeling can be obtained by choosing any point $\mathbf{s} = (0, a)$ from the remaining points on the axis [0] satisfying the condition in the Coordinatisation Step 9.

5.4 Common Properties of the PTRs

The PTRs representing the Figueroa plane $\mathscr{P}$ of order 27 obtained from our coordinatisation process have some interesting properties. Although the PTRs are not linear, the measure of linearity is high. Some properties common to all the PTRs we obtained are listed below. For every PTR $(\mathcal{R}, T)$,

(i) The domain of linearity (see Definition 4.1.2) is of order 14067. For reference, the order of $\mathcal{R}^3$ is 19683. So the relation $T(x, y, z) = xy + z$ holds true on more than 71% of the domain $\mathcal{R}^3$.

(ii) The linear relation i.e. $T(x, y, z) = xy + z$ holds trivially for all $y, z \in \mathcal{R}$ and $x \in \{0, 1\}$. By using $\Gamma((0), [0])$ to label (2) as the image of (1), we have achieved $T(2, y, z) = 2y + z \ \forall y, z \in \mathcal{R}$ as well (Theorem 1.10.8). Note the values of x correspond to parallel classes in the plane defined by the PTR $(\mathcal{R}, T)$.

(iii) The linear relation fails to hold by an equal amount in every other parallel class. For every parallel class of the plane corresponding to $x \in \mathcal{R} \setminus \{0, 1, 2\}$ i.e. values of x not in the prime subfield $\mathbb{F}_3$ of $\mathbb{F}_{27}$, the linear relation $T(x, y, z) = xy + z$ fails in exactly 234 counts. Note there are 24 affine parallel classes other than the ones corresponding to $x \in \mathbb{F}_3$ and $234 \cdot 24 = 5616 = 19683 - 14067 =$

$|\mathcal{R}^3| - |\text{domain of linearity}|$ is the total number of tuples (x, y, z) in $\mathcal{R}^3$ where $T(x, y, z) \neq xy + z$. Finally, the number of points (x, y) on any line $[m, k]$ where the incidence is not given by a linear relation i.e. $T(m, x, y) \neq k$ is one of 0, 3, 9, or 12 averaging to 8 points per y value in any parallel class. Once again, $8 \cdot 26 = 234$ and we discount $y = 0$ since $T(x, 0, z) = z$ holds trivially. All this suggests an extensive level of regularity in the way the PTR deviates from a linear property.

5.5 Further Optimisation of the PTRs

The coordinatisation method achieves the optimal values for all measures of the algebraic properties of the corresponding PTR since $(\mathcal{R}, \oplus, \odot)$ coincides fully with $(\mathbb{F}_{27}, +, \cdot)$. Therefore, we aim to optimise the PTR further by having an appropriate set of elements in the special subsets of $\mathcal{R}$ defined in Section 1.10.1 and then using various results given in Section 1.10.2 to include suitably chosen subsets of $\mathcal{R}^3$ in the domain of linearity. The process is explained in detail next.

We already have $T(x, y, z) = xy + z$ for all $y, z \in \mathcal{R}$ and $x \in \mathbb{F}_3$ (Section 5.4, Property 2). In the Coordinatisation Step 8 we chose any $\mathbf{u} = (0, 1)$ on the vertical axis. Suppose instead we make the choice by defining $(0, 1) = (0, 0)^\alpha$ for some $1 \neq \alpha \in \Gamma((\infty), [\infty])$. Then we have, $(0, 2) = (0, 1)^\beta$ for the remaining non-trivial elation β in $\Gamma((\infty), [\infty])$ (which is of order 3). To see this, recall the relation between $\Gamma((\infty), [\infty])$-elations and the PTR addition $\oplus$ discussed in Section 1.11. Following the discussion, $(0, 1)^\alpha = (0, 1 \oplus 1) = (0, 2)$ since we fully optimise the PTR addition in the labeling process. By Theorem 1.10.11, we have $2 \in \mathcal{S} = \{s : (0, 0)^\omega = (0, s) \text{ for some } \omega \in \Gamma((\infty), [\infty])\}$. Since $|\,\Gamma((\infty), [\infty])\,| = 3$, there are exactly two non-trivial elations in the group and the conclusion follows easily. By another statement in the same Theorem 1.10.11, we have $T(x,y,z) = xy + z$ for all $x, y \in \mathcal{R}$ and $z \in \mathbb{F}_3$.

The above choice of $\mathbf{u} = (0,1)$ also achieves another aspect of the PTR optimisation. By following similar arguments, it can be shown that $(0,0)^\alpha = (1,0)$ and $(0,0)^\beta = (2,0)$ where $1 \neq \alpha \neq \beta,\ \alpha, \beta \in \Gamma((0), [\infty])$ are the two non-trivial elations in the group. This means, Theorem 1.10.7 can be invoked to claim $T(x, y, x \odot w) = x \odot (y \oplus w)$ for all $x, w \in \mathcal{R},\ y \in \mathbb{F}_3$. But the PTR is distributive so that $T(x, y, x \odot w) = (x \odot y) \oplus (x \odot w)$ for all $x, w \in \mathcal{R},\ y \in \mathbb{F}_3$. We can vary $x \odot w$ in $\mathcal{R}$ for every fixed $x \in \mathcal{R}$ simply by varying w in $\mathcal{R}$. Therefore, we have shown $T(x,y,z) = (x \odot y) \oplus z = xy + z$ for all $x, z \in \mathcal{R},\ y \in \mathbb{F}_3$.

5.6 Optimal PTR Polynomials

The general form of a PTR polynomial of the Figueroa plane without any optimisation in the coordinatisation process is given by Equation 1.16.1 since the plane is of LB type I.1. In our coordinatisation we coordinatise a subplane of order 3 by $(\mathbb{F}_3, +, \cdot)$. Besides, the PTR $(\mathcal{R}, T)$ is fully optimised in both the PTR addition and multiplication. The subplane of order 3 is necessarily Desarguesian and therefore the restriction of the PTR polynomial to $\mathbb{F}_3 \times \mathbb{F}_3 \times \mathbb{F}_3$ must be $T(X,Y,Z) = XY + Z$. Since the function $t_3 : \mathbb{F}_{27} \to \mathbb{F}_{27}$ (recall the definition of $t_k(x)$ from Section 1.15.3) vanishes on $\mathbb{F}_3$, we can claim a form

$$T(X,Y,Z) = XY + Z + p(X,Y,Z)\left(X^3 - X\right) + q(X,Y,Z)\left(Y^3 - Y\right), \qquad (5.6.1)$$

where $p(X,Y,Z),\ q(X,Y,Z) \in \mathbb{F}_{27}[X,Y,Z]$.

When the PTR is further optimised as described in Section 5.5, we see that the PTR function $T : \mathcal{R}^3 \to \mathcal{R}$ satisfies the property $T(x,y,z) = xy + z$ whenever any of x, y, or z is in $\mathbb{F}_3$. For such optimised PTRs, we obtain the form

$$T(X,Y,Z) = XY + Z + r(X,Y,Z)\left(X^3 - X\right)\left(Y^3 - Y\right)\left(Z^3 - Z\right), \qquad (5.6.2)$$

where $r(X,Y,Z) \in \mathbb{F}_{27}[X,Y,Z]$.

In fact, every PTR polynomial we obtained by optimising the choice of the point $\mathbf{u} = (0,1)$ as described in Section 5.5 has the coefficient polynomial $r(X,Y,Z)$ in $\mathbb{F}_3[X,Y,Z]$. That is, we have the optimal PTR polynomial with coefficients entirely in $\mathbb{F}_3$. We do not have a theoretical proof for the observation at the time of writing.

The PTR polynomials have some other interesting properties too. Every PTR polynomial obtained by the coordinatisation method has a length 1188. This is unusual because we have obtained PTR polynomials ranging widely in lengths for PTRs holding identical measures of algebraic properties in the case of every non-Desarguesian plane of order 16. We note a simultaneous occurrence of this regularity in the lengths of distinct PTR polynomials and the full optimisation of both the PTR addition and the PTR multiplication in the coordinatisation. Given below is one of the PTR polynomials we have obtained:

$$T(X,Y,Z) = XY + Z - r(X,Y,Z)\left(X^3 - X\right)^4\left(Y^3 - Y\right)\left(Z^3 - Z\right), \qquad (5.6.3)$$

where $r(X,Y,Z) \in \mathbb{F}_{27}[X,Y,Z]$ is irreducible and has a length 414. The polynomial $r(X,Y,Z)$ is given in expanded form in Appendix B.

We have not further analysed the polynomial $r(X,Y,Z)$ at the time of writing. The exercise could reveal more properties of the PTR. For example, the form of the polynomial in (5.6.3) gives a lower bound for the domain of linearity as $3 \cdot 27 \cdot 27 + 24 \cdot 3 \cdot 27 + 24 \cdot 24 \cdot 3 = 5859$, a gross undercount given the actual order is 14067 (see

Section 5.4).

Chapter 6

A CLASS OF PTR POLYNOMIALS OF THE HALL PLANES

We obtained three distinct PTR polynomials of similar shapes representing the Hall plane of order 16 in Section 3.4.3. The polynomials were obtained through computations based on the optimal coordinatisations of the plane. In this chapter, we obtain the shape of the PTR polynomials in Section 3.4.3 with a theoretical approach, thus establishing a general shape for the Hall planes of any order.

Refer to Section 1.18 for an introduction to the Hall planes. While the original construction given by Hall is based on a family of quasifields called the Hall quasifields, the result in this chapter is based on a different method of constructing the planes—the derivation (Section 1.17). The Hall planes are constructed by derivation of the Desarguesian planes. See [17], Chapter X.

6.1 The Hall Planes as Derived Planes

Consider the Desarguesian projective plane $\mathcal{P}$ of order q^2 for some prime power $q = p^e$. Fix any line of $\mathcal{P}$ as the line at infinity $[\infty]$ and define the affine Desarguesian plane $\mathcal{A} = \mathcal{P}^{[\infty]}$. First, we want to obtain a derivation set for the affine plane $\mathcal{A}$. Consider the equation $x^{q+1} - 1 = 0$. The $q + 1$ roots of the equation in $\mathbb{F}_{q^2}$ are,

$$\mathcal{S} = \left\{1,\, a^{q-1},\, a^{2(q-1)}, \dots,\, a^{q(q-1)}\right\},$$

where a is the generator of the cyclic multiplicative group $\left(\mathbb{F}^*_{q^2}, \cdot\right)$.

By [17], Theorem 10.11, the set $\mathcal{H} = \{(m) \in [\infty] \mid m \in \mathcal{S}\}$ is a derivation set of the Desarguesian affine plane $\mathcal{A}$. The affine plane $\mathcal{A}_{\mathcal{H}}$ derived from the Desarguesian affine plane $\mathcal{A}$ using the set $\mathcal{H}$ as the derivation set is a Hall plane of order q^2 (Corollary to [17], Theorem 10.13).

Remark 6.1.1. *There are many other derivation sets of the plane $\mathcal{A}$. For example, see [17], Theorem 10.11. But Lemma 10.10 in the same chapter in [17] states that all derived planes of the affine Desarguesian plane $\mathcal{A}$ obtained from the different derivation sets are isomorphic. Hence, $\mathcal{A}_{\mathcal{H}}$ is the Hall plane of order q^2 up to isomorphism. We have chosen the derivation set $\mathcal{H}$ containing the specific points to facilitate the development of a particular form of the PTR polynomial of the Hall plane $\mathcal{A}_{\mathcal{H}}$.* $\square$

Some properties of the set $\mathcal{S}$ based on which the derivation set $\mathcal{H}$ is defined are listed next, for later use. Let $s_k = a^{k(q-1)}$ where $0 \leq k \leq q+1$ is an integer. The following statements about the set $\mathcal{S}$, any two integers k, l such that $0 \leq k, l \leq q+1$, and the elements $s_k, s_l \in \mathcal{S}$ are a consequence of the properties of $(\mathbb{F}_{q^2}, +, \cdot)$:

(i) We have $s_0 = 1$ and $\mathcal{S} = \{s_0,\, s_1,\, \dots,\, s_q\}$.

(ii) Consider the subfield $\mathbb{F}_q = \left\{0, 1, a^{q-1}, a^{2(q-1)}, \dots, a^{(q-2)(q-1)}\right\}$ of $\mathbb{F}_{q^2}$. Then, $\mathcal{S} \cap \mathbb{F}_q = \{1, p-1\}$. Note p is the characteristic of $\mathbb{F}_{q^2}$ and $\mathcal{S} \cap \mathbb{F}_q = \{1\}$ for even characteristic.

(iii) We have $s_k^{q+1} = a^{k(q^2-1)} = 1$ and $s_k s_l = a^{k(q-1)} a^{l(q-1)} = a^{(k+l)(q-1)} = s_{k+l}$.

(iv) From (iii), $s_k^q s_k = s_k^{q+1} = 1$ and $s_{q+1} = a^{(q+1)(q-1)} = a^{q^2-1} = 1$.

Noting that $s_{q+1} = s_0 = 1$, we extend the definition of s_k to negative subscripts and define $s_{-k} = a^{-k(q-1)} = \left(a^{k(q-1)}\right)^{-1} = s_k^{-1}$. We then have $s_k^q = s_k^{-1} = s_{-k}$. This notation will be useful for computations.

6.2 Coordinatisation of the Hall Plane Based on the Coordinatisation of the Parent Desarguesian Plane

Consider an optimal coordinatisation of the Desarguesian plane $\mathcal{P}$ with the finite field $\mathbb{F}_{q^2}$ as the coordinatising set. Let the frame of coordinatisation be $\{\mathbf{y}, \mathbf{x}, \mathbf{O}, \mathbf{u}, \mathbf{J}\}$ with the usual labeling of the points so that $\overline{\mathbf{x}\,\mathbf{y}} = [\infty]$, $\mathbf{J} = (1) \in \mathcal{H}$ and so on. As an example, recall the coordinatisation of the Desarguesian plane of order 16 in Section 3.1. The optimal PTR polynomial of $\mathcal{P}$ is,

$$T(X, Y, Z) = (X \odot Y) \oplus Z = XY + Z.$$

where $\oplus$ and $\odot$ are the PTR addition and multiplication coinciding with the field addition and multiplication as shown provided the plane is coordinatised optimally.

To obtain a coordinatisation of the Hall plane $\mathcal{A}_{\mathcal{H}}$, we consider the plane obtained from a derivation of the Desarguesian plane $\mathcal{P}$ with $\mathcal{H}$ as the derivation set (Section 1.17.1) with a focus on coordinates of the points and lines in the original plane and changes required in the new plane.

The lines $[0]$ and $[0, 0]$ are retained in the derived plane as the lines $[0]'$ and $[0, 0]'$ respectively since they meet the line $[\infty]$ at the points (∞) and (0) not in the derivation set $\mathcal{H}$. This means, by choosing to retain the labeling of the affine points $\mathbf{a}$ and $\mathbf{b}$ from the optimal coordinatisation of the Desarguesian plane $\mathcal{P}$, we can use the same labeling of the points on the vertical axis $[0]$ by the elements of $\mathbb{F}_{q^2}$ as in the original coordinatisation to coordinatise the derived plane. However, the PTR addition $\oplus'$ and the PTR multiplication $\odot'$ in the derived plane may no longer coincide with the finite field addition and multiplication for every pair $x, y \in \mathbb{F}_{q^2}$ as the labeling of the ideal points in the derivation set $\mathcal{H}$ can change. We will clarify this as we go through the derivation process.

Let us take a closer look at the labelling of the points corresponding to the derivation set $\mathcal{H}$ on the line $[\infty]'$ of the derived plane. For every $s_k \in \mathcal{S}$, we label the ideal point in $[\infty]'$ that corresponds to the parallel class containing the line $[(0, s_k)', (1, 0)']$ by $(s_k)'$. As a consequence of the labeling, we have $s_k \odot' 1 = s_k$ for all integers k which implies, in turn, $x \odot' 1 = x$ for all $x \in \mathbb{F}_{q^2}$. Since the three axes $[0]'$, $[0, 0]'$, and $[\infty]'$ of coordinatisation as well as three vertices of the triangle of coordinatisation, namely $(0)'$, $(0, 0)'$, and $(\infty)'$ retain their labeling in the derivation process, it follows $x \odot' 0 = 0$ and $0 \odot' x = 0$ for all $x \in \mathbb{F}_{q^2}$. In summary, the coordinatisation of the derived plane will have the following three properties compared to the original coordinatisation of the Desarguesian plane:

- The same frame of coordinatisation.
- The same labeling for the three axes of coordinatisation as well as the points on the three axes except possibly for the points in the derivation set $\mathcal{H}$.
- Relabeling of the points in the derivation set $\mathcal{H}$ such that $x \odot' 1 = x$, $x \odot' 0 = 0$, and $0 \odot' x = 0$ holds for all $x \in \mathbb{F}_{q^2}$.

Remark 6.2.1. *At this point, we do not know if $1 \odot' x = x$ holds for $x \notin \{0, 1\}$ since the ideal point (1) is in the derivation set $\mathcal{H}$. More on this in Section 6.5.*

6.3 The new PTR addition and multiplication

Our next goal is to study the behaviour of the new PTR addition $\oplus'$ and the new PTR multiplication $\odot'$ via their actions on the vertical axis $[0]'$ of the derived plane or the Hall plane $\mathcal{A}_{\mathcal{H}}$.

Recall the action of the PTR addition on the vertical axis $[0]$ in Section 1.7.1. The sum $x \oplus' y$ on the vertical axis $[0]'$ is given by the intersection $\overline{\mathbf{J}'(x, y)'} \cap [0]'$. The point $(x, y)'$ is labeled based on the relation $(x, y)' = [x]' \cap [0, y]'$. From the details of the derivation process described above we know the points $(x, 0), (0, y)$, and (0) retain their respective labeling i.e. $(x, 0)' = (x, 0)$, $(0, y)' = (0, y)$, and $(0)' = (0)$. Therefore, $[x]' = [x]$ and $[0, y]' = [0, y]$ so that $(x, y)' = (x, y)$ as well. We have shown the coordinates or the labeling of all affine points in the derived plane is the same as the labeling in the parent Desarguesian plane. The line containing the points $(0, x + y)$ and (x, y) in the parallel class corresponding to $\mathbf{J} = (1)$ in the plane $\mathscr{A}$ is replaced by the line $\overline{\mathbf{J}'(x, y)'} = \overline{\mathbf{J}'(x, y)}$ in the derived plane $\mathscr{A}_{\mathscr{H}}$ which also contains the point $(0, x + y)' = (0, x + y)$. Therefore, with the new $\mathbf{J}' = (1)'$, the new PTR addition $\oplus'$ coincides with the PTR addition in the optimal coordinatisation of the Desarguesian plane $\mathscr{P}$. In other words, the PTR addition in the derived plane is also fully optimised. We will thus use $+$ denoting the finite field addition in place of the PTR addition $\oplus'$ for the remainder of this chapter.

Consider next the action of the new PTR multiplication $\odot'$ on the vertical axis $[0]'$. The product $x \odot' y$ depends, as will be proven shortly, on whether the ideal point (x) is in the derivation set $\mathscr{H}$ or its compliment $[\infty] \setminus \mathscr{H}$.

- **Case I:** Suppose $(x) \in [\infty] \setminus \mathscr{H}$. Then, the line $[x, x \odot y]$ is retained in the derivation process for every $y \in \mathbb{F}_{q^2}$. Therefore, $[x, x \odot' y]' = [x, x \odot y] = [x, xy]$. The line $[x, x \odot' y]'$ contains the points $(x)'$, $(y, 0)'$, and $(0, x \odot' y)'$. So, $x \odot' y = xy$ for all $x \in \mathbb{F}_{q^2} \setminus \mathcal{S}$, $y \in \mathbb{F}_{q^2}$.

- **Case II:** Suppose $(x) \in \mathscr{H}$. As noted in the derivation process, $x \odot' y = xy$ for $y \in \{0, 1\}$ and any $x \in \mathbb{F}_{q^2}$. Suppose then $y \in \mathbb{F}_{q^2} \setminus \{0, 1\}$. Let $x = s_k$ for some $k \in \{0, 1, \dots, q\}$. We will identify the parallel class in the plane $\mathscr{A}_H$ to which the line $\mathcal{M}'$ containing points $(s_k)'$ and $(y, 0)'$ belongs. The line $\mathcal{M}'$ in the derived plane $\mathscr{A}_H$ contains the affine points of a Baer subplane $\mathscr{B}$ of the parent plane $\mathscr{A}$. The Baer subplane $\mathscr{B}$ contains the derivation set $\mathscr{H}$. Therefore, $\mathcal{M}' \cap [0]' = (0,\ s_l y)'$ for some $s_l \in \mathcal{S}$ such that $s_k \odot' y = s_l y$.

We consider two Baer subplanes of the Desarguesian plane $\mathscr{A}$:

(i) $\mathscr{B}_1$ containing $\mathscr{H}$, $(0,\ s_k)$, and $(1, 0)$.

(ii) $\mathscr{B}_2$ containing $\mathscr{H}$, $(0,\ s_j y)$, and $(y, 0)$ for some $s_j \in \mathcal{S}$.

The orders of the Baer subplanes are q each and the set $\mathscr{H}$ contains $q+1$ points. Therefore, both for $\mathscr{B}_1$ and $\mathscr{B}_2$, the points of intersection with the line $[\infty]$ in the plane $\mathscr{P}$ are exactly the points of the derivation set $\mathscr{H}$.

Suppose now $\mathscr{B}_1^{[\infty]} \cap \mathscr{B}_2^{[\infty]}$ is non-empty. By [17] , Lemma 10.1 they have exactly one point in common, say (x, y). Consider the following two statements about the point (x, y):

(i) The point (x, y) lies on a line in the pencil of lines through $(0, s_k)$ in $\mathscr{B}_1$. Therefore,

$$\begin{aligned} & (x, y) \in [s_{k_1}, s_k] \\ \Longrightarrow\ & T(s_{k_1}, x, y) = s_k \\ \Longrightarrow\ & s_{k_1} x + y = s_k \end{aligned} \tag{6.3.1}$$

for some $s_{k_1} \in \mathscr{H}$.

(ii) The point (x, y) lies on a line in the pencil of lines through $(1, 0)$ in $\mathscr{B}_1$. Therefore,

$$\begin{aligned} & (x, y) \in [s_{k_2}, s_{k_2}] \\ \Longrightarrow\ & T(s_{k_2}, x, y) = s_{k_2} \end{aligned}$$

$$\Longrightarrow s_{k_2}x + y = s_{k_2} \tag{6.3.2}$$

for some $s_{k_2} \in \mathcal{S}$.

Solving for x from (6.3.1) and (6.3.2), we get

$$x = \frac{s_k - s_{k_2}}{s_{k_1} - s_{k_2}} = \frac{s_{k-k_2} - 1}{s_{k_1-k_2} - 1}. \tag{6.3.3}$$

Repeating the argument with (x, y) as a point of the subplane $\mathcal{B}_2$, we get

$$\begin{aligned} s_{j_1}x + y &= s_j y, \text{ and} \\ s_{j_2}x + y &= s_{j_2} y, \end{aligned} \tag{6.3.4}$$

for some s_{j_1}, $s_{j_2} \in \mathcal{S}$ so that by solving for x, we have

$$x = y\left(\frac{s_{j-j_2} - 1}{s_{j_1-j_2} - 1}\right). \tag{6.3.5}$$

Equating x in (6.3.3) and (6.3.5), we get

$$y = \left(\frac{s_{k-k_2} - 1}{s_{k_1-k_2} - 1}\right)\left(\frac{s_{j_1-j_2} - 1}{s_{j-j_2} - 1}\right). \tag{6.3.6}$$

We raise both sides of (6.3.6) to the q^{th} power. Since q is a power of the characteristic p of the field $(\mathbb{F}_{q^2}, +, \cdot)$, $(s_l - 1)^q = s_l^q - 1$ for any integer $1 \leq l \leq q + 1$. Using the properties of the elements s_k in Section 6.1, we get

$$y^q = \left(\frac{s_{k-k_2}^q - 1}{s_{k_1-k_2}^q - 1}\right)\left(\frac{s_{j_1-j_2}^q - 1}{s_{j-j_2}^q - 1}\right)$$

$$
\begin{aligned}
&= \left(\frac{s_{-k+k_2}-1}{s_{-k_1+k_2}-1}\right)\left(\frac{s_{-j_1+j_2}-1}{s_{-j+j_2}-1}\right) \\
&= \left(\frac{s_{-k+k_2}-1}{s_{-k_1+k_2}-1}\cdot\frac{s_{k-k_2}}{s_{k_1-k_2}}\right)\left(\frac{s_{-j_1+j_2}-1}{s_{-j+j_2}-1}\cdot\frac{s_{j_1-j_2}}{s_{j-j_2}}\right)\cdot\frac{s_{k_1-k_2}}{s_{k-k_2}}\cdot\frac{s_{j-j_2}}{s_{j_1-j_2}} \\
&= \left(\frac{1-s_{k-k_2}}{1-s_{k_1-k_2}}\right)\left(\frac{1-s_{j_1-j_2}}{1-s_{j-j_2}}\right)\cdot s_{k_1-k_2}s_{k-k_2}^{-1}s_{j-j_2}s_{j_1-j_2}^{-1} \\
&= \left(\frac{s_{k-k_2}-1}{s_{k_1-k_2}-1}\right)\left(\frac{s_{j_1-j_2}-1}{s_{j-j_2}-1}\right)\cdot s_{k_1-k_2}s_{k_2-k}s_{j-j_2}s_{j_2-j_1} \\
&= ys_{j-k+k_1-j_1}.
\end{aligned}
$$

Multiplying both sides by s_k, we get

$$
s_k y^q = y s_{j+k_1-j_1}. \tag{6.3.7}
$$

If $k_1 = j_1$ then from (6.3.1) and (6.3.4), we get $s_k = s_{k_1}x + y = s_{j_1}x + y = s_j y$. Substituting in (6.3.7), $s_k y^q = s_j y = s_k \implies y^q = 1$ since $s_k \neq 0$. In the finite field $(\mathbb{F}_{q^2}, +, \cdot)$, this implies $y = 1$. This is a contradiction since $y \in \mathbb{F}_{q^2} \setminus \{0, 1\}$ by hypothesis. We conclude $\mathcal{B}_1^{[\infty]} \cap \mathcal{B}_2^{[\infty]} \neq \emptyset \implies k_1 \neq j_1$. Taking the contra-positive of the statement, $k_1 = j_1 \implies \mathcal{B}_1^{[\infty]} \cap \mathcal{B}_2^{[\infty]} = \emptyset$.

Since $s_k \odot' 1 = s_k$ and $s_k \odot' y = s_l y$, the Baer subplanes $\mathcal{B}_1^{[\infty]}$ and the subplane containing $(0,\, s_l y)'$, $(y, 0)'$ and $\mathcal{H}$, call it $\mathcal{B}_{s_l}^{[\infty]}$, belong to the same parallel class of the derived plane $\mathcal{A}_{\mathcal{H}}$ i.e. their intersection is empty. Finally, since $s_i \odot' y$ are all distinct and of the form $s_{i_l} y$ for some $s_{i_l} \in \mathcal{S}$, it follows the subplane $\mathcal{B}_{s_l}^{[\infty]}$ is unique. Therefore, $\mathcal{B}_{s_l}^{[\infty]} = \mathcal{B}_2^{[\infty]}$ which means, $s_j y = s_l y$. From (6.3.7), we get $s_l = s_k y^{q-1}$ so that $s_k \odot' y = s_l y = s_k y^q$.

Combining the results from the two cases above,

$$x \odot' y = \begin{cases} xy & \text{if } x \in \mathbb{F}_{q^2} \setminus \mathcal{S}, \quad \text{and} \\ \\ xy^q & \text{if } x \in \mathcal{S}. \end{cases} \tag{6.3.8}$$

For $y \in \mathbb{F}_q$, we have $y^q = y$ so that $x \odot' y = xy$ for any $x \in \mathbb{F}_{q^2}$, $y \in \mathbb{F}_q$. Using this information to update (6.3.8), we have

$$x \odot' y = \begin{cases} xy & \text{if } x \in \mathbb{F}_{q^2} \setminus \mathcal{S} \text{ or } y \in \mathbb{F}_q, \quad \text{and} \\ \\ xy^q & \text{if } x \in \mathcal{S} \text{ and } y \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q. \end{cases} \tag{6.3.9}$$

We would like to mention the above rule of multiplication is given as an Andre quasifield multiplication for the case $q = 4$ in [24], page 5 and for the general case in [37], page 125. In [37], the approach is based on constructing quasifields as vector spaces over a field via what are called *replaceable partial spreads* in the corresponding finite field. While the replaceable partial spreads are analogues of the affine lines and the Baer subplanes that we use in above derivation, our approach is tied closely to the main work in this dissertation. That is, we use the coordinatisations of the projective planes and analyse the entire derivation process from the perspective of original labeling and the changes in labeling of the points and lines.

6.4 The PTR Polynomial

The two rules for the PTR multiplication $\odot'$ given by (6.3.9) in two parts of the domain $\mathbb{F}_{q^2} \times \mathbb{F}_{q^2}$ can be used to define $\odot'$ by a polynomial in $\mathbb{F}_{q^2}[X, Y]$.

Claim:

$$X \odot' Y = XY - \prod_{u \in \mathbb{F}_{q^2} \setminus \mathcal{S}} (X - u) \prod_{v \in \mathbb{F}_q} (Y - v). \tag{6.4.1}$$

Proof of the claim: The first product on the right side vanishes for $x \in \mathbb{F}_{q^2} \setminus \mathcal{S}$ while the second product vanishes for $y \in \mathbb{F}_q$. Hence, the PTR multiplication polynomial gives the required value of the product $x \odot' y$ for all $x, y \in \mathbb{F}_{q^2}$ such that either $x \in \mathbb{F}_{q^2} \setminus \mathcal{S}$ or $y \in \mathbb{F}_q$. As for the other rule, that is, for $x \in \mathcal{S}$ and $y \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$, consider again the two products on the right side of (6.4.1). The second product is the polynomial $t_q(Y) = Y^q - Y$. To evaluate the first product, consider the following identities in $(\mathbb{F}_{q^2}, +, \cdot)$:

$$\begin{aligned} \prod_{u \in \mathcal{S}} (x - u) &= x^{q+1} - 1, \text{ so that} && (6.4.2) \\ x^{q^2} - x &= \prod_{u \in \mathbb{F}_{q^2}} (x - u) \\ &= \left(\prod_{u \in \mathbb{F}_{q^2} \setminus \mathcal{S}} (x - u) \right) \left(\prod_{u \in \mathcal{S}} (x - u) \right) \\ &= \left(\prod_{u \in \mathbb{F}_{q^2} \setminus \mathcal{S}} (x - u) \right) \left(x^{q+1} - 1 \right). && (6.4.3) \end{aligned}$$

On the other hand,

$$\begin{aligned} x^{q^2} - x &= x \left(\left(x^{q+1} \right)^{q-1} - 1 \right) \\ &= \left(x \sum_{k=0}^{q-2} \left(x^{q+1} \right)^k \right) (x^{q+1} - 1). && (6.4.4) \end{aligned}$$

From (6.4.3) and (6.4.4), we have for all $x \in \mathbb{F}_{q^2}$,

$$\prod_{u \in \mathbb{F}_{q^2} \setminus \mathcal{S}} (x-u) = x \sum_{k=0}^{q-2} \left(x^{q+1}\right)^k . \tag{6.4.5}$$

We are interested in the case $x \in \mathcal{S}$ where the product on the left does not vanish. By definition, $x \in \mathcal{S} \implies x^{q+1} = 1$. Hence, $\prod_{u \in \mathbb{F}_{q^2} \setminus \mathcal{S}} (x-u) = x \underbrace{(1+1+\cdots+1)}_{q-1 \text{ times}} = -x$. Therefore in (6.4.1), we have

$$x \odot' y = xy - (-x)\left(y^q - y\right) = xy^q,$$

for $x \in \mathcal{S}$ and $y \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$.

Finally, the PTR of the Hall plane $\mathcal{A}_{\mathcal{H}}$ corresponding to its coordinatisation as a derived plane is linear because the plane is $((\infty), [\infty])$-transitive under the coordinatisation. The PTR is fully optimised for the addition $\oplus'$ (Section 6.3) and the multiplication $\odot'$ is given by (6.4.1). Combining all this, we obtain the following polynomial form:

$$T'(X,Y,Z) = XY - \prod_{x \in \mathbb{F}_{q^2} \setminus \mathcal{S}} (X-x) \prod_{y \in \mathbb{F}_q} (Y-y) + Z. \tag{6.4.6}$$

By virtue of (6.4.5) and the definition of t_k polynomials, we can rewrite the PTR polynomial form (6.4.6) by

$$\boxed{T'(X,Y,Z) = XY - Xh_{q-2}\left(X^{q+1}\right)t_q(Y) + Z}, \tag{6.4.7}$$

as a general PTR polynomial representation of the Hall planes of order q^2.

It is interesting to note the polynomial $h_{q-2}\left(X^{q+1}\right)$ appears in the polynomial representation of a general element in a cyclic subgroup of order $q+1$ in $(\mathbb{F}_{q^2}, +, \cdot)$ in [4], Theorem 2.3.2. Observe that the set $\mathcal{S}$ and its multiplicative cosets are subgroups of order $q+1$ of $(\mathbb{F}^*_{q^2}, \cdot)$.

6.5 Some Additional Notes

(i) For $q = 4$ i.e. $n = q^2 = 16$, Equation 6.4.6 gives the following PTR polynomial representing the Hall plane of order 16:

$$T'(X,Y,Z) = XY - X\left(X^5 - a^5\right)\left(X^5 - a^{10}\right)\left(Y^4 - Y\right) + Z \tag{6.5.1}$$

which has the same shape as the PTR polynomials obtained in Section 3.4.3.

(ii) The multiplicative identity 1 of $(\mathbb{F}^*_{q^2}, \cdot)$ is not the identity 1 in $(\mathcal{R}^*, \odot)$ of the PTR $(\mathcal{R}, T)$ we have obtained for the Hall plane. The multiplicative behavior of $1 \in \mathcal{R}$ in $(\mathcal{R}, \oplus, \odot)$ is: $y \odot' 1 = y$ and $1 \odot' y = y^q$ for all $y \in \mathbb{F}_{q^2}$.

(iii) The system $(\mathcal{R}, \oplus, \odot)$ is therefore a left pre-quasifield with a *zero* 0 and a *right unit* 1. Recall from Section 1.17 such a pre-quasifield is isotopic with a quasifield and the two coordinatise isomorphic planes.

(iv) Different PTR polynomials in the shape given by (6.4.6) can be obtained by choosing different derivation sets $\mathcal{H}$ on $[\infty]$.

(v) The points of the Hall plane $\mathcal{A}_{\mathcal{H}}$ with coordinates in $\mathbb{F}_q \times \mathbb{F}_q$ do not correspond to the *kernel subplane* of $\mathcal{A}_{\mathcal{H}}$. Refer to [17], Sections VII.3-VII.4 for definition and properties of the kernel in a quasifield (plane). The points with coordinates in $\mathbb{F}_q \times \mathbb{F}_q$ correspond to a kernel subplane of $\mathcal{A}_{\mathcal{H}}$ if the derivation set $\mathcal{H}$ is chosen as the set of points on $[\infty]$ labeled by $\mathbb{F}_q$ and (∞).

Chapter 7

A PTR POLYNOMIAL REPRESENTATION OF THE SEMI4 PLANE FROM A STUDY OF THE SEMIFIELD MULTIPLICATION

We coordinatised the SEMI4 plane $\mathscr{P}$ of order 16 in Section 3.2. Three distinct PTR polynomials representing the plane and obtained from our coordinatisations were given at the end of the section. The approach taken in Section 3.2 was to utilise the geometric structure of the SEMI4 plane to define an optimal coordinate system for the plane and then to compute a PTR polynomial from an algebraic object, the PTR $(\mathcal{R}, T)$, obtained from the coordinate system. Based on some known properties of the SEMI4 plane we can expect the PTR polynomials representing the plane to possess some properties. For example, the plane is constructed from the semifield of order 16 with a kernel $GF(4)$. Since the semifields have both the distributive properties, and our coordinatisation is optimal, the PTR polynomials must be linearized in both X and Y. The polynomials we obtain have this property. The coordinatisation involves coordinatising a subplane of order 4 optimally and so the PTR polynomial T must reduce to $T(X, Y, Z) = XY + Z$ with $X \odot Y = XY$ and $Y \oplus Z = Y + Z$ in $\mathbb{F}_4 \times \mathbb{F}_4$. This too can be verified easily for the PTR polynomials in Section 3.2. Additionally, some of the polynomials display a stronger property of the corresponding PTR multiplication as we pointed out in Section 3.2.4.1. The PTR multiplication coincides with the field multiplication on a larger subset of $\mathcal{R} \times \mathcal{R}$ than $\mathbb{F}_4 \times \mathbb{F}_4$. The observation prompted us to study the PTR polynomial representation of the SEMI4 plane from a theoretical perspective. This chapter is a summary of our findings from the study.

7.1 Introduction

The SEMI4 plane is a semifield plane of order 16 (LB type V.1). Let $(\mathcal{S}, +, *)$ be a corresponding semifield given by the system V in [26], Section 2.2 (pages 184-185). Also recall from the note on the history of the SEMI4 and SEMI2 planes in Section 1.18 that 23 non-isomorphic semifields of order were enumerated by [25]. This was an exhaustive listing and 5 of the 23 non-isomorphic semifields were proper semifields with a kernel $GF(4)$. Hence, the system V in [26] is one of these 5 non-isomorphic semifields. Now, $(\mathcal{S}, +, *)$ is a vector space of dimension 2 over $(\mathbb{F}_4, +, \cdot)$. Letting $\mathcal{S}$ be the set $\mathbb{F}_{16}$ of the elements of the finite field $(\mathbb{F}_{16}, +, \cdot)$, we choose a $\lambda = a$ in $\mathcal{S} \setminus \mathbb{F}_4$ and fix it for the remainder of this chapter. Recall that a is a generator of the multiplicative group $(\mathbb{F}_{16}^*, \cdot)$. See Section 1.14 for notations and an introduction to the finite fields. Now set

$$\mathcal{S} = \{u + \lambda v : u, v \in \mathbb{F}_4\}.$$

The addition $+$ in $(\mathcal{S}, +, *)$ is the same as the addition in $(\mathbb{F}_{16}, +, \cdot)$. The multiplication $*$ in $(\mathcal{S}, +, *)$ is given by,

$$(u + \lambda v) * (r + \lambda s) = (ur + \omega v^2 s) + \lambda(vr + u^2 s), \tag{7.1.1}$$

where juxtaposition stands for the field multiplication in $(\mathbb{F}_{16}, +, \cdot)$ and ω is any generator of $(\mathbb{F}_4^*, \cdot)$. Since a is the generator of $(\mathbb{F}_{16}^*, \cdot)$, we have $\omega = a^5$ or $\omega = a^{10}$. We fix $\omega = a^5$ for the sequel.

7.2 PTR Function and PTR Polynomial of an Optimal SEMI4 PTR

Let $(\mathcal{S}, T)$ be an optimal PTR representation of the SEMI4 plane with the PTR addition $+$ and the PTR multiplication $*$ as defined above. Then, the PTR $(\mathcal{S}, T)$ is linear and the tri-variate PTR function $T(x, y, z) : \mathcal{S}^3 \to \mathcal{S}$ is given by,

$$T(x, y, z) = (x * y) + z. \tag{7.2.1}$$

The linear PTR $(\mathcal{S}, T)$ of the SEMI4 plane in the previous section is obtained by coordinatising the plane optimally with the semifield $(\mathcal{S}, +, *)$. Consider and fix the coordinates of the points of the SEMI4 plane thus coordinatized. We want to obtain a PTR polynomial representation of the function T in $\mathbb{F}_{16}[X, Y, Z]$. In other words, we want to obtain a polynomial $T' : \mathbb{F}_{16}^3 \to \mathbb{F}_{16}$ such that,

$$T'(x, y, z) = T(x, y, z), \tag{7.2.2}$$

identically in $\mathbb{F}_{16}$.

7.3 Multiplications in the Semifield and the Field

Let $x, y \in \mathcal{S}$ be any two elements. Then, $x = u + \lambda v$, and $y = r + \lambda s$ for some $u, v, r, s \in \mathbb{F}_4$. We have,

$$x * y = (u + \lambda v) * (r + \lambda s) = (ur + \omega v^2 s) + \lambda(vr + u^2 s), \quad \text{and} \tag{7.3.1}$$

$$xy = (u + \lambda v)(r + \lambda s) = ur + \lambda us + \lambda vr + \lambda^2 vs \tag{7.3.2}$$

We want to know the different cases in which the semifield product coincides with the field product of x and y, that is, $x * y = xy$. When this is true, from (7.3.1) and (7.3.2), we get

$$\omega v^2 s + \lambda u^2 s = \lambda u s + \lambda^2 v s. \tag{7.3.3}$$

Let us consider the various possibilities for (7.3.3) to hold:

- **Case I:** $\underline{s = 0}$. Then, $y = r + \lambda(0) \in \mathbb{F}_4$. We verify

$$x * y = (u + \lambda v) * r = ur + \lambda vr = (u + \lambda v)r = xy.$$

- **Case II:** $\underline{s \neq 0}$. Then, (7.3.3) reduces to

$$\begin{aligned} \omega v^2 + \lambda u^2 &= \lambda u + \lambda^2 v \\ \implies v^2 + \lambda^2\omega^2 v + \lambda\omega^2(u^2 + u) &= 0. \end{aligned} \tag{7.3.4}$$

  We consider two sub-cases:

  (i) $\underline{\text{Suppose } u \in \{0, 1\}}$. Then $u^2 + u = 0$. From (7.3.4), we have

  $$v(v + \lambda^2\omega^2) = 0.$$

  Either $v = 0$ or $v = \lambda^2\omega^2$. But $\lambda^2\omega^2 = a^2(a^5)^2 = a^{12} \notin \mathbb{F}_4$. Therefore, $u = 0$ or 1 and $v = 0 \implies x = 0$ or 1.

  (ii) $\underline{\text{Suppose } u \in \{a^5, a^{10}\}}$. Then $u^2 + u = 1$. From (7.3.4), we have

  $$v^2 + \lambda^2\omega^2 v + \lambda\omega^2 = 0. \tag{7.3.5}$$

  The existence and the values of the solutions v of (7.3.5) depend on the choice of λ. By exhaustive search, v takes one of the values $1, a^5$, or a^{10}

for certain choices of λ while (7.3.5) has no roots in $\mathbb{F}_{16}$ for other choices. For the value $\lambda = a$ we have chosen, we get

$$\begin{aligned} & v^2 + a^{12}v + a^{11} = 0 \\ \Longrightarrow\ & v^2 + (1 + a^{11})v + a^{11} = 0 \\ \Longrightarrow\ & v = 1 \text{ or } a^{11}. \end{aligned}$$

Since $v \in \mathbb{F}_4$, we have $v \neq a^{11}$. Finally, $v = 1 \implies x = a^5 + a = a^2$ or $x = a^{10} + a = a^8$.

We summarise the analysis in this section by giving the following description of the particular semifield multiplication $*$ obtained for the semifield of the SEMI4 plane (note the rule is for the case $\lambda = a$ only):

$$x * y = \begin{cases} xy & \text{if } x \in \{0, 1, a^2, a^8\} \text{ or } y \in \mathbb{F}_4, \quad \text{and} \\ f(x, y) & \text{for some } f \in \mathbb{F}_{16}[X, Y] \text{ otherwise.} \end{cases} \tag{7.3.6}$$

Compare the multiplication rule (7.3.6) with the rule (3.2.3) we determined for a SEMI4 PTR polynomial in Section 3.2.4.1. The rules are identical on $\mathcal{U} \times \mathbb{F}_{16}$ and $\mathbb{F}_{16} \times \mathbb{F}_4$ where $\mathcal{U} = \{0, 1, a^2, a^8\}$ ignoring the interchange of x and y, a non-issue for a self-dual plane and fully distributive PTRs.

Remark 7.3.1. *We can obtain a similar description of a different semifield multiplication by taking different values of λ and/or by considering the sub-cases with values of v instead of s in the Case I and Case II above. The resulting semifield multiplication coincides with the field multiplication if either $x \in \mathbb{F}_4$, $y \in \mathcal{V}$ or $y \in \mathbb{F}_4$, $x \in \mathcal{U}_1$ for some suitable subgroups $(\mathcal{V}, +)$ and $(\mathcal{U}_1, +)$ of order 4 of $(\mathbb{F}_{16}, +)$.*

7.4 The PTR Polynomial

Based on (7.3.6), the semifield multiplication $*$ in $(\mathcal{S}, +, *)$ can be given by the following polynomial in $\mathbb{F}_{16}[X, Y]$:

$$\begin{aligned} X * Y &= XY - p(X,Y) \left(\prod_{x \in \{0,1,a^2,a^8\}} X - x \right) \left(\prod_{y \in \mathbb{F}_4} Y - y \right) \\ &= XY - p(X,Y) \left(X^4 - a^5 X^2 + a^{10} X \right) \left(Y^4 - Y \right), \end{aligned}$$

for some polynomial $p(X, Y)$ in $\mathbb{F}_{16}[X, Y]$. Clearly, the polynomial $p(X, Y)$ depends on the function $f(X, Y)$ in (7.3.6).

Hence, the PTR function $T'(x, y, z)$ in (7.2.2) can be represented by the PTR polynomial of the form

$$T'(X,Y,Z) = XY - p(X,Y) \left(X^4 - a^5 X^2 + a^{10} X \right) \left(Y^4 - Y \right) + Z. \tag{7.4.1}$$

By Remark 7.3.1, a different PTR polynomial of the SEMI4 plane can have a form

$$T'(X,Y,Z) = XY + q(X,Y) \left(X^4 - X \right) \left(Y^4 - a^5 Y^2 + a^{10} Y \right) + Z, \tag{7.4.2}$$

for some polynomial $q(X, Y)$ in $\mathbb{F}_{16}[X, Y]$.

Remark 7.4.1. *Since (7.4.1) and (7.4.2) represent PTR polynomials of a semifield plane, the polynomials $T'(X, Y, Z)$ must be linearised in both X and Y to reflect the right and the left distributive properties of the semifield respectively. This condition gives a restriction on the form of the coefficient polynomials $p(X, Y)$ and $q(X, Y)$. Now, the product of linearised polynomials is not linearised in general. One of the* *cases in which the product is a linearised polynomial is when the product is also a composition of linearised polynomials (not necessarily the same polynomials as in the product). To clarify this, we will show the calculations for the composition for the PTR polynomial listed in Section 3.2.4. We gave the following PTR polynomial for the SEMI4 plane in Section 3.2.4:*

$$T'(X, Y, Z) = XY + a^{11}\left(X^4 - X\right)\left(Y^8 - a^{10}Y^2 + a^5Y\right) + Z,$$

which has the form given by (7.4.2). Now, define $q_1(Y) = Y^4 - a^5Y^2 + a^{10}Y$ *and* $q_2(Y) = Y^2 + a^{10}Y$*. It is easily checked* $q(Y) = q_2\left(q_1(Y)\right)$*.*

Chapter 8

CONSTRUCTING PTR MULTIPLICATION TABLE OF PLANES OF LB TYPE II.1 OR ABOVE

In this chapter, we outline an approach to constructing projective planes of LB type II or above by constructing a multiplication table of a PTR representing the plane. The existence of linear PTRs for the planes of LB type I.2 or above (when coordinatised optimally) is essential to our method of construction. Besides, we utilize the same results from Chapters 3 and 2 that we used to obtain the optimal coordinatisations of projective planes in Chapters 3 and 4. In fact, the intuition for the constructions of planes in this chapter comes from the coordinatisations in Chapters 3 and 4.

We have used this approach to construct all planes of order 16 type II.1 or above and many planes in order 25. Further, the method of construction enables a search for projective planes with a given set of properties in an efficient and exhaustive manner. The ultimate goal of our method is to obtain new planes of finite prime power order.

We begin by proving a property of the PTR multiplication tables of the planes of LB type II.1 or above. The result is a restatement of [9], Section 3.1, (28) on page 129. The statement and the proof of the result we give here will serve as a tool for eliminating elements of $\mathcal{R}$ from a pool of possible values of an entry in the PTR multiplication table.

Theorem 8.0.1. *Let $\mathcal{P}$ be a projective plane of Lenz Barlotti type at least II.1. Let $(\mathcal{R}, T)$ be a Cartesian group representation of $\mathcal{P}$. Consider the resulting ternary system $(\mathcal{R}, \oplus, \odot)$. Let $-x$ denote the additive inverse of x in the group $(\mathcal{R}, \oplus)$. Then for all $m_1, m_2, x_1, x_2 \in \mathcal{R}^*$, $m_1 \neq m_2, x_1 \neq x_2$,*

$$(m_1 \odot x_1) - (m_1 \odot x_2) \neq (m_2 \odot x_1) - (m_2 \odot x_2).$$

Proof. Assume to the contrary

$$(m_1 \odot x_1) - (m_1 \odot x_2) = (m_2 \odot x_1) - (m_2 \odot x_2) = c. \quad \text{(say)} \tag{8.0.1}$$

Let $c = y_2 - y_1$ for some $y_1, y_2 \in \mathcal{R}$. Then, from the first and the third expressions in (8.0.1), we have

$$\begin{aligned} &(m_1 \odot x_1) - (m_1 \odot x_2) = y_2 - y_1 \\ \Longrightarrow &(m_1 \odot x_1) \oplus y_1 = (m_1 \odot x_2) \oplus y_2 = k_1 \quad \text{(say)} \\ \Longrightarrow &(x_1, y_1) \in [m_1, k_1] \text{ and } (x_2, y_2) \in [m_1, k_1]. \end{aligned} \tag{8.0.2}$$

We have utilised the fact $(\mathcal{R}, \oplus)$ is a group to change the sides of the terms in an equation and associate the addition in a chosen way to obtain the first implication. For the second implication, recall the PTR is linear i.e. $T(m, x, y) = (m \odot x) \oplus y$ for all $m, x, y \in \mathcal{R}$ and $(x, y) \in [m, k] \iff T(m, x, y) = k$.

Similarly, from the second and the third expressions in (8.0.1), we get

$$\begin{aligned} &(m_2 \odot x_1) - (m_2 \odot x_2) = y_2 - y_1 \\ \Longrightarrow\ &(m_2 \odot x_1) \oplus y_1 = (m_2 \odot x_2) \oplus y_2 = k_2 \quad \text{(say)} \end{aligned}$$

$$\Longrightarrow (x_1, y_1) \in [m_2, k_2] \text{ and } (x_2, y_2) \in [m_2, k_2]. \tag{8.0.3}$$

From equations (8.0.2) and (8.0.3), it follows $(x_1, y_1) \in [m_1, k_1] \cap [m_2, k_2]$ and $(x_2, y_2) \in [m_1, k_1] \cap [m_2, k_2]$. However, $m_1 \neq m_2$ so that the lines $[m_1, k_1]$ and $[m_2, k_2]$ are (necessarily) distinct. Since two distinct lines intersect at a unique point, we must have $(x_1, y_1) = (x_2, y_2) = [m_1, k_1] \cap [m_2, k_2]$. But then, $x_1 = x_2$, a contradiction. □

This result is a useful tool for the construction of a PTR multiplication table in an efficient way since $m_2 \odot x_2 \neq -m_1 \odot x_1 + m_1 \odot x_2 + m_2 \odot x_1$ eliminates an increasingly larger number of elements from possible values of $m_2 \odot x_2$ as more entries in the table are filled.

Next we describe how the various nuclei (Section 1.10.1) can be used to determine multiple entries in a multiplication table based on the entry at one position.

Lemma 8.0.2. *Let $\mathcal{P}$ be a projective plane and $(\mathcal{R}, T)$ a PTR obtained from a co-ordinatisation of $\mathcal{P}$. Consider the system $(\mathcal{R}, \oplus, \odot)$ with $\oplus$ and $\odot$ defined by $(\mathcal{R}, T)$. Let $\mathcal{N}_m, \mathcal{N}_l,$ and $\mathcal{N}_r$ respectively be the middle, left, and right nuclei of the PTR. Suppose for some $x, y \in \mathcal{R}$, $x \odot y = c$. Denoting by s^{-1} the inverse of an element s in a subgroup of $(\mathcal{R}^*, \odot)$, we have*

(i) $\forall\ s \in \mathcal{N}_m$, $(x \odot s^{-1}) \odot (s \odot y) = c$,

(ii) $\forall\ s \in \mathcal{N}_l$, $(s \odot x) \odot y = s \odot c$, *and*

(iii) $\forall\ s \in \mathcal{N}_r$, $x \odot (y \odot s) = c \odot s$. □

To appreciate the utility of the above lemma, for example, suppose that the left multiplication tables of the elements of the left nuclei $\mathcal{N}_l$ have been determined i.e. $s \odot x$ is known for all $s \in \mathcal{N}_l$ and all $x \in \mathcal{R}$. Suppose $|\mathcal{N}_m| = e$. Then with every new assignment $x \odot y = c$ for some pair $x, y \neq 0$, we can determine e number of entries in the PTR multiplication table. To see this, consider $(x \odot d) \odot (d^{-1} \odot y) = x \odot (d \odot (d^{-1} \odot y))$ for every $d \in \mathcal{N}_m$. Here, d^{-1} is the inverse of d in the group $(\mathcal{N}_m, \odot)$. Since $d^{-1} \in \mathcal{N}_m$ as well, we have $d \odot (d^{-1} \odot y) = (d \odot d^{-1}) \odot y = y$ so that $(x \odot d) \odot (d^{-1} \odot y) = x \odot y = c$.

The construction of a PTR multiplication table for planes of higher LB types like IVa or above is further aided by the use of properties like the distributive properties. For instance, in a left distributive PTR, if the values of $s \odot y$ are known for a given y and each $s \in \mathcal{S} \subset \mathcal{R}$, then setting $s \odot x = c$ for some $x \notin \mathcal{S}$ yields $s \odot (x \oplus y) = (s \odot x) \oplus (s \odot y) = c \oplus (s \odot y)$ for all $s \in \mathcal{S}$ assuming the PTR addition table is known.

8.1 Construction of Two Translation Planes of Order 16

Let $\mathcal{P}$ be a translation plane of order 16. So, there exists a PTR $(\mathcal{R}, T)$ representing $\mathcal{P}$ such that $(\mathcal{R}, \oplus, \odot)$ is a left quasifield.

Recalling the optimal coordinatising strategies of the translation planes of order 16 in Chapter 3, we know the additive loop $(\mathcal{R}, \oplus)$ is an elementary group and the addition can be fully optimised in an optimal PTR. We assume the PTR $(\mathcal{R}, T)$ being constructed comes from one such coordinatisation. Hence, the set $\mathcal{R}$ is the set $\mathbb{F}_{16}$ of the elements of the field $(\mathbb{F}_{16}, +, \cdot)$ and we replace the PTR addition $\oplus$ by $+$ in the remainder of this section.

Consider the multiplicative loop $(\mathcal{R}^*, \odot)$. From the results in Section 1.10.2, the principal homology groups of $\mathcal{P}$ correspond to subgroups of $(\mathcal{R}^*, \odot)$. Since the orders of homology groups of $\mathcal{P}$ must divide $n - 1 = 15$, the possible values are 1, 3, 5, or 15. From the classification of translation planes of order 16, we know there exist no planes of LB type IVa.2 or IVa.3 (proper nearfield planes) in order 16. See, for example, the discussion on the known planes of order 16 in Section 1.18. So, we rule out the value 15.

Next, we take a closer look at the possible orders of the three nuclei $\mathcal{N}_l, \mathcal{N}_m$, and $\mathcal{N}_r$. Since the PTR $(\mathcal{R}, T)$ is linear, by Theorem 1.10.8, the middle nucleus $\mathcal{N}_m$ equals the set $\mathcal{S}$ in the theorem. This means, the order of the middle nucleus $\mathcal{N}_m$ is the same as $|\,\Gamma((0),[0])\,|$ in an optimal coordinatisation. Similarly, by Theorem 1.10.9, the order of the left nucleus $\mathcal{N}_l$ equals $|\,\Gamma((\infty),[0,0])\,|$ as the PTR is left distributive too. Therefore, we consider the left and the middle nuclei of orders 1, 3, or 5 in our construction. As for the right nucleus, by Theorem 1.10.10, the minimum order of $\mathcal{N}_r$ (not including 0) is that of $\Gamma((0,0),[\infty])$. We cannot claim an equality unlike in the case of left nucleus because the PTR is not right distributive.

We now show the construction of two translation planes of order 16.

8.1.1 The Hall Plane

Suppose $\mathcal{P}$ admits a PTR $(\mathcal{R}, T)$ with $|\mathcal{N}_m| = |\mathcal{N}_l| = 5$, and $|\mathcal{N}_r| \geq 3$, if such plane exists. So, there exist subgroups $(\mathcal{N}_m, \odot)$ and $(\mathcal{N}_l, \odot)$ of $(\mathcal{R}^*, \odot)$ of order 5 and a subgroup $(\mathcal{S}, \odot)$ of $(\mathcal{R}^*, \odot)$ of order 3 such that $\mathcal{S} \subseteq \mathcal{N}_r$. The set $\mathcal{S}$ in this case is the set in Theorem 1.10.8. Since the order of the groups are prime, we must have $(\mathcal{N}_m, \odot) \cong C_5$, $(\mathcal{N}_l, \odot) \cong C_5$, and $(\mathcal{S}, \odot) \cong C_3$, where C_k is the cyclic group of order k. Therefore, we have the option to choose two (possibly identical) subsets of $\mathcal{R}^*$ of order 5 to identify as $\mathcal{N}_m$ and $\mathcal{N}_l$, and a subset of order 3 for $\mathcal{S} \subseteq \mathcal{N}_r$.

In a trial and error approach, we first let $\mathcal{N}_m = \mathcal{N}_l$. Given the coordinatising set $\mathcal{R}$ is the set $\mathbb{F}_{16}$, it is intuitive to let $\mathcal{N}_m = \mathcal{N}_l = \{1, a^3, a^6, a^9, a^{12}\}$, and $\mathcal{S} = \{1, a^5, a^{10}\}$. Up to isomorphism, there is only one multiplication table each for the two subsets of $\mathcal{R}$, as the subgroups are of prime order (cyclic). At this point, we can construct the following partial PTR multiplication tables:

- For any $x \in \mathcal{R}$, $0 \odot x = x \odot 0 = 0$ and $1 \odot x = x \odot 1 = x$.
- For all $0 \leq i, j \leq 4$, $a^{3i} \odot a^{3j} = a^{(3i+3j) \bmod 15}$. Note that $a^{15} = a^0 = 1$.
- For all $0 \leq i, j \leq 2$, $a^{5i} \odot a^{5j} = a^{(5i+5j) \bmod 15}$.

Table 8.1 shows the partial tables in an incomplete PTR multiplication table. All the entries so far coincide the multiplication in $(\mathbb{F}_{16}^*, \cdot)$.

	0	1	a^3	a^6	a^9	a^{12}	a	a^4	a^7	a^{10}	a^{13}	a^2	a^5	a^8	a^{11}	a^{14}
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	a^3	a^6	a^9	a^{12}	a	a^4	a^7	a^{10}	a^{13}	a^2	a^5	a^8	a^{11}	a^{14}
a^3	0	a^3	a^6	a^9	a^{12}	1										
a^6	0	a^6	a^9	a^{12}	1	a^3										
a^9	0	a^9	a^{12}	1	a^3	a^6										
a^{12}	0	a^{12}	1	a^3	a^6	a^9										
a	0	a														
a^4	0	a^4														
a^7	0	a^7														
a^{10}	0	a^{10}								a^5			1			
a^{13}	0	a^{13}														
a^2	0	a^2														
a^5	0	a^5								1			a^{10}			
a^8	0	a^8														
a^{11}	0	a^{11}														
a^{14}	0	a^{14}														

Table 8.1: Step I of Multiplication Table of a PTR with $\mathcal{N}_m \cong \mathcal{N}_l \cong C_5$, and $(\mathcal{S}, \odot) \cong C_3$, $\mathcal{S} \subseteq \mathcal{N}_r$.

The rows and the columns of the PTR multiplication table will be referred by the corresponding elements of $\mathcal{R}$. First, we apply the left distributive property of $(\mathcal{R}, \oplus, \odot)$ on the rows a^3, a^6, a^9, and a^{12} of Table 8.1. By theorem 1.10.10, the set $\mathcal{S} = \{1, a^5, a^{10}\} \subseteq \mathcal{D}_r$. That is, right distributive property $(x+y) \odot s = x \odot s + y \odot s$ holds whenever $s \in \mathcal{S}$. Hence, after applying the left distributive property as described above, we use the right distribution on the columns a^5 and a^{10}. We get (new entries determined in any step are colored dark gray):

	0	1	a^3	a^6	a^9	a^{12}	a	a^4	a^7	a^{10}	a^{13}	a^2	a^5	a^8	a^{11}	a^{14}
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	a^3	a^6	a^9	a^{12}	a	a^4	a^7	a^{10}	a^{13}	a^2	a^5	a^8	a^{11}	a^{14}
a^3	0	a^3	a^6	a^9	a^{12}	1	a^4	a^7	a^{10}	a^{13}	a	a^5	a^8	a^{11}	a^{14}	a^2
a^6	0	a^6	a^9	a^{12}	1	a^3	a^7	a^{10}	a^{13}	a	a^4	a^8	a^{11}	a^{14}	a^2	a^5
a^9	0	a^9	a^{12}	1	a^3	a^6	a^{10}	a^{13}	a	a^4	a^7	a^{11}	a^{14}	a^2	a^5	a^8
a^{12}	0	a^{12}	1	a^3	a^6	a^9	a^{13}	a	a^4	a^7	a^{10}	a^{14}	a^2	a^5	a^8	a^{11}
a	0	a	**c**							a^{11}			a^6			
a^4	0	a^4								a^{14}			a^9			
a^7	0	a^7								a^2			a^{12}			
a^{10}	0	a^{10}								a^5			1			
a^{13}	0	a^{13}								a^8			a^3			
a^2	0	a^2								a^{12}			a^7			
a^5	0	a^5								1			a^{10}			
a^8	0	a^8								a^3			a^{13}			
a^{11}	0	a^{11}								a^6			a			
a^{14}	0	a^{14}								a^9			a^4			

Table 8.2: After applying the left distribution where possible and right distribution on the columns of $\mathcal{S}$.

The table so far still coincides with the finite field multiplication table. In fact, it is deliberate since we seek to obtain a PTR whose polynomial representation is as nice as possible. We will later give an example of a table constructed in a more random way, and that represents the same plane, but has a PTR polynomial that gives very little information readily.

Having filled the entries of the table that come freely with the desired properties of the plane plus our choice of the elements in the subgroups of $(\mathcal{R}^*, \odot)$ associated with the principal homology groups, we proceed to fill the remaining entries in the table. Consider the entry $a \odot a^3$ (marked **c** in Table 8.2). The set of possible values of **c** is determined using Theorem 8.0.1: $\mathbf{c} \notin \{-m_1 \odot x_1 + m_1 \odot a^3 + a \odot x_1\}$ for $m_1 \in \{0, 1, a^3, a^6, a^9, a^{12}\}$ and $x_1 \in \{0, 1\}$. This leaves $\mathbf{c} \in \{a^{13}, a^4\}$.

Suppose we choose $a \odot a^3 = a^{13}$. Based on this choice, we can compute a number of other entries using Lemma 8.0.2 and the left distributivity:

- $a^4 \odot a^3 = (a^3 \odot a) \odot a^3 = a^3 \odot (a \odot a^3) = a^3 \odot a^{13} = a \quad [\because a^3 \in \mathcal{N}_l\ ]$,
- $a \odot a^{12} = a \odot (a^3 + a^{10}) = a \odot a^3 + a \odot a^{10} = a^{13} + a^{11} = a^4$

  [$\because$ Left distributivity],
- $a \odot a^{13} = a \odot (a^3 \odot a^{10}) = (a \odot a^3) \odot a^{10} = a^{13} \odot a^{10} = a^8 \quad [\because a^3 \in \mathcal{N}_m\ ]$,

and so on. When all possible entries in the PTR multiplication table are updated using the above mentioned results, we get

	0	1	a^3	a^6	a^9	a^{12}	a	a^4	a^7	a^{10}	a^{13}	a^2	a^5	a^8	a^{11}	a^{14}
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	a^3	a^6	a^9	a^{12}	a	a^4	a^7	a^{10}	a^{13}	a^2	a^5	a^8	a^{11}	a^{14}
a^3	0	a^3	a^6	a^9	a^{12}	1	a^4	a^7	a^{10}	a^{13}	a	a^5	a^8	a^{11}	a^{14}	a^2
a^6	0	a^6	a^9	a^{12}	1	a^3	a^7	a^{10}	a^{13}	a	a^4	a^8	a^{11}	a^{14}	a^2	a^5
a^9	0	a^9	a^{12}	1	a^3	a^6	a^{10}	a^{13}	a	a^4	a^7	a^{11}	a^{14}	a^2	a^5	a^8
a^{12}	0	a^{12}	1	a^3	a^6	a^9	a^{13}	a	a^4	a^7	a^{10}	a^{14}	a^2	a^5	a^8	a^{11}
a	0	a	a^{13}	a^{10}	a^7	a^4	a^5	a^2	a^{14}	a^{11}	a^8	a^9	a^6	a^3	1	a^{12}
a^4	0	a^4	a	a^{13}	a^{10}	a^7	a^8	a^5	a^2	a^{14}	a^{11}	a^{12}	a^9	a^6	a^3	1
a^7	0	a^7	a^4	a	a^{13}	a^{10}	a^{11}	a^8	a^5	a^2	a^{14}	1	a^{12}	a^9	a^6	a^3
a^{10}	0	a^{10}	a^7	a^4	a	a^{13}	a^{14}	a^{11}	a^8	a^5	a^2	a^3	1	a^{12}	a^9	a^6
a^{13}	0	a^{13}	a^{10}	a^7	a^4	a	a^2	a^{14}	a^{11}	a^8	a^5	a^6	a^3	1	a^{12}	a^9
a^2	0	a^2	$\mathbf{c}'$							a^{12}			a^7			
a^5	0	a^5								1			a^1			
a^8	0	a^8								a^3			a^{13}			
a^{11}	0	a^{11}								a^6			a			
a^{14}	0	a^{14}								a^9			a^4			

Table 8.3: New entries based on $a \odot a^3 = a^{13}$.

There is also a different interpretation for the part of the Table 8.3 filled in at this step. Based on the choice $a \odot a^3 = a^{13}$, and the previously known entries in the table, the row $a \odot y$ for $y \in \mathcal{R}$ can be completely determined first (using Lemma 8.0.2 and the left distributivity). That is:

y	0	1	a^3	a^6	a^9	a^{12}	a	a^4	a^7	a^{10}	a^{13}	a^2	a^5	a^8	a^{11}	a^{14}
$a \odot y$	0	a	a^{13}	a^{10}	a^7	a^4	a^5	a^2	a^{14}	a^{11}	a^8	a^9	a^6	a^3	1	a^{12}

Now, the middle nucleus is $\mathcal{N}_m = \{1, a^3, a^6, a^9, a^{12}\}$. We have discussed earlier that $(\mathcal{N}_m, \odot) \cong \Gamma((0), [0])$. Consider the orbit of the point (a) on the line at infinity under the action of $\Gamma((0), [0])$. Let $s \in \mathcal{N}_m$. Then, there is a homology $\gamma_s \in \Gamma((0), [0])$ such that $(1, 0)^{\gamma_s} = (s, 0)$. Let $(a)^{\gamma_s} = (m)$. Take the equation $a \odot 1 = a$ which is equivalent to $\overline{(a)(1, 0)} \cap [0] = (0, a)$. Since γ_s fixes $[0]$, we get $\overline{(a)(1, 0)}^{\gamma_s} \cap [0] = (0, a) \iff \overline{(a)^{\gamma_s}(1, 0)^{\gamma_s}} \cap [0] = (0, a) \iff \overline{(m)(s, 0)} \cap [0] = (0, a)$ which is equivalent to $m \odot s = a$. Multiplying by s^{-1} on the right, we have $(m \odot s) \odot s^{-1} = a \odot s^{-1}$. Here, s^{-1} is the inverse of $s \in \mathcal{R}$ in $(\mathbb{F}_{16}^*, \cdot)$. By our choice of the multiplication in $(\mathcal{N}_m, \odot)$, the inverse of $s \in \mathcal{N}_m$ in $(\mathbb{F}_{16}^*, \cdot)$ is the same as its inverse in the group $(\mathcal{N}_m, \odot)$. Finally, since $s \in \mathcal{N}_m$, by definition, $(m \odot s) \odot s^{-1} = m \odot (s \odot s^{-1}) = m$. Therefore, using the notation $\Gamma(\mathbf{p}, \mathcal{L})(\mathbf{s})$ to denote the orbit of the point $\mathbf{s}$ under the action of the central collineation group $\Gamma(\mathbf{p}, \mathcal{L})$, we have

$$\begin{aligned}\Gamma((0), [0])\,((a)) &= \left\{\left(a \odot x^{-1}\right)\ : x \in \mathcal{N}_m\right\}\\ &= \left\{(a), \left(a^4\right), \left(a^7\right), \left(a^{10}\right), \left(a^{13}\right)\right\}.\end{aligned}$$

We observe that in Table 8.3, the multiplication tables of the elements corresponding to the $\Gamma((0), [0])$ orbit of (a) have been determined. In fact,

$$\forall s \in \mathcal{N}_m,\ y \in \mathcal{R},\quad (a \odot s) \odot y = a \odot (s \odot y). \tag{8.1.1}$$

Since the tables for $s \odot y$ i.e. the values $s \in \mathcal{N}_m, y \in \mathcal{R}$ are known, (8.1.1) yields the tables for $a \odot s$ for each $s \in \mathcal{N}_m$.

Proceeding with the construction, again by using Theorem 8.0.1 we determine $\mathbf{c}' = a^2 \odot a^3 \in \{a^{14}, a^5\}$. Suppose we choose $a^2 \odot a^3 = a^5$. The rest follows as before and we obtain the complete PTR multiplication table.

	0	1	a^3	a^6	a^9	a^{12}	a	a^4	a^7	a^{10}	a^{13}	a^2	a^5	a^8	a^{11}	a^{14}
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	a^3	a^6	a^9	a^{12}	a	a^4	a^7	a^{10}	a^{13}	a^2	a^5	a^8	a^{11}	a^{14}
a^3	0	a^3	a^6	a^9	a^{12}	1	a^4	a^7	a^{10}	a^{13}	a	a^5	a^8	a^{11}	a^{14}	a^2
a^6	0	a^6	a^9	a^{12}	1	a^3	a^7	a^{10}	a^{13}	a	a^4	a^8	a^{11}	a^{14}	a^2	a^5
a^9	0	a^9	a^{12}	1	a^3	a^6	a^{10}	a^{13}	a	a^4	a^7	a^{11}	a^{14}	a^2	a^5	a^8
a^{12}	0	a^{12}	1	a^3	a^6	a^9	a^{13}	a	a^4	a^7	a^{10}	a^{14}	a^2	a^5	a^8	a^{11}
a	0	a	a^{13}	a^{10}	a^7	a^4	a^5	a^2	a^{14}	a^{11}	a^8	a^9	a^6	a^3	1	a^{12}
a^4	0	a^4	a	a^{13}	a^{10}	a^7	a^8	a^5	a^2	a^{14}	a^{11}	a^{12}	a^9	a^6	a^3	1
a^7	0	a^7	a^4	a	a^{13}	a^{10}	a^{11}	a^8	a^5	a^2	a^{14}	1	a^{12}	a^9	a^6	a^3
a^{10}	0	a^{10}	a^7	a^4	a	a^{13}	a^{14}	a^{11}	a^8	a^5	a^2	a^3	1	a^{12}	a^9	a^6
a^{13}	0	a^{13}	a^{10}	a^7	a^4	a	a^2	a^{14}	a^{11}	a^8	a^5	a^6	a^3	1	a^{12}	a^9
a^2	0	a^2	a^5	a^8	a^{11}	a^{14}	a^3	a^6	a^9	a^{12}	1	a^4	a^7	a^{10}	a^{13}	a
a^5	0	a^5	a^8	a^{11}	a^{14}	a^2	a^6	a^9	a^{12}	1	a^3	a^7	a^{10}	a^{13}	a	a^4
a^8	0	a^8	a^{11}	a^{14}	a^2	a^5	a^9	a^{12}	1	a^3	a^6	a^{10}	a^{13}	a	a^4	a^7
a^{11}	0	a^{11}	a^{14}	a^2	a^5	a^8	a^{12}	1	a^3	a^6	a^9	a^{13}	a	a^4	a^7	a^{10}
a^{14}	0	a^{14}	a^2	a^5	a^8	a^{11}	1	a^3	a^6	a^9	a^{12}	a	a^4	a^7	a^{10}	a^{13}

Table 8.4: New entries based on $a^2 \odot a^3 = a^5$.

The system $(\mathcal{R}, \oplus, \odot)$ is completely determined. By using magma software to test for isomorphism, it was determined the corresponding projective plane $\mathcal{P}$ is the Hall plane of order 16. We use polynomial interpolation with $x \odot y$ values from Table 8.4 and $x + y$ values from 1.6 to obtain the following polynomial in $\mathbb{F}_{16}[X, Y]$ representing the multiplication $X \odot Y$.

$$X \odot Y = a^5 X^{11} Y^4 - a^5 X^{11} Y + a^{10} X^6 Y^4 - a^{10} X^6 Y + XY^4$$

$$\implies X \odot Y = XY - a^5 X^6 \left(X^5 - a^5\right) \left(Y^4 - Y\right).$$

Together with the usual field (or quasifield) addition, we now have a PTR representation of the Hall plane of order 16:

$$T(X, Y, Z) = XY - a^5 X^6 \left(X^5 - a^5\right) \left(Y^4 - Y\right) + Z.$$

We end the discussion on this construction of a PTR multiplication table with some notes.

- The number of choices of $\mathbf{c}$ and $\mathbf{c}'$ is two each. It turns out choosing the alternative in the first entry again gives rise to two choices for the second entry. Thus, there are total four distinct completions of the multiplication table under the assumptions of the principal homology groups in this construction that are PTR multiplication tables of a projective plane.

- Of the four tables, one is $(\mathbb{F}_{16}, \cdot)$, and corresponds to the Desarguesian plane. The other three correspond to the Hall plane.

- The PTR multiplication polynomials for the Hall planes obtained from the other two tables are:

$$X \odot Y = XY - a^{10} X^6 \left(X^5 - a^{10}\right) \left(Y^4 - Y\right), \text{ and}$$
$$X \odot Y = XY - X^6 \left(X^5 - 1\right) \left(Y^4 - Y\right).$$

- As mentioned at the beginning of this construction, we can choose different subsets that form the nuclei (of appropriate orders though) and construct a table as above. Here is a PTR polynomial obtained from a different set of nuclei:

$$\begin{aligned}X \odot Y = a^{10}X^{11}Y^4 + a^{10}X^{11}Y + a^5X^9Y^4 + a^5X^9Y + a^5X^6Y^4 \\ + a^5X^6Y + a^{10}X^5Y^4 + a^{10}X^5Y + X^4Y^4 + X^4Y \\ + a^5X^3Y^4 + a^5X^3Y + a^{10}X^2Y^4 + a^{10}X^2Y + XY + Z,\end{aligned}$$

with $\mathcal{N}_m = \mathcal{N}_l = \{1, a^{11}, a^{13}, a^{14}, a^7\}$, and $\mathcal{N}_r = \{1, a^5, a^{10}\}$.

- The different systems $(\mathcal{R}, \oplus, \odot)$ that give the above PTR polynomials are different isotopes of the Hall quasifield.

- It is not necessary that every PTR $(\mathcal{R}, T)$ of a plane have the nuclei of same orders. For example, when a plane of LB type I.2 or above is coordinatised to obtain a linear PTR, the order of the middle nucleus depends on the choice of the coordinatising frame $\{\mathbf{y}, \mathbf{x}, \mathbf{O}, \mathbf{u}, \mathbf{J}\}$ as we have $|\mathcal{N}_m| = |\,\Gamma((0), [0])\,|$.

- In the case of translation planes, the lower bound for the order of the right nucleus $\mathcal{N}_r$ is an isotope invariant. This follows easily from, for example, Lemma 2.4.1 and Theorem 1.10.10.

- All PTRs we obtained for the Hall plane of order 16 have a right nucleus $\mathcal{N}_r$ of order 3. However, the middle and left nuclei are either trivial or of order 5. This follows from Theorems 1.10.10 and 1.10.9 and the fact that in the Hall plane, all nontrivial homologies with centers on the translation line are of order 5. Recall the description the central collineation groups of the Hall plane in Section 3.4.1.

- Below, we give a PTR multiplication table of the Hall plane with trivial $\mathcal{N}_m$ and $\mathcal{N}_l$. The corresponding PTR polynomial is also given.

The following multiplication table of a PTR of the Hall plane has a center of order 4. Specifically, the center (including 0) coincides with the right nucleus, both being equal to $\mathbb{F}_4$.

1	0	1	a	a^2	a^3	a^4	a^5	a^6	a^7	a^8	a^9	a^{10}	a^{11}	a^{12}	a^{13}	a^{14}
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	a	a^2	a^3	a^4	a^5	a^6	a^7	a^8	a^9	a^{10}	a^{11}	a^{12}	a^{13}	a^{14}
a	0	a	a^8	a^{14}	a^2	a^{10}	a^6	a^{13}	a^4	a^7	1	a^{11}	a^3	a^9	a^{12}	a^5
a^2	0	a^2	a^3	a^4	a^5	a^6	a^7	a^8	a^9	a^{10}	a^{11}	a^{12}	a^{13}	a^{14}	1	a
a^3	0	a^3	a^{14}	a^6	a^{12}	1	a^8	a^4	a^{11}	a^2	a^5	a^{13}	a^9	a	a^7	a^{10}
a^4	0	a^4	a^{10}	a^{13}	a^6	a^2	a^9	1	a^3	a^{11}	a^7	a^{14}	a^5	a^8	a	a^{12}
a^5	0	a^5	a^6	a^7	a^8	a^9	a^{10}	a^{11}	a^{12}	a^{13}	a^{14}	1	a	a^2	a^3	a^4
a^6	0	a^6	a^2	a^9	1	a^3	a^{11}	a^7	a^{14}	a^5	a^8	a	a^{12}	a^4	a^{10}	a^{13}
a^7	0	a^7	a^{13}	a	a^9	a^5	a^{12}	a^3	a^6	a^{14}	a^{10}	a^2	a^8	a^{11}	a^4	1
a^8	0	a^8	a^9	a^{10}	a^{11}	a^{12}	a^{13}	a^{14}	1	a	a^2	a^3	a^4	a^5	a^6	a^7
a^9	0	a^9	a^{12}	a^5	a	a^8	a^{14}	a^2	a^{10}	a^6	a^{13}	a^4	a^7	1	a^{11}	a^3
a^{10}	0	a^{10}	a^{11}	a^{12}	a^{13}	a^{14}	1	a	a^2	a^3	a^4	a^5	a^6	a^7	a^8	a^9
a^{11}	0	a^{11}	a^4	1	a^7	a^{13}	a	a^9	a^5	a^{12}	a^3	a^6	a^{14}	a^{10}	a^2	a^8
a^{12}	0	a^{12}	1	a^8	a^4	a^{11}	a^2	a^5	a^{13}	a^9	a	a^7	a^{10}	a^3	a^{14}	a^6
a^{13}	0	a^{13}	a^5	a^{11}	a^{14}	a^7	a^3	a^{10}	a	a^4	a^{12}	a^8	1	a^6	a^9	a^2
a^{14}	0	a^{14}	a^7	a^3	a^{10}	a	a^4	a^{12}	a^8	1	a^6	a^9	a^2	a^{13}	a^5	a^{11}

Table 8.5: A PTR Multiplication Table of the Hall Plane with $\mathcal{Z} = \{0, 1, a^5, a^{10}\}$.

The multiplication polynomial:

$$X \odot Y = XY - \left(Y^4 - Y\right)\left(X^4 - X\right)^2\left(X^2 + X + a^{10}\right)$$
$$= XY - t_4(Y)\,t_4(X)^2\left(h_2(X) + a^5\right).$$

From the polynomial representation it is clear the multiplication coincides with the field multiplication when either x or y is in the center (or right nucleus).

8.1.2 The DSFP Plane

Next we show the steps of construction of the PTR multiplication table of another left quasifield plane. Suppose a translation plane $\mathcal{P}$ of order 16 exists such that $\mathcal{P}$ admits a PTR $(\mathcal{R}, T)$ with $|\mathcal{N}_l| = 3$, while the middle and right nuclei are trivial. Let $\mathcal{N}_l = \{1, a^5, a^{10}\}$. The steps of construction of the multiplication table are given below.

Initial multiplication table with $(\mathcal{N}_l, \odot)$ in $(\mathcal{R}^*, \odot)$:

	0	1	a^5	a^{10}	a	a^6	a^{11}	a^2	a^7	a^{12}	a^3	a^8	a^{13}	a^4	a^9	a^{14}
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	a^5	a^{10}	a	a^6	a^{11}	a^2	a^7	a^{12}	a^3	a^8	a^{13}	a^4	a^9	a^{14}
a^5	0	a^5	a^{10}	1	$\mathbf{c}_1$											
a^{10}	0	a^{10}	1	a^5												
a	0	a														
a^6	0	a^6														
a^{11}	0	a^{11}														
a^2	0	a^2														
a^7	0	a^7														
a^{12}	0	a^{12}														
a^3	0	a^3														
a^8	0	a^8														
a^{13}	0	a^{13}														
a^4	0	a^4														
a^9	0	a^9														
a^{14}	0	a^{14}														

Table 8.6: Initial Multiplication Table with $(\{1, a^5, a^{10}\}, \odot) \cong C_3$ in $(\mathcal{R}^*, \odot)$.

Possible values of $\mathbf{c}_1 = a^5 \odot a$ obtained by applying Theorem 8.0.1 are

$$\mathbf{c}_1 \in \{a^{11}, a^{12}, a^{13}, a^3, a^{14}, a^6, a^7, a^9\}.$$

Suppose we choose: $\mathbf{c}_1 = a^5 \odot a = a^6$.

New entries in the table are determined by virtue of left distribution:

$a^5 \odot (s + a) = (a^5 \odot s) + (a^5 \odot a) = a^5 s + a^6$ for $s \in \mathcal{N}_l = \mathbb{F}_4$.

	0	1	a^5	a^{10}	a	a^6	a^{11}	a^2	a^7	a^{12}	a^3	a^8	a^{13}	a^4	a^9	a^{14}
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	a^5	a^{10}	a	a^6	a^{11}	a^2	a^7	a^{12}	a^3	a^8	a^{13}	a^4	a^9	a^{14}
a^5	0	a^5	a^{10}	1	a^6	$\mathbf{c}_2$		a^7				a^{13}		a^9		
a^{10}	0	a^{10}	1	a^5												
a	0	a														
a^6	0	a^6														
a^{11}	0	a^{11}														
a^2	0	a^2														
a^7	0	a^7														
a^{12}	0	a^{12}														
a^3	0	a^3														
a^8	0	a^8														
a^{13}	0	a^{13}														
a^4	0	a^4														
a^9	0	a^9														
a^{14}	0	a^{14}														

Table 8.7: Multiplication Table after Choosing $a^5 \odot a = a^6$ and Updating with Left Distribution.

Possible values of $\mathbf{c}_2$ are $\{a^{11}, a^{12}, a^3, a^{14}\}$.

Value chosen: $\mathbf{c}_2 = a^5 \odot a^6 = a^{11}$.

In addition to the left distribution, new entries determined by virtue of the left nucleus $\mathcal{N}_l$ as: $a^{10} \odot a = (a^5 \odot a^5) \odot a = a^5 \odot (a^5 \odot a) = a^5 \odot a^6 = a^{11}$ and so on.

	0	1	a^5	a^{10}	a	a^6	a^{11}	a^2	a^7	a^{12}	a^3	a^8	a^{13}	a^4	a^9	a^{14}
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	a^5	a^{10}	a	a^6	a^{11}	a^2	a^7	a^{12}	a^3	a^8	a^{13}	a^4	a^9	a^{14}
a^5	0	a^5	a^{10}	1	a^6	a^{11}	a	a^7	a^{12}	a^2	a^8	a^{13}	a^3	a^9	a^{14}	a^4
a^{10}	0	a^{10}	1	a^5	a^{11}	a	a^6	a^{12}	a^2	a^7	a^{13}	a^3	a^8	a^{14}	a^4	a^9
a	0	a	$\mathbf{c}_3$													
a^6	0	a^6														
a^{11}	0	a^{11}														
a^2	0	a^2														
a^7	0	a^7														
a^{12}	0	a^{12}														
a^3	0	a^3														
a^8	0	a^8														
a^{13}	0	a^{13}														
a^4	0	a^4														
a^9	0	a^9														
a^{14}	0	a^{14}														

Table 8.8: Multiplication Table after Choosing $a^5 \odot a^6 = a^{11}$ and Updating with Left Distribution and Applying Left Nucleus Property.

Possible values of $\mathbf{c}_3$ are $\{a^{12}, a^{13}, a^4, a^5, a^{10}\}$.

Value chosen: $\mathbf{c}_3 = a \odot a^5 = a^{10}$.

After two more steps, we have the following table.

	0	1	a^5	a^{10}	a	a^6	a^{11}	a^2	a^7	a^{12}	a^3	a^8	a^{13}	a^4	a^9	a^{14}
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	a^5	a^{10}	a	a^6	a^{11}	a^2	a^7	a^{12}	a^3	a^8	a^{13}	a^4	a^9	a^{14}
a^5	0	a^5	a^{10}	1	a^6	a^{11}	a	a^7	a^{12}	a^2	a^8	a^{13}	a^3	a^9	a^{14}	a^4
a^{10}	0	a^{10}	1	a^5	a^{11}	a	a^6	a^{12}	a^2	a^7	a^{13}	a^3	a^8	a^{14}	a^4	a^9
a	0	a	a^{11}	a^6	a^{10}	a^5	1	a^{14}	a^9	a^4	a^{12}	a^7	a^2	a^8	a^3	a^{13}
a^6	0	a^6	a	a^{11}	1	a^{10}	a^5	a^4	a^{14}	a^9	a^2	a^{12}	a^7	a^{13}	a^8	a^3
a^{11}	0	a^{11}	a^6	a	a^5	1	a^{10}	a^9	a^4	a^{14}	a^7	a^2	a^{12}	a^3	a^{13}	a^8
a^2	0	a^2														
a^7	0	a^7														
a^{12}	0	a^{12}														
a^3	0	a^3														
a^8	0	a^8														
a^{13}	0	a^{13}														
a^4	0	a^4														
a^9	0	a^9														
a^{14}	0	a^{14}														

Table 8.9: Multiplication Table at the Completion of 7 Rows.

The table is being filled in orbits of elements determined by the action of the homology group associated with the left nucleus, i.e. $(\mathcal{N}_l, \odot) \cong \Gamma((\infty), [0,0])$. In fact, for any $x, y \in \mathcal{R}, s \in \mathcal{N}_l$, we have $(s \odot x_0) \odot y = s \odot (x_0 \odot y)$.

Hence the tables for $s \odot x$ are determined for every $s \in \mathcal{N}_l$ once the table for x is known i.e. $x \odot y$ is known for all $y \in \mathcal{R}$. Continuing this way, and making choices from available values at various positions in the table, many PTR multiplication tables are obtained. One such completed multiplication table is given below.

	0	1	a^5	a^{10}	a	a^6	a^{11}	a^2	a^7	a^{12}	a^3	a^8	a^{13}	a^4	a^9	a^{14}
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	a^5	a^{10}	a	a^6	a^{11}	a^2	a^7	a^{12}	a^3	a^8	a^{13}	a^4	a^9	a^{14}
a^5	0	a^5	a^{10}	1	a^6	a^{11}	a	a^7	a^{12}	a^2	a^8	a^{13}	a^3	a^9	a^{14}	a^4
a^{10}	0	a^{10}	1	a^5	a^{11}	a	a^6	a^{12}	a^2	a^7	a^{13}	a^3	a^8	a^{14}	a^4	a^9
a	0	a	a^{11}	a^6	a^{10}	a^5	1	a^{14}	a^9	a^4	a^{12}	a^7	a^2	a^8	a^3	a^{13}
a^6	0	a^6	a	a^{11}	1	a^{10}	a^5	a^4	a^{14}	a^9	a^2	a^{12}	a^7	a^{13}	a^8	a^3
a^{11}	0	a^{11}	a^6	a	a^5	1	a^{10}	a^9	a^4	a^{14}	a^7	a^2	a^{12}	a^3	a^{13}	a^8
a^2	0	a^2	a^{12}	a^7	a^3	a^{13}	a^8	a^{10}	a^5	1	a^9	a^4	a^{14}	a^6	a	a^{11}
a^7	0	a^7	a^2	a^{12}	a^8	a^3	a^{13}	1	a^{10}	a^5	a^{14}	a^9	a^4	a^{11}	a^6	a
a^{12}	0	a^{12}	a^7	a^2	a^{13}	a^8	a^3	a^5	1	a^{10}	a^4	a^{14}	a^9	a	a^{11}	a^6
a^3	0	a^3	a^{13}	a^8	a^4	a^{14}	a^9	a^{11}	a^6	a	a^{10}	a^5	1	a^7	a^2	a^{12}
a^8	0	a^8	a^3	a^{13}	a^9	a^4	a^{14}	a	a^{11}	a^6	1	a^{10}	a^5	a^{12}	a^7	a^2
a^{13}	0	a^{13}	a^8	a^3	a^{14}	a^9	a^4	a^6	a	a^{11}	a^5	1	a^{10}	a^2	a^{12}	a^7
a^4	0	a^4	a^{14}	a^9	a^2	a^{12}	a^7	a^{13}	a^8	a^3	a	a^{11}	a^6	a^{10}	a^5	1
a^9	0	a^9	a^4	a^{14}	a^7	a^2	a^{12}	a^3	a^{13}	a^8	a^6	a	a^{11}	1	a^{10}	a^5
a^{14}	0	a^{14}	a^9	a^4	a^{12}	a^7	a^2	a^8	a^3	a^{13}	a^{11}	a^6	a	a^5	1	a^{10}

Table 8.10: A Completed PTR Multiplication Table.

The PTR multiplication polynomial obtained from the above table is

$$X \odot Y = XY + Xt_4\left(X\right)^3 t_2\left(Y\right) + \left(X^3 - a^{10}\right) t_4\left(X\right) t_4\left(Y\right)^2 .$$

By using isomorphism test in magma, it is found the corresponding plane is the derived semifield plane (DSFP) of order 16.

8.2 Construction of the LB type II Planes

In this section, we consider the construction of PTR multiplication tables for a type of strict Cartesian groups. A Cartesian group is a linear PTR with an associative addition but no distributive property. The additive loop $(\mathcal{R}, \oplus)$ is a group in a Cartesian group $(\mathcal{R}, T)$. We will consider the case where the group is elementary abelian. Further, we make no assumption of a transitive homology group in the corresponding plane. So, the multiplicative loop $(\mathcal{R}^*, \odot)$ is not assumed to be a group. This means, we are aiming to construct PTRs of planes of LB type II.1 or above. To clarify this, we note that even though no assumption is made regarding the group structure of the multiplicative loop, it is possible some PTRs of the planes obtained from this construction method admit such structure. Geometrically, the planes may admit homology groups of various orders including transitive homology groups. Also, we recall Remark 1.16.4 in this context.

Firstly, a full left or right distribution can no longer be applied to the construction. We rely on other results that relate to non-transitive principal elation or homology groups. See, for example, the results in Section 1.10.2.

As an illustration, we will show the construction of a PTR multiplication table of a LB type at least II.1 plane of order 16. We assume the plane admits no homologies and so, we cannot assume any non-trivial nuclei guaranteed by a theorem in Section 1.10.2. We assume a left middle distributor $\mathcal{D}$ and a right middle distributor $\mathcal{D}'$ of orders 2 each. For convenience of computation, let $\mathcal{D} = \mathcal{D}' = \{0, 1\}$. The result of this assumption is that we have the following relation hold for all $x, y \in \mathcal{R}$, $u, v \in \mathbb{F}_2$, we have

$$(x + u) \odot (v + y) = xv + (x \odot y) + uv + uy \tag{8.2.1}$$

$$= uv + xv + uy + (x \odot y).$$

Note the equation uses the assumption of an elementary abelian addition. Clearly, the hypothesis we have here closely relates to the observations made in the coordinatisation of the Mathon plane in Section 3.9. We repeat this exercise is intended to show the method of construction and we have deliberately chosen to illustrate an example of a known plane.

The usefulness of (8.2.1) in constructing a PTR multiplication table is that assigning the value of the product $x \odot y$ to one ordered pair (x, y) yields the products for a total of four orders pairs viz. $(x, y), (1 + x, y), (x, 1 + y)$, and $(1 + x, 1 + y)$. In a visual sense, we can fill the PTR multiplication table 2 by 2 blocks at a time. Of course, this is under the assumption the field is of even order and no non-trivial homologies are assumed.

Finally, if we assume a subplane of a certain type like a Desarguesian subplane exists in a plane, and there exists a coordinatisation of the plane such that the subplane is coordinatised optimally, then we can incorporate the information in the PTR multiplication table at the beginning. For instance, if we assume that the plane of order 16 we want to construct admits a subplane of order 4, then we can embed the table of $(\mathbb{F}_4^*, \cdot)$ in the PTR multiplication table of the plane of order 16. See Table 8.11. We now give the first two steps of the construction and the final PTR multiplication table of the Mathon plane:

	0	1	a^5	a^{10}	a	a^4	a^2	a^8	a^3	a^{14}	a^{11}	a^{12}	a^9	a^7	a^6	a^{13}
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	a^5	a^{10}	a	a^4	a^2	a^8	a^3	a^{14}	a^{11}	a^{12}	a^9	a^7	a^6	a^{13}
a^5	0	a^5	a^{10}	1	$\mathbf{c}$											
a^{10}	0	a^{10}	1	a^5												
a	0	a														
a^4	0	a^4														
a^2	0	a^2														
a^8	0	a^8														
a^3	0	a^3														
a^{14}	0	a^{14}														
a^{11}	0	a^{11}														
a^{12}	0	a^{12}														
a^9	0	a^9														
a^7	0	a^7														
a^6	0	a^6														
a^{13}	0	a^{13}														

Table 8.11: Initial PTR Multiplication Table with a Sub-PTR $\cong (\mathbb{F}_4, +, \cdot)$.

From Theorem 8.0.1, the possible values of $\mathbf{c}$ are $\{a^3, a^{14}, a^{11}, a^{12}, a^9, a^7, a^6, a^{13}\}$. We choose $\mathbf{c} = a^5 \odot a = a^{11}$. Using (8.2.1), we determine the values of $a^{10} \odot a$, $a^5 \odot a^4$, and $a^{10} \odot a^4$ as well. For example,

$$\begin{aligned} a^{10} \odot a^4 &= (1 + a^5) \odot (1 + a) \\ &= 1 + a + a^5 + (a^5 \odot a) \end{aligned}$$

$$= a^4 + a^5 + a^{11}$$

$$= a^8 + a^{11}$$

$$= a^7.$$

The resulting table:

	0	1	a^5	a^{10}	a	a^4	a^2	a^8	a^3	a^{14}	a^{11}	a^{12}	a^9	a^7	a^6	a^{13}
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	a^5	a^{10}	a	a^4	a^2	a^8	a^3	a^{14}	a^{11}	a^{12}	a^9	a^7	a^6	a^{13}
a^5	0	a^5	a^{10}	1	a^{11}	a^3	$\mathbf{c}_1$									
a^{10}	0	a^{10}	1	a^5	a^6	a^7										
a	0	a														
a^4	0	a^4														
a^2	0	a^2														
a^8	0	a^8														
a^3	0	a^3														
a^{14}	0	a^{14}														
a^{11}	0	a^{11}														
a^{12}	0	a^{12}														
a^9	0	a^9														
a^7	0	a^7														
a^6	0	a^6														
a^{13}	0	a^{13}														

Table 8.12: PTR Multiplication Table after Assigning Value to $a^5 \odot a$.

Continuing this way, many PTR multiplication tables are obtained. Among the tables, the ones which are neither left distributive nor right distributive correspond to a plane of LB type II.1 which we have found to be always either the Mathon plane or dual Mathon plane. We have verified this exhaustively. One such table and the corresponding PTR polynomial are given next.

	0	1	a^5	a^{10}	a	a^4	a^2	a^8	a^3	a^{14}	a^{11}	a^{12}	a^9	a^7	a^6	a^{13}
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	a^5	a^{10}	a	a^4	a^2	a^8	a^3	a^{14}	a^{11}	a^{12}	a^9	a^7	a^6	a^{13}
a^5	0	a^5	a^{10}	1	a^{11}	a^3	a^{14}	a^{12}	a^6	a^9	a^7	a^{13}	a	a^2	a^8	a^4
a^{10}	0	a^{10}	1	a^5	a^6	a^7	a^{13}	a^9	a^2	a^4	a^8	a	a^3	a^{12}	a^{14}	a^{11}
a	0	a	a^{12}	a^{13}	a^2	a^{10}	a^7	a^3	a^{14}	a^6	a^5	a^4	1	a^8	a^{11}	a^9
a^4	0	a^4	a^{14}	a^9	a^{10}	a^8	a^{11}	a^6	a^5	a	a^{12}	a^7	a^{13}	a^3	a^2	1
a^2	0	a^2	a^3	a^6	a^9	a^{12}	a	a^{10}	a^8	a^5	a^{13}	a^{11}	a^4	1	a^7	a^{14}
a^8	0	a^8	a^{11}	a^7	a^{14}	a^{13}	a^{10}	a^4	a^9	a^3	a^2	a^5	a^{12}	a^6	1	a
a^3	0	a^3	a^{13}	a^8	1	a^{14}	a^6	a^2	a^{12}	a^{10}	a	a^9	a^{11}	a^5	a^4	a^7
a^{14}	0	a^{14}	a^7	a	a^4	a^9	a^3	1	a^{10}	a^{11}	a^6	a^8	a^2	a^{13}	a^{12}	a^5
a^{11}	0	a^{11}	a^9	a^2	a^{12}	1	a^8	a^7	a^4	a^{13}	a^{14}	a^{10}	a^6	a	a^5	a^3
a^{12}	0	a^{12}	a^6	a^4	a^{13}	a	1	a^{11}	a^7	a^2	a^{10}	a^3	a^5	a^{14}	a^9	a^8
a^9	0	a^9	a^4	a^{14}	a^8	a^{11}	a^5	a^{13}	a	a^{12}	1	a^6	a^7	a^{10}	a^3	a^2
a^7	0	a^7	a^8	a^{11}	a^5	a^6	a^4	a^{14}	a^{13}	1	a^3	a^2	a^{10}	a^9	a	a^{12}
a^6	0	a^6	a^2	a^3	a^7	a^5	a^{12}	a	a^{11}	a^8	a^9	1	a^{14}	a^4	a^{13}	a^{10}
a^{13}	0	a^{13}	a	a^{12}	a^3	a^2	a^9	a^5	1	a^7	a^4	a^{14}	a^8	a^{11}	a^{10}	a^6

Table 8.13: Completed PTR Multiplication Table of a LB Type II.1 Plane.

The following PTR polynomial is obtained from interpolation of $T(x, y, z) =$

$(x \odot y) + z$ over $\mathcal{R}^3$ where $\oplus$ is given by Table 8.13 and $+$ by Table 1.6:

$$T(X,Y,Z) = XY + p(X,Y) + Z,$$

where $p(X,Y) = r(Y)t_4(X)^2 + s(X)t_4(Y)^2 + u(X,Y)t_4(X) + v(X,Y)t_4(Y)$ with

$$\begin{aligned}
r(Y) &= (a^5Y^8 + a^8Y^4 + a^{10}Y^2 + a^4Y)\, t_4(Y) + a^{10}t_2(Y),\\
s(X) &= a^{10}t_2(X),\\
u(X,Y) &= \left(Y^8 + a^5Y^2 + Y\right) t_4(Y) + a^8t_2(Y)\\
&\qquad + X^8\left(Y^8 + a^2Y^4 + a^8Y\right) + a^4X^2\left(Y^4 + a^{12}Y\right), \text{ and}\\
v(X,Y) &= X^{10} + a^5X^9 + X^5 + a^5X^3 + a^2X^2 + a^2X.
\end{aligned}$$

8.3 Some Planes of Small Orders Obtained by Constructing PTR Tables

We have used the method of constructing the PTR multiplication tables described in this chapter to obtain planes of LB type II.1 or above in orders 16, 25, 27, 32, 49, 64, and 81. Some of the planes we constructed are given in this section. Except for orders 16 and 27, we have not checked the isomorphism of the planes listed in this section with the known planes of the respective orders. Since most planes are translation planes, we suspect all the planes listed here belong to some known class. With this method, we have constructed all planes of LB type II.1 or above in order 16 and 27, many non-isomorphic planes in order 25, 49, and 81, and some non-isomorphic planes in orders 32 and 64. For each plane $\mathcal{P}$, the following data are given:

- the order,
- the LB type,

- the p-rank,
- the order of Aut $\mathcal{P}$,
- the distribution of the orders of central collineation groups $\Gamma(\mathcal{L})$ for all lines of $\mathcal{P}$ as multisets $\mathcal{C}(\Gamma(\mathcal{L}))$ where $\Gamma(\mathcal{L})$ is the group of all central collineations with axis $\mathcal{L}$ (see description below), and
- a PTR polynomial representing the plane in the form

$$T(X, Y, Z) = XY + p(X, Y) + Z,$$

where $p(x, y) = -xy + (x \odot y)\ \forall x, y \in \mathcal{R}$.

We have refrained from giving the entire multiplication tables for the planes of higher orders as we did for the planes of order 16 to save on space and also since the tables can be readily obtained from the PTR polynomials by $x \odot y = T(x, y, 0)$ for all $x, y \in \mathcal{R}$. The planes are named as 'PnA', 'PnB' etc where 'n' is the order of the plane. The multisets $\mathcal{C}(\Gamma(\mathcal{L}))$ of the orders of central collineation groups $\Gamma(\mathcal{L})$ of $\mathcal{P}$ are interpreted as follows: $\{1^{150}, 3^{500}, 2500^{1}\}$ means there are 150 lines $\mathcal{L}_1$ in $\mathcal{P}$ with a trivial $\Gamma(\mathcal{L}_1)$, 500 lines $\mathcal{L}_2$ with $|\Gamma(\mathcal{L}_2)| = 3$, and a line $\mathcal{L}_3$ with $|\Gamma(\mathcal{L}_3)| = 2500$.

(i) **Plane P25A:** is a plane of order 25 LB type IVa.1, p-rank = 239, $|\text{Aut }\mathcal{P}|$ $= 1,800,000$, $\mathcal{C}(\Gamma(\mathcal{L})) = \{1^{150}, 3^{500}, 2500^{1}\}$, and

$$T(X, Y, Z) = XY - \left(\prod_{i=1}^{5} p_i(X)\right) t_5(Y) + Z,$$

where

$p_1(X) = X(X-1)(X+a^3)(X+a^4)(X+a^9),$

$p_2(X) = X^3 + X + a^{23},$

$p_3(X) = X^3 + a^{21}X^2 + a^{23}X + 4,$

$p_4(X) = X^4 + a^8X^3 + a^{23}X^2 + a^{10}X + a^{10}$, and

$p_5(X) = X^4 + a^{17}X^3 + a^8X^2 + a^7X + a^{17}.$

(ii) **Plane P25B:** is a plane of order 25 LB type IVa.1, p-rank 251, $|\text{Aut } \mathcal{P}| = 3,600,000$, $\mathcal{C}(\Gamma(\mathcal{L})) = \{2^{150}, 6^{500}, 2500^1\}$, and

$$T(X,Y,Z) = XY - 2\left(X^6 - 3\right)\left(X^{13} - X\right)t_5(Y) + Z.$$

(iii) **Plane P25C:** is a plane of order 25 LB type II.1, p-rank $= 262$, $|\text{Aut } \mathcal{P}| = 50,000$, $\mathcal{C}(\Gamma(\mathcal{L})) = \{1^{625}, 2^{25}, 125^1\}$, and

$$T(X,Y,Z) = XY - a^{22}p(X,Y)t_2(X)\,t_5(Y) + Z,$$

where

$$
\begin{aligned}
p(X,Y) = {} & X^{15}Y^{10} + 3X^{15}Y^{6} + a^{2}1X^{15}Y^{5} + X^{15}Y^{2} + a^{9}X^{15}Y + a^{20}X^{4} \\
& + X^{15} + X^{14}Y^{10} + 3X^{14}Y^{6} + a^{22}X^{14}Y^{5} + X^{14}Y^{2} + a^{10}X^{14}Y \\
& + a^{2}1X^{14} + X^{13}Y^{10} + 3X^{13}Y^{6} + a^{22}X^{13}Y^{5} + X^{13}Y^{2} + a^{10}X^{13}Y \\
& + a^{2}X^{13} + X^{12}Y^{10} + 3X^{12}Y^{6} + a^{22}X^{12}Y^{5} + X^{12}Y^{2} + a^{10}X^{12}Y \\
& + a^{2}X^{12} + a^{17}X^{11}Y^{10} + a^{11}X^{11}Y^{6} + a^{20}X^{11}Y^{5} + a^{17}X^{11}Y^{2} \\
& + a^{8}X^{11}Y + a^{8}X^{11} + a^{17}X^{10}Y^{10} + a^{11}X^{10}Y^{6} + a^{15}X^{10}Y^{5} + a^{4}X^{5}Y \\
& + a^{17}X^{10}Y^{2} + a^{3}X^{10}Y + a^{9}X^{10} + a^{17}X^{9}Y^{10} + a^{11}X^{9}Y^{6} + a^{15}X^{9}Y^{5} \\
& + a^{17}X^{9}Y^{2} + a^{3}X^{9}Y + a^{19}X^{9} + a^{17}X^{8}Y^{10} + a^{11}X^{8}Y^{6} + a^{15}X^{8}Y^{5} \\
& + a^{17}X^{8}Y^{2} + a^{3}X^{8}Y + a^{19}X^{8} + 3X^{7}Y^{10} + 4X^{7}Y^{6} + a^{11}X^{7}Y^{5} \\
& + 3X^{7}Y^{2} + a^{2}3X^{7}Y + a^{2}1X^{7} + 3X^{6}Y^{10} + 4X^{6}Y^{6} + a^{16}X^{6}Y^{5} \\
& + 3X^{6}Y^{2} + a^{4}X^{6}Y + 3X^{5}Y^{10} + 4X^{5}Y^{6} + a^{16}X^{5}Y^{5} + 3X^{5}Y^{2} \\
& + a^{20}X^{5} + 3X^{4}Y^{10} + 4X^{4}Y^{6} + a^{16}X^{4}Y^{5} + 3X^{4}Y^{2} + a^{4}X^{4}Y \\
& + 4X^{3}Y^{10} + 2X^{3}Y^{6} + 2X^{3}Y^{5} + 4X^{3}Y^{2} + 3X^{3}Y + a^{19}X^{3} + 4X^{2}Y^{10} \\
& + 2X^{2}Y^{6} + a^{10}X^{2}Y^{5} + 4X^{2}Y^{2} + a^{22}X^{2}Y + 3X^{2} + 4XY^{10} + 2XY^{6} \\
& + a^{10}XY^{5} + 4XY^{2} + a^{22}XY + a^{14}X + 4Y^{10} + 2Y^{6} + a^{10}Y^{5} \\
& + 4Y^{2} + a^{22}Y + a^{14}.
\end{aligned}
$$

(iv) **Plane P32A:** is a plane of order 32 LB type V.1, p-rank $= 342$, $|\mathrm{Aut}\ \mathcal{P}| = 163,840$, $\mathcal{C}(\Gamma(\mathcal{L})) = \{1^{1024}, 32^{32}, 1024^{1}\}$, and

$$T(X,Y,Z) = XY - p(X,Y) + Z,$$

where $p(X,Y) = a^5X^{16}Y^{16} + a^5X^{16}Y^8 + a^{30}X^{16}Y^4 + a^{14}X^{16}Y^2$
$$+ a^{23}X^{16}Y + a^5X^8Y^{16} + a^9X^8Y^8 + a^{14}X^2Y^{16} + a^8X^2Y^8$$
$$+ a^{19}X^2Y^4 + a^{13}X^2Y^2 + a^6X^2Y + a^{23}XY^{16} + a^{23}XY^8$$
$$+ a^8XY^4 + a^6XY^2 + a^{11}XY.$$

(v) **Plane P49A:** is a plane of order 49 LB type IVa.1, p-rank $= 907$, $|\mathrm{Aut}\ \mathcal{P}|$ $= 4,840,416$, $\mathcal{C}\left(\Gamma(\mathcal{L})\right) = \{1^{2058}, 2^{343}, 14^{49}, 14406^1\}$, and

$$T(X,Y,Z) = XY - \left(\prod_{i=1}^{4} p_i(X)\right) t_7\left(X\right) t_7\left(Y\right) + Z,$$

where

$p_1(X) = t_7\left(X\right) + a^3$,

$p_2(X) = t_7\left(X\right) + a^{11}$,

$p_3(X) = t_7\left(X\right) + a^{31}$, and

$p_4(X) = t_7\left(X\right) + a^{41}$.

(vi) **Plane P49B:** is a plane of order 49 LB type IVa.1, p-rank $= 905$, $|\mathrm{Aut}\ \mathcal{P}|$ $= 22,127,616$, $\mathcal{C}\left(\Gamma(\mathcal{L})\right) = \{2^{2352}, 48^{98}, 14406^1\}$, and

$$T(X,Y,Z) = XY - \left(X^6+1\right)\left(X^{12} - a^{18}\right)\left(X^{12} - a^{19}\right) t_7\left(X\right) t_7\left(Y\right) + Z.$$

The factors X^6+1 and $X^{12} - a^{18}$ can be further factorised using

$x^6 - \lambda^6 = (x+\lambda)(x+2\lambda)(x+3\lambda)(x+4\lambda)(x+5\lambda)(x+6\lambda)$ for any $\lambda \in \mathbb{F}_{49}$ while the factor $X^{12} - a^{19}$ is irreducible.

(vii) **Plane P49C:** is a plane of order 49 LB type IVa.1, p-rank = 855, $|\mathrm{Aut}\ \mathcal{P}|$ = $77,446,656$, $\mathcal{C}\left(\Gamma(\mathcal{L})\right) = \{2^{392}, 8^{2058}, 14406^{1}\}$, and

$$\begin{aligned} T(X,Y,Z) &= XY - 3\left(X^{16}+1\right)\left(X^{16}+2\right)\left(X^{25}-X\right)t_7\left(Y\right) + Z \\ &= XY - 3\left(X^{16}+1\right)\left(X^{16}+2\right)t_7\left(X\right)t_7\left(Y\right)h_3(X^6) + Z. \end{aligned}$$

The middle expression of the right hand side splits into linear factors in $\mathbb{F}_{49}[X,Y]$.

(viii) **Plane P64A:** is a plane of order 64 LB type IVa.1, p-rank = 1066, $|\mathrm{Aut}\ \mathcal{P}|$ = $38,535,168$, $\mathcal{C}\left(\Gamma(\mathcal{L})\right) = \{1^{4096}, 8^{64}, 28672^{1}\}$, and

$$T(X,Y,Z) = XY - \left(\prod_{i=1}^{4} p_i(X)\right) t_8\left(X\right) t_8\left(Y\right) + Z,$$

where

$p_1(X) = \prod_{x_1 \in \mathcal{S}_1} \left(X^2 + X + x_1\right)$ where $\mathcal{S}_1 = \{a^{13}, a^{24}, a^{43}, a^{44}\}$,

$p_2(X) = \prod_{x_2 \in \mathcal{S}_2} \left(X^2 + a^{27}X + x_2\right)$ where $\mathcal{S}_2 = \{a^{2}, a^{10}, a^{49}, a^{60}\}$,

$p_3(X) = \prod_{x_3 \in \mathcal{S}_3} \left(X^2 + a^{45}X + x_3\right)$ where $\mathcal{S}_3 = \{a, a^{5}, a^{30}, a^{56}\}$, and

$p_4(X) = \prod_{x_4 \in \mathcal{S}_4} \left(X^2 + a^{54}X + x_4\right)$ where $\mathcal{S}_4 = \{a^{6}, a^{25}, a^{26}, a^{58}\}$.

(ix) **Plane P81A:** is a plane of order 81 LB type IVa.1, p-rank = 1843, $|\mathrm{Aut}\ \mathcal{P}|$ = $18,895,680$, $\mathcal{C}\left(\Gamma(\mathcal{L})\right) = \{1^{6561}, 9^{81}, 52488^{1}\}$, and

$$T(X,Y,Z) = XY - \left(\prod_{i=1}^{4} p_i(X)\right) t_9\left(X\right) t_9\left(Y\right) + Z,$$

where

$$
\begin{aligned}
p_1(X) &= t_9\,(X) + a^{19}, \\
p_2(X) &= X^{15} + X^{13} + a^9 X^{12} + X^{11} + a^{49} X^{10} + 2X^9 + 2X^7 \\
&\qquad + a^{78} X^6 + 2X^5 + a^{43} X^4 + a^{29} X^3 + a^{57} X^2 + a^{69} X + a^9, \\
p_3(X) &= X^{15} + X^{13} + a^{54} X^{12} + X^{11} + a^{14} X^{10} + a^{74} X^9 + 2X^7 \\
&\qquad + a^{78} X^6 + 2X^5 + a^{36} X^4 + a^{23} X^3 + a^{61} X^2 + aX + a^{62}, \text{ and} \\
p_4(X) &= X^{15} + X^{13} + a^{68} X^{12} + X^{11} + a^{28} X^{10} + a^{48} X^9 + 2X^7 \\
&\qquad + a^{78} X^6 + 2X^5 + a^{58} X^4 + a^{77} X^3 + a^8 X^2 + a^{25} X + a^{48}.
\end{aligned}
$$

8.4 Search for Planes of LB type II.1

We originally developed the method of constructing planes by completing the PTR multiplication tables to search for new planes of LB type II.1. In the course of this research, we have

- conducted an exhaustive search of planes of orders 16 and 25 admitting an elementary abelian addition, and
- verified exhaustively that Mathon plane (and its dual) are the only planes of LB type II.1 with an elementary abelian additive group $(\mathcal{R}, \oplus)$.

No new planes were found in orders 16 and 25. Besides these two orders, the search so far has been unsuccessful in constructing LB type II.1 planes of other small orders like 27, 32, 49, 64, or 81.

Our technique assumes an elementary abelian addition in $(\mathcal{R}, \oplus)$ which is not proved in theory in the general case. Specifically, if a plane admits a unique transitive elation group $\Gamma(\mathbf{p}, \mathcal{L})$, and $\mathcal{L}$ is not an axis of any non-trivial elation not in $\Gamma(\mathbf{p}, \mathcal{L})$, then it is not proven, to our best knowledge, that $\Gamma(\mathbf{p}, \mathcal{L})$ is necessarily elementary abelian.

Chapter 9

FUTURE WORK

We consider some extensions of the works in this dissertation.

In Chapters 3 and 4, some coordinatisations of the planes of order 16 were obtained. The PTR polynomials obtained from coordinatisations were presented in forms that showed some properties of the planes. An extension of this exercise will be to obtain such coordinatisations of planes of higher orders and find common patterns possibly leading to generalisations. In fact, the generalised polynomial representation of the Hall planes obtained in Chapter 6 was motivated by a pattern seen in the polynomials obtained from our coordinatisations.

For the planes of order 16 LB type I.1, we obtained coordinatisations that resulted in a system $(\mathcal{R}, \oplus, \odot)$ having stronger properties like left distributivity or a nucleus larger than guaranteed by its central collineation groups. A study of theoretical reasons behind these findings and whether the results can be generalised is warranted.

In the coordinatisations obtained for the planes of order 16 LB type I.1, the PTR addition is fully optimised as is guaranteed by [5] for all derivable planes. We also obtained a PTR with fully optimised PTR addition and multiplication for the Figueroa plane of order 27 which is not derivable. It is of interest to study if such coordinatisations exist for other planes of LB type I.n and the theoretical reasons behind the results.

As mentioned in Chapter 8, an incomplete part of our research is the construction of new planes of LB type II.1 by completing a PTR multiplication table. In particular, work is needed in the direction of designing algorithms that can systematically and exhaustively complete the tables for a given order in a computationally efficient manner.

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Appendix A

UNPUBLISHED ARTICLE OF DR. ROBERT S. COULTER

Algebraic substructures of planar ternary rings*

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August 19, 2023

Abstract

It has been known for many years that when the group G of $((\infty), [\infty])$ elations (resp. of $((0), [0])$ homologies) is maximal, the additive (resp. multiplicative) loop of a coordinatised plane will form a group isomorphic to G. Here we derive results on the additive and multiplicative loops in those cases where the respective groups are not necessarily maximal, thereby extending these well-known historical results.

1 Central collineations and the Lenz-Barlotti classification

A *collineation* (or automorphism) γ of a projective plane $\mathcal{P}$ is a bijective map of points onto points and lines onto lines that preserves incidence. The collineation γ has a *centre* $\mathbf{p}$ if $\mathcal{L}^\gamma = \mathcal{L}$ for every line $\mathcal{L}$ through the point $\mathbf{p}$. Dually, γ has an *axis* $\mathcal{L}$ if $\mathbf{p}^\gamma = \mathbf{p}$ for every point $\mathbf{p}$ on the line $\mathcal{L}$. We have the following result, see Pickert, [13], Section 3.1.

Lemma 1. *Any non-identity collineation γ of a projective plane $\mathcal{P}$ has at most one centre and one axis. Moreover, γ has a centre if and only if it has an axis.*

A *central* collineation is any collineation which has at least one centre. Central collineations naturally split into two cases: Given a central collineation γ with centre $\mathbf{p}$ and axis $\mathcal{L}$, γ is called *an elation* when $\mathbf{p}$ is incident with $\mathcal{L}$, and *a homology* when $\mathbf{p}$ is not incident with $\mathcal{L}$.

Given a point-line flag $(\mathbf{p}, \mathcal{L})$, it is clear that the set $\Gamma(\mathbf{p}, \mathcal{L})$ of all central collineations with centre $\mathbf{p}$ and axis $\mathcal{L}$ forms a group. Perhaps less clear is...

Lemma 2. *Let $\mathcal{P}$ be a projective plane of order n and $(\mathbf{p}, \mathcal{L})$ a point-line flag of $\mathcal{P}$. Then either*

(i) $|\Gamma(\mathbf{p}, \mathcal{L})|$ *divides n and $\mathbf{p}$ is incident with $\mathcal{L}$; or*

(ii) $|\Gamma(\mathbf{p}, \mathcal{L})|$ *divides $n-1$ and $\mathbf{p}$ is not incident with $\mathcal{L}$.*

See, for example, Hughes and Piper [10], Exercise 4.7. When $|\Gamma(\mathbf{p}, \mathcal{L})|$ attains either of these maximal orders, the plane $\mathcal{P}$ is $(\mathbf{p}, \mathcal{L})$*-transitive*. The use of the word transitive here is explained by the fact that a central collineation is uniquely determined by the image of any one of its non-fixed points.

*Dedicated to Günter Pickert (1917–2015).
This work constitutes an unpublished draft which has not been made publicly available. Some of these results can be found in other works, notably in a German book, but the author is unaware of a unified treatment like that given here.

Lemma 3 (Pickert, [13], page 66). *Let $\mathcal{P}$ be a projective plane and $(\mathbf{p}, \mathcal{L})$ be a point-line flag of $\mathcal{P}$. Let $\mathbf{q}, \mathbf{r}$ be any two points collinear with $\mathbf{p}$, both distinct from $\mathbf{p}$, and both not incident with $\mathcal{L}$. Then there exists at most one $\gamma \in \Gamma(\mathbf{p}, \mathcal{L})$ satisfying $\mathbf{q}^\gamma = \mathbf{r}$.*

One sees immediately that the plane $\mathcal{P}$ is $(\mathbf{p}, \mathcal{L})$-transitive if for any two points $\mathbf{q}, \mathbf{r}$ satisfying the conditions of the lemma, there must be exactly one central collineation in $\Gamma(\mathbf{p}, \mathcal{L})$ which maps one to the other.

The concept of $(\mathbf{p}, \mathcal{L})$-transitive forms the basis for the Lenz-Barlotti classification system for projective planes. There are 7 main classes developed by Lenz [12], each refined by Barlotti [4] into a series of subclasses. Broadly speaking, the higher the Lenz-Barlotti class, the more structure and the larger the automorphism group of the plane, with LB type I.1 having no point-line transitivities, while LB type VII.2 is the Desarguesian plane, which is flag transitive for every possible flag. Some of the original classes have been shown to be empty, while some classes remain empty but open.

To expand on this last comment, let $\mathcal{T}$ denote the set of all point-line flags $(\mathbf{p}, \mathcal{L})$ for which a given projective plane $\mathcal{P}$ is $(\mathbf{p}, \mathcal{L})$-transitive. In the following table, The classification, restricted to those cases for which either there exist non-trivial examples (type IVa.3 occurs only for order 9) or the question of existence remains open, is outlined in the following table.

The Lenz-Barlotti Classification

Lenz-Barlotti type	Form of $\mathcal{T}$	Existence finite case	Existence infinite case
I.1	$\varnothing$	yes	yes
I.2	$\{(\mathbf{p}, \mathcal{L})\}$ with $\mathbf{p} \notin \mathcal{L}$	open	yes
I.3	$\{(\mathbf{p}, \mathcal{L}), (\mathbf{q}, \mathcal{M})\}$ with $\mathbf{p} \in \mathcal{M} \setminus \mathcal{L}$, $\mathbf{q} \in \mathcal{L} \setminus \mathcal{M}$	open	yes
I.4	$\{(\mathbf{p}, \mathcal{L}), (\mathbf{q}, \mathcal{M}), (\mathbf{r}, \mathcal{N})\}$ consisting of vertices and opposite sides of a triangle	open	yes
II.1	$\{(\mathbf{p}, \mathcal{L})\}$ with $\mathbf{p} \in \mathcal{L}$	yes	yes
II.2	$\{(\mathbf{p}, \mathcal{L}), (\mathbf{q}, \mathcal{M})\}$ with $\mathbf{p} = \mathcal{L} \cap \mathcal{M}$, $\mathbf{q} \in \mathcal{L} \setminus \mathcal{M}$	open	yes
III.1	$\{(\mathbf{q}, \overline{\mathbf{p}\,\mathbf{q}}) \,:\, \text{all } \mathbf{q} \in \mathcal{L}\}$ with $\mathbf{p} \notin \mathcal{L}$	no	yes
III.2	$\{(\mathbf{p}, \mathcal{L})\} \cup \{(\mathbf{q}, \overline{\mathbf{p}\,\mathbf{q}}) \,:\, \text{all } \mathbf{q} \in \mathcal{L}\}$ with $\mathbf{p} \notin \mathcal{L}$	no	yes
IVa.1	$\{(\mathbf{p}, \mathcal{L}) \,:\, \text{all } \mathbf{p} \in \mathcal{L}\}$	yes	yes
IVa.2	$\{(\mathbf{r}, \mathcal{L}) \,:\, \text{all } \mathbf{r} \in \mathcal{L}\} \cup \{(\mathbf{p}, \mathcal{M}) \,:\, \text{all } \mathcal{M} \text{ through } \mathbf{q}\}$ $\cup\{(\mathbf{q}, \mathcal{N}) \,:\, \text{all } \mathcal{N} \text{ through } \mathbf{p}\}$ with $\mathbf{p} \neq \mathbf{q}$ and $\mathbf{p}, \mathbf{q} \in \mathcal{L}$	yes	yes
V.1	$\{(\mathbf{q}, \mathcal{L}) \,:\, \text{all } \mathbf{q} \in \mathcal{L}\} \cup \{(\mathbf{p}, \mathcal{M}) \,:\, \text{all } \mathcal{M} \text{ through } \mathbf{p}\}$ with $\mathbf{p} \in \mathcal{L}$	yes	yes
VII.1	$\{(\mathbf{p}, \mathcal{L}) \,:\text{all } \mathcal{L} \text{ and all } \mathbf{p} \in \mathcal{L}\}$	no	yes
VII.2	$\{(\mathbf{p}, \mathcal{L}) \,:\text{all } \mathbf{p}, \text{ all } \mathcal{L}\}$	yes	yes

As the table shows, in the infinite case, there are examples of each of these classes known. However, for LB types I.2, I.3, I.4 and II.1, no finite examples are known, but noone has yet managed to exclude them as possibilities; these cases remain among the more important open problems in projective geometry. In particular, in the 1970s it was widely expected that finite examples of LB types I.2 and II.2 would be produced, and yet, four decades later, we still don't know of a single example of either type.

2 The coordinatisation method

There are at least three standard methods for introducing coordinates to a projective plane. They are basically equivalent, except that they produce slightly different properties in the resulting PTRs. Throughout, we use the method outlined by Hughes and Piper in [10], Chapter 5 – the two other methods are given at the end of that chapter. Here we give a description of the method.

Let $\mathcal{P}$ be a projective plane of order n and let $\mathcal{R}$ be any set of cardinality n – this set along with the symbol ∞ will be all that is required to produce a coordinate system for the plane. We designate two special elements of $\mathcal{R}$ by 0 and 1 for reasons which will become clear. We now proceed to coordinatise $\mathcal{P}$.

- Choose any triangle in the plane $\mathbf{O}, \mathbf{x}, \mathbf{y}$. Label $\mathbf{O} = (0,0)$, $\mathbf{x} = (0)$ and $\mathbf{y} = (\infty)$ – by doing so we have now determined the "line at infinity" $\overline{\mathbf{x}\,\mathbf{y}} = [\infty]$. We also set $[0] = \overline{\mathbf{O}\,\mathbf{y}}$ and $[0,0] = \overline{\mathbf{O}\,\mathbf{x}}$.
- A fourth point, $\mathbf{I}$, not collinear with any two of $\mathbf{O}, \mathbf{x}, \mathbf{y}$ is now chosen and labelled $\mathbf{I} = (1,1)$.
- To finalise the initialisation process, we label some obvious intersection points:
  - Set $\overline{\mathbf{x}\,\mathbf{I}} \cap [0] = (0,1)$.
  - Set $\overline{\mathbf{y}\,\mathbf{I}} \cap [0,0] = (1,0)$.
  - Set $\overline{(1,0)(0,1)} \cap [\infty] = \mathbf{J} = (1)$.

At this point, we have labelled 3 of the $n+1$ points of both of the lines $[0]$ and $[0,0]$. One may now label the remaining $n-2$ points of $[0]$ as $(0,a)$ in an arbitrary way using the remaining $n-2$ elements $a \in \mathcal{R} \setminus \{0,1\}$. This is the last remaining freedom of choice in the process, as from this stage onwards, the coordinates of all points and lines are totally determined.

We now proceed to label all points and lines of the plane, starting with the remaining points.

- To label the remaining points of $[0,0]$ we set $\overline{(0,a)\,\mathbf{J}} \cap [0,0] = (a,0)$.
- To label the remaining points of $[\infty]$ we set $\overline{(0,a)(1,0)} \cap [\infty] = (a)$.
- To label the remaining "affine" points we set $\overline{(a,0)\,\mathbf{y}} \cap \overline{(0,b)\,\mathbf{x}} = (a,b)$.

With a labelling of the points complete, it remains only to give a labelling of the lines.

- To label the "vertical" lines we set $\overline{(a,0)\,\mathbf{y}} = [a]$.
- To label the "lines of slope m" we set $\overline{(m)(0,k)} = [m,k]$.

This completes the introduction of coordinates to the plane. From this coordinatisation, one now defines a tri-variate function T on $\mathcal{R}$, called a *planar ternary ring (PTR)*, by setting $T(m,x,y) = k$ if and only if $(x,y) \in [m,k]$. This PTR will exhibit certain properties and is actually equivalent to the projective plane as any three variable function exhibiting those properties can be used to define a projective plane. More precisely, we have the following important result, essentially due to Hall [9]; see also Hughes and Piper, [10], Theorem 5.1.

Lemma 4 (Hall, [9], Theorem 5.4)**.** *Let $\mathcal{P}$ be a projective plane of n and $\mathcal{R}$ be any set of cardinality n. Let $T : \mathcal{R}^3 \to \mathcal{R}$ be a PTR obtained from coordinatising $\mathcal{P}$. Then T must satisfy the following properties:*

(a) $T(a,0,z) = T(0,b,z) = z$ *for all* $a,b,z \in \mathcal{R}$.

(b) $T(x,1,0) = x$ *and* $T(1,y,0) = y$ *for all* $x,y \in \mathcal{R}$.

(c) If $a,b,c,d \in \mathcal{R}$ *with* $a \neq c$*, then there exists a unique* x *satisfying* $T(x,a,b) = T(x,c,d)$.

(d) If $a,b,c \in \mathcal{R}$*, then there is a unique* z *satisfying* $T(a,b,z) = c$.

(e) If $a, b, c, d \in \mathcal{R}$ with $a \neq c$, then there is a unique pair (y, z) satisfying $T(a, y, z) = b$ and $T(c, y, z) = d$.

Conversely, any tri-variate function T defined on $\mathcal{R}$ which satisfies Properties (c) through (e) can be used to define an affine plane $\mathcal{A}_T$ of order q as follows:

- *the points of $\mathcal{A}$ are (x, y), with $x, y \in \mathcal{R}$;*
- *the lines of $\mathcal{A}$ are the symbols $[m, a]$, with $m, a \in \mathcal{R}$, defined by*

$$[m, a] = \{(x, y) \in \mathcal{R} \times \mathcal{R} \,:\, a = T(m, x, y)\},$$

and the symbols $[c]$, with $c \in \mathcal{R}$, defined by

$$[c] = \{(c, y) \,:\, y \in \mathbb{F}_q\}.$$

Following Hall, it is customary to define an addition $\oplus$ and multiplication $\odot$ on $\mathcal{R}$ by

$$x \oplus y = T(1, x, y),$$
$$x \odot y = T(x, y, 0),$$

for all $x, y \in \mathcal{R}$. It is well known that the properties of the plane guarantee that both $\oplus$ and $\odot$ are loop operations with identities 0 and 1 over $\mathcal{R}$ and $\mathcal{R}^*$, respectively. A PTR is called *linear* over $\mathcal{R}$ if $T(x, y, z) = (x \odot y) \oplus z$ for all $x, y, z \in \mathcal{R}$ – that is, if T can be reconstructed from only knowing the operations $\oplus$ and $\odot$. If the PTR is linear and $(\mathcal{R}, \oplus)$ is a group, then T is called a *Cartesian group.* A variety of sets related to the loop operations $\oplus$ and $\odot$ will arise in what follows. Firstly, there are the two sets $\mathcal{A}_l$ and $\mathcal{A}_r$ which measure the amount of left and right associativity of the additive loop; specifically,

$$\mathcal{A}_l = \{a \in \mathcal{R} \,:\, a \oplus (x \oplus y) = (a \oplus x) \oplus y \text{ for all } x, y \in \mathcal{R}\}, \text{ and}$$
$$\mathcal{A}_r = \{a \in \mathcal{R} \,:\, x \oplus (y \oplus a) = (x \oplus y) \oplus a \text{ for all } x, y \in \mathcal{R}\}.$$

Similarly, we have the nuclei of the PTR. The left, middle, and right nucleus of the PTR measure how much the multiplicative loop associates on the left, middle, and right, respectively. These are thus defined:

$$\mathcal{N}_l = \{a \in \mathcal{R} \,:\, a \odot (x \odot y) = (a \odot x) \odot y \text{ for all } x, y \in \mathcal{R}\},$$
$$\mathcal{N}_m = \{a \in \mathcal{R} \,:\, x \odot (a \odot y) = (x \odot a) \odot y \text{ for all } x, y \in \mathcal{R}\}, \text{ and}$$
$$\mathcal{N}_r = \{a \in \mathcal{R} \,:\, x \odot (y \odot a) = (x \odot y) \odot a \text{ for all } x, y \in \mathcal{R}\}.$$

The nuclei have played a significant and central role in the study of semifields for many years, for examples see the works of Knuth [11], Cohen and Ganley [5], Ball and Lavrauw [3], Coulter and Henderson [6], and Ebert, Marino, Polverino, and Trombetti [8]. Surprisingly, our results will provide a new insight into their geometric meaning for arbitrary planes. **Note:** André gives a geometrical interpretation of the nuclei (exactly corresponding to the collineation groups I'm connecting them to below) for the quasifield case in [1] and [2] – see Dembowski 3.1.28 through 3.1.30 and comment following 5.3.5; also Hughes and Piper Theorem 8.2.

Finally, there are three sets which measure the amount by which the distributive laws hold. We have

$$\mathcal{D}_l = \{a \in \mathcal{R} \,:\, a \odot (x \oplus y) = (a \odot x) \oplus (a \odot y) \text{ for all } x, y \in \mathcal{R}\},$$
$$\mathcal{D}_r = \{a \in \mathcal{R} \,:\, (x \oplus y) \odot a = (x \odot a) \oplus (y \odot a) \text{ for all } x, y \in \mathcal{R}\}, \text{ and}$$
$$\mathcal{D} = \{a \in \mathcal{R} \,:\, x \odot (a \oplus y) = (x \odot a) \oplus (x \odot y) \text{ for all } x, y \in \mathcal{R}\}.$$

The set $\mathcal{D}$ arises in the study of translation planes, where it is called the *distributor*; see André [2], Dembowski [7], page 134. The set $\mathcal{N}_l \cap \mathcal{D}_l$ also arises in the study of translation planes; it is known as the *kernel* of a quasifield and is necessarily a field in the translation plane case, see [7], page 132, especially 3.1.24. Both the distributor and the kernel will appear in our work, again giving geometric interpretations which are more general than previously seen in the literature.

3 A comment on Pickert's results

Pickert's results from [13] tie the behaviour of the loop operations of the PTR with certain flag-transitivities of the plane. More specifically, they give necessary and sufficient conditions for the additive or multiplicative loops to be groups, or for the left or right distributive laws to hold in full in the PTR, in terms of flag-transitivities involving the points $\mathbf{O}, \mathbf{x}, \mathbf{y}$ and the lines $[0], [0,0], [\infty]$. To produce his results, Pickert used the original coordinatisation method of Hall, while in this article we use the variant described by Hughes and Piper. Consequently, when we derive Pickert's results below, we give them in the equivalent forms from Chapter 6 of [10].

4 The influence of elations on the PTR

Theorem 5. *Let $T : \mathcal{R}^3 \to \mathcal{R}$ be a planar ternary ring obtained from the projective plane $\mathcal{P}$ and let $\Gamma = \Gamma((\infty), [\infty])$. Set*

$$\mathcal{S} = \{s \in \mathcal{R} \,:\, (0,0)^\gamma = (0,s) \text{ for some } \gamma \in \Gamma\}.$$

The following statements hold.

(i) $\mathcal{S} \subseteq \mathcal{A}_r$.

(ii) $(\mathcal{S}, \oplus)$ *is a group isomorphic to* Γ.

(iii) For all $m, x \in \mathcal{R}$, $s \in \mathcal{S}$, *we have* $T(m,x,s) = (m \odot x) \oplus s$.

(iv) If T is linear, then $\mathcal{S} = \mathcal{A}_r$.

Proof. By Lemma 3, any $\gamma \in \Gamma$ is uniquely defined by the image of one point off the axis line $[\infty]$. For $s \in \mathcal{S}$, we use γ_s to denote the unique element of Γ that maps $(0,0)$ to $(0,s)$. As $\gamma_s \in \Gamma((\infty),[\infty])$, it fixes line-wise each point (m), $m \in \mathcal{R}$. In particular, we must have $[0,k]^{\gamma_s} = [0,k^{\sigma_s}]$ for some permutation function σ_s on $\mathcal{R}$. In addition, each line $[m]$ through (∞) is fixed line-wise; i.e. $[m]^{\gamma_s} = [m]$. As $(x,y) = [x] \cap [0,y]$, we see

$$(x,y)^{\gamma_s} = [x] \cap [0, y^{\sigma_s}] = (x, y^{\sigma_s})$$

for all $x, y \in \mathcal{R}$. This completely determine how γ_s acts on the plane $\mathcal{P}$:

$$\begin{aligned} (m) &\mapsto (m) \\ (x,y) &\mapsto (x, y^{\sigma_s}) \\ [m] &\mapsto [m] \\ [m,k] &\mapsto [m, k^{\sigma_s}]. \end{aligned}$$

The last identity follows from the fact we must have $(0,k)^{\gamma_s}$ on $[m,k]^{\gamma_s}$. We now examine this more closely. We have $(x,y) \in [m,k]$ if and only if $(x, y^{\sigma_s}) \in [m, k^{\sigma_s}]$. Equivalently $k = T(m,x,y)$ if and only if $k^{\sigma_s} = T(m,x,y^{\sigma_s})$. Thus

$$T(m,x,y^{\sigma_s}) = T(m,x,y)^{\sigma_s} \tag{1}$$

holds for all $m,x,y \in \mathcal{R}$. Fixing $m = 1, y = 0$ in (1), we find

$$T(1,x,s) = T(1,x,0)^{\sigma_s},$$

from which we deduce $x \oplus s = x^{\sigma_s}$. Returning to (1) with $m = 1$, we have for all $x, y \in \mathcal{R}$ and $s \in \mathcal{S}$,

$$\begin{aligned} x \oplus (y \oplus s) &= x \oplus y^{\sigma_s} \\ &= T(1,x,y^{\sigma_s}) \\ &= T(1,x,y)^{\sigma_s} \end{aligned}$$

$$
\begin{aligned}
&= (x \oplus y)^{\sigma_s} \\
&= (x \oplus y) \oplus s,
\end{aligned}
$$

which establishes the (i).

Associativity of $\oplus$ on $\mathcal{S}$ now follows at once. Clearly $0 \in \mathcal{S}$ and $0 \oplus x = x \oplus 0 = x$ for all $x \in \mathcal{R}$, so that we clearly have an identity. For closure, choose $s, t \in \mathcal{S}$. Then

$$
\begin{aligned}
(0, s \oplus t) &= (0, s^{\sigma_t}) \\
&= (0, s)^{\gamma_t} \\
&= (0, 0)^{\sigma_s \sigma_t} \\
&= (0, 0)^{\sigma_u} \quad \text{for some } u \in \mathcal{S} \\
&= (0, u),
\end{aligned}
$$

and so $s \oplus t = u \in \mathcal{S}$ and $(\mathcal{S}, \oplus)$ is closed. Set $\gamma_s^{-1} = \gamma_t$ for some $t \in \mathcal{S}$. Then

$$
(0, s \oplus t) = (0, 0)^{\sigma_s \sigma_t} = (0, 0) = (0, 0)^{\sigma_t \sigma_s} = (0, t \oplus s).
$$

Hence $s \oplus t = 0 = t \oplus s$, confirming the existence of inverses. We have proved $(\mathcal{S}, \oplus)$ is a group, and it is easily verified the map $s \mapsto \gamma_{-s}$ is an isomorphism from $(\mathcal{S}, \oplus)$ to Γ, thus establishing (ii).

Returning once more to (1), and setting $y = -s$, we find

$$
T(m, x, (-s)^{\sigma_s}) = T(m, x, 0) = m \odot x = T(m, x, -s) \oplus s.
$$

Finally, adding $-s$ to both sides (using $\oplus$), and using $\mathcal{S} \subseteq \mathcal{A}_r$, shows $T(m, x, -s) = (m \odot x) \oplus (-s)$ for all $m, x \in \mathcal{R}$ and $s \in \mathcal{S}$, and this is equivalent to (iii).

Finally, for (iv), assume T is linear. For each $a \in \mathcal{A}_r$ we construct an elation $\phi_a \in \Gamma$. Set ϕ_a to be the map

$$
\begin{aligned}
(\infty) &\mapsto (\infty) \\
(m) &\mapsto (m) \\
(x, y) &\mapsto (x, y \oplus a) \\
[m] &\mapsto [m] \\
[m, k] &\mapsto [m, k \oplus a].
\end{aligned}
$$

We need to only verify that if $(x, y) \in [m, k]$, then $(x, y)^{\phi_a} \in [m, k]^{\phi_a}$. Since T is linear, we have $k = (m \odot x) \odot y$, from which we have

$$
\begin{aligned}
k \oplus a &= ((m \odot x) \oplus y) \oplus a \\
&= (m \odot x) \oplus (y \oplus a),
\end{aligned}
$$

as $a \in \mathcal{A}_r$. Hence $(x, y \oplus a) \in [m, k \oplus a]$ as desired, and $\phi_a \in \Gamma$. Since $(0, 0)^{\phi_a} = (0, a)$, we see $a \in \mathcal{S}$ and (iv) is established. □

From Theorem 5 we can obtain the first of Pickert's results.

Corollary 6 ([10], Theorem 6.2)**.** *Let $T : \mathcal{R}^3 \to \mathcal{R}$ be a planar ternary ring obtained from the projective plane $\mathcal{P}$. Then T is a Cartesian group if and only if $\mathcal{P}$ is $((\infty), [\infty])$-transitive.*

Proof. Suppose T is a Cartesian group. Then T is linear and $(R, \oplus)$ is a group. By the latter, $\mathcal{A}_r = \mathcal{R}$, while the former implies the set $\mathcal{S}$ as defined in Theorem 5 satisfies $\mathcal{S} = \mathcal{A}_r$. Thus $\Gamma((\infty), [\infty])$ is maximal and $\mathcal{P}$ is necessarily $((\infty), [\infty])$-transitive.

Conversely, suppose $\mathcal{P}$ is $((\infty), [\infty])$-transitive. Then $\mathcal{S} = \mathcal{R}$ and $(\mathcal{R}, \oplus)$ is thus a group. Moreover, Theorem 5 (iii) shows T is also linear, so that T is a Cartesian group. □

Our second main result sees the distributor arise for any linear PTR, thereby providing a more general geometric explanation of its role.

Theorem 7. *Let $T : \mathcal{R}^3 \to \mathcal{R}$ be a planar ternary ring obtained from the projective plane $\mathcal{P}$ and let $\Gamma = \Gamma((0), [\infty])$. Set*

$$\mathcal{S} = \{s \in \mathcal{R} \,:\, (0,0)^\gamma = (s,0) \text{ for some } \gamma \in \Gamma\}.$$

The following statements hold.

(i) $\mathcal{S} \subseteq \mathcal{A}_l$.

(ii) $(\mathcal{S}, \oplus)$ *is a group isomorphic to* Γ.

(iii) For all $m, x \in \mathcal{R}$, $s \in \mathcal{S}$, *we have* $T(m, s, m \odot x) = m \odot (s \oplus x)$.

(iv) If T *is linear, then* $\mathcal{S} \subseteq \mathcal{A}_l \cap \mathcal{D}$.

(v) If T *is a Cartesian group, then* $\mathcal{S} = \mathcal{D}$.

Proof. As in the previous proof, we know from Lemma 3 that any $\gamma \in \Gamma$ is uniquely determined by the image of any one point not on the axis $[\infty]$. Let γ_s be the unique element of γ mapping $(0,0)$ to $(s,0)$. Now $[0,k]^{\gamma_s} = [0,k]$ for all $k \in \mathcal{R}$ as $[0,k]$ is incident with the centre (0). Also, since (∞) lies on the axis $[\infty]$, there must exist a permutation σ_s of $\mathcal{R}$ satisfying $[m]^{\gamma_s} = [m^{\sigma_s}]$. Now $(x,y) = [x] \cap [0,y]$, and so

$$\begin{aligned}(x,y)^{\gamma_s} &= [x]^{\gamma_s} \cap [0,y] \\ &= (x^{\sigma_s}, y).\end{aligned}$$

Note $0^{\sigma_s} = s$. Each point (m) lies on the axis $[\infty]$, and so is fixed by γ_s. As $[m,k] = \overline{(m)(0,k)}$, we also have

$$\begin{aligned}[m,k]^{\gamma_s} &= \overline{(m)^{\gamma_s}(0,k)^{\gamma_s}} \\ &= \overline{(m)(s,k)} \\ &= [m, T(m,s,k)].\end{aligned}$$

Thus the mapping γ_s is completely described:

$$\begin{aligned}(m) &\mapsto (m) \\ (x,y) &\mapsto (x^{\sigma_s}, y) \\ [m] &\mapsto [m^{\sigma_s}] \\ [m,k] &\mapsto [m, T(m,s,k)].\end{aligned}$$

Thus $k = T(m,x,y)$ if and only if $T(m,s,k) = T(m, x^{\sigma_s}, y)$. In particular, we have

$$T(m, s, T(m,x,y)) = T(m, x^{\sigma_s}, y) \tag{2}$$

for all $m, x, y \in \mathcal{R}$. Set $m = 1$. Then

$$s \oplus (x \oplus y) = x^{\sigma_s} \oplus y. \tag{3}$$

Setting $y = 0$ yields $x^{\sigma_s} = s \oplus x$. Returning to (3) we find

$$s \oplus (x \oplus y) = (s \oplus x) \oplus y$$

for all $x, y \in \mathcal{R}$ and $s \in \mathcal{S}$, establishing (i).

The proof that $(\mathcal{S}, \oplus)$ is a group now follows in almost identical fashion to the previous proof, while the map $s \mapsto \gamma_s$ acts as an isomorphism between $(\mathcal{S}, \oplus)$ and Γ.

Part (iii) follows from (2) with $y = 0$, while $\mathcal{S} \subseteq \mathcal{D}$ is merely (iii) in the case T is linear, and this proves (iv).

It remains to establish (v). As with the proof of Theorem 5 (iv), we proceed by constructing a $\phi_a \in \Gamma$ for each $a \in \mathcal{D}$. Define the map ϕ_a by

$$\begin{aligned}(\infty) &\mapsto (\infty)\\ (m) &\mapsto (m)\\ (x,y) &\mapsto (a \oplus x, y)\\ [m] &\mapsto [a \oplus m]\\ [m,k] &\mapsto [m, (m \odot a) \oplus k].\end{aligned}$$

Again, we need only confirm that if $(x, y) \in [m, k]$, then $(x, y)^{\phi_a} \in [m, k]^{\phi_a}$. Under the hypothesis, T is linear with $(\mathcal{R}, \oplus)$ a group. Thus, if $(x, y) \in [m, k]$, then $k = (m \odot x) \oplus y$. Hence

$$\begin{aligned}(m \odot a) \oplus k &= (m \odot a) \oplus ((m \odot x) \oplus y)\\ &= ((m \odot a) \oplus (m \odot x)) \oplus y\\ &= (m \odot (a \oplus x)) \oplus y,\end{aligned}$$

from which we see ϕ_a preserves incidence and is thus an element of Γ. Since $(0,0)^{\phi_a} = (a, 0)$, $a \in \mathcal{S}$, completing the proof. □

From this result we obtain the second of Pickert's results.

Corollary 8 ([10], Theorem 6.3)**.** *Let $T : \mathcal{R}^3 \to \mathcal{R}$ be a Cartesian group obtained from the projective plane $\mathscr{P}$. Then the full left distributive law holds in T – that is, $x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z)$ for all $x, y, z \in \mathcal{R}$ – if and only if $\mathscr{P}$ is $((0), [\infty])$-transitive.*

Proof. Under the conditions of the statement, we know $\mathcal{S} = \mathcal{D}$ from Theorem 7.

Suppose the full left distributive law holds in T. Then $\mathcal{D} = \mathcal{R}$, implying $\Gamma((0), [\infty])$ is maximal by Theorem 7 (ii). Hence $\mathscr{P}$ is $((0), [\infty])$-transitive.

Now suppose $\mathscr{P}$ is $((0), [\infty])$-transitive. Then $\mathcal{S} = \mathcal{R} = \mathcal{D}$ and so the full left distributive law holds in T. □

An extension of the distributive law result of the previous corollary can be made.

Lemma 9. *Let $T : \mathcal{R}^3 \to \mathcal{R}$ be a planar ternary ring obtained from the projective plane $\mathscr{P}$. Set*

$$\begin{aligned}\mathcal{S}_0 &= \{s \in \mathcal{S} \,:\, (0,0)^\gamma = (s, 0) \textit{ for some } \gamma \in \Gamma((0), [\infty])\}, \textit{ and}\\ \mathcal{S}_\infty &= \{s \in \mathcal{S} \,:\, (0,0)^\gamma = (0, s) \textit{ for some } \gamma \in \Gamma((\infty), [\infty])\}.\end{aligned}$$

Then

$$m \odot (s \oplus x) = (m \odot s) \oplus (m \odot x)$$

whenever $s \in \mathcal{S}_0$ and either T is linear or $m \odot x \in \mathcal{S}_\infty$.

Proof. By Theorem 7, we have

$$T(m, s, m \odot x) = m \odot (s \oplus x)$$

for all $m, x \in \mathcal{R}$, $s \in \mathcal{S}_0$. If T is linear, then we immediately have the claim. Otherwise, Theorem 5 tells us $T(m, x, y)$ is linear provided $y \in \mathcal{S}_\infty$. Thus, if $m \odot x \in \mathcal{S}_\infty$, then $T(m, s, m \odot x) = (m \odot s) \oplus (m \odot x)$, completing the proof. □

5 The influence of homologies on the PTR

Theorem 10. *Let $T : \mathcal{R}^3 \to \mathcal{R}$ be a planar ternary ring obtained from the projective plane $\mathcal{P}$ and let $\Gamma = \Gamma((0), [0])$. Set*

$$\mathcal{S} = \{s \in \mathcal{R} \,:\, (1,0)^\gamma = (s,0) \text{ for some } \gamma \in \Gamma\}.$$

The following statements hold.

(i) $\mathcal{S} \subseteq \mathcal{N}_m$.

(ii) $(\mathcal{S}, \odot)$ *is a group isomorphic to* Γ.

(iii) For all $x, y \in \mathcal{R}$, $s \in \mathcal{S}$, *we have* $T(s,x,y) = (s \odot x) \oplus y$.

(iv) If T *is linear, then* $\mathcal{S} = \mathcal{N}_m$.

Proof. From Lemma 3, any $\gamma \in \Gamma$ is uniquely determined by the image of any one point not the centre (0) nor on the axis $[0]$. For $s \in \mathcal{S}$, denote by γ_s the unique element of Γ that maps $(1,0)$ to $(s,0)$. As (∞) lies on the axis $[0]$, we have $[m]^{\gamma_s} = [m^{\beta_s}]$ for some permutation β_s of $\mathcal{R}$ satisfying $0^{\beta_s} = 0$. Now $[0,y]$ is incident with the centre (0), and so is fixed by γ_s. Since $(x,y) = [0,y] \cap [x]$, we have

$$(x,y)^{\gamma_s} = [0,y] \cap [x^{\beta_s}] = (x^{\beta_s}, y)$$

for all $x, y \in \mathcal{R}$. This also shows $1^{\beta_s} = s$. Additionally, as $[\infty] = \overline{(0)(\infty)}$, and both points are fixed by γ_s, there must exist some bijection α_s of $\mathcal{R}$ satisfying $0^{\alpha_s} = 0$ and $(m)^{\gamma_s} = (m^{\alpha_s})$. We thus have the following decription for γ_s:

$$\begin{aligned}
(m) &\mapsto (m^{\alpha_s}) \\
(x,y) &\mapsto (x^{\beta_s}, y) \\
[m] &\mapsto [m^{\beta_s}] \\
[m,k] &\mapsto [m^{\alpha_s}, k].
\end{aligned}$$

(The last identity follows from the fact $(0,k) \in [m,k]$ and $(0,k)$ is fixed by γ_s.) In terms of the PTR T, we thus have

$$k = T(m,x,y) = T(m^{\alpha_s}, x^{\beta_s}, y). \tag{4}$$

Set $y = 0$. Then

$$m \odot x = m^{\alpha_s} \odot x^{\beta_s} \tag{5}$$

for all $m, x \in \mathcal{R}$. Next now set $x = 1$ to find

$$m = m^{\alpha_s} \odot s. \tag{6}$$

Setting $m = s$ now yields $s = s^{\alpha_s} \odot s$, and so $s^{\alpha_s} = 1$ as $\odot$ is a loop operation. Returning to Equation 5 and setting $m = s$, we find

$$s \odot x = 1 \odot x^{\beta_s} = x^{\beta_s}.$$

Set $m = u \odot s$ and note $m^{\alpha_s} = u$ from (6). As $\odot$ is a loop operation, m ranges over all of $\mathcal{R}$ as u does. Combining all of our gathered facts now yields

$$\begin{aligned}
(u \odot s) \odot x &= m \odot x \\
&= m^{\alpha_s} \odot x^{\beta_s} \\
&= u \odot (s \odot x),
\end{aligned}$$

which establishes (i).

Associativity of $(\mathcal{S}, \odot)$ now follows at once. Clearly $1 \in \mathcal{S}$ and $1 \odot x = x \odot 1 = x$, so $\mathcal{S}$ contains an identity. For $s, t \in \mathcal{S}$,

$$\begin{aligned}
(s \odot t, 0) &= (t^{\beta_s}, 0) \\
&= (t, 0)^{\gamma_s} \\
&= (1, 0)^{\gamma_t \gamma_s} \\
&= (1, 0)^{\gamma_u} \quad \text{for some } u \in \mathcal{S} \\
&= (u, 0).
\end{aligned}$$

Hence $s \odot t \in \mathcal{S}$ and $(\mathcal{S}, \odot)$ is closed. Finally, let $\gamma_s^{-1} = \gamma_t$. Then

$$(s \odot t, 0) = (1, 0)^{\gamma_t \gamma_s} = (1, 0) = (1, 0)^{\gamma_s \gamma_t} = (t \odot s, 0).$$

Thus $s \odot t = 1 = t \odot s$ and $(\mathcal{S}, \odot)$ has inverses, and so $(\mathcal{S}, \odot)$ is a group. To complete the proof of (ii), we note that the map $s \mapsto \gamma_s$ acts as an isomorphism between $(\mathcal{S}, \odot)$ and Γ.

To prove (iii), note that from $s^{\alpha_s} = 1$ and (4) we can deduce

$$\begin{aligned}
T(s, x, y) &= T(s^{\alpha_s}, x^{\beta_s}, y) \\
&= T(1, s \odot x, y) \\
&= (s \odot x) \oplus y,
\end{aligned}$$

for all $x, y \in \mathcal{R}$.

It remains to establish (iv). For each $a \in \mathcal{N}_m$, we define a mapping ϕ_a as follows:

$$\begin{aligned}
(\infty) &\mapsto (\infty) \\
(m \odot a) &\mapsto (m) \\
(x, y) &\mapsto (a \odot x, y) \\
[m] &\mapsto [a \odot m] \\
[m \odot a, k] &\mapsto [m, k].
\end{aligned}$$

It suffices to prove $\phi_a \in \Gamma$, as $(1, 0)^{\phi_a} = (a, 0)$. To show ϕ_a preserves incidence, we need only prove that if $(x, y) \in [m \odot a, k]$, then $(x, y)^{\phi_a} \in [m \odot a, k]^{\phi_a}$. By hypothesis, T is linear. Suppose $(x, y) \in [m \odot a, k]$. Then

$$\begin{aligned}
k &= ((m \odot a) \odot x) \oplus y \\
&= (m \odot (a \odot x)) \oplus y,
\end{aligned}$$

as $a \in \mathcal{N}_m$. Hence $(x, y)^{\phi_a} \in [m \odot a, k]^{\phi_a}$, and incidence is preserved. Clearly ϕ_a has centre (0) and axis $[0]$, and so $\phi_a \in \Gamma$. □

Theorem 10 yields another of Pickert's results.

Corollary 11 ([10], Theorem 6.5)**.** *Let $T : \mathcal{R}^3 \to \mathcal{R}$ be a planar ternary ring obtained from the projective plane $\mathcal{P}$. Then T is linear with associative multiplication if and only if $\mathcal{P}$ is $((0), [0])$-transitive.*

Proof. Suppose T is linear with associative multiplication, so that $\mathcal{N}_m = \mathcal{R}$. By Theorem 10 (iv), we have $\mathcal{N}_m = \mathcal{S} = \mathcal{R}$, and so Theorem 10 (ii) now shows $\Gamma((0), [0])$ is maximal. Hence $\mathcal{P}$ is $((0), [0])$-transitive.

Now suppose $\mathcal{P}$ is $((0), [0])$-transitive. Then $\mathcal{S} = \mathcal{R}$. Parts (ii) and (iii) of Theorem 10 now show T is linear with associative multiplication. □

Our fourth main theorem includes a geometric interpretation of the kernel of the plane in any case where the PTR is linear; this extends our understanding of the kernel from translation planes to any plane not of LB type I.1.

Theorem 12. *Let $T : \mathcal{R}^3 \to \mathcal{R}$ be a planar ternary ring obtained from the projective plane $\mathcal{P}$ and let $\Gamma = \Gamma((\infty), [0, 0])$. Set*

$$\mathcal{S} = \{s \in \mathcal{R} \,:\, (0, 1)^{\gamma} = (0, s) \text{ for some } \gamma \in \Gamma\}.$$

The following statements hold.

(i) $\mathcal{S} \subseteq \mathcal{N}_l$.

(ii) $(\mathcal{S}, \odot)$ *is a group isomorphic to* Γ.

(iii) For all $m, x, y \in \mathcal{R}$, $s \in \mathcal{S}$, *we have* $T(s \odot m, x, s \odot y) = s \odot T(m, x, y)$.

(iv) If T *is linear, then* $\mathcal{S} = \mathcal{N}_l \cap \mathcal{D}_l$.

Proof. Any $\gamma \in \Gamma$ is uniquely determined by the image of any one point not the centre (∞) nor on the axis $[0, 0]$, by Lemma 3. Let γ_s denote the unique element of Γ that maps $(0, 1)$ to $(0, s)$. Each line $[m]$ is fixed by γ_s as $[m]$ is incident with the centre (∞). Additionally, as (0) is on the axis $[0, 0]$, there must exist a bijection β_s of $\mathcal{R}$ satisfying $[0, k]^{\gamma_s} = [0, k^{\beta_s}]$ and $0^{\beta_s} = 0$. As $(x, y) = [x] \cap [0, y]$, we see that

$$(x, y)^{\gamma_s} = (x, y^{\beta_s}).$$

In particular, $1^{\beta_s} = s$. Now $(m) \in \overline{(\infty)(0)}$, a line fixed by γ_s. Thus there exists a second bijection α_s satisfying $(m)^{\gamma_s} = (m^{\alpha_s})$ and $0^{\alpha_s} = 0$. We can now describe γ_s completely:

$$\begin{aligned}(m) &\mapsto (m^{\alpha_s})\\ (x, y) &\mapsto (x, y^{\beta_s})\\ [m] &\mapsto [m]\\ [m, k] &\mapsto [m^{\alpha_s}, k^{\beta_s}],\end{aligned}$$

with the last identity following from $[m, k] = \overline{(m)(0, k)}$. In terms of the PTR T, we thus have

$k = T(m, x, y)$ if and only if $k^{\beta_s} = T(m^{\alpha_s}, x, y^{\beta_s})$.

Thus, for all $m, x, y \in \mathcal{R}$,

$$T(m, x, y)^{\beta_s} = T(m^{\alpha_s}, x, y^{\beta_s}). \tag{7}$$

Note that Equation 7 gives (iii). Setting $y = 0$ in this equation yields

$$(m \odot x)^{\beta_s} = m^{\alpha_s} \odot x, \tag{8}$$

for all $m, x \in \mathcal{R}$. Now setting $x = 1$ we find $m^{\alpha_s} = m^{\beta_s}$; i.e. $\alpha_s = \beta_s$. Next, by setting $m = 1$ in (8), we obtain

$$x^{\beta_s} = 1^{\beta_s} \odot x = s \odot x.$$

Equation 8 now states that, for all $m, x \in \mathcal{R}$ and $s \in \mathcal{S}$,

$$s \odot (m \odot x) = (s \odot m) \odot x$$

holds. Hence $\mathcal{S} \subseteq \mathcal{N}_l$.

Having established the claimed associativity statement of the theorem, proceeding to prove $(\mathcal{S}, \odot)$ is a group isomorphic to Γ follows in much the same fashion as the proof of Theorem 10.

For (iv), assume T is linear. Then $\mathcal{S} \subseteq \mathcal{D}_l$ follows from (iii) with $m = 1$. For the converse containment, let $a \in \mathcal{N}_l \cap \mathcal{D}_l$. We define a mapping ϕ_a on $\mathcal{P}$ by

$$\begin{aligned}
(\infty) &\mapsto (\infty) \\
(m) &\mapsto (a \odot m) \\
(x, y) &\mapsto (x, a \odot y) \\
[m] &\mapsto [m] \\
[m, k] &\mapsto [a \odot m, a \odot k],
\end{aligned}$$

To prove ϕ_a is a collineation, we need to prove preservation of incidence for the lines $[m, k]$. Suppose $(x, y) \in [m, k]$. Then $k = (m \odot x) \oplus y$, and

$$\begin{aligned}
a \odot k &= a \odot ((m \odot x) \oplus y) \\
&= (a \odot (m \odot x) \oplus (a \odot y) \\
&= ((a \odot m) \odot x) \oplus (a \odot y).
\end{aligned}$$

So $(x, y)^{\phi_a} \in [m, k]^{\phi_a}$ and ϕ_a is a collineation. Clearly, $\phi_a \in \Gamma$, and $(0, 1)^{\phi_a} = (0, a)$, so that $a \in \mathcal{S}$. Hence $\mathcal{N}_l \cap \mathcal{D}_l \subseteq \mathcal{S}$ and (iv) is established. $\square$

The final result of Pickert's now follows.

Corollary 13 ([10], Theorem 6.6)**.** *Let $T : \mathcal{R}^3 \to \mathcal{R}$ be a linear planar ternary ring obtained from the projective plane $\mathcal{P}$ and where multiplication is associative. Then the full left distributive law holds in T – that is, $x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z)$ for all $x, y, z \in \mathcal{R}$ – if and only if $\mathcal{P}$ is $((\infty), [0, 0])$-transitive.*

Proof. Under hypothesis, $\mathcal{S} = \mathcal{D}_l$ by Theorem 12 (v).

Suppose the full left distributive law holds in T. Then $\mathcal{D}_l = \mathcal{R}$, implying $\Gamma((\infty), [0, 0])$ is maximal via Theorem 12 (ii). Hence $\mathcal{P}$ is $((\infty), [0, 0])$-transitive.

Now suppose $\mathcal{P}$ is $((\infty), [0, 0])$-transitive. Then $\mathcal{S} = \mathcal{R} = \mathcal{D}_l$, and so the full left distributive law holds in T. $\square$

As with the elations case and Lemma 9, the distributive law can be extended in a small way.

Lemma 14. *Let $T : \mathcal{R}^3 \to \mathcal{R}$ be a planar ternary ring obtained from the projective plane $\mathcal{P}$. Set*

$$\begin{aligned}
\mathcal{S}_0 &= \{s \in \mathcal{S} \,:\, (1, 0)^\gamma = (s, 0) \text{ for some } \gamma \in \Gamma((0), [0])\}, \\
\mathcal{S}_\infty &= \{s \in \mathcal{S} \,:\, (0, 1)^\gamma = (0, s) \text{ for some } \gamma \in \Gamma((\infty), [0, 0])\},
\end{aligned}$$

and $\mathcal{S} = \mathcal{S}_0 \cap \mathcal{S}_\infty$. Then for $s \in \mathcal{S}$, and all $x, y \in \mathcal{R}$,

$$s \odot (x \oplus y) = (s \odot x) \oplus (s \odot y).$$

Proof. Effectively, we need combine Theorem 10 (iii) with Theorem 12 (iii). From the latter, we have

$$T(s \odot m, x, s \odot y) = s \odot T(m, x, y) \tag{9}$$

for all $s \in \mathcal{S}_\infty$ and $m, x, y \in \mathcal{R}$. Note that $(\mathcal{S}, \odot)$ is a group. Let $s, t \in \mathcal{S}$. Then as $t \in \mathcal{S}_0$, the right hand side of (9) with $m = t$ is

$$s \odot T(t, x, y) = s \odot ((t \odot x) \oplus y).$$

by Theorem 10 (iii). As $s \odot t \in \mathcal{S}_0$ also, we may similarly appeal to the Theorem 10 (iii) to manipulate the left hand side of (9), obtaining

$$\begin{aligned}
T(s \odot t, x, s \odot y) &= ((s \odot t) \odot x) \oplus (s \odot y) \\
&= (s \odot (t \odot x)) \oplus (s \odot y),
\end{aligned}$$

where the last step follows from Theorem 12 (i), as $s, t \in \mathcal{S}_\infty$. The result now follows from the observation that, as x ranges over all of $\mathcal{R}$, so too does $t \odot x$. $\square$

Our next theorem is similar in style to our previous results. Nevertheless, there is one important difference we wish to highlight. The results concerning the distributive laws given in Theorems 7 and 12 are generalisations of the previously known results of Pickert. However part (i) of the following result concerning the right distributive law appears to be entirely new; the author is unaware of any result showing a partial distributive law in the PTR is implied from only one group of central collineations.

Theorem 15. *Let $T : \mathcal{R}^3 \to \mathcal{R}$ be a planar ternary ring obtained from the projective plane $\mathcal{P}$ and let $\Gamma = \Gamma((0,0),[\infty])$. Set*

$$\mathcal{S} = \{s \in \mathcal{R} \,:\, (0,1)^\gamma = (0,s) \text{ for some } \gamma \in \Gamma\}.$$

The following statements hold.

(i) $\mathcal{S} \subseteq \mathcal{N}_r \cap \mathcal{D}_r$.

(ii) $(\mathcal{S}, \odot)$ *is a group isomorphic to* Γ.

(iii) For all $m, x, y \in \mathcal{R}$, $s \in \mathcal{S}$, *we have* $T(m, x \odot s, y \odot s) = T(m,x,y) \odot s$.

(iv) If T *is linear, then* $\mathcal{S} = \mathcal{N}_r \cap \mathcal{D}_r$.

Proof. From Lemma 3, we know any $\gamma \in \Gamma$ is uniquely determined by the image of any one point not the centre $(0,0)$ nor on the axis $[\infty]$. For $s \in \mathcal{S}$, denote by γ_s the unique element of γ that maps $(0,1)$ to $(0,s)$. Now (m), (∞) and $(0,0)$ are fixed by γ_s. As $[0] = \overline{(0,0)(\infty)}$, we see $[0]$ is also fixed by γ_s. Consequently, there exists a bijection σ_s of $\mathcal{R}$ satisfying $(0,y)^{\gamma_s} = (0, y^{\sigma_s})$ with $0^{\sigma_s} = 0$ and $1^{\sigma_s} = s$. Thus

$$[m,k]^{\gamma_s} = \overline{(m)^{\gamma_s}(0,k)^{\gamma_s}} = [m, k^{\sigma_s}].$$

Further, since $\overline{(1)(0,x)} \cap [0,0] = (x,0)$, $(x,0)^{\gamma_s} = (x^{\sigma_s}, 0)$. Hence $[x]^{\gamma_s} = [x^{\sigma_s}]$. Finally, we see

$$(x,y)^{\gamma_s} = [x]^{\gamma_s} \cap [0,y]^{\gamma_s} = (x^{\sigma_s}, y^{\sigma_s}).$$

This completes the description of how γ_s acts on the plane:

$$\begin{aligned} (m) &\mapsto (m) \\ (x,y) &\mapsto (x^{\sigma_s}, y^{\sigma_s}) \\ [m] &\mapsto [m^{\sigma_s}] \\ [m,k] &\mapsto [m, k^{\sigma_s}]. \end{aligned}$$

It follows that $T(m,x,y) = k$ if and only if $T(m, x^{\sigma_s}, y^{\sigma_s}) = k^{\sigma_s}$. Consequently, we have the identity

$$T(m,x,y)^{\sigma_s} = T(m, x^{\sigma_s}, y^{\sigma_s}) \tag{10}$$

for all $m,x,y \in \mathcal{R}$. Setting $y = 0$ and $x = 1$ yields $m^{\sigma_s} = m \odot s$ for all $m \in \mathcal{R}$. Combining this fact with (10) proves (iii).

Returning to (10) with $y = 0$ we find

$$(m \odot x) \odot s = m \odot (x \odot s)$$

for all $m, x \in \mathcal{R}$ and $s \in \mathcal{S}$. Thus $\mathcal{S} \subseteq \mathcal{N}_r$. Again we return to (10), this time setting $m = 1$, to find

$$T(1,x,y) \odot s = (x \oplus y) \odot s = T(1, x \odot s, y \odot s) = (x \odot s) \oplus (y \odot s),$$

which proves $\mathcal{S} \subseteq \mathcal{D}_r$, and (i) is established.

The proof of (ii) proceeds in almost identical fashion to the corresponding part of the proof of Theorem 10.

Now let $a \in \mathcal{N}_r \cap \mathcal{D}_r$ and suppose T is linear. We define a map ϕ_a on $\mathcal{P}$ by

$$\begin{aligned}(\infty) &\mapsto (\infty)\\ (m) &\mapsto (m)\\ (x,y) &\mapsto (x \odot a, y \odot a)\\ [m] &\mapsto [m \odot a]\\ [m,k] &\mapsto [m, k \odot a].\end{aligned}$$

Suppose $(x,y) \in [m,k]$, so that $k = (m \odot x) \oplus y$. Then

$$\begin{aligned}k \odot a &= ((m \odot x) \oplus y) \odot a\\ &= ((m \odot x) \odot a) \oplus (y \odot a)\\ &= (m \odot (x \odot a)) \oplus (y \odot a),\end{aligned}$$

meaning $(x,y)^{\phi_a} \in [m,k]^{\phi_a}$. Thus $\phi_a \in \Gamma$. Since $(0,1)^{\phi_a} = (0,a)$, $a \in \mathcal{S}$, and so $\mathcal{S} = \mathcal{N}_r \cap \mathcal{D}_r$, which is the final claim. □

Theorem 15 yields a new result in the style of Pickert.

Corollary 16. *Let $T : \mathcal{R}^3 \to \mathcal{R}$ be a linear planar ternary ring obtained from the projective plane $\mathcal{P}$. Then $\mathcal{P}$ is $((0),[0])$-transitive and the full right distributive law holds in T – that is, $(x \oplus y) \odot z = (x \odot z) \oplus (y \odot z)$ for all $x, y, z \in \mathcal{R}$ – if and only if $\mathcal{P}$ is $((0,0),[\infty])$-transitive.*

Proof. By hypothesis, Theorem 15 (iv) tells us $\mathcal{S} = \mathcal{N}_r \cap \mathcal{D}_r$.

Suppose $\mathcal{P}$ is $((0),[0])$-transitive and T exhibits the full right distributive law. Then the latter implies $\mathcal{D}_r = \mathcal{R}$, while the former implies, via Theorem 10, that $(\mathcal{R}^*, \odot)$ is a group, and so $\mathcal{N}_r = \mathcal{R}$ also. Hence $\mathcal{S} = \mathcal{R}$, and $\Gamma((0,0),[\infty])$ is maximal, from which we conclude $\mathcal{P}$ is $((0,0),[\infty])$-transitive.

Now suppose $\mathcal{P}$ is $((0,0),[\infty])$-transitive. Then $\mathcal{R} = \mathcal{S} = \mathcal{N}_r \cap \mathcal{D}_r \subseteq \mathcal{R}$. Firstly, this implies the full right distributive law holds in T. Secondly, since $\mathcal{N}_r = \mathcal{R}$, we have $(\mathcal{R}^*, \odot)$ is a group. Now Theorem 10 tells us $\Gamma((0),[0])$ is maximal, and so $\mathcal{P}$ is $((0),[0])$-transitive. □

It should be pointed out that the corollary's hypothesis is met only when the plane is $((0),[0])$-transitive or $((\infty),[\infty])$-transitive. In the latter case, the Lenz-Barlotti classification tells us that the plane must be Desarguesian – type VII.2 – if it is both $((\infty),[\infty])$- and $((0,0),[\infty])$-transitive. Consequently, it could be argued that the more interesting case is really the situation where the plane is assumed to be $((0),[0])$-transitive, whereby the corollary could be modified accordingly. This is basically the reason we have not previously seen a partial distributive law arising from a single group before.

Epilogue

There are no doubt more results to be derived here, but the author was discouraged from continuing further by a colleague in projective geometry who saw an earlier version of the current draft, mostly on the basis that it was all known. The author has not managed to corroborate this in its entirety, but certainly some of the results herein are known and published elsewhere.

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Appendix B

THE COEFFICIENT POLYNOMIAL IN THE PTR POLYNOMIAL OF THE FIGUEROA PLANE

The coefficient polynomial $r(X, Y, Z)$ in Equation (5.6.3):

$$
\begin{aligned}
&X^{11}Y^{18}Z^{15} + X^{11}Y^{18}Z^{13} + X^{11}Y^{18}Z^{11} + 2X^{11}Y^{18}Z^{9} + 2X^{11}Y^{18}Z^{7} \\
&\quad + 2X^{11}Y^{18}Z^{5} + 2X^{11}Y^{18}Z^{3} + X^{11}Y^{18}Z + X^{11}Y^{12}Z^{15} + X^{11}Y^{12}Z^{13} \\
&\quad + X^{11}Y^{12}Z^{11} + X^{11}Y^{12}Z^{9} + 2X^{11}Y^{12}Z^{7} + 2X^{11}Y^{12}Z^{5} + X^{11}Y^{12}Z^{3} \\
&\quad + X^{11}Y^{10}Z^{9} + X^{11}Y^{10}Z^{3} + X^{11}Y^{10}Z + 2X^{11}Y^{6}Z^{9} + X^{11}Y^{6}Z^{3} + 2X^{11}Y^{4}Z^{15} \\
&\quad + 2X^{11}Y^{4}Z^{13} + 2X^{11}Y^{4}Z^{11} + X^{11}Y^{4}Z^{9} + X^{11}Y^{4}Z^{7} + X^{11}Y^{4}Z^{5} + 2X^{11}Y^{2}Z^{15} \\
&\quad + 2X^{11}Y^{2}Z^{13} + 2X^{11}Y^{2}Z^{11} + 2X^{11}Y^{2}Z^{9} + X^{11}Y^{2}Z^{7} + X^{11}Y^{2}Z^{5} \\
&\quad + X^{11}Y^{2}Z^{3} + X^{11}Y^{2}Z + 2X^{11}Z^{15} + 2X^{11}Z^{13} + 2X^{11}Z^{11} + X^{11}Z^{7} + X^{11}Z^{5} \\
&\quad + X^{11}Z^{3} + 2X^{10}Y^{21}Z^{18} + X^{10}Y^{21}Z^{12} + X^{10}Y^{21}Z^{10} + 2X^{10}Y^{21}Z^{4} + X^{10}Y^{19}Z^{18} \\
&\quad + 2X^{10}Y^{19}Z^{12} + 2X^{10}Y^{19}Z^{10} + X^{10}Y^{19}Z^{4} + X^{10}Y^{15}Z^{18} + X^{10}Y^{15}Z^{10} \\
&\quad + X^{10}Y^{15}Z^{2} + 2X^{10}Y^{15} + X^{10}Y^{13}Z^{12} + 2X^{10}Y^{13}Z^{10} + 2X^{10}Y^{13}Z^{4} \\
&\quad + X^{10}Y^{13}Z^{2} + 2X^{10}Y^{13} + 2X^{10}Y^{11}Z^{18} + 2X^{10}Y^{11}Z^{12} + X^{10}Y^{11}Z^{4} \\
&\quad + X^{10}Y^{11}Z^{2} + 2X^{10}Y^{11} + 2X^{10}Y^{9}Z^{6} + 2X^{10}Y^{9}Z^{4} + 2X^{10}Y^{9}Z^{2} + X^{10}Y^{9} \\
&\quad + 2X^{10}Y^{7}Z^{18} + 2X^{10}Y^{7}Z^{10} + 2X^{10}Y^{7}Z^{2} + X^{10}Y^{7} + X^{10}Y^{5}Z^{18} + X^{10}Y^{5}Z^{12} \\
&\quad + 2X^{10}Y^{5}Z^{4} + 2X^{10}Y^{5}Z^{2} + X^{10}Y^{5} + X^{10}Y^{3}Z^{6} + X^{10}Y^{3}Z^{4} + X^{10}Y^{3}Z^{2} \\
&\quad + 2X^{10}YZ^{12} + X^{10}YZ^{10} + X^{10}YZ^{4} + 2X^{10}YZ^{2} + X^{9}Y^{18}Z^{21} + 2X^{9}Y^{18}Z^{19}
\end{aligned}
$$

$$+ X^9Y^{18}Z^{15} + 2X^9Y^{18}Z^{13} + 2X^9Y^{18}Z^7 + X^9Y^{18}Z^3 + 2X^9Y^{12}Z^{21}$$
$$+ X^9Y^{12}Z^{19} + X^9Y^{12}Z^{15} + 2X^9Y^{12}Z^{11} + X^9Y^{12}Z^9 + 2X^9Y^{12}Z^7$$
$$+ X^9Y^{12}Z^5 + 2X^9Y^{12}Z^3 + 2X^9Y^{10}Z^{21} + X^9Y^{10}Z^{19} + 2X^9Y^{10}Z^{13}$$
$$+ X^9Y^{10}Z^{11} + 2X^9Y^{10}Z^9 + 2X^9Y^{10}Z^5 + 2X^9Y^{10}Z^3 + 2X^9Y^6Z^3 + X^9Y^6Z$$
$$+ X^9Y^4Z^{21} + 2X^9Y^4Z^{19} + 2X^9Y^4Z^{15} + X^9Y^4Z^{11} + 2X^9Y^4Z^9 + X^9Y^4Z^7$$
$$+ 2X^9Y^4Z^5 + X^9Y^4Z + 2X^9Y^2Z^{15} + 2X^9Y^2Z^{13} + 2X^9Y^2Z^{11} + X^9Y^2Z^9$$
$$+ X^9Y^2Z^7 + X^9Y^2Z^5 + 2X^9Y^2Z^3 + X^9Y^2Z + 2X^9Z^{15} + 2X^9Z^{13} + 2X^9Z^{11}$$
$$+ X^9Z^9 + X^9Z^7 + X^9Z^5 + X^9Z^3 + 2X^9Z + X^8Y^9 + 2X^8Y^3Z^6 + 2X^8Y^3Z^4$$
$$+ 2X^8Y^3Z^2 + 2X^8Y^3 + X^8YZ^6 + X^8YZ^4 + X^8YZ^2 + 2X^7Y^{18}Z^{15} + X^7Y^{18}Z^7$$
$$+ 2X^7Y^{18}Z^{13} + 2X^7Y^{18}Z^{11} + X^7Y^{18}Z^5 + X^7Y^{18}Z^3 + X^7Y^{12}Z^{15} + X^7Y^{12}Z^{13}$$
$$+ X^7Y^{12}Z^{11} + X^7Y^{12}Z^9 + 2X^7Y^{12}Z^7 + 2X^7Y^{12}Z^5 + 2X^7Y^{12}Z^3 + 2X^7Y^{12}Z$$
$$+ X^7Y^{10}Z^{15} + X^7Y^{10}Z^{13} + X^7Y^{10}Z^{11} + 2X^7Y^{10}Z^9 + 2X^7Y^{10}Z^7$$
$$+ 2X^7Y^{10}Z^5 + 2X^7Y^{10}Z^3 + X^7Y^{10}Z + 2X^7Y^4Z^{15} + 2X^7Y^4Z^{13} + 2X^7Y^4Z^{11}$$
$$+ 2X^7Y^4Z^9 + X^7Y^4Z^7 + X^7Y^4Z^5 + X^7Y^4Z^3 + X^7Y^4Z + X^7Y^2Z^9$$
$$+ 2X^7Y^2Z + 2X^7Z^9 + 2X^7Z^3 + 2X^7Z + 2X^6Y^{15}Z^{18} + X^6Y^{15}Z^{12} + X^6Y^{15}Z^{10}$$
$$+ 2X^6Y^{15}Z^4 + 2X^6Y^{13}Z^{18} + X^6Y^{13}Z^{12} + X^6Y^{13}Z^{10} + 2X^6Y^{13}Z^4$$
$$+ 2X^6Y^{11}Z^{18} + X^6Y^{11}Z^{12} + X^6Y^{11}Z^{10} + 2X^6Y^{11}Z^4 + X^6Y^9Z^{12}$$
$$+ 2X^6Y^9Z^{10} + 2X^6Y^9Z^4 + X^6Y^9Z^2 + 2X^6Y^9 + X^6Y^7Z^{18} + 2X^6Y^7Z^{12}$$
$$+ 2X^6Y^7Z^{10} + X^6Y^7Z^4 + X^6Y^5Z^{18} + 2X^6Y^5Z^{12} + 2X^6Y^5Z^{10} + X^6Y^5Z^4$$
$$+ X^6Y^3Z^{18} + 2X^6Y^3Z^{12} + 2X^6Y^3Z^{10} + X^6Y^3Z^4 + 2X^6Y^3 + 2X^6YZ^{12}$$
$$+ X^6YZ^{10} + X^6YZ^4 + 2X^6YZ^2 + 2X^6Y + 2X^5Y^{18}Z^{15} + 2X^5Y^{18}Z^{13}$$
$$+ 2X^5Y^{18}Z^{11} + X^5Y^{18}Z^7 + X^5Y^{18}Z^5 + X^5Y^{18}Z^3 + X^5Y^{12}Z^{15} + X^5Y^{12}Z^{13}$$

$$
\begin{aligned}
&+ X^5Y^{12}Z^{11} + X^5Y^{12}Z^9 + 2X^5Y^{12}Z^7 + 2X^5Y^{12}Z^5 + 2X^5Y^{12}Z^3 + 2X^5Y^{12}Z \\
&+ X^5Y^{10}Z^{15} + X^5Y^{10}Z^{13} + X^5Y^{10}Z^{11} + 2X^5Y^{10}Z^9 + 2X^5Y^{10}Z^7 + X^5Y^{10}Z \\
&+ 2X^5Y^{10}Z^5 + 2X^5Y^{10}Z^3 + 2X^5Y^4Z^{15} + 2X^5Y^4Z^{13} + 2X^5Y^4Z^{11} + 2X^5Y^4Z^9 \\
&+ X^5Y^4Z^7 + X^5Y^4Z^5 + X^5Y^4Z^3 + X^5Y^4Z + X^5Y^2Z^9 + 2X^5Y^2Z \\
&+ 2X^5Z^9 + 2X^5Z^3 + 2X^5Z + 2X^4Y^{15}Z^{18} + X^4Y^{15}Z^{12} + X^4Y^{15}Z^{10} + 2X^4Y^{15}Z^4 \\
&+ 2X^4Y^{13}Z^{18} + X^4Y^{13}Z^{12} + X^4Y^{13}Z^{10} + 2X^4Y^{13}Z^4 + 2X^4Y^{11}Z^{18} \\
&+ X^4Y^{11}Z^{12} + X^4Y^{11}Z^{10} + 2X^4Y^{11}Z^4 + X^4Y^9Z^{12} + 2X^4Y^9Z^{10} + 2X^4Y^9Z^4 \\
&+ X^4Y^9Z^2 + 2X^4Y^9 + X^4Y^7Z^{18} + 2X^4Y^7Z^{12} + 2X^4Y^7Z^{10} + X^4Y^7Z^4 \\
&+ X^4Y^5Z^{18} + 2X^4Y^5Z^{12} + 2X^4Y^5Z^{10} + X^4Y^5Z^4 + X^4Y^3Z^{18} + 2X^4Y^3Z^{12} \\
&+ 2X^4Y^3Z^{10} + X^4Y^3Z^4 + 2X^4Y^3 + 2X^4YZ^{12} + X^4YZ^{10} + X^4YZ^4 + 2X^4YZ^2 \\
&+ 2X^4Y + 2X^3Y^6Z^3 + X^3Y^6Z + 2X^3Y^4Z^3 + X^3Y^4Z + 2X^3Y^2Z^3 + X^3Y^2Z \\
&+ X^3Z^9 + 2X^3Z^3 + X^2Y^{21}Z^{18} + 2X^2Y^{21}Z^{12} + 2X^2Y^{21}Z^{10} + X^2Y^{21}Z^4 \\
&+ 2X^2Y^{19}Z^{18} + X^2Y^{19}Z^{12} + X^2Y^{19}Z^{10} + 2X^2Y^{19}Z^4 + X^2Y^{15}Z^{18} \\
&+ X^2Y^{15}Z^{12} + 2X^2Y^{15}Z^4 + 2X^2Y^{15}Z^2 + 2X^2Y^{15} + 2X^2Y^{13}Z^{18} + 2X^2Y^{13}Z^{10} \\
&+ 2X^2Y^{13}Z^2 + 2X^2Y^{13} + 2X^2Y^{11}Z^{12} + X^2Y^{11}Z^{10} + X^2Y^{11}Z^4 + 2X^2Y^{11}Z^2 \\
&+ 2X^2Y^{11} + X^2Y^9Z^{12} + 2X^2Y^9Z^{10} + 2X^2Y^9Z^4 + X^2Y^9Z^2 + X^2Y^9 + XY^{18}Z^5 \\
&+ 2X^2Y^7Z^{18} + 2X^2Y^7Z^{12} + X^2Y^7Z^4 + X^2Y^7Z^2 + X^2Y^7 + X^2Y^5Z^{12} \\
&+ 2X^2Y^5Z^{10} + 2X^2Y^5Z^4 + X^2Y^5Z^2 + X^2Y^5 + X^2Y^3Z^{18} + 2X^2Y^3Z^{12} \\
&+ 2X^2Y^3Z^{10} + 2X^2Y^3Z^6 + 2X^2Y^3Z^2 + X^2Y^3 + X^2YZ^6 + X^2YZ^4 + X^2YZ^2 \\
&+ 2X^2Y + 2XY^{18}Z^{21} + XY^{18}Z^{19} + XY^{18}Z^{15} + 2XY^{18}Z^{11} + 2XY^{18}Z^7 \\
&+ XY^{12}Z^{21} + 2XY^{12}Z^{19} + XY^{12}Z^{13} + 2XY^{12}Z^{11} + XY^{12}Z^5 + 2XY^{12}Z \\
&+ XY^{10}Z^{21} + 2XY^{10}Z^{19} + XY^{10}Z^{15} + 2XY^{10}Z^{13} + 2XY^{10}Z^7 + XY^{10}Z
\end{aligned}
$$

$$+ 2XY^6Z^9 + XY^6Z^3 + 2XY^4Z^{21} + XY^4Z^{19} + 2XY^4Z^{13} + XY^4Z^{11} + 2XY^4Z^9$$
$$+ 2XY^4Z^5 + XY^4Z^3 + XY^4Z + XY^2Z^{15} + XY^2Z^{13} + XY^2Z^{11} + 2XY^2Z^9$$
$$+ 2XY^2Z^7 + 2XY^2Z^5 + XY^2Z^3 + 2XY^2Z + 2XZ^{15} + 2XZ^{13} + 2XZ^{11} + XZ^9$$
$$+ XZ^7 + XZ^5 + Y^{15}Z^{18} + Y^{15}Z^{12} + 2Y^{15}Z^4 + 2Y^{15}Z^2 + 2Y^{15} + Y^{13}Z^{18}$$
$$+ Y^{13}Z^{12} + 2Y^{13}Z^4 + 2Y^{13}Z^2 + 2Y^{13} + Y^{11}Z^{18} + Y^{11}Z^{12} + 2Y^{11}Z^4 + 2Y^{11}Z^2$$
$$+ 2Y^{11} + 2Y^9Z^{18} + Y^9Z^{12} + Y^9Z^{10} + 2Y^9Z^6 + Y^9Z^4 + 2Y^9Z^2 + 2Y^7Z^{18}$$
$$+ 2Y^7Z^{12} + Y^7Z^4 + Y^7Z^2 + Y^7 + 2Y^5Z^{18} + 2Y^5Z^{12} + Y^5Z^4 + Y^5Z^2 + Y^5$$
$$+ 2Y^3Z^{18} + Y^3Z^{12} + Y^3Z^{10} + Y^3Z^6 + Y^3Z^2 + Y^3 + YZ^{18} + YZ^{10} + YZ^2$$