Version of Record at: https://doi.org/10.1371/journal.pone.0285322

PLOS ONE

RESEARCH ARTICLE

A traveler-centric mobility game: Efficiency and stability under rationality and prospect theory

Ioannis Vasileios Chremos*, Andreas A. Malikopoulos

Department of Mechanical Engineering, University of Delaware, Newark, Delaware, United States of America

* ichremos@udel.edu

OPEN ACCESS

Citation: Chremos IV, Malikopoulos AA (2023) A traveler-centric mobility game: Efficiency and stability under rationality and prospect theory. PLoS ONE 18(5): e0285322. https://doi.org/10.1371/journal.pone.0285322

Editor: Praveen Kumar Donta, TU Wien: Technische Universitat Wien, AUSTRIA

Received: February 8, 2023

Accepted: April 19, 2023

Published: May 5, 2023

Data Availability Statement: All relevant data to reproduce the figures are within the paper.

Funding: This work has been supported by the US National Science Foundation under Grant CNS-2149520 (https://www.nsf.gov/). The funders had no role in study design, data collection and analysis, decision to publish, or preparation of the manuscript.

Competing interests: The authors have declared that no competing interests exist.

Abstract

In this paper, we study a routing and travel-mode choice problem for mobility systems with a multimodal transportation network as a "mobility game" with coupled action sets. We formulate an atomic routing game to focus on the travelers' preferences and study the impact on the efficiency of the travelers' behavioral decision-making under rationality and prospect theory. To control the innate inefficiencies, we introduce a mobility "pricing mechanism," in which we model traffic congestion using linear cost functions while also considering the waiting times at different transport hubs. We show that the travelers' selfish actions lead to a pure-strategy Nash equilibrium. We then perform a Price of Anarchy and Price of Stability analysis to establish that the mobility system's inefficiencies remain relatively low and the social welfare at a NE remains close to the social optimum as the number of travelers increases. We deviate from the standard game-theoretic analysis of decision-making by extending our mobility game to capture the subjective behavior of travelers using prospect theory. Finally, we provide a detailed discussion of implementing our proposed mobility game.

Introduction

Motivation

Commuters in big cities have continuously experienced the frustration of congestion and traffic jams [1]. Travel delays, accidents, and road altercations have consistently impacted the economy, society, and the natural environment regarding energy and pollution [2]. One of the pressing challenges of our time is the increasing demand for energy, which requires us to make fundamental transformations in how our societies use and access transportation [3]. Thanks to the technological evolution of mobility (e.g., electrification of vehicles, smart mobility with self-driving cars, and improved vehicle sensor technology [4]) it is highly expected that we will be able to eliminate congestion while significantly increasing mobility efficiency in terms of energy and travel time [5]. Several studies have shown the benefits of *emerging mobility systems* (EMS) (e.g., ride-hailing, on-demand mobility services, shared vehicles, self-driving cars) in reducing energy and alleviating traffic congestion in a number of different transportation scenarios [6–10]. For a thorough review of models and the possible methods and techniques for smart mobility-on-demand systems, see [11].

The cyber-physical nature (e.g., data and shared information) of EMS is associated with significant control challenges and gives rise to a new level of complexity in modeling and control [12]. Research efforts over the last twenty years have tended to focus on the technological dimension. What is missing is a complementary theoretical study of the broader social implications of smart mobility. The impact of selfish social behavior in routing networks of regular and autonomous vehicles has been studied in [13–15]. Other efforts have addressed "how people learn and make routing decisions" with behavioral dynamics [16, 17]. A game-theoretic framework using sequential games was proposed to study the socioeconomic interactions and the different tradeoffs that emerge between the mobility stakeholders of a mobility "ecosystem" [18]. It seems, though, that the problem of how automation in mobility will affect the tendency to travel and decision-making has not been adequately approached yet. In a recent study [19], it was shown that when daily commuters were offered a convenient and affordable taxi service for their travels, a change of behavior was noticed; the commuters adjusted their travel behavior and activities and used the taxi service considerably more often leading to an 83% overall increase in vehicle miles traveled. Along with other similar studies [20, 21], this shows that EMS most probably may affect people's tendency to travel and incentivize them to use cars more frequently, which potentially can also lead to a shift away from public transit.

In this paper, we are interested in one open question: *Can we devise appropriate mobility prices in a game-theoretic framework for a multimodal transportation system that considers rational or subjective decision-making?* Ideally, we want these prices to lead to an equilibrium that ensures an efficient mobility system. To address this question, we first need to understand the behavioral interactions of travelers with different modes of transportation along with the implications to system efficiency. Thus, we study the game-theoretic interactions of travelers seeking to travel in a transportation network comprised of roads used by different modes of transportation (e.g., cars, buses, light rail, and bikes). A key characteristic of our approach is that we adopt the Mobility-as-a-Service (MaaS) concept, i.e., a multimodal mobility system that centrally handles the travelers' information and provides travel services (e.g., navigation, location, booking, payment). We aim to provide a game-theoretic framework that captures the most significant factors of a traveler's decision-making in a transportation network under two different behavioral models.

Literature review

One of the standard approaches to alleviate congestion in a transportation system has been through "congestion pricing." With this approach, we can manage the travelers' demand while also considering the scarcity of road space or limited public transit supply [22, 23]. Congestion pricing has been instrumental in intelligent and scalable traffic routing, in which the aim is to optimize routing decisions and "guide" travelers via appropriate prices [24–26]. To understand the innate competition over the scarce resources in transportation and the impact of congestion, pricing game theory has been one of the standard tools that can help us investigate the impact of selfishness and traffic efficiency [27, 28]. By adopting a game-theoretic approach, advanced systems have been proposed to assign travelers concrete routes or minimize all travelers' travel time while studying the system's Nash equilibrium (NE) under different mechanisms of congestion pricing [29–37]. Key theoretical games used for this approach include *routing/congestion games* [14, 38–41], which are a generalization of the standard resource-sharing game of an arbitrary number of resources in a network with a finite number of travelers. For example, each traveler may contribute a certain amount of traffic from a source to a destination and affect the overall congestion on a route, thus increasing the travel time for all other travelers.

Another important class of games is *potential games*, first introduced in [42]. In a potential game, the incentive of all players to change their strategy can be expressed using a single global function called the "potential" function, which depends on the action sets of all players. As it capture the changes of utility as the actions vary, potential games have been used extensively in wide-ranging applications (taxation of public goods [43], economics of shallow lakes [44], and electricity markets [45]. Routing/congestion and potential games have played an instrumental role in understanding competition over shared resources. Both classes of games have been studied in multiple disciplines to model transportation and communication networks [23, 46–48] and common-pool resource games in economics [49]. Thus, for our purposes, it is most appropriate to formulate our problem as a routing game and show that there exists a potential function. This way, we can keep the mathematical analysis tractable and still be able to draw key insights on the impact of the travelers' decision-making. We deviate from the current state of the art in transportation with game theory by considering *negative congestion externalities*. Namely, we suppose that if the number of co-travelers that utilize the same route or mode of transportation increases, then a traveler's utility decreases too [50–52].

So far, most of the existing game-theoretical literature in transportation and routing/congestion games assumes that players' are *rational and intelligent*, i.e., each player is a risk-neutral, selfish, utility maximizer. This makes transportation models quite unrealistic, as unexpected travel delays can lead to uncertainty in a traveler's utility. Irrational decision-making over uncertainties and risks in utility can play a significant role, and its study can help us understand how large-scale systems perform inefficiently. To support this assertion, empirical experiments were conducted that showed how a human's choices or preferences systematically deviate from the choice or preferences of a rational game-theoretic player [53, 54]. For example, humans compare the outcomes of their choices to a known expected amount of utility (called reference) and decide, based on that reference, whether their utility is a gain or loss. *Prospect theory* has laid down the theoretical foundations to study such biases and the subjective perception of risk in the utility of humans [55]. Prospect theory has been recognized as a closer-to-reality behavioral model for the decision-making of humans and has been used in a wide range of applications and fields [56], including recent studies in engineering [57–59]. There has also been considerable work at the intersection of transportation studies and prospect theory [60–62]. Thus, in this paper, we offer a game-theoretic formulation for the routing and travel-mode choice problem and propose a pricing mechanism that successfully leads to an efficient equilibrium under the standard assumption of rationality and prospect theory.

Contribution of this paper

In this paper, we formulate a mobility game for the travelers' routing and travel-mode choices in a multimodal transportation network. We study the existence of a NE and the resulting inefficiencies of the travelers' decision-making. Our main contribution is to show that although we cannot guarantee equilibrium uniqueness, our pricing mechanism allows us to control the inefficiencies that arise at equilibrium and we can then derive an upper bound for the PoA. In particular, our mobility game considers the impacts of negative congestion externalities and waiting costs in travelers' decision-making. That way, we offer an improved look at the socio-economic factors that can affect the efficient and sustainable distribution of travel demand in a transportation network with multiple different modes of transportation (e.g., car, bus, light rail, bike). Moreover, we study the travelers' decision-making under two behavioral models: (1) rational, where players are selfish and seek to maximize only their own utility; and (2) prospect theory, where the players' biases and subjectivity are taken into account when decisions are made under risk.

The features that distinguish our work from the state of the art are as follows:

1. we model the interactions between travelers using a mobility game, a combination of an atomic routing and potential game with travel-mode choices and coupled action sets (see Section "Modeling Framework");
2. we take into account the traffic congestion cost factors using linear cost functions and the waiting time of travelers at different transport hubs; each transport hub allows a traveler to choose any of the available modes of transportation to utilize for their travel needs (see Section "Modeling Framework");
3. we introduce a mobility pricing mechanism to control travel demand (both regarding the route and mobility service) and study the inefficiencies at a NE by showing that a NE exists (Theorem 1) and deriving a bound that remains small enough as the number of travelers increases (Theorem 2); in addition, we also upper bound the ratio of the maximum social welfare that can be achieved by all travelers to the maximum social welfare that is achieved at a NE (Theorem 3), and
4. we incorporate prospect theory for the travelers' decision-making under the uncertainty of the transport hubs' budgets (Theorem 4).

Organization of the paper

The remainder of the paper is structured as follows. First, we present the mathematical formulation of our mobility game, which forms the basis of our theoretical study in this paper. Then, we derive the properties of our mobility game, i.e., we show NE existence, and we bound the Price of Anarchy (PoA) and the Price of Stability (PoS). Then we prove that a NE exists under prospect theory. Finally, we draw conclusions and offer a discussion of future research.

Modeling framework

We consider a mobility system of two finite, disjoint, and non-empty sets, (1) the set of travelers $\mathcal{I}$, $|\mathcal{I}| = I \in \mathbb{N}_{\geq 2}$, and (2) the set of mobility services by $\mathcal{J}$, $|\mathcal{J}| = J \in \mathbb{N}$. For example, $j \in \mathcal{J}$ can represent either a car, a bus, a light rail vehicle, or a bike. We consider that in our mobility game, $I < J$. The set of all mobility services $\mathcal{J}$ can be partitioned to a finite number of disjoint subsets, each representing a specific type of a mobility service, i.e., $\mathcal{J} = \bigcup_{h=1}^{H} \mathcal{J}_h$, where $H \in \mathbb{N}$ is the total number of subsets of $\mathcal{J}$. For example, if there are only two modes of transportation, say cars and buses, then $\mathcal{J} = \mathcal{J}_1 \cup \mathcal{J}_2$, where $\mathcal{J}_1$ represents the subset of all available cars, and $\mathcal{J}_2$ represents the subset of all available buses.

Definition 1. The set of all different types of services is $\mathcal{H} = \{1, \ldots, H\}$, $H \in \mathbb{N}$, where each element $h \in \mathcal{H}$ represents a possible travel option. We denote the type of service *j* used by traveler *i* by $h_i \in \mathcal{H}$.

For example, suppose $H = 4$. Then each element $h \in \mathcal{H}$ can be associated one-to-one to the elements of the set {car, bus, light rail, bike}.

Naturally, each service can accommodate up to a finite number of travelers, that is different for each type of service. So, we expect the "physical traveler capacity" of each service to vary significantly.

Definition 2. Each service $j \in \mathcal{J}$ is characterized by a current *physical traveler capacity*, i.e., $\varepsilon_j \in \{0, 1, 2, \ldots, \bar{\varepsilon}_j\}$, where $\bar{\varepsilon}_j \in \mathbb{N}$ denotes the maximum traveler capacity of service *j*.

For example, one bus can provide travel services up to eighty travelers (seated and standing) compared to a bike-sharing company's bike (one bike per traveler).

Travelers seek to travel in a transportation network represented by a directed multigraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where each node in $\mathcal{V}$ represents a city area (neighborhood) with a "transport hub," i.e., a central place where travelers can use different modes of transportation. Each edge $\mathcal{E}$ represents a sequence of city roads with public transit lanes. For our purposes, we think of $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ as a representation of a smart city network with a road and public transit infrastructure. In network $\mathcal{G}$, any traveler $i \in \mathcal{I}$ seeks to travel from an origin $o \in \mathcal{V}$ to a destination $d \in \mathcal{V}$ while making optional stops at a self-chosen transport hub $v_i \in \mathcal{V}$. So, on one hand, all travelers are associated with the same origin-destination pair (o, d). On the other hand, travelers can make a stop along their route. Next, each type of mobility services $h \in \mathcal{H}$ is associated with a sequence of edges, i.e., a route that connects at least two nodes (or transport hubs). We say that there exists a set of routes for each traveler i where each route connects their origin-destination pair (o, d) and can be traveled by any mobility service. Formally, we have $\mathcal{P}^{(o,d)} \subset 2^{\mathcal{E}}$ to denote the set of routes available to traveler i in origin-destination (o, d), where each route in $\mathcal{P}^{(o,d)}$ consists of a set of edges connecting o to d.

Each traveler $i \in \mathcal{I}$ seeks to travel in network $\mathcal{G}$ using one of the available mobility services $j \in \mathcal{J}$ of type $h \in \mathcal{H}$. Thus, any traveler can choose the type of mobility service they prefer for their specific travel needs. Note that any $j \in \mathcal{J}$ can use any edge. Thus, travelers compete with each other for the available services in the transportation network. For the purposes of this work, we restrict our attention to all available modes of transportation that use the road infrastructure. In addition, each transport hub (including the ones at (o, d) allows travel by any mobility service (any mode of transportation), thus a traveler's travel preferences or needs can be satisfied by many and different mobility services (as one expects from a multimodal transportation network).

Selfish behavior, however, may lead to inefficiencies. Therefore, as part of our efforts to *control* the inefficiencies that arise from the travelers' selfishness (and thus control the emergence of rebound effects), in our mobility game, we introduce the idea of a "mobility pricing mechanism" to incentivize travelers to use services in public transit for their travel needs. Informally, each transport hub starts with a budget, collects payments for services, and then provides monetary incentives (pricing mechanism) to travelers to ensure a *socially-efficient* utilization of services in the network. By "socially-efficient," we mean that the endmost collective travel outcomes must achieve two objectives: (i) respect and satisfy the travelers' preferences regarding mobility, and (ii) ensure the alleviation of congestion in the system. We formalize the idea of our mobility pricing mechanism in the following definitions.

Definition 3. Each traveler i starts with a *mobility wallet* represented in monetary units by $\theta_i \in [0, \bar{\theta}_i]$, where $\bar{\theta}_i \in \mathbb{R}_{>0}$ is the maximum amount of traveler i's monetary units. Traveler i uses their wallet θ_i to pay for their travels in-network $\mathcal{G}$.

The mobility wallet for each traveler represents the financial resources available to them, i.e., the amount of money they have to spend on transportation. In addition, by introducing mobility wallets, we can realistically model the travelers' cost-constrained decision-making, where different transportation options have different costs.

Definition 4. Any traveler i is required to pay a fee, called "mobility payment," for using a mobility service in network $\mathcal{G}$. This mobility payment is given by function $\pi_i : \mathcal{H} \times \mathbb{N} \to \mathbb{R}_{\geq 0}$, where $0 \leq \pi_i(h_i, \varepsilon_j) \leq \theta_i$.

Note that $\pi_i(h_i, \varepsilon_j)$ has the same monetary units as θ_i in Definition 3. Intuitively, a traveler i pays π_i for using mobility service j of type h_i. The mobility payment π_i of any traveler i varies extensively for each type of service h_i and increases fast as ε_j tends to $\bar{\varepsilon}_j$. To ensure our exposition is compact, we omit the arguments of $\pi_i(h_i, \varepsilon_j)$, and simply write π_i.

