NON-DESTRUCTIVE TESTING OF ANISOTROPIC MATERIALS

by

Isaac Harris

A dissertation submitted to the Faculty of the University of Delaware in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Applied Mathematics

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Sir Isaac Newton once wrote “If I have seen further it is by standing on the shoulders of giants.”-even though this quote was intended to be an insult to Robert Hooke it still relays a beautiful sentiment. Here is where I thank the giants whose shoulders I have been standing on. I start by thanking my loving mother Patricia Harris, without her love and support this thesis would not exists. I would also like to thank my amazing grandmother Evelyn Harris for everything she has done for me. I also want to thank my entire family for their support throughout the years.

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ABSTRACT

In the thesis we consider *the inverse problem of detecting the shape, size, position and some information about the material properties of a (possibly anisotropic) penetrable defective region in a known anisotropic material.* First we consider the problem of detecting defects by using the measured transmission eigenvalues, which are related to non-scattering frequencies. In particular we consider the transmission eigenvalue problem corresponding to the scattering problem for an anisotropic magnetic material with voids, i.e. subregions with refractive index the same as the background, restricting ourselves to the scalar case of TE-polarization for electromagnetic waves or acoustic waves. Under appropriate assumptions on the material properties, we show that the transmission eigenvalues can be determined from the far field measurements, and we prove the existence of at least one real transmission eigenvalue for sufficiently small voids. We also show that the first transmission eigenvalue can be used to provide qualitative information about the size of the void.

Even though the transmission eigenvalues can be used to determine the size of a defective region, *to reconstruct the shape and position of the defect* we need to use different techniques. To this end, we develop the Factorization Method (FM) which provides an indicator function for the defective region. The FM connects the support of the defective region to the range of a compact operator which is known from physical experiments. Hence, evaluating the indicator function derived from the FM amounts to applying Picard's criteria, which only requires the singular values

and vectors of a known compact operator. Since evaluating the indicator function needs the far-field pattern of the background Green's function, we prove a mixed reciprocity result connecting the far-field pattern of the Greens function to the total field of the unperturbed material.

We then consider the inverse scattering problem for an anisotropic media with small homogeneous penetrable defects. We considered the transmission eigenvalue problem for the perturbed media as well as derive a MUSIC algorithm to reconstruct the locations of the small defects. For the corresponding transmission eigenvalue problem we study the convergence and convergence rate of the transmission eigenvalues and construct appropriate corrector terms for the transmission eigenvalues as the size of the defects tends to zero. Using the corrector and the knowledge of the location of the inclusions one can derive an algorithm to reconstruct the constitutive parameters of the inclusions.

In the same spirit as in using the transmission eigenvalues to determine information about a defective region, in the next project we use the transmission eigenvalues for *parameter identification*. In particular we consider the interior transmission problem associated with the scattering by an inhomogeneous (possibly anisotropic) highly oscillating periodic media. We show that, under appropriate assumptions, the solution of the interior transmission problem converges to the solution of a homogenized problem as the period goes to zero. Furthermore, we prove that the associated real transmission eigenvalues converge to transmission eigenvalues of the homogenized problem. In our investigation of the convergence, we construct boundary corrections for the anisotropic case, which are used to determine the convergence rate for the interior transmission problem. Finally we show how to use the first transmission eigenvalue of the period media, to obtain information about constant effective material properties of the periodic media.

Chapter 1

INTRODUCTION

The development of *qualitative methods* (otherwise known as non-iterative or direct or sampling methods) in inverse acoustic and electromagnetic scattering is a very active field of research. The qualitative methods are a late arrival to the theory of inverse scattering. Since their first appearance in 1996 in [36] this direction has had an explosion of interest in both the theory and application of these methods. The vast available literature is a representative of the myriad directions that this research has taken (see e.g. [14], [27], [33], [55], [34], [58], [72], [76] and the references therein). Qualitative methods have been used to solve multiple inverse problems such as parameter identification and shape reconstruction. For example in scattering theory, these methods aim to not only detect an object but also to identify the unknown object through the use of acoustic or electromagnetic waves. The advantages of using qualitative methods rather than non-linear optimization techniques are: first the optimization methods require a priori information that may not be readily available, second these methods can be computationally expensive. On the other hand, non-linear optimization techniques seek to reconstruct all the unknown parameters, while qualitative methods seek to only reconstruct partial information with very limited a priori knowledge on the topology and physics of the unknown object(s). Hence one can obtain partial information in a computationally inexpensive way that is rigorously justified and in many applications partial information about the unknown target will suffice.

In inverse scattering theory, methods that seek to provide (somewhat limited) information about the scattering object in a computationally simple way are in general considered to be qualitative methods. The two most well known examples of such methods are the Linear Sampling Method(LSM) and Factorization Method(FM) which primarily seek to determine an approximation to the shape of the scattering object. Typically these methods allow one to construct an indicator function for the support of the scattering object(s) here called D.

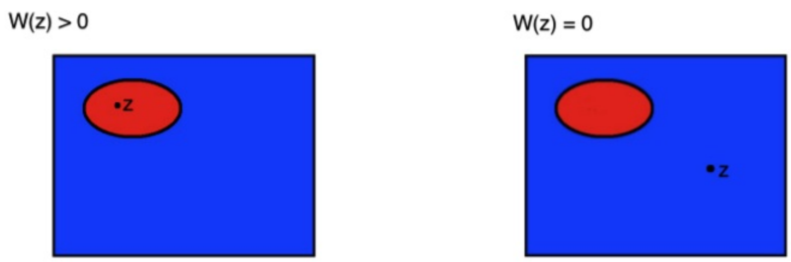

Figure 1.1: A depiction of how the linear sampling and factorization methods work. $W(z)$ is the indicator function which is non-zero if and only if the sampling point z is inside the scatterer.

The LSM connects the support of D to the solution of an ill-posed linear integral equation that involves a smoothing operator obtained from the measured scattered field. To this end one must use a sufficient regularization technique to solve the ill-posed problem, but in general there is no theoretical guarantee that the regularized solution inherits the desired properties. On the other hand, in the case of far field data, the FM connects the support D to the range of an operator derived from the measured far-field (or scattering) operator (see [4] and [5]). This not only provides a rigorous characterization of the object D, it also provides a theoretically justified numerical algorithm to reconstruct the scattering object. On the other hand the solution provided by the LSM can be related to a boundary value problem defined in the support of the scatterer, which can be used to obtain information about the material properties of the scattering object(s) in addition to the support [14]. Furthermore the LSM can be used for a broader class of scattering problems than the FM that requires more restrictive assumptions on the scatterer. Both the above methods require multi-static data, i.e. data collected simultaneously on an array of receivers due to an array of transmitters. In recent years efforts have been made to construct a bridge between the LSM and FM. As a result the Generalized Linear Sampling Method developed in [6] provides a rigorous characterization of the support of the scatterer as well as inherits the advantages of the LSM.

To motivate our problem, consider electromagnetic waves propagating in an inhomogeneous anisotropic dielectric magnetic medium in $\mathbb{R}^3$ with electric permittivity $\epsilon = \epsilon(x)$ and magnetic permeability $\mu = \mu(x)$. For time harmonic electromagnetic waves of the form

$$\mathcal{E}(x,t) = \tilde{E}(x)e^{-i\omega t}, \qquad \mathcal{H}(x,t) = \tilde{H}(x)e^{-i\omega t}$$

with frequency $\omega > 0$, we deduce that the complex valued space dependent parts $\tilde{E}$ and $\tilde{H}$ satisfy

$$\nabla \times \tilde{E} - i\omega\mu(x)\tilde{H} = 0 \quad \text{and} \quad \nabla \times \tilde{H} + i\omega\epsilon(x)\tilde{E} = 0.$$

Now let us suppose that the inhomogeneity occupies an infinitely long cylinder with cross section D having piece-wise smooth boundary ∂D with ν being the unit outward normal to ∂D. We assume that the axis of the cylinder coincides with the z-axis. We further assume that the conductor is imbedded in a non-conducting homogeneous background, i.e. the electric permittivity $\epsilon_0 > 0$ and the magnetic permeability $\mu_0 > 0$ of the background medium. For an orthotropic medium we have that the relative electric permittivity and relative magnetic permeability matrices $\mathcal{A}$ and $\mathcal{N}$ respectively are independent of the z-coordinate and are of the form

$$\mathcal{A} := \frac{\epsilon(x)}{\epsilon_0} = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & a \end{pmatrix} \qquad \mathcal{N} := \frac{\mu(x)}{\mu_0} = \begin{pmatrix} n_{11} & n_{12} & 0 \\ n_{21} & n_{22} & 0 \\ 0 & 0 & n \end{pmatrix}.$$

Then it is well-known (see [14]) that the only component u of the total magnetic field $\tilde{H} = (0, 0, u)$ polarized perpendicular to the axis of the cylinder satisfies

$$\nabla \cdot A(x)\nabla u + k^2 n(x) u = 0$$

where

$$A := \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^{-1},$$

and analogously the scattered field u^s satisfies $\Delta u^s + k^2 u^s = 0$ outside the scatter D with the wavenumber $k = \omega/\sqrt{\epsilon_0 \mu_0}$.

The direct scattering problem reads: find the total field $u = u^s + u^i$ with incident field u^i (which is typically an entire solution of Helmholtz equation or a point source) and scattered field u^s that satisfies

$$\Delta u^s + k^2 u^s = 0 \text{ in } \mathbb{R}^2 \setminus \overline{D} \quad \text{and} \quad \nabla \cdot A\nabla u + k^2 n u = 0 \quad \text{in} \quad D$$
$$u = u^s + u^i \quad \text{and} \quad \frac{\partial u}{\partial \nu_A} = \frac{\partial}{\partial \nu}(u^s + u^i) \quad \text{on} \quad \partial D$$

with $\frac{\partial u}{\partial \nu_A} = \nu \cdot A\nabla u$ and the scattered field satisfies the Sommerfeld radiation condition,

$$r^{1/2}\left(\frac{\partial u^s}{\partial r} - iku^s\right) \to 0 \qquad \text{as} \qquad r = |x| \to \infty$$

uniformly with respect to $\hat{x} = x/|x|$. The Sommerfeld radiation condition imposed on the scattered files implies that u^s has the asymptotic expansion

$$u^s(x,d) = \frac{e^{ik|x|}}{|x|^{1/2}}\left\{u^\infty(\hat{x},d) + \mathcal{O}\left(\frac{1}{|x|}\right)\right\} \text{ as } |x| \to \infty,$$

where $u^\infty(\hat{x},d)$ is called the *far-field pattern.*

There are special frequencies k that play an important role in inverse scattering, that are related to non-scattering frequencies and are called *transmission eigenvalues.* The transmission eigenvalue problem belongs to a new class of eigenvalue problems that are nonlinear and non self-adjoint, hence are not covered by the standard theory ofelliptic eigenvalue problems. The transmission eigenvalue problem for an inhomogeneous anisotropic media is given by: find the wavenumber k and eigenfunctions w and v such that

$$\nabla \cdot A(x)\nabla w + k^2 n(x) w = 0 \quad \text{and} \quad \Delta v + k^2 v = 0 \quad \text{in} \quad D \tag{1.1}$$
$$w - v = 0 \quad \text{and} \quad \frac{\partial w}{\partial \nu_A} - \frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial D. \tag{1.2}$$

For the direct scattering problem with $u = u^s + u^i$ if the wavenumber k and incident field u^i are constructed such that $u^s = 0$ outside of D (non-scattering) then $w = u\big|_D$ and $v = u^i\big|_D$ are the corresponding transmission eigenfunctions corresponding to the transmission eigenvalue k. Conversely if k is a transmission eigenvalue and if the eigenfunction v can be extended to a solution of the Helmholtz equation outside D then this incident field does not scatter. Unfortunately in the general case this is impossible, in [9] it is proven that an inhomogeneity with support having a right angle will always produce a scattered field. Thus the transmission eigenvalues in general are not non-scattering frequencies, however one can construct an incident field that produces an arbitrary small scattered field. As we will see later the transmission eigenvalues can be seen in the far-field data.

Generally speaking, the *inverse problem* in consideration is: from a knowledge of the measured far-field pattern $u^\infty(\hat{x}, d)$ for a set of observation directions $\hat{x}_i$ and incident directions d_j and a range of wave numbers $k \in [k_{min}, k_{max}]$, detect/reconstruct perturbations in the known inhomogeneity D, A and n. Note that in general, the above data does not ensure unique determination of the (perturbed) A and n for a matrix valued function A. However, it is possible to uniquely determine the support of the perturbations as well as some of it's physical properties. This is exactly what a qualitative approach seeks to reconstruct.

In this thesis we investigate four model problems related to the inverse scattering for non-destructive testing of anisotropic materials.

We start by considering the *transmission eigenvalue problem for an anisotropic magnetic materials (where the contrast is in both the electric permeability and magnetic permeability) with a void(s) (i.e. subregions where $A = I$ and $n = 1$).* The goal is to understand how the real transmission eigenvalues are related to the geometry of the void(s) and use this knowledge to determine information about the defect. The transmission eigenvalue problem for the same type of anisotropic materials but without a void(s) was considered in [29]. In Chapter 2 we generalize the analysis presented in [29] to prove the existence of real transmission eigenvalues for anisotropic magnetic materials with a void(s). We also show that the first transmission eigenvalue is an increasing function of the defective region D_0, which motivates a way to use it as a target signature for voids. Lastly we extended the result in [20] to our case, in particular we show that the far-field pattern can determine the transmission eigenvalues. Related work on this problem can be found in [14], [42] and [62].

The next problem we investigate is *reconstructing the support of a defective region D_0 in a known anisotropic material from a knowledge of the far-field pattern.* To this end in Chapter 3 we develop the factorization method (FM), and this study is the first extension of the FM to the case of an anisotropic inhomogeneous background. The analysis depends on a variational approach to define the operators involved in the factorization instead of a boundary/volume integral approach developed in [10] and [46]. As aleady known the FM connects the support of the defect to the range of a compact operator defined by the far-field data and hence one can construct an indicator function using Picard's criteria. In order to evaluate the indicator function one needs the far-field pattern of the background Green's function. To facilitate computation for the case of a homogenous anisotropic background we prove a mixed reciprocity result connecting the far-field pattern of the Greens function to the total field due to the unperturbed material. This makes the FM computationally cheap to implement while being an analytically rigorous way to reconstruct the support of D_0. We briefly analyze the Generalized Linear Sampling Method (GLSM) developed in [6], and show the under the same assumptions as the FM that the GLSM can be used to reconstruct the support of D_0. The GLSM is being studied as an alternative to the FM that is formulated in the same spirit as the LSM. This direction is being developed with the goal of providing a rigorous solution to the inverse problem under less restrictive assumptions than the FM.

In Chapters 4 we *investigate the inverse problem of finding small volume defects in an anisotropic material.* This chapter can be seen as a combination of the ideas of Chapters 2 and 3 applied to small inhomogeneities. First we review the derivation of the Multiple Signal Classification(MUSIC) algorithm for detecting the location of small perturbations in an anisotropic scatterer. Although the derivation is standard we did not find a comprehensive study of the MUSIC algorithm for an anisotropic background in the literature. The MUSIC algorithm can be seen as a discrete version of the FM which gives a rigorous characterization of the location(s) of the perturbation(s) in the scatterer using the so-called "multi-static response matrix" that is derived from the asymptotic expansions of the far-field pattern [3] and [48]. Then we proceed with the study of how the presence of these small defects affect the transmission eigenvalues. Since the transmission eigenvalue problem for an anisotropic media can not be reduced to a linear eigenvalue problem as in the isotropic case(see [31] and [32]) to study this problem we must appeal to perturbation theory for non-linear eigenvalues problems(e.g. see [64]). In particular we study: the convergence with convergence rates of the transmission eigenvalues as the volume of the inhomogeneities tend to zero and construct appropriate first order corrector term for the transmission eigenvalues.

Lastly in Chapter 5 we considered the *scattering problem for a highly oscillating periodic media.* The governing equations (1.1)-(1.2) have rapidly oscillating periodic coefficients $A(x/\epsilon)$ and $n(x/\epsilon)$,which for $\epsilon \ll 1$ typically model the wave propagation through composite materials with a fine microstructure. Using a homogenization approach we arrived at the homogenized interior transmission problem and show that it is an approximation of the problem for a periodic media. To prove strong convergence in $H^1(D)$ we construct the so-called bulk-corrector which involves the solution to a partial differential equation in a cell (i.e. the period of the coefficients A and n). We have also shown that as $\epsilon \to 0$ the real transmission eigenvalues of the periodic media converge to the real transmission eigenvalues of the homogenized problem. Since the transmission eigenvalue problem is non-linear and non-selfadjoint that makes the analysis interesting mathematically but also challenging. In our investigation of the convergence, we have constructed boundary corrections for the case with $A \neq I$, which is used to determine the convergence rate of the interior transmission problem and the transmission eigenvalues. Our analysis shows that the first transmission eigenvalue of the period media, which can be measurable from scattering data can be used to obtain information about the effective constitutive parameters of the periodic media. This project is the first work that has considered the transmission eigenvalue problem for periodic media and is a preliminary study which still leaves many open questions.

Some of the work presented in this thesis has been published in the following papers:

1. F. Cakoni, H. Haddar and I. Harris "Homogenization approach for the transmission eigenvalue problem for periodic media and application to the inverse problem". *Inverse Problems and Imaging (Accepted)*

2. F. Cakoni and I. Harris "The factorization method for a defective region in an anisotropic material". *Inverse Problems*, **31** 025002 (2015)

3. I. Harris, F. Cakoni and J. Sun "Transmission eigenvalues and non-destructive testing of anisotropic magnetic materials with voids". *Inverse Problems*, **30** 035016 (2014)

1.1 Basic Mathematical Tools

We now rigorously formulate the central scattering problem for an anisotropic media with a penetrable (possibly anisotropic) defect in $\mathbb{R}^m$(for $m = 2$ or $m = 3$). Let

$D \subset \mathbb{R}^m$ be a bounded simply connected open set with piece-wise smooth boundary ∂D. Furthermore assume that we have a symmetric matrix valued function $A(x)$ and scalar function $n(x)$ for $x \in D$. We consider the scattering of an incident plane wave $u^i(x,d) = e^{ikx\cdot d}$ (with $|d| = 1$ being the probing direction) by an anisotropic penetrable inhomogeneous media D with (possibly) a defective region $\overline{D}_0 \subset D$, where the scatterer D is embedded in a homogeneous background. The constitutive material properties of the healthy media is given by A and n, are extended outside of the scatterer by $A = I$ and $n = 1$. Inside the scatterer we have that $A \in C^1(D, \mathbb{C}^{m\times m})$ and $n \in L^\infty(D)$. The defective media has constitutive material properties given by

$$A_\delta = A + (A_0 - A)\chi_{D_0} \quad \text{and} \quad n_\delta = n + (n_0 - n)\chi_{D_0}$$

with $A_0 \in C^1(D_0, \mathbb{C}^{m\times m})$ and $n_0 \in L^\infty(D_0)$, where we assume that $A_0 \neq A$ and $n_0 \neq n$. This gives rise to the scattered field $u^s(x,d)$, and the corresponding total field $u(x,d) = u^s(x,d) + u^i(x,d)$ that satisfies

$$\Delta u^s + k^2 u^s = 0 \text{ in } \mathbb{R}^m \setminus \overline{D} \quad \text{and} \quad \nabla \cdot A_\delta \nabla u + k^2 n_\delta u = 0 \quad \text{in} \quad D \tag{1.3}$$

$$u = u^s + u^i \quad \text{and} \quad \frac{\partial u}{\partial \nu_{A_\delta}} = \frac{\partial}{\partial \nu}(u^s + u^i) \quad \text{on} \quad \partial D \tag{1.4}$$

where u^s satisfies the Sommerfeld radiation condition uniformly with respect to $\hat{x} = x/|x|$

$$\lim_{r\to\infty} r^{\frac{m-1}{2}} \left(\frac{\partial u^s}{\partial r} - iku^s \right) = 0 \tag{1.5}$$

Assumption 1.1.1. *Assume that A_δ and n_δ are such that:*

1. $\overline{\xi} \cdot \Re(A_\delta)\xi \geq \alpha|\xi|^2$ *for all* $\xi \in \mathbb{C}^m$ *and for almost every* $x \in D$
2. $\overline{\xi} \cdot \Im(A_\delta)\xi \leq 0$ *for all* $\xi \in \mathbb{C}^m$ *and for almost every* $x \in D$
3. $\Re(n_\delta) \geq n_{min} > 0$ *and* $\Im(n_\delta) \geq 0$ *for almost every* $x \in D$.

Notice that the scattered field satisfies

$$\nabla \cdot A_\delta \nabla u^s + k^2 n_\delta u^s = \nabla \cdot (I - A_\delta)\nabla u^i + k^2(1 - n_\delta)u^i \quad \text{in } \mathbb{R}^m \tag{1.6}$$

Our aim in this section is to establish the existence of a unique solution $u^s \in H^1_{loc}(\mathbb{R}^m)$ to (1.6). To this end we will rely on variational approach and sketch the main points of the proof.

Definition 1.1.1. *The Dirichlet-to-Neumann map $\mathbb{T}_k : H^{1/2}(\partial B_R) \longmapsto H^{-1/2}(\partial B_R)$ is defined by*

$$\mathbb{T}_k : v \to \frac{\partial v}{\partial \nu} \quad \text{on } \partial B_R$$

where v is a radiating (i.e. satisfying Sommerfeld radiation condition) solution to the Helmholtz equation $\Delta v + k^2 v = 0$ in $\mathbb{R}^m \setminus \overline{B}_R$ and ν is the outward unit normal to ∂B_R, where B_R is the ball centered at the origin of radius R.

It has been shown in [14] for $m = 2$ and in [36] for $m = 3$ that $\mathbb{T}_k - \mathbb{T}_0$ is compact and

$$-\int_{\partial B_R} \overline{v}\, \mathbb{T}_0 v \, ds \geq 0 \quad \text{for all} \quad v \in H^{1/2}(\partial B_R).$$

For later we note that

$$\Re\left(\int_{\partial B_R} \overline{v}\, \mathbb{T}_k v \, ds\right) \leq 0 \quad \text{and} \quad \Im\left(\int_{\partial B_R} \overline{v}\, \mathbb{T}_k v \, ds\right) \geq 0 \tag{1.7}$$

for all $v \in H^{1/2}(\partial B_R)$ (see [67] for details). Now with the help of the Dirichlet-to-Neumann mapping we can write (1.6) in the following equivalent variational form in the truncated domain B_R as find $u^s \in H^1(B_R)$ such that for all $\varphi \in H^1(B_R)$

$$\int_{B_R} A_\delta \nabla u^s \cdot \nabla \overline{\varphi} - k^2 n_\delta u^s \, \overline{\varphi} \, dx - \int_{\partial B_R} \overline{\varphi} \, \mathbb{T}_k u^s \, ds = \int_D (I - A_\delta) \nabla u^i \cdot \nabla \overline{\varphi} - k^2 (1 - n_\delta) u^i \, \overline{\varphi} \, dx. \tag{1.8}$$

Therefore u^s solves the variational problem: find $u^s \in H^1(B_R)$ such that

$$\mathcal{A}(u^s, \varphi) - \mathcal{B}(u^s, \varphi) = L(\varphi) \;\text{ for all } \varphi \in H^1(B_R)$$

where the bounded sesquilinear forms $\mathcal{A}(u^s, \varphi)$, $\mathcal{B}(u^s, \varphi)$ on $H^1(B_R) \times H^1(B_R)$ and the conjugate linear functional $L(\varphi)$ on $H^1(B_R)$ are given by

$$\begin{aligned}
\mathcal{A}(u^s, \varphi) &:= \int_{B_R} A_\delta \nabla u^s \cdot \nabla \overline{\varphi} + u^s \, \overline{\varphi} \, dx - \int_{\partial B_R} \overline{\varphi} \, \mathbb{T}_0 u^s \, ds, \\
\mathcal{B}(u^s, \varphi) &:= \int_{B_R} (k^2 n_\delta + 1) u^s \, \overline{\varphi} \, dx + \int_{\partial B_R} \overline{\varphi} \, (\mathbb{T}_k - \mathbb{T}_0) u^s \, ds, \\
L(\varphi) &:= \int_D (I - A_\delta) \nabla u^i \cdot \nabla \overline{\varphi} - k^2 (1 - n_\delta) u^i \, \overline{\varphi} \, dx.
\end{aligned}$$

Theorem 1.1.1. *Let the sesquilinear forms $\mathcal{A}(\cdot,\cdot)$, $\mathcal{B}(\cdot,\cdot) : H^1(B_R) \times H^1(B_R) \longmapsto \mathbb{C}$ and the conjugate linear functional $L(\cdot) : H^1(B_R) \longmapsto \mathbb{C}$ be as defined above, then we have the following*

1. *$\mathcal{A}(\cdot,\cdot)$ can be represented by an operator with a bounded inverse.*
2. *$\mathcal{B}(\cdot,\cdot)$ can be represented by a compact operator*
3. *$L(\cdot)$ is a bounded conjugate linear functional*

Proof. (i) We prove the result by applying the Lax-Milgram Lemma to the sesquilinear forms $\mathcal{A}(w, \varphi)$. By the Riesz Representation Theorem $\exists \mathbb{A} : H^1(B_R) \mapsto H^1(B_R)$ such that $(\mathbb{A}w, \varphi)_{H^1(B_R)} = \mathcal{A}(w, \varphi)$. Therefore by the assumptions on the coefficients that

$$\big|(\mathbb{A}w, w)_{H^1(B_R)}\big| \geq \Re\big(\mathcal{A}(w, w)\big) \geq \min\{1, \alpha\}||w||^2_{H^1(B_R)}$$

giving the result.
(ii) Similarly by the Riesz Representation Theorem $\exists \mathbb{B} : H^1(B_R) \mapsto H^1(B_R)$ that represent the sesquilinear forms $\mathcal{B}(w, \varphi)$ that is compact by the compact embedding of $H^1(B_R)$ into $L^2(B_R)$ and compactness of the operator $\mathbb{T}_k - \mathbb{T}_0$.
(iii) The proof for this is a simple application of the Cauchy-Schwarz inequality. □

Notice that Theorem 1.1.1 shows that the variational problem (1.8) has the Fredholm property and therefore is well posed provided we have uniqueness. We now show that (1.8) has at most one solution, and to this end taking the imaginary part (1.8) with $u^i = 0$ gives that

$$\Im\left(\int\limits_{\partial B_R} \overline{u^s}\, \frac{\partial u^s}{\partial \nu}\, ds\right) = \int\limits_{B_R} \Im(A_\delta)\nabla u^s \cdot \nabla \overline{u^s} - k^2 \Im(n_\delta)|u^s|^2 dx \leq 0$$

where we have used that $\mathbb{T}_k u^s = \frac{\partial u^s}{\partial \nu}$ on ∂B_R. Therefore Theorem 3.6 of [14] and the unique continuation principal give that $u^s = 0$ proving uniqueness. The Fredholm Alternative now gives that there is a unique solution to (1.8) such that $\|u^s\|_{H^1(B_R)} \leq C\|u^i\|_{H^1(D)}$ where the constant $C > 0$ independent of u^i. The Sommerfeld radiation condition implies that the scattered field u^s has the asymptotic expansion

$$u^s(x, d) = \frac{e^{ik|x|}}{|x|^{(m-1)/2}} \left\{ u^\infty(\hat{x}, d) + \mathcal{O}\left(\frac{1}{|x|}\right)\right\} \quad \text{as} \quad |x| \to \infty, \tag{1.9}$$

where we recall that $u^\infty(\hat{x}, d)$ is called the far-field pattern.

In the proceeding sections we will assume that we know $u^\infty(\hat{x}, d)$ for all $\hat{x}$ and d in the unit disk/sphere $\mathbb{S}$, incident direction d and observation direction $\hat{x}$. Using Green's Representation theorem one can show that the far field pattern is given by

$$u^\infty(\hat{x}, d) = \gamma_m \int\limits_{\partial\Omega} u^s(y, d) \frac{\partial}{\partial \nu_y} e^{-ik\hat{x}\cdot y} - \frac{\partial u^s(y, d)}{\partial \nu_y} e^{-ik\hat{x}\cdot y} \, ds_y \tag{1.10}$$

where the constant γ_m is given by $\gamma_2 = \frac{e^{i\pi/4}}{\sqrt{8\pi k}}$ and $\gamma_3 = \frac{1}{4\pi}$, and the region Ω is any subset of $\mathbb{R}^m$ such that $D \subseteq \Omega$. Rellich's Lemma (see [14] and [36]) implies that if $u^\infty = 0$ then $u^s = 0$.

The following result is reciprocity relationship for the far-field pattern.

Theorem 1.1.2 (Theorem 4.2 in [14])**.** *Let $u^\infty(\hat{x}, d)$ be the far-field pattern defined by* (1.10) *then $u^\infty(\hat{x}, d) = u^\infty(-d, -\hat{x})$.*

We can define the far-field operator that plays an important role in our analysis of the aforementioned inverse problem and in reconstruction techniques (see [14], [37] for the connection between the far-field operator and the scattering operator) as $F : L^2(\mathbb{S}) \longmapsto L^2(\mathbb{S})$

$$(Fg)(\hat{x}) := \int_{\mathbb{S}} u^\infty(\hat{x}, d) g(d) \, ds(d). \tag{1.11}$$

We now introduce the Herglotz wave functions that are defined as

$$v_g(x) := \int_{\mathbb{S}} e^{ikx\cdot d} g(d) \, ds(d). \tag{1.12}$$

Note that by superposition it can be shown that Fg is the far-field pattern to the scattered field corresponding to v_g replacing $e^{ikx\cdot d}$ as the incident field.

Theorem 1.1.3. *Let $g, h \in L^2(\mathbb{S})$ and let v_g and v_h be the Herglotz wave functions with kernels g and h respectively. Then if u_g^s and u_h^s are the solutions of the scattering problem corresponding to the incident field $u^i := v_g$ and $u^i := v_h$ respectively, we have that*

$$-\int_D \Im(A_\delta)\nabla u_g \cdot \nabla \overline{u_h}\, dx - k^2 \Im(n_\delta) u_g \overline{u_h}\, dx = 2\pi(Fg, h) - 2\pi(g, Fh) - ik(Fg, Fh).$$

Proof. We give the proof in $\mathbb{R}^3$(similar arguments hold in $\mathbb{R}^2$). Let $u_g = u_g^s + v_g$ and $u_h = u_h^s + v_h$ be the total fields in $\mathbb{R}^3 \setminus \overline{D}$. Then using transmission conditions, the divergence theorem along with the symmetry of A_δ and the equations in D we have

$$\begin{aligned}
& \int_{|x|=a} \left(u_g \frac{\partial \overline{u_h}}{\partial \nu} - \overline{u_h} \frac{\partial u_g}{\partial \nu} \right) ds = \int_{\partial D} \left(u_g \frac{\partial \overline{u_h}}{\partial \nu} - \overline{u_h} \frac{\partial u_g}{\partial \nu} \right) ds \\
= & \int_{\partial D} \left(u_g \overline{A_\delta \nabla u_h} \cdot \nu - \overline{u_h} A_\delta \nabla u_g \cdot \nu \right) ds = \int_D \left(\nabla \cdot (u_g \overline{A_\delta \nabla u_h}) - \nabla \cdot (\overline{u_h} A_\delta \nabla u_g) \right) dx \\
= & \int_D \left(\nabla u_g \cdot \overline{A_\delta \nabla u_h} - \nabla \overline{u_h} \cdot A_\delta \nabla u_g \right) dx + \int_D \left(u_g \nabla \cdot \overline{A_\delta \nabla u_h} - \overline{u_h} \cdot \nabla A_\delta \nabla u_g \right) dx \\
= & \int_D \left(\nabla u_g \cdot \overline{A_\delta \nabla u_h} - \nabla \overline{u_h} \cdot A_\delta \nabla u_g \right) dx + k^2 \int_D \left(\overline{u_h} n_\delta u_g - u_g \overline{n_\delta u_h} \right) dx
\end{aligned}$$

Hence we have that

$$\begin{aligned}
& \int_{|x|=a} \left(u_g \frac{\partial \overline{u_h}}{\partial \nu} - \overline{u_h} \frac{\partial u_g}{\partial \nu} \right) ds \\
& \qquad = -2i \int_D \Im(A_\delta) \nabla u_g \cdot \nabla \overline{u_h}\, dx + 2ik^2 \int_D \Im(n_\delta) u_g \overline{u_h}\, dx.
\end{aligned} \tag{1.13}$$

Using the Sommerfeld radiation condition along with (1.10) we obtain that

$$\begin{aligned}
& \int_{|x|=a} \left(u_g \frac{\partial \overline{u_h}}{\partial \nu} - \overline{u_h} \frac{\partial u_g}{\partial \nu} \right) ds \\
& \qquad = 4\pi(Fg, h) - 4\pi(g, Fh) - 2ik(Fg, Fh).
\end{aligned} \tag{1.14}$$

Combining (1.13) and (1.14) yield the result $\square$

Theorem 1.1.4. *Assume that $\Im(A_\delta) = 0$ and $\Im(n_\delta) = 0$. Then far-field operator is normal, i.e. $F^*F = FF^*$, and the scattering operator $\mathcal{S} = I + 2ik\gamma_m F$ is unitary, i.e. $\mathcal{S}\mathcal{S}^* = \mathcal{S}^*\mathcal{S} = I$.*

Proof. We give the proof in $\mathbb{R}^3$(similar arguments hold in $\mathbb{R}^2$), since $\Im(A_\delta) = 0$ and $\Im(n_\delta) = 0$ we have that

$$ik(Fg, Fh) = 2\pi\left[(Fg, h) - (g, Fh)\right] \tag{1.15}$$

for $g, h \in L^2(\mathbb{S})$. By reciprocity we have that

$$\begin{aligned}
(F^*g)(\hat{x}) &= \int_{\mathbb{S}} \overline{u_\infty(d, \hat{x})} g(d)\, ds(d) \\
&= \int_{\mathbb{S}} \overline{u_\infty(-\hat{x}, -d)} g(d)\, ds(d) \\
&= \overline{\int_{\mathbb{S}} u_\infty(-\hat{x}, d) \overline{g(-d)}\, ds(d)},
\end{aligned}$$

i.e. $F^*g = \overline{RFR\overline{g}}$ where $(Rh)(\hat{x}) := h(-\hat{x})$. Since $(Rg, Rh) = (g, h) = (\overline{g}, \overline{h})$, we have from (1.15) that

$$\begin{aligned}
ik(F^*h, F^*g) &= ik(RFR\overline{g}, RFR\overline{h}) \\
&= ik(FR\overline{g}, FR\overline{h}) \\
&= 2\pi(FR\overline{g}, R\overline{h}) - 2\pi(R\overline{g}, FR\overline{h}) \\
&= 2\pi(RFR\overline{g}, \overline{h}) - 2\pi(\overline{g}, RFR\overline{h}) \\
&= 2\pi(h, F^*g) - 2\pi(F^*h, g) \\
&= 2\pi(Fh, g) - 2\pi(h, Fg) \\
&= ik(Fh, Fg)
\end{aligned}$$

and hence $F^*F = FF^*$. Finally, (1.15) implies that

$$-(g, ikF^*Fh) = 2\pi\big(g, (F^* - F)h\big),$$

i.e. $ikF^*F = 2\pi(F - F^*)$. This, together with $F^*F = FF^*$, implies that $\mathcal{S}^*\mathcal{S} = \mathcal{S}\mathcal{S}^* = I$ by direct substitution. □

Properties such as injectivity and the characterization of the range of F are closely related to the study of the transmission eigenvalue problem. We want to investigate if there exists values of k for which the incident wave does not scatter. This corresponds to the injectivity of the far field operator. The following result gives a characterization of the injectivity of the far field operator.

Theorem 1.1.5. *The far field operator corresponding to the scattering problem* (1.3)-(1.4) *is injective with dense range if and only if* $\nexists$ *nontrivial* (u, v_g) *solving:*

$$\nabla \cdot A_\delta \nabla u + k^2 n_\delta u = 0 \quad \textit{and} \quad \Delta v_g + k^2 v_g = 0 \qquad \textit{in} \quad D \tag{1.16}$$

$$u = v_g \quad \textit{and} \quad \frac{\partial u}{\partial \nu_A} = \frac{\partial v_g}{\partial \nu} \qquad \textit{on} \quad \partial D \tag{1.17}$$

Proof. We first note that it can be proven that u^∞ is analytic therefore we have that F and F^* are continuous operators. So to prove that the range is dense we only need to show the adjoint operator F^* is also injective, since $\mathcal{N}(F^*)^\perp = \overline{\mathcal{R}(F)}$, and $L^2(\mathbb{S}) = \mathcal{N}(F^*) \oplus \mathcal{N}(F^*)^\perp$. We have that $(F^* g)(\hat{x}) = \overline{(Fh)(-\hat{x})}$ where $h(d) = \overline{g(-d)}$, therefore F^* is injective if and only if F is injective. We now investigate the injectivity of the far field operator. So assume that $\mathcal{N}(F) \neq \{0\}$ then $\exists\, g \neq 0$ such that $Fg = 0$ where $\exists\, v_g \neq 0$ for which the far field pattern is zero. Since the far field pattern is zero this says $u^s = 0$ for the scattering problem with v_g replacing $e^{ikx \cdot d}$ in the boundary data, and v_g solves Helmholtz equation, which proves the result. $\square$

Chapter 2

TRANSMISSION EIGENVALUES FOR ANISOTROPIC MEDIA WITH VOIDS

The non-destructive testing of composite materials using electromagnetic waves is an important problem in engineering. A number of such problems involve complicated materials, in particular anisotropic, hence many methods of reconstructing the matrix refractive index are either unfeasible or computationally expensive. On the other hand for practical purposes it suffices to obtain some partial information on the refractive index in order to evaluate the integrity of the material. As mentioned in the introduction the so-called qualitative methods in inverse scattering do just this (see e.g. [14]). In this chapter we consider the problem of detecting voids in a known anisotropic dielectric material from electromagnetic measurements in the frequency domain for a range of frequencies. Our inversion method is based on quantifying the effect that the presence of voids have on the so-called *transmission eigenvalues*, which are detectable from the scattering data which we will prove in this chapter. Transmission eigenvalues have been used to determine material properties of the scattering media starting with [24] for isotropic inhomogeneities. The work in this Chapter has been publish as the article I. Harris, F. Cakoni and J. Sun "Transmission eigenvalues and non-destructive testing of anisotropic magnetic materials with voids" *Inverse Problems*, **30** 035016 (2014).

2.1 Existence of Transmission Eigenvalues for Domains with Voids

In this section we prove that real transmission eigenvalues exists for anisotropic magnetic dielectric media with voids. The existing results on this question [18] and [42] include only the case of non-magnetic material, i.e. when the magnetic permeability of the media is the same as of the background and the approach used in these chapter rely heavily on the fact that the contrast is only on one constitutive parameters of the medium. Our approach to proving the existence of transmission eigenvalues follows the formulation introduced in [29] with appropriate modifications to allow for the presence of voids. Furthermore, we show that the first transmission eigenvalue can be used to determining material properties and provide qualitative information about the size of the void(s). In addition, we show that the real transmission eigenvalues can be determined from the scattering data. Some numerical examples are given to demonstrate the feasibility of our theoretical results.