In mobility game $\mathcal{M}$, each traveler i pays for using a mobility service j of type h_i on route ρ_i with origin-destination pair (o, d) making an optional stop at transport hub v_i. At each transport hub, available funds can be offered to incentivize travelers to use public transit services. Our mobility game can be thought of as a static game that is played repeatedly [63], thus travelers are assumed to take different actions multiple times. Therefore, the pricing mechanism needs to consider both the payments of all travelers for each type of service and the available funds at each transport hub. We formalize this idea for the allocation of mobility payments for each traveler i by stating the following definition.

Definition 5. Suppose traveler $i \in \mathcal{I}$ chooses route $\rho_i \in \mathcal{P}^{(o,d)}$ and makes a stop at transport hub v_i along that route using some service $j \in \mathcal{J}$ of type $h_i \in \mathcal{H}$. Then, the set of co-travelers at $v_i \in \mathcal{V}$ is $\mathcal{S}_{v_i} = \{k \in \mathcal{I} \mid v_k = v_i\}$.

In words, $\mathcal{S}_{v_i}$ groups all travelers including traveler i who have made a stop at transport hub v_i. Next, we formally define the available budget at transport hub v_i.

Definition 6. Let $b(v_i) \in \mathbb{R}$ be the number of funds available for transactions at traveler i's transport hub $v_i \in \mathcal{V}\backslash\{(o, d)\}$ over all types of services $h \in \mathcal{H}$.

Intuitively, $b(v_i)$ represents the available funds (e.g., after covering all expenses), in the same monetary units as θ_i and π_i, that transport hub v_i may allocate to travelers. Practically, even though our proposed mobility game is not dynamic, $b(v_i)$ can be computed based on historical data (e.g., along similar lines presented in [64]), and thus capture the "demand" of services at a particular v_i. Each traveler $i \in \mathcal{I}$ starts with a mobility wallet θ_i and pays π_i while they make a stop at transport hub v_i. Note that we are interested in modeling the travelers' decision-making in regards to commuting with cars and public transportation. That is why, in our model, each traveler is required to pass through a specific vertex, such as a transfer station (e.g., business district). Our justification is that many public transportation trips involve transferring between different modes of transportation or commuters must pass through a specific locations as part of their daily trip.

We can capture the travelers' preferences of different outcomes using a "utility function." Travelers are expected to act as utility maximizers. Thus, we can influence the travelers' behavior by introducing a *control input* in the utility function. In our mobility game, we consider a *mobility pricing mechanism*, as a control input, that aims to reward or penalize each traveler i (either by increasing a traveler's utility or decreasing it). We offer here an informal description of our pricing mechanism. The total excess amount of mobility funds is $b(v_i) - \sum_{i \in \mathcal{S}_{v_i}} \pi_i$. The total excess amount of mobility funds at transport hub v_i excluding traveler i is $b(v_i) - \sum_{k \in \mathcal{S}_{v_i} \backslash \{i\}} \pi_k$. Given the available mobility funds already present at v_i, we can redistribute the "mobility wealth" based on the types of services and roads utilized by the travelers as follows: we consider a quadratic-based pricing mechanism $\tau(v_i, \pi_i)$, defined formally next, which is the same for all travelers. Under this pricing mechanism, we observe the following two interesting properties: for high values of π_i, τ is strictly decreasing; for low values of π_i, τ is strictly increasing. Thus, if traveler i pays a high payment π_i (e.g., which implies traveler i uses a car), then disincentive is also high to use this mobility service. Thus, this serves as an indirect incentive for a traveler to use public transit or a different transport hub (this becomes clear in (4)). Furthermore, the pricing mechanism τ can take negative values, and actually strictly decreases fast as π_i takes high values for any traveler i. So, travelers can get penalized if they choose a "high-demand" type of service (thus, leading to a high valued $\sum_{k \in \mathcal{S}_{v_i} \backslash \{i\}} \pi_k$). Even if a traveler has the means to pay (i.e., the traveler has a large mobility wallet), the pricing mechanism can penalize the traveler with hefty fees, thus all travelers have the incentive to minimize the penalties and choose public transit services or less congested transport hubs. For example, when a traveler uses a bike, their mobility payment will be low and so they can earn (instead of paying for the service). This incentivizes a sustainable allocation of services to all travelers. We offer the formal definition of the pricing mechanism next for the allocation of the mobility funds and payments.

Definition 7. The pricing mechanism is a multivariable function $\tau \mapsto \mathbb{R}$ that depends on a traveler i's transport hub v_i and mobility payment π_i, and is explicitly given by

$$\tau(v_i, \pi_i) = \left(b(v_i) - \sum_{k \in \mathcal{S}_{v_i} \setminus \{i\}} \pi_k \right)^2 - \left(b(v_i) - \sum_{i \in \mathcal{S}_{v_i}} \pi_i \right)^2. \tag{1}$$

Recall that the term $b(v_i)$ captures the demand of a transport hub v_i based on what types of services in general travelers have been using (e.g., if a transport hub has a lot of money, it means travelers use cars significantly).

Remark 1. *If we expand* (1) *and simplify, we obtain the following relation*

$$\tau(v_i, \pi_i) = 2\pi_i \left(b(v_i) - \frac{\pi_i}{2} - \sum_{k \in \mathcal{S}_{v_i} \setminus \{i\}} \pi_k \right). \tag{2}$$

The behavior of (1) *is rather interesting. Obviously, as traveler i*'s *payment increases, then* (1) *decreases. However, given that* $b(v_i) > \sum_{k \in \mathcal{S}_{v_i} \setminus \{i\}} \pi_k$, *for small values of* π_i, τ *increases up to a maximum point, and then starts to decrease. This characteristic of* (1) *serves as a strong incentive for travelers to choose services that are "cheap" (bikes) or uncongested (buses) since then* π_i *will be small. Otherwise,* τ *can take very high negative values as* π_i *increases.*

As long as $b(v_i)$ is higher than $\sum_{k \in \mathcal{S}_{v_i} \setminus \{i\}} \pi_k$, then the pricing mechanism (1) redistributes wealth back to each traveler i based on what is available on the self-chosen transport hub v_i and how much travelers pay by taking into consideration traveler i's contribution at transport hub v_i.

Since the travelers' objective is to maximize their payoff, we need a way to "incentivize" travelers to avoid decisions that may lead to an empty mobility wallet. Thus, we introduce an empty wallet "disincentive" for an arbitrary traveler i.

Definition 8. Given the current amount of mobility wallet θ_i of any traveler i, the *disincentive* of having an empty wallet is a decreasing function $g : [0, \bar{\theta}_i] \to \mathbb{R}$ given by

$$g(\theta_i) = \frac{\bar{\theta}_i}{\theta_i + \eta_i \pi_i}, \tag{3}$$

where $\eta_i \in (0, 1)$ is a socioeconomic characteristic of traveler i and affects the impact of how much they choose to spend or save in terms of their mobility wallet.

Definition 8 establishes mathematically a disincentive as a function where $\bar{\theta}_i$ is proportional to the sum of the current mobility wallet θ_i and the weighted mobility payment $\eta_i \pi_i$. We expect each traveler to avoid as much as possible an empty wallet; hence, (3) ensures to "penalize" travelers with a low mobility wallet θ_i while choosing to spend $\eta_i \pi_i \approx \theta_i$. Thus, (3) grows fast as θ_i decreases. We offer now the formal definition of a traveler's action set.

Definition 9. For an arbitrary traveler $i \in \mathcal{I}$, the action set is $\mathcal{A}_i = \mathcal{P}^{(o,d)} \times \mathcal{V} \times \mathbb{R}_{\geq 0}$, where $\mathcal{P}^{(o,d)}$ is the set of routes that connects traveler i's origin-destination pair (o, d), $\mathcal{V}$ is the set of nodes in network $\mathcal{G}$ that includes all possible transports hubs v_i, and $\pi_i \in \mathbb{R}_{\geq 0}$ is the mobility payment of traveler i.

Note that, by Definition 9, the action set $\mathcal{A}_i$ of an arbitrary traveler i is a coupled set with discrete values (route, transport hub, source/destination pair), and continuous values (mobility payment). Thus, the action profile $a_i \in \mathcal{A}_i$ of traveler $i \in \mathcal{I}$ is a vector of discrete and continuous values. We write $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2 \times \ldots \times \mathcal{A}_I$ for the Cartesian product of all the travelers' action sets. We write $a_{-i} = (a_1, a_2, \ldots, a_{i-1}, a_{i+1}, \ldots, a_I)$ for the action profile that excludes traveler $i \in \mathcal{I}$. Next, for the aggregate action profile, we write $a = (a_i, a_{-i})$, $a \in \mathcal{A}$. We also denote by a^*, a^{Nash} an action profile at a social optimum and at a NE, respectively.

Next, we introduce a travel time latency function to capture the congestion cost that travelers may experience.

Definition 10. Let the total number of services of all types $h = 1, \ldots, H$ on road $e \in \mathcal{E}$ with at least one traveler be $J_e = \sum_{h\in\mathcal{H}} \omega_h |\mathcal{J}_{e,h}|$, where $\mathcal{J}_{h,e}$ is the set of all services on road e of type h, and $(\omega_h)_{h\in\mathcal{H}}$, $\omega_h \in [0, 1]$ are weight parameters that depend on the type h to capture the different impact of services on the traffic. Then, the travel time latency function is a strictly increasing linear function $c_e : \mathbb{N} \to \mathbb{R}$, with explicit form $c_e(J_e) = \xi_1 J_e + \xi_2$, where ξ_1, ξ_2 are constants.

Notice that we assume linearity in the travel time latency functions c_e, which is not unique in the literature [62, 65, 66]. The justification behind linearity is that it is the simplest yet most useful way for a mathematical analysis to capture the travel costs in terms of distance or road capacity and the traffic congestion costs. The choice of the constants ξ_1 and ξ_2 play an important role, namely ξ_2 can represent the length of road e and ξ_1 normalizes the number of services on road e so that both components of c_e have the same units.

We can now formally define the utility of any traveler $i \in \mathcal{I}$.

Definition 11. The utility $u_i : \mathcal{A} \to \mathbb{R}$ of traveler $i \in \mathcal{I}$ is what traveler i receives under the risk-neutral setting given by

$$u_i(a) = \tau(v_i, \pi_i) - g(\theta_i) - \zeta_1 \left(\sum_{e\in\rho_i:\rho_i\in\mathcal{P}^{(o,d)}} c_e(J_e) \right) - \zeta_2 \left(\frac{|\mathcal{S}_{v_i}|}{\sigma(v_i, h_i)} \right), \tag{4}$$

where $\sigma(v_i, h_i)$ is the rate of travel service at transport hub v_i for type of service h_i (how many travelers per hour can travel using type of service h_i from transport hub v_i), and $\zeta_1, \zeta_2 \in \mathbb{R}$ are unit parameters that transform time to money (that way the units of (4) are consistent).

Note that both constants ζ_1, ζ_2 get absorbed by the constants of function c_e (as defined in Definition 10) and parameter σ, respectively. So, we can safely omit them from the mathematical analysis. In (4), the first term represents the pricing mechanism and is the amount of mobility funds redistributed to traveler i for paying π_i. The second term is the disincentive as defined in Definition 8 and the third term is the congestion cost of traveler i due to traffic on road e. Finally, the last term in (4) is a waiting cost for joining transport hub v_i, where the number of travelers at transport hub v_i is proportional to the rate of travel service at transport hub v_i.

Next, we characterize fully and formally the mobility game.

Definition 12. The mobility game is fully characterized by the tuple

$$\mathcal{M} = \langle \mathcal{I}, \mathcal{J}, (\mathcal{A}_i)_{i\in\mathcal{I}}, (u_i)_{i\in\mathcal{I}} \rangle, \tag{5}$$

a collection of sets of travelers, mobility services, actions, and a profile of utilities.

The mobility game $\mathcal{M}$ is a non-cooperative repeated routing game with a multimodal transportation network and coupled action sets. Travelers have a travel-mode choice to make that will satisfy their travel needs. The benefit of ensuring that our mobility game $\mathcal{M}$ is a repeated game is that it eliminates the possibility of unassigned travelers. At this point also, we clarify the *information structure* of the mobility game $\mathcal{M}$ ("who knows what?" [67]). All travelers have complete knowledge of the mobility system (network, travel time latencies, waiting costs, and utility functions). Each traveler knows their own information (action and utility) as well as the information of other travelers. For the purposes of our work, we observe that a NE is most fitting to apply as a solution concept as it requires complete information.

Before we continue to the analysis of our mobility game $\mathcal{M}$, we summarize the notation that has been introduced so far in our paper.

Analysis and properties

In this section, our goal is to establish the existence of at least one NE in the mobility game $\mathcal{M}$, derive an upper bound for the PoA, and perform a prospect theory analysis.

We start our exposition by summarizing two necessary preliminary concepts and results of game theory that we use throughout the paper.

Definition 13. A game $\mathcal{M}$ is an *exact potential game* if there exists a potential function $\Phi : \mathcal{A} \to \mathbb{R}$ such that for all $i \in \mathcal{I}$, for all a_{-i}, and for all $a_i, a_i' \in \mathcal{A}_i$, we have

$$\Phi(a_i, a_{-i}) - \Phi(a_i', a_{-i}) = u_i(a_i, a_{-i}) - u_i(a_i', a_{-i}). \tag{6}$$

Definition 14. An action profile $a^{\text{Nash}} = (a_i^{\text{Nash}}, a_{-i}^{\text{Nash}})$ is called a pure-strategy *Nash equilibrium* for game $\mathcal{M}$ if, for all $i \in \mathcal{I}$, we have $u_i(a_i^{\text{Nash}}, a_{-i}^{\text{Nash}}) \geq u_i(a_i', a_{-i}^{\text{Nash}})$, for all $a_i' \in \mathcal{A}_i$.

The potential function Φ is a useful tool in showing whether a game has a NE and analyzing its properties. This is because, by construction, the effect on the utility of any traveler's action is expressed by one function common for all travelers.

Existence of a nash equilibrium

In this subsection, we prove that for the mobility game $\mathcal{M}$, as defined in Definition 12 (please also see Table 1), there exists at least one NE. The key idea of our proof is the use of a potential function, as defined in Definition 13, that captures the changes in utility of an arbitrary traveler that deviates in their action.

Theorem 1. *The mobility game $\mathcal{M}$ admits a pure-strategy NE.*

As it is standard in the existence of NE results for potential games (see Chapter 2 [68]), we start by stating the explicit form of the potential function for the mobility game $\mathcal{M}$, i.e.,

$$\Phi(a) = \sum_{v\in\mathcal{V}} \left(b(v) - \sum_{i\in\mathcal{S}_v} \pi_i \right)^2 - \sum_{i\in\mathcal{I}} \frac{\bar{\theta}_i}{\theta_i + \eta_i \pi_i} - \sum_{e\in\mathcal{E}} \sum_{k=1}^{J_e} c_e(k) - \sum_{v\in\mathcal{V}} \sum_{i\in\mathcal{I}} \frac{|\mathcal{S}_v|(|\mathcal{S}_v| + 1)}{2\sigma(v, h_i)}, \tag{7}$$

Our goal now is to verify Definition 13, thus showing that $\mathcal{M}$ is a potential game. Mathematically, for an arbitrary traveler i and for any two actions $a_i = (\rho_i \in \mathcal{P}^{(o,d)}, v_i, \pi_i)$ and $a_i' = (\rho_i' \in \mathcal{P}^{(o,d)}, v_i', \pi_i')$, we need to show

$$\Phi(a_i, a_{-i}) - \Phi(a_i', a_{-i}) = u_i(a_i, a_{-i}) - u_i(a_i', a_{-i}). \tag{8}$$

Note here that any traveler i that deviates in their action a_i to a_i' by changing their route ρ_i to ρ_i' that does not require an additional service j on any new roads $e \in \rho_i'$, then traveler i's impact to congestion is negligent. Thus, traveler i can change routes and still travel along an existing service j in road $e \in \rho_i'$ if that service j has not reached its maximum capacity $\bar{\varepsilon}_j$. If traveler i changes their route from ρ_i to ρ_i' and the traveler capacities of all services on that

Version of Record at: https://doi.org/10.1371/journal.pone.0285322

Table 1. A summary of our notation.

Symbol	Description
$\mathcal{I}$	Set of travelers
$\mathcal{J}$	Set of mobility services
$\mathcal{J}_h$	Set of mobility services of type h
$\mathcal{H}$	Set of different types of services
ε_j	Physical traveler capacity for service $j \in \mathcal{J}$
$\bar{\varepsilon}_j$	Maximum traveler capacity of service $j \in \mathcal{J}$
$\mathcal{G}$	Network with set of edges $\mathcal{E}$ and set of nodes $\mathcal{V}$
v_i	Node in network $\mathcal{G}$ that represents a transport hub
$\mathcal{P}^{(o,d)}$	Set of routes that connect the origin o to destination d
θ_i	Mobility wallet for traveler $i \in \mathcal{I}$
$\bar{\theta}_i$	Maximum amount of travelers i's monetary units for the wallet θ_i
π_i	Mobility payment of traveler $i \in \mathcal{I}$
ρ_i	Route chosen by traveler $i \in \mathcal{I}$
$\mathcal{S}_{v_i}$	Set of co-travelers at transport hub v_i for traveler $i \in \mathcal{I}$
$b(v_i)$	Amount of funds available for transactions at transport hub for traveler $i \in \mathcal{I}$
τ	Pricing mechanism
g	Empty wallet disincentive
η_i	Socioeconomic characteristic of traveler $i \in \mathcal{I}$
$\mathcal{A}_i$	Set of actions for traveler $i \in \mathcal{I}$
$\mathcal{A}$	Cartesian product of all action sets
J_e	Total number of services of all types on road $e \in \mathcal{E}$
c_e	Travel time latency function
σ	Rate of travel service

https://doi.org/10.1371/journal.pone.0285322.t001

route are not maxed out, then the number of services J_e does not change in the roads that remain the same along both routes (any $e \in \rho_i \cap \rho_i'$). However, the number of services J_e increases by one in any road $e \in \rho_i' \backslash \rho_i$ since we require an additional service j in road e for traveler i. Thus, we have $\sum_{e\in\rho_i'} c_e(J_e) = \sum_{e\in\rho_i'\cap\rho_i} c_e(J_e) + \sum_{e\in\rho_i'\backslash\rho_i} c_e(J_e+1)$.

If traveler i changes their transport hub v_i to v_i', then their new waiting cost is $\frac{|\mathcal{S}_{v_i'}|+1}{\sigma(v_i',h)}$, where $\mathcal{S}_{v_i'} = \{k \in \mathcal{I}\backslash\{i\} \mid v_k = v_i'\}$. We make a similar argument for π_i and π_i', and thus, we can write

$$u_i(a_i', a_{-i}) = \left(b(v_i') - \sum_{k\in\mathcal{S}_{v_i'}\backslash\{i\}} \pi_k\right)^2 - \left(b(v_i') - \pi_i' - \sum_{k\in\mathcal{S}_{v_i'}\backslash\{i\}} \pi_k\right)^2 - \frac{\bar{\theta}_i}{\theta_i + \eta_i\pi_i'} - \frac{|\mathcal{S}_{v_i'}|}{\sigma(v_i', h_i)} - \left[\sum_{e\in\rho_i'\cap\rho_i} c_e(J_e) + \sum_{e\in\rho_i'\backslash\rho_i} c_e(J_e+1)\right], \tag{9}$$

where the first component represents the squared remaining cost that traveler i has to pay for deviating to the alternative transport hub v_i' without taking into account their π_i'. Thus, it accounts for the base cost of the deviation in the transport hub and mobility payment and the sum of prices paid by other the travelers. The second component represents the squared remaining cost that traveler i has to pay for the alternative transport hub v_i' when actually they do consider their π_i', along with the sum of prices paid by other travelers. We note here that the difference between these two terms highlights the impact of traveler i's price π_i' and v_i' on their u_i when they deviate and choose an alternative action. If traveler i chooses a higher payment, the remaining cost decreases, and this difference will have a negative impact on their utility. Conversely, if traveler i chooses a lower payment, the remaining cost increases, and this difference will have a positive impact on their utility. Next, we subtract (9) from (4) to get

$$\begin{aligned} u_i(a_i, a_{-i}) - u_i(a_i', a_{-i}) = & \left(b(v_i) - \sum_{k\in\mathcal{S}_{v_i}\setminus\{i\}} \pi_k\right)^2 - \left(b(v_i) - \sum_{i\in\mathcal{S}_{v_i}} \pi_i\right)^2 \\ & - \left(b(v_i') - \sum_{k\in\mathcal{S}_{v_i'}\setminus\{i\}} \pi_k\right)^2 + \left(b(v_i') - \pi_i' - \sum_{k\in\mathcal{S}_{v_i'}\setminus\{i\}} \pi_k\right)^2 - \sum_{e\in\rho_i\setminus\rho_i'} c_e(J_e) \\ & + \sum_{e\in\rho_i'\setminus\rho_i} c_e(J_e+1) - \frac{|\mathcal{S}_{v_i}|}{\sigma(v_i, h_i)} + \frac{|\mathcal{S}_{v_i'}|+1}{\sigma(v_i', h_i)} + \frac{\eta_i\bar{\theta}_i(\pi_i - \pi_i')}{(\theta_i + \eta_i\pi_i)(\theta_i + \eta_i\pi_i')}, \end{aligned} \tag{10}$$

where $\sum_{e\in\rho_i\setminus\rho_i'} c_e(J_e) = \sum_{e\in\rho_i} c_e(J_e) - \sum_{e\in\rho_i'\cap\rho_i} c_e(J_e)$. Now, we denote all four components of (7) as follows: $\phi_1 = -\sum_{v\in\mathcal{V}}(b(v) - \sum_{i\in\mathcal{S}_v}\pi_i)^2$, $\phi_2 = -\sum_{i\in\mathcal{I}}\frac{\bar{\theta}_i}{\theta_i+\eta_i\pi_i}$, $\phi_3 = -\sum_{e\in\mathcal{E}}\sum_{k=1}^{J_e} c_e(k)$, and $\phi_4 = -\sum_{v\in\mathcal{V}}\sum_{i\in\mathcal{I}}\frac{|\mathcal{S}_v|(|\mathcal{S}_v|+1)}{2\sigma(v,h_i)}$. Step by step, we compute the difference of all four different ϕ's under a_i and a_i' as follows

$$\begin{aligned} & \varphi_1(a_i, a_{-i}) - \varphi_1(a_i', a_{-i}) = \\ & \left(b(v_i') + \sum_{i\in\mathcal{S}_{v_i'}} \pi_i'\right)^2 - \sum_{v\in\mathcal{V}}\left(b(v) - \sum_{i\in\mathcal{S}_v}\pi_i\right)^2 - \sum_{v\in\mathcal{V}\setminus\{v_i'\}}\left(b(v) - \sum_{i\in\mathcal{S}_v}\pi_i\right)^2 \\ & = \left(b(v_i) - \sum_{k\in\mathcal{S}_{v_i}\setminus\{i\}}\pi_k\right)^2 - \left(b(v_i) - \sum_{i\in\mathcal{S}_{v_i}}\pi_i\right)^2 \\ & - \left(b(v_i') - \sum_{k\in\mathcal{S}_{v_i'}\setminus\{i\}}\pi_k\right)^2 + \left(b(v_i') - \pi_i' - \sum_{k\in\mathcal{S}_{v_i'}\setminus\{i\}}\pi_k\right)^2, \end{aligned} \tag{11}$$

where we note that in $\sum_{v\in\mathcal{V}}$ both v_i and v_i' are included, so if we expand the summations that involve the unrelated nodes in $\mathcal{V}$, then most terms cancel out (what remains are only the terms that involve the deviations of traveler i, thus we get (11)). Next, we have

$$\begin{aligned}\varphi_2(a_i, a_{-i}) - \varphi_2(a_i', a_{-i}) &= -\sum_{i\in\mathcal{I}} \frac{\bar{\theta}_i}{\theta_i + \eta_i \pi_i} + \left[\sum_{k\in\mathcal{I}\setminus\{i\}} \frac{\bar{\theta}_k}{\theta_k + \eta_k \pi_k} + \frac{\bar{\theta}_i}{\theta_i + \eta_i \pi_i'}\right] \\ &= \sum_{i\in\mathcal{I}} \frac{\eta_i \bar{\theta}_i (\pi_i - \pi_i')}{(\theta_i + \eta_i \pi_i)(\theta_i + \eta_i \pi_i')}.\end{aligned} \tag{12}$$

$$\begin{aligned}\varphi_3(a_i, a_{-i}) - \varphi_3(a_i', a_{-i}) &= -\sum_{e\in\mathcal{E}} \sum_{k=1}^{J_e} c_e(k) + \sum_{e\in\mathcal{E}\setminus\{e\in\rho_i'\}} \sum_{k=1}^{J_e-1} c_e(k) + \sum_{e\in\mathcal{E}\setminus\{e\in\rho_i\}} \sum_{k=1}^{J_e+1} c_e(J_e) \\ &= -\sum_{e\in\rho_i\setminus\rho_i'} c_e(J_e) + \sum_{e\in\rho_i'\setminus\rho_i} c_e(J_e+1),\end{aligned} \tag{13}$$

$$\begin{aligned}\varphi_4(a_i, a_{-i}) - \varphi_4(a_i', a_{-i}) = \sum_{v\in\mathcal{V}} \sum_{i\in\mathcal{I}} \frac{|\mathcal{S}_v|(|\mathcal{S}_v|+1)}{2\sigma(v, h_i)} \\ - \sum_{v\in\mathcal{V}\setminus\{v_i\}\cup\{v_i'\}} \sum_{k\in\mathcal{I}\setminus\{i\}} \left[\frac{|\mathcal{S}_v|(|\mathcal{S}_v|+1)}{2\sigma(v, h_k)}\right] - \frac{|\mathcal{S}_{v_i}|\left(|\mathcal{S}_{v_i}|-1\right)}{2\sigma(v_i, h_i)} \\ - \frac{\left(|\mathcal{S}_{v_i'}|+1\right)\left(|\mathcal{S}_{v_i'}|+2\right)}{2\sigma(v_i', h_i)} = \frac{|\mathcal{S}_{v_i}|}{\sigma(v_i, h_i)} - \frac{|\mathcal{S}_{v_i'}|+1}{\sigma(v_i', h_i)},\end{aligned} \tag{14}$$

We define $\Phi(a_i, a_{-i}) - \Phi(a_i', a_{-i}) = \sum_{k=1}^{4}[\phi_k(a_i, a_{-i}) - \phi_k(a_i', a_{-i})]$. We take the sum of (11)–(14) and thus, we obtain $\Phi(a_i, a_{-i}) - \Phi(a_i', a_{-i}) = u_i(a_i, a_{-i}) - u_i(a_i', a_{-i})$. This proves that the mobility game $\mathcal{M}$ is a potential game, and so following from key results (see [68]), we conclude that $\mathcal{M}$ admits a pure-strategy NE.

Note that this existence result is not straightforward as the action set of any traveler is a coupled set composed of countable and uncountable subsets.

Corollary 1. *If the mobility game $\mathcal{M}$ is played repeatedly, then the travelers' actions converge to a pure-strategy NE in finite time.*

Proof. We aim to show that if the mobility game $\mathcal{M}$ is played repeatedly, the travelers' actions will converge to a pure-strategy NE in finite time. This proof relies on Theorem 1 and Theorem 2.6 (pp. 33) from [68]. The mobility game $\mathcal{M}$ can be classified as a repeated routing game with complete information since each traveler $i \in \mathcal{I}$ has full knowledge of the mobility system (travel time latencies, network congestion, and other relevant parameters) when making their decisions. We demonstrate convergence to a pure-strategy NE by considering the decision-making process of the travelers; so we outline the iterative process: (i) Traveler $i \in \mathcal{I}$ chooses their initial action a_i based on the current state of the mobility system. (ii) After observing traveler i's action, all other travelers $k \in \mathcal{I}, k \neq i$ choose their actions a_k accordingly, taking into account the updated state of the mobility system. (iii) The mobility system's state is updated again, reflecting the actions of all travelers in the current round. (iv) Traveler i now evaluates their action a_i and decides whether to deviate to a_i' based on the updated state of the mobility system. (v) These steps are repeated for all travelers until no traveler has an incentive to change their action, leading to a convergence to a pure-strategy NE. we observe that travelers compete and as a result, each traveler's actions are influenced by the actions of others. So, continuous deviations eventually will lead to a pure strategy NE by an iterative process (see Theorem 2.6 from [68]). In conclusion, by showing that the mobility game $\mathcal{M}$ is a repeated routing game with complete information and that the iterative decision-making process of the travelers leads to a stable equilibrium, we have demonstrated that the travelers' actions will converge to a pure-strategy NE in finite time.

Price of anarchy and stability analysis

An existence result (Theorem 1) leads to the problem of multiple NE and raises questions about the efficiency of each equilibrium. For example, an important concern is the efficiency of the equilibrium that the travelers will reach (as it is guaranteed by Corollary 1). To address this concern, we provide an analysis based on the *Price of Anarchy* (PoA) [69], which is one of the most widely-used metrics to measure the inefficiency in a system and provides an understanding of how the travelers' decision-making affect the overall performance of the system. We provide the formal definition of the PoA.

Definition 15. Let the *social welfare* of the mobility game $\mathcal{M}$ be represented by $J(a) = \sum_{i\in\mathcal{I}} u_i(a)$. Then, the PoA is the ratio of the maximum optimal social welfare over the minimum social welfare at a NE, i.e.,

$$\mathrm{PoA} = \frac{\max_{a\in\mathcal{A}} \sum_{i\in\mathcal{I}} u_i(a)}{\min_{a\in\mathcal{A}^{\mathrm{Nash}}} \sum_{i\in\mathcal{I}} u_i(a)} \geq 1, \tag{15}$$

where $\mathcal{A}^{\mathrm{Nash}}$ is the set of NE, which is guaranteed to be non-empty according to Theorem 1.

Next, we show that, for the mobility game $\mathcal{M}$, (15) is as low as possible at an arbitrary NE. Thus, it follows that our PoA result yields an upper bound for the inefficiencies at a NE of the mobility game $\mathcal{M}$.

But first, we prove two useful lemmata that are necessary for our work.

Lemma 1. *Let* $a_i^* = (\rho_i^*, v_i^*, \pi_i^*)$ *denote the optimal action of traveler* $i \in \mathcal{I}$, *define* $\tilde{b}^2 = \sum_{v\in\mathcal{V}} \left(\sum_{h\in\mathcal{H}} b(v,h)\right)^2 = \sum_{v\in\mathcal{V}} (b(v))^2$, *and at a NE*: $J_3(a) = \sum_{i\in\mathcal{I}} \left[\left(b(v_i^*) - \sum_{i\in\mathcal{S}_{v_i^*}} \pi_i\right)^2 - \left(b(v_i^*) - \pi_i^* - \sum_{k\in\mathcal{S}_{v_i^*}\setminus\{i\}} \pi_k\right)^2 - \frac{\bar{\theta}_i}{\theta_i + \eta_i \pi_i^*}\right]$. *Then, we have*

$$\begin{aligned}
&\sum_{i\in\mathcal{I}} \left[\left(b(v_i^*) - \sum_{k\in\mathcal{S}_{v_i^*}\setminus\{i\}} \pi_k\right)^2 - \left(b(v_i^*) - \pi_i^* - \sum_{k\in\mathcal{S}_{v_i^*}\setminus\{i\}} \pi_k\right)^2 - \frac{\bar{\theta}_i}{\theta_i + \eta_i \pi_i^*}\right] \\
&\leq J_3(a^*) - \sqrt{\left(\tilde{b}^2 + 2\left(J_3(a^{\mathrm{Nash}}) - I\bar{\theta}_i\right)\right)\left(\tilde{b}^2 + 2\left(J_3(a^*) - I\bar{\theta}_i\right)\right)} - 4I\bar{\theta}_i - \tilde{b}^2 \\
&\quad - \tilde{b}\left(\sqrt{\tilde{b}^2 + 2\left(J_3(a^{\mathrm{Nash}}) - I\bar{\theta}_i\right)} + \sqrt{\tilde{b}^2 + 2\left(J_3(a^*) - I\bar{\theta}_i\right)}\right).
\end{aligned} \tag{16}$$

Proof. At social optimum, the pricing mechanism is given by

$$\begin{aligned}\tau^*\left(v_i^*,\pi_i^*\right) &= \left(b\left(v_i^*\right)-\sum_{k\in\mathcal{S}^*_{v_i^*}\setminus\{i\}}\pi_k^*\right)^2-\left(b\left(v_i^*\right)-\sum_{i\in\mathcal{S}^*_{v_i^*}}\pi_i^*\right)^2\\ &= \left(b\left(v_i^*\right)-\sum_{k\in\mathcal{S}^*_{v_i^*}\setminus\{i\}}\pi_k^*\right)^2-\left(b\left(v_i^*\right)-\pi_i^*-\sum_{k\in\mathcal{S}^*_{v_i^*}\setminus\{i\}}\pi_k^*\right)^2\\ &= 2\pi_i^*b\left(v_i^*\right)-\left(\pi_i^*\right)^2-2\pi_i^*\sum_{k\in\mathcal{S}^*_{v_i^*}\setminus\{i\}}\pi_k^*.\end{aligned} \tag{17}$$

Summing over all travelers now gives

$$\begin{aligned}&\sum_{i\in\mathcal{I}}\left[2\pi_i^*b\left(v_i^*\right)-\left(\pi_i^*\right)^2-2\pi_i^*\sum_{k\in\mathcal{S}^*_{v_i^*}\setminus\{i\}}\pi_k^*\right]\\ &=2\sum_{i\in\mathcal{I}}\pi_i^*b\left(v_i^*\right)-\sum_{v\in\mathcal{V}}\left(2\left(\sum_{i\in\mathcal{S}^*_{v^*}}\pi_i^*\right)^2\right)+\sum_{i\in\mathcal{I}}\left(\pi_i^*\right)^2\\ &=2\sum_{v\in\mathcal{V}}b(v)\sum_{i\in\mathcal{S}^*_{v_i^*}}\pi_i^*-\sum_{v\in\mathcal{V}}\left(2\left(\sum_{i\in\mathcal{S}^*_{v^*}}\pi_i^*\right)^2\right)+\sum_{i\in\mathcal{I}}\left(\pi_i^*\right)^2.\end{aligned} \tag{18}$$

So, we use the Cauchy-Schwarz inequality to bound (18), i.e.,

$$J_3(a^*)-\sum_{i\in\mathcal{I}}\frac{\bar{\theta}_i}{\theta_i+\eta_i\pi_i^*}\leq 2\sqrt{\sum_{v\in\mathcal{V}}(b(v))^2\sum_{v\in\mathcal{V}}\left(\sum_{i\in\mathcal{S}^*_{v^*}}\pi_i^*\right)^2}-2\sum_{v\in\mathcal{V}}\left(\sum_{i\in\mathcal{S}^*_{v^*}}\pi_i^*\right)^2+\sum_{i\in\mathcal{I}}\left(\pi_i^*\right)^2. \tag{19}$$

For any traveler *i*, it is always true that $\frac{\bar{\theta}_i}{\theta_i+\eta_i\pi_i^*}>0$, $\pi_i^*\in[0,\bar{\theta}_i]$, and also $\tilde{b}^2=\sum_{v\in\mathcal{V}}(b(v))^2$. Thus, (19) simplifies to

$$\sum_{v\in\mathcal{V}}\left(\sum_{i\in\mathcal{S}^*_{v^*}}\pi_i^*\right)^2-\tilde{b}\sqrt{\sum_{v\in\mathcal{V}}\left(\sum_{i\in\mathcal{S}^*_{v^*}}\pi_i^*\right)^2}-\frac{1}{2}\left(J_3(a^*)-I\cdot\bar{\theta}_i\right)\leq 0, \tag{20}$$

Note that (20) is a second-order polynomial with respect to $\sqrt{\sum_{v\in\mathcal{V}}\left(\sum_{i\in\mathcal{S}^*_{v^*}}\pi_i^*\right)^2}$. Thus, we compute the discriminant $\Delta^*=\tilde{b}^2+2\left(J_3(a^*)-I\cdot\bar{\theta}_i\right)$, where Δ^* denotes the discriminant at the social optimum, so clearly $\Delta^*\geq 0$. So, from (20), we get

$$2\sqrt{\sum_{v\in\mathcal{V}}\left(\sum_{i\in\mathcal{S}^*_{v^*}}\pi_i^*\right)^2}\leq\tilde{b}+\sqrt{\Delta^*}. \tag{21}$$

We need to follow the same steps to obtain a similar relation as (21) for a NE. Hence, we have

$$2\sqrt{\sum_{v\in\mathcal{V}}\left(\sum_{k\in\mathcal{S}_v}\pi_k\right)^2} \leq \tilde{b} + \sqrt{\Delta}, \tag{22}$$

where $\Delta = \tilde{b}^2 + 2\big(J_3(a^{\text{Nash}}) - I\cdot\bar{\theta}_i\big)$. Next, the LHS of (16), if expanded, can be simplified as follows:

$$\begin{aligned}
&\sum_{i\in\mathcal{I}}\left[\left(b(v_i^*) - \sum_{k\in\mathcal{S}_{v_i^*}\setminus\{i\}}\pi_k\right)^2 - \left(b(v_i^*) - \pi_i^* - \sum_{k\in\mathcal{S}_{v_i^*}\setminus\{i\}}\pi_k\right)^2 - \frac{\bar{\theta}_i}{\theta_i + \eta_i\pi_i^*}\right] \\
&\quad = 2\sum_{i\in\mathcal{I}}\left(b(v_i^*)\pi_i^* - \pi_i^*\sum_{k\in\mathcal{S}_{v_i^*}\setminus\{i\}}\pi_k - \frac{1}{2}(\pi_i^*)^2\right) - \sum_{i\in\mathcal{I}}\frac{\bar{\theta}_i}{\theta_i + \eta_i\pi_i^*}.
\end{aligned} \tag{23}$$

$$\begin{aligned}
&J_3(a^*) - 2\sum_{i\in\mathcal{I}}\pi_i^*\sum_{k\in\mathcal{S}_{v_i^*}\setminus\{i\}}\pi_k + 2\sum_{i\in\mathcal{I}}\pi_i^*\sum_{k\in\mathcal{S}^*_{v_i^*}\setminus\{i\}}\pi_k^* \\
&= J_3(a^*) - 2\sum_{i\in\mathcal{I}}\pi_i^*\left[\sum_{k\in\mathcal{S}_{v_i^*}\setminus\{i\}}\pi_k - \sum_{k\in\mathcal{S}^*_{v_i^*}\setminus\{i\}}\pi_k^*\right] \\
&= J_3(a^*) - 2\sum_{i\in\mathcal{I}}\pi_i^*\left(\sum_{k\in\mathcal{S}_{v_i^*}}\pi_k^* - \pi_i(v_i^*) - \sum_{k\in\mathcal{S}^*_{v_i^*}}\pi_k^* + \pi_i^*\right),
\end{aligned} \tag{24}$$

where $\pi_i(v_i^*)$ denotes traveler i's payment at an optimal v_i^*, and thus, (24) can be simplified by noting that

$$\begin{aligned}
&2\sum_{i\in\mathcal{I}}\pi_i^*\left(\sum_{k\in\mathcal{S}_{v_i^*}}\pi_k^* - \pi_i(v_i^*) - \sum_{k\in\mathcal{S}^*_{v_i^*}}\pi_k^* + \pi_i^*\right) \\
&= 2\sum_{i\in\mathcal{I}}[(\pi_i^*)^2 - \pi_i^*\pi_i(v_i^*)] + 2\sum_{v\in\mathcal{V}}\sum_{k\in\mathcal{S}^*_{v^*}}\pi_k^*\sum_{k\in\mathcal{S}_v}\pi_k - 2\sum_{v\in\mathcal{V}}\left(\sum_{k\in\mathcal{S}^*_{v^*}}\pi_k^*\right)^2,
\end{aligned} \tag{25}$$

Using (25), we impose an upper bound to (24) as follows

$$\begin{aligned}
&J_3(a^*) - 2\sum_{i\in\mathcal{I}}\pi_i^*\left(\sum_{k\in\mathcal{S}_{v_i^*}}\pi_k^* - \pi_i(v_i^*) - \sum_{k\in\mathcal{S}^*_{v_i^*}}\pi_k^* + \pi_i^*\right) \\
&\qquad \leq J_3(a^*) - 4I\bar{\theta}_i - 2\sum_{v\in\mathcal{V}}[\sum_{k\in\mathcal{S}^*_{v^*}}\pi_k^*\sum_{k\in\mathcal{S}_v}\pi_k].
\end{aligned} \tag{26}$$

We continue by upper bounding the summations in (26):

$$\begin{aligned}\sum_{v\in\mathcal{V}}\left[\sum_{k\in\mathcal{S}^*_{v^*}}\pi^*_k\sum_{k\in\mathcal{S}_v}\pi_k\right] &\leq \sqrt{\sum_{v\in\mathcal{V}}\left(\sum_{k\in\mathcal{S}^*_{v^*}}\pi^*_k\right)^2\sum_{v\in\mathcal{V}}\quad\sum_{k\in\mathcal{S}_v}\pi_k\Bigr)^2} \\ &\leq \frac{1}{2}\left(\sqrt{\tilde{b}^2}+\sqrt{\Delta}\right)\left(\sqrt{\tilde{b}^2}+\sqrt{\Delta^*}\right) \\ &= \frac{1}{2}\sqrt{\Delta\Delta^*}+\frac{\tilde{b}}{2}\left(\sqrt{\Delta}+\sqrt{\Delta^*}\right)+\frac{\tilde{b}^2}{2},\end{aligned} \tag{27}$$

by the Cauchy-Schwartz inequality and relations (21) and (22). Finally, we substitute $\Delta = \tilde{b}^2 + 2\left(J_3(a^{\text{Nash}}) - I\bar{\theta}_i\right)$ and $\Delta^* = \tilde{b}^2 + 2\left(J_3(a^*) - I\bar{\theta}_i\right)$ into (27) and with (26) we obtain the desired bound.

Lemma 2. *We have*

$$\begin{aligned}\frac{J(a^*)}{J(a^{\text{Nash}})} \leq 1 - &\sqrt{\left(\frac{\tilde{b}^2}{I}+\frac{2J(a^*)}{J(a^{\text{Nash}})}\right)\left(\frac{\tilde{b}^2}{I}+2\right)} - \frac{\tilde{b}^2}{I} - 4\bar{\theta}_i \\ &- \frac{\tilde{b}}{\sqrt{I}}\left(\sqrt{\frac{\tilde{b}^2}{I}+\frac{2J(a^*)}{J(a^{\text{Nash}})}}+\sqrt{\frac{\tilde{b}^2}{I}+2}\right).\end{aligned} \tag{28}$$

Proof. We can show this result by expanding and rearranging (28) to obtain a simplified relation. So, we have

$$\frac{J(a^*)}{J(a^{\text{Nash}})} - \quad 1+\frac{\tilde{b}^2}{I}-4\bar{\theta}_i\Bigr) + \frac{\tilde{b}}{\sqrt{I}}\sqrt{\frac{\tilde{b}^2}{I}+2} \leq -\left(\sqrt{\frac{\tilde{b}^2}{I}+2}+\frac{\tilde{b}}{\sqrt{I}}\right)\sqrt{\frac{\tilde{b}^2}{I}+\frac{2J(a^*)}{J(a^{\text{Nash}})}}. \tag{29}$$

We seek to solve for $\frac{J(a^*)}{J(a^{\text{Nash}})}$, so we remove the square roots by squaring both sides of (29), i.e.,

$$\frac{J(a^*)}{J(a^{\text{Nash}})} - 2(D+E^2)\frac{J(a^*)}{J(a^{\text{Nash}})} + \quad D^2 - \frac{\tilde{b}^2E^2}{I}\Bigr) \leq 0, \tag{30}$$

where $E = \sqrt{\frac{\tilde{b}^2}{I}+2}+\frac{\tilde{b}}{\sqrt{I}}$ and $D = \left(1+\frac{\tilde{b}^2}{I}-4\bar{\theta}_i\right) - \frac{\tilde{b}}{\sqrt{I}}\sqrt{\frac{\tilde{b}^2}{I}+2}$. We solve (30) by noting the positivity of the coefficients to obtain

$$\frac{J(a^*)}{J(a^{\text{Nash}})} \leq E^2 + D + E\sqrt{E^2+2D+\frac{\tilde{b}^2}{I}}. \tag{31}$$

We observe that $E^2 + 2D + \frac{\tilde{b}^2}{I} \leq \left(E + \frac{D}{E} + \frac{\tilde{b}^2}{2EI}\right)^2$, and so, an upper bound exists for (31). Thus, we have $\frac{J(a^*)}{J(a^{\text{Nash}})} \leq 2E^2 + 2D + \frac{\tilde{b}^2}{2I}$. We substitute back both E and D and get

$$\begin{aligned}\frac{J(a^*)}{J(a^{\text{Nash}})} &\leq 2 + \frac{3\tilde{b}^2}{2I} + \frac{3\tilde{b}}{\sqrt{I}}\sqrt{\frac{\tilde{b}^2}{I} + 2} + \frac{5\tilde{b}^2}{2I} \\ &= 2 + \frac{4\tilde{b}^2}{I} + \frac{\tilde{b}}{\sqrt{I}}\sqrt{\frac{\tilde{b}^2}{I} + 2}.\end{aligned} \tag{32}$$

Hence, since $\frac{\tilde{b}^2}{I} + 2 \leq \left(\frac{\tilde{b}}{\sqrt{I}} + \sqrt{I}\tilde{b}\right)^2$, the result follows.

We are ready now to state and prove our PoA result.

Theorem 2. *Any inefficiencies of any NE of the mobility game $\mathcal{M}$ remain low as close to a constant as the number of travelers $|\mathcal{I}| = I$ tends to infinity. Mathematically, we have*

$$\text{PoA} \leq 2 + \frac{5}{I}\sum_{v\in\mathcal{V}}\left(\sum_{h\in\mathcal{H}} b(v,h)\right)^2. \tag{33}$$

Proof. From Definition 14, for some arbitrary traveler $i \in \mathcal{I}$, it is clear that $u_i\left(a_i^{\text{Nash}}, a_{-i}\right) \leq u_i\left(a_i^*, a_{-i}\right)$, so if we expand the RHS of it, we get

$$\begin{aligned}u_i\left(a_i^*, a_{-i}\right) = &\left(b\left(v_i^*\right) - \sum_{k\in\mathcal{S}_{v_i^*}\setminus\{i\}}\pi_k\right)^2 - \left(b\left(v_i^*\right) - \pi_i^* - \sum_{k\in\mathcal{S}_{v_i^*}\setminus\{i\}}\pi_k\right)^2 \\ &- \sum_{e\in\rho_i^*\setminus\rho_i} c_e(J_e + 1) - \sum_{e\in\rho_i^*\cap\rho_i} c_e(J_e) - \frac{|\mathcal{S}_{v_i^*}|}{\sigma\left(v_i^*, h_i\right)} - \frac{\bar{\theta}_i}{\theta_i + \eta_i\pi_i^*},\end{aligned} \tag{34}$$

where $b\left(v_i^*\right)$ is the budget at an optimal v_i^* and $\mathcal{S}_{v_i^*} = \{k \in \mathcal{I} \mid v_k^* = v_i^*\}$. Summing over all travelers in (34), and keeping note of $u_i\left(a_i^{\text{Nash}}, a_{-i}\right) \leq u_i\left(a_i^*, a_{-i}\right)$ yields

$$J(a^{\text{Nash}}) \leq \sum_{i\in\mathcal{I}} u_i\left(a_i^*, a_{-i}\right). \tag{35}$$

At this point, we recall that the travel time latency functions are linear. Thus, we can write

$$J_1(a^{\text{Nash}}) = \sum_{i\in\mathcal{I}}\sum_{e\in\rho_i^{\text{Nash}}} c_e\left(J_e^{\text{Nash}}\right) = \sum_{e\in\mathcal{E}} J_e^{\text{Nash}}\left(\xi_1 J_e^{\text{Nash}} + \xi_2\right), \tag{36}$$

$$J_1(a^*) = \sum_{i\in\mathcal{I}}\sum_{e\in\rho_i^*} c_e\left(J_e^*\right) = \sum_{e\in\mathcal{E}} J_e^*\left(\xi_1 J_e^* + \xi_2\right), \tag{37}$$

where J_e^{Nash} and J_e^* denote the number of services on road e at a NE and social optimum, respectively. Inspired from [70], we impose an upper bound on each component of the RHS of (35), and thus we have the following

$$\begin{aligned}\sum_{i\in\mathcal{I}}\left(\sum_{e\in\rho_i^*\setminus\rho_i}c_e(J_e+1)+\sum_{e\in\rho_i^*\cap\rho_i}c_e(J_e)\right)&\leq\sum_{i\in\mathcal{I}}\sum_{e\in\rho_i^*}c_e(J_e+1)\\&=\sum_{e\in\mathcal{E}}\xi_1J_e^*J_e+J_e^*(\xi_1+\xi_2)\\&\leq\sqrt{\sum_{e\in\mathcal{E}}\xi_1J_e^2\xi_1\left(J_e^*\right)^2}+\sum_{e\in\mathcal{E}}J_e^*\left(\xi_1J_e^*+\xi_2\right)\\&\leq\sqrt{\sum_{e\in\mathcal{E}}\left(\xi_1J_e^2+\xi_2J_e\right)\left(\xi_1\left(J_e^*\right)^2+\xi_2J_e^*\right)}+\sum_{e\in\mathcal{E}}J_e^*\left(\xi_1J_e^*+\xi_2\right)\\&=\sqrt{J_1(a^{\text{Nash}})\times J_1(a^*)}+J_1(a^*),\end{aligned}\tag{38}$$

where we simplified the notation as $J_e^{\text{Nash}}=J_e$, used $c_e(J_e)\leq c_e(J_e+1)$ for each $e\in\rho_i^*\cap\rho_i$, and applied the Cauchy-Schwarz inequality twice. Note that $\xi_1J_e^*+\xi_2\leq\xi_1\left(J_e^*\right)^2+\xi_2J_e^*$ at any $e\in\mathcal{E}$. Next, we introduce notation: $J_2(a^{\text{Nash}})=\sum_{i\in\mathcal{I}}\frac{|\mathcal{S}_{v_i}^{\text{Nash}}|}{\sigma\left(v_i^{\text{Nash}},h_i\right)}$ and $J_2(a^*)=\sum_{i\in\mathcal{I}}\frac{|\mathcal{S}_{v_i^*}|}{\sigma\left(v_i^*,h_i\right)}$. We have

$$\sum_{i\in\mathcal{I}}\frac{|\mathcal{S}_{v_i^*}|}{\sigma\left(v_i^*,h_i\right)}\leq\sqrt{J_2(a^*)J_2(a^{\text{Nash}})}+J_2(a^*).\tag{39}$$

Now we introduce the following notation:

$$\begin{aligned}J_3(a^{\text{Nash}})=\sum_{i\in\mathcal{I}}\Bigg[&\left(b\left(v_i^{\text{Nash}}\right)-\sum_{k\in\mathcal{S}_{v_i}^{\text{Nash}}\setminus\{i\}}\pi_k^{\text{Nash}}\right)^2-\left(b\left(v_i^{\text{Nash}}\right)-\sum_{i\in\mathcal{S}_{v_i}^{\text{Nash}}}\pi_i^{\text{Nash}}\right)^2\\&-\frac{\bar{\theta}_i}{\theta_i+\eta_i\pi_i^{\text{Nash}}}\Bigg],\end{aligned}\tag{40}$$

where $b\left(v_i^{\text{Nash}}\right)$ is the budget at a v_i^{Nash}, and

$$J_3(a^*)=\sum_{i\in\mathcal{I}}\left[\left(b\left(v_i^*\right)-\sum_{k\in\mathcal{S}^*_{v_i^*}\setminus\{i\}}\pi_k^*\right)^2-\left(b\left(v_i^*\right)-\sum_{i\in\mathcal{S}^*_{v_i^*}}\pi_i^*\right)^2-\frac{\bar{\theta}_i}{\theta_i+\eta_i\pi_i^*}\right].\tag{41}$$

By Lemma 1, we have a bound for the first two components of (35), thus

$$\begin{aligned}&\sum_{i\in\mathcal{I}}\left[\left(b\left(v_i^*\right)-\sum_{k\in\mathcal{S}_{v_i^*}\setminus\{i\}}\pi_k\right)^2-\left(b\left(v_i^*\right)-\pi_i^*-\sum_{k\in\mathcal{S}_{v_i^*}\setminus\{i\}}\pi_k\right)^2-\frac{\bar{\theta}_i}{\theta_i+\eta_i\pi_i^*}\right]\\&\leq J_3(a^*)-\sqrt{\left(\tilde{b}^2+2\left(J_3(a^{\text{Nash}})-I\bar{\theta}_i\right)\right)\left(\tilde{b}^2+2\left(J_3(a^*)-I\bar{\theta}_i\right)\right)}-4I\bar{\theta}_i-\tilde{b}^2\\&\quad-\tilde{b}\left(\sqrt{\tilde{b}^2+2\left(J_3(a^{\text{Nash}})-I\bar{\theta}_i\right)}+\sqrt{\tilde{b}^2+2\left(J_3(a^*)-I\bar{\theta}_i\right)}\right),\end{aligned}\tag{42}$$

We combine all relations together (38), (39), and (42) and substitute them into (35) to get

$$\begin{aligned} J(a^{\text{Nash}}) \leq & \sqrt{J_1(a^{\text{Nash}})J_1(a^*)} + J_1(a^*) + \sqrt{J_2(a^*)J_2(a^{\text{Nash}})} + J_2(a^*) - \tilde{b}^2 + J_3(a^*) \\ & + \sqrt{(\tilde{b}^2 + 2\big(J_3(a^{\text{Nash}}) - I\bar{\theta}_i\big)\Big(\tilde{b}^2 + 2\big(J_3(a^*) - I\bar{\theta}_i\big)\Big)} \\ & - \tilde{b}\left(\sqrt{\tilde{b}^2 + 2\big(J_3(a^{\text{Nash}}) - I\bar{\theta}_i\big)} + \sqrt{\tilde{b}^2 + 2\big(J_3(a^*) - I\bar{\theta}_i\big)}\right) \\ \leq & J(a^*) - \tilde{b}\left(\sqrt{\tilde{b}^2 + 2J_3(a^{\text{Nash}})} + \sqrt{\tilde{b}^2 + 2J_3(a^*)}\right) \\ & - \sqrt{\left(\tilde{b}^2 + 2\left(\sum_{k=1}^{3} J_k(a^*) - I\bar{\theta}_k\right)\right)\left(\tilde{b}^2 + 2\left(\sum_{k=1}^{3} J_k\big(a^{\text{Nash}}\big) - I\bar{\theta}_k\right)\right)} - 4I\bar{\theta}_i - \tilde{b}^2, \end{aligned} \tag{43}$$

where we have used the fact that for any four numbers $(\gamma_k \in \mathbb{R}_{>0})$, $k = 1, 2, 3, 4$, we have $\sqrt{\gamma_1\gamma_2} + \sqrt{\gamma_3\gamma_4} \leq \sqrt{(\gamma_1 + \gamma_3)(\gamma_2 + \gamma_4)}$. Then (43) leads to

$$\begin{aligned} & J(a^*) - \sqrt{\Big(\tilde{b}^2 + 2J(a^*)\Big)\Big(\tilde{b}^2 + 2J(a^{\text{Nash}})\Big)} \\ & - \tilde{b}\left(\sqrt{\tilde{b}^2 + 2J_3(a^{\text{Nash}})} + \sqrt{\tilde{b}^2 + 2J_3(a^*)}\right) - 4I\bar{\theta}_i - \tilde{b}^2 \\ \leq & J(a^*) - \sqrt{\Big(\tilde{b}^2 + 2J(a^*)\Big)\Big(\tilde{b}^2 + 2J(a^{\text{Nash}})\Big)} \\ & - \tilde{b}\left(\sqrt{\tilde{b}^2 + 2J(a^{\text{Nash}})} + \sqrt{\tilde{b}^2 + 2J(a^*)}\right) - 4I\bar{\theta}_i - \tilde{b}^2. \end{aligned} \tag{44}$$

So, we have the following after a simple rearrangement

$$\begin{aligned} J(a^*) \leq & J(a^{\text{Nash}}) - \sqrt{\Big(\tilde{b}^2 + 2J(a^*)\Big)\Big(\tilde{b}^2 + 2J(a^{\text{Nash}})\Big)} \\ & - \tilde{b}\left(\sqrt{\tilde{b}^2 + 2J(a^{\text{Nash}})} + \sqrt{\tilde{b}^2 + 2J(a^*)}\right) - 4I\bar{\theta}_i - \tilde{b}^2. \end{aligned} \tag{45}$$

We divide both sides of (45) by $J(a^{\text{Nash}})$ to obtain

$$\begin{aligned} \frac{J(a^*)}{J(a^{\text{Nash}})} \leq & 1 - \sqrt{\left(\frac{\tilde{b}^2}{I} + \frac{2J(a^*)}{J(a^{\text{Nash}})}\right)\left(\frac{\tilde{b}^2}{I} + 2\right)} - \frac{\tilde{b}^2}{I} - 4\bar{\theta}_i \\ & - \frac{\tilde{b}}{\sqrt{I}}\left(\sqrt{\frac{\tilde{b}^2}{I} + \frac{2J(a^*)}{J(a^{\text{Nash}})}} + \sqrt{\frac{\tilde{b}^2}{I} + 2}\right), \end{aligned} \tag{46}$$

By Lemma 2, we reach the desired bound.

Next, we discuss the intuition behind Theorem 2. The travelers of the mobility game $\mathcal{M}$ are considered selfish and non-cooperative. Thus, one important question is what will be the impact of selfishness on the efficiency of the mobility system. Since the existence of a NE is guaranteed by Theorem 1 and the mobility game $\mathcal{M}$ converges to at least one NE, we can compare the level of inefficiency at a NE to the social optimum (this is exactly what the PoA does). The bound we have derived in (33) under certain conditions ensures the mobility system's inefficiency is guaranteed to remain within a constant, and as the number of travelers increases, this bound becomes smaller and smaller. So, our mobility game $\mathcal{M}$ can ensure in a realistic setting (big city with road infrastructure) with a large number of travelers a sufficiently efficient operation of the mobility system as it could ideally be operated by a central authority "ordering" the travelers how to travel. Our bound in (33) is strong in the sense that it excludes any other possibility of an improvement in efficiency compared to what we can achieve at a NE.

It happens that we can also upper bound using the "potential function method" (as used in [66]) the *Price of Stability* (PoS) for our mobility game $\mathcal{M}$. The PoS is defined as a ratio comparing social optimum and the best possible social welfare at a NE, i.e.,

$$\text{PoS} = \frac{\max_{a\in\mathcal{A}} \sum_{i\in\mathcal{I}} u_i(a)}{\max_{a\in\mathcal{A}^{\text{Nash}}} \sum_{i\in\mathcal{I}} u_i(a)}. \tag{47}$$

Theorem 3. *With linear travel time latency functions and pricing (7), the PoS for the mobility game $\mathcal{M}$ is upper bounded, i.e.,*

$$\text{PoS} = \frac{J(a^*)}{J(\tilde{a}^{\text{Nash}})} \leq \frac{1}{2} + \max_{i\in\mathcal{I}} \left[\bar{\theta}_i(\bar{\theta}_i - 1)\right] + \frac{1}{I}\sum_{v\in\mathcal{V}} (b(v))^2. \tag{48}$$

Proof. Recall that the *social welfare* of the mobility game $\mathcal{M}$ is represented by $J(a) = \sum_{i\in\mathcal{I}} u_i(a)$. We aim to compare the $J(a)$ with the potential function $\Phi(a)$. So, we have

$$\Phi(a) = \sum_{v\in\mathcal{V}} \left(b(v) - \sum_{i\in\mathcal{S}_v} \pi_i\right)^2 - \sum_{i\in\mathcal{I}} \frac{\bar{\theta}_i}{\theta_i + \eta_i\pi_i} - \sum_{e\in\mathcal{E}} \sum_{k=1}^{J_e} (\xi_1 k + \xi_2) - \sum_{v\in\mathcal{V}} \sum_{i\in\mathcal{I}} \frac{|\mathcal{S}_v|(|\mathcal{S}_v| + 1)}{2\sigma(v, h_i)}, \tag{49}$$

where we have substituted $c_e(J_e) = \xi_1 J_e + \xi_2$. Next, we have

$$\begin{aligned}\Phi(a) = &\sum_{v\in\mathcal{V}} (b(v))^2 - 2\sum_{v\in\mathcal{V}} \left(b(v)\sum_{i\in\mathcal{S}_v} \pi_i\right) + \sum_{v\in\mathcal{V}} \left(\sum_{i\in\mathcal{S}_v} \pi_i\right)^2 - \sum_{i\in\mathcal{I}} \frac{\bar{\theta}_i}{\theta_i + \eta_i\pi_i} \\ &- \sum_{e\in\mathcal{E}} (\xi_1 J_e^2 + (\xi_1 + 2\xi_2) J_e) - \sum_{v\in\mathcal{V}} \sum_{i\in\mathcal{I}} \frac{|\mathcal{S}_v|(|\mathcal{S}_v| + 1)}{2\sigma(v, h_i)}.\end{aligned} \tag{50}$$

We also have the following

$$\begin{aligned}J(a) = \sum_{i\in\mathcal{I}} u_i(a) = &\sum_{i\in\mathcal{I}} 2\pi_i \left(b(v_i) - \frac{\pi_i}{2} - \sum_{k\in\mathcal{S}_{v_i}\setminus\{i\}} \pi_k\right) - \sum_{i\in\mathcal{I}} \frac{\bar{\theta}_i}{\theta_i + \eta_i\pi_i} \\ &- \sum_{i\in\mathcal{I}} \sum_{e\in\rho_i:\rho_i\in\mathcal{P}^{(o,d)}} (\xi_1 J_e + \xi_2) - \sum_{i\in\mathcal{I}} \frac{|\mathcal{S}_{v_i}|}{\sigma(v_i, h_i)},\end{aligned} \tag{51}$$

which leads to

$$\begin{aligned}J(a) = &\sum_{i\in\mathcal{I}} 2\pi_i \left(b(v_i) - \frac{\pi_i}{2} - \sum_{k\in\mathcal{S}_{v_i}\setminus\{i\}} \pi_k\right) - \sum_{i\in\mathcal{I}} \frac{\bar{\theta}_i}{\theta_i + \eta_i\pi_i} \\ &- \sum_{e\in\mathcal{E}} (\xi_1 J_e^2 + \xi_2 J_e) - \sum_{i\in\mathcal{I}} \frac{|\mathcal{S}_{v_i}|}{\sigma(v_i, h_i)},\end{aligned} \tag{52}$$

By comparing now (50) and (52), we note that

$$\frac{1}{2}J(a) \leq \Phi(a) \leq J(a) + \sum_{v\in\mathcal{V}} (b(v))^2 + 2\sum_{i\in\mathcal{I}} \sum_{k\in\mathcal{S}_{v_i}\setminus\{i\}} \pi_i \pi_k \tag{53}$$

$$\leq J(a) + 2I\max_{i\in\mathcal{I}}[\bar{\theta}_i(\bar{\theta}_i - 1)] + \sum_{v\in\mathcal{V}} (b(v))^2. \tag{54}$$

Our next step is to define a NE that happens to maximize the potential function, say $\tilde{a}^{\text{Nash}}$ to get

$$J(\tilde{a}^{\text{Nash}}) \leq \Phi(\tilde{a}^{\text{Nash}}) \leq \Phi(a^*) \leq J(a^*) + 2I\max_{i\in\mathcal{I}}[\bar{\theta}_i(\bar{\theta}_i - 1)] + \sum_{v\in\mathcal{V}} (b(v))^2. \tag{55}$$

Therefore, if we divide by $J(\tilde{a}^{\text{Nash}})$, we get the following relation

$$\text{PoS} = \frac{J(a^*)}{J(\tilde{a}^{\text{Nash}})} \leq \frac{1}{2} + \max_{i\in\mathcal{I}}[\bar{\theta}_i(\bar{\theta}_i - 1)] + \frac{1}{I}\sum_{v\in\mathcal{V}} (b(v))^2. \tag{56}$$

Theorem 3 helps us understand the mobility game's NE and provides a metric for how close any NE might be to the social optimum. We can ensure that as the number of travelers increases, the smaller the PoS becomes, guaranteeing that the closer the mobility game's NE is to the social optimum.

Prospect theory analysis

In this subsection, we introduce prospect theory and its main concepts [56, 71]. We then incorporate prospect theory to our game $\mathcal{M}$. One of the main questions prospect theory attempts to answer is how a decision-maker may evaluate different possible actions/outcomes under uncertain and risky circumstances. Thus, prospect theory is a descriptive behavioral model and focuses on three main behavioral factors:

1. *Reference dependence*: decision makers make decisions based on their utility, which is measured from the "gains" or "losses." However, the utility is a gain or loss relative to a reference point that may be unique to each decision-maker. It has been shown in experimental studies [56] the reference dependence captures the tendency of a decision-maker to be more affected in their decisions by the *changes in attributes* than by the *absolute magnitudes*. For example, the shortest/average travel time between two locations.
2. *Diminishing sensitivity*: changes in value have a greater impact near the reference point than away from the reference point. For example, an individual is highly likely to discriminate between a 1 and 2 hours travel time but not very likely to notice the difference between 18 and 19 hours travel time.
3. *Loss aversion*: decision-makers are more conservative in gains and riskier in losses. For example, a traveler may prefer to secure a 45 min commute rather than risking for a 1.5 hours commute.

One way to mathematize the above behavioral factors (1)—(3) is to consider an action by a decision-maker as a "gamble" with objective utility value $z \in \mathbb{R}$ (e.g., money). We say that this decision maker *perceives* z subjectively using a *value function* [54, 72]

$$v(z) = \begin{cases} (z - z_0)^{\beta_1}, & \text{if } z \geq z_0, \\ -\lambda(z_0 - z)^{\beta_2}, & \text{if } z < z_0, \end{cases} \tag{57}$$

where z_0 represents a reference point, $\beta_1, \beta_2 \in (0, 1)$ are parameters that represent the diminishing sensitivity. Both β_1, β_2 shape (57) in a way that the changes in value have a greater impact near the reference point than away from the reference point. We observe that (57) is concave in the domain of gains and convex in the domain of losses. Moreover, $\lambda \geq 1$ reflects the level of loss aversion of decision makers (see Fig 1).

Remark 2 *To the best of our knowledge, there does not exist a widely-agreed theory that determines and defines the reference dependence* [53, 73, 74]. *In engineering* [62, 75], *it is assumed that* $z_0 = 0$ *captures a decision maker's expected status-quo level of the resources.*

As we discussed earlier in this subsection, prospect theory models the subjective behavior of decision-makers under uncertainty and risk. Each objective utility $z \in \mathbb{R}$ is associated with a probabilistic occurrence, say $p \in [0, 1]$. Decision makers, though, are subjective and perceive p in different ways depending on its value. To capture this behavior, we introduce a strictly increasing function $w : [0, 1] \rightarrow \mathbb{R}$ with $w(0) = 0$ and $w(1) = 1$ called the *probability weighting function*. This function allows us to model how decision-makers may overestimate small probabilities of objective utilities, i.e., $w(p) > p$ if p is close to 0, or underestimate high probabilities, i.e., $w(p) < p$ if p is close to 1 (see Fig 2). For the purposes of this work, we use the probability

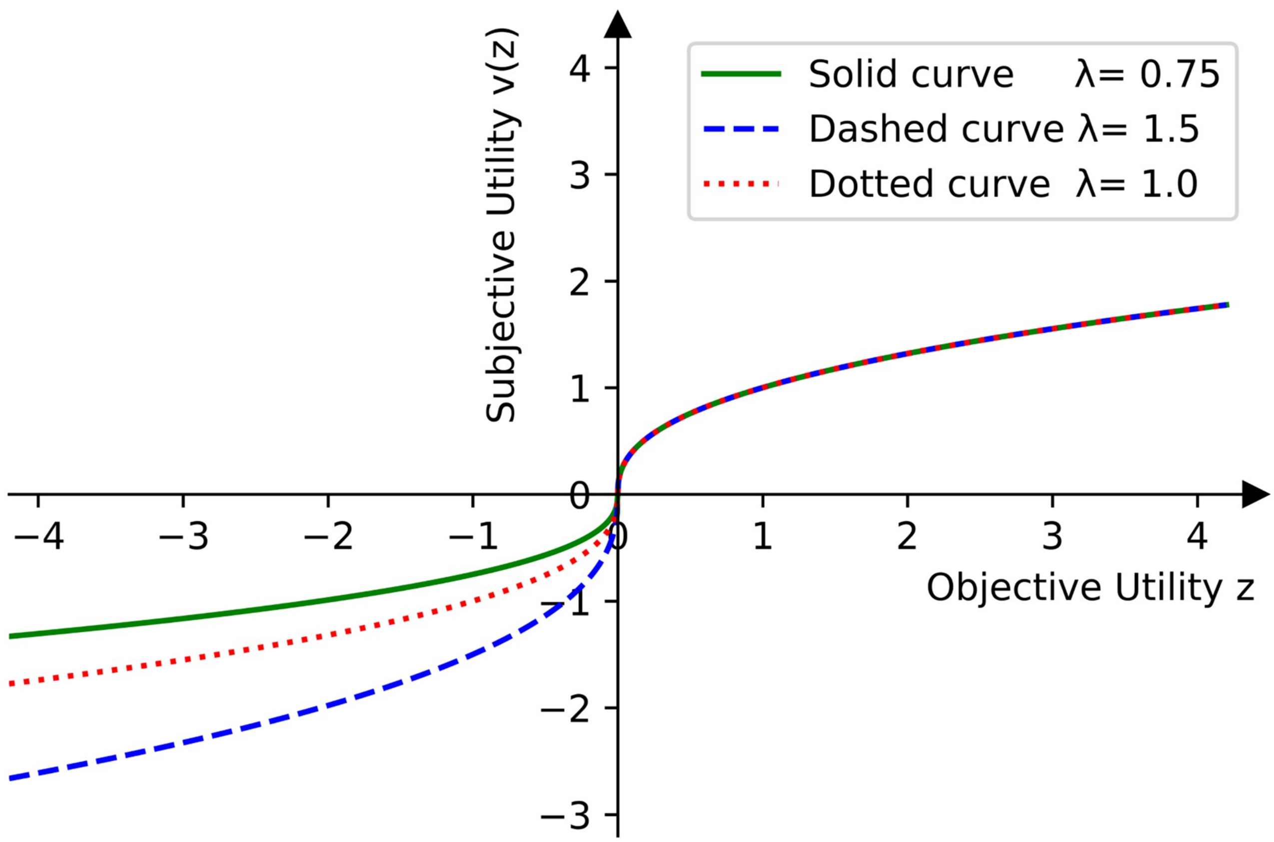

Fig 1. The value function for three different values of λ.

https://doi.org/10.1371/journal.pone.0285322.g001

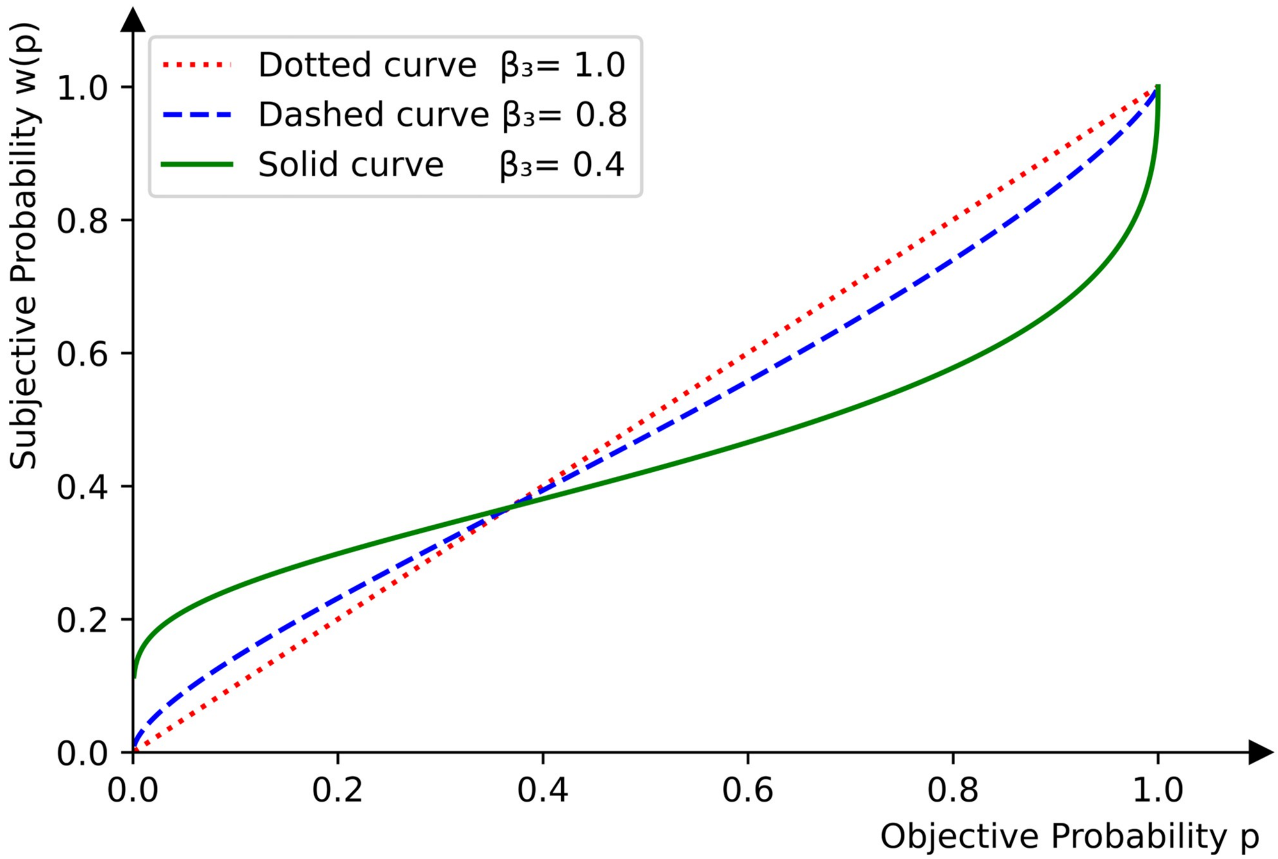

Fig 2. The probability weighting function for three different values of the rational index β_3.

https://doi.org/10.1371/journal.pone.0285322.g002

weighting function first introduced in [76],

$$w(p) = \exp\Big(-(-\log(p))^{\beta_3}\Big), \quad p \in [0, 1], \tag{58}$$

where $\beta_3 \in (0, 1)$ represents a *rational index*, i.e., the distortion of a decision-makers probability perceptions. Mathematically, β_3 controls the curvature of the weighting function (see Fig 2). Although there are many different formulations for the probability weighting function, we use (58) defined in [76] as a single-parameter function and easier computationally to estimate.

Next, we define a *prospect* which is a tuple of the objective utility (gain or loss) and its probability of happening.

Definition 16. Suppose that there are $K \in \mathbb{N}$ possible outcomes available to a decision-maker and $z_k \in \mathbb{R}$ is the kth gain/loss of objective utility. Then a prospect ℓ_k is a tuple of the utilities and their respective probabilities

$$\ell_k = (z_0, z_1, z_2, \cdots, z_K; p_0, p_1, p_2, \cdots, p_K), \tag{59}$$

where $k = 1, 2, \ldots, K$. We denote the kth prospect more compactly as $\ell_k = (z_k, p_k)$. We have that $\sum_{k=0}^{K} p_k = 1$ and ℓ_k is well-ordered, i.e., $z_0 \leq x_1 \leq \cdots \leq z_K$. Under prospect theory, the decision-maker evaluates their "subjective utility" as $u(\ell) = \Sigma_{0 \leq k \leq K} v(z_k) w(p_k)$, where $\ell = (\ell_k)_{k=1}^{K}$ is the profile of prospects of K outcomes.

In the remainder of this subsection, we apply the prospect theory to our game, clearly define the mobility outcomes (objective and subjective utilities), and then show that the prospect-theoretic mobility game $\mathcal{M}$ admits a NE.

Travelers may be uncertain about the available amount of mobility funds at any transport hub, that is why we define a *mobility prospect* to represent as a random variable Z with objective utilities $z_1, z_2, \ldots, z_K$ and their probabilities $p_1, p_2, \ldots, p_K$. Each z_k now represents the uncertain $b(v_i)$. In addition, the reference dependence of each traveler i is represented by $z_i^0 \in \mathbb{R}$. For any traveler i, the probability weighting function is $w_i : [0, 1] \to \mathbb{R}$ and the value function is $v_i(z_k, z_i^0) : \mathbb{R}^2 \to \mathbb{R}$, $k = 1, 2, \ldots, K$. Thus, we have

$$\mathbb{E}[Z] = \sum_{k=1}^{K} v_i(z_k, z_i^0) w_i(p_k), \tag{60}$$

where $w_i(p_k)$ is given by (58), and

$$v_i(z_k, z_i^0) = \begin{cases} (z_k - z_i^0)^{\beta}, & \text{if } z_k \geq z_i^0, \\ -\lambda (z_i^0 - z_k)^{\beta}, & \text{if } z_k < z_i^0, \end{cases} \tag{61}$$

where $\beta = \beta_1 = \beta_2$. We can justify $\beta_1 = \beta_2$ in the above definition as it has been verified to produce extremely good results, and the outcomes are consistent with the original data [54]. Next, we explicitly define the reference point for the mobility game $\mathcal{M}$ as follows $z_i^0 = \left(\sum_{k\in\mathcal{S}_{v_i}\setminus\{i\}} \pi_k\right)^2 - \left(\sum_{i\in\mathcal{S}_{v_i}} \pi_i\right)^2$ where z_i^0 represents the ideal redistribution of wealth to traveler i (since no transport hub v_i should make a profit, i.e., $b(v_i) = 0$). For the random variable Z, we assume a continuous distribution F with zero mean and a probability density function f, and explicitly have

$$Z = \left(F - \sum_{k\in\mathcal{S}_{v_i}\setminus\{i\}} \pi_k\right)^2 - \left(F - \sum_{i\in\mathcal{S}_{v_i}} \pi_i\right)^2. \tag{62}$$

So, by (60), we have $\mathbb{E}[Z] = \sum_{n\in\mathbb{R}} v_i(z(n), z_i^0) w_i(f(n))$, where $z(n)$ represents at each transport hub v_i of an arbitrary traveler i the realization of Z with $n \in \mathbb{R}$ available mobility funds. The total utility now under prospect theory for a traveler i is

$$u_i^{\mathrm{PT}}(a) = z_i^0 + \mathbb{E}[Z] - \frac{\bar{\theta}_i}{\theta_i + \eta_i \pi_i} - \sum_{e\in\rho_i:\rho_i\in\mathcal{P}^{(o,d)}} c_e(J_e) - \frac{|\mathcal{S}_{v_i}|}{\sigma(v_i, h_i)}. \tag{63}$$

Next, we show that our mobility game $\mathcal{M}$ under prospect theory is guaranteed to have at least one NE.

Theorem 4. *The mobility game $\mathcal{M}$ under prospect theory admits a pure-strategy NE.*

proof. We expand $z(n)$ and subtract z_i^0 and simplify to get

$$\begin{aligned} z(n) &= \left(n - \sum_{k\in\mathcal{S}_{v_i}\setminus\{i\}} \pi_k\right)^2 - \left(n - \sum_{i\in\mathcal{S}_{v_i}} \pi_i\right)^2 \\ z(n) - z_i^0 &= 2n\Big[\sum_{k\in\mathcal{S}_{v_i}\setminus\{i\}} \pi_k - \sum_{i\in\mathcal{S}_{v_i}} \pi_i\Big] = 2n\pi_i. \end{aligned} \tag{64}$$

where $z_i^0 = \left(\sum_{k\in\mathcal{S}_{v_i}\setminus\{i\}}\pi_k\right)^2 - \left(\sum_{i\in\mathcal{S}_{v_i}}\pi_i\right)^2$. Substituting (64) into (63) yields

$$u_i^{\mathrm{PT}}(a) = z_i^0 + \sum_{n\in\mathbb{R}} v_i \cdot (2n\pi_i)\cdot w_i(f(n)) - \sum_{e\in\rho_i:\rho_i\in\mathcal{P}^{(o,d)}} c_e(J_e) - \frac{|\mathcal{S}_{v_i}|}{\sigma(v_i,h_i)} - \frac{\bar{\theta}_i}{\theta_i+\eta_i\pi_i}, \tag{65}$$

where v_i is given by (61). The next step is explicitly defining a new potential function under prospect theory. We have

$$\begin{aligned}\Psi(a) = &\sum_{i\in\mathcal{I}}\sum_{n\in\mathbb{R}} v_i\cdot(2n\pi_i)\cdot w_i(f(n)) - \sum_{e\in\mathcal{E}}\sum_{k=1}^{J_e} c_e(k)\\ &-\sum_{v\in\mathcal{V}}\frac{|\mathcal{S}_v|(|\mathcal{S}_v|+1)}{2\sigma(v,h_i)} - \sum_{i\in\mathcal{I}}\frac{\bar{\theta}_i}{\theta_i+\eta_i\pi_i} + \sum_{v\in\mathcal{V}}\left(\sum_{k\in\mathcal{S}_v}\pi_k\right)^2.\end{aligned} \tag{66}$$

Next, we show that Ψ as given in (66) is an exact potential function. We notice that $\sum_{n\in\mathbb{R}} v_i\cdot(2n\pi_i)\cdot w_i(f(n))$ does not depend on a_{-i}, i.e., the actions of the other travelers except traveler i. Hence, following similar arguments as in Theorem 1, we obtain $u_i^{\mathrm{PT}}(a_i,a_{-i}) - u_i^{\mathrm{PT}}(a_i',a_{-i}) = \Psi(a_i,a_{-i}) - \Psi(a_i',a_{-i})$. Hence, Ψ is indeed an exact potential function for the mobility game $\mathcal{M}$ under prospect theory. Therefore, since any action profile that minimizes Ψ results in a NE, the mobility game $\mathcal{M}$ admits a NE under prospect theory.

Corollary 2. *For the mobility game $\mathcal{M}$ under prospect theory, the sequence of best responses of an arbitrary traveler $i\in\mathcal{I}$ converges to a NE.*

Proof. It is sufficient to note that the action set $\mathcal{A}_i$ of any traveler i is compact. Thus, it follows from the results in [42] that the sequence of best responses of any traveler $i\in\mathcal{I}$ converges to a NE.

Both Theorem 4 and Corollary 2 ensure that the mobility game $\mathcal{M}$ under the prospect-theoretic behavioral model admits a NE and prospect-based travelers will eventually converge to it. Both results establish that we can still ensure that an equilibrium can be reached under certain conditions for the cost and pricing functions.

Concluding remarks

In this paper, we proposed a mobility game to study the behavioral interactions of travelers in a multimodal transportation network. First, we formulated a repeated non-cooperative routing game with a finite number of travelers. In our first result, we showed that the mobility game admits a NE under the assumption of rationality. In our second result, we derived a bound for the PoA. Although we cannot have uniqueness at an equilibrium, our upper bound guarantees that the inefficiencies are as low as possible if the number of travelers is large enough (which is naturally expected in a mobility system). We also derived an upper bound for the PoS, showing that the greater the number of travelers, the close some NE can be to the social optimum. Next, we extended our game to consider the subjective behavior of travelers under prospect theory and showed that our mobility game admits a NE.

Implementation

In this subsection, we outline how our proposed mobility game can be potentially implemented. We consider a major metropolitan area with an extensive road and public transit infrastructure; a good example is Boston. Several key areas in Boston are connected by roads, buses, light rail, and bikes. These areas can serve as transport hubs from which travelers can utilize any of the available modes of transportation. We can apply the MaaS concept and offer on each transport hub travel services (e.g., navigation, location, booking, payment) to all passing travelers. Information can be shared among all travelers via a "mobility app," which allows travelers to access the services on the transport hubs. Using this app, travelers can pay for their travel needs and, at the same time, receive mobility payments. For example, a traveler who informs the app and uses a bike multiple times (per day or per week) can receive mobility payments. In addition, travelers travel multiple times and interact with each other more than once. So, travelers seek to move from one place to another while competing with many other travelers, use the transport hubs to access their preferred mode of transportation, and pay using a mobility app. Each mode of transportation offers different benefits in utility; for example, a car is more convenient than a bus and is expected to be in high demand. This naturally will lead to inefficiency and congestion.

The mobility game $\mathcal{M}$ with the utility structure defined in (4) captures the key factors that may play a role in a traveler's decision-making. It can be seen by Theorem 1 and Corollary 1 that an equilibrium exists and can be reached by the travelers without direct intervention from a central authority. The particular pricing mechanism we have proposed in (1) ensures all travelers with the computational power of their cellphone can quickly derive the NE strategy (route, transport hub, payment). This is important as we can avoid solving a mixed-integer nonlinear program for all the travelers in the mobility system. In addition, by Theorem 2 we can guarantee that the inefficiency of the mobility system stays low as long as the number of travelers remains large (something that is expected in a typical mobility system). We can guarantee that a NE will stay close to the social optimum as the number of travelers increases. Thus, even though we cannot guarantee the uniqueness of a NE, we can ensure that all NE are similarly efficient and nearly as efficient as the social optimum as long as the number of travelers is high.

Using prospect theory, we can also consider how a traveler can feel uncertain about whether they may receive mobility payments for choosing a more sustainable mode of transportation (e.g., bike). Under certain conditions, we show that indeed a NE exists (Theorem 4) and it can be reached by the travelers as they can travel from the hub that is nearest to their home to the hub that is nearest to their work (Corollary 2). Thus, our game $\mathcal{M}$ framework is proved to lead to a NE under two different behavioral models and capture the impact of the travelers' decision-making.

Technical discussion and a numerical example

In this subsection, we discuss in more detail the technical implementation of our mobility game. So, we can compute a NE using the potential function, leveraging the fact that our game is a potential game with a finite set of travelers (as shown in Theorem 1). Note that this means that there exists a potential function (given by (7)) that maps each strategy profile to a real value. Intuitively, any change in any traveler's utility that unilaterally deviates from a strategy is equal to the change in our potential function. Hence, we can find the strategy profiles that simply maximize our potential function. One approach to achieve this for mixed-integer optimization problems is to use the branch-and-bound algorithm [77]. So, by Corollary 1, convergence to a pure strategy NE is guaranteed, thus we can find a NE, compute it, and use for our PoA analysis.

Typically, we solve numerically the optimization problem that arises from the routing game. We can either employ a gradient-based methods or a learning algorithm (e.g., fictitious play). The gradient-based method involves updating the travelers' strategies by moving in the direction of the gradient of the potential function. However, since the action sets are coupled and include route choices, the optional stop at a transport hub along the route, and the payment for the mobility service, the standard way solve this problem is as a mixed-integer nonlinear program (MINLP) (in such a case use the branch-and-bound or branch-and-cut algorithm). Alternatively, we can first use the specialized algorithm Dijkstra and find the best route and transport hub through the network based on criteria such as shortest distance and least time. Once we have this optimal route, we then use "fmincon" to compute the optimal value of the mobility payments along that route and transport hub.

We now offer a numerical example with a simple transportation network that has one unique origin-destination (OD) pair. In this network, there are three routes, namely route 1, route 2, and route 3. We assume that there are two mobility services: car and bike. The travel time on each route depends on the volume of traffic and the service. We have the following

$$\text{Route } 1: t_1(\text{car}) = 10 + x_1, \quad t_1(\text{bike}) = 8, \tag{67}$$

$$\text{Route } 2: t_2(\text{car}) = 5 + 2x_2, \quad t_2(\text{bike}) = 7, \tag{68}$$

$$\text{Route } 3: t_3(\text{car}) = 15, \quad t_3(\text{bike}) = 5 + x_3, \tag{69}$$

where x_k denotes the fraction of travelers choosing either of routes $k = 1, 2, 3$ using a car. We also consider that the pricing functions for the two modes: car and bike take the explicit form as $\tau(\text{car}) = 10$ and $\tau(\text{bike}) = -2$. So, travelers receive a $2 incentive for choosing a bike. Suppose we have 50 travelers in total, with 30 travelers preferring route 1 and route 2 (we call this Group A), while the remaining 20 travelers prefer route 2 and route 3 (we call this Group B). Now, say that Group A chooses to utilize route 1 with a car, and Group B chooses to utilize route 3 using a car. We can now compute the utilities for each traveler:

$$\text{Route } 1: t_1(\text{car}) = 10 + 30 = 40, \quad \text{utility} = -40 - 10 = -50, \tag{70}$$

$$\text{Route } 2: t_2(\text{car}) = 5, \quad \text{utility} = -5 - 10 = -15, \tag{71}$$

$$\text{Route } 3: t_3(\text{car}) = 15, \quad \text{utility} = -15 - 10 = -25. \tag{72}$$

So, for Group A the utility for route 2 (car) is higher than route 1 (car), thus all travelers in Group A will deviate to route 2 (car). Similarly, in Group B the utility for route 3 (bike) is higher than Route 3 (car), and so all travelers will deviate to route 3 (bike). Let us now compute the utilities for any arbitrary traveler, i.e.,

$$\text{Route } 1: t_1(\text{car}) = 10, \quad u = -10 - 10 = -20, \tag{73}$$

$$\text{Route } 1: t_1(\text{bike}) = 8, \quad u = -8 + 2 = -6, \tag{74}$$

$$\text{Route } 2: t_2(\text{car}) = 5 + 2(30) = 65, \quad u = -65 - 10 = -75, \tag{75}$$

$$\text{Route } 2: t_2(\text{bike}) = 7, \quad u = -7 + 2 = -5, \tag{76}$$

$$\text{Route } 3: t_3(\text{car}) = 15, \quad u = -15 - 10 = -25, \tag{77}$$

$$\text{Route } 3: t_3(\text{bike}) = 5, \quad u = -5 + 2 = -3. \tag{78}$$

We continue our equilibrium analysis as follows: for Group A, the utility for route 1 (bike) is higher than Route 2 (car), so travelers will deviate to route 1 (bike). Next, Group B will not deviate as route 3 (bike) (already highest utility). Hence, we have reached the point in which no traveler can deviate and receive better utility; thus we have a NE. We notice that all travelers from Group A and Group B choose route 1 and route 3 utilizing bikes, respectively. We conclude that it is possible as we showed in our theoretical analysis for a NE to exist and it is easily converged to based on our pricing mechanism in this simple multimodal transportation network. Next, at this Ne, all travelers have chosen to utilize the mobility service: bike, which is a sustainable and environmentally-friendly mode of transportation. Our pricing mechanism naturally will favor such modes of transportation and provide incentives to travelers for better utilization. Thus, we can control travel demand and effectively reduce any inefficiencies that may arise from congestion or higher pollution levels caused by car usage. One last note: in this example, we used a similar method to the best-response dynamics approach and found the NE using only a few iterations. Although our example quickly leads to an equilibrium solution, for larger and more complex transportation networks, we cannot draw the same conclusion and thus, it remains future work to adopt more advanced optimization and algorithmic techniques in order to find and compute our mobility game's NE.

Limitations and future work

One important limitation of our framework is the assumption of complete information. Realistically, we cannot expect travelers to have accurate and complete knowledge of other travelers or the system's capabilities (network, road capacities). A potential direction for future research should relax this assumption by only allowing travelers to know their own actions and utilities. In the literature, attempts have been made to investigate the emergence of cooperation among selfish travelers and how to bound rationality/irrationality in travel-choice problems [78–81]. A standard technique is Bayesian game-theoretic analyses, and recently, and techniques to learn representations of unknown information from observed data [82–86]. Another interesting direction for future research is to expand the current framework by explicitly designing the socially-efficient pricing functions to achieve the best possible equilibrium in the mobility system using techniques from mechanism design. Furthermore, to showcase the benefits of the proposed game-theoretic approach, a necessary extension of our work is using machine learning techniques [87–91] with real-life data. Finally, future work may explore how to perform a comparative analysis on models focusing on real-time efficacy for similar problems like ours using techniques similar to other fields [92–95].

Author Contributions

Conceptualization: Ioannis Vasileios Chremos, Andreas A. Malikopoulos.

Formal analysis: Ioannis Vasileios Chremos, Andreas A. Malikopoulos.

Investigation: Ioannis Vasileios Chremos.

Supervision: Andreas A. Malikopoulos.

Writing – original draft: Ioannis Vasileios Chremos.

Writing – review & editing: Ioannis Vasileios Chremos, Andreas A. Malikopoulos.

References

1. Varaiya P. Smart cars on smart roads: problems of control. IEEE Transactions on Automatic Control. 1993; 38(2):195–207. https://doi.org/10.1109/9.250509
2. Colini-Baldeschi R, Cominetti R, Mertikopoulos P, Scarsini M. The asymptotic behavior of the price of anarchy. In International Conference on Web and Internet Economics. 2017; p. 133–145. https://doi.org/10.1007/978-3-319-71924-5_10

3. Zhao L, Malikopoulos AA. Enhanced mobility with connectivity and automation: A review of shared autonomous vehicle systems. IEEE Intelligent Transportation Systems Magazine. 2022; 14(1):87–102. https://doi.org/10.1109/MITS.2019.2953526
4. Colombo A, de Campos GR, Rossa FD. Control of a City Road Network: Distributed Exact Verification of Traffic Safety. IEEE Transactions on Automatic Control. 62(10);2017:4933–4948. https://doi.org/10.1109/TAC.2017.2676464
5. Sarkar R, Ward J. DOE SMART mobility: Systems and modeling for accelerated research in transportation. In: Meyer G, Beiker S, editors. Road Vehicle Automation. Springer; 2016. p. 39–52.
6. Papageorgiou M, Kotsialos A. Freeway ramp metering: an overview. IEEE Transactions on Intelligent Transportation Systems. 2002; 3(4):271–281. https://doi.org/10.1109/TITS.2002.806803
7. Zardini G, Lanzetti N, Salazar M, Censi A, Frazzoli E, Pavone M. Towards a co-design framework for future mobility systems. Annual Meeting of the Transportation Research Board. 2020;.
8. Chalaki B, Malikopoulos AA. Optimal Control of Connected and Automated Vehicles at Multiple Adjacent Intersections. IEEE Transactions on Control Systems Technology. 2022; 30(3):972–984. https://doi.org/10.1109/TCST.2021.3082306
9. Ntousakis IA, Nikolos IK, Papageorgiou M. Optimal vehicle trajectory planning in the context of cooperative merging on highways. Transportation Research Part C: Emerging Technologies. 2016; 71:464–488. https://doi.org/10.1016/j.trc.2016.08.007
10. Malikopoulos AA, Beaver LE, Chremos IV. Optimal Time Trajectory and Coordination for Connected and Automated Vehicles. Automatica. 2021; 125(109469). https://doi.org/10.1016/j.automatica.2020.109469
11. Zardini G, Lanzetti N, Pavone M, Frazzoli E. Analysis and control of autonomous mobility-on-demand systems. Annual Review of Control, Robotics, and Autonomous Systems. 2022; 5:633–658. https://doi.org/10.1146/annurev-control-042920-012811
12. Ferrara A, Sacone S, Siri S. Freeway Traffic Modeling and Control. Springer Nature; 2018.
13. Mehr N, Horowitz R. How will the presence of autonomous vehicles affect the equilibrium state of traffic networks? IEEE Transactions on Control of Network Systems. 2019; 7(1):96–105. https://doi.org/10.1109/TCNS.2019.2918682
14. Lazar DA, Coogan S, Pedarsani R. Routing for Traffic Networks With Mixed Autonomy. IEEE Transactions on Automatic Control. 2021; 66(6):2664–2676. https://doi.org/10.1109/TAC.2020.3020059
15. Bıyık E, Lazar D, Pedarsani R, Sadigh D. Incentivizing efficient equilibria in traffic networks with mixed autonomy. IEEE Transactions on Control of Network Systems. 2021; 8(4):1717–1729. https://doi.org/10.1109/TCNS.2021.3084045
16. Krichene W, Drighès B, Bayen AM. Online learning of Nash equilibria in congestion games. SIAM Journal on Control and Optimization. 2015; 53(2):1056–1081. https://doi.org/10.1137/140980685
17. Krichene W, Castillo MS, Bayen AM. On social optimal routing under selfish learning. IEEE Transactions on Control of Network Systems. 2016; 5(1):479–488. https://doi.org/10.1109/TCNS.2016.2619910
18. ZardiniG, LanzettiN, GuerriniL, FrazzoliE, DörflerF. Game theory to study interactions between mobility stakeholders. In IEEE International Intelligent Transportation Systems Conference (ITSC). 2021; p. 2054–2061.
19. Harb M, Xiao Y, Circella G, Mokhtarian PL, Walker JL. Projecting travelers into a world of self-driving vehicles: estimating travel behavior implications via a naturalistic experiment. Transportation. 2018; 45(6):1671–1685. https://doi.org/10.1007/s11116-018-9937-9
20. Auld J, Sokolov V, Stephens TS. Analysis of the effects of connected-automated vehicle technologies on travel demand. Journal of the Transportation Research Board. 2017; 2625:1–8. https://doi.org/10.3141/2625-01
21. Bissell D, Birtchnell T, Elliott A, Hsu EL. Autonomous automobilities: The social impacts of driverless vehicles. Current Sociology. 2018;.
22. Pigou AC. The Economics of Welfare. Macmillan London; 2013.
23. Ferguson BL, Brown PN, Marden JR. The Effectiveness of Subsidies and Tolls in Congestion Games. IEEE Transactions on Automatic Control. 2021 (early access);.
24. Bui KT, Huynh VA, Frazzoli E. Dynamic traffic congestion pricing mechanism with User-Centric considerations. In: 2012 IEEE 15th International Conference on Intelligent Transportation Systems (ITSC); 2012. p. 147–154.
25. Levinson D. Micro-foundations of congestion and pricing: A game theory perspective. Transportation Research Part A: Policy and Practice. 2005; 39(7-9):691–704.

26. Tian D, Zhou J, Wang Y, Sheng Z, Duan X, Leung VC. Channel access optimization with adaptive congestion pricing for cognitive vehicular networks: An evolutionary game approach. IEEE Transactions on Mobile Computing. 2019; 19(4):803–820. https://doi.org/10.1109/TMC.2019.2901471

27. Marden JR, Shamma JS. Game theory and distributed control. In: Peyton Young H, Zamir S, editors. Handbook of Game Theory With Economic Applications. vol. 4. Elsevier; 2015. p. 861–899.

28. Hota AR, Sundaram S. Controlling Human Utilization of Failure-Prone Systems via Taxes. IEEE Transactions on Automatic Control. 2020;early access.

29. Brown PN, Marden JR. Optimal mechanisms for robust coordination in congestion games. IEEE Transactions on Automatic Control. 2017; 63(8):2437–2448. https://doi.org/10.1109/TAC.2017.2768901

30. Chandan R, Paccagnan D, Ferguson BL, Marden JR. Computing optimal taxes in atomic congestion games. In: Proceedings of the 14th Workshop on the Economics of Networks, Systems and Computation; 2019.

31. Salazar M, Lanzetti N, Rossi F, Schiffer M, Pavone M. Intermodal autonomous mobility-on-demand. IEEE Transactions on Intelligent Transportation Systems. 2019; 21(9):3946–3960. https://doi.org/10.1109/TITS.2019.2950720

32. Chremos IV, Beaver LE, Malikopoulos AA. A Game-Theoretic Analysis of the Social Impact of Connected and Automated Vehicles. In: 2020 23rd International Conference on Intelligent Transportation Systems (ITSC). IEEE; 2020. p. 2214–2219.

33. Chremos IV, Malikopoulos AA. Social Resource Allocation in a Mobility System with Connected and Automated Vehicles: A Mechanism Design Problem. In: Proceedings of the 59th IEEE Conference on Decision and Control (CDC), 2020; 2020. p. 2642–2647.

34. Chremos IV, Malikopoulos AA. Design and Stability Analysis of a Shared Mobility Market. In: 2021 European Control Conference (ECC); 2021. p. 375–380.

35. Chremos IV, Malikopoulos AA. In: Murphey YL, Kolmanovsky I, Watta P, editors. Socioeconomic Impact of Emerging Mobility Markets and Implementation Strategies. Springer International Publishing; 2023. p. 481–510.

36. Dave A, Chremos IV, Malikopoulos AA. Social Media and Misleading Information in a Democracy: A Mechanism Design Approach. IEEE Transactions on Automatic Control. 2022; 67(5):2633–2639. https://doi.org/10.1109/TAC.2021.3087466

37. Chremos IV, Malikopoulos AA. An Analytical Study of a Two-Sided Mobility Game. In: 2022 American Control Conference (ACC); 2022. p. 1254–1259.

38. Rosenthal R. A Class of Games Possessing Pure-Strategy Nash Equilibria. International Journal of Game Theory. 1973; 2:65–67. https://doi.org/10.1007/BF01737559

39. Cominetti R, Correa JR, Stier-Moses NE. The impact of oligopolistic competition in networks. Operations Research. 2009; 57(6):1421–1437. https://doi.org/10.1287/opre.1080.0653

40. Hao Y, Pan S, Qiao Y, Cheng D. Cooperative Control via Congestion Game Approach. IEEE Transactions on Automatic Control. 2018; 63(12):4361–4366. https://doi.org/10.1109/TAC.2018.2824978

41. Tanaka T, Nekouei E, Pedram AR, Johansson KH. Linearly Solvable Mean-Field Traffic Routing Games. IEEE Transactions on Automatic Control. 2021; 66(2):880–887. https://doi.org/10.1109/TAC.2020.2986195

42. Monderer D, Shapley L. Potential Games. Games and Economic Behavior. 1996; 14:124–143. https://doi.org/10.1006/game.1996.0044

43. Konishi H, Le Breton M, Weber S. Equilibrium in a Finite Local Public Goods Economy. Journal of Economic Theory. 1998; 79:224–244. https://doi.org/10.1006/jeth.1997.2386

44. Mäler K, Xepapadeas A, Zeeuw A. The Economics of Shallow Lakes. Environmental and Resource Economics. 2003; 26:603–624.

45. Garcia A, Arbeláez L. Market Power Analysis for the Colombian Electricity Market. Energy Economics. 2002; 24:217–229. https://doi.org/10.1016/S0140-9883(02)00015-4

46. Orda A, Rom R, Shimkin N. Competitive routing in multiuser communication networks. IEEE/ACM Transactions on networking. 1993; 1(5):510–521. https://doi.org/10.1109/90.251910

47. Roughgarden T, Schoppmann F. Local smoothness and the price of anarchy in splittable congestion games. Journal of Economic Theory. 2015; 156:317–342. https://doi.org/10.1016/j.jet.2014.04.005

48. Sekar S, Zheng L, Ratliff LJ, Zhang B. Uncertainty in Multicommodity Routing Networks: When Does It Help? IEEE Transactions on Automatic Control. 2020; 65(11):4600–4615. https://doi.org/10.1109/TAC.2019.2962102

49. Ostrom E, Gardner R, Walker J. Rules, games, and common-pool resources. University of Michigan Press; 1994.

50. Amin S, Schwartz GA, Sastry SS. On the interdependence of reliability and security in networked control systems. In: IEEE 50th Annual Conference on Decision and Control (CDC) and Control and European Control Conference (ECC); 2011. p. 4078–4083.

51. Zhong R, Xu R, Sumalee A, Ou S, Chen Z. Pricing Environmental Externality in Traffic Networks Mixed With Fuel Vehicles and Electric Vehicles. IEEE Transactions on Intelligent Transportation Systems. 2020; 22(9):5535–5554. https://doi.org/10.1109/TITS.2020.2987832

52. Barrera J, Garcia A. Dynamic incentives for congestion control. IEEE Transactions on Automatic Control. 2014; 60(2):299–310. https://doi.org/10.1109/TAC.2014.2348197

53. Kahneman D, Tversky A. Prospect theory: An analysis of decision under risk. Econometrica. 1979; 47(2):363–391. https://doi.org/10.2307/1914185

54. Tversky A, Kahneman D. Advances in prospect theory: Cumulative representation of uncertainty. Journal of Risk and Uncertainty. 1992; 5(4):297–323. https://doi.org/10.1007/BF00122574

55. Camerer CF, Loewenstein G, Rabin M. Advances in Behavioral Economics. Princeton University Press; 2004.

56. Barberis NC. Thirty years of prospect theory in economics: A review and assessment. Journal of Economic Perspectives. 2013; 27(1):173–196. https://doi.org/10.1257/jep.27.1.173

57. El Rahi G, Sanjab A, Saad W, Mandayam NB, Poor HV. Prospect theory for enhanced smart grid resilience using distributed energy storage. In: IEEE 54th Annual Allerton Conference on Communication, Control, and Computing (Allerton); 2016. p. 248–255.

58. Nar K, Ratliff LJ, Sastry S. Learning prospect theory value function and reference point of a sequential decision maker. In: IEEE 56th Annual Conference on Decision and Control (CDC); 2017. p. 5770–5775.

59. Etesami SR, Saad W, Mandayam NB, Poor HV. Stochastic games for the smart grid energy management with prospect prosumers. IEEE Transactions on Automatic Control. 2018; 63(8):2327–2342. https://doi.org/10.1109/TAC.2018.2797217

60. de Palma A, Picard N. Equilibria and information provision in risky networks with risk averse drivers. Transportation Science. 2006; 40(4):393–408. https://doi.org/10.1287/trsc.1060.0152

61. Lam TC, Small KA. The value of time and reliability: Measurement from a value pricing experiment. Transportation Research E. 2001; 37:231–251. https://doi.org/10.1016/S1366-5545(00)00016-8

62. Etesami SR, Saad W, Mandayam NB, Poor HV. Smart routing of electric vehicles for load balancing in smart grids. Automatica. 2020; 120(109148). https://doi.org/10.1016/j.automatica.2020.109148

63. Webster TJ. Analyzing Strategic Behavior in Business and Economics: A Aame Theory Primer. Lexington Books; 2014.

64. Bandi C, Bertsimas D. Optimal Design for Multi-Item Auctions: A Robust Optimization Approach. Mathematics of Operations Research. 2014; 39(4):1012–1038. https://doi.org/10.1287/moor.2014.0645

65. Roughgarden T, Tardos E. How bad is selfish routing? Journal of the ACM (JACM). 2002; 49(2):236–259. https://doi.org/10.1145/506147.506153

66. Roughgarden T. Routing Games. In: Nisan N, Roughgarden T, Tardos E, Vazirani VV, editors. Algorithmic Game Theory. vol. 1. Cambridge University Press; 2007. p. 459–484.

67. Malikopoulos AA. On Team Decision Problems with Nonclassical Information Structures. IEEE Transactions on Automatic Control. 2023 (early access);.

68. Lã QD, Chew YH, Soong BH. Potential Game Theory. Springer; 2016.

69. Koutsoupias E, Papadimitriou C. Worst-case equilibria. In Annual Symposium on Theoretical Aspects of Computer Science. 1999; p. 404–413.

70. Awerbuch B, Azar Y, Epstein A. The price of routing unsplittable flow. In: Proceedings of the thirty-seventh annual ACM symposium on Theory of computing; 2005. p. 57–66.

71. Wakker PP. Prospect Theory: For Risk and Ambiguity. Cambridge University Press; 2010.

72. Al-Nowaihi A, Bradley I, Dhami S. A note on the utility function under prospect theory. Economics Letters. 2008; 99(2):337–339. https://doi.org/10.1016/j.econlet.2007.08.004

73. Kőszegi B, Rabin M. Reference-dependent risk attitudes. American Economic Review. 2006; 97 (4):1047–1073.

74. Baucells M, Weber M, Welfens F. Reference-point formation and updating. Management Science. 2011; 57(3):506–519. https://doi.org/10.1287/mnsc.1100.1286

75. Hota AR, Garg S, Sundaram S. Fragility of the commons under prospect-theoretic risk attitudes. Games and Economic Behavior. 2016; 98:135–164. https://doi.org/10.1016/j.geb.2016.06.003

76. Prelec D. The probability weighting function. Econometrica. 1998; 66(3):497–527. https://doi.org/10.2307/2998573

77. Ross GT, Soland RM. A branch and bound algorithm for the generalized assignment problem. Mathematical programming. 1975; 8(1):91–103. https://doi.org/10.1007/BF01580430

78. Farokhi F, Johansson KH. A study of truck platooning incentives using a congestion game. IEEE Transactions on intelligent transportation systems. 2014; 16(2):581–595.

79. Innocenti A, Lattarulo P, Pazienza MG. Car stickiness: Heuristics and biases in travel choice. Transport Policy. 2013; 25:158–168. https://doi.org/10.1016/j.tranpol.2012.11.004

80. Klein I, Ben-Elia E. Emergence of cooperation in congested road networks using ICT and future and emerging technologies: A game-based review. Transportation Research Part C: Emerging Technologies. 2016; 72:10–28. https://doi.org/10.1016/j.trc.2016.09.005

81. Di X, Liu HX. Boundedly rational route choice behavior: A review of models and methodologies. Transportation Research Part B: Methodological. 2016; 85:142–179. https://doi.org/10.1016/j.trb.2016.01.002

82. Baisero A, Amato C. Learning internal state models in partially observable environments. In: Reinforcement Learning under Partial Observability, NeurIPS Workshop; 2018.

83. Subramanian J, Sinha A, Seraj R, Mahajan A. Approximate Information State for Approximate Planning and Reinforcement Learning in Partially Observed Systems. J Mach Learn Res. 2022; 23:1–83.

84. Malikopoulos AA. Separation of Learning and Control for Cyber-Physical Systems. Automatica. 2023; 151(110912). https://doi.org/10.1016/j.automatica.2023.110912

85. Venkatesh N, Le VA, Dave A, Malikopoulos AA. Connected and Automated Vehicles in Mixed-Traffic: Learning Human Driver Behavior for Effective On-Ramp Merging. arXiv preprint arXiv:230400397. 2023.

86. Dave A, Venkatesh N, Malikopoulos AA. On Robust Control of Partially Observed Uncertain Systems with Additive Costs. In: 2023 American Control Conference (ACC); 2023 (to appear).

87. Balcan MF, Blum A, Hartline JD, Mansour Y. Mechanism design via machine learning. In 46th Annual IEEE Symposium on Foundations of Computer Science (FOCS'05). 2005; p. 605–614. https://doi.org/10.1109/SFCS.2005.50

88. Qian T, Shao C, Shi D, Wang X, Wang X. Automatically Improved VCG Mechanism for Local Energy Markets via Deep Learning. IEEE Transactions on Smart Grid. 2021; 13(2):1261–1272. https://doi.org/10.1109/TSG.2021.3128182

89. Vineyard CM, Heileman GL, Verzi SJ, Jordan R. Game theoretic mechanism design applied to machine learning classification. In 2012 3rd International Workshop on Cognitive Information Processing (CIP). 2012; p. 1–5.

90. Lim WYB, Xiong Z, Miao C, Niyato D, Yang Q, Leung C, et al. Hierarchical incentive mechanism design for federated machine learning in mobile networks. IEEE Internet of Things Journal. 2020; 7(10):9575–9588. https://doi.org/10.1109/JIOT.2020.2985694

91. Kim S. Incentive design and differential privacy based federated learning: A mechanism design perspective. IEEE Access. 2020; 8:187317–187325. https://doi.org/10.1109/ACCESS.2020.3030888

92. Mahajan S, Abualigah L, Pandit AK. Hybrid arithmetic optimization algorithm with hunger games search for global optimization. Multimedia Tools and Applications. 2022; 81(20):28755–28778. https://doi.org/10.1007/s11042-022-12922-z

93. Mahajan S, Abualigah L, Pandit AK, Al Nasar MR, Alkhazaleh HA, Altalhi M. Fusion of modern meta-heuristic optimization methods using arithmetic optimization algorithm for global optimization tasks. Soft Computing. 2022; 26(14):6749–6763. https://doi.org/10.1007/s00500-022-07079-8

94. Amiri M, Sobhani A, Al Osman H, Shirmohammadi S. Sdn-enabled game-aware routing for cloud gaming datacenter network. IEEE Access. 2017; 5:18633–18645. https://doi.org/10.1109/ACCESS.2017.2752643

95. Khan AA, Abolhasan M, Ni W. An evolutionary game theoretic approach for stable and optimized clustering in VANETs. IEEE Transactions on Vehicular Technology. 2018; 67(5):4501–4513. https://doi.org/10.1109/TVT.2018.2790391