To this end we recall transmission eigenvalues for an anisotropic with a void(s) as the values of $k \in \mathbb{C}$ such that there exists nontrivial $(w, v) \in H^1(D) \times H^1(D)$ such that

$$\Delta w + k^2 w = 0 \quad \text{in} \quad D_0 \tag{2.1}$$

$$\nabla \cdot A \nabla w + k^2 n w = 0 \quad \text{in} \quad D \setminus \overline{D}_0 \tag{2.2}$$

$$\Delta v + k^2 v = 0 \quad \text{in} \quad D \tag{2.3}$$

$$w^- = w^+ \text{ and } \frac{\partial w^-}{\partial \nu} = \frac{\partial w^+}{\partial \nu_A} \quad \text{on} \quad \partial D_0 \tag{2.4}$$

$$w = v \text{ and } \frac{\partial w}{\partial \nu_A} = \frac{\partial v}{\partial \nu} \quad \text{on} \quad \partial D. \tag{2.5}$$

We start this chapter by investigating the transmission eigenvalue problem (2.1)-(2.1). We apply similar analysis as in [29] to prove the existence of real transmission eigenvalues, but the techniques in there must be modified to account for the void D_0 where $A_0 = I$ and $n_0 = 1$. Let us introduce the following notations

$$\inf_{x \in D \setminus \overline{D}_0} \inf_{|\xi|=1} \overline{\xi} \cdot A(x)\xi = A_{min} \quad \text{and} \quad \inf_{x \in D \setminus \overline{D}_0} n(x) = n_{min}$$
$$\sup_{x \in D \setminus \overline{D}_0} \sup_{|\xi|=1} \overline{\xi} \cdot A(x)\xi = A_{max} \quad \text{and} \quad \sup_{x \in D \setminus \overline{D}_0} n(x) = n_{max},$$

and at this point, we consider the case $A_{min} > 1$ and $n_{max} < 1$, or $A_{max} < 1$ and $n_{min} > 1$(i.e. when the contrast has different sign). Note that in the context of Maxwell's equations this is the practical case since A is the inverse of the electric permeability. Our goal is to prove the existence of real transmission eigenvalues, hence we assume that $k^2 \geq 0$. To this end, we formulate the transmission eigenvalue problem (2.1)-(2.5) as a problem for the difference $u := v - w \in H_0^1(D)$. By subtracting the partial differential equations and boundary conditions for v and w we have that the boundary value problem for v and u is given by

$$\nabla \cdot A\nabla u + k^2 nu = \nabla \cdot (A - I)\nabla v + k^2(n - 1)v \qquad \text{in} \quad D \setminus \overline{D}_0 \tag{2.6}$$
$$\frac{\partial u}{\partial \nu_A} = \frac{\partial v}{\partial \nu_A} - \frac{\partial v}{\partial \nu} \qquad \text{on} \quad \partial D \tag{2.7}$$

Notice that from $\partial w^+/\partial \nu_A = \partial w^-/\partial \nu$ and the continuity of $\partial v^+/\partial \nu = \partial v^-/\partial \nu$ across ∂D_0 we have that

$$\frac{\partial u^+}{\partial \nu_A} - \frac{\partial u^-}{\partial \nu} = \frac{\partial v^+}{\partial \nu_A} - \frac{\partial v^+}{\partial \nu} \quad \text{on} \quad \partial D_0 \tag{2.8}$$

where the superscripts + and − indicate approaching the boundary from outside and inside D_0 respectively. Following [29], we will consider (2.6)-(2.8) as a Neuman boundary value problem for v which is defined in $D \setminus \overline{D}_0$ where we must incorporate the fact that u is a solution to the Helmholtz equation in D_0. To this end, we need to assume that k^2 is not a Dirichlet eigenvalue for $-\Delta$ in D_0 and define the interior Dirichlet to Neumann mapping $T_k : H^{1/2}(\partial D_0) \to H^{-1/2}(\partial D_0)$ by

$$T_k : u\big|_{\partial D_0} \longmapsto \frac{\partial u}{\partial \nu}\Big|_{\partial D_0} \qquad \text{where} \quad \Delta u + k^2 u = 0, \qquad \text{in } \; D_0. \tag{2.9}$$

With the help of T_k we are able to go from boundary terms on ∂D_0 to terms defined in D_0. In particular integration by parts gives that

$$\int_{\partial D_0} \overline{\varphi}\, T_k u \, ds = \int_{D_0} \nabla u \cdot \nabla \overline{\varphi} - k^2 u \overline{\varphi}\, dx \qquad \forall \varphi \in H^1(D_0). \tag{2.10}$$

(If D_0 has multiple simply connected components then we define the Dirichlet to Neumann operator component wise). Then for a given $u \in H_0^1(D)$ satisfying the Helmholtz equation inside D_0, we see (2.6)-(2.8) as a Neumann boundary value problem for v which can be written in an equivalent variational form as follows

$$\begin{aligned} \int_{D \setminus \overline{D}_0} (A - I)\nabla v \cdot \nabla \overline{\varphi} - k^2(n-1) v \overline{\varphi}\, dx &= \int_{D \setminus \overline{D}_0} A \nabla u \cdot \nabla \overline{\varphi} - k^2 n u \overline{\varphi}\, dx \\ &\quad + \int_{\partial D_0} \overline{\varphi}\, T_k u \, ds \qquad \forall \varphi \in H^1\left(D \setminus \overline{D}_0\right). \end{aligned} \tag{2.11}$$

We use the variational formulation to define a bounded linear operator that maps $u \in H_0^1(D) \mapsto v_u \in H^1\left(D \setminus \overline{D}_0\right)$. To this end let us define the bounded sesqulinear form and the bounded conjugate linear functional from the variational formulation as

$$\mathcal{B}_k(v, \varphi) := \int_{D \setminus \overline{D}_0} (A - I)\nabla v \cdot \nabla \overline{\varphi} - k^2(n-1) v \overline{\varphi}\, dx,$$

$$f_u(\varphi) := \int\limits_{D\setminus\overline{D}_0} A\nabla u\cdot\nabla\overline{\varphi} - k^2 n u\overline{\varphi}\, dx + \int\limits_{\partial D_0} \overline{\varphi}\, T_k u\, ds.$$

and consider the variational problem of finding $v \in H^1\left(D \setminus \overline{D}_0\right)$ such that

$$\mathcal{B}_k(v,\varphi) = f_u(\varphi) \quad \text{ for all } \quad \varphi \in H^1\left(D \setminus \overline{D}_0\right). \tag{2.12}$$

We split the solution $v = \widehat{v} + c$ where c is a constant and

$$\widehat{v} \in \widehat{H}^1\left(D \setminus \overline{D}_0\right) := \left\{ \widehat{v} \in H^1\left(D \setminus \overline{D}_0\right) \;\Big|\; \int_{D\setminus\overline{D}_0} (n-1)\widehat{v} dx = 0 \right\}$$

equipped with the $H^1\left(D \setminus \overline{D}_0\right)$ inner-product. It can be shown that functions in $\widehat{H}^1\left(D \setminus \overline{D}_0\right)$ satisfy the Poincaré inequality, that is $||\widehat{v}||^2_{L^2(D\setminus\overline{D}_0)} \leq C_p||\nabla\widehat{v}||^2_{L^2(D\setminus\overline{D}_0)}$ for all $\widehat{v} \in \widehat{H}^1(D \setminus \overline{D}_0)$ (e.g see [45]). Now letting $\varphi = 1$ for $k^2 \neq 0$ we have that

$$k^2 \int\limits_{D\setminus\overline{D}_0} (n-1)v\, dx = k^2 \int\limits_{D\setminus\overline{D}_0} nu\, dx + \int\limits_{\partial D_0} T_k u\, ds = k^2 \int\limits_{D\setminus\overline{D}_0} nu\, dx - k^2 \int\limits_{D_0} u\, dx$$

where the latter equality holds due to the fact that u solves the Helmholtz equation in D_0. Using this along with $v = \widehat{v} + c$ we have that

$$c = \frac{1}{\int_{D\setminus\overline{D}_0}(n-1)\, dx} \left(\int\limits_{D\setminus\overline{D}_0} nu\, dx - \int\limits_{D_0} u\, dx \right).$$

If $k^2 = 0$ we require c to still be defined as in the non-zero case. Now we show the variational problem is well posed in the space $\widehat{H}^1(D \setminus \overline{D}_0)$ by proving that $\pm\mathcal{B}_k(\widehat{v},\widehat{\varphi})$ is $\widehat{H}^1(D \setminus \overline{D}_0)$-coercive, when $A_{min} - 1 > 0$ and $n_{max} - 1 < 0$, or $A_{max} - 1 < 0$ and

$n_{min} - 1 > 0$ respectively. If $A_{min} - 1 > 0$ and $n_{max} - 1 < 0$

$$\begin{aligned}
\mathcal{B}_k(\widehat{v}, \widehat{v}) &= \int\limits_{D \setminus \overline{D}_0} (A - I)\nabla \widehat{v} \cdot \nabla \overline{\widehat{v}} - k^2(n-1)|\widehat{v}|^2 \, dx \\
&\geq \int\limits_{D \setminus \overline{D}_0} (A_{min} - 1)\nabla \widehat{v} \cdot \nabla \overline{\widehat{v}} + k^2(1 - n_{max})|\widehat{v}|^2 \, dx \\
&\geq (A_{min} - 1)||\nabla \widehat{v}||^2_{L^2\left(D \setminus \overline{D}_0\right)} \geq C||\widehat{v}||^2_{H^1\left(D \setminus \overline{D}_0\right)}
\end{aligned}$$

where we have used the Poincaré inequality for $\widehat{H}^1\left(D \setminus \overline{D}_0\right)$. Similarly we can show that if $A_{max} - 1 < 0$ and $n_{min} - 1 > 0$, then $-\mathcal{B}_k(\cdot, \cdot)$ is coercive. Having $v_u \in H^1\left(D \setminus \overline{D}_0\right)$ defined in the annulus $D \setminus \overline{D}_0$ for any $u \in H^1_0(D)$, since the transmission eigenfunction v solves the Helmholtz equation in the domain D, we insure that v_u can be extended to a solution of the Helmholtz equation in D. Using the Riesz representation theorem we can now define $\mathbb{L}_k u$ by

$$(\mathbb{L}_k u, \varphi)_{H^1\left(D \setminus \overline{D}_0\right)} = \int\limits_{D \setminus \overline{D}_0} \nabla v_u \cdot \nabla \overline{\varphi} - k^2 v_u \overline{\varphi} \, dx + \int\limits_{\partial D_0} \overline{\varphi} \, T_k v_u \, ds \quad \forall \varphi \in H^1_0\left(D \setminus \overline{D}_0, \partial D\right), \tag{2.13}$$

where

$$H^1_0\left(D \setminus \overline{D}_0, \partial D\right) = \left\{u \in H^1\left(D \setminus \overline{D}_0\right) : \; u = 0 \text{ on } \partial D\right\}.$$

Notice that the mapping $k \mapsto \mathbb{L}_k$ is continuous for $k \in \mathbb{R}$ and k^2 not a Dirichlet eigenvalue. We can now connect the kernel of the operator $\mathbb{L}_k : H^1_0\left(D \setminus \overline{D}_0, \partial D\right) \to H^1_0\left(D \setminus \overline{D}_0, \partial D\right)$ to the set of transmission eigenfunctions.

Theorem 2.1.1. *Assume that k^2 is not a Dirichlet eigenvalue for $-\Delta$ in D_0 and assume that either $A_{min} > 1$ and $n_{max} < 1$, or $A_{max} < 1$ and $n_{min} > 1$. If $w, v \in H^1(D)$ solves (2.1)-(2.5) then $u|_{D \setminus \overline{D}_0} = w - v \in H^1_0\left(D \setminus \overline{D}_0, \partial D\right)$ is such that $\mathbb{L}_k u =$*

0. *Conversely, if* $\mathbb{L}_k u = 0$ *for* $u \in H_0^1(D \setminus \overline{D}_0, \partial D)$ *then* v_u *and* u *can be extended to solution* $v, u \in H^1(D)$ *of Helmholtz equation in* D_0 *and the pair* $(u+v, v)$ *solves* *(2.1)-(2.5).*

Proof. The first part of the theorem is by construction. Obviously $\mathbb{L}_k u = 0$ since v_u satisfies the Helmholtz equation in D. Conversely, let $\mathbb{L}_k u = 0$ and define $v := v_u \in H^1\left(D \setminus \overline{D}_0\right)$ as above and in D_0 by

$$\Delta v + k^2 v = 0 \qquad \text{in } D_0, \qquad v = v_u^+ \qquad \text{on } \partial D_0. \tag{2.14}$$

Since $\mathbb{L}_k u = 0$, (2.13) implies that $v \in H^1(D)$ and satisfies the Helmholtz equation in D. Furthermore, extending u in D_0 by

$$\Delta u + k^2 u = 0 \qquad \text{in } D_0, \qquad u = u^+ \qquad \text{on } \partial D_0,$$

then (2.11) implies that $(u+v, v)$ solves (2.1)-(2.5). $\square$

The following lemma states some properties of the operator $\mathbb{L}_k$.

Lemma 2.1.1. *Assume that* k^2 *is not a Dirichlet eigenvalue of* $-\Delta$ *in* D_0. *For the operator* $\mathbb{L}_k$ *we have the following:*

1. $\mathbb{L}_k : H_0^1\left(D \setminus \overline{D}_0, \partial D\right) \longmapsto H_0^1\left(D \setminus \overline{D}_0, \partial D\right)$ *is a self-adjoint operator for all* $k \in \mathbb{R}_{\geq 0}$
2. $\pm\mathbb{L}_0$ *is a coercive operator when* $(A_{min} - 1) > 0$, *or* $(A_{max} - 1) < 0$ *respectively*
3. $\mathbb{L}_k - \mathbb{L}_0$ *is a compact operator in* $H_0^1\left(D \setminus \overline{D}_0, \partial D\right)$

Proof. (i) Let u_1 and u_2 be given in $H_0^1\left(D \setminus \overline{D}_0, \partial D\right)$ and consider v_1 and v_2 in $H^1\left(D \setminus \overline{D}_0\right)$ satisfying (2.11) extended inside D as solutions to the Helmholtz equation by (2.14). Thus, for these functions we have

$$\int\limits_{D\setminus\overline{D}_0} (A-I)\nabla v_i \cdot \nabla\overline{\varphi} - k^2(n-1)v_i\overline{\varphi}\,dx = \int\limits_{D\setminus\overline{D}_0} A\nabla u_i \cdot \nabla\overline{\varphi} - k^2 n u_i\overline{\varphi}\,dx + \int\limits_{\partial D_0} \overline{\varphi}\,T_k u_i\,ds \qquad \forall\varphi \in H^1(D) \tag{2.15}$$

By the definition of $\mathbb{L}_k$ we have that

$$\begin{aligned}(\mathbb{L}_k u_1, u_2)_{H^1\left(D\setminus\overline{D}_0\right)} &= \int\limits_{D\setminus\overline{D}_0} \nabla v_1 \cdot \nabla\overline{u_2} - k^2 v_1\overline{u_2}\,dx + \int\limits_{\partial D_0} \overline{u_2}\,T_k v_1\,ds \\ &= -\int\limits_{D\setminus\overline{D}_0} (A-I)\nabla v_1 \cdot \nabla\overline{u_2} - k^2(n-1)v_1\overline{u_2}\,dx \\ &+ \int\limits_{D\setminus\overline{D}_0} A\nabla v_1 \cdot \nabla\overline{u_2} - k^2 n v_1\overline{u_2}\,dx + \int\limits_{\partial D_0} \overline{u_2}\,T_k v_1\,ds \end{aligned} \tag{2.16}$$

Taking $i = 2$ and $\varphi = v_1$ and then $i = 1$ and $\varphi = u_2$ in (2.15) we obtain

$$(\mathbb{L}_k u_1, u_2)_{H^1\left(D\setminus\overline{D}_0\right)} = -\int\limits_{D\setminus\overline{D}_0} A\nabla u_1 \cdot \nabla\overline{u_2} - k^2 n u_1\overline{u_2}\,dx - \int\limits_{\partial D_0} \overline{u_2}\,T_k u_1\,ds \tag{2.17}$$

$$\begin{aligned} &+ \int\limits_{D\setminus\overline{D}_0} (A-I)\nabla v_1 \cdot \nabla\overline{v_2} - k^2(n-1)v_1\overline{v_2}\,dx - \int\limits_{\partial D_0} v_1 T_k\overline{u_2} - \overline{u_2}\,T_k v_1\,ds \\ &= -\int\limits_{D\setminus\overline{D}_0} A\nabla u_1 \cdot \nabla\overline{u_2} - k^2 n u_1\overline{u_2}\,dx - \int\limits_{D_0} \nabla u_1\,\nabla\overline{u_2} - k^2 u_1\overline{u_2}\,dx \\ &+ \int\limits_{D\setminus\overline{D}_0} (A-I)\nabla v_1 \cdot \nabla\overline{v_2} - k^2(n-1)v_1\overline{v_2}\,dx = (u_1, \mathbb{L}_k u_2)_{H^1\left(D\setminus\overline{D}_0\right)}. \end{aligned}$$

Obviously, the right hand side is a selfadjoint expression of u_1 and u_2 giving that $\mathbb{L}_k$ is a selfadjoint operator and the boundary terms cancel thanks to (2.10).

(ii) To prove that $\pm\mathbb{L}_0$ is coercive we first assume that $A_{min} - 1 > 0$ and therefore we consider the operator $\mathbb{L}_0$. Letting $v - u = w$ we have that

$$\begin{aligned}(\mathbb{L}_0 u, u)_{H^1(D\setminus\overline{D}_0)} &= \int_{D\setminus\overline{D}_0} \nabla v \cdot \nabla \overline{u}\, dx + \int_{\partial D_0} \overline{u}\, T_0 v\, ds \\ &= \int_{D\setminus\overline{D}_0} |\nabla u|^2\, dx + \int_{D\setminus\overline{D}_0} \nabla w \cdot \nabla \overline{u}\, dx + \int_{\partial D_0} \overline{u}\, T_0 u\, ds + \int_{\partial D_0} \overline{u}\, T_0 w\, ds.\end{aligned}$$

Using (2.11) for $k^2 = 0$ and $\varphi = w$ and taking the conjugate, we see that

$$\int_{D\setminus\overline{D}_0} (A - I)\nabla w \cdot \nabla \overline{v}\, dx = \int_{D\setminus\overline{D}_0} A\nabla w \cdot \nabla \overline{u}\, dx + \int_{\partial D_0} w\, T_0 \overline{u}\, ds$$

Now once again using that $v = u + w$ we see that

$$\int_{D} \nabla w \cdot \nabla \overline{u}\, dx = \int_{D\setminus\overline{D}_0} (A - I)\nabla w \cdot \nabla \overline{w}\, dx - \int_{\partial D_0} w\, T_0 \overline{u}\, ds.$$

Using the latter equation we see that

$$\begin{aligned}(\mathbb{L}_0 u, u)_{H^1(D\setminus\overline{D}_0)} &= \int_{D\setminus\overline{D}_0} |\nabla u|^2\, dx + \int_{D\setminus\overline{D}_0} (A - I)\nabla w \cdot \nabla \overline{w}\, dx - \int_{\partial D_0} w\, T_0 \overline{u}\, ds \\ &+ \int_{\partial D_0} \overline{u}\, T_0 w\, ds + \int_{\partial D_0} \overline{u}\, T_0 u\, ds\end{aligned}$$

where from

$$\int_{\partial D_0} \overline{u}\, T_0 w\, ds = \int_{D_0} \nabla w \cdot \nabla \overline{u}\, dx = \int_{\partial D_0} w\, T_0 \overline{u}\, ds$$

the boundary terms involving w cancel. Now using that $A_{min} - 1 > 0$ we have that

$$\int_{D \setminus \overline{D}_0} (A - I)\nabla w \cdot \nabla \overline{w}\, dx \geq (A_{min} - 1) \int_{D \setminus \overline{D}_0} |\nabla w|^2\, dx \geq 0.$$

Also notice that integration by parts gives that

$$\int_{\partial D_0} \overline{u}\, T_0 u\, ds = \int_{D_0} |\nabla u|^2\, dx \geq 0.$$

Therefore

$$(\mathbb{L}_0 u, u)_{H^1\left(D \setminus \overline{D}_0\right)} \geq \int_{D \setminus \overline{D}_0} |\nabla u|^2\, dx + \int_{\partial D_0} \overline{u}\, T_0 u\, ds \geq \int_{D \setminus \overline{D}_0} |\nabla u|^2\, dx$$

proving the coercivity due to the zero boundary condition on ∂D.

Next, assume that $A_{max} - 1 < 0$, therefore considering the operator $-\mathbb{L}_0$. From (2.17) we have that

$$(-\mathbb{L}_0 u, u)_{H^1\left(D \setminus \overline{D}_0\right)} = -\int_{D \setminus \overline{D}_0} (A - I)\nabla v \cdot \nabla \overline{v}\, dx + \int_{D_0} |\nabla u|^2\, dx + \int_{D \setminus \overline{D}_0} A \nabla u \cdot \nabla \overline{u}\, dx$$

Now since $A_{max} - 1 < 0$ we have that

$$-\int_{D \setminus \overline{D}_0} (A - I)\nabla v \cdot \nabla \overline{v}\, dx \geq (1 - A_{max}) \int_{D \setminus \overline{D}_0} |\nabla v|^2\, dx \geq 0.$$

Therefore $(-\mathbb{L}_0 u, u)_{H^1(D\setminus\overline{D}_0)} \geq A_{min} \int\limits_{D\setminus\overline{D}_0} |\nabla u|^2 \, dx$ proving coercivity in this case.

(iii) We now show the compactness of $\mathbb{L}_k - \mathbb{L}_0$. Assume that the sequence $u^j \rightharpoonup 0$ in $H_0^1\left(D \setminus \overline{D}_0, \partial D\right)$ and therefore we have the existence of $v_k^j \rightharpoonup 0$ and $v_0^j \rightharpoonup 0$ in $H^1\left(D \setminus \overline{D}_0\right)$, corresponding to solutions of (2.15). Recall that (2.13) defines $(\mathbb{L}_k - \mathbb{L}_0)u^j$ in terms of v_k^j and v_0^j. Since zero and k^2 are not Dirichlet eigenvalues, we have that their extension as solution to the Helmholtz equation inside D_0 converge weakly to 0 in D. From the Rellich's embedding theorem, a subsequence of the aforementioned sequences, still denoted by v_k^j and v_0^j converge strongly to zero in $L^2(D)$. We see that the sequences v_k^j and v_0^j satisfy

$$\int\limits_{D\setminus\overline{D}_0} (A-I)\nabla v_k^j \cdot \nabla\overline{\varphi} - k^2(n-1)v_k^j\overline{\varphi}\, dx = \int\limits_{D\setminus\overline{D}_0} A\nabla u^j \cdot \nabla\overline{\varphi} - k^2 n u^j \overline{\varphi}\, dx + \int\limits_{\partial D_0} \overline{\varphi}\, T_k u^j \, ds$$

and

$$\int\limits_{D\setminus\overline{D}_0} (A-I)\nabla v_0^j \cdot \nabla\overline{\varphi}\, dx = \int\limits_{D\setminus\overline{D}_0} A\nabla u^j \cdot \nabla\overline{\varphi}\, dx + \int\limits_{\partial D_0} \overline{\varphi}\, T_0 u^j \, ds$$

for all $\varphi \in H^1(D)$. Now using that

$$\int\limits_{\partial D_0} \overline{\varphi}\, T_k u^j \, ds = \int\limits_{D_0} \nabla u^j \cdot \nabla\overline{\varphi} - k^2 u^j \overline{\varphi}\, dx$$

and letting $\tilde{v}^j := v_k^j - v_0^j$ we have that

$$\int\limits_{D\setminus\overline{D}_0} (A-I)\nabla \tilde{v}^j \cdot \nabla\overline{\varphi}\, dx = k^2 \int\limits_{D\setminus\overline{D}_0} (n-1)v_k^j\overline{\varphi} - n u^j\overline{\varphi}\, dx + k^2 \int\limits_{D_0} u^j\overline{\varphi}\, dx \quad \forall\, \varphi \in H^1(D).$$

Letting $\varphi = \tilde{v}^j$ and for either $A - I$ positive or negative definite we obtain that $\tilde{v}^j \to 0$ in $H^1\left(D \setminus \overline{D}_0\right)$. Now we have that

$$\Delta \tilde{v}^j = -k^2 v_k^j \quad \text{in } D_0 \quad \text{and} \quad \tilde{v}^j = v_k^j - v_0^j \quad \text{on} \quad \partial D_0.$$

Therefore

$$||\tilde{v}^j||_{H^1(D_0)} \leq C\left(||v_k^j - v_0^j||_{H^1\left(D\setminus\overline{D}_0\right)} + ||v_k^j||_{L^2(D)}\right) \longrightarrow 0$$

where we have used the trace theorem on ∂D_0. Now

$$\begin{aligned}\left((\mathbb{L}_k - \mathbb{L}_0)u^j, \varphi\right)_{H^1\left(D\setminus\overline{D}_0\right)} &= \int\limits_{D\setminus\overline{D}_0} \nabla\tilde{v}^j \cdot \nabla\overline{\varphi} - k^2 v_k^j \overline{\varphi}\, dx + \int\limits_{\partial D_0} \overline{\varphi}\left(T_k v_k^j - T_0 v_0^j\right)\, ds \\ &= \int\limits_{D} \nabla\tilde{v}^j \cdot \nabla\overline{\varphi} - k^2 v_k^j \overline{\varphi}\, dx\end{aligned}$$

therefore by the using the Cauchy-Schwartz inequality we have that

$$||(\mathbb{L}_k - \mathbb{L}_0)u^j||_{H^1\left(D\setminus\overline{D}_0\right)} \leq C\Big(||\tilde{v}^j||_{H^1(D)} + ||v_k^j||_{L^2(D)}\Big).$$

Which proves the claim since the right hand side tends to zero. □

Notice that the second part of this theorem says that for $k = 0$ the operator $\pm\mathbb{L}_k$ is positive. We now prove that $\pm\mathbb{L}_k$ is positive for a range of values, which gives a lower bound on the transmission eigenvalues.

Theorem 2.1.2. *Let $\lambda_1(D)$ be the first Dirichlet eigenvalue of $-\Delta$ in D and let k^2 be a real transmission eigenvalue:*

1. *If $A_{min} > 1$ and $n_{max} < 1$, then we have that $k^2 \geq \lambda_1(D)$.*
2. *If $A_{max} < 1$ and $n_{min} > 1$, then we have that $k^2 \geq \frac{A_{min}}{n_{max}}\lambda_1(D)$.*

Proof. (i) Assume that $A_{min} - 1 > 0$ and $n_{max} - 1 < 0$, we have that if u is the difference of eigenfunctions then $(\mathbb{L}_k u, u)_{H^1\left(D\setminus\overline{D}_0\right)} = 0$. So by the definition of $\mathbb{L}_k$ and by using that $v = u + w$ we have that:

$$
\begin{aligned}
(\mathbb{L}_k u, u)_{H^1\left(D\setminus\overline{D}_0\right)} &= \int\limits_{D\setminus\overline{D}_0} \nabla v \cdot \nabla \overline{u} - k^2 v\overline{u}\, dx + \int\limits_{\partial D_0} \overline{u}\, T_k v\, ds = \int\limits_{D} \nabla v \cdot \nabla \overline{u} - k^2 v\overline{u}\, dx \\
&= \int\limits_{D} |\nabla u|^2 - k^2 |u|^2\, dx + \int\limits_{D} \nabla w \cdot \nabla \overline{u} - k^2 w\overline{u}\, dx.
\end{aligned}
$$

Now we use the variational form (2.11) for $\varphi = w$ which gives that

$$
\begin{aligned}
\int\limits_{D\setminus\overline{D}_0} (A - I)\nabla v \cdot \nabla \overline{w} - k^2(n-1)v\overline{w}\, dx \quad &- \int\limits_{D\setminus\overline{D}_0} A\nabla u \cdot \nabla \overline{w} - k^2 n u\overline{w}\, dx \\
&= \int\limits_{D_0} \nabla u \cdot \nabla \overline{w} - k^2 u\overline{w}\, dx.
\end{aligned}
$$

On the left hand side we once again use that $v = w + u$ and combine the integrals involving both u and w giving that

$$
\int\limits_{D} \nabla u \cdot \nabla \overline{w} - k^2 u\overline{w}\, dx = \int\limits_{D\setminus\overline{D}_0} (A - I)\nabla w \cdot \nabla \overline{w} - k^2(n-1)|w|^2\, dx
$$

Now we look at $(\mathbb{L}_k u, u)_{H^1(D\setminus\overline{D}_0)}$ and use the fact that under the assumptions on the coefficients that

$$
\int\limits_{D\setminus\overline{D}_0} (A - I)\nabla w \cdot \nabla \overline{w} - k^2(n-1)|w|^2\, dx \geq 0.
$$

Therefore we have that

$$\begin{aligned}
(\mathbb{L}_k u, u)_{H^1\left(D\setminus\overline{D}_0\right)} &= \int\limits_D |\nabla u|^2 - k^2|u|^2\,dx + \int\limits_{D\setminus\overline{D}_0} (A-I)\nabla w\cdot\nabla\overline{w} - k^2(n-1)|w|^2\,dx \\
&\geq \int\limits_D |\nabla u|^2 - k^2|u|^2\,dx \qquad (2.18) \\
&\geq \left[\lambda_1(D) - k^2\right]\int\limits_D |u|^2\,dx.
\end{aligned}$$

So if $(\lambda_1(D) - k^2) > 0$, we have that $(\mathbb{L}_k u, u)_{H^1(D\setminus\overline{D}_0)} > 0$ which contradicts the fact that $\mathbb{L}_k u = 0$. Which implies all transmission eigenvalues satisfy $k^2 \geq \lambda_1(D)$.

(ii) Assume that $A_{max} - 1 < 0$ and $n_{min} - 1 > 0$, we have that if u is the difference of eigenfunctions then $(-\mathbb{L}_k u, u)_{H^1(D\setminus\overline{D}_0)} = 0$. We know by the previous theorem that:

$$\begin{aligned}
(-\mathbb{L}_k u, u)_{H^1\left(D\setminus\overline{D}_0\right)} &= -\int\limits_{D\setminus\overline{D}_0} (A-I)\nabla v\cdot\nabla\overline{v} - k^2(n-1)|v|^2\,dx + \int\limits_{D_0} \nabla|u|^2 - k^2|u|^2\,dx \\
&+ \int\limits_{D\setminus\overline{D}_0} A\nabla u\cdot\nabla\overline{u} - k^2 n|u|^2\,dx
\end{aligned}$$

Notice that under the assumptions on the coefficients that

$$-\int\limits_{D\setminus\overline{D}_0} (A-I)\nabla v\cdot\nabla\overline{v} - k^2(n-1)|v|^2\,dx \geq 0.$$

Therefore we have that:

$$
\begin{aligned}
(-\mathbb{L}_k u, u)_{H^1(D\setminus\overline{D}_0)} &\geq \int\limits_{D_0} |\nabla u|^2 - k^2|u|^2\, dx + \int\limits_{D\setminus\overline{D}_0} A\nabla u\cdot\nabla\overline{u} - k^2 n|u|^2\, dx \\
&\geq \int\limits_{D_0} |\nabla u|^2 - k^2|u|^2\, dx + A_{min}\int\limits_{D\setminus\overline{D}_0} |\nabla u|\, dx - k^2 n_{max}\int\limits_{D_2} |u|^2\, dx \\
&\geq A_{min}\int\limits_{D} |\nabla u|\, dx - k^2 n_{max}\int\limits_{D} |u|^2\, dx \\
&\geq \Big[A_{min}\lambda_1(D) - k^2 n_{max}\Big]\int\limits_{D} |u|^2\, dx.
\end{aligned}
$$

So if $(A_{min}\lambda_1(D) - k^2 n_{max}) > 0$, we have that $(-\mathbb{L}_k u, u)_{H^1(D_2)} > 0$ which contradicts the fact that $\mathbb{L}_k u = 0$. Which implies all transmission eigenvalues satisfy $k^2 \geq \frac{A_{min}}{n_{max}}\lambda_1(D)$. □

The previous result shows that the operator $\pm\mathbb{L}_k$ is positive for a range of k values. We next show that the operator is non-positive for some k on a subset of $H_0^1\big(D\setminus\overline{D}_0, \partial D\big)$.

Theorem 2.1.3. *Provided that the measure of each component of the void D_0 is sufficiently small, there exists a $k > 0$ such that $\mathbb{L}_k$, or $-\mathbb{L}_k$ for $(A_{min} - 1) > 0$ and $(n_{max} - 1) < 0$, or $(A_{max} - 1) < 0$ and $(n_{min} - 1) > 0$ respectively, is non-positive on a subspace of $H_0^1\big(D\setminus\overline{D}_0, \partial D\big)$.*

Proof. Assume that $(A_{min} - 1) > 0$ and $(n_{max} - 1) < 0$, and look at the operator $\mathbb{L}_k$. We denote by B_r the ball of radius r. Let R and ϵ be positive numbers such that $\overline{B_R} \subset D$, $\overline{D}_0 \subset B_\epsilon$ and $R > \epsilon$. By using separation of variables one can see that there exists transmission eigenvalues for the system.

$$\begin{aligned} \Delta \hat{w} + \tau^2 \hat{w} = 0 \quad &\text{in} \quad B_\epsilon \\ \nabla \cdot A_{min} \nabla \hat{w} + \tau^2 n_{max} \hat{w} = 0 \quad &\text{in} \quad B_R \setminus \overline{B_\epsilon} \\ \Delta \hat{v} + \tau^2 \hat{v} = 0 \quad &\text{in} \quad B_R \\ \hat{w}^- = \hat{w}^+ \text{ and } \frac{\partial \hat{w}^-}{\partial \nu} = \frac{\partial \hat{w}^+}{\partial \nu_{A_{min}}} \quad &\text{on} \quad \partial B_\epsilon \\ \hat{w} = \hat{v} \text{ and } \frac{\partial \hat{w}}{\partial \nu_{A_{min}}} = \frac{\partial \hat{v}}{\partial \nu} \quad &\text{on} \quad \partial B_R \end{aligned} \tag{2.19}$$

Now recall we can only define $\mathbb{L}_k$ when k^2 is not a Dirichlet eigenvalue of $-\Delta$ in D_0, so we denote the first eigenvalue as $\lambda_1(D_0)$. Since $\lambda_1(D_0) \to \infty$ as $|D_0| \to 0^+$ we can insure that there is at least one transmission eigenvalue of the form $\tau^2 = k^2(B_R, B_\epsilon, A_{min}, n_{max}) < \lambda_1(D_0)$ provided that the measure of each components of D_0 is sufficiently small. So let $\hat{u}$ be the difference of these eigenfunctions with eigenvalue τ^2 giving that from (2.18)

$$\int_{B_R} |\nabla \hat{u}|^2 - \tau^2 |\hat{u}|^2 \, dx + \int_{B_R \setminus \overline{B_\epsilon}} (A_{min} - 1) \nabla \hat{w} \cdot \nabla \overline{\hat{w}} - \tau^2 (n_{max} - 1) |\hat{w}|^2 \, dx = 0$$

Therefore $\hat{u} \in H_0^1(B_R)$, so let the extension by zero of $\hat{u}$ to the whole domain D be denoted $\widetilde{u}$. Now since $\overline{D}_0 \subset B_\epsilon$ we have that $\Delta \widetilde{u} + k^2 \widetilde{u} = 0$ in D_0. Since $A_{min} - 1 > 0$ and $n_{max} - 1 < 0$, we can construct nontrivial $\widetilde{v} \in H^1 \left(D \setminus \overline{D}_0 \right)$ that solve (2.11) with coefficients A, n in the domain D with void D_0 and let $\widetilde{w} = \widetilde{v} - \widetilde{u}$. Hence from

(2.11) and using that $\widetilde{w} = \widetilde{v} - \widetilde{u}$ we have that

$$\int_{D\setminus\overline{D}_0} (A-I)\nabla\widetilde{w}\cdot\nabla\overline{\varphi} - \tau^2(n-1)\widetilde{w}\overline{\varphi}\,dx = \int_{B_R} \nabla\hat{u}\cdot\nabla\overline{\varphi} - \tau^2\hat{u}\overline{\varphi}\,dx$$
$$= \int_{B_R\setminus\overline{B_\epsilon}} (A_{min}-1)\nabla\hat{w}\cdot\nabla\overline{\varphi} - \tau^2(n_{max}-1)\hat{w}\overline{\varphi}\,dx \tag{2.20}$$

Therefore for $\varphi = \tilde{w}$ using (2.20) and the Cauchy-Schwartz inequality we have that

$$\int_{D\setminus\overline{D}_0}(A-I)\nabla\widetilde{w}\cdot\nabla\overline{\widetilde{w}} - \tau^2(n-1)\widetilde{w}\overline{\widetilde{w}}\,dx = \int_{B_R\setminus\overline{B_\epsilon}}(A_{min}-1)\nabla\hat{w}\cdot\nabla\overline{\widetilde{w}} - \tau^2(n_{max}-1)\hat{w}\overline{\widetilde{w}}\,dx \le$$
$$\left[\int_{B_R\setminus\overline{B_\epsilon}}(A_{min}-1)|\nabla\hat{w}|^2 - \tau^2(n_{max}-1)|\hat{w}|^2\,dx\right]^{\frac{1}{2}}\left[\int_{B_R\setminus\overline{B_\epsilon}}(A_{min}-1)|\nabla\widetilde{w}|^2 - \tau^2(n_{max}-1)|\widetilde{w}|^2\,dx\right]^{\frac{1}{2}}$$

and using the bounds on the coefficients we obtain that

$$\int_{D\setminus\overline{D}_0} (A-I)\nabla\widetilde{w}\cdot\nabla\overline{\widetilde{w}} - \tau^2(n-1)|\widetilde{w}|^2\,dx \le \int_{B_R\setminus\overline{B_\epsilon}} (A_{min}-1)|\nabla\hat{w}|^2 - \tau^2(n_{max}-1)|\hat{w}|^2\,dx$$

Now we use the definition (2.18) for the operator $\mathbb{L}_\tau$ with the functions $\widetilde{u}$ and $\widetilde{w}$ to conclude

$$(\mathbb{L}_\tau\widetilde{u},\widetilde{u})_{H^1(D\setminus\overline{D}_0)} = \int_D |\nabla\widetilde{u}|^2 - \tau^2|\widetilde{u}|^2\,dx + \int_{D\setminus\overline{D}_0} (A-I)\nabla\widetilde{w}\cdot\nabla\overline{\widetilde{w}} - \tau^2(n-1)|\widetilde{w}|^2\,dx$$
$$\le \int_{B_R} |\nabla\hat{u}|^2 - \tau^2|\hat{u}|^2\,dx + \int_{B_R\setminus\overline{B_\epsilon}} (A_{min}-1)\nabla\hat{w}\cdot\nabla\overline{\hat{w}} - \tau^2(n_{max}-1)|\hat{w}|^2\,dx$$
$$= 0$$

So the operator $\mathbb{L}_\tau$ is non-positive on this one dimensional subspace.

Alternatively, we can construct a finite dimensional subspace of $H_0^1\left(D \setminus \overline{D}_0, \partial D\right)$ where $\mathbb{L}_\tau$ is non-positive by considering small balls $B_\delta \subset D \setminus \overline{D}_0$. In this case let κ, $\hat{w}$ and $\hat{v}$ are the first transmission eigenvalue and corresponding eigenvectors of

$$\begin{aligned} \nabla \cdot A_{min} \nabla \hat{w} + \kappa^2 n_{max} \hat{w} = 0 \quad &\text{in} \quad B_\delta \\ \Delta \hat{v} + \kappa^2 \hat{v} = 0 \quad &\text{in} \quad B_\delta \\ \hat{w} = \hat{v} \text{ and } \frac{\partial \hat{w}}{\partial \nu_{A_{min}}} = \frac{\partial \hat{v}}{\partial \nu} \quad &\text{on} \quad \partial B_\delta. \end{aligned} \tag{2.21}$$

Now provided that the measure of each component of D_0 is small enough such that κ^2 is smaller then the first corresponding Dirichlet eigenvalue for $-\Delta$, we can use $\hat{u} = \hat{v} - \hat{w} \in H_0^1(B_\delta)$ its extension by zero $\tilde{u}$ to the whole domain and the corresponding $\tilde{w}$ and $\tilde{v}$ exactly as above to show that $\mathbb{L}_\kappa$ is non-positive in a m-dimensional subspace of $H_0^1\left(D \setminus \overline{D}_0, \partial D\right)$ where m is the number of balls of radius δ included in $D \setminus \overline{D}_0$

The same result can be proven for $-\mathbb{L}_k$ exactly in a similar way for the case when $A_{max} - 1 < 0$ and $n_{min} - 1 > 0$. $\square$

Now we uses a similar result as in theorem 2.6 [29] to prove the existence of transmission eigenvalues.

Theorem 2.1.4. *Let $\mathbb{L}_k : H_0^1\left(D \setminus \overline{D}_0, \partial D\right) \mapsto H_0^1\left(D \setminus \overline{D}_0, \partial D\right)$ be as defined above. If*

1. *there exists $k_{min} \geq 0$ such that $\sigma \mathbb{L}_{k_{min}}$ is positive on $H_0^1\left(D \setminus \overline{D}_0, \partial D\right)$*
2. *there exists $k_{max} < \lambda_1(D_0)$ such that $\sigma \mathbb{L}_{k_{max}}$ is non-positive on a m-dimensional subspace of $H_0^1\left(D \setminus \overline{D}_0, \partial D\right)$*

then there exists m transmission eigenvalues in $[k_{min}, k_{max}]$, where $\sigma = 1$ or $\sigma = -1$ provided $A_{min} > 1$ and $n_{max} < 1$, or $A_{max} < 1$ and $n_{min} > 1$, respectively.

In particular the result can be obtained by using min-max condition for the auxiliary eigenvalue problem for the self adjoint compact operator:

$$\mathbb{I} - \lambda(k)(\sigma\mathbb{L}_0)^{-1/2}\sigma(\mathbb{L}_k - \mathbb{L}_0)(\sigma\mathbb{L}_0)^{-1/2}.$$

Theorem 2.1.5. *Assume that either $A_{min} > 1$ and $n_{max} < 1$, or $A_{max} < 1$ and $n_{min} > 1$. If the first transmission eigenvalue $\tau > 0$ of (2.19) is smaller than the first Dirichlet eigenvalue for each of the components of D_0, then there exists one transmission eigenvalue in the interval $(0, \tau)$. If the first transmission eigenvalue $\kappa > 0$ of (2.21) is smaller than the first Dirichlet eigenvalue for each of the components of D_0 then there exits $m := m(\delta)$ transmission eigenvalue (counting multiplicity) in the interval $(0, \kappa)$, where m is the number of balls of radius $\delta > 0$ that can fit in $D \setminus \overline{D}_0$.*

For sake of completeness we prove the discreteness of the transmission eigenvalues. This result is proven in a more general case in [11]. To this end we recall the Analytic Fredholm Theorem, which will be used to prove the the set of real transmission eigenvalues is at most discrete.

Theorem 2.1.6 (Analytic Fredholm Theorem)**.** *Assume that $\Omega \subseteq \mathbb{C}$ is an open connected set. Let $\mathbb{K}_\lambda : \Omega \mapsto \mathcal{L}(X)$ with X a Hilbert Space and where the operator valued function $\mathbb{K}_\lambda$ depends analytically on $\lambda \in \Omega$ then either*

1. *$(\mathbb{I} - \mathbb{K}_\lambda)^{-1}$ Does not exist for any $\lambda \in \Omega$*
2. *$(\mathbb{I} - \mathbb{K}_\lambda)^{-1}$ exists except for at most a discrete set in Ω,*

provided that $\mathbb{K}_\lambda$ is a compact operator for all $\lambda \in \Omega$.

Theorem 2.1.7. *Assume that either $A_{min} > 1$ and $n_{max} < 1$, or $A_{max} < 1$ and $n_{min} > 1$. Then the set of real transmission eigenvalues is at most discrete and furthermore $+\infty$ is the only accumulation point for the set of real transmission eigenvalues.*

Proof. Notice that by the definition of $\mathbb{L}_k$ we have that the mapping $k \mapsto \mathbb{L}_k$ is analytic in the set $\{z \in \mathbb{C} : \Re(z^2) > 0\} \setminus \left\{\lambda_j(D)\right\}_{j\in\mathbb{N}}$ where $\lambda_j(D)$ is the j-th eigenvalue of $-\Delta$ in D. It is clear the k is a transmission eigenvalue if and only if the null space of the operator $\mathbb{I} - (\sigma\mathbb{L}_0)^{-1/2}\sigma(\mathbb{L}_k - \mathbb{L}_0)(\sigma\mathbb{L}_0)^{-1/2}$ is non-trivial. The result then follows from Theorem 2.1.2 and the Analytic Fredholm Theorem. □

Remark 2.1.1. If $A_{min} - 1 > 0$ and $n_{max} - 1 > 0$, or $A_{max} - 1 < 0$ and $n_{min} - 1 < 0$ it is now obvious how to modify the approach of [29] to prove the existence of transmission eigenvalue. In this case in addition to assuming that the voids are small enough it is necessary to assume that $|n - 1|$ is also small. We do not include a detailed discussion here.

As a by product of theorems 2.1.2 and 2.1.3 we have that the first transmission eigenvalue has the following upper and lower bounds.

Corollary 2.1.1. *Let $k_1^2(D, D_0, A, n)$, be the first transmission eigenvalue of the given media with voids and the measure of each component of D_0 is small enough (as discussed above). Then the following inequalities hold:*

1. *If $(A_{min} - 1) > 0$ and $(n_{max} - 1) < 0$, then*

$$\lambda_1(D) \leq k_1^2(D, D_0, A, n) \leq \min\left\{k_1^2(B_R, B_\epsilon, A_{min}, n_{max}), k_1^2(B_\delta, A_{min}, n_{max})\right\}$$

   $k_1^2(B_R, B_\epsilon, A_{min}, n_{max})$ and $k_1^2(B_\delta, A_{min}, n_{max})$ are the first transmission eigenvalue corresponding to (2.19) and the first transmission eigenvalue corresponding to (2.21), respectively.

2. if $(A_{max}-1)<0$ and $(n_{min}-1)>0$, then

$$\frac{A_{min}}{n_{max}}\lambda_1(D) \le k_1^2(D,D_0,A,n) \le \min\left\{k^2(B_R,B_\epsilon,A_{max},n_{min}),k^2(B_\delta,A_{max},n_{min})\right\}$$

where $k_1^2(B_R,B_\epsilon,A_{max},n_{min})$ and $k_1^2(B_\delta,A_{min},n_{max})$ the first transmission eigenvalue corresponding to (2.19) and the first transmission eigenvalue corresponding to (2.21), respectively, with A_{max} replaced by A_{min} and n_{min} replaced n_{max}.

Here $\lambda_1(D)$ is the first Dirichlet eigenvalue of $-\Delta$ in D.

We now prove a monotonicity result for the first transmission eigenvalue with respect to the size of D_0, which can be useful in non-destructive testing.

Theorem 2.1.8. *Let $D_0 \subseteq D_1 \subset D$ then we have that $k_1(D_0) \le k_1(D_1)$ where $k_1(\Omega)$ is the first transmission eigenvalue corresponding to D with void Ω*

Proof. Assume that $(A_{min}-1)>0$ and $(n_{max}-1)<0$, and that $\tilde{v}$ and $\tilde{w}$ are the transmission eigenfunctions corresponding to the transmission eigenvalue $k_1(D_1)=\tilde{k}$. Now let $\tilde{u}=\tilde{v}-\tilde{w}$, therefore we have the existence of $v \in H^1(D\setminus\overline{D}_0)$ that solves (2.12) and define $w=v-\tilde{u}$. Therefore we have from (2.18) that

$$\int_D |\nabla\tilde{u}|^2 - \tilde{k}^2|\tilde{u}|^2\,dx + \int_{D\setminus\overline{D}_1} (A-I)\nabla\tilde{w}\cdot\nabla\overline{\tilde{w}} - \tilde{k}^2(n-1)|\tilde{w}|^2\,dx = 0.$$

By the definition of w and (2.12) we obtain that

$$\begin{aligned}\int_{D\setminus\overline{D}_0} (A-I)\nabla w\cdot\nabla\overline{\varphi} - \tilde{k}^2(n-1)w\overline{\varphi}\,dx &= \int_D \nabla\tilde{u}\cdot\nabla\overline{\varphi} - \tilde{k}^2\tilde{u}\overline{\varphi}\,dx \qquad (2.22)\\ &= \int_{D\setminus\overline{D}_1} (A-I)\nabla\tilde{w}\cdot\nabla\overline{\varphi} - \tilde{k}^2(n-1)\tilde{w}\overline{\varphi}\,dx.\end{aligned}$$

Therefore letting $\varphi = w$ in (2.11) and using the Cauchy-Schwartz inequality (in the same way as the equations below (2.20)) along with $D_0 \subseteq D_1$ we have that

$$\int_{D\setminus\overline{D}_0} (A-I)\nabla w \cdot \nabla \overline{w} - \tilde{k}^2(n-1)|w|^2\, dx \leq \int_{D\setminus\overline{D}_1} (A-I)|\nabla \tilde{w}|^2 - \tilde{k}^2(n-1)|\tilde{w}|^2\, dx.$$

Now we use the definition (2.18) for the operator $\mathbb{L}_{\tilde{k}}$ with the functions $\tilde{u}$ to conclude that

$$\begin{aligned}
(\mathbb{L}_{\tilde{k}}\tilde{u}, \tilde{u})_{H^1(D\setminus\overline{D}_0)} &= \int_D |\nabla \tilde{u}|^2 - \tilde{k}^2|\tilde{u}|^2\, dx + \int_{D\setminus\overline{D_0}} (A-I)\nabla w \cdot \nabla \overline{w} - \tilde{k}^2(n-1)|w|^2\, dx \\
&\leq \int_D |\nabla \tilde{u}|^2 - \tilde{k}^2|\tilde{u}|^2\, dx + \int_{D\setminus\overline{D}_1} (A-I)\nabla \tilde{w} \cdot \nabla \overline{\tilde{w}} - \tilde{k}^2(n-1)|\tilde{w}|^2\, dx \\
&= 0
\end{aligned}$$

where $\mathbb{L}_{\tilde{k}}$ is the operator corresponding to the problem with void D_0. Since $\mathbb{L}_{\tilde{k}}$ is nonpositive on the subspace spanned by $\tilde{u}$ it means that there is an eigenvalue corresponding to D_0 in $(0, \tilde{k}]$. Therefore the first transmission eigenvalue $k_1(D_0)$ must satisfy $k_1(D_0) \in (0, \tilde{k}]$ which proves the claim. A similar argument holds for when $A_{max} - 1 < 0$ and $n_{min} - 1 > 0$, by looking at the operator $-\mathbb{L}_{\tilde{k}}$. $\square$

In a similar manner one can prove the following monotonicity results in terms of the material parameters.

Theorem 2.1.9. *Assume that* $\forall\, \xi \in \mathbb{C}^2$ *we have* $\overline{\xi} \cdot A_1\xi \leq \overline{\xi} \cdot A_2\xi$ *and* $n_1 \leq n_2$ *with* D *and* D_0 *fixed. Then we have that:*

1. *If* $A_\ell - I$ *is positive definite and* $n_\ell - 1 < 0$ *then we have that:*

$$k_1(A_2, n_1) \leq k_1(A_1, n_2)$$

2. If $A_\ell - I$ is negative definite and $n_\ell - 1 > 0$ then we have that:

$$k_1(A_1, n_2) \leq k_1(A_2, n_1)$$

Remark 2.1.2. *The above monotonicity results for the coefficients still hold in the case where the domain does not have a void.*

2.2 Determination of Transmission Eigenvalues from Scattering Data

The goal of this section is to show that real transmission eigenvalues can be determine for the far-field pattern $u^\infty(\hat{x}, d)$ for $\hat{x}, d \in \mathbb{S}$ (or possibly in a subset of $\mathbb{S}$) following the approach in [19] where the same result is proven for the case of isotropic media (i.e. where $A = I$). To this end, recall the scattering of a plane wave $u^i := e^{ikx\cdot d}$ by this anisotropic inhomogeneous media with voids which is given by (1.3)-(1.4) where $A_0 = I$ and $n_0 = 1$. Recall that the far-field patterns for all incident directions d defines the far field operator $F : L^2(\mathbb{S}) \longmapsto L^2(\mathbb{S})$ by

$$(Fg)(\hat{x}) := \int_{\mathbb{S}} u^\infty(\hat{x}, d; k) g(d)\, ds(d).$$

Here we have indicated the dependance on the wave number k in the far-field pattern. It is also well-known that the far-field operator is injective if and only if there does not exist a nontrivial (w, v) solving (2.1)-(2.5) such that v takes the form of a Herglotz function (1.12). We now introduce the far field equation

$$(Fg_z)(\hat{x}) = \Phi^\infty(\hat{x}, z), \qquad z \in D, \quad \hat{x} \in \mathbb{S} \tag{2.23}$$

where $\Phi^\infty(\hat{x}, z) = \gamma_m e^{-ik\hat{x}\cdot z}$, is the far-field pattern of the radiating fundamental solution to the Helmholtz equation. Let F^δ be the far-field operator corresponding to the noisy measurements $u^\infty_\delta(\hat{x}, d)$ satisfying $||u^\infty_\delta(\hat{x}, d; k) - u^\infty(\hat{x}, d; k)||_{L^2} \leq \delta$. We find the Tikhonov regularized solution $g_{z,\delta} := g^{\delta}_{z,\epsilon(\delta)}$ of the far-field equation defined as the unique minimizer of

$$\|F^{\delta}g - \Phi^{\infty}(\cdot, z)\|^2_{L^2(\mathbb{S})} + \epsilon\|g\|^2_{L^2(\mathbb{S})}$$

where the regularization parameter $\epsilon := \epsilon(\delta) \to 0$ as $\delta \to 0$. Provided that F has dense range (which is true in general except for the case when the transmission eigenfunction v takes the form of Herglotz function; see Appendix of [19]) the regularized solution $g_{z,\delta}$ is such that

$$\lim_{\delta\to 0}\|F^{\delta}g_{z,\delta} - \Phi^{\infty}(\cdot, z)\|_{L^2(\mathbb{S})} = 0. \tag{2.24}$$

Following [4] (note that our far-field operator in normal) and using the results developed in Section 7.2 in [14] it is possible to show that if k is not a transmission eigenvalue and $z \in D$ then the Herglotz function $v_{g_{z,\delta}}$ converges in the $H^1(D)$-norm to v where (v, w) solves

$$\nabla \cdot A\nabla w + k^2 nw = 0 \quad \text{in } D \tag{2.25}$$

$$\Delta v + k^2 v = 0 \quad \text{in } D \tag{2.26}$$

$$w - v = \Phi(\cdot, z) \quad \text{on } \partial D \tag{2.27}$$

$$\frac{\partial w}{\partial \nu_A} - \frac{\partial v}{\partial \nu} = \frac{\partial \Phi(\cdot, z)}{\partial \nu} \quad \text{on } \partial D, \tag{2.28}$$

provided that the solution exists (which is satisfied if (2.25)-(2.28) is Fredholm with index zero). Let us recall an equivalent variational formulation for the above interior transmission problem analyzed in [11]. To this end, we define a lifting of the essential boundary data into the domain D. Thus we let $\phi_z \in H^1(D)$ be such that $\phi_z = \Phi(\cdot, z)$

on ∂D and attempt to find a solution of the interior transmission problem where $v = v_0 - \phi_z$, and the pair $(w, v_0) \in X(D) := \{w, v_0 \in H^1(D) \,|\, w - v_0 \in H_0^1(D)\}$ satisfies

$$\mathcal{A}_k\big((w, v_0); (\varphi_1, \varphi_2)\big) = \ell(\varphi_1, \varphi_2) \quad \text{for all} \quad (\varphi_1, \varphi_2) \in X(D) \tag{2.29}$$

where the sesqulinear form $\mathcal{A}_k(\cdot\,;\cdot) : X(D) \times X(D) \mapsto \mathbb{C}$ and the conjugate linear functional $\ell(\cdot) : X(D) \mapsto \mathbb{C}$ are given by

$$\mathcal{A}_k\big((w, v_0); (\varphi_1, \varphi_2)\big) := \int\limits_D A\nabla w \cdot \nabla\overline{\varphi_1} - \nabla v_0 \cdot \nabla\overline{\varphi_2}\, dx - k^2 \int\limits_D nw\overline{\varphi_1} - v_0\overline{\varphi_2}\, dx,$$

$$\ell(\varphi_1, \varphi_2) := \int\limits_{\partial D} \overline{\varphi}_1 \frac{\partial}{\partial \nu}\Phi(x, z)\, ds_x - \int\limits_D \nabla\phi_z \cdot \nabla\overline{\varphi}_2 - k^2\phi_z\overline{\varphi}_2\, dx.$$

Let us introduce the following notation:

$$\inf_{x\in\mathcal{N}(\partial D)} \inf_{|\xi|=1} \overline{\xi} \cdot A(x)\xi = A_\star \quad \text{and} \quad \inf_{x\in\mathcal{N}(\partial D)} n(x) = n_\star$$
$$\sup_{x\in\mathcal{N}(\partial D)} \sup_{|\xi|=1} \overline{\xi} \cdot A(x)\xi = A^\star \quad \text{and} \quad \sup_{x\in\mathcal{N}(\partial D)} n(x) = n^\star$$

where $\mathcal{N}(\partial D)$ is a fixed neighborhood of the boundary. It has been proven in [11] that the variational problem (2.29) satisfies the Fredholm property provided $A_\star > 1$, $n_\star > 1$, or $A^\star < 1$, $n^\star < 1$. In particular, if k is a transmission eigenvalue with transmission eigenfunctions (w_k, v_k) then (2.29) has a solution if and only if the following solvability condition is satisfied

$$\ell(w_k, v_k) = \int\limits_{\partial D} \overline{w}_k \frac{\partial}{\partial \nu}\Phi(x, z)\, ds_x - \int\limits_D \nabla\phi_z \cdot \nabla\overline{v}_k - k^2\phi_z\overline{v}_k\, dx = 0. \tag{2.30}$$

Theorem 2.2.1. *Let k be a real transmission eigenvalue and assume that $A_\star > 1$, $n_\star > 1$, or $A^\star < 1$, $n^\star < 1$. Then for $z \in D$ (except for possibly a nowhere dense set of points), $||v_{g_{z,\delta}}||_{H^1(D)}$ can not be bounded as $\delta \to 0$, where $g_{z,\delta}$ satisfies (2.24).*

Proof. Assume there is a set of positive measure such that $||v_{g_{z,\delta}}||_{H^1(D)}$ is bounded. Hence, a subsequence $v_{g_{z,\delta_n}}$ converges weakly to a $v \in H^1(D)$ satisfying $\Delta v + k^2 v = 0$ in D. Since $Fg_{z,\delta} \to \Phi^\infty(\cdot, z)$, Rellich's lemma implies that $u^s = \Phi(\cdot, z)$ outside of D, where u^s is the scattered field with the far-field pattern $Fg_{z,\delta}$. Now, the corresponding total field w in D and v satisfy the interior transmission problem (2.25)-(2.28), which gives that there is a solution to the variational problem $\mathcal{A}_k\big((w, v_0); (\varphi_1, \varphi_2)\big) = \ell(\varphi_1, \varphi_2)$. Using integration by parts on the Fredholm solvability condition (2.30) and using that $\phi_z = \Phi(\cdot, z)$ on ∂D and $\Delta v_k + k^2 v_k = 0$ in D, we have that

$$\int_{\partial D} \overline{w}_k \frac{\partial}{\partial \nu}\Phi(x, z) - \frac{\partial \overline{v}_k}{\partial \nu}\Phi(x, z)\, ds_x = 0$$

Notice that since $w_k = v_k$ on ∂D, Green's representation theorem and the unique continuation principle implies that $v_k = 0$ in D. So v_k has zero Cauchy data on ∂D, which implies that $w_k = 0$ and $\frac{\partial w_k}{\partial \nu_A} = 0$ on ∂D, whence $w_k = 0$ in D. Therefore $(w_k, v_k) = (0, 0)$ which contradicts the fact that (w_k, v_k) are eigenfunctions. $\square$

The above analysis indicates that when plotting the $||v_{g_{z,\delta}}||_{H^1(D)}$ against k, where $g_{z,\delta}$ is the Tikhonov regularized solution to the far field equation, the transmission eigenvalues will appear as sharp peaks in the graph. We will we present numerical examples that show the viability of this approach to determine real transmission eigenvalues from far-field data.

Remark 2.2.1. *Theorem 2.2.1 is valid for any assumptions on A and n where* (2.29) *satisfies the Fredholm alternative.*

In [62] for the case where $n = 1$ the real transmission eigenvalues are related to the eigenvalues of the far-field operator. This is done for more restrictive assumptions on A and does not include the case of voids.

2.3 Numerical Validation

In this section, we show some numerical examples to show that the first transmission eigenvalue can give information about the voids. We shall address the following issues:

1. We check if the transmission eigenvalues can be determined from scattering data for the case of an anisotropic magnetic materials with voids based on the discussion in these chapter (see e.g. [78] for near field data). We confirm that the eigenvalues determined from the far-field data are actually the transmission eigenvalue. In particular, we consider a special case when the scattering object and the void are concentric disks in which case the transmission eigenvalues can be obtained analytically. For general geometry we compute the transmission eigenvalues using a continuous finite element method with eigenvalues searching technique described in [39] (see also [77] and [79]).
2. We numerically study how the size, location and geometry of voids affects the first transmission eigenvalue.
3. We numerically study the inverse problem of estimating the size of the void(s) using the first transmission eigenvalue. Numerical results indicate that qualitative information can be obtained on the size of the void(s).

Theorem 2.2.1 suggests that if we solve the far-field equation and plot the L^2 norm of the solution g against a range of k values, at a transmission eigenvalue (TEV) the norm of g "blows up", which should look like a spike in the graph. Below is the numerical procedure with simulated far-field data:

(a) Solve the direct problem using a cubic finite element method with a perfectly matched layer, for a range of k values.

(b) Evaluate an approximate u^∞ with 1% random noise added (unless otherwise stated).

(c) Using the approximated u^∞ to solve the far field equation $Fg_z = \Phi^\infty(\cdot, z)$ for 25 random locations of z in D by a Tikhonov-Morozov regularization strategy.

(d) Plot $||g||_{L^2(0,2\pi)}$ averaged over z versus k. Note that from the estimates derived in the previous section we have a priori knowledge of the interval where the first transmission eigenvalue lies, and we consider this information when choosing the range of k in our computations.

In the following calculations we use N different incident direction ϕ_j and N observation directions θ_i that are uniformly spaced in $[0, 2\pi)$,where $N = 25$ unless specified otherwise. The far field pattern $u^\infty(\theta_i, \phi_j)$ is obtained from solving the direct problem. This corresponds to a $N \times N$ discretized version of the far-field operator. We then solve the discretized far field equation for 25 random point in the domain D. Once we have solved this linear systems for $\vec{g}$ which has components $g_i \approx g(\theta_i)$, we plot the average approximation of $||g||_{L^2(0,2\pi)}$ over a range of k values.

2.3.1 Comparison with Separation of Variables

To gain an explicit understanding of the behavior of the transmission eigenvalues (TEVs) we now consider a TEV problem with constant coefficients. For this we assume that $A = \alpha I$ for some constant $\alpha > 0$ and let n be constant such that $n > 0$. Furthermore assume that $D = B_R$ and $D_0 = B_\epsilon$ where $0 < \epsilon < R$. Under these assumptions we have the following TEV problem formulated in the disk: find nontrivial $(w, v) \in H^1(B_R) \times H^1(B_R)$ such that

$$\Delta w + k^2 w = 0 \quad \text{in} \quad B_\epsilon \tag{2.31}$$

$$\alpha \Delta w + k^2 n w = 0 \quad \text{in} \quad B_R \setminus \overline{B_\epsilon} \tag{2.32}$$

$$\Delta v + k^2 v = 0 \quad \text{in} \quad B_R \tag{2.33}$$

$$w^- = w^+ \text{ and } \frac{\partial w^-}{\partial r} = \alpha \frac{\partial w^+}{\partial r} \quad \text{on} \quad \partial B_\epsilon \tag{2.34}$$

$$w = v \text{ and } \alpha \frac{\partial w}{\partial r} = \frac{\partial v}{\partial r} \quad \text{on} \quad \partial B_R. \tag{2.35}$$

It can be shown that trying to find transmission eigenfunctions of the form $w(r, \theta) = w_m(r) e^{im\theta}$ and $v(r, \theta) = v_m(r) e^{im\theta}$ with $m \in \mathbb{Z}$ gives that the transmission eigenvalues are given by the roots of $d_m(k)$, where $d_m(k)$ is defined as

$$d_m(k) := \det \begin{pmatrix} \mathrm{J}_m(k\epsilon) & -\mathrm{J}_m(k\sqrt{\frac{n}{\alpha}}\epsilon) & -\mathrm{Y}_m(k\sqrt{\frac{n}{\alpha}}\epsilon) & 0 \\ -\mathrm{J}'_m(k\epsilon) & \sqrt{n\alpha}\,\mathrm{J}'_m(k\sqrt{\frac{n}{\alpha}}\epsilon) & \sqrt{n\alpha}\,\mathrm{Y}'_m(k\sqrt{\frac{n}{\alpha}}\epsilon) & 0 \\ 0 & \mathrm{J}_m(k\sqrt{\frac{n}{\alpha}}R) & \mathrm{Y}_m(k\sqrt{\frac{n}{\alpha}}R) & -\mathrm{J}_m(kR) \\ 0 & -\sqrt{n\alpha}\,\mathrm{J}'_m(k\sqrt{\frac{n}{\alpha}}R) & -\sqrt{n\alpha}\,\mathrm{Y}'_m(k\sqrt{\frac{n}{\alpha}}R) & \mathrm{J}'_m(kR) \end{pmatrix}$$

with $\mathrm{J}_p(t)$ and $\mathrm{Y}_p(t)$ are the Bessel functions of the first and second kind (see [36]). It is well known that TEV problems are non-selfadjoint eigenvalue problem, so it is possible to have complex TEVs. The existence of complex eigenvalues is proven for the spherically symmetric case for $A = I$ in [35]. Through numerical calculations it is possible to compute the complex TEVs for this particular TEV problem. Using standard root finding methods it is possible to find the roots of $d_m(k)$, and doing so we see that $d_0(k)$ has complex roots $k \approx 3.8405 \pm 0.5805i$ where $\alpha = 1/5$, $n = 1$, $\epsilon = 0.1$ in a unit ball.

Now let the voids D_0 and D_1 satisfy $D_0 \subseteq D_1$, with A and n fixed then we

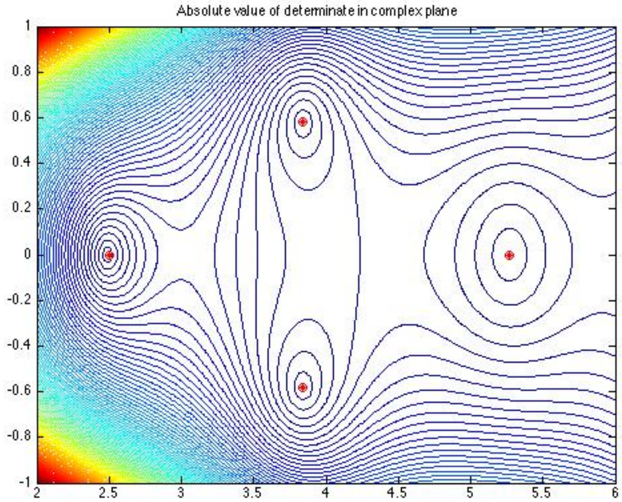

Figure 2.1: Contour plot of $|d_0(k)|$ in the complex plane, red dots are the roots. We let $\alpha = 1/5$, $n = 1$, $\epsilon = 0.1$ and $R = 1$

have shown in Theorem 2.1.8 that the first TEV k_1 is a monotonic function of the size of the void. Figure 2.2 is a plot that shows the monotonicity of the first root of $d_0(k)$ in terms of the size of the circular void ϵ were $\alpha = 1/5$, $n = 1$ and $R = 1$.

From Theorem 2.1.9 we also know that the first TEV is a monotonic function of the coefficient. We can easily check this property for the simple case of problem (2.31)-(2.35) and the results presented on Figure 2.3 confirm it.

In the previous section we have shown that the TEV can be determined from the Far-Field Equation(FFE). To see if the FFE will actually capture the TEVs for domains with voids we apply the method discussed to (2.31)-(2.35). We expect to see spikes in the average L^2-norm of g_z (the average is over 25 randomly chosen sampling points $z \in D$) at the known TEVs that are the roots of $d_0(k)$. In Figure 2.4 we plot the average $||g_z||_{L^2(0,2\pi)}$ for the parameters $\alpha = 1/5$, $n = 1$ and $R = 1$ with

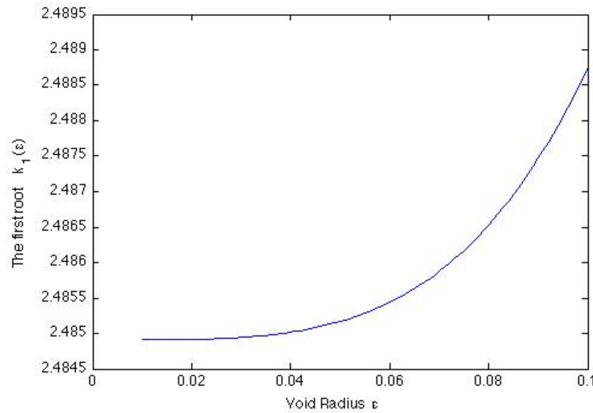

Figure 2.2: Graph of $k_1(\epsilon)$ v.s. $\epsilon \in [0.01, 0.1]$ to show the monotonicity of the first TEV with respect to the size of the void

$\epsilon = 0.1$. Using Newton's method with a centered finite difference approximation for the derivative we can compute the first two roots of $d_0(k)$ given by $k \approx 2.48$, 5.27.

We let $A_\alpha = \alpha I$, $n = 1$ and compare the roots of $d_0(k)$ to the spikes in the graph for $||g||_{L^2(0,2\pi)}$ for various values of α and ϵ, where we let the outer radius $R = 1$. There results are shown in Table 2.1. The values agree very well.

Table 2.1: Root Finding v.s. Far-Field Eqation

α	ϵ	1st root of d_0	spike in the $\vert\vert g\vert\vert_{L^2(0,2\pi)}$
1/2	0.01	7.99	7.80
1/4	0.1	2.91	2.92
1/10	0.05	1.67	1.68

We now do a more detailed study on using the Far Field equation (FFE) approach for seeing TEVs in the scattering data for an anisotropic medium. For this we will use the far field pattern to investigate the dependance of the known TEVs

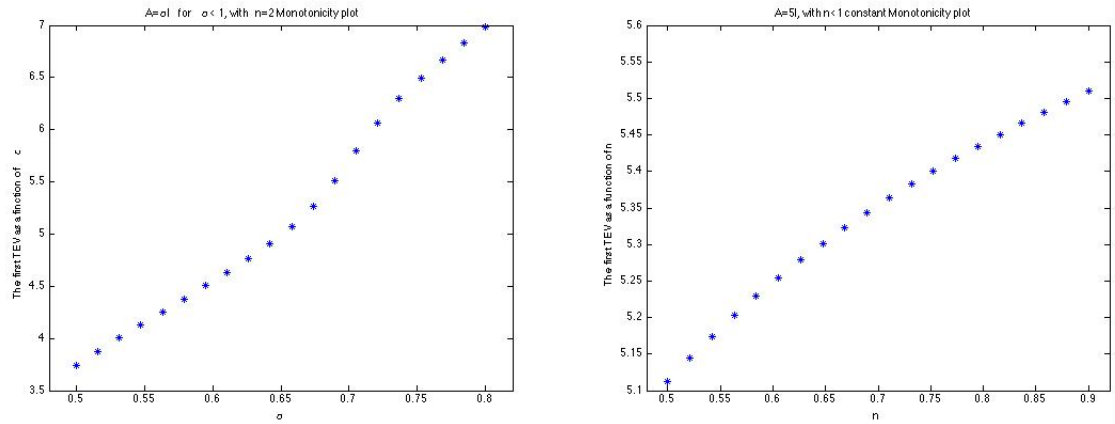

Figure 2.3: Plots to show monotonicity of the 1st TEV w.r.t. the coefficients. On the left $A = \alpha I$ for $0.5 \leq \alpha \leq 0.8$ with $n = 2$ and $\epsilon = 0.2$. While on the right $0.5 \leq n \leq 0.9$ with $A = 5I$ and $\epsilon = 0.2$

0n the number of incident directions and the shape/position of the void.

To this end we consider the matrices $A_a = \frac{1}{a}I$ for $a = 4, 9$ in the TEV problem (2.31)-(2.35). We now investigate how the number of incident and observation directions affects the accuracy of the FFE approach to finding TEVs. It is recommended that one should take $N > 2kR$ were k is the wave number and R radius of the circumscribing circle for the domain D.

Table 2.2: Dependace on N

Matrix	Root Finding (k_1)	N= 20	N= 10	N= 5
A_4	2.91	2.92	2.92	2.90
A_9	1.77	1.80	1.80	1.78

This table shows results for the TEV problem (2.31)-(2.35) with $n = 1$, $R = 1$ and $\epsilon = 0.1$. We see for A_4 that $N > 2k_1 \approx 6$ and for A_9 that $N > 2k_1 \approx 4$, were k_1

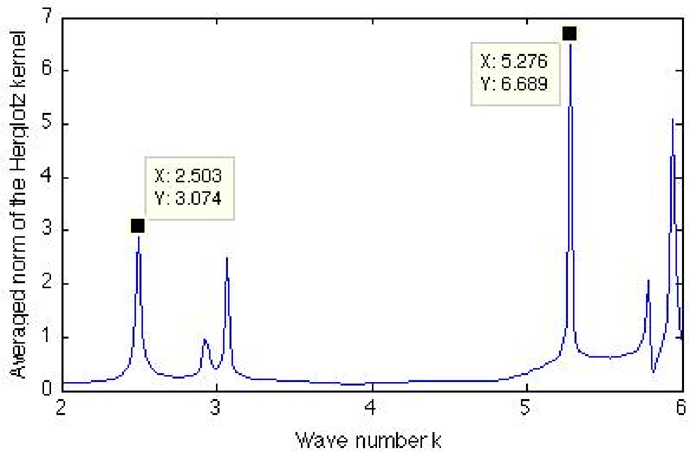

Figure 2.4: Notice that there are spikes at $k = 2.50$, 5.27 on the graph while the first two roots of $d_0(k)$ are 2.48, 5.27. The other spikes in the graph corresponds to roots for $d_m(k)$ where $m \neq 0$

is the first root of $d_0(k)$. From the table we see that taking N much larger that $2k_1$ does not seem to add more accuracy to the FFE approach.

Now we wish to investigate if the location of the void will noticeably change the first TEV. To this end we use the same algorithm used in the previous examples to see compute the first TEV from the far field measurements. We now consider the domain $D = B_1$ with $n = 1$, where the void is a ball of radius $\epsilon = 0.1$ that is centered at (x_1, x_2). In the table below we see little to no difference if the location is changed.

Table 2.3: Dependence of first transmission eigenvalue on void's position

location	(0, 0)	(0.6, 0)	(0.3, 0.7)	(-0.2, 0.4)	(0.6, 0.6)
A_4; k_1	2.90	2.92	2.92	2.96	2.92
A_9; k_1	1.77	1.80	1.78	1.80	1.78

We will now see if the shape of the void will change the known TEVs. To investigate this we choose two domains that have the same area as B_ϵ. Let $S_\epsilon = \Big(-\epsilon\sqrt{\pi}/2, \epsilon\sqrt{\pi}/2\Big)^2$ and $E_\epsilon = \Big\{x \in \mathbb{R}^2 : \left(\frac{x_1}{2\epsilon}\right)^2 + \left(\frac{x_2}{\epsilon/2}\right)^2 < 1\Big\}$, notice that these are both contained in B_1 with area$= \pi\epsilon^2$.

Table 2.4: Dependace of first transmission eigenvalue on void's shape

Matrix	$k_1(B_\epsilon)$	$k_1(S_\epsilon)$	$k_1(E_\epsilon)$
A_4	2.91	2.92	2.92
A_9	1.77	1.78	1.78

Given the data in this table it would seem that the shape of the void does not seem to affect the first TEV, provided that the voids have the same area.

2.3.2 Comparison to the Finite Element Method

Now we consider non-circular domains. Provided we know the domain we can use a Finite Element Method (FEM) to compute the TEVs. We now compare the reconstructed TEVs using the Far-Field Equation (FFE) with the FEM. We fix $A = \text{Diag}(5, 6)$ and $n = 2$. The direct computation by the FEM in Table 2.5 is done by a continuous FEM using the linear Lagrange elements with the mesh size $h \approx 0.01$.

Table 2.5: Comparison of FFE Computation v.s. FEM Calculations

Method	Domain	1st TEV	2nd TEV
FFE	square (2×2)	1.84	6.60
FEM	square (2×2)	1.84	6.63
FFE	circle $(R = 1)$	1.98	7.23
FEM	circle $(R = 1)$	1.98	7.13

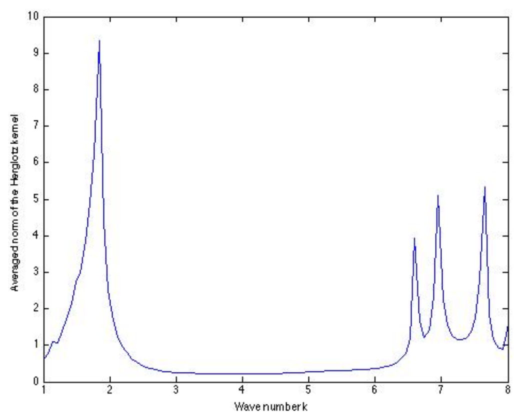

Figure 2.5: The plot of the average $||g||_{L^2(0,2\pi)}$ for the square (2×2) with no void

We now look at the question of partial aperture in using the far field data to compute the TEVs. Partial aperture is where the angles ϕ and θ are not distributed over the entire interval $[0, 2\pi)$, but rather some subset. So in the table below we use $N = 20$ angles distributed uniformly over $[0, \pi)$. It is known that the smaller the aperture the more unstable the Far Field equation method. To test this we decrease the amount of random noise added to the calculations to see if the first spike computed with partial aperture coincides with our first FEM computed TEV for sufficiently small noise. The results are shown in Table 2.6.

Table 2.6: Limited aperture for Disk with $R = 1$. Exact TEV 1.98

Noise	1st spike
10^{-3}	1.84
10^{-6}	1.91
10^{-9}	1.98

2.3.3 Determination of the Area from Far Field Measurements

We now consider the inverse spectral problem of determining information about the void D_0 form the first TEV. Recall that if we fix D, $A = \text{Diag}(5,6)$ and $n = 2$ then we have that the first TEV is an increasing function of the void size. On the other hand we noticed that the first TEV is not affected by the shape and location of the void. This suggests that the first TEV could hold information about the size of the void. If the first TEV is known but the size of the void isn't then we wish to find information about the size of the void. We suppose the first TEV of a domain with possible voids inside is known, for example, estimated using the Far Field Measurements as previously discussed, and we consider the inverse problem of finding the area of a void(s) from the first TEV. To do so we try to find an r such that a void of the form $B(\mathbf{0}\,;r)$ satisfies $k_1(\text{void(s)}) \approx k_1(B(\mathbf{0}\,;r))$. We first compute the first TEV for the domains, a disk of radius 1 and a 2×2 square $[-1,1]\times[-1,1]$ with a void of the form $B(\mathbf{0}\,;\epsilon)$ for various ϵ and the results are shown in Table 2.7.

Now we try to reconstruct the area of multiple voids by using the first TEV computed by the FEM. To do so we put two circular voids in the domains considered above. The voids both have radii 0.1 and are centered at $(0, 0)$ and $(0.5, 0.5)$ respectively. The inversion algorithm to reconstruct the area of a void is given by:

- Find $k_1(B(\mathbf{0}\,; r))$ for various radii r using the FEM
- Construct a polynomial $P(t)$ s.t. $P(\text{Area}(B_r)) \approx k_1(B_r)$
- Use Far Field measurements to compute $k_1(D_0)$
- Solve: $P(t) = k_1(D_0)$
- Then use $\text{Area}(B_r)$ to approximate $\text{Area}(D_0)$

We then compute the first TEV, then find the area of a single disk that has the same first TEV. Note that the total area of the two voids is approximately 0.063. The area of the single void $B(\mathbf{0}\,; r)$ is 0.0607 for the unit disk and 0.0775 for the square.

Table 2.7: First TEV for various void sizes computed by the FEM

ϵ	0.2	0.19	0.18	0.17	0.16	0.15	0.14	0.13	0.12	0.11	0.1
Circle	9.53	9.27	9.02	8.77	8.54	8.31	8.08	7.86	7.64	7.43	7.22
Square	7.76	7.57	7.39	7.21	7.04	6.87	6.70	6.53	6.37	6.21	6.05

Figure 2.6 and the above calculations give numerical evidence that the first TEVs can be used to gain qualitative information about the size of the void(s). In those calculations we used the “exact” TEV computed by the FEM. For this to be useful for industrial applications we wish to see if the TEVs computed for the FFE will also give an approximation of the size of the void(s).

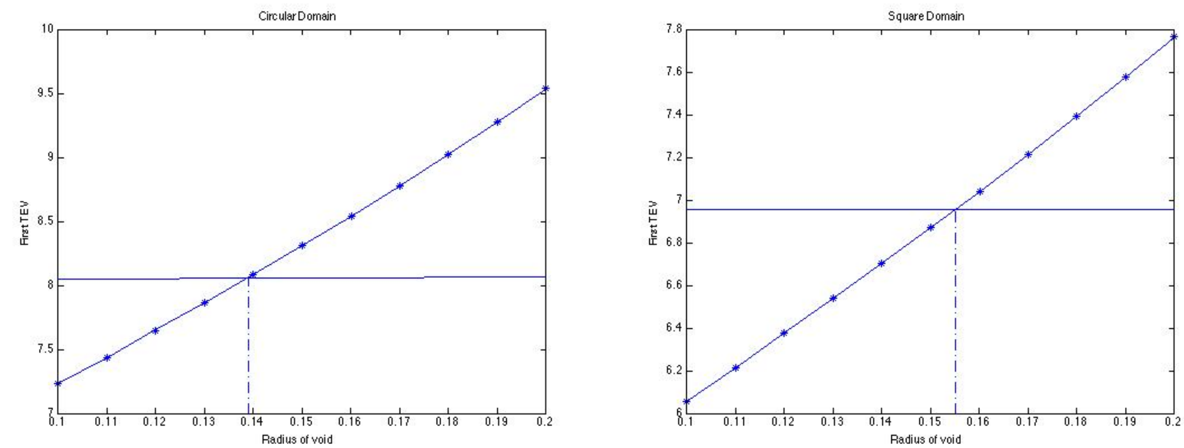

Figure 2.6: The first TEV of the two domains with a single void v.s the radius of the void. The horizontal lines are the k_1's for the two domains with two voids. The vertical dotted lines are the approximated values of r such that a void of the form $B(\mathbf{0}\,;r)$ gives the same TEV approximately.

Table 2.8: Qualitative Reconstruction of Area from FF-measurements

D	D_0	$\lvert B(\mathbf{0}\,;r)\rvert$	$\lvert D_0\rvert$	Percent Error
Disk $R = 1$	Disk	0.0328	0.0314	4.46%
	Square	0.0303	0.0300	1.00%
$[-1,1] \times [-1,1]$	Ellipse	0.0613	0.0628	2.39%
	Square	0.0749	0.1256	40.37%

The numerical experiments presented here are satisfactory but it is desirable to find a way to use more transmission eigenvalues in order to obtain addition information about voids, such as the number of voids and there locations.

Chapter 3

THE FACTORIZATION METHOD FOR A DEFECTIVE REGION IN AN ANISOTROPIC MATERIAL

In this chapter we consider the inverse acoustic scattering (in $\mathbb{R}^3$) or electromagnetic scattering (in $\mathbb{R}^2$, for the scalar TE-polarization case) problem of reconstructing the support of possibly multiple defective (possibly anisotropic) penetrable regions in a known anisotropic material of compact support. We develop the factorization method for the case of a non-absorbing anisotropic background media containing anisotropic (possibly absorbing) penetrable defects. In particular, under appropriate assumptions on the anisotropic material properties of the media we develop a rigorous characterization for the support of the defective regions from the given far field measurements. We present some numerical examples in the two dimensional case to demonstrate the feasibility of our reconstruction method including examples for the case when the defects are voids (i.e. subregions with refractive index the same as the background outside the inhomogeneous hosting media). The work on the factorization method that is presented here was published in the paper F. Cakoni and I. Harris "The factorization method for a defective region in an anisotropic material" *Inverse Problems*, **31** 025002 (2015). We remark that in Chapter 2 we showed how to detect the presence of voids from the knowledge of the first real transmission eigenvalue of the defective material. This is done without any computation of the Green's Function for the anisotropic background. As it will become clear later, to reconstruct the support of the defect one actually needs to solve the partial differential equations corresponding to the healthy material.

3.1 Formulation of the Inverse Problem

The inverse problem that we are interested in is to determine the shape and position of a defect in a known anisotropic material. It has been shown in chapter 2 that one can qualitatively obtain information about the size of the void(s) from the far field data. In this chapter we wish to use the theory in [55], [61] and [6] to develop a qualitative method to reconstruct the defective region. A similar problem was considered in [10] where it is assumed that the healthy material is homogeneous and isotropic with a sound-soft and/or sound-hard obstacles embedded in it. Also in [46] the factorization method was developed for the non-absorbing isotropic piecewise homogeneous media. The factorization method gives a rigorous characterization of the support of the defect in terms of the far field operator provided that the background is known hence providing also a uniqueness result. Note that for anisotropic defects the unique determination of the support is the best we can hope, since in general it is well known that the matrix-valued refractive index is not uniquely determined. We will construct a factorization of an operator $\tilde{F}$ to be defined later in terms of the measured data corresponding to a time-harmonic acoustic scattering (in $\mathbb{R}^3$) or electromagnetic scattering (in $\mathbb{R}^2$, for the scalar TE-polarization case) problem, and derive an indicator function for the support of the defect D_0 embedded in a non-absorbing scattering object with support D. This problem is motivated from non-destructive testing of an anisotropic material where one wants to detect the support of penetrable defective regions. For this complicated background, our analysis is based on variational formulation of the involved operators as oppose to integral operator formulation as in [10]. We note that, the factorization method for this configuration involves the computation of the far field pattern of Green's function for the inhomogeneous background media. However for the case of anisotropic homogeneous media we extend the result in [10] and provide a simple formula to compute the far field pattern of the background Green's function in terms of the total field due to the background. As a particular application of this study, we consider the determination of the support of voids inside a known anisotropic media.

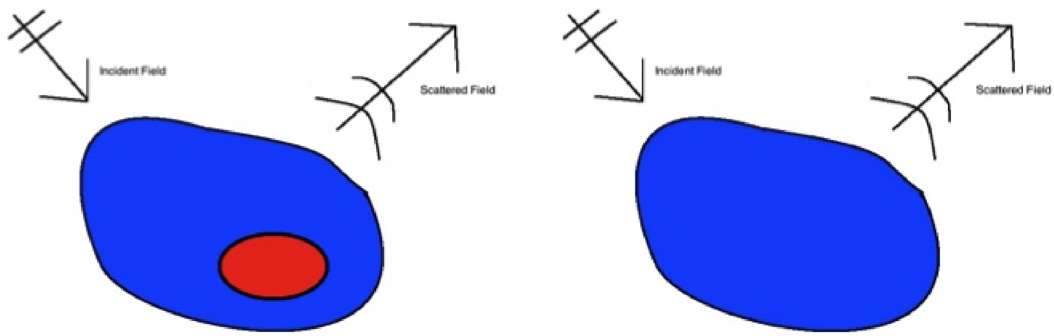

Figure 3.1: Example geometry of the scattering of a medium with and without a defective region.

To this end, let $D \subset \mathbb{R}^m$ be a bounded simply connected open set having piecewise smooth boundary ∂D with ν being the unit outward normal to the boundary. For convenience we use a slightly different notation than in the Introduction for the material parameters. We assume that the constitutive parameters of the media in D are represented by a real-valued symmetric matrix $\tilde{A} \in C^1\left(D, \mathbb{R}^{m\times m}\right)$ and a real valued function $\tilde{n} \in L^\infty(D)$ such that $\overline{\xi} \cdot \tilde{A}(x)\xi \geq a_{min}|\xi|^2 > 0$ and $\tilde{n}(x) \geq n_{min} > 0$ for almost all $x \in D$ and all $\xi \in \mathbb{C}^m$. Outside D the background media is homogeneous isotropic with refractive index scaled to one. We denote by A and n the constitutive parameters of the anisotropic background $\mathbb{R}^m$ given by

$$A(x) := \begin{cases} \tilde{A}(x) & x \in D \\ I & x \in \mathbb{R}^m \setminus \overline{D} \end{cases} \qquad n(x) := \begin{cases} \tilde{n}(x) & x \in D \\ 1 & x \in \mathbb{R}^m \setminus \overline{D} \end{cases}$$

where I is the identity matrix. Note that the support of $A - I$ and $n - 1$ is $\overline{D}$. Now the scattering of an incident plane wave $e^{ikx\cdot d}$, where d is a unit vector, by the "healthy" anisotropic material (i.e. without defects) is mathematically formulated as: find $u_b \in H^1_{loc}(\mathbb{R}^m)$ with $u_b = u_b^s + e^{ikx\cdot d}$ such that

$$\nabla \cdot A(x)\nabla u_b + k^2 n(x) u_b = 0 \quad \text{in} \quad \mathbb{R}^m \tag{3.1}$$

$$\lim_{r\to\infty} r^{\frac{m-1}{2}} \left(\frac{\partial u_b^s}{\partial r} - iku_b^s \right) = 0 \tag{3.2}$$

where the radiation condition (3.2) is satisfied uniformly with respect to $\hat{x} = x/|x|$. We recall that (3.1) implies that across the interface ∂D we have

$$\frac{\partial u_b^-}{\partial \nu_A} = \frac{\partial u_b^+}{\partial \nu} \qquad \text{on } \partial D$$

where the superscripts $+$ and $-$ for a generic function indicates the trace on the boundary taken from the exterior or interior of its surrounding domain, respectively. Here u_b is the total field in the background (including the homogeneous part and the anisotropic media of compact support D) and u_b^s is the scattered field due to the anisotropic region D of the background. As we have already seen the scattered field $u_b^s(\cdot, d)$ which depends on the incident direction d, and the corresponding far field pattern $u_b^\infty(\hat{x}, d)$, which depends on the incident direction d and observation direction $\hat{x}$. The far field pattern is given by the integral representation (1.10) with

u^s replaced by u_b^s. We now define the far field operator for the background scattering problem as $F_b : L^2(\mathbb{S}) \longmapsto L^2(\mathbb{S})$

$$(F_b g)(\hat{x}) := \int_{\mathbb{S}} u_b^\infty(\hat{x}, d) g(d)\, ds(d), \quad g \in L^2(\mathbb{S})$$

where $\mathbb{S} = \{x \in \mathbb{R}^m \,:\, |x| = 1\}$ is the unit circle or sphere. For later use we recall the scattering operator associated with this scattering problem, which plays an essential role in our factorization in the follow section.

Definition 3.1.1. *The scattering operator $\mathcal{S}_b : L^2(\mathbb{S}) \to L^2(\mathbb{S})$ for (3.1)-(3.2) is defined by*

$$\mathcal{S}_b = I + 2ik\gamma_m F_b. \tag{3.3}$$

Since A and n are real valued, the scattering operator is unitary, i.e. $\mathcal{S}_b\mathcal{S}_b^* = \mathcal{S}_b^*\mathcal{S}_b = I$ (see Theorem 7.32 in [14] in $\mathbb{R}^2$; and Theorem 1.1.4 in Introduction).

Next we assume that inside the anisotropic material D there is a defect (possibly anisotropic and/or absorbing) occupying the subregion D_0 such that $\overline{D}_0 \subset D$ having piecewise smooth boundary ∂D_0 (see Figure 3.1). Note that D_0 can be of multiple components with connected complement. We denote by $\tilde{A}_0$ and $\tilde{n}_0$ the material properties of the medium in D_0. We further assume that the symmetric matrix-valued function $\tilde{A}_0$ is such that $\tilde{A}_0 \in C^1\left(D_0, \mathbb{C}^{m\times m}\right)$, $\xi \cdot \Re(\tilde{A}_0(x))\xi \geq \alpha_0|\xi|^2$, $\xi \cdot \Im(\tilde{A}_0(x))\xi \leq 0$ for all $\xi \in \mathbb{C}^m$ and for all $x \in D_0$, whereas the scalar-valued function $\tilde{n}_0$ is such that $\tilde{n}_0 \in L^\infty(D_0)$, $\Re(\tilde{n}_0(x)) \geq c_0 > 0$ and $\Im(\tilde{n}_0(x)) \geq 0$ for all

$x \in D_0$. Let us denote by A_0 and n_0 the extensions

$$A_0(x) := \begin{cases} \tilde{A}_0(x) & x \in D_0 \\ A & x \in \mathbb{R}^m \setminus \overline{D}_0 \end{cases} \qquad n_0(x) := \begin{cases} \tilde{n}_0(x) & x \in D_0 \\ n & x \in \mathbb{R}^m \setminus \overline{D}_0 \end{cases}.$$

Obviously, $A_0(x)$ and $n_0(x)$ are such that $A - A_0$ and $n - n_0$ are supported on $\overline{D}_0$. Notice that a specific case of a defect is a void with $\tilde{A}_0 = I$ and $\tilde{n}_0 = 1$. The scattering problem for the anisotropic media with the defective region D_0 now reads: find $u_0 \in H^1_{loc}(\mathbb{R}^m)$ with $u_0 = u_0^s + e^{ikx\cdot d}$ such that

$$\nabla \cdot A_0(x)\nabla u_0 + k^2 n_0(x) u_0 = 0 \quad \text{in} \quad \mathbb{R}^m \tag{3.4}$$

$$\lim_{r\to\infty} r^{\frac{m-1}{2}} \left(\frac{\partial u_0^s}{\partial r} - iku_0^s \right) = 0 \tag{3.5}$$

where again the radiation condition (3.2) is satisfied uniformly with respect to $\hat{x} = x/|x|$. Once again we recall that across the interfaces ∂D and ∂D_0 we have that

$$\frac{\partial u_0^-}{\partial \nu_A} = \frac{\partial u_0^+}{\partial \nu} \quad \text{on } \partial D \qquad \frac{\partial u_0^-}{\partial \nu_{A_0}} = \frac{\partial u_0^+}{\partial \nu_A} \quad \text{on } \partial D_0.$$

Similarly since u_0^s is a radiating solution to the Helmholtz equation in $\mathbb{R}^m \setminus \overline{D}$, we have that its corresponding far field pattern $u_0^\infty(\hat{x}, d)$ is given by (1.10) where u^s is replaced with u_0^s. The far field operator $F_0 : L^2(\mathbb{S}) \longmapsto L^2(\mathbb{S})$ for the defective anisotropic media is now defined by

$$(F_0 g)(\hat{x}) := \int_{\mathbb{S}} u_0^\infty(\hat{x}, d) g(d)\, ds(d) \quad \text{where} \quad g(d) \in L^2(\mathbb{S}).$$

The *inverse problem* we consider here is to determine the support of D_0 from a knowledge of F_0, i.e. from a knowledge of the measured far field pattern $u_0^\infty(\hat{x}, d)$ for all $d, \hat{x} \in \mathbb{S}$, provided that A, n and D are known.

One can see that, if we take the incident field in (3.4)-(3.5) to be $u_b(\cdot, d) = u_b^s + e^{ikx\cdot d}$ then the resulting scattered field $u^s = u_0^s - u_b^s$ is due to the defect D_0. Note that the scattered field u^s due to the incident field $u_b(\cdot, d) = u_b^s + e^{ikx\cdot d}$ satisfies the source problem

$$\nabla \cdot A_0 \nabla u^s + k^2 n_0 u^s = \nabla \cdot (A - A_0)\nabla u_b + k^2(n - n_0)u_b \quad \text{in } \mathbb{R}^m. \tag{3.6}$$

together with the Sommerfeld radiation condition, which coincides with the equation for $u_0^s - u_b^s$ by linearity and (3.1) and (3.4). Therefore the relative far-field operator associated with the scattered field due to the defect is given by

$$(Fg)(\hat{x}) := \int_{\mathbb{S}} \left[u_0^\infty(\hat{x}, d) - u_b^\infty(\hat{x}, d) \right] g(d)\, ds(d) \quad \text{where} \quad g(d) \in L^2(\mathbb{S}),$$

which is $F = F_0 - F_b$. Note that F_0 is what we measure and F_b is computable since A, n and D are known, hence we can assume that we know F.

Remark 3.1.1. *The smoothness of the coefficients A_0 and A in our analysis can be relaxed to e.g. Lipshitz continuous or as regular as it is needed to apply unique continuation to the solution of the direct scattering problems.*

3.2 Factorization of the Far Field Operator

Our goal in the current section is to construct a factorization of the relative far field operator $F = F_0 - F_b$ in such a way as to use the factorization method in [61], [55], in order to develop a range test for the support D_0 of the defect in terms of the measured far field operator. To this end motivated by the expression (3.6) for the scattered field due to the defect, we consider the problem of finding $u \in H^1_{loc}(\mathbb{R}^m)$ for a given $v \in H^1(D_0)$ such that

$$\nabla \cdot A_0 \nabla u + k^2 n_0 u = \nabla \cdot (A - A_0)\nabla v + k^2 (n - n_0) v \quad \text{in } \mathbb{R}^m \tag{3.7}$$
$$\lim_{r\to\infty} r^{\frac{m-1}{2}} \left(\frac{\partial u}{\partial r} - iku \right) = 0.$$

At this point let us recall the exterior Dirichlet-to-Neumann(DtN) operator $\mathbb{T}_k : H^{1/2}(\partial B_R) \mapsto H^{-1/2}(\partial B_R)$ given by $\mathbb{T}_k f = \frac{\partial \varphi}{\partial \nu}$ on ∂B_R where

$$\Delta\varphi + k^2\varphi = 0 \qquad \text{in } \mathbb{R}^m \setminus \overline{B}_R$$
$$\varphi = f \qquad \text{on } \partial B_R$$
$$\lim_{r\to\infty} r^{\frac{m-1}{2}} \left(\frac{\partial \varphi}{\partial r} - ik\varphi \right) = 0$$

(see the Introduction for properties of the DtN operator) with $B_R = \{x \in \mathbb{R}^m : |x| < R\}$. With help of DtN operator we can write (3.7) in the following equivalent variational form: find $u \in H^1(B_R)$ such that

$$\int_{B_R} A_0 \nabla u \cdot \nabla \overline{\varphi} - k^2 n_0 u \overline{\varphi}\, dx - \int_{\partial B_R} \mathbb{T}_k u \, \overline{\varphi}\, ds =$$
$$\int_{D_0} (A - A_0) \nabla v \cdot \nabla \overline{\varphi} - k^2 (n - n_0) v \overline{\varphi}\, dx, \quad \forall \varphi \in H^1(B_R) \tag{3.8}$$

which will be used frequently in what follows. It is standard to shown that the above problem is well-posed (see discussion in the Introduction), and furthermore if $v = u_b|_{D_0}$ we see that the scattered field $u^s = u_0^s - u_b^s$ (where u_b^s and u_0^s are the scattered fields for (3.1)-(3.2) and (3.4)-(3.5), respectively) must coincide with u

given by (3.7). We now define the source-to-far field pattern operator as

$$G : H^1(D_0) \longmapsto L^2(\mathbb{S}) \quad \text{given by} \quad Gv := u^\infty.$$

In addition, let us define

$$v_g^b(x) := \int_{\mathbb{S}} u_b(x,d) g(d)\, ds(d), \qquad\qquad g \in L^2(\mathbb{S}), \;\; x \in \mathbb{R}^m$$

where $u_b(x,d) = u_b^s(x,d) + e^{ikx\cdot d}$ solves (3.1)-(3.2) and consider the bounded linear operator

$$H : L^2(\mathbb{S}) \mapsto H^1(D_0), \quad \text{defined by} \quad Hg := v_g^b|_{D_0}.$$

Obviously $F = GH$. To further factorize the operator F we first need to compute the adjoint $H^* : H^1(D_0) \to L^2(\mathbb{S})$ of the operator H defined above.

Lemma 3.2.1. *The operator $H^* : H^1(D_0) \longmapsto L^2(\mathbb{S})$ is given by*

$$-\gamma_m H^* v = \mathcal{S}_b^* \tilde{v}^\infty$$

where $\tilde{v}^\infty$ is the far field pattern of the radiating field $\tilde{v} \in H^1_{loc}(\mathbb{R}^m)$ satisfying

$$-\int\limits_{B_R} A\nabla\tilde{v}\cdot\nabla\overline{\varphi} - k^2 n\tilde{v}\overline{\varphi}\, dx + \int\limits_{\partial B_R} \overline{\varphi}\mathbb{T}_k\tilde{v}\, ds = (v,\varphi)_{H^1(D_0)}, \quad \forall\varphi \in H^1(B_R). \tag{3.9}$$

Proof. Let $v \in H^1(D_0)$ be given then we can construct a unique radiating field $\tilde{v} \in H^1_{loc}(\mathbb{R}^m)$ that satisfies (3.9) (see Chapter 5 of [14] and Introduction). Now we have that integration by parts gives

$$
\begin{aligned}
(H^*v, g)_{L^2(\mathbb{S})} &= (v, Hg)_{H^1(D_0)} \\
&= -\int\limits_{B_R} \left(A\nabla\tilde{v} \cdot \nabla\overline{v_g^b} - k^2 n\tilde{v}\overline{v_g^b} \right) \, dx + \int\limits_{\partial B_R} \overline{v_g^b} \mathbb{T}_k \tilde{v} \, ds \\
&= \int\limits_{\partial B_R} \left(\overline{v_g^b} \frac{\partial \tilde{v}}{\partial \nu} - \tilde{v} \frac{\partial \overline{v_g^b}}{\partial \nu} \right) \, ds + \int\limits_{B_R} \tilde{v}(\nabla \cdot A\nabla\overline{v_g^b} + k^2 n\overline{v_g^b}) \, dx
\end{aligned}
$$

where we recall that $v_g^b(x) = \int_{\mathbb{S}} \left(u_b^s(x,d) + e^{ikx\cdot d} \right) g(d) \, ds(d)$ for all of $x \in \mathbb{R}^m$. Using that the matrix A is real symmetric along with $\Im\{n(x)\} = 0$ and that

$$
\nabla \cdot A\nabla v_g^b + k^2 n v_g^b = 0 \quad \text{in} \quad \mathbb{R}^m
$$

gives that the integral over B_R is zero. Now by using the definition of v_g^b and changing the order of integration we have that

$$
\begin{aligned}
(H^*v, g)_{L^2(\mathbb{S})} &= \int\limits_{\partial B_R} \left(\overline{v_g^b} \frac{\partial \tilde{v}}{\partial \nu} - \tilde{v} \frac{\partial \overline{v_g^b}}{\partial \nu} \right) \, ds_x \\
&= \int\limits_{\mathbb{S}} \overline{g(d)} \left[\int\limits_{\partial B_R} \left(\frac{\partial \tilde{v}}{\partial \nu} e^{-ikx\cdot d} - \tilde{v} \frac{\partial e^{-ikx\cdot d}}{\partial \nu} \, ds_x \right) \right] ds(d) \\
&\quad + \int\limits_{\mathbb{S}} \overline{g(d)} \left[\int\limits_{\partial B_R} \left(\overline{u_b^s(x,d)} \frac{\partial \tilde{v}}{\partial \nu} - \tilde{v} \frac{\partial \overline{u_b^s(x,d)}}{\partial \nu} \right) \, ds_x \right] ds(d). \quad (3.10)
\end{aligned}
$$

We notice that (1.10) gives that

$$
\int\limits_{\mathbb{S}} \overline{g(d)} \left[\int\limits_{\partial B_R} \left(\frac{\partial \tilde{v}}{\partial \nu} e^{-ikx\cdot d} - \tilde{v} \frac{\partial e^{-ikx\cdot d}}{\partial \nu} \right) \, ds_x \right] ds(d) = -\frac{1}{\gamma_m} (\tilde{v}^\infty, g)_{L^2(\mathbb{S})}. \quad (3.11)
$$

Using the asymptotic behavior of a radiating solution to Helmholtz equation and its derivative (see [14] for the case of $\mathbb{R}^2$ and [37] for the case of $\mathbb{R}^3$) and letting $R \to \infty$ the second integral in (3.10) becomes

$$\int_{\mathbb{S}} \overline{g(d)} \left[2ik \int_{\mathbb{S}} \tilde{v}^\infty(\hat{x}) \overline{u_b^\infty(\hat{x}, d)}\, ds(\hat{x}) \right] ds(d).$$

Using the reciprocity identity $u_b^\infty(\hat{x}, d) = u_b^\infty(-d, -\hat{x})$ (see Theorem 1.1.2) and making the change of variables $\hat{x} \longmapsto -\hat{x}$ we obtain

$$\int_{\mathbb{S}} \overline{g(d)} \left[2ik \int_{\mathbb{S}} \tilde{v}^\infty(-\hat{x}) \overline{u_b^\infty(-d, \hat{x})}\, ds(\hat{x}) \right] ds(d) = 2ik (F_b^* \tilde{v}^\infty, g)_{L^2(\mathbb{S})}. \tag{3.12}$$

Finally, combining (3.11) and (3.12) we have that

$$H^* v = \left(-\frac{1}{\gamma_m} I + 2ik F_b^* \right) \tilde{v}^\infty$$

giving the result by multiplying by $-\gamma_m$ and by Definition 3.1.1. $\square$

Now for any given $\phi \in H^1(D_0)$ we can construct a function $w_\phi \in H^1_{loc}(\mathbb{R}^m)$ that satisfies

$$\nabla \cdot A \nabla w_\phi + k^2 n w_\phi = \nabla \cdot (A - A_0) \nabla \phi + k^2 (n - n_0) \phi \ \text{ in } \ \mathbb{R}^m \tag{3.13}$$
$$\lim_{r \to \infty} r^{\frac{m-1}{2}} \left(\frac{\partial w_\phi}{\partial r} - ik w_\phi \right) = 0$$

and then let $u \in H^1_{loc}(\mathbb{R}^m)$ be the unique solution to (3.7) for a given $v \in H^1(D_0)$. Now by letting $\phi = v + u|_{D_0}$ and the corresponding $w := w_\phi$, we observe that this w

satisfies the variational problem

$$
\begin{aligned}
-\int\limits_{B_R} A\nabla w\cdot\nabla\overline{\varphi} - k^2 n w\overline{\varphi}\,dx + \int\limits_{\partial B_R} \overline{\varphi}\mathbb{T}_k w\,ds = -\int\limits_{D_0} (A-A_0)\nabla(v+u)\cdot\nabla\overline{\varphi}\,dx \\
+k^2\int\limits_{D_0}(n-n_0)(v+u)\overline{\varphi}\,dx \qquad \forall\varphi\in H^1(B_R).
\end{aligned}
$$

Next by means of Riesz representation theorem, we define the bounded linear operator $T : H^1(D_0) \longmapsto H^1(D_0)$ such that for all $\varphi \in H^1(D_0)$

$$
(Tv,\varphi)_{H^1(D_0)} = -\int\limits_{D_0}(A-A_0)\nabla(v+u)\cdot\nabla\overline{\varphi}\,dx + \int\limits_{D_0} k^2(n-n_0)(v+u)\overline{\varphi}\,dx. \quad (3.14)
$$

Notice that the function u defined by solving (3.7) satisfies

$$
\nabla\cdot A\nabla u + k^2 n u = \nabla\cdot(A-A_0)\nabla(v+u) + k^2(n-n_0)(v+u) \text{ in } \mathbb{R}^m \qquad (3.15)
$$

together with the Sommerfeld radiation condition, which gives that $u = w$ in $\mathbb{R}^m$ since (3.13) is well-posed. Therefore we conclude that $u^\infty = w^\infty$. Now by the definition of the operators G we have that $u^\infty = Gv$ while using the definition of H^* and T we have that $w^\infty = -\gamma_m\mathcal{S}_b H^* T v$. We now conclude that $Gv = -\gamma_m\mathcal{S}_b H^* T v$. From the above analysis and the fact that $F = GH$ we have the following factorization.

Theorem 3.2.1. *The far field operator $F : L^2(\mathbb{S}) \longmapsto L^2(\mathbb{S})$ associated with (3.7) can be factorized as $F = -\gamma_m\mathcal{S}_b H^* T H$.*

3.3 The Factorization Method

In this section we connect the support of the defect D_0 to the range of an operator defined by the measured far field operator based on the factorization method discussed in [55] or [61]. We make this connection by analyzing the factorization of the far field operator developed in the previous section. Defining $\tilde{F} := \gamma_m^{-1}\mathcal{S}_b^* F$, we recall from the previous section that we have the following factorization $\tilde{F} = -H^*TH$. Under appropriate assumptions on the operators H and T the factorization method states that the range of the operators $H^* : H^1(D_0) \mapsto L^2(\mathbb{S})$ and $\tilde{F}_\sharp^{1/2} : L^2(\mathbb{S}) \mapsto L^2(\mathbb{S})$ coincide, where $\tilde{F}_\sharp = |\Re(\tilde{F})| + |\Im(\tilde{F})|$.

To arrive at the above range test we use the abstract theorems proven in [55] and [61] on the range identities. To this end, we recall that for a generic bounded linear operator $B : X \to Y$, where X and Y are Banach spaces, we define the real and imaginary part selfajoint operators by

$$\Re(B) = \frac{B + B^*}{2} \quad \text{and} \quad \Im(B) = \frac{B - B^*}{2i}.$$

Furthermore for a generic self-adjoint compact operator B on a Hilbert space U, $|B|$ is defined in terms of the spectral decomposition as $|B|x = \sum |\lambda_j|(x, \psi_j)\psi_j$ for all $x \in U$ where $(\lambda_j, \psi_j) \in \mathbb{R} \times U$ is the orthonormal eigensystem of B. Now, let $X \subset U \subset X^*$ be a Gelfand triple with a Hilbert space U and a reflexive Banach space X such that the embedding is dense. Furthermore, let Y be a second Hilbert space and let $\tilde{F} : Y \mapsto Y$, $H : Y \mapsto X$ and $T : X \mapsto X^*$ be linear bounded operators such that $\tilde{F} = H^*TH$. Depending on the properties of the operators H and T we will use either one of the following abstract theorems.

Theorem 3.3.1. *(Theorem 2.15 in [55]) Assume that*

1. *H^* is compact with dense range.*
2. *There exists $t \in [0, 2\pi]$ such that $\Re(e^{it}T)$ is the sum of a compact operator and a self adjoint coercive operator.*
3. *$\Im(T)$ is compact and non-negative on the range $\mathcal{R}(H)$ of H.*

4. $\Re(e^{it}T)$ *is injective or* $\Im(T)$ *is strictly positive on the closure* $\overline{\mathcal{R}(H)}$.

Then the operator $\tilde{F}_\sharp = |\Re(e^{it}\tilde{F})| + \Im(\tilde{F})$ *is positive, and the range of the operators* $H^* : X^* \mapsto Y$ *and* $\tilde{F}_\sharp^{1/2} : Y \mapsto Y$ *coincide.*

Theorem 3.3.2. *(Theorem 2.1 in [61]) Assume that*

1. *H is compact and injective.*

2. $\Re(T)$ *is the sum of a compact operator and a self adjoint coercive operator.*

3. $\Im(T)$ *is non-negative on* X.

Moreover assume that either of the following is satisfied:

4. *T is injective.*

5. $\Im(T)$ *is strictly positive on the (finite dimensional) null space of* $\Re(T)$.

Then the operator $\tilde{F}_\sharp = |\Re(\tilde{F})| + \Im(\tilde{F})$ *is positive, and the range of the operators* $H^* : X^* \mapsto Y$ *and* $\tilde{F}_\sharp^{1/2} : Y \mapsto Y$ *coincide.*

We note that just as in the remark after Theorem 2.15 in [55] we have that if $\Im(T)$ is non-positive then both theorems hold for $\tilde{F}_\sharp = |\Re(\tilde{F})| - \Im(\tilde{F})$, hence in either case we can use $|\Re(\tilde{F})| + |\Im(\tilde{F})|$ in the range test.

We dedicate this section to showing that H and T satisfy the necessary conditions to apply any of the above range tests. To this end, let's define the interior transmission eigenvalue problem in the defective region D_0 as finding a pair $(w, v) \in H^1(D_0) \times H^1(D_0)$ such that for given $(f, h) \in H^{1/2}(\partial D_0) \times H^{-1/2}(\partial D_0)$ satisfies

$$\nabla \cdot A_0 \nabla w + k^2 n_0 w = 0 \quad \text{and} \quad \nabla \cdot A \nabla v + k^2 n v = 0 \quad \text{in} \quad D_0 \tag{3.16}$$

$$w - v = f \quad \text{and} \quad \frac{\partial w}{\partial \nu_{A_0}} - \frac{\partial v}{\partial \nu_A} = h \quad \text{on} \quad \partial D_0 \tag{3.17}$$

Definition 3.3.1. The values of $k \in \mathbb{R}$ for which the homogeneous interior transmission problem, i.e. (3.16)-(3.17) with $(f, h) = (0, 0)$, has nontrivial solutions are called transmission eigenvalues for D_0.

The following results are know if $A_0 = I$ and $n_0 = 1$. The proofs can be readily extended to the current case. We state the results and give the corresponding reference for the proof in the case of $A_0 = I$ and $n_0 = 1$.

Theorem 3.3.3. *Assume that $A - \Re(A_0)$ is positive definite or negative definite. Then (3.16)-(3.17) satisfies the Fredholm alternative, i.e if k is not a transmission eigenvalue there exits a unique solution to (3.16)-(3.17) that depends continuously on the data (f, h).*

See [14] for the proof.

Theorem 3.3.4.

1. *If $\Im(A_0) < 0$ and/or $\Im(n_0) > 0$ in D_0 then there are no real transmission eigenvalues.*
2. *Assume that $\Im(A_0) = 0$ and $\Im(n_0) = 0$. Then the set of real transmission eigenvalues is at most discrete with $+\infty$ as the only possible accumulation point provided:*
   (a) $A - A_0$ is positive or negative definite uniformly in D_0 and $\int_{D_0} (n - n_0)\, dx \neq 0$,
   (b) $A - A_0$ is positive or negative definite uniformly in D_0 and $n \equiv n_0$.

See [14], Chapter 6 for the proof of parts (i) and $(ii)(b)$, and [11] for the proof of part $(ii)(a)$.

Assumption 3.3.1. *The wavenumber k is not a transmission eigenvalues for D_0.*

We call $\mathbb{G}(\cdot,\cdot)$ the Green's function of the background media, i.e. $\mathbb{G}(\cdot,z) \in H^1_{loc}(\mathbb{R}^m \setminus \{z\})$ which solves

$$\nabla \cdot A\nabla \mathbb{G}(\cdot,z) + k^2 n \mathbb{G}(\cdot,z) = -\delta(\,\cdot\, - z) \;\text{ in } \mathbb{R}^m \setminus \{z\}$$
$$\lim_{r\to\infty} r^{\frac{m-1}{2}} \left(\frac{\partial \mathbb{G}(\cdot,z)}{\partial r} - ik\mathbb{G}(\cdot,z) \right) = 0$$

Outside of the scattering object D we have that, for a fixed $z \in \mathbb{R}^m$, $\mathbb{G}(\,\cdot\,,z)$ is a radiating solution to Helmholtz equation in $\mathbb{R}^m \setminus \overline{B}_R$ for some R sufficiently large. So we let $\mathbb{G}^\infty(\,\cdot\,,z) \in L^2(\mathbb{S})$ be the far field pattern of $\mathbb{G}(\,\cdot\,,z)$.

Theorem 3.3.5. *The operator $H^* : H^1(D_0) \longmapsto L^2(\mathbb{S})$ defined in Lemma 3.2.1 satisfies the following:*

1. *H^* is compact with dense range (or in other words H is compact and injective).*
2. *$\mathcal{S}_b^* \mathbb{G}^\infty(\cdot,z) \in \mathcal{R}(H^*)$ if and only if $z \in D_0$.*

Proof. (i) H^* is compact due to the fact that the mapping $v \mapsto \tilde{v}$ is bounded from $H^1(D_0)$ to $H^1_{loc}(\mathbb{R}^m)$ and $\tilde{v} \mapsto \tilde{v}^\infty$ is compact from $H^1_{loc}(\mathbb{R}^m)$ to $L^2(\mathbb{S})$. We have also used that the scattering operator is bounded. Now to prove that H^* has dense range it is sufficient to prove that H is injective. So assume that $g \in L^2(\mathbb{S})$ is such that $Hg = 0$, then v_g^b defined by $v_g^b := \displaystyle\int_{\mathbb{S}} u_b(x,d)g(d)\,ds(d)$ for all $x \in \mathbb{R}^m$ therefore we have that $v_g^b = 0$ in D_0. Therefore since v_g^b satisfies

$$\nabla \cdot A\nabla v_g^b + k^2 n v_g^b = 0 \quad \text{in} \quad \mathbb{R}^m$$

we have that, by a unique continuation argument, $v_g^b = 0$ in any large ball B_R of arbitrary radius R. But $v_g^b = u_g^s + u_g^i$ where

$$u_g^s = \int_{\mathbb{S}} u_b^s(x,d)g(d)\,ds(d) \quad \text{and} \quad u_g^i = \int_{\mathbb{S}} e^{ikx\cdot d} g(d)\,ds(d).$$

We now observe that u_g^s is a radiating solution to the Helmholtz equation whereas u_g^i is an entire solution to the Helmholtz equation. Hence $u_g^s = u_g^i = 0$ in $\mathbb{R}^m$ which implies $g = 0$ in $L^2(\mathbb{S})$.

(ii) Let $z \in \mathbb{R}^m \setminus \overline{D}_0$ and assume that there is some $v \in H^1(D_0)$ such that $H^*v = \mathcal{S}_b^*\mathbb{G}^\infty(\cdot, z)$. We can then conclude by the definition of H^* that there is a $\tilde{v} \in H^1_{loc}(\mathbb{R}^m)$ satisfying (3.9) and therefore $\nabla \cdot A\nabla\tilde{v} + k^2 n\tilde{v} = 0$ in $\mathbb{R}^m \setminus \overline{D}_0$ and $\tilde{v}^\infty = \mathbb{G}^\infty(\cdot, z)$. Therefore Rellich's lemma and unique continuation gives that $\tilde{v} = \mathbb{G}(\cdot, z)$ in $\mathbb{R}^m \setminus (\overline{D}_0 \cup \{z\})$, which is a contradiction since $|\mathbb{G}(x, z)| \to \infty$ as $x \to z$ and $|\tilde{v}(x)| = \mathcal{O}(1)$ as $x \to z$.

Now let $z \in D_0$ then we have that $\mathbb{G}(\cdot, z) \in H^1_{loc}(\mathbb{R}^m \setminus \overline{D}_0)$. Since k is not a transmission eigenvalue in D_0 we can construct (w_z, v_z) that solve the interior transmission problem (3.16)-(3.17) with $(f, h) = \left(\mathbb{G}(\cdot\,, z)\,,\, \frac{\partial}{\partial \nu_A}\mathbb{G}(\cdot\,, z)\right)$. Now let

$$u_z := \begin{cases} w_z - v_z & \text{in} \quad D_0 \\ \mathbb{G}(\cdot\,, z) & \text{in} \quad \mathbb{R}^m \setminus \overline{D}_0 \end{cases}$$

therefore we have that $u_z \in H^1_{loc}(\mathbb{R}^m)$ with $u_z^\infty = \mathbb{G}^\infty(\cdot, z)$ such that

$$\nabla \cdot A\nabla u_z + k^2 n u_z = \nabla \cdot (A - A_0)\nabla w_z + k^2(n - n_0)w_z \;\text{ in }\; \mathbb{R}^m.$$

The latter implies that for all $\varphi \in H^1(B_R)$

$$-\int\limits_{B_R} A\nabla u_z \cdot \nabla\overline{\varphi} - k^2 n u_z \overline{\varphi}\, dx + \int\limits_{\partial B_R} \overline{\varphi}\mathbb{T}_k u_z\, ds = \\ -\int\limits_{D_0} (A - A_0)\nabla w_z \cdot \nabla\overline{\varphi}\, dx + \int\limits_{D_0} k^2(n - n_0)w_z\overline{\varphi}\, dx. \qquad (3.18)$$

Let $\theta_z \in H^1(D_0)$ be defined from the right hand side of (3.18) by means of the Riesz representation theorem, hence we have

$$-\int\limits_{B_R} A\nabla u_z \cdot \nabla\overline{\varphi} - k^2 n u_z \overline{\varphi}\, dx + \int\limits_{\partial B_R} \overline{\varphi}\mathbb{T}_k u_z\, ds = (\theta_z, \varphi)_{H^1(D_0)}.$$

Thus we now conclude that $-\gamma_m H^* \theta_z = \mathcal{S}_b^* \mathbb{G}^\infty(\cdot, z)$ by the definition of H^* giving the result. □

Next we analyze the properties of the middle operator T defined by (3.14).

Theorem 3.3.6. *The operator $T : H^1(D_0) \longmapsto H^1(D_0)$ is injective provided that either one of the following conditions are satisfied:*

1. *$\Im(A_0) < 0$ in D_0 and $\int_D (n - n_0)\, dx \neq 0$.*
2. *$\Im(A_0) \leq 0$ and $\Im(n_0) > 0$ in D_0.*
3. *$\Im(A_0) = 0$, $\Im(n_0) = 0$ in D_0 and either $A - A_0 > 0$ and $n - n_0 < 0$ or $A - A_0 < 0$ and $n - n_0 > 0$ in D_0.*

Proof. Assume that $Tv = 0$, therefore $u \in H^1_{loc}(\mathbb{R}^m)$ defined by solving (3.7) satisfies for all $\varphi \in H^1(B_R)$

$$-\int\limits_{B_R} A\nabla u \cdot \nabla\overline{\varphi} - k^2 n u \overline{\varphi}\, dx + \int\limits_{\partial B_R} \overline{\varphi}\mathbb{T}_k u\, ds = (Tv, \varphi)_{H^1(D_0)} = 0.$$

which implies that $u = 0$ and therefore we have that for all $\varphi \in H^1(D_0)$

$$\int\limits_{D_0} (A - A_0)\nabla v \cdot \nabla\overline{\varphi} - k^2 (n - n_0) v \overline{\varphi}\, dx = 0 \tag{3.19}$$

Letting $\varphi := v$, parts (i) and (ii) of the proof follow by taking the imaginary part of (3.19) (note that $\Im(n_0) \geq 0$) whereas part (iii) is obvious from the assumptions. □

Theorem 3.3.7. *The imaginary part of the operator $T : H^1(D_0) \longmapsto H^1(D_0)$ satisfies the following properties:*

1. $(\Im(T)v, v)_{H^1(D_0)} \leq 0$.
2. *If k is not a transmission eigenvalue for D_0 then $(\Im(T)v, v)_{H^1(D_0)} < 0$ for $v \in \overline{\mathcal{R}(H)}$.*
3. *If $\Im(A_0) = 0$ then $\Im(T)$ is compact.*

Proof. (i) Recall that for any $v_j \in H^1(D_0)$ there is a unique $u_j \in H^1_{loc}(\mathbb{R}^m)$ that is a solution to (3.7). Now we let $\phi_j = v_j + u_j$, therefore using (3.14) we have that

$$(Tv_1, v_2)_{H^1(D_0)} = -\int_{D_0} (A - A_0)\nabla\phi_1 \cdot \nabla\overline{(\phi_2 - u_2)} - k^2(n - n_0)\phi_1\overline{(\phi_2 - u_2)}\, dx$$
$$= -\int_{D_0} (A - A_0)\nabla\phi_1 \cdot \nabla\overline{\phi_2} - k^2(n - n_0)\phi_1\overline{\phi_2}\, dx$$
$$+ \int_{D_0} (A - A_0)\nabla\phi_1 \cdot \nabla\overline{u_2} - k^2(n - n_0)\phi_1\overline{u_2}\, dx.$$

Now using that

$$\nabla \cdot A\nabla u_1 + k^2 n u_1 = \nabla \cdot (A - A_0)\nabla\phi_1 + k^2(n - n_0)\phi_1 \quad \text{in} \quad \mathbb{R}^m$$

multiplying by $\overline{u_2}$ and integrating by parts over B_R such that $D \subset B_R$ we have that

$$-\int_{B_R} A\nabla u_1 \cdot \nabla\overline{u_2} - k^2 n u_1\overline{u_2}\, dx + \int_{\partial B_R} \overline{u_2}\frac{\partial u_1}{\partial \nu}\, ds = -\int_{D_0} (A - A_0)\nabla\phi_1 \cdot \nabla\overline{u_2}\, dx$$
$$+k^2 \int_{D_0} (n - n_0)\phi_1\overline{u_2}\, dx.$$

This gives that

$$(Tv_1, v_2)_{H^1(D_0)} = -\int\limits_{D_0} (A - A_0)\nabla\phi_1 \cdot \nabla\overline{\phi_2} - k^2(n - n_0)\phi_1\overline{\phi_2}\, dx + \int\limits_{B_R} A\nabla u_1 \cdot \nabla\overline{u_2} - k^2 n u_1 \overline{u_2}\, dx - \int\limits_{\partial B_R} \overline{u_2}\frac{\partial u_1}{\partial \nu}\, ds. \tag{3.20}$$

Now taking the imaginary part of (3.20) where we substitute v_2 by v_1, using the fact that A and A_0 are symmetric matrices, A and n are real valued and letting $R \to \infty$ we obtain

$$(\Im(T)v_1, v_1)_{H^1(D_0)} = \int\limits_{D_0} \Im(A_0)|\nabla\phi_1|^2 - k^2\Im(n_0)|\phi_1|^2\, dx - k\int\limits_{\mathbb{S}} |u_1^\infty|^2\, ds(\hat{x}) \tag{3.21}$$

where u_1^∞ is defined by the asymptotic expansion of the radiating field u_1

$$u_1(x) = \frac{e^{ikr}}{r^{(m-1)/2}} u_1^\infty(\hat{x}) + O\left(r^{-(m+1)/2}\right), \qquad r = |x|, \;\; \hat{x} = x/|x|,$$

(see [37] in $\mathbb{R}^3$ and [14] in $\mathbb{R}^2$), which gives that $\Im(T)$ is non-positive.

(ii) Now let $v \in \overline{\mathcal{R}(H)}$ and assume that $(\Im(T)v, v)_{H^1(D_0)} = 0$. Then there is a sequence $v_\ell \in \mathcal{R}(H)$ such that $v_\ell \to v$ in $H^1(D_0)$, and let $u_\ell \in H^1_{loc}(\mathbb{R}^m)$ be the sequence of the corresponding solutions of (3.7). Since u_ℓ is bounded in $H^1_{loc}(\mathbb{R}^m)$ by the well-posedness of (3.7), we can conclude that $u_\ell \rightharpoonup u$ weakly in $H^1_{loc}(\mathbb{R}^m)$ which implies that

$$\lim_{\ell\to\infty} \int\limits_{\mathbb{R}^m} A_0\nabla u_\ell \cdot \nabla\overline{\varphi} - k^2 n_0 u_\ell \overline{\varphi}\, dx = \int\limits_{\mathbb{R}^m} A_0\nabla u \cdot \nabla\overline{\varphi} - k^2 n_0 u\overline{\varphi}\, dx, \qquad \varphi \in H^1(\mathbb{R}^m).$$

Hence, this limit u is a week solution of

$$\nabla \cdot A_0 \nabla u + k^2 n_0 u = \nabla \cdot (A - A_0)\nabla v + k^2 (n - n_0) v \quad \text{in } \mathbb{R}^m$$
$$\nabla \cdot A \nabla v + k^2 n v = 0 \quad \text{in } D_0.$$

Furthermore, since $(\Im(T)v_\ell, v_\ell)_{H^1(D_0)} \to 0$, from (3.21) we conclude that $u^\infty = 0$ whence by Rellich' lemma and unique continuation u is zero outside of D_0. So we have that $u^+ = 0$ and $\frac{\partial u^+}{\partial \nu_A} = 0$ on ∂D_0 therefore the pair $(u + v, v)$ are transmission eigenfunctions for D_0 but since k is not a transmission eigenvalue we have that $v = 0$.

(iii) If $\Im(A_0) = 0$ then

$$(-\Im(T)v_1, v_2)_{H^1(D_0)} = \int_{D_0} k^2 \Im(n_0) \phi_1 \overline{\phi_2}\, dx + k \int_{\mathbb{S}} u_1^\infty \overline{u_2^\infty}\, ds(\hat{x}),$$

now using that the mapping $v \mapsto \tilde{v}$ is bounded from $H^1(D_0)$ to $H^1_{loc}(\mathbb{R}^m)$ and $\tilde{v} \mapsto \tilde{v}^\infty$ is compact from $H^1_{loc}(\mathbb{R}^m)$ to $L^2(\mathbb{S})$, we can conclude that the second term in the variational form given above is compact. Furthermore from the fact that $H^1(D_0)$ is compactly embedded in $L^2(D_0)$, we can finally conclude that $\Im(T)$ is compact. $\square$

Theorem 3.3.8. *The real part of the operator T satisfies the following property:*

1. *If $\Re(A_0) - A$ is positive definite in D_0 then $\Re(T)$ is the sum of a compact operator and a self-adjoint coercive operator.*
2. *If $(A - \Re(A_0) - \alpha|\Im(A_0)|) > 0$ uniformly in D_0 and $(\Re(A_0) - \frac{1}{\alpha}|\Im(A_0)|) \geq 0$ for some constant $\alpha > 0$ then $-\Re(T)$ is the sum of a compact operator and a self-adjoint coercive operator.*

Proof. (i) Assume first that $\Re(A_0) - A$ is positive definite. Now by using the variational form (3.20) for T and the Dirichlet to Neumann operator $\mathbb{T}_k$ we have that

$$(Tv_1, v_2)_{H^1(D_0)} = -\int_{D_0} (A - A_0)\nabla\phi_1 \cdot \nabla\overline{\phi_2} - k^2(n - n_0)\phi_1\overline{\phi_2}\, dx + \int_{B_R} A\nabla u_1 \cdot \nabla\overline{u_2} - k^2 n u_1\overline{u_2}\, dx - \int_{\partial B_R} \overline{u_2}\mathbb{T}_k u_1 ds. \tag{3.22}$$

Now define the bounded linear operators S and $K : H^1(D_0) \longmapsto H^1(D_0)$ by the Riesz representation theorem such that

$$(Sv_1, v_2)_{H^1(D_0)} = \int_{D_0} (A_0 - A)\nabla\phi_1 \cdot \nabla\overline{\phi_2} + \phi_1\overline{\phi_2}\, dx + \int_{B_R} A\nabla u_1 \cdot \nabla\overline{u_2} + u_1\overline{u_2}\, dx - \int_{\partial B_R} \overline{u_2}\mathbb{T}_k u_1\, ds$$

$$(-Kv_1, v_2)_{H^1(D_0)} = k^2 \int_{D_0} (n - n_0)\phi_1\overline{\phi_2}\, dx + \int_{D_0} \phi_1\overline{\phi_2}\, dx + \int_{B_R} (k^2 n + 1)u_1\overline{u_2}\, dx.$$

By the definition of T we have that $T = S + K$. By the compact embedding of $H^1(D_0)$ into $L^2(D_0)$ and $H^1(B_R)$ into $L^2(B_R)$ we have that K is a compact operator which implies that $\Re(K)$ is also compact. We now show that $\Re(S)$ is self-adjoint and coercive on $H^1(D_0)$. Notice that since A is a real symmetric matrix we have that

$$(\Re(S)v_1, v_2)_{H^1(D_0)} = \int_{D_0} (\Re(A_0) - A)\nabla\phi_1 \cdot \nabla\overline{\phi_2} + \phi_1\overline{\phi_2}\, dx + \int_{B_R} A\nabla u_1 \cdot \nabla\overline{u_2} + u_1\overline{u_2}\, dx - \int_{\partial B_R} \overline{u_2}\Re(\mathbb{T}_k)u_1\, ds$$

which gives that $\Re(S)$ is self-adjoint. To prove coercivity we write

$$(\Re(S)v_1, v_1)_{H^1(D_0)} = \int_{D_0} (\Re(A_0) - A)|\nabla(v_1 + u_1)|^2 + |(v_1 + u_1)|^2 \, dx + \int_{B_R} A|\nabla u_1|^2 + |u_1|^2 \, dx - \int_{\partial B_R} \overline{u_1} \Re(\mathbb{T}_k) u_1 \, ds.$$

Using the fact that the real part of the DtN operator $\Re(\mathbb{T}_k)$ is non-positive (see Introduction) we obtain that

$$(\Re(S)v_1, v_1)_{H^1(D_0)} \geq \alpha \|v_1\|^2_{H^1(D_0)}$$

from a contradiction argument, namely by considering a sequence $v^n \in H^1(D_0)$ and the corresponding u_n such that $\|v^n\|_{H^1(D_0)} = 1$ for which $(\Re(S)v^n, v^n)_{H^1(D_0)} \to 0$ we arrive at the contradiction that $v^n \to 0$ in $H^1(D_0)$. This proves the claim when $\Re(A_0) - A$ is positive definite.

(ii) We now assume that $A - \Re(A_0)$ is positive definite. Unfortunately due to incompatible signs for $A - \Re(A_0)$ and the real part of the DtN operator we can not work with (3.22) for the operator T. To derive an appropriate expression for T, we use (3.14) and letting $\phi_j = v_j + u_j$, we arrive at

$$\begin{aligned} (Tv_1, v_2)_{H^1(D_0)} &= -\int_{D_0} (A - A_0)\nabla\phi_1 \cdot \nabla\overline{v_2} - k^2(n - n_0)\phi_1\overline{v_2} \, dx \\ &= -\int_{D_0} (A - A_0)\nabla v_1 \cdot \nabla\overline{v_2} - k^2(n - n_0)v_1\overline{v_2} \, dx \\ &\quad - \int_{D_0} (A - A_0)\nabla u_1 \cdot \nabla\overline{v_2} - k^2(n - n_0)u_1\overline{v_2} \, dx \end{aligned}$$

Now recall that for a given $v_2 \in H^1(D_0)$ we have that $u_2 \in H^1_{loc}(\mathbb{R}^m)$ satisfies

$$\nabla \cdot A_0 \nabla u_2 + k^2 n_0 u_2 = \nabla \cdot (A - A_0)\nabla v_2 + k^2(n - n_0)v_2 \quad \text{in } \mathbb{R}^m.$$

Hence multiplying the above equation by $\overline{u_1}$ and integrating by parts over B_R such $D \subset B_R$ we have that

$$-\int_{B_R} A_0 \nabla u_2 \cdot \nabla \overline{u_1} - k^2 n_0 u_2 \overline{u_1}\, dx + \int_{\partial B_R} \overline{u_1} \frac{\partial u_2}{\partial \nu}\, ds = \\ -\int_{D_0} (A - A_0)\nabla v_2 \cdot \nabla \overline{u_1} - k^2 \int_{D_0} (n - n_0) v_2 \overline{u_1}\, dx. \tag{3.23}$$

By taking the conjugate of the above expression and using the Dirichlet to Neumann operator $\mathbb{T}_k$ we have that

$$\begin{aligned}(-Tv_1, v_2)_{H^1(D_0)} = &\int_{D_0} (A - A_0)\nabla v_1 \cdot \nabla \overline{v_2} - k^2(n - n_0)v_1 \overline{v_2}\, dx \\ &+ \int_{B_R} A_0 \nabla u_1 \cdot \nabla \overline{u_2} - k^2 n_0 u_1 \overline{u_2}\, dx - \int_{\partial B_R} u_1 \overline{\mathbb{T}_k u_2}\, ds \\ &- \int_{D_0} (A_0 - \overline{A_0})\nabla u_1 \cdot \nabla \overline{v_2} - k^2(n_0 - \overline{n_0}) u_1 \overline{v_2}\, dx.\end{aligned}$$

In order to analyze $\Re(T)$ we first compute $(T^* v_1, v_2)_{H^1(D_0)}$

$$(-T^* v_1, v_2)_{H^1(D_0)} = \overline{(-Tv_2, v_1)}_{H^1(D_0)}$$

we therefore have that

$$
\begin{aligned}
(-T^*v_1, v_2)_{H^1(D_0)} = &\int\limits_{D_0} (A - \overline{A_0})\nabla v_1 \cdot \nabla \overline{v_2} - k^2(n - \overline{n_0})v_1\overline{v_2}\,dx \\
&+ \int\limits_{B_R} \overline{A_0}\nabla u_1 \cdot \nabla \overline{u_2} - k^2\overline{n_0}u_1\overline{u_2}\,dx - \int\limits_{\partial B_R} \overline{u_2}\mathbb{T}_k u_1\,ds \\
&- \int\limits_{D_0} (\overline{A_0} - A_0)\nabla \overline{u_2} \cdot \nabla v_1 - k^2(\overline{n_0} - n_0)\overline{u_2}v_1\,dx
\end{aligned}
$$

and then computing $\frac{1}{2}\big((T + T^*)v_1, v_2\big)_{H^1(D_0)}$ to obtain

$$
\begin{aligned}
(-\Re(T)v_1, v_2)_{H^1(D_0)} = &\int\limits_{D_0} (A - \Re(A_0))\nabla v_1 \cdot \nabla \overline{v_2} - k^2(n - \Re(n_0))v_1\overline{v_2}\,dx \\
+ &\int\limits_{B_R} \Re(A_0)\nabla u_1 \cdot \nabla \overline{u_2} - k^2\Re(n_0)u_1\overline{u_2}\,dx - \int\limits_{\partial B_R} u_1\overline{\Re(\mathbb{T}_k)u_2}\,ds \\
-i &\int\limits_{D_0} \Im(A_0)\nabla u_1 \cdot \nabla \overline{v_2} - k^2\Im(n_0)u_1\overline{v_2}\,dx + i\int\limits_{D_0} \Im(A_0)\nabla v_1 \cdot \nabla \overline{u_2} - k^2\Im(n_0)v_1\overline{u_2}\,dx.
\end{aligned}
$$

(Note that it is easy to see that the above expression is self-adjoint despite the appearance of the complex i in front of complex-valued mixed terms.)

Now define the bounded linear operators S and $K : H^1(D_0) \longmapsto H^1(D_0)$ by the Riesz representation theorem such that

$$
\begin{aligned}
(Sv_1, v_2)_{H^1(D_0)} = &\int\limits_{D_0} (A - \Re(A_0))\nabla v_1 \cdot \nabla \overline{v_2} + v_1\overline{v_2}\,dx + \int\limits_{B_R} \Re(A_0)\nabla u_1 \cdot \nabla \overline{u_2}\,dx \\
&- \int\limits_{\partial B_R} u_1\overline{\Re(\mathbb{T}_k)u_2}\,ds - i\int\limits_{D_0} \Im(A_0)\nabla u_1 \cdot \nabla \overline{v_2}\,dx + i\int\limits_{D_0} \Im(A_0)\nabla v_1 \cdot \nabla \overline{u_2}\,dx
\end{aligned}
$$

and $(Kv_1, v_2)_{H^1(D_0)} = (-\Re(T)v_1, v_2)_{H^1(D_0)} - (Sv_1, v_2)_{H^1(D_0)}$. Note that in the definition of K there are only L^2-terms, hence K is a compact operator due to the compact embedding of $H^1(D_0)$ into $L^2(D_0)$ and $H^1(B_R)$ into $L^2(B_R)$. Now, using that $A - \Re(A_0) > 0$ and $\Re(A_0) > 0$ along with the fact that the real part of the DtN mapping $\Re(\mathbb{T}_k)$ is non-positive (see Introduction) and applying Young's inequality we have

$$(Sv_1, v_1)_{H^1(D_0)} \geq \Big((A - \Re(A_0) - \alpha|\Im(A_0)|)\nabla v_1, \nabla v_1\Big)_{L^2(D_0)} + (v_1, v_1)_{L^2(D_0)}$$
$$+ \left((\Re(A_0) - \frac{1}{\alpha}|\Im(A_0)|)\nabla u_1, \nabla u_1\right)_{L^2(D_0)} \geq C\|v_1\|_{H^1(D_0)}.$$

Provided α is such that $(A - \Re(A_0) - \alpha|\Im(A_0)|) > 0$ uniformly in D and $(\Re(A_0) - \frac{1}{\alpha}|\Im(A_0)|) \geq 0$ which prove the second part of the theorem. □

Now we are ready to state the main theorem of the chapter which characterizes the support of defective region D_0 in terms of the range of the operator $\tilde{F}_\sharp^{1/2}$, where we define

$$\tilde{F}_\sharp := \left|\Re\big(\gamma_m^{-1}\mathcal{S}_b^* F\big)\right| + \left|\Im\big(\gamma_m^{-1}\mathcal{S}_b^* F\big)\right| : L^2(\mathbb{S}) \longmapsto L^2(\mathbb{S}).$$

We assume that the coefficients A, A_0, n and n_0 satisfy the assumptions stated in Section 3.1.

Theorem 3.3.9. *Assume that k is not a transmission eigenvalue for D_0 where $\Im(A_0) = 0$ and $\Im(n_0) = 0$ otherwise the assumptions of Theorem 3.3.6 hold. Furthermore assume that either $\Re(A_0) - A > 0$ uniformly in D_0, or $A - A_0 > 0$ uniformly in D_0 or there is some constant $\alpha > 0$ such that $A - \Re(A_0) - \alpha|\Im(A_0)| > 0$ uniformly in D_0 and $\Re(A_0) - \frac{1}{\alpha}|\Im(A_0)| \geq 0$ in D_0. For any $z \in \mathbb{R}^m$ we define*

$\phi_z := \mathcal{S}_b^* \mathbb{G}^\infty(\cdot, z) \in L^2(\mathbb{S})$, *then*

$$z \in D_0 \quad \textit{if and only if} \quad \phi_z \in \mathcal{R}(\tilde{F}_\sharp^{1/2}).$$

Proof. Combining Theorems 3.3.5, 3.3.6, 3.3.7 and 3.3.8 the result follows by applying Theorem 2.15 in [55] if $\Im(A_0) = 0$ in D_0 or Theorem 2.1 in [61] if $\Im(A_0) < 0$ in D_0 to the operator $\tilde{F}_\sharp$. □

Notice that from the statement of Theorem 3.3.9 as a byproduct we have the following uniqueness result.

Corollary 3.3.1. *Assume that A, n and D are fixed, let there be two defective regions $D_0^{(1)}$ and $D_0^{(2)}$ both included in D with coefficients $A_0^{(1)}$, $n_0^{(1)}$ and $A_0^{(2)}$, $n_0^{(2)}$ respectively that satisfy the assumptions in Section 3.1 and Theorem 3.3.9. If there is a fixed wave number k that is not a transmission eigenvalue for which the far field patterns coincide for all incident directions then $D_0^{(1)} = D_0^{(2)}$.*

The result in Theorem 3.3.9 implies that the linear problem

$$\tilde{F}_\sharp^{1/2} g_z = \mathcal{S}_b^* \mathbb{G}^\infty(\cdot, z) \quad \text{for} \quad z \in \mathbb{R}^m \tag{3.24}$$

is solvable if and only if $z \in D_0$. Since the operator $\tilde{F}_\sharp^{1/2} : L^2(\mathbb{S}) \mapsto L^2(\mathbb{S})$ is compact equation (3.24) is ill-possed and therefore we can take the unique minimizer of the Tikhonov functional

$$\|\tilde{F}_\sharp^{1/2} g_z^\alpha - \mathcal{S}_b^* \mathbb{G}^\infty(\cdot, z)\|_{L^2(\mathbb{S})}^2 + \alpha \|g_z^\alpha\|_{L^2(\mathbb{S})}^2 \tag{3.25}$$

where the regularization parameter α is chosen by Morozov discrepancy principal. Therefore we have that $\|g_z^\alpha\|_{L^2(\mathbb{S})}$ is finite if and only if $z \in D_0$ as the regularization parameter $\alpha := \alpha(\epsilon) \to 0$ as $\epsilon \to 0$.

We can also obtain an indicator function for D_0 by using Picard's criterion. To this end let $(\lambda_i, \psi_i) \in \mathbb{R}^+ \times L^2(\mathbb{S})$ be an orthonormal eigensystem of $\tilde{F}_\sharp$ then by appealing to Picard's criterion (see e.g. Theorem 2.7 of [14]) we have the following characterization of the support of the defect D_0.

Corollary 3.3.2. *Assume that k is not a transmission eigenvalue of D_0 and A, A_0, n and n_0 satisfy the assumptions of Theorem 3.3.9. Then for $\phi_z := \mathcal{S}_b^* \mathbb{G}^\infty(\cdot, z)$*

$$z \in D_0 \quad \text{if and only if} \quad \sum_{i=1}^{\infty} \frac{|(\phi_z, \psi_i)|^2}{\lambda_i} < \infty.$$

This analysis gives that to reconstruct the defect D_0 we can use as an indicator function

$$W(z) = \left[\sum_{i=1}^{\infty} \frac{|(\phi_z, \psi_i)|^2}{\lambda_i} \right]^{-1}.$$

3.3.1 Remarks on the Generalized Linear Sampling Method

Recently, an alternative mathematically rigorous sampling method has been introduced in [6] referred to as Generalized Linear Sampling Method (GLSM). The GLSM is designed to bridge the gap between the Linear Sampling Method and the Factorization Method. From the above factorizations we can connect the support of D_0 to the solution of a minimization problem. In particular the GLSM consider finding a minimizer over all $g \in L^2(\mathbb{S})$ for the functional

$$\mathcal{J}(g; \alpha, \epsilon) = \alpha \left(|\langle Bg, g \rangle| + \epsilon \|g\|^2 \right) + \|Fg - \mathbb{G}^\infty(\cdot, z)\|^2 \tag{3.26}$$

where the regularization parameter is chosen such that $\alpha := \alpha(\epsilon) \to 0$ as $\epsilon \to 0$. In order to connect the support of the defect to the behavior of the minimizer we need the following theoretical result.

Theorem 3.3.10. *(Theorem 4 of [6]) Let X and Y be two reflexive Banach Spaces where $H : X \mapsto Y$, $T : Y \mapsto Y^*$ and $G : \overline{\mathcal{R}(H)} \subset Y \mapsto X^*$ are bounded linear operators. Assume that the following factorizations hold $B = H^*TH$ and $F = GH$. Moreover we assume that*

1. *G is compact and F has dense range*
2. *T is a coercive operator on $\mathcal{R}(H)$, i.e. $|\langle T\phi, \phi\rangle| \geq \alpha\|\phi\|^2 \quad \forall \phi \in \mathcal{R}(H)$*
3. *B is a compact operator.*

Then if $g_z^{\alpha(\epsilon)}$ is the minimizer of $\mathcal{J}(g;\alpha,\epsilon)$ defined by (3.26) then $\mathbb{G}^\infty(\cdot, z) \in \mathcal{R}(G)$ if and only if $\liminf_{\alpha\to 0} \liminf_{\epsilon\to 0} |\langle Bg_z^{\alpha(\epsilon)}, g_z^{\alpha(\epsilon)}\rangle| < \infty$.

We see that the above result gives a characterization of the defective region by adding the regularization term $|\langle Bg_z^{\alpha(\epsilon)}, g_z^{\alpha(\epsilon)}\rangle|$ to the Tikhonov regularized solution of the far-field equation. Using the factorizations $F = GH$ and $F = -\gamma_m \mathcal{S}_b H^*TH$ we will develop a new indicator function to test for a defect using the GLSM under different assumptions on the material parameters than in the previous section. To this end, let $B = H^*\Im(T)H$ and we will prove that $\Im(T)$ is a coercive operator provided that the defective region is absorbing, also notice that $B = \Im\,(\gamma_m^{-1}\mathcal{S}_b^* F)$.

Theorem 3.3.11. *Assume $\xi \cdot \Im(A_0(x))\xi < 0$ and $\Im(n_0) > 0$ uniformly in D_0 then the operator $\Im(T)$ is coercive on $H^1(D_0)$.*

Proof. We are going to prove coercivity by using a contradiction argument, therefore assume that $\exists\, v_\ell \in H^1(D_0)$ such that

$$\|v_\ell\|_{H^1(D_0)} = 1 \quad \text{where} \quad |(\Im(T)v_\ell, v_\ell)_{H^1(D_0)}| \to 0 \quad \text{as} \quad \ell \to \infty.$$

This gives that u_ℓ defined by solving (3.7) with source $v = v_\ell$ is bounded in $H^1_{loc}(\mathbb{R}^m)$. Recall that

$$\Im(Tv_\ell, v_\ell)_{H^1(D_0)} = \int\limits_{D_0} \Im(A_0)|\nabla\phi_\ell|^2 - k^2\Im(n_0)|\phi_\ell|^2 \, dx - k\int\limits_{\mathbb{S}} |u_\ell^\infty|^2 \, ds(\hat{x})$$

and we have that $(\Im(T)v_\ell, v_\ell)_{H^1(D_0)} \to 0$ as $\ell \to \infty$. Using that $\xi \cdot \Im(A_0(x))\xi < 0$ and $\Im(n_0(x)) > 0$ we conclude that $\phi_\ell \to 0$ in $H^1(D_0)$. By the well-posedness of (3.13) we have that

$$\|u_\ell\|_{H^1(B_R)} \leq C\|\phi_\ell\|_{H^1(D_0)} \longrightarrow 0 \quad \text{as} \quad \ell \to \infty.$$

Now since $v_\ell = \phi_\ell - u_\ell$ we have that $v_\ell \to 0$ in $H^1(D_0)$ which gives that result by contradiction since $||v_\ell||_{H^1(D_0)} = 1$. □

With this we now give another indicator function that characterizes the support of D_0. First note that using similar arguments as in Theorem 3.3.5 we have that if k is not a transmission eigenvalue of D_0 then $\mathbb{G}^\infty(\cdot, z) \in \mathcal{R}(G)$ if and only if $z \in D_0$. We also have that F has a dense range when k is not a transmission eigenvalue of D_0. Now using Theorem 3.3.10 and Theorem 3.3.11 we have the following result.

Theorem 3.3.12. *Let $B = \Im\left(\gamma_m^{-1}\mathcal{S}_b^* F\right)$ with A, A_0, n and n_0 satisfying the assumptions in Section 3.1 moreover assume that $\xi \cdot \Im(A_0(x))\xi < 0$ and $\Im(n_0) > 0$. Then if $g_z^{\alpha(\epsilon)}$ is the minimizer of $\mathcal{J}(g; \alpha, \epsilon)$ defined by (3.26) then $z \in D_0$ if and only if*

$$\liminf_{\alpha \to 0} \liminf_{\epsilon \to 0} \left| \left(\Im\left(\gamma_m^{-1}\mathcal{S}_b^* F\right) g_z^{\alpha(\epsilon)}, g_z^{\alpha(\epsilon)} \right) \right| < \infty.$$

One can take the indicator function to be

$$P(z) = \frac{1}{\left\| \Im \left(\gamma_m^{-1} \mathcal{S}_b^* F\right)^{1/2} g_z^{\alpha(\epsilon)} \right\|^2 + \epsilon \left\| g_z^{\alpha(\epsilon)} \right\|^2}$$

with $g_z^{\alpha(\epsilon)}$ being the minimizer of (3.26) which can be shown to be the solution of (see [6])

$$\alpha\big(\Im\left(\gamma_m^{-1}\mathcal{S}_b^* F\right) g + \epsilon g\big) + F^* F g = F^* \mathbb{G}^\infty(\cdot, z).$$

Notice that this criteria does not require any assumptions on the real or imaginary parts of the matrix valued contrasts as in the previous section , however the assumptios on the imaginary part in D_0 is strengthened.

To detect the defective region we can also take $B = \tilde{F}_\sharp$. Therefore since $\tilde{F}_\sharp = H^* T_\sharp H$ with $T_\sharp = \big|\Re\big(T\big)\big| + \big|\Im\big(T\big)\big|$ and $T_\sharp$ being coercive provide that (see proof of Theorem 2.1 in [61])

1. $\Re(T)$ is the sum of a coercive and compact operators
2. $\Im(T)$ is non-positive (or non-negative)
3. T is injective.

This implies that when the factorization method as discussed in the previous section is valid we can apply the generalized linear sampling method.

Theorem 3.3.13. *Let $B = \tilde{F}_\sharp$ with k not a transmission eigenvalue of D_0 and A, A_0, n, n_0 satisfy the assumptions of Theorem 3.3.9. Then if $g_z^{\alpha(\epsilon)}$ is the minimizer of $\mathcal{J}(g; \alpha, \epsilon)$ defined by (3.26) then $z \in D_0$ if and only if*

$$\liminf_{\alpha \to 0} \liminf_{\epsilon \to 0} \Big| \left(\tilde{F}_\sharp\, g_z^{\alpha(\epsilon)}, g_z^{\alpha(\epsilon)}\right) \Big| < \infty.$$

The indicator function for the GLSM is given by

$$Q(z) = \frac{1}{\left\| \tilde{F}_{\sharp}^{1/2} g_z^{\alpha(\epsilon)} \right\|^2 + \epsilon \left\| g_z^{\alpha(\epsilon)} \right\|^2}$$

where $g_z^{\alpha(\epsilon)} = \arg\min \mathcal{J}(g; \alpha, \epsilon)$. It is easy to see that the minimizer for the GLSM functional is given by solving

$$\alpha \left(\tilde{F}_{\sharp} g + \epsilon g \right) + F^* F g = F^* \mathbb{G}^{\infty}(\cdot, z).$$

One useful advantage of the GLSM is that for noisy data such that

$$||F - F^{\delta}|| \leq \delta \quad \text{and} \quad ||B - B^{\delta}|| \leq \delta$$

then we minimize

$$\mathcal{J}(g; \alpha, \delta) = \alpha \left(\left\| (\tilde{F}_{\sharp}^{1/2})^{\delta} g \right\|_{L^2(\mathbb{S})}^2 + \delta \|g\|_{L^2(\mathbb{S})}^2 \right) + \|F^{\delta} g - \mathbb{G}^{\infty}(\cdot, z)\|_{L^2(\mathbb{S})}^2.$$

There is still the question of how should one choose the regularization parameter α. One choice of picking α is such that it is fixed by using Morozov discrepancy principle for the Tikhonov regularization functional. Once α has been fixed you can minimize the discretized GLSM functional. For a more detailed discussion on the implementation of the GLSM see [6]

3.4 Numerical Validation of the Factorization Method

In this section we show numerical examples in $\mathbb{R}^2$, where a defective region is reconstructed from simulated far-field data. To simulate the data, we solve the direct scattering problems using a cubic finite element method with a perfectly matched layer and from this we will evaluate approximated u_0^∞ and u_b^∞. In the following calculations we use N different incident and observation directions $d_j = \hat{x}_j = \big(\cos(\theta_j), \sin(\theta_j)\big)$ where θ_j are uniformly spaced points in $[0, 2\pi)$. This leads to discretized far field operators

$$\mathbf{F_0} = \big[u_0^\infty(\hat{x}_i, d_j)\big]_{i,j=1}^N, \quad \mathbf{F_b} = \big[u_b^\infty(\hat{x}_i, d_j)\big]_{i,j=1}^N \quad \text{and} \quad \mathbf{F} = \mathbf{F_0} - \mathbf{F_b}$$

where we can apply the Picard's criterion in Corollary 3.3.2. Even though the scattering operator $\mathcal{S}_b$ is unitary, due to approximation error in the discretized operator $\mathbf{S}_b$ we use $\mathbf{S}_b^{-1}$, instead of its adjoint $\mathbf{S}_b^*$ in order to minimize the error (in all our examples we observe that $\|\mathbf{S}_b^*\mathbf{S}_b - I\|/\|\mathbf{S}_b^*\|^2 \approx 1.0014$). Hence we let

$$\tilde{\mathbf{F}}_\sharp = \big|\Re\big(\gamma_m^{-1}\mathbf{S}_b^{-1}\mathbf{F}\big)\big| + \big|\Im\big(\gamma_m^{-1}\mathbf{S}_b^{-1}\mathbf{F}\big)\big|$$

in the calculations along with $\phi_z = [\mathbf{S}_b^{-1}\mathbb{G}^\infty(\hat{x}_j, z)]_{j=1}^N$ where $\mathbf{S}_b^{-1}$ is computed by a LU decomposition. The application of the factorization method requires the computation of the far field pattern $\mathbb{G}^\infty(\hat{x}, z)$ of the background Green's function $\mathbb{G}(\hat{x}, z)$. In order to avoid dealing with singularity at the point z, for the case of piecewise homogeneous isotropic background in Theorem 2.1 of [10] the authors provide a relation between the far field pattern of the background Green's function and the total field due to the background media extending the mixed reciprocity relation known for homogeneous background [37]. We use this relation in our examples for piecewise homogeneous background. In the case of anisotropic homogenous media in D we provide a partial result of mixed reciprocity relation for $z \in D$ extending the result in [10] (for problems in nondestructive testing when D is known, it is reasonable to consider the sampling points z inside D). To this end let us first assume that $A \neq I$ is constant matrix and $n \neq 1$ is constant in D. The fundamental solution of the differential operator $Lu := \nabla \cdot A\nabla u + k^2 nu$ is given by

$$\Phi_b(x,y) = \frac{i}{4\sqrt{\det A}} H_0^{(1)}(k\sqrt{n}|x-y|_A) \quad \text{in} \quad \mathbb{R}^2$$

$$\Phi_b(x,y) = \frac{1}{4\pi\sqrt{\det A}} \frac{\exp(ik\sqrt{n}|x-y|_A)}{|x-y|_A} \quad \text{in} \quad \mathbb{R}^3$$

where $|x-y|_A^2 = (x-y)^\top A^{-1}(x-y)$.

Theorem 3.4.1. *Assume that A is a constant positive definite matrix and $n > 0$ constant. Then for $\hat{x} \in \mathbb{S}$ and $z \in D$ we have that*

$$\mathbb{G}^\infty(\hat{x}, z) = \gamma_m u_b(z, -\hat{x})$$

with $u_b(z, -\hat{x})$ is that solution of (3.1)-(3.2).

Proof. Assume that $z \in D$ therefore we have that $\mathbb{G}(y,z)$ is a smooth radiating solution to Helmholtz equation in $\mathbb{R}^m \setminus \overline{D}$. So by (1.10) we have that

$$\mathbb{G}^\infty(\hat{x}, z) = \gamma_m \int\limits_{\partial D} \left(\mathbb{G}(y,z)^+ \frac{\partial}{\partial \nu_y} e^{-ik\hat{x}\cdot y} - \frac{\partial}{\partial \nu_y} \mathbb{G}(y,z)^+ \, e^{-ik\hat{x}\cdot y} \right) ds_y.$$

Now from Green's second identity we have that for $z \in D$

$$u_b(z, -\hat{x}) = \int\limits_{\partial D} \frac{\partial}{\partial \nu_{A_y}} u_b^-(y, -\hat{x})\, \Phi_b(y,z) - u_b^-(y, -\hat{x}) \frac{\partial}{\partial \nu_{A_y}} \Phi_b(y,z)\, ds_y. \tag{3.27}$$

Noting that the difference $\mathbb{G}(y,z) - \Phi_b(y,z)$ is a smooth solution of (3.1) in D and using again Green's second identity implies that

$$0 = \int\limits_{\partial D} \frac{\partial}{\partial \nu_{A_y}} u_b^-(y,-\hat{x}) \left[\mathbb{G}(y,z)^- - \Phi_b(y,z)\right] - u_b^-(y,-\hat{x}) \frac{\partial}{\partial \nu_{A_y}} \left[\mathbb{G}(y,z)^- - \Phi_b(y,z)\right] ds_y.$$

By adding this identity with (3.27) gives that

$$\begin{aligned} u_b(z,-\hat{x}) &= \int\limits_{\partial D} \left(\frac{\partial}{\partial \nu_{A_y}} u_b^-(y,-\hat{x})\, \mathbb{G}(y,z)^- - u_b^-(y,-\hat{x}) \frac{\partial}{\partial \nu_{A_y}} \mathbb{G}(y,z)^- \right) ds_y \\ &= \int\limits_{\partial D} \left(\frac{\partial}{\partial \nu_y} u_b^+(y,-\hat{x})\, \mathbb{G}(y,z)^+ - u_b^+(y,-\hat{x}) \frac{\partial}{\partial \nu_y} \mathbb{G}(y,z)^+ \right) ds_y \qquad (3.28) \end{aligned}$$

where the second equality is due to the continuity conditions of the Cauchy data across ∂D. Now since $u_b(z,-\hat{x}) = u_b^s(z,-\hat{x}) + e^{-ik\hat{x}\cdot z}$ with $u_b^s(z,-\hat{x})$ being a radiating solution to Helmholtz equation in $\mathbb{R}^m \setminus \overline{D}$, once more an application of Green's second identity yields that

$$0 = \int\limits_{\partial D} \left(\frac{\partial}{\partial \nu_y} u_b^s(y,-\hat{x})\, \mathbb{G}(y,z)^+ - u_b^s(y,-\hat{x}) \frac{\partial}{\partial \nu_y} \mathbb{G}(y,z)^+ \right) ds_y$$

Therefore we have that

$$u_b(z,-\hat{x}) = \int\limits_{\partial D} \left(\mathbb{G}(y,z)^+ \frac{\partial}{\partial \nu_y} e^{-ik\hat{x}\cdot y} - \frac{\partial}{\partial \nu_y} \mathbb{G}(y,z)^+\, e^{-ik\hat{x}\cdot y} \right) ds_y.$$

which proves the result. □

Remark 3.4.1. The proof of Theorem 3.4.1 holds true for non-constant media in $\mathbb{R}^2$ or $\mathbb{R}^3$ as long as one can define the corresponding fundamental solution $\Phi_b(\cdot,\cdot)$ of the operator $Lu := \nabla \cdot A \nabla u + k^2 n u$ (see e.g. [63]).

The above result gives that the $\mathbb{G}^\infty(\hat{x}, z)$ can be approximated using the same cubic finite element method with a perfectly matched layer that is used to compute the scattered field u_b^s. In particular this way we compute $\mathbb{G}^\infty(\hat{x}, z_p)$ at the sampling points z_p being the mesh points in the finite element mesh. The defective region D_0 is visualized by plotting the indicator function

$$W_N(z) = \left[\sum_{i=1}^{N} \frac{\left| \left(\mathbf{S}_b^{-1} u_b(z, -\hat{x}), \psi_i \right) \right|_{\ell^2}^2}{|\lambda_i|} \right]^{-1} \qquad \text{for} \qquad z \in D$$

where $(\lambda_i, \psi_i) \in \mathbb{R}^+ \times \mathbb{C}^N$ is the eigensystem for the discretized operator $\tilde{\mathbf{F}}_\sharp$ defined by the discretized far field operators and scattering operator.

Example 1. We consider $D = [-2, 2]^2$ where the defective region is a void D_0 (i.e $A_0 = I$ and $n_0 = 1$ in D_0) embedded in isotropic media. The coefficients in D are given by $A = 0.5I$ and $n = 3$. We consider four examples of the void region D_0, namely the ball centered at the origin with radius $R = 1$, the square $D_0 = [-1, 1]^2$, the ellipse centered at $(0.5, 1)$ with axis $a = 0.5$ and $b = 0.3$, and two circular voids with radius 0.3 centered at $(-1, 1)$ and $(1, -1)$, respectively. Reconstructions are shown in Figure 3.2 and Figure 3.3. In all our examples, we use $N = 32$, i.e. 32 incident directions and observation directions.

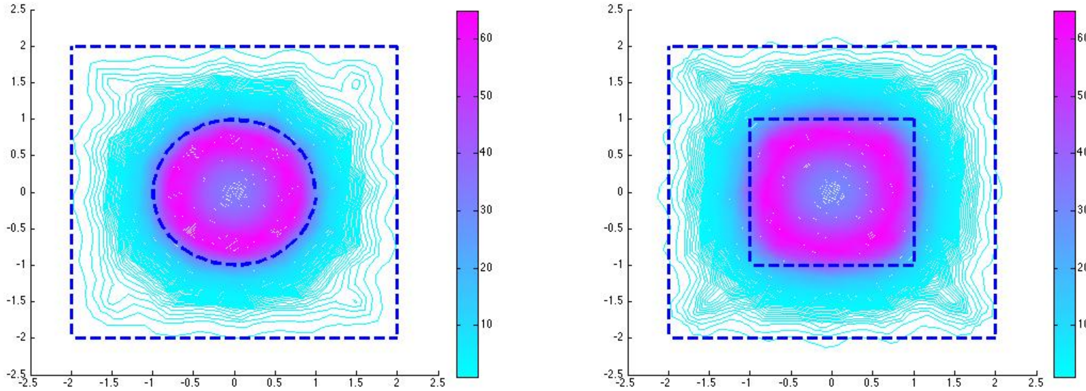

Figure 3.2: On the left is the reconstruction of the circular void and on the right the square void. The defective region is a void so the coefficients are given by $A_0 = I$ and $n_0 = 1$ in D_0 for wavenumber $k = 1$. Dashed line: exact boundaries of the scatterer D and void(s) D_0. No added noise.

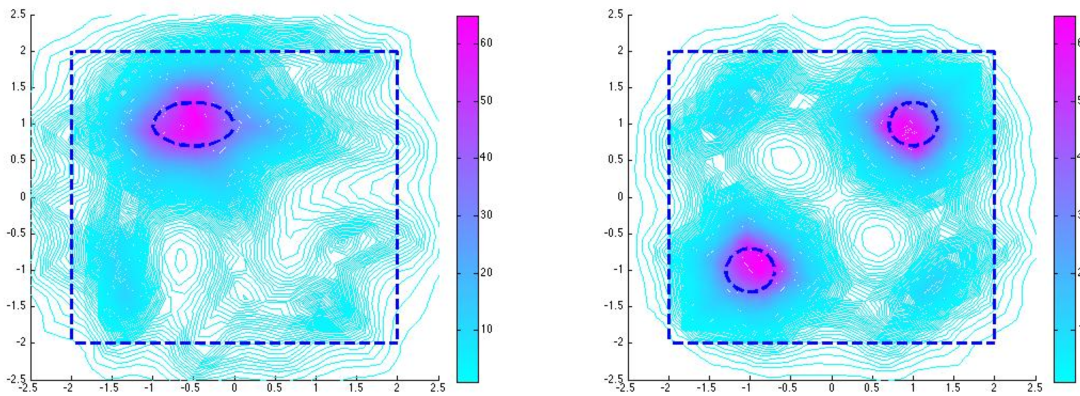

Figure 3.3: Reconstruction of the ellipse void on the left and of the 2 circular voids on the right using the factorization method. The wavenumber in both examples is $k = 1$. Dashed line: exact boundaries of the scatterer D and void(s) D_0. 2% noise.

Example 2. For this example we now reconstruct a circular void of radius 1 centered at the origin and two small circular voids in an anisotropic square scatterer $D = [-2,2]^2$. As in the previous example the two circular voids both have radius 0.3 and they are centered at $(-1,1)$ and $(1,-1)$ respectively. The coefficients in D are chosen to be given by

$$A = \begin{pmatrix} 0.6022 & 0.1591 \\ 0.1591 & 0.7478 \end{pmatrix}$$

and $n = 3$ with $N = 64$. The reconstructions are presented in Figure 3.4.

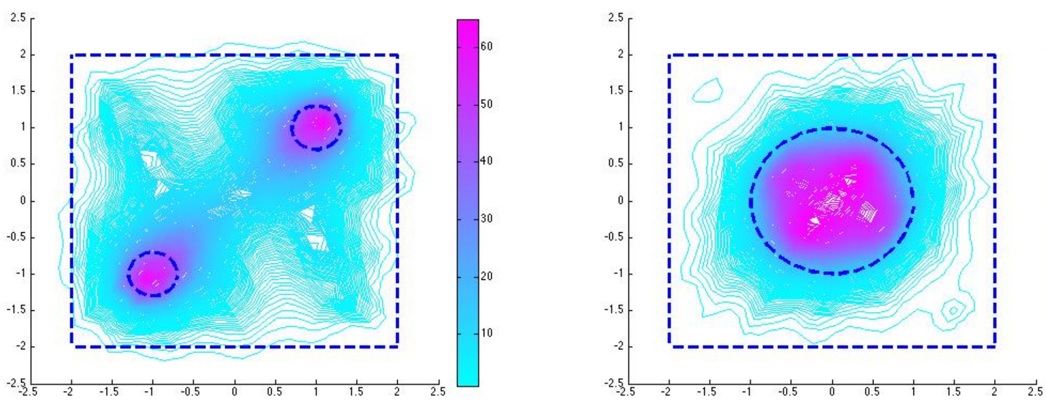

Figure 3.4: On the left is the reconstruction of the 2 circular. While on the right is the reconstruction of the a circular void of radius 1, where the wavenumber is $k = 1$. Dashed line: exact boundaries of the scatterer and void(s). No added noise.

Example 3. For our next example we now consider anisotropic defects embedded in anisotropic material. In particular, we reconstruct the two small circular defects and the ellipse inside the square $D = [-2,2]^2$. The coefficients are chosen in D and

D_0 to be given respectively by $n = n_0 = 3$

$$A = \begin{pmatrix} 0.6022 & 0.1591 \\ 0.1591 & 0.7478 \end{pmatrix} \quad \text{and} \quad A_0 = \begin{pmatrix} 0.1673 & -0.0308 \\ -0.0308 & 0.2030 \end{pmatrix}$$

with $N = 64$ in both cases. The reconstructions are shown in Figure 3.5.

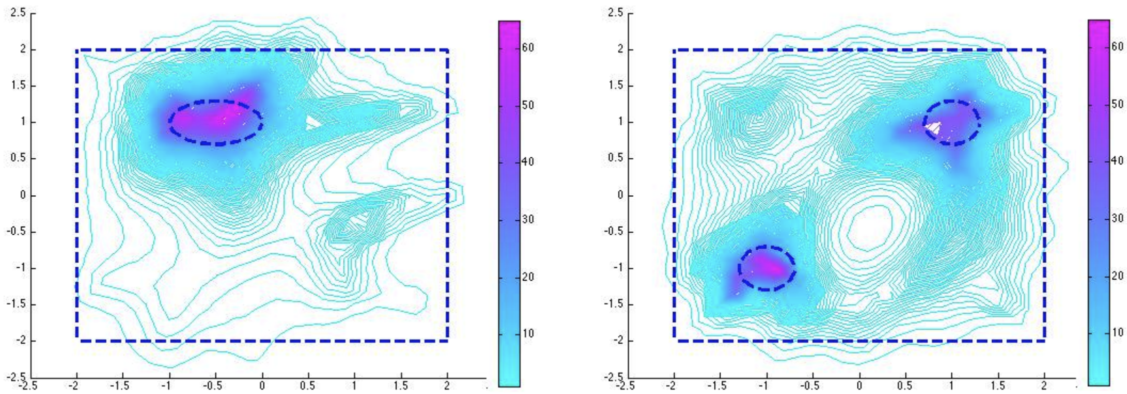

Figure 3.5: On the left is the reconstruction of the ellipse, while on the right is the reconstruction of the two discs. The wavenumber is $k = 1$. Dashed line: exact boundaries of the scatterer D and defect(s) D_0. 4% noise.

Chapter 4

ASYMPTOTIC METHODS FOR SMALL VOLUME DEFECTS IN AN ANISOTROPIC MEDIA

In this Chapter we consider the inverse problem of reconstructing small volume penetrable defects in an anisotropic media. We derive a MUSIC algorithm to reconstruct the locations of the defects using specially polarized time-harmonic electromagnetic waves in $\mathbb{R}^2$ and the acoustic waves in $\mathbb{R}^3$. The derivation of the multi-static response matrix makes use of an asymptotic expansion for the scattered field with respect to a small parameter that characterizes the size of the defects. We also present some numerical examples in the two dimensional case to demonstrate the feasibility of the reconstruction method. Next we consider the transmission eigenvalue problem for an anisotropic media with small volume inclusions. As discussed in Chapter 2, the transmission eigenvalues have been proven to hold information about the material properties of a media, and can be determined from the scattering data, hence they can play an important role in a variety of inverse problems for non-destructive testing. The goal of our study is to understand how the presence of these small defects affect the transmission eigenvalues and use this to obtain information on the strength of the defects. Since the transmission eigenvalue problem for magnetic materials is inherently non-linear we can not appeal to standard analytical techniques. The eigenvalue problem for small inhomogeneities has been studied for isotropic inhomogeneous media in [31] and [32] where the convergence as the volume tends to zero is proven and explicit asymptotic expansions for the transmission eigenvalues are also given.

4.1 Formulation of the Inverse Problem for Small Volume Defects

We start by investigating the electromagnetic inverse problem of locating small volume penetrable defects in an anisotropic background media. To reconstruct the locations of the defects we derive a **MU**ltiple **SI**gnal **C**lassification (MUSIC) algorithm which can be seen as a discrete version of the factorization method considered in Chapter 3. In this study we wish to extend the applicability of the MUSIC algorithm to imaging within an anisotropic media, being motivated by nondestructive testing using electromagnetic/acoustic waves for determining the location/support of small penetrable defective regions embedded in a known anisotropic media. MUSIC provides a rigorous characterization of the defects by a range test of the multi-static response matrix, which is given by the far field pattern of the scattered field for a finite number of sources and receivers, while the factorization method gives a rigorous characterization of the defective regions by a range test for a compact operator obtained from the knowledge of the scattered field due to a continuum of sources and receiver.

The multi-static response matrix is derived by exploiting the assumption that the defects are 'small', and using the asymptotic formulas derived in [3] and [48] for the far-field pattern in the presents of small defects in an anisotropic media. Similar problems have been considered in [71] where the authors derive an algorithm for reconstruction thin penetrable inclusion in homogenous free space, and in [44] where the MUSIC algorithm is extended to elastic waves. To obtain a range test for the defects, we follow the analysis in Section 4.1 of [55] and standard arguments from linear algebra. This analytical framework will give an indicator function for the locations of the defects but our numerical examples show that the size of the defect can be obtained for a defective region with sufficiently small components.

To avoid technical difficulties with asymptotic expansions, we will consider the problem of reconstructing small volume penetrable defects in a known homogenous media. To this end let $D \subset \mathbb{R}^d$ (for $d = 2$ or 3) be a bounded domain with piecewise smooth boundary. Assume that we have A and n constant being the constitutive parameters for the unperturbed ("healthy") media. Without loss of generality we assume that outside the scatterer D the background media has refractive index scaled to one, therefore we define

$$A(x) = \begin{cases} I & \text{for } x \in \mathbb{R}^d \setminus \overline{D} \\ A & \text{for } x \in D \end{cases} \quad \text{and} \quad n = \begin{cases} 1 & \text{for } x \in \mathbb{R}^d \setminus \overline{D} \\ n & \text{for } x \in D. \end{cases}$$

Now the scattering of an incident plane wave $e^{ikx\cdot\hat{y}}$, where $\hat{y} \in \mathbb{S}$ = unit circle/sphere, by the unperturbed media (i.e. without defects) is mathematically formulated as: find $u_b \in H^1_{loc}(\mathbb{R}^d)$ with $u_b = u_b^s + e^{ikx\cdot\hat{y}}$ such that

$$\nabla \cdot A(x)\nabla u_b + k^2 n(x) u_b = 0 \quad \text{in} \quad \mathbb{R}^d \tag{4.1}$$

$$\lim_{r\to\infty} r^{\frac{d-1}{2}} \left(\frac{\partial u_b^s}{\partial r} - iku_b^s \right) = 0. \tag{4.2}$$

Here u_b is the total field in the background (including the homogeneous part and the media of compact support $\overline{D}$) and u_b^s is the scattered field due to the region D. Let $u_b^\infty(\hat{x}, \hat{y})$ be the far field pattern for the scattered field u_b^s, which depend on the incident direction $\hat{y}$ and observation direction $\hat{x}$.

Now consider the small defective regions that are given by $z_m + \epsilon B_m$ where

B_m is a smooth deformations of a ball centered at the origin. Now let A_m and n_m be constant constitutive parameters for the defective regions given by $z_m + \epsilon B_m$. We assume that

$$|z_i - z_j| \geq c_0 > 0 \quad \text{for all} \quad i \neq j \quad \text{with} \quad i, j = 1, 2, \dots M$$

and

$$\text{dist}(z_m\,, \partial D) \geq c_0 > 0 \quad \text{for all} \quad m = 1, 2, \dots M.$$

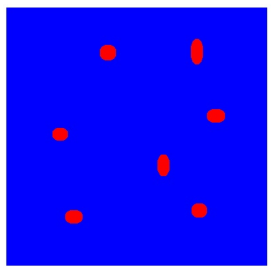

Figure 4.1: An example of an anisotropic media with finitely many small volume inhomogeneities

The union of the defective regions is denoted by $D_\epsilon = \bigcup_{m=1}^{M} (z_m + \epsilon B_m)$ and we let

$$A_\epsilon(x) = \begin{cases} A_m & \text{for } x \in (z_m + \epsilon B_m) \\ A(x) & \text{for } x \in \mathbb{R}^d \setminus \overline{D}_\epsilon \end{cases} \tag{4.3}$$

and

$$n_\epsilon(x) = \begin{cases} n_m & \text{for } x \in (z_m + \epsilon B_m) \\ n(x) & \text{for } x \in \mathbb{R}^d \setminus \overline{D}_\epsilon. \end{cases} \tag{4.4}$$

The scattering problem for the media with the defective region D_ϵ now reads: find $u_\epsilon \in H^1_{loc}(\mathbb{R}^d)$ with $u_\epsilon = u^s_\epsilon + e^{ikx\cdot\hat{y}}$ such that

$$\nabla \cdot A_\epsilon(x)\nabla u_\epsilon + k^2 n_\epsilon(x) u_\epsilon = 0 \quad \text{in} \quad \mathbb{R}^d \tag{4.5}$$

$$\lim_{r\to\infty} r^{\frac{d-1}{2}} \left(\frac{\partial u^s_\epsilon}{\partial r} - iku^s_\epsilon \right) = 0. \tag{4.6}$$

Similarly since u^s_ϵ is a radiating solution to the Helmholtz equation in $\mathbb{R}^m \setminus \overline{D}$, we have that its corresponding far field pattern $u^\infty_\epsilon(\hat{x}, \hat{y})$.

We now derive a multi-static response matrix by exploiting the fact that the each of the defective regions has small volume as in [71], which will be used to reconstruct the defective regions. To this end we first recall $\mathbb{G}(\cdot, \cdot)$ the Green's function for the background layered media, i.e. the solution of

$$\nabla \cdot A(x)\nabla \mathbb{G}(\cdot, z) + k^2 n(x)\mathbb{G}(\cdot, z) = -\delta(\,\cdot - z) \;\text{ in }\; \mathbb{R}^d$$
$$\lim_{r\to\infty} r^{\frac{d-1}{2}} \left(\frac{\partial \mathbb{G}(\cdot, z)}{\partial r} - ik\mathbb{G}(\cdot, z) \right) = 0.$$

Let $\mathbb{G}^\infty(\cdot\,, z) \in L^2(\mathbb{S})$ be it's far field pattern. Since $A(x)$ is a symmetric constant positive definite matrix for $x \in D$ and $n(x)$ is a positive constant for $x \in D$ we have by Theorem 3.4.1that

$$\mathbb{G}^\infty(\hat{x}, z) = \gamma u_b(z, -\hat{x}) \quad \text{where} \quad \gamma = \frac{e^{i\pi/4}}{\sqrt{8\pi k}} \;\text{ in }\; \mathbb{R}^2 \quad \text{and} \quad \gamma = \frac{1}{4\pi} \;\text{ in }\; \mathbb{R}^3.$$

It can be shown (see [3]) by using Green's identities and the Sommerfeld radiation condition that u_ϵ satisfies the Lippmann-Schwinger representation formula given by

$$u_\epsilon(x,\hat{y}) = u_b(x,\hat{y}) + \sum_{m=1}^{M} k^2 \int\limits_{z_m+\epsilon B_m} (n_m - n)\mathbb{G}(x,z)u_\epsilon(z,\hat{y})\,dz + \int\limits_{z_m+\epsilon B_m} (A - A_m)\nabla_z \mathbb{G}(x,z)\cdot\nabla u_\epsilon(z,\hat{y})\,dz.$$

By linearity it is clear that the scattered field $u^s = u^s_\epsilon - u^s_b$ is due to the defective regions D_ϵ. This gives that scattered field u^s is given by

$$u^s(x,\hat{y}) = \sum_{m=1}^{M} k^2 \int\limits_{z_m+\epsilon B_m} (n_m - n)\mathbb{G}(x,z)u_\epsilon(z,\hat{y})\,dz + \int\limits_{z_m+\epsilon B_m} (A - A_m)\nabla_z \mathbb{G}(x,z)\cdot\nabla u_\epsilon(z,\hat{y})\,dz \tag{4.7}$$

From (4.7) an asymptotic expansion for $u^\infty(\hat{x},\hat{y})$ can be obtained by combining the asymptotic results from [3] and [48] as well as Theorem 3.4.1, namely

$$u^\infty(\hat{x},\hat{y}) = \gamma\epsilon^d k^2 \sum_{m=1}^{M} |B_m|\big(n_m - n\big)u_b(z_m,-\hat{x})u_b(z_m,\hat{y}) + \gamma\epsilon^d \sum_{m=1}^{M} \mathbf{M}^{(m)}\nabla u_b(z_m,-\hat{x})\cdot\nabla u_b(z_m,\hat{y}) + o(\epsilon^d), \tag{4.8}$$

where the polarization tensor $\mathbf{M}^{(m)}$ is given by

$$\mathbf{M}^{(m)}_{i,j} = e_i\cdot(A_m - A)e_j + \int\limits_{\partial B_m} [\nu(y)\cdot(A_m - A)e_j]\,\phi^+_i(y)\,ds_y$$

with e_i being the ith basis vector in $\mathbb{R}^d$ and ϕ_i is the solution to

$$\nabla \cdot A(x)\nabla\phi_i = 0 \ \text{ in } \ \mathbb{R}^d \setminus \overline{B}_m$$
$$\nabla \cdot A_m\nabla\phi_i = 0 \ \text{ in } \ B_m$$
$$\phi_i^- - \phi_i^+ = x_i \ \text{ on } \ \partial B_m$$
$$\frac{\partial}{\partial\nu_{A_m}}\phi_i^- - \frac{\partial}{\partial\nu_A}\phi_i^+ = \frac{\partial}{\partial\nu_A}x_i \ \text{ on } \ \partial B_m$$
$$\phi_i(x) = \mathcal{O}\left(\frac{1}{|x|^{d-1}}\right)$$

We now wish to use the leading term in (4.8) to determine the location of the defective regions $z_m + \epsilon B_m$. To this end assume that there are N incident and observation directions given by $\hat{y}_j, \hat{x}_i \in \mathbb{S}$ for $i, j = 1, 2, \ldots N$. Now we define the multi-static response matrix $\mathbf{F} \in \mathbb{C}^{N\times N}$ given by

$$\mathbf{F}_{i,j} = \gamma\epsilon^d \sum_{m=1}^{M} k^2|B_m|\big(n_m - n\big)u_b(z_m, -\hat{x}_i)u_b(z_m, \hat{y}_j) + \\ +\gamma\epsilon^d \sum_{m=1}^{M} \mathbf{M}^{(m)}\nabla u_b(z_m, -\hat{x}_i)\cdot\nabla u_b(z_m, \hat{y}_j). \qquad (4.9)$$

The *inverse problem* we consider here is to determine the set $\{z_m : m = 1, \ldots, M\}$ (i.e. the location of D_ϵ) from a knowledge of the measured far field pattern u_ϵ^∞ for given $\hat{y}_j, \hat{x}_i \in \mathbb{S}$ for $i, j = 1, 2, \ldots N$, provided that A, n and D are known. Since A, n and D are known we can compute u_b^∞, therefore from (4.8) we have that $\mathbf{F}_{i,j} \approx u_\epsilon^\infty(\hat{x}_i, \hat{y}_j) - u_b^\infty(\hat{x}_i, \hat{y}_j)$ up to $o(\epsilon^d)$, hence we assume that $\mathbf{F}$ is known for the measurements.

Remark 4.1.1. *The asymptotic expansion in* (4.8) *as well as the multi-static response matrix* $\mathbf{F}$ *given by* (4.9) *can be constructed for the more general case of an* *inhomogeneous background (i.e. where $A(x)$ is a matrix valued function and $n(x)$ is a scalar function in D). In this case A and n are replaced by $A(z_m)$ and $n(z_m)$ as well as replacing $\gamma u_b(z,-\hat{x})$ with $\mathbb{G}^\infty(\hat{x},z)$ in the asymptotic expansion.*

4.2 MUSIC Algorithm for Small Volume Defects

In this section we connect the locations of the defects $\{z_m : m = 1, \dots, M\}$ to the range of a matrix defined by the multi-static response matrix $\mathbf{F}$ to arrive at a MUSIC algorithm, which can be considered as a discrete analogue of the factorization method.

4.2.1 The case when $A = A_m$ and $n \neq n_m$

Assume we have a finite number of incident and observation directions where $N \geq M$, with M being the number of small defects and N being the number of incident and observation directions. We will also assume that we have full aperture data and that the incident and observation directions are given by the same equally distributed points on $\mathbb{S}$(i.e $\hat{y} = \hat{x}$). Also let the contracts $\big(n_m - n\big) \neq 0$ for all m. Notice that by (4.9) we have that the multi-static response matrix is given by

$$\mathbf{F}_{i,j} = \gamma\epsilon^d \sum_{m=1}^{M} k^2|B_m|\big(n_m - n\big)u_b(z_m, -\hat{x}_i)u_b(z_m, \hat{x}_j).$$

We now factorize that multi-static response matrix. To this end we define the matrices $\mathbf{U} \in \mathbb{C}^{N\times M}$, $\mathbf{V} \in \mathbb{C}^{N\times M}$ and $\mathbf{\Sigma} \in \mathbb{C}^{M\times M}$, where the columns of $\mathbf{U}$ are given by

$$\mathbf{U}_m = \big(u_b(z_m, -\hat{x}_1),\, \dots\, , u_b(z_m, -\hat{x}_N)\big)^\top$$

and the columns of $\mathbf{V}$ are given by

$$\mathbf{V}_m = \Big(\overline{u_b(z_m, \hat{x}_1)},\, \dots\, , \overline{u_b(z_m, \hat{x}_N)}\Big)^\top,$$

with $\mathbf{\Sigma} = \text{diag}(\sigma_m)$ where $\sigma_m = \gamma\epsilon^d k^2 |B_m| \big(n_m - n\big) \neq 0$. Therefore we have that $\mathbf{F} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^*$ and it follows that

$$\mathbf{F}\mathbf{F}^* = \mathbf{U}\widetilde{\mathbf{\Sigma}}\mathbf{U}^* \quad \text{with} \quad \widetilde{\mathbf{\Sigma}} = \mathbf{\Sigma}\mathbf{V}^*\mathbf{V}\mathbf{\Sigma}^*. \tag{4.10}$$

Now define the vector $\mathbf{g}_z \in \mathbb{C}^N$ for any point $z \in \mathbb{R}^d$ by

$$\mathbf{g}_z = \big(\, u_b(z, -\hat{x}_1),\, \dots, u_b(z, -\hat{x}_N)\,\big)^\top.$$

The goal of this section is to show that $\mathbf{g}_z$ is in the range of the matrix $\mathbf{F}\mathbf{F}^*$ if and only if $z \in \{z_m \, : \, m = 1, \dots, M\}$. Since we are interested in finding defects in a known scatterer D it is sufficient to prove the result only for values of $z \in D$. To do so we will follow the analytic framework as described in Section 4.1 of [55]. We now give a result that can be proven by using standard arguments from linear algebra (see proof of Theorem 3.1 in [44] for details).

Lemma 4.2.1. *Let the matrix* $\mathbf{F}$ *have the following factorization* $\mathbf{F} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^*$ *where* $\mathbf{U} \in \mathbb{C}^{N\times M}$, $\mathbf{V} \in \mathbb{C}^{N\times M}$ *and* $\mathbf{\Sigma} \in \mathbb{C}^{M\times M}$*. Assume that*

1. *The matrices* $\mathbf{U}$ *and* $\mathbf{V}$ *have full rank* M
2. *The matrix* $\mathbf{\Sigma}$ *is invertible*

then Range$\big(\mathbf{U}\big)$ =*Range*$\big(\mathbf{F}\mathbf{F}^*\big)$.

We now wish to construct an indicator function derived from a range test using Lemma 4.2.1 to detect the locations of the defects. For each sampling point

$z \in D$ we need to show that the vector $\mathbf{g}_z$ is in the range of $\mathbf{FF}^*$ if and only if $z \in \{z_m \, : \, m = 1, \ldots, M\}$. We now prove an auxiliary result that connects the location of the defects to the range of the matrix $\mathbf{U}$.

Theorem 4.2.1. *Assume that the set $S = \{\hat{x}_i \, : \, i \in \mathbb{N}\}$ is dense in $\mathbb{S}$ such that any analytic function that vanishes on S also vanishes on $\mathbb{S}$. Let $z \in D$ then there is a number $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$ we have that the matrices $\mathbf{U}$ and $\mathbf{V}$ have full rank, moreover $\mathbf{g}_z \in$ Range$\big(\mathbf{U}\big)$ if and only if $z \in \{z_m \, : \, m = 1, \ldots, M\}$.*

Proof. It is clear that $\mathbf{g}_{z_m}$ is in the range of $\mathbf{U}$ since $\mathbf{g}_{z_m}$ is the m-th column of $\mathbf{U}$. To prove that for $z \in D \setminus \{z_m \, : \, m = 1, \ldots, M\}$ that $\mathbf{g}_z$ is not in the range of $\mathbf{U}$ for some N sufficiently large we proceed by way of contradiction. Now assume that there does not exist such a N_0, then we have that there exists α_m^N, $\alpha^N \in \mathbb{C}$ such that

$$|\alpha^N| + \sum_{m=1}^{M} |\alpha_m^N| = 1 \tag{4.11}$$

and

$$\alpha^N u_b(z, -\hat{x}_i) + \sum_{m=1}^{M} \alpha_m^N u_b(z_m, -\hat{x}_i) = 0 \quad \text{for all} \quad 1 \leq i \leq N.$$

We then conclude that(up to a subsequence) α_m^N, $\alpha^N \to \alpha_m$, α as $N \to \infty$. This gives that

$$\alpha u_b(z, -\hat{x}_i) + \sum_{m=1}^{M} \alpha_m u_b(z_m, -\hat{x}_i) = 0 \quad \text{for all} \quad i \in \mathbb{N}.$$

Due to the density of S and since $u_b(z_m, -\hat{x}_i)$ is analytic with respect to $\hat{x}$ we have that

$$\alpha u_b(z, -\hat{x}) + \sum_{m=1}^{M} \alpha_m u_b(z_m, -\hat{x}) = 0 \quad \text{for all} \quad \hat{x} \in \mathbb{S}, \tag{4.12}$$

therefore since $\gamma u_b(z, -\hat{x}) = \mathbb{G}^{\infty}(\hat{x}, z)$ for $z \in D$ we can apply Rellich's lemma and the unique continuation principle to conclude that

$$\alpha \mathbb{G}(x, z) + \sum_{m=1}^{M} \alpha_m \mathbb{G}(x, z_m) = 0 \quad \text{for all} \quad x \in D \setminus \{z\} \cup \{z_m \,:\, m = 1, \dots, M\}.$$

Letting $x \to z, z_1, \dots, z_M$ gives that $\alpha_m = \alpha = 0$ for all m, but by (4.11) we have that

$$|\alpha| + \sum_{m=1}^{M} |\alpha_m| = 1$$

which gives a contradiction. Moreover the fact that $\mathbf{U}$ and $\mathbf{V}$ have full rank is a consequence of the given arguments. $\square$

Now by combining this with Lemma 4.2.1 and Theorem 4.2.1 we have the following range test.

Theorem 4.2.2. *Assume that the set $S = \{\hat{x}_i \,:\, i \in \mathbb{N}\}$ is dense in $\mathbb{S}$ such that any analytic function that vanishes on S also vanishes on $\mathbb{S}$. If $z \in D$ then there is a number $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$*

$$\mathbf{g}_z \in \textit{Range}\big(\mathbf{F}\mathbf{F}^*\big) \quad \textit{if and only if} \quad z \in \{z_m \,:\, m = 1, \dots, M\}.$$

Now we are ready to construct an indicator function which characterizes the locations of the defective regions. Let $\mathbf{P} : \mathbb{C}^N \mapsto \text{Null}\big(\mathbf{F}\mathbf{F}^*\big)$ be the orthogonal projection onto $\text{Null}\big(\mathbf{F}\mathbf{F}^*\big)$. Therefore by Theorem 4.2.2 we have that $\mathbf{P}\mathbf{g}_z = 0$ if and only if $z \in \{z_m \,:\, m = 1, \dots, M\}$. Notice that since $\mathbf{F}\mathbf{F}^*$ is a self-adjoint matrix we have that it is orthogonally diagonalizable. So we let $\mathbf{w}_j$ be the j-th orthonormal eigenvector of $\mathbf{F}\mathbf{F}^*$, we also let r=Rank$\big(\mathbf{F}\mathbf{F}^*\big)$. This gives that $\mathbf{w}_j$ for $r+1 \leq j \leq N$

is an orthonormal basis for $\text{Null}\big(\mathbf{F}\mathbf{F}^*\big)$ and therefore we have that $\mathbf{P}$ is given by

$$\mathbf{P}\mathbf{g}_z = \sum_{j=r+1}^{N} \big(\mathbf{g}_z, \mathbf{w}_j\big)_{\ell^2} \mathbf{w}_j.$$

This now leads to the following result.

Lemma 4.2.2. *Let $\mathbf{w}_j$ be the j-th orthonormal eigenvector of $\mathbf{F}\mathbf{F}^*$ and let r=Rank$\big(\mathbf{F}\mathbf{F}^*\big)$. Assume that the set $S = \{\hat{x}_i \, : \, i \in \mathbb{N}\}$ is dense in $\mathbb{S}$ such that any analytic function that vanishes on S also vanishes on $\mathbb{S}$. If $z \in D$ then there is a number $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$*

$$\mathcal{I}(z) = \left[\sum_{j=r+1}^{N} \big| \big(\mathbf{g}_z, \mathbf{w}_j\big) \big|^2_{\ell^2} \right]^{-1} < \infty \quad \text{if and only if} \quad z \in \{z_m \, : \, m = 1, \dots, M\}. \tag{4.13}$$

4.2.2 The case when $A \neq A_m$ and $n = n_m$

Next we consider the case where the defective regions have a different anisotropic constitutive parameter than the background media. We recall that by (4.9) the multi-static response matrix is given by

$$\mathbf{F}_{i,j} = \gamma \epsilon^d \sum_{m=1}^{M} \mathbf{M}^{(m)} \nabla u_b(z_m, -\hat{x}_i) \cdot \nabla u_b(z_m, \hat{x}_j).$$

We have that the matrix $\mathbf{M}^{(m)}$ is symmetric positive/negative definite provided that $A_m - A$ is symmetric positive/negative definite (see Theorems 3.3 and 3.4 in [48]). Just as in the previous case we wish to factorize the matrix $\mathbf{F}$ and use Lemma 4.2.1 to construct an indicator function for the locations defective regions. Without loss of generality let $d = 2$ and the analysis is similar for $d = 3$. Therefore we have that $\mathbf{M}^{(m)} \in \mathbb{C}^{2\times 2}$ is unitarily diagonalizable which gives that there are eigenvectors $\mathbf{v}_1^{(m)}$ and $\mathbf{v}_2^{(m)}$ that form an orthonormal basis for $\mathbb{C}^2$. Now denote the non-zero eigenvalues of $\mathbf{M}^{(m)}$ by $\lambda_1^{(m)}$ and $\lambda_2^{(m)}$. This then gives that

$$\mathbf{M}^{(m)}\nabla u_b(z_m, -\hat{x}_i)\cdot\nabla u_b(z_m, \hat{x}_j) = \sum_{k=1}^{2} \lambda_k^{(m)} \left(\overline{\mathbf{v}_k^{(m)}} \cdot \nabla u_b(z_m, -\hat{x}_i)\right) \left(\mathbf{v}_k^{(m)} \cdot \nabla u_b(z_m, \hat{x}_j)\right).$$

Therefore the multi-static response matrix can be written as

$$\mathbf{F}_{i,j} = \gamma\epsilon^2 \sum_{m=1}^{M}\sum_{k=1}^{2} \lambda_k^{(m)} \left(\overline{\mathbf{v}_k^{(m)}} \cdot \nabla u_b(z_m, -\hat{x}_i)\right) \left(\mathbf{v}_k^{(m)} \cdot \nabla u_b(z_m, \hat{x}_j)\right).$$

Now to factorize $\mathbf{F}$ where we assume that $N > 2M$ and define the matrices $\mathbf{W} \in \mathbb{C}^{N\times 2M}$, $\mathbf{Q} \in \mathbb{C}^{N\times 2M}$ and $\mathbf{T} \in \mathbb{C}^{2M\times 2M}$, where the columns of $\mathbf{W}$ are given by

$$\begin{aligned}\mathbf{W}_{2m-1} &= \left(\overline{\mathbf{v}_1^{(m)}} \cdot \nabla u_b(z_m, -\hat{x}_1), \, \dots \, , \overline{\mathbf{v}_1^{(m)}} \cdot \nabla u_b(z_m, -\hat{x}_N)\right)^\top \\ \mathbf{W}_{2m} &= \left(\overline{\mathbf{v}_2^{(m)}} \cdot \nabla u_b(z_m, -\hat{x}_1), \, \dots \, , \overline{\mathbf{v}_2^{(m)}} \cdot \nabla u_b(z_m, -\hat{x}_N)\right)^\top,\end{aligned}$$

the columns of $\mathbf{Q}$ are given by

$$\begin{aligned}\mathbf{Q}_{2m-1} &= \left(\overline{\mathbf{v}_1^{(m)} \cdot \nabla u_b(z_m, \hat{x}_1)}, \, \dots \, , \overline{\mathbf{v}_1^{(m)} \cdot \nabla u_b(z_m, \hat{x}_N)}\right)^\top \\ \mathbf{Q}_{2m} &= \left(\overline{\mathbf{v}_2^{(m)} \cdot \nabla u_b(z_m, \hat{x}_1)}, \, \dots \, , \overline{\mathbf{v}_2^{(m)} \cdot \nabla u_b(z_m, \hat{x}_N)}\right)^\top,\end{aligned}$$

the diagonal matrix

$$\mathbf{T} = \text{diag}(\tau_j) \quad \text{where} \quad \tau_{2m-1} = \gamma\epsilon^2\lambda_1^{(m)} \quad \text{and} \quad \tau_{2m} = \gamma\epsilon^2\lambda_2^{(m)}.$$

Therefore just as in the previous case we have that $\mathbf{F} = \mathbf{W}\mathbf{T}\mathbf{Q}^*$ and it follows that $\mathbf{F}\mathbf{F}^* = \mathbf{W}\widetilde{\mathbf{T}}\mathbf{W}^*$ with $\widetilde{\mathbf{T}} = \mathbf{T}\mathbf{Q}^*\mathbf{Q}\mathbf{T}^*$.

Now define the vector $\mathbf{g}_{z,b} \in \mathbb{C}^N$ for any point $z \in \mathbb{R}^2$ and $b \neq 0 \in \mathbb{C}^2$ by

$$\mathbf{g}_{z,b} = \big(b \cdot \nabla u_b(z, -\hat{x}_1),\, \dots\, , b \cdot \nabla u_b(z, -\hat{x}_N)\big)^\top.$$

Just as in the previous section our goal is to show that $\mathbf{g}_{z,b}$ is in the range of the matrix $\mathbf{F}\mathbf{F}^*$ if and only if $z \in \{z_m \,:\, m = 1, \dots, M\}$, and once again since we are interested in finding defects in a known scatterer D we prove the result for $z \in D$. Following in the same way as in Theorem 4.2.1 we can prove the following result.

Theorem 4.2.3. *Assume that the set $S = \{\hat{x}_i \,:\, i \in \mathbb{N}\}$ is dense in $\mathbb{S}$ such that any analytic function that vanishes on S also vanishes on $\mathbb{S}$. Let $z \in D$ then there is a number $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$ we have that the matrices $\mathbf{W}$ and $\mathbf{Q}$ have full rank, moreover $\mathbf{g}_{z,b} \in$ Range$\big(\mathbf{W}\big)$ if and only if $z \in \{z_m \,:\, m = 1, \dots, M\}$.*

Proof. It is clear that when $z = z_m$ that $\mathbf{g}_{z_m,b} \in \text{Range}\big(\mathbf{W}\big)$, since $\mathbf{v}_1^{(m)}$ and $\mathbf{v}_2^{(m)}$ form a basis for $\mathbb{C}^2$ for any m. Therefore $\mathbf{g}_{z_m,b}$ is a linear combination of two columns of $\mathbf{U}$ by expressing b as a combination of $\overline{\mathbf{v}_1^{(m)}}$ and $\overline{\mathbf{v}_2^{(m)}}$. Now we prove by way of contradiction that if $z \neq z_m$ then $\mathbf{g}_{z,b}$ can not be a linear combination of the columns of $\mathbf{U}$ which proves that result. Assume that there does not exist such a N_0, then we have that there exists α_j^N, $\alpha^N \in \mathbb{C}$ such that

$$|\alpha^N| + \sum_{j=1}^{2M} |\alpha_j^N| = 1 \tag{4.14}$$

and

$$\alpha^N b \cdot \nabla u_b(z, -\hat{x}_i) + \sum_{m=1}^{M} \alpha^N_{2m-1} \overline{\mathbf{v}_1^{(m)}} \cdot \nabla u_b(z_m, -\hat{x}_i) + \alpha^N_{2m} \overline{\mathbf{v}_2^{(m)}} \cdot \nabla u_b(z_m, -\hat{x}_i) = 0$$

for all $1 \leq i \leq N$. We then conclude that (up to a subsequence) α^N_j, $\alpha^N \to \alpha_j$, α as $N \to \infty$. This gives that just as in Theorem 4.2.1

$$\alpha b \cdot \nabla \mathbb{G}(x, z) + \sum_{m=1}^{M} \alpha_{2m-1} \overline{\mathbf{v}_1^{(m)}} \cdot \nabla \mathbb{G}(x, z_m) + \alpha_{2m} \overline{\mathbf{v}_2^{(m)}} \cdot \nabla \mathbb{G}(x, z_m) = 0$$

for all $x \in D \setminus \{z\} \cup \{z_m \,:\, m = 1, \ldots, M\}$. Letting $x \to z, z_1, \ldots, z_M$ gives that $\alpha_j = \alpha = 0$ for all j, which gives the result by way of contradiction. □

Similarly as in the other case we can use the eigenvector of $\mathbf{FF}^*$ to construct an indicator function, doing so gives the following result.

Lemma 4.2.3. *Let $\mathbf{w}_j$ be the j-th orthonormal eigenvector of $\mathbf{FF}^*$ and let r=Rank$\big(\mathbf{FF}^*\big)$. Assume that the set $S = \{\hat{x}_i \,:\, i \in \mathbb{N}\}$ is dense in $\mathbb{S}$ such that any analytic function that vanishes on S also vanishes on $\mathbb{S}$. If $z \in D$ then there is a number $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$ we have that*

$$\mathcal{I}_b(z) = \left[\sum_{j=r+1}^{N} \left| \big(\mathbf{g}_{z,b}, \mathbf{w}_j \big) \right|^2_{\ell^2} \right]^{-1} < \infty \quad \textit{if and only if} \quad z \in \{z_m \,:\, m = 1, \ldots, M\}. \tag{4.15}$$

4.2.3 The case when $A \neq A_m$ and $n \neq n_m$

For this case (4.9) gives that the multi-static response matrix is given by $\mathbf{F} = \mathbf{U\Sigma V}^* + \mathbf{WTQ}^*$. Now just as in [71] we write $\mathbf{F}$ in a block matrix form such that $\mathbf{F} = (\mathbf{U} \;\; \mathbf{W})\, \mathbf{\Pi}\, (\mathbf{V} \;\; \mathbf{Q})^* \in \mathbb{C}^{N\times N}$ with

$$\mathbf{\Pi} = \begin{pmatrix} \mathbf{\Sigma} & \mathbf{0} \\ \mathbf{0} & \mathbf{T} \end{pmatrix} \in \mathbb{C}^{(d+1)M\times(d+1)M}$$

where the matrices are as defined in the previous sections. Assume that $N > (d+1)M$, we wish to conclude a similar result as in Lemma 4.2.2 and 4.2.3. Now define a vector such that for any point $z \in \mathbb{R}^d$ and $b \in \mathbb{C}^d$ is given by

$$\mathbf{g}_{z,(1,b)} = \mathbf{g}_z + \mathbf{g}_{z,b}$$

where the vectors $\mathbf{g}_z$ and $\mathbf{g}_{z,b}$ are as defined in the previous sections. Notice that by combining the proofs of Theorems 4.2.1 and 4.2.3 we have that following result.

Theorem 4.2.4. *Assume that the set $S = \{\hat{x}_i \,:\, i \in \mathbb{N}\}$ is dense in $\mathbb{S}$ such that any analytic function that vanishes on S also vanishes on $\mathbb{S}$. Let $z \in D$ then there is a number $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$ we have that the matrices $(\mathbf{U} \;\; \mathbf{W})$ and $(\mathbf{V} \;\; \mathbf{Q})$ have full rank, moreover $\mathbf{g}_{z,(1,b)} \in Range\big\{\, (\mathbf{U} \;\; \mathbf{W})\,\big\}$ if and only if $z \in \{z_m \,:\, m = 1, \ldots, M\}$.*

This give the following lemma.

Lemma 4.2.4. *Let $\mathbf{w}_j$ be the j-th orthonormal eigenvector of $\mathbf{FF}^*$ and let $r = Rank\big(\mathbf{FF}^*\big)$. Assume that the set $S = \{\hat{x}_i \,:\, i \in \mathbb{N}\}$ is dense in $\mathbb{S}$ such that any analytic function that vanishes on S also vanishes on $\mathbb{S}$. If $z \in D$ then there is a number $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$ we have that*

$$\mathcal{I}_{(1,b)}(z) = \left[\sum_{j=r+1}^{N} \left|\big(\mathbf{g}_{z,(1,b)}, \mathbf{w}_j\big)\right|^2_{\ell^2}\right]^{-1} < \infty \quad \textit{if and only if} \quad z \in \{z_m \,:\, m = 1, \ldots, M\}. \tag{4.16}$$

4.2.4 Numerical Validation of the MUSIC Algorithm

In this section we discuss the numerical implementation of the MUSIC algorithm derived in the previous section for the $\mathbb{R}^2$ case. We note that the algorithm in the $\mathbb{R}^3$ case is similar to what is presented here. To this end, we use simulated far-field data to reconstruct the defects in a square scatterer. The simulated data comes form solving the direct scattering problems (4.1)-(4.2) and (4.5)-(4.6) using a cubic finite element method with a perfectly matched layer. From this we will have the approximated scattered fields $u^s_\epsilon(\cdot\,,\hat{x})$ and $u^s_b(\cdot\,,\hat{x})$. The multi-static response matrix will be defined as

$$\mathbf{F} = \left[u^\infty_\epsilon(\hat{x}_i,\hat{x}_j) - u^\infty_b(\hat{x}_i,\hat{x}_j)\right]^N_{i,j=1},$$

where the far-field patterns are given by the solutions of the direct problems using the finite element method. In the following we use N different directions on the unit circle given by

$$\hat{x}_i = \Big(\cos\left(2\pi(i-1)/N\right), \sin\left(2\pi(i-1)/N\right)\Big) \quad \text{for} \quad i = 1,\dots,N.$$

In all examples we take $D = [-2,2]^2$ and we fix the wave number $k = 1$ unless otherwise stated.

We want to illustrate the performance of the MUSIC algorithm in reconstructing the defective regions $z_m + \epsilon B_m$ inside D. We give examples with random noise added to the simulated data for $u^\infty_\epsilon(\hat{x}_i,\hat{x}_j)$. The random noise level is given by δ where the noise is added to the far-field data $u^\infty_\epsilon(\hat{x}_i,\hat{x}_j) + \delta E_{i,j}$ and the random matrix E is such that $\|E\|_2 = 1$. Since we have that A, n and D are known for non-destructive testing we can assume that the far-field pattern $u^\infty_b(\hat{x}_i,\hat{x}_j)$ is known by direct computations as presented in this section. To evaluate the indicator function we need the total field $u_b(z, \hat{x})$ at sampling points inside of D. To this end we use the mesh points in the finite element method z_p as the sampling points.

Example 1. We reconstruct two defective regions that are given by circles centered at $(1, 1)$ and $(-1, -1)$ with radius $\epsilon = 0.3$. We let $A = A_m = I$ where the refractive index in the scatterer D is given by $n = 0.5$ and $n_{1,2} = 3$ are the refractive indices in defective region D_ϵ. In Figure 4.2 we give the reconstruction with 5% and 10% random noise added for $N = 32$.

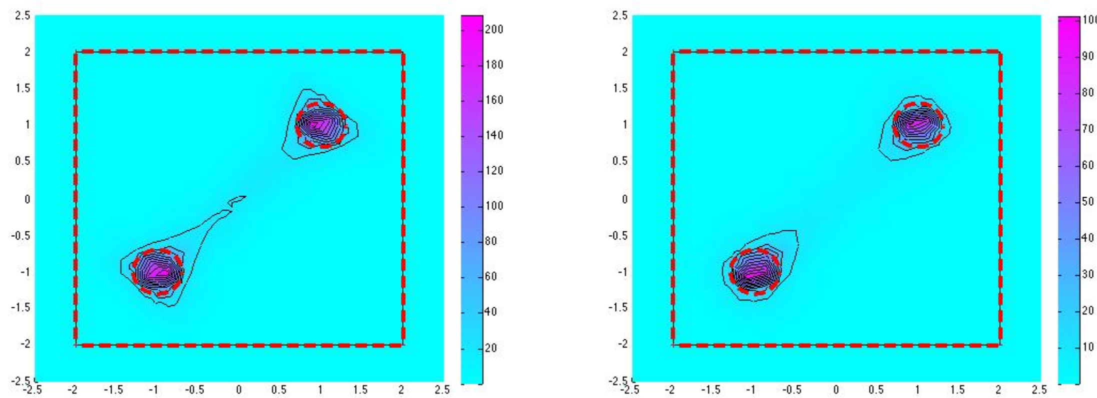

Figure 4.2: On the left is the reconstruction with 5% noise and on the right is with 10% noise, of 2 small circular defects in an isotropic media.

Example 2. We reconstruct the an ellipse centered at $(0.5, -1)$ with axes equal 0.5 and 0.3 in the scatterer. The material parameters are $A = 0.5I$ and $n = 5$ in the scatterer D. While the parameters are $A_1 = I$ and $n_1 = 1$ in the defective region D_ϵ. In Figure 4.3 we give the reconstruction without any added noise and with 10% random noise added for $N = 64$.

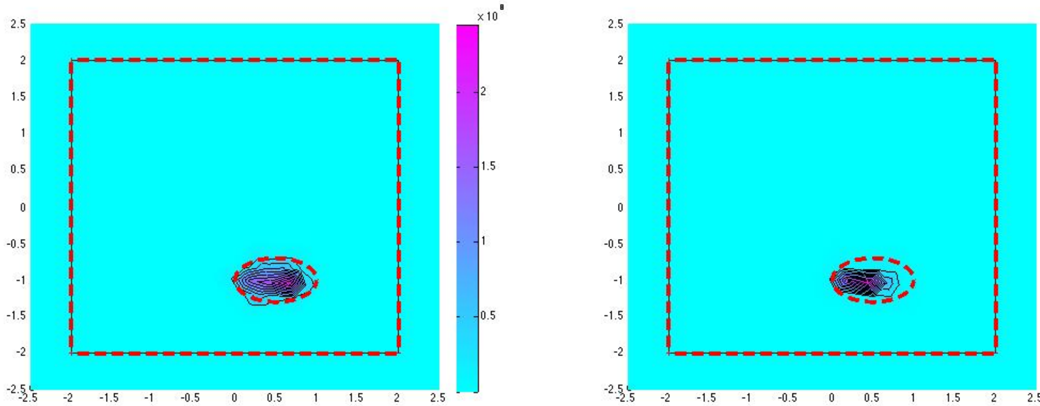

Figure 4.3: On the left is the reconstruction without noise and on the right is with 10% noise, of an ellipse shaped void.

Example 3. We reconstruct two defective regions that are given by circles centered at $(1, 1)$ and $(-1, -1)$ with radius $\epsilon = 0.3$. The material parameters are $A = 0.5I$ and $n = 5$ in the scatterer D. While the parameters are $A_{1,2} = I$ and $n_{1,2} = 1$ in the defective region D_ϵ. In Figure 4.4 we give the reconstruction without any added noise and with 10% random noise added for $N = 64$.

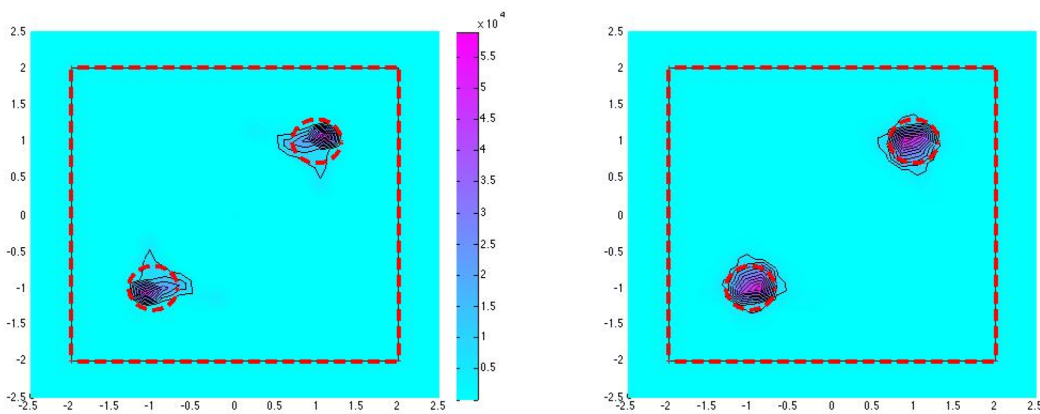

Figure 4.4: On the left is the reconstruction without noise and on the right is with 10% noise, of 2 circular voids in a layered media.

Example 4. We reconstruct two defective regions that are given by circles centered at $(-1, 1)$ and $(1, -1)$ with radius $\epsilon = 0.3$. The material parameters are

$$A = \begin{pmatrix} 10 & 1 \\ 1 & 10 \end{pmatrix} \quad \text{and} \quad n = 5$$

in the scatterer D. While the parameters are $A_{1,2} = I$ and $n_{1,2} = 1$ in the defective region D_ϵ. In Figure 4.5 we give the reconstruction without any added noise and with 1% random noise added for $N = 64$.

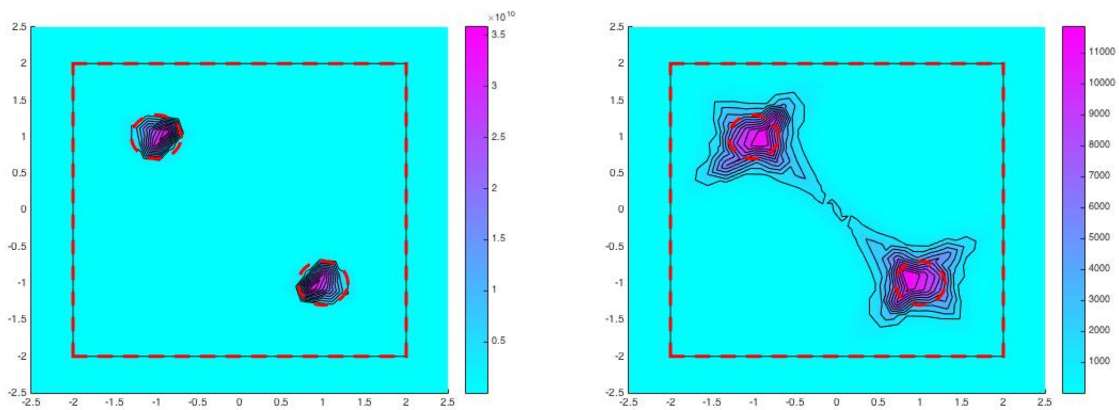

Figure 4.5: On the left is the reconstruction without noise and on the right is with 1% noise, of 2 circular voids in an anisotropic media.

4.3 The Transmission Eigenvalue Problem For Small Volume Defects

Having reconstructed the location of the small defects we would like to obtain additional information from the transmission eigenvalues. Since the multi-static data used for the MUSIC algorithm can determine the transmission eigenvalues (see Theorem 2.2.1) we now investigate how the small defects affect the transmission eigenvalues. Therefore we consider the transmission eigenvalue problem for an anisotropic media with small penetrable defects. Referring to the configuration in the beginning of this Chapter we allow the coefficients of the unperturbed media to be non-constant. Therefore we assume that for the convergence analysis that $A(x) \in C^1(D, \mathbb{R}^{d\times d})$ with $A = A^\top$ and $n(x) \in C^1(D)$ being the given coefficients for an unperturbed medium. The coefficients for the perturbed media are given by A_ϵ and n_ϵ defined by (4.3) and (4.4).

Definition 4.3.1. *The transmission eigenvalues are the values $k_\epsilon \in \mathbb{C}$ such that there is a non-trivial solution $(w, v) \in H^1(D) \times H^1(D)$ such that*

$$\nabla \cdot A_\epsilon \nabla w + k_\epsilon^2 n_\epsilon w = 0 \quad \textit{and} \quad \Delta v + k_\epsilon^2 v = 0 \qquad \textit{in } D \tag{4.17}$$

$$w = v \quad \textit{and} \quad \frac{\partial w}{\partial \nu_{A_\epsilon}} = \frac{\partial v}{\partial \nu} \qquad \textit{on } \partial D \tag{4.18}$$

With the continuity condition

$$\frac{\partial w^+}{\partial \nu_A} = \frac{\partial w^-}{\partial \nu_{A_m}} \quad \text{on} \quad \partial(z_m + \epsilon B_m).$$

Let us define the constants

$$\inf_{x\in D} \inf_{|\xi|=1} \overline{\xi} \cdot A(x)\xi = A_{min} > 0 \qquad \text{and} \qquad \sup_{x\in D} \sup_{|\xi|=1} \overline{\xi} \cdot A(x)\xi = A_{max} < \infty$$

$$\min_{m=1\ldots M} \inf_{|\xi|=1} \overline{\xi} \cdot A_m\xi = a_{min} > 0 \qquad \text{and} \qquad \max_{m=1\ldots M} \sup_{|\xi|=1} \overline{\xi} \cdot A_m\xi = a_{max} < \infty.$$

In this study we are interested in showing that the eigenvalues k_ϵ for the perturbed media converge to the eigenvalues for the unperturbed media as $\epsilon \to 0$ and derive an asymptotic formula that can be used to obtain more information about the small defects. The transmission eigenvalue problem problem for the unperturbed media satisfies (4.17)-(4.18) with coefficients A and n. To analyze this problem we define the variational space

$$X(D) := \big\{(w,v) :\ w,v \in H^1(D)\,|\, w-v \in H^1_0(D)\big\}$$

equipped with the $H^1(D) \times H^1(D)$ inner product. It is clear that the variational form of (4.17)-(4.18) is given by

$$\int\limits_D A_\epsilon \nabla w \cdot \nabla \overline{\varphi}_1 - \nabla v \cdot \nabla \overline{\varphi}_2 - k_\epsilon^2 (n_\epsilon w \overline{\varphi}_1 - v\overline{\varphi}_2)\, dx = 0 \qquad \text{for all } (\varphi_1, \varphi_2) \in X(D).$$

For connivence we define the bounded sesquilinear forms

$$\mathcal{A}_\epsilon\big((w,v);(\varphi_1,\varphi_2)\big) := \int\limits_D A_\epsilon \nabla w \cdot \nabla \overline{\varphi}_1 + A_{min} w \overline{\varphi}_1\, dx - \int\limits_D \nabla v \cdot \nabla \overline{\varphi}_2 + v\overline{\varphi}_2\, dx,$$

$$\mathcal{B}_\epsilon\big((w,v);(\varphi_1,\varphi_2)\big) := \int\limits_D n_\epsilon w \overline{\varphi}_1 \; - v\overline{\varphi}_2\, dx,$$

and

$$\mathcal{C}\big((w,v);(\varphi_1,\varphi_2)\big) := \int\limits_D A_{min} w \overline{\varphi}_1 \; - v\overline{\varphi}_2\, dx.$$

Therefore we have that (4.17)-(4.18) can be written as for all $(\varphi_1, \varphi_2) \in X(D)$

$$\mathcal{A}_\epsilon\big((w,v);(\varphi_1,\varphi_2)\big) - k_\epsilon^2 \mathcal{B}_\epsilon\big((w,v);(\varphi_1,\varphi_2)\big) - \mathcal{C}\big((w,v);(\varphi_1,\varphi_2)\big) = 0. \tag{4.19}$$

Let us define by $\mathbf{A}_\epsilon$, $\mathbf{B}_\epsilon$ and $\mathbf{C} : X(D) \to X(D)$ the bounded linear operators defined from $\mathcal{A}_\epsilon\big(\cdot\,;\cdot\big)$, $\mathcal{B}_\epsilon\big(\cdot\,;\cdot\big)$ and $\mathcal{C}\big(\cdot\,;\cdot\big)$ by means of the Riesz representation theorem. For $\epsilon = 0$ we define the operators corresponding to the unperturbed media. It can be shown using $\mathbb{T}$-coercivity then if A_{min} and $a_{min} > 1$ that $\mathbf{A}_\epsilon$ is invertible with the norm of the inverse independent of $\epsilon \geq 0$. To this end we consider the isomorphism $\mathbb{T}(w,v) = (w, -v+2w) : X(D) \mapsto X(D)$ (it is easy to check that $\mathbb{T} = \mathbb{T}^{-1}$). Using this we now show that $\mathcal{A}_\epsilon\big((w,v);\mathbb{T}(w,v)\big)$ is coercive in $X(D)$. To this end we have that

$$\begin{aligned}\Big|\mathcal{A}_\epsilon\big((w,v);\mathbb{T}(w,v)\big)\Big| \;&\geq\; \int\limits_D A_\epsilon \nabla w \cdot \nabla \overline{w} + A_{min}|w|^2 \, dx + \int\limits_D |\nabla v|^2 + |v|^2 \, dx \\ &-\; 2\left|\int\limits_D \nabla v_\epsilon \cdot \nabla \overline{w}_\epsilon + v_\epsilon \overline{w}_\epsilon \, dx\right|\end{aligned}$$

and by Young's inequality we obtain that

$$\Big|\mathcal{A}_\epsilon\big((w,v);\mathbb{T}(w,v)\big)\Big| \;\geq\; \left(\alpha - \frac{1}{\delta}\right)||w_\epsilon||^2_{H^1(D)} + (1-\delta)||v_\epsilon||^2_{H^1(D)}.$$

where we let $\alpha = \min\{A_{min},\, a_{min}\}$. Therefore we have proven that $\mathcal{A}_\epsilon\big((w,v);\mathbb{T}(w,v)\big)$ is coercive provided that $\delta \in (1/\alpha,\, 1)$. Similar arguments hold for A_{max} and $a_{max} < 1$ where A_{min} is replaced by A_{max} in $\mathcal{A}_\epsilon\big(\cdot\,;\cdot\big)$ and $\mathcal{C}\big(\cdot\,;\cdot\big)$ with $\mathbb{T}(w,v) = (w-2v, -v)$. It is clear that in either case that $\mathbf{B}_\epsilon$ and $\mathbf{C}$ are compact operators by appealing to the compact embedding of $H^1(D)$ in $L^2(D)$. Now by (4.19) it is clear that (w,v) are eigenfunctions corresponding to the eigenvalue k_ϵ provided that

$$\big(\mathbf{I} - k_\epsilon^2 \mathbf{A}_\epsilon^{-1}\mathbf{B}_\epsilon - \mathbf{A}_\epsilon^{-1}\mathbf{C}\big)(w,v) = (0,0). \tag{4.20}$$

Lets denote the eigenvalue parameter $\tau_\epsilon = k_\epsilon^2$ and define

$$\mathbf{T}_\epsilon(\tau_\epsilon) := \mathbf{A}_\epsilon^{-1}\mathbf{B}_\epsilon + \frac{1}{\tau_\epsilon}\mathbf{A}_\epsilon^{-1}\mathbf{C} : X(D) \to X(D). \tag{4.21}$$

We can now rephrase (4.19) as a non-linear eigenvalue problem

$$\tau_\epsilon \mathbf{T}_\epsilon(\tau_\epsilon)(w, v) = (w, v). \tag{4.22}$$

It is clear that $\mathbf{T}_\epsilon(\tau)$ depends analytically on τ in any subset of the complex plane that does not include the origin. The operator $\mathbf{T}_\epsilon(\tau)$ corresponds to the perturbed transmission eigenvalue problem with coefficients A_ϵ and n_ϵ, while the operator $\mathbf{T}_0(\tau)$ corresponds to the unperturbed transmission eigenvalues problem with coefficients A and n. Similarly we can rephrase the unperturbed transmission eigenvalue problem as

$$\tau \mathbf{T}_0(\tau)(w, v) = (w, v). \tag{4.23}$$

In order to prove the convergence of the transmission eigenvalues for the perturbed problem to the unperturbed problem we need to show that $\mathbf{T}_\epsilon(\tau)$ converges to $\mathbf{T}_0(\tau)$ in the operator norm.

4.3.1 Convergence of the Spectrum

In this section we will study the convergence of $\mathbf{T}_\epsilon(\tau)$ in the operator norm to the unperturbed operator $\mathbf{T}_0(\tau)$ and then use results from [64] to prove convergence for the transmission eigenvalues and eigenfunctions. To this end notice that since $\mathbf{B}_\epsilon$ and $\mathbf{C}$ are compact operators along with using that $\|\mathbf{A}_\epsilon^{-1}\|$ is uniformly bounded we can conclude that $\mathbf{T}_\epsilon(\tau)$ is compact for all $\epsilon \geq 0$, where in this section $\|\cdot\|$ will refer to the operator norm from $X(D)$ to itself and $(\cdot\,;\cdot)$ will refer to that inner product on $X(D)$. The convergence of $\mathbf{T}_\epsilon(\tau)$ would then imply the convergence of the transmission eigenvalues. We start by studying the convergence of the operator

$\mathbf{B}_\epsilon$ to $\mathbf{B}_0$, where the operators are defined by

$$\big(\mathbf{B}_\epsilon(w,v);(\varphi_1,\varphi_2)\big) = \int\limits_D n_\epsilon w\overline{\varphi}_1 \; - v\overline{\varphi}_2 \, dx \tag{4.24}$$

and

$$\big(\mathbf{B}_0(w,v);(\varphi_1,\varphi_2)\big) = \int\limits_D n w\overline{\varphi}_1 \; - v\overline{\varphi}_2 \, dx. \tag{4.25}$$

Theorem 4.3.1. *Let the operators $\mathbf{B}_\epsilon$ and $\mathbf{B}_0$ be defined by the variational form (4.24) and (4.25) respectively. Then $\mathbf{B}_\epsilon \to \mathbf{B}_0$ in norm, moreover for some $\alpha \in (0,1)$ we have that $\|\mathbf{B}_\epsilon - \mathbf{B}_0\| \leq C\epsilon^\alpha$ in $\mathbb{R}^d$ for some C independent of ϵ, $d = 2,3$.*

Proof. By subtracting (4.24) from (4.25) we have that

$$\begin{aligned} \big|\big(\mathbf{B}_\epsilon(w,v);(\varphi_1,\varphi_2)\big) - \big(\mathbf{B}_0(w,v);(\varphi_1,\varphi_2)\big)\big| &= \left| \int\limits_{D_\epsilon} (n_\epsilon - n) w\overline{\varphi}_1 \, dx \right| \\ &\leq ||(n_\epsilon - n)w||_{L^2(D_\epsilon)} ||(\varphi_1,\varphi_2)||_{X(D)}. \end{aligned}$$

Therefore we have that $\big\|\big(\mathbf{B}_\epsilon - \mathbf{B}_0\big)(w,v)\big\|_{X(D)} \leq ||(n_\epsilon - n)w||_{L^2(D_\epsilon)}$. Now since $w \in H^1(D)$ we have from Sobolev's embedding in $\mathbb{R}^2$ or $\mathbb{R}^3$ that $w \in L^p(D)$ for some $p \geq 2$ (see e.g. [8] for embedding results). We then conclude that $|w|^2 \in L^{p/2}(D)$. Now let q be defined by $\frac{1}{p/2} + \frac{1}{q} = 1$ notice that $\frac{1}{q} = \frac{p-2}{p}$. Therefore by using the duality between $L^{p/2}(D)$ and $L^q(D)$ along with Sobolev's embedding we have that

$$\begin{aligned}\big\|\big(\mathbf{B}_\epsilon - \mathbf{B}_0\big)(w,v)\big\|^2_{X(D)} &\leq ||(n_\epsilon - n)||^2_\infty ||w||^2_{L^2(D_\epsilon)} \\ &\leq C||\,|w|^2||_{L^{p/2}(D)}||\chi_{D_\epsilon}||_{L^q(D)} \\ &= C|D_\epsilon|^{1/q}||w||^2_{L^p(D)} \\ &\leq C\epsilon^{d/q}||(w,v)||^2_{X(D)}.\end{aligned}$$

Hence we have that

$$||\mathbf{B}_\epsilon - \mathbf{B}_0|| \leq C\epsilon^{d/2q} \quad \text{for } \ d = 2,3$$

where the constant C incorporates the norm of the contrasts but is independent of ϵ. Now for the $\mathbb{R}^2$ for any choice of $p > 2$ we have that $\frac{1}{q} < 1$ giving the result. For the case in $\mathbb{R}^3$ we can choose $p < 6$ giving that $\frac{1}{q} < 2/3$ and therefore $d/2q < 1$, which gives the result in $\mathbb{R}^3$. □

We are now interested in the convergence of $\mathbf{A}_\epsilon^{-1}\mathbf{B}_\epsilon$ and $\mathbf{A}_\epsilon^{-1}\mathbf{C}$ as ϵ tends to zero. The operator $\mathbf{A}_\epsilon$ and $\mathbf{A}_0$ are defined by the Riesz representation theorem such that

$$\begin{aligned}\big(\mathbf{A}_\epsilon(w,v);(\varphi_1,\varphi_2)\big) = &\int_D A_\epsilon \nabla w\cdot\nabla\overline{\varphi}_1 + A_{min} w\overline{\varphi}_1\,dx \\ &- \int_D \nabla v\cdot\nabla\overline{\varphi}_2 + v\overline{\varphi}_2\,dx \qquad (4.26)\end{aligned}$$

and

$$\big(\mathbf{A}_0(w,v);(\varphi_1,\varphi_2)\big) = \int\limits_D A\nabla w\cdot\nabla\overline{\varphi}_1 + A_{min} w\overline{\varphi}_1\,dx$$
$$-\int\limits_D \nabla v\cdot\nabla\overline{\varphi}_2 + v\overline{\varphi}_2\,dx. \qquad (4.27)$$

As we have mentioned, due to $\mathbb{T}$-coercivity $\mathbf{A}_\epsilon^{-1}$ exists as a bounded linear operator for all $\epsilon \geq 0$ where the norm of $\mathbf{A}_\epsilon^{-1}$ is uniformly bounded with respect to ϵ. To study the convergence of $\mathbf{A}_\epsilon^{-1}\mathbf{B}_\epsilon$ and $\mathbf{A}_\epsilon^{-1}\mathbf{C}$ we first need some regularity results pertaining to $\mathbf{B}_0$ and $\mathbf{C}$. Notice that by the variational definition of $\mathbf{B}_0$ we have that for any $(f,g)\in X(D)$ if we denote $\mathbf{B}_0(f,g)=(w,v)$ then

$$-\Delta w + w = nf \quad \text{and} \quad -\Delta v + v = g \quad \text{in } D. \qquad (4.28)$$

Therefore by elliptic regularity we have that w and v are in $H^3_{loc}(D)$ provided that n is continuously differentiable, and for any $\Omega\subset D$

$$\|w\|_{H^3(\Omega)} + \|v\|_{H^3(\Omega)} \leq C\left(\|f\|_{H^1(D)} + \|g\|_{H^1(D)}\right).$$

Next, recall that the operator $\mathbf{C}$ is defined via the Riesz Representation Theorem from the variational form

$$\big(\mathbf{C}(w,v);(\varphi_1,\varphi_2)\big) = \int\limits_D A_{min} w\overline{\varphi}_1 - v\overline{\varphi}_2\,dx.$$

Therefore we have that for any $(f,g)\in X(D)$ if we denote $\mathbf{C}(f,g)=(w,v)$ then

$$-\Delta w + w = A_{min} f \quad \text{and} \quad -\Delta v + v = g \quad \text{in } D,$$

and we have the elliptic regularity estimates for any $\Omega \subset D$

$$\|w\|_{H^3(\Omega)} + \|v\|_{H^3(\Omega)} \leq C \left(\|f\|_{H^1(D)} + \|g\|_{H^1(D)} \right).$$

Theorem 4.3.2. *Let the operators $\mathbf{A}_\epsilon$ and $\mathbf{A}_0$ be defined by the variational form (4.26) and (4.27) respectively. Then we have that*

$$\mathbf{A}_\epsilon^{-1}\mathbf{B}_\epsilon \to \mathbf{A}_0^{-1}\mathbf{B}_0 \quad \textit{and} \quad \mathbf{A}_\epsilon^{-1}\mathbf{C} \to \mathbf{A}_0^{-1}\mathbf{C}$$

in the operator norm as $\epsilon \to 0$.

Proof. Consider the pair (w_ϵ, v_ϵ) and (w, v) in $X(D)$ defined by

$$(w_\epsilon, v_\epsilon) = \mathbf{A}_\epsilon^{-1}(f, g) \quad \text{and} \quad (w, v) = \mathbf{A}_0^{-1}(f, g)$$

for any $(f, g) \in X(D)$. Using (4.26) and (4.27) we have that

$$\mathcal{A}_\epsilon\big((w - w_\epsilon, v - v_\epsilon); (\varphi_1, \varphi_2)\big) = \int\limits_{D_\epsilon} (A_\epsilon - A)\nabla w \cdot \nabla \overline{\varphi}_1 \, dx,$$

where $\mathcal{A}_\epsilon\big(\cdot\,;\cdot\big)$ is the sesquilinear form that defines $\mathbf{A}_\epsilon$. Now by using the $\mathbb{T}$-coercivity along with the definitions of (w_ϵ, v_ϵ) and (w, v) we conclude that

$$\left\| \big(\mathbf{A}_\epsilon^{-1} - \mathbf{A}_0^{-1}\big)(f, g) \right\|_{X(D)} \leq C||(A_\epsilon - A)\nabla w||_{L^2(D_\epsilon)}.$$

The dominated convergence theorem implies that the right hand side of the inequality tends to zero for any fixed $(f, g) \in X(D)$. Recalling that for $\mathbf{B}_0(f, g) = (p, q)$ we have $\|p\|_{H^3(\Omega)} + \|q\|_{H^3(\Omega)} \leq C \left(\|f\|_{H^1(D)} + \|g\|_{H^1(D)} \right)$ where $\Omega \subset D$, now consider

$\left\|\left(\mathbf{A}_\epsilon^{-1}-\mathbf{A}_0^{-1}\right)\mathbf{B}_0(f,g)\right\|_{X(D)}$. Hence we now have that $\mathbf{A}_0^{-1}\mathbf{B}_0(f,g)=(w,v)$ due to the variational form of $\mathbf{A}_0$ satisfies

$$-\nabla\cdot A(x)\nabla w+A_{min}w=-\Delta p+p \quad \text{and} \quad -\Delta v+v=-\Delta q+q \quad \text{in } D.$$

Therefore by elliptic regularity given any $\Omega'\subset\Omega\subset D$ we have that

$$\|w\|_{H^3(\Omega')}+\|v\|_{H^3(\Omega')}\leq C\left(\|p\|_{H^3(\Omega)}+\|q\|_{H^3(\Omega)}\right)\leq C||(f,g)||_{X(D)}.$$

Therefore we fix Ω' and Ω such that $D_\epsilon\subset\Omega'\subset\Omega\subset D$ for all ϵ sufficiently small. Now using that $H^3(\Omega')\subset C^1(\Omega')$ we have the following estimates

$$\begin{aligned}\left\|\left(\mathbf{A}_\epsilon^{-1}-\mathbf{A}_0^{-1}\right)\mathbf{B}_0(f,g)\right\|_{X(D)} &\leq C||w||_{C^1(\Omega')}||\chi_{D_\epsilon}||_{L^2(D)}\\ &\leq C\epsilon^{d/2}||w||_{C^1(\Omega')}.\end{aligned}$$

Now appealing to the continuity of the embedding of $H^3(\Omega')$ into $C^1(\Omega')$ and the regularity estimate we have that

$$\left\|\left(\mathbf{A}_\epsilon^{-1}-\mathbf{A}_0^{-1}\right)\mathbf{B}_0(f,g)\right\|_{X(D)}\leq C\epsilon^{d/2}||(f,g)||_{X(D)}.$$

Using that

$$\left\|\mathbf{A}_\epsilon^{-1}\mathbf{B}_\epsilon-\mathbf{A}_0^{-1}\mathbf{B}_0\right\|\leq\left\|\mathbf{A}_\epsilon^{-1}\left(\mathbf{B}_\epsilon-\mathbf{B}_0\right)\right\|+\left\|\left(\mathbf{A}_\epsilon^{-1}-\mathbf{A}_0^{-1}\right)\mathbf{B}_0\right\|$$

along with the uniform boundedness of $||\mathbf{A}_\epsilon^{-1}||$ and the norm convergence of $\mathbf{B}_\epsilon$ to $\mathbf{B}_0$ implies that $\mathbf{A}_\epsilon^{-1}\mathbf{B}_\epsilon\to\mathbf{A}_0^{-1}\mathbf{B}_0$ in norm. The same arguments work for showing that $\mathbf{A}_\epsilon^{-1}\mathbf{C}\to\mathbf{A}_0^{-1}\mathbf{C}$ in norm. □

Now by the proofs of Theorem 4.3.1 and 4.3.2 we obtain the follow rate of convergence for the relevant operators.

Corollary 4.3.1. *Let the operators* $\mathbf{A}_\epsilon$*,* $\mathbf{A}_0$*,* $\mathbf{B}_\epsilon$*,* $\mathbf{B}_0$ *and* $\mathbf{C}$ *be defined by the variational forms given above. Then we have that for* $d = 2, 3$

$$\big\|\big(\mathbf{A}_\epsilon^{-1} - \mathbf{A}_0^{-1}\big)\mathbf{B}_0\big\| = \mathcal{O}(\epsilon^{d/2}), \qquad \big\|\big(\mathbf{A}_\epsilon^{-1} - \mathbf{A}_0^{-1}\big)\mathbf{C}\big\| = \mathcal{O}(\epsilon^{d/2}),$$

and $\big\|\mathbf{A}_\epsilon^{-1}\big(\mathbf{B}_\epsilon - \mathbf{B}_0\big)\big\| = \mathcal{O}(\epsilon^{\alpha})$ *for some* $\alpha \in (0, 1)$ *.*

Remark 4.3.1. *The convergence of the operators presented here still holds for* $d > 3$ *but since the proofs use Sobolev's embedding results the convergence and rates of convergence must be derived separately for higher dimensions.*

With the use of the above convergence results we are ready to state the convergence result for the operator $\mathbf{T}_\epsilon(\tau)$ corresponding to the non-linear eigenvalue problem (4.22).

Theorem 4.3.3. *Let the operator* $\mathbf{T}_\epsilon(\tau)$ *be as defined in* (4.21) *and* $\tau \in U$ *with* U *being any bounded subset of* $\mathbb{C}$ *with zero not a limit point of* U*. Then we have that*

$$\|\mathbf{T}_\epsilon(\tau) - \mathbf{T}_0(\tau)\| \to 0 \quad \text{as} \quad \epsilon \to 0.$$

Moreover if $n_\epsilon = n$ *for all* $\epsilon \geq 0$ *then we have that*

$$\|\mathbf{T}_\epsilon(\tau) - \mathbf{T}_0(\tau)\| = \mathcal{O}(\epsilon^{d/2}).$$

Proof. The theorem is a direct consequence of the triangle inequality and using the fact that U is a bounded set with zero not a limit point. □

Since we have proven the convergence of the operator $\mathbf{T}_\epsilon(\tau)$ we now study the convergence of the transmission eigenvalues. To this end we will need some well known bounds on the transmission eigenvalues and results pertaining to perturbations of non-linear eigenvalue problem for compact operators. Here we give an abstract result for convergence of non-linear eigenvalues on a Hilbert space proven in [64].

Lemma 4.3.1. *Let τ be a non-linear eigenvalue of $\mathbf{T}_0$ and assume that $\mathbf{T}_0(\tau)$ and $\mathbf{T}_\epsilon(\tau)$ are both meromorphic in some region U of $\mathbb{C}$ containing τ. Also assume that $\mathbf{T}_\epsilon(\tau) \to \mathbf{T}_0(\tau)$ in the operator norm. Then for any ball around τ there exists a $\epsilon_0 > 0$ such that $\mathbf{T}_\epsilon$ has a non-linear eigenvalue in the ball for all $\epsilon < \epsilon_0$. Conversely if τ_ϵ is a sequence of non-linear eigenvalues of $\mathbf{T}_\epsilon$ that converges as $\epsilon \to 0$, then the limit τ is a non-linear eigenvalue of $\mathbf{T}_0$.*

By Theorem 4.3.3 we have that $\mathbf{T}_\epsilon(\tau) \to \mathbf{T}_0(\tau)$ in the operator norm in any in region U of $\mathbb{C} \setminus \{0\}$ and from the definition of the operator $\mathbf{T}_\epsilon(\tau)$ we have that it depends analytically on τ in any subset of the complex plane that does not include the origin. We now recall an important results pertaining to bounds of the real transmission eigenvalues. The following existence result is proven in [22] and [29] while the boundedness can be obtained by modifying the proof of Theorem 2.6 and Theorem 2.10 in [29] in a similar way as in the proof of Corollary 2.6 in [22].

Lemma 4.3.2. *The following holds:*

1. *Assume that either $A_{min} > 1$ and $0 < n_\epsilon < 1$ or $0 < A_{max} < 1$ and $n_\epsilon > 1$. There exists a infinite sequence of real transmission eigenvalues $k_{\epsilon,j}$, $j \in \mathbb{N}$ of (4.17)-(4.18) accumulating at $+\infty$ such that*

$$
\begin{aligned}
&k_j\left(A_{max}, \inf(n_\epsilon), D\right) \le k_{\epsilon,j} < k_j\left(A_{min}, \sup(n_\epsilon), D\right) \quad \text{if} \quad A_{min} > 1,\ 0 < n_\epsilon < 1 \\
&k_j\left(A_{min}, \sup(n_\epsilon), D\right) \le k_{\epsilon,j} < k_j\left(A_{max}, \inf(n_\epsilon), D\right) \quad \text{if} \quad 0 < A_{max} < 1,\ n_\epsilon > 1
\end{aligned}
$$

where $k_j(a,b,D)$ denotes an eigenvalue of (4.17)-(4.18) where we let $A_\epsilon = aI$ and $n_\epsilon = b$.

2. *Assume that $A_{min} > 1$ or $0 < A_{max} < 1$ and $n_\epsilon = 1$. There exists infinitely many real transmission eigenvalues $k_{\epsilon,j}$, $j \in \mathbb{N}$ of (4.17)-(4.18) accumulating at $+\infty$ such that*

$$\begin{aligned} 0 < k_{\epsilon,j} < k_j(A_{min}, D) \quad &\text{if} \quad A_{min} > 1 \\ 0 < k_{\epsilon,j} < k_j(A_{max}, D) \quad &\text{if} \quad 0 < A_{max} < 1 \end{aligned}$$

   where $k_j(a,D)$ denotes the an eigenvalue of (4.17)-(4.18) with $A_\epsilon = aI$ and $n_\epsilon = 1$.

Here j counts the eigenvalue in the sequence under consideration which may not necessarily be the j-th transmission eigenvalue. In particular the first transmission eigenvalue satisfies the above estimates.

Here the real transmission eigenvalue k_ϵ correspond to eigenvalues $\tau_\epsilon := k_\epsilon^2$ of the (4.22). The above bounds proves that there exists a limit point τ for the set $\{\tau_\epsilon\}_{\epsilon>0}$, and hence Lemma 4.3.1 implies that the limit point is a eigenvalue of (4.23) which gives the following result.

Theorem 4.3.4. *Assume that for all $x \in D$ either:*

1. *$n_\epsilon = 1$ for all $\epsilon \geq 0$ and $A_\epsilon - I$ is uniformly positive or negative definite, or*
2. *$A_\epsilon - I$ is uniformly positive(negative) definite and $n_\epsilon - 1$ is negative(positive).*

Then there are infinitely many real transmission eigenvalues $\tau_{\epsilon,j}$ of (4.22) that (up to a subsequence) converge to τ_j corresponding to a real transmission eigenvalue of (4.23). Moreover for any τ_j being a transmission eigenvalue corresponding to (4.23) there is a sequence $\tau_{\epsilon,j}$ that are non-linear eigenvalues of (4.22) such that $\tau_{\epsilon,j} \to \tau_j$.

4.3.2 Correction for the Operator $\mathbf{A}_\epsilon^{-1} - \mathbf{A}_0^{-1}$

Since we have proven the convergence of the transmission eigenvalues we now want to obtain asymptotic formula for the real transmission eigenvalues. To this end, we need to construct an appropriate corrector that will give an explicit formula for the first term in the asymptotic expansion for the transmission eigenvalues. The goal is to construct the right corrector to get an appropriate asymptotic formula for the operator $\mathbf{A}_\epsilon^{-1}$. The corrector will be derived for a homogeneous anisotropic media and the results can be generalized for an inhomogeneous media as in [32]. Hence in this section we again assume that the coefficients A and n are constant in D.

Consider the pair (w_ϵ, v_ϵ) and (w, v) in $X(D)$ defined by

$$(w_\epsilon, v_\epsilon) = \mathbf{A}_\epsilon^{-1}(f, g) \quad \text{and} \quad (w, v) = \mathbf{A}_0^{-1}(f, g) \tag{4.29}$$

and we assume that w is a smooth function. Assume that the defective region is of the form ϵB where B is the unit ball centered at the origin with constant matrix A_1 being the constitutive parameter. We make the scaling $y = x/\epsilon$ and $\tilde{D} = \frac{1}{\epsilon} D$ and let $w_\epsilon^{(1)}(y) \in H_0^1(\tilde{D})$ be the unique solution to

$$\int_{\tilde{D}} \tilde{A} \nabla_y w_\epsilon^{(1)} \cdot \nabla_y \overline{\varphi} + A_{min} w_\epsilon^{(1)} \overline{\varphi} \, dy = \int_{\partial B} \big[(A_1 - A)\nabla_x w(0) \cdot \nu\big] \overline{\varphi} \, ds_y \tag{4.30}$$

with $\tilde{A} = A_1 \chi_B + A(1 - \chi_B)$.

Theorem 4.3.5. *Assume that (w_ϵ, v_ϵ) and (w, v) are defined by (4.29) with w being a smooth function, then we have that*

$$\|w_\epsilon(x) - w(x) - \epsilon w(0) w_\epsilon^{(1)}(x/\epsilon)\|_{H^1(D)} + \|v_\epsilon(x) - v(x)\|_{H^1(D)} = \mathcal{O}(\epsilon^{d/2+1}). \tag{4.31}$$

Proof. Recall that $x = \epsilon y$ and we define the error functions in $X(\tilde{D})$ (note that $w_\epsilon^{(1)}(y) \in H_0^1(\tilde{D})$)

$$e_\epsilon^w = w_\epsilon(\epsilon y) - w(\epsilon y) - \epsilon w(0) w_\epsilon^{(1)}(y) \quad \text{and} \quad e_\epsilon^v = v_\epsilon(x) - v(x).$$

Now let $(\varphi_1, \varphi_2) \in X(\tilde{D})$ and define the sesquilinear form

$$\tilde{\mathcal{A}}_\epsilon\big((e_\epsilon^w, e_\epsilon^v); (\varphi_1, \varphi_2)\big) := \int_{\tilde{D}} \tilde{A} \nabla_y e_\epsilon^w \cdot \nabla_y \overline{\varphi}_1 + A_{min} e_\epsilon^w \overline{\varphi}_1 \, dy - \int_{\tilde{D}} \nabla_y e_\epsilon^v \cdot \nabla_y \overline{\varphi}_2 + e_\epsilon^v \overline{\varphi}_2 \, dx$$

Using (4.29) we have that

$$\tilde{\mathcal{A}}_\epsilon\big((e_\epsilon^w, e_\epsilon^v); (\varphi_1, \varphi_2)\big) = \int\limits_B (A_1 - A) \nabla_y w(\epsilon y) \cdot \nabla_y \overline{\varphi}_1 \, dy$$
$$-\epsilon w(0) \int\limits_{\tilde{D}} \tilde{A} \nabla_y w_\epsilon^{(1)} \cdot \nabla_y \overline{\varphi}_1 + A_{min} w_\epsilon^{(1)} \overline{\varphi}_1 \; dy.$$

Using integration by parts and (4.30) gives that

$$\begin{aligned}\tilde{\mathcal{A}}_\epsilon\big((e_\epsilon^w, e_\epsilon^v); (\varphi_1, \varphi_2)\big) \\ = \epsilon^2 \int\limits_B \overline{\varphi}_1 \nabla_x \cdot (A - A_1) \nabla_x w(\epsilon y) \, dy \\ + \epsilon w(0) \int\limits_{\partial B} \Big[(A_1 - A)(\nabla_x w(\epsilon y) - \nabla_x w(0)) \cdot \nu \Big] \overline{\varphi}_1 \; ds_y.\end{aligned}$$

Recall that w is smooth, therefore $\nabla_x \cdot (A - A_1) \nabla_x w(\epsilon y)$ is bounded in B. Also notice that by Taylor's expansion we have that the term $(\nabla_x w(\epsilon y) - \nabla_x w(0)) = \mathcal{O}(\epsilon)$.

Therefore we can conclude that there is a constant C independent of ϵ such that

$$\left|\tilde{\mathcal{A}}_\epsilon\big((e^w_\epsilon, e^v_\epsilon);(\varphi_1,\varphi_2)\big)\right| \leq C\epsilon^2\|(\varphi_1,\varphi_2)\|_{H^1(\tilde{D})\times H^1(\tilde{D})}$$

Using the $\mathbb{T}$-coercivity of the sesquilinear form $\tilde{\mathcal{A}}_\epsilon\big(\cdot\,;\cdot\big)$ in $X(\tilde{D})$ gives that

$$\|w_\epsilon(\epsilon y) - w(\epsilon y) - \epsilon w(0) w^{(1)}_\epsilon(y)\|_{H^1(\tilde{D})} + \|v_\epsilon(\epsilon y) - v(\epsilon y)\|_{H^1(\tilde{D})} \leq C\epsilon^2, \qquad (4.32)$$

and the result follows from scaling. □

Notice that from (4.30) we have that $\|w^{(1)}_\epsilon(y)\|_{H^1(\tilde{D})}$ is bounded independently of ϵ by the Lax-Milgram lemma. Therefore by scaling we have that $\|w^{(1)}_\epsilon(x/\epsilon)\|_{H^1(D)} \leq C\epsilon^{d/2-1}$ with C independent of ϵ, which gives the following result.

Corollary 4.3.2. *Assume that (w_ϵ, v_ϵ) and (w, v) are defined by (4.29) with w being a smooth function then we have that*

$$\|w_\epsilon(x) - w(x)\|_{H^1(D)} + \|v_\epsilon(x) - v(x)\|_{H^1(D)} = \mathcal{O}(\epsilon^{d/2}). \qquad (4.33)$$

Notice that the corrector $w^{(1)}_\epsilon(y)$ depends on ϵ, hence we now wish to construct a corrector that is independent of the small parameter ϵ. To this end, we define the function $w^{(1)}(y) \in H^1(\mathbb{R}^d)$ such that for all $\varphi \in H^1(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} \tilde{A}\nabla_y w^{(1)} \cdot \nabla_y \overline{\varphi} + A_{min} w^{(1)} \overline{\varphi}\, dy = \int_{\partial B} \big[(A_1 - A)\nabla_x w(0)\cdot \nu\big]\overline{\varphi}\, ds_y \qquad (4.34)$$

Notice that the variational problem (4.34) implies that

$$-\nabla_y \cdot \tilde{A}\nabla_y w^{(1)} + A_{min} w^{(1)} = 0 \quad \text{in} \quad \mathbb{R}^d \setminus \partial B.$$

Therefore we have that $\left|w^{(1)}\right| \to 0$ as $|y| \to \infty$, exponentially fast. This gives that $\nabla_y w^{(1)}$ decays faster than the gradient of a solution to Laplace's equation, therefore we have that

$$\|\nabla_y w^{(1)}(x/\epsilon)\|_{L^\infty(\partial D)} = o(\epsilon^d) \quad \text{for} \quad d = 2, 3.$$

Theorem 4.3.6. *Let* $w_\epsilon^{(1)}$ *and* $w^{(1)}$ *be defined as the solutions to* (4.30) *and* (4.34) *respectively, then we have that*

$$\|w_\epsilon^{(1)}(x/\epsilon) - w^{(1)}(x/\epsilon)\|_{H^1(D)} = o(\epsilon^{d/2+2}) \quad \textit{for} \;\; d = 2,$$
$$\|w_\epsilon^{(1)}(x/\epsilon) - w^{(1)}(x/\epsilon)\|_{H^1(D)} = o(\epsilon^{d/2+5/2}) \quad \textit{for} \;\; d = 3.$$

Proof. Let $u_\epsilon = w_\epsilon^{(1)}(y) - w^{(1)}(y)$, there exists a constant $\alpha > 0$ such that

$$\begin{aligned}
\alpha\|u_\epsilon\|^2_{H^1(\tilde{D})} &\leq \int_{\tilde{D}} \tilde{A}\nabla_y u_\epsilon \cdot \nabla_y \overline{u}_\epsilon + A_{min}|u_\epsilon|^2 \, dy \\
&= \int_{\tilde{D}} \tilde{A}\nabla_y w_\epsilon^1 \cdot \nabla_y \overline{u}_\epsilon + A_{min} w_\epsilon^1 \overline{u}_\epsilon \, dy - \int_{\tilde{D}} \tilde{A}\nabla_y w^{(1)} \cdot \nabla_y \overline{u}_\epsilon + A_{min} w^{(1)} \overline{u}_\epsilon \, dy \\
&= \int_{\partial B} \left[(A_1 - A)\nabla_x w(0) \cdot \nu\right] \overline{u}_\epsilon \, ds_y - \int_{\tilde{D}} \tilde{A}\nabla_y w^{(1)} \cdot \nabla_y \overline{u}_\epsilon + A_{min} w^{(1)} \overline{u}_\epsilon \, dy
\end{aligned}$$

Notice that the variational form (4.34) implies that

$$\left(A\frac{\partial w^{(1)}}{\partial \nu_y}\right)^+ - \left(A_1\frac{\partial w^{(1)}}{\partial \nu_y}\right)^- = (A_1 - A)\nabla_x w(0) \cdot \nu \quad \text{on} \quad \partial B.$$

Therefore integration by parts gives that

$$\int_{\partial B} \big[(A_1 - A)\nabla_x w(0)\cdot\nu\big]\overline{u}_\epsilon \; ds_y - \int_{\tilde{D}} \tilde{A}\nabla_y w^{(1)}\cdot\nabla_y \overline{u}_\epsilon + A_{min} w^{(1)}\overline{u}_\epsilon \; dy$$

$$= \int_{\partial B} \big[(A_1 - A)\nabla_x w(0)\cdot\nu\big]\overline{u}_\epsilon \; ds_y + \int_{\tilde{D}} \overline{u}_\epsilon(\nabla_y\cdot\tilde{A}\nabla_y w^{(1)} - A_{min} w^{(1)}) \; dy$$

$$- \int_{\partial B} \left[\left(A\frac{\partial w^{(1)}}{\partial \nu_y}\right)^+ - \left(A_1\frac{\partial w^{(1)}}{\partial \nu_y}\right)^-\right]\overline{u}_\epsilon \; ds_y + \int_{\partial\tilde{D}} A\frac{\partial w^{(1)}}{\partial \nu_y}\overline{u}_\epsilon \; ds_y.$$

Now by using the boundary value problem for $w^{(1)}$ we have that

$$\alpha\|u_\epsilon\|^2_{H^1(\tilde{D})} \le \left|\int_{\partial\tilde{D}} A\frac{\partial w^{(1)}}{\partial \nu_y}\overline{u}_\epsilon \; ds_y\right|$$

$$= \epsilon^{1-d}\left|\int_{\partial D} (A\nabla_y w^{(1)}(x/\epsilon)\cdot\nu)\overline{u}_\epsilon(x/\epsilon) \; ds_x\right|$$

$$\le C\epsilon^{1-d}\|\nabla_y w^{(1)}(x/\epsilon)\|_{L^\infty(\partial D)}\|u_\epsilon(x/\epsilon)\|_{H^1(D)}.$$

therefore by the scaling we have that

$$\|u_\epsilon\|^2_{H^1(\tilde{D})} \le C\epsilon^{1-d/2}\|\nabla_y w^{(1)}(x/\epsilon)\|_{L^\infty(\partial D)}\|u_\epsilon(x/\epsilon)\|_{H^1(\tilde{D})}.$$

Since

$$\|\nabla_y w^{(1)}(x/\epsilon)\|_{L^\infty(\partial D)} = o(\epsilon^d) \quad \text{for} \quad d = 2, 3$$

we can conclude that

$$\|u_\epsilon\|_{H^1(\tilde{D})} = o(\epsilon^2) \;\; \text{for} \;\; d=2 \;\; \text{and} \;\; \|u_\epsilon\|_{H^1(\tilde{D})} = o(\epsilon^{5/2}) \;\; \text{for} \;\; d=3,$$

which gives the result by scaling the norm back to the domain D. □

By appealing to the triangle inequality we have the following result.

Corollary 4.3.3. *Let $w^{(1)}$ be the solutions to (4.34), also assume that (w_ϵ, v_ϵ) and (w, v) are defined by (4.29) with w being a smooth function then we have that*

$$\|w_\epsilon(x) - w(x) - \epsilon w(0) w^{(1)}(x/\epsilon)\|_{H^1(D)} = \mathcal{O}(\epsilon^{d/2+1}). \tag{4.35}$$

The arguments used in this section carry over to the case of multiple inhomogeneities. Indeed, for multiple inhomogeneities centered at z_m with anisotropic material parameter A_m we have that by using translation and summing over a finite number of inhomogeneities gives that the corrector takes the form

$$\tilde{w}^{(1)}(x/\epsilon) = \sum_{m=1}^{M} w(z_m) w_m^{(1)}(x/\epsilon)$$

where $w_m^{(1)}(x/\epsilon)$ is the solution to

$$\int_{\mathbb{R}^d} \tilde{A}_m \nabla_y w_m^{(1)} \cdot \nabla_y \overline{\varphi} + A_{min} w_m^{(1)} \overline{\varphi} \, dy = \int_{\partial B_m} \big[(A_m - A)\nabla_x w(z_m) \cdot \nu\big] \overline{\varphi} \, ds_y$$

for all $\varphi \in H^1(\mathbb{R}^d)$ with $\tilde{A}_m = A_m \chi_{B_m} + A(1 - \chi_{B_m})$. The convergence results in this section still hold for $w(0)w^{(1)}(x/\epsilon)$ replaced by $\tilde{w}^{(1)}(x/\epsilon)$.

4.3.3 Asymptotic Formula for the Transmission Eigenvalues

In this section we give an asymptotic formula for the transmission eigenvalues using the results in [64]. To do so we assume that contrast in the defect is only in the matrix valued material parameter (i.e. $n_\epsilon = n$ for all $\epsilon > 0$), and we still take A and A_m constant matrices. Under this assumption we have that the operator $\mathbf{T}_\epsilon(\tau) = \mathbf{A}_\epsilon^{-1}\mathbf{B}_0 + \frac{1}{\tau}\mathbf{A}_\epsilon^{-1}\mathbf{C}$ converges in the operator norm.

We now recall Theorem 4.1 of [64] which is a generalization of Osborn's Theorem (see [69] for Osborn's result) to nonlinear eigenvalue problems.

Theorem 4.3.7. *Let X be a Hilbert space and $\mathbf{T}_\epsilon(\tau) : X \to X$ be a compact operator valued functions of τ which are analytic in a region U of the complex plane, such that $\|\mathbf{T}_\epsilon(\tau) - \mathbf{T}_0(\tau)\| \to 0$ for all $\tau \in U$. Now assume that τ is a simple nonlinear eigenvalue of $\mathbf{T}_0(\tau)$ with normalized eigenfunction ϕ. Then if*

$$\tau^2 \left(\frac{d}{d\tau} \mathbf{T}_0(\tau)\phi, \phi \right) \neq -1$$

we have that

$$\tau_\epsilon = \tau + \tau^2 \frac{\big((\mathbf{T}_0(\tau) - \mathbf{T}_\epsilon(\tau))\phi, \phi\big)}{1 + \tau^2 \left(\frac{d}{d\tau}\mathbf{T}_0(\tau)\phi, \phi\right)} + \mathcal{O}\left(\sup_{\tau \in U} \|(\mathbf{T}_\epsilon(\tau) - \mathbf{T}_0(\tau))\phi\| \, \|(\mathbf{T}_\epsilon^*(\tau) - \mathbf{T}_0^*(\tau))\phi\| \right)$$

with τ_ϵ is a nonlinear eigenvalue for $\mathbf{T}_\epsilon(\tau)$.

Theorem 4.3.7 only holds for simple eigenvalues. Notice that we have established the order of convergence of the operator defined by the transmission eigenvalue problem. In particular, the results in the previous section (see equation (4.33)) gives that

$$\|\mathbf{T}_\epsilon(\tau)(w_\tau, v_\tau) - \mathbf{T}_0(\tau)(w_\tau, v_\tau)\| = \mathcal{O}(\epsilon^{d/2}).$$

We now consider the point wise convergence for the adjoint operator.

Lemma 4.3.3. *Let $(w_\tau, v_\tau) \in X(D)$ be the smooth eigenfunction corresponding to* *the eigenvalue τ of the operator $\mathbf{T}_0(\tau)$, then we have that*

$$\|\mathbf{T}^*_\epsilon(\tau)(w_\tau, v_\tau) - \mathbf{T}^*_0(\tau)(w_\tau, v_\tau)\| = \mathcal{O}(\epsilon^{d/2+1}).$$

Proof. Notice that $\mathbf{T}^*_\epsilon(\tau) = \mathbf{B}_0\mathbf{A}_\epsilon^{-1} + \mathbf{C}\mathbf{A}_\epsilon^{-1}$ where we define $(w, v) = \mathbf{A}_0^{-1}(w_\tau, v_\tau)$ and $(w_\epsilon, v_\epsilon) = \mathbf{A}_\epsilon^{-1}(w_\tau, v_\tau)$. Now for any $(\varphi_1, \varphi_2) \in X(D)$

$$\big(\mathbf{B}_0(\mathbf{A}_\epsilon^{-1} - \mathbf{A}_0^{-1})(w_\tau, v_\tau); (\varphi_1, \varphi_2)\big) = \mathcal{B}_0\big((w_\epsilon - w, v_\epsilon - v); (\varphi_1, \varphi_2)\big).$$

Since the sesqulinear form $\mathcal{B}_0$ only has $L^2(D)$ terms, we have that

$$\big|\big(\mathbf{B}_0(\mathbf{A}_\epsilon^{-1} - \mathbf{A}_0^{-1})(w_\tau, v_\tau); (\varphi_1, \varphi_2)\big)\big| \leq C\|(w_\epsilon - w, v_\epsilon - v)\|_{L^2(D)}\|(\varphi_1, \varphi_2)\|_{X(D)}$$

By rescaling the L^2 norm in equation (4.32) gives that

$$\|w_\epsilon(x) - w(x)\|_{L^2(D)} + \|v_\epsilon(x) - v(x)\|_{L^2(D)} = \mathcal{O}(\epsilon^{d/2+1}).$$

therefore $\big\|\mathbf{B}_0(\mathbf{A}_\epsilon^{-1} - \mathbf{A}_0^{-1})(w_\tau, v_\tau)\big\|_{X(D)} = \mathcal{O}(\epsilon^{d/2+1})$. A similar argument gives that $\big\|\mathbf{C}(\mathbf{A}_\epsilon^{-1} - \mathbf{A}_0^{-1})(w_\tau, v_\tau)\big\|_{X(D)} = \mathcal{O}(\epsilon^{d/2+1})$, proving that claim. □

Remark 4.3.2. *This result shows why the case where $n_\epsilon \neq n$ can not be handled by this analytic framework. In particular, the rate of convergence in Theorem 4.3.1 for $\mathbf{B}_0 - \mathbf{B}_\epsilon$ is not fast enough to provide an improved convergence rate for $\mathbf{T}^*_\epsilon(\tau) - \mathbf{T}^*_0(\tau)$ which is necessary to apply Theorem 4.3.7.*

We have just shown that the remainder term for the non-linear eigenvalue corrector formula is of the order ϵ^{d+1}. To construct an asymptotic formula for the transmission eigenvalues we need to construct an asymptotic formula for

$$\big(\mathbf{T}_0(\tau)(w_\tau, v_\tau) - \mathbf{T}_\epsilon(\tau)(w_\tau, v_\tau); (w_\tau, v_\tau)\big)_{X(D)}$$

where (w_τ, v_τ) are the eigenfunctions for $\epsilon = 0$. By equation (4.19) we have that $\mathbf{B}_0(w, v) + \frac{1}{\tau}\mathbf{C}(w, v) = \mathbf{A}_0(w, v)$. Now since the operator $\mathbf{A}_\epsilon$ is self-adjoint for all $\epsilon \geq 0$ the definition of $\mathbf{T}_\epsilon(\tau)$ in (4.21) gives that

$$\big(\mathbf{T}_\epsilon(\tau)(w_\tau, v_\tau) - \mathbf{T}_0(\tau)(w_\tau, v_\tau); (w_\tau, v_\tau)\big)_{X(D)} = \frac{1}{\tau}\big(\mathbf{A}_0(w_\tau, v_\tau); \left(\mathbf{A}_\epsilon^{-1} - \mathbf{A}_0^{-1}\right)(w_\tau, v_\tau)\big)_{X(D)}.$$

This gives that we only need to construct an asymptotic formula for

$$\mathcal{A}_0\big((w_\tau, v_\tau); \left(\mathbf{A}_\epsilon^{-1} - \mathbf{A}_0^{-1}\right)(w_\tau, v_\tau)\big).$$

We now derive an asymptotic formula for $\mathbf{A}_\epsilon^{-1} - \mathbf{A}_0^{-1}$ with respect to the sesquilinear form $\mathcal{A}_0\big(\cdot\,;\cdot\big)$, and to this end we obtain the following result.

Theorem 4.3.8. *Let (w_τ, v_τ) be the eigenfunctions for the unperturbed media where $\epsilon = 0$ with transmission eigenvalue τ and define $(w, v) = \mathbf{A}_0^{-1}(w_\tau, v_\tau)$, then we have that*

$$\mathcal{A}_0\big((w_\tau, v_\tau); \left(\mathbf{A}_\epsilon^{-1} - \mathbf{A}_0^{-1}\right)(w_\tau, v_\tau)\big) = \epsilon^d \sum_{m=1}^{M} (A - A_m)|B_m| \nabla w_\tau(z_m) \cdot \nabla \overline{w(z_m)}$$
$$+\epsilon^d \sum_{m=1}^{M} w_\tau(z_m)\overline{w(z_m)} \int\limits_{\partial B_m} \Big[(A - A_m)\nabla \overline{w_m^{(1)}}(y) \cdot \nu_y\Big]\, ds_y + o(\epsilon^d).$$

Proof. We will prove the result for a single defect centered at the origin then by using translation and summing a finite number of such inhomogeneities, the asymptotic result follows. Letting $(w_\epsilon, v_\epsilon) = \mathbf{A}_\epsilon^{-1}(w_\tau, v_\tau)$, we have that

$$\begin{aligned}\mathcal{A}_0\big((w_\tau, v_\tau); (w_\epsilon - w, v_\epsilon - v\big) &= (\mathcal{A}_0 - \mathcal{A}_\epsilon)\big((w_\tau, v_\tau); (w_\epsilon, v_\epsilon)\big)\\ &= (\mathcal{A}_0 - \mathcal{A}_\epsilon)\big((w_\tau, v_\tau); (w_\epsilon - w - \epsilon w(0)w^{(1)}, v_\epsilon - v)\big)\\ &\quad + (\mathcal{A}_0 - \mathcal{A}_\epsilon)\big((w_\tau, v_\tau); (w + \epsilon w(0)w^{(1)}, v)\big).\end{aligned} \tag{4.36}$$

Recall that by elliptic regularity we have the for any Ω such that $\epsilon B \subset \Omega \subset D$ the eigenfunctions are in $C^1(\Omega)$. Using this along with the support of $A - A_\epsilon$ and Corollary 4.3.3 we can now estimate the first term

$$\begin{aligned}&\big|(\mathcal{A}_0 - \mathcal{A}_\epsilon)\big((w_\tau, v_\tau); (w_\epsilon - w - \epsilon w(0)w^{(1)}, v_\epsilon - v)\big)\big|\\ &\qquad = \left|\int_{\epsilon B} (A - A_1)\nabla w_\tau \cdot \nabla\overline{w_\epsilon - w - \epsilon w(0)w^{(1)}}\, dx\right|\\ &\qquad \leq C||w_\tau||_{H^1(\epsilon B)}||w_\epsilon - w - \epsilon w(0)w^{(1)}||_{H^1(D)}\\ &\qquad \leq C\epsilon^{d/2+1}||\chi_{\epsilon B}||_{L^2(D)}||w_\tau||_{C^1(\Omega)}\\ &\qquad \leq C\epsilon^{d+1}||w_\tau||_{C^1(\Omega)}.\end{aligned}$$

We now consider the second term of (4.37) which is given by

$$\begin{aligned}(\mathcal{A}_0 - \mathcal{A}_\epsilon)\big((w_\tau, v_\tau); (w + \epsilon w(0)w^{(1)}, v)\big) &= \int_{\epsilon B} (A - A_1)\nabla w_\tau \cdot \nabla\overline{w + \epsilon w(0)w^{(1)}}\, dx\\ &= \int_{\epsilon B} (A - A_1)\nabla w_\tau \cdot \nabla\overline{w}\, dx + \epsilon\overline{w(0)}\int_{D_\epsilon} (A - A_1)\nabla w_\tau \cdot \nabla\overline{w^{(1)}}\, dx\\ &= \epsilon^d (A - A_1)|B|\nabla w_\tau(0) \cdot \nabla\overline{w(0)}\\ &\qquad + \epsilon\overline{w(0)}\int_{\epsilon B} (A - A_1)\nabla w_\tau \cdot \nabla\overline{w^{(1)}}\, dx + o(\epsilon^d)\end{aligned}$$

where we have used Taylor's expansion about the origin to estimate the first integral. Now by the divergence theorem we have that the volume integral involving the eigenfunction and the corrector is given by

$$\epsilon \int_{\epsilon B} (A - A_1)\nabla w_\tau \cdot \nabla \overline{w^{(1)}}\, dx = \epsilon \int_{\epsilon B} w_\tau(x) \nabla \cdot (A - A_1)\nabla \overline{w^{(1)}}(x/\epsilon)\, dx$$

$$+\epsilon \int_{\partial(\epsilon B)} w_\tau(x) \left[(A - A_1)\nabla \overline{w^{(1)}}(x/\epsilon) \cdot \nu_x \right] ds_x.$$

Now by rescaling the second integral for $x = \epsilon y$ and using a Taylor's expansion we have that integration is given by

$$\epsilon \int_{\epsilon B} (A - A_1)\nabla w_\tau \cdot \nabla \overline{w^{(1)}}\, dx = \epsilon^{d+1} \int_{B} w_\tau(\epsilon y) \nabla \cdot (A - A_1)\nabla \overline{w^{(1)}}(y)\, dy$$

$$+\epsilon^d w_\tau(0) \int_{\partial B} \left[(A - A_1)\nabla \overline{w^{(1)}}(y) \cdot \nu_y \right] ds_y + o(\epsilon^d)$$

proving the result. □

Now we have all we need for an asymptotic formula for simple transmission eigenvalues. Notice that $\frac{d}{d\tau}\mathbf{T}_0(\tau) = -\frac{1}{\tau^2}\mathbf{A}_0^{-1}\mathbf{C}$, therefore we have that

$$\tau^2 \left(\frac{d}{d\tau}\mathbf{T}_0(\tau)(w_\tau, v_\tau), (w_\tau, v_\tau) \right) = -\mathcal{C}\big((w_\tau, v_\tau); \mathbf{A}_0^{-1}(w_\tau, v_\tau) \big).$$

For convenience let the constant

$$q_m = \int_{\partial B_m} \left[(A - A_m)\nabla \overline{w_m^{(1)}}(y) \cdot \nu_y \right] ds_y \tag{4.37}$$

Therefore we have that simple transmission eigenvalues have the expansion.

Theorem 4.3.9. *Let (w_τ, v_τ) be the eigenfunctions for the unperturbed media where $\epsilon = 0$ with simple transmission eigenvalue τ and define $(w, v) = \mathbf{A}_0^{-1}(w_\tau, v_\tau)$, then we have that*

$$\tau_\epsilon = \tau + \tau\epsilon^d \sum_{m=1}^{M} \frac{(A_m - A)|B_m|\nabla w_\tau(z_m)\cdot\nabla\overline{w(z_m)} + q_m w_\tau(z_m)\overline{w(z_m)}}{1 - \mathcal{C}\big((w_\tau, v_\tau);(w, v)\big)} + o(\epsilon^d)$$

where q_m is given by (4.37) and

$$\mathcal{C}\big((w_\tau, v_\tau);(w, v)\big) = \int\limits_D A_{min} w_\tau \overline{w} \; - v_\tau \overline{v}\, dx.$$

The asymptotic formula given in Theorem 4.3.9 can potentially be used to determine the strength of the small defective region(s). Notice that the MUSIC algorithm discussed at the beginning of the chapter gives the location of the defect(s) and recall that by Theorem 2.2.1 we have that the transmission eigenvalues for the perturbed media τ_ϵ can be measured from scattering data, where as the transmission eigenvalues τ and eigenfunctions (w_τ, v_τ) for the unperturbed media can be computed since A and n are assumed to be known. To use the asymptotic formula in Theorem 4.3.9 one also needs the functions $(w, v) = \mathbf{A}_0^{-1}(w_\tau, v_\tau)$ which can be solved for(e.g. using the FEM) since A and (w_τ, v_τ) are known. Having identified the location of the defect(s) from the MUSIC algorithm (i.e. the points z_m are known) one could devise a Least Squares Method to reconstruct the strength of the defect(s) which is given by $(A_m - A)|B_m|$ and q_m. Notice that the strength of the defect(s) only depend on the constitutive coefficients and geometry of the defect(s).

For completeness we now consider a few examples to illustrate the convergence of the transmission eigenvalues as $\epsilon \to 0$. To this end, let the matrix valued coefficient for the unperturbed media be given by

$$A = \begin{pmatrix} 10 & 1 \\ 1 & 10 \end{pmatrix}.$$

We will denote the first transmission eigenvalue for the unperturbed media by k_1 and the first transmission eigenvalue for the perturbed media by $k_1(\epsilon)$. To compute the transmission eigenvalues we use a continuous finite element method with the eigenvalue searching technique described in [39] (see also [77] and [79]) just as in Chapter 2.

Example 1. Let the domain D be a disk of radius 1 and the defect D_0 be a disk of radius ϵ both centered at the origin. For this case we take $n = n_\epsilon = 1$ for all ϵ. Below in Table 4.1 we display the first transmission eigenvalue for the perturbed media where $A_1 = (12.5)I$. In Figure 4.6 we also give the plot of the norm of the solution to the far-field equation (2.23) for $k \in [4, 5]$ with $\epsilon = 0.5$.

Table 4.1: Convergence of the first TE, $k_1 = 4.4734$. The order of convergence in this example is approximately 1.48.

ϵ	0.5	0.4	0.3	0.2	0.1
$k_1(\epsilon)$	4.4182	4.4356	4.4509	4.4629	4.470

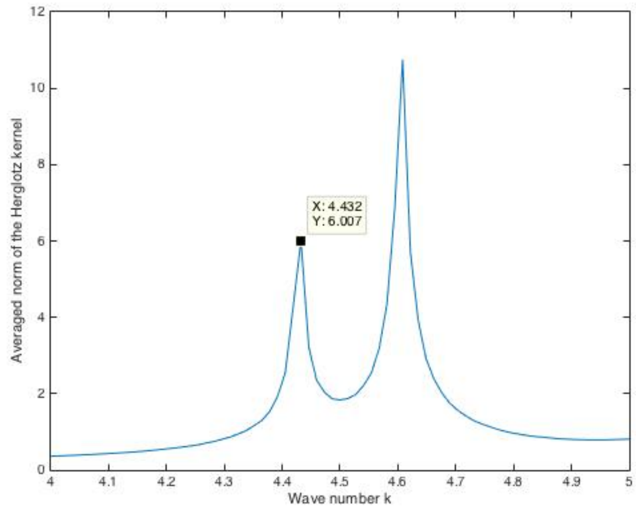

Figure 4.6: The first spike in the graph seems to matches up with the first transmission eigenvalue for the perturbed media.

Example 2. Here we let the domain $D = [-3, 3]^2$ and the defects are two disks of radius ϵ centered at $(-1, 1)$ and $(1, -1)$. For this case we take $n = 0.5$, $n_{1,2} = 0.15$ and $A_{1,2} = (1.5)I$. Below in Table 4.2 we show the first transmission eigenvalue for the perturbed media. We also give the plot of the norm of the solution to the far-field equation (2.23) for $k \in [1, 2]$ with $\epsilon = 0.5$ in Figure 4.7.

Table 4.2: Convergence of the first TE, $k_1 = 1.3156$. The order of convergence in this example is approximately 2.13.

ϵ	0.5	0.4	0.3	0.2	0.1
$k_1(\epsilon)$	1.3312	1.3252	1.3209	1.3178	1.3161

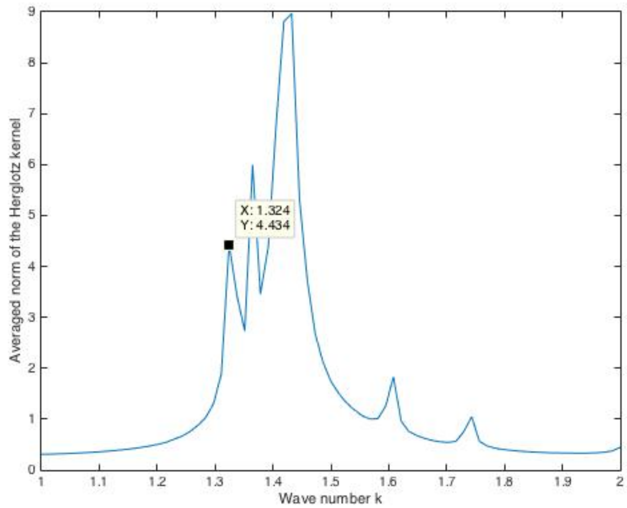

Figure 4.7: The first spike in the graph seems to matches up with the first transmission eigenvalue for the perturbed media.

4.3.4 Remarks

In the previous sections we have shown the convergence of all the transmission eigenvalues(both real and complex). The asymptotic formula given in Theorem 4.3.9 can only be applied to simple transmission eigenvalues due to the analysis of the non-linear eigenvalue problem. It is advantageous to construct an asymptotic formula that is valid for all of the transmission eigenvalues. We suggest a new form of the transmission eigenvalue problem given by a linear eigenvalue problem for a shifted eigenvalue parameter using which would avoid the non-linearity. In this new formulation the transmission eigenvalue problem is: find $\tau_\epsilon \in \mathbb{C}$ and nontrivial $(w_\epsilon, v_\epsilon) \in X(D)$ such that

$$\mathcal{A}_\epsilon\big((w_\epsilon, v_\epsilon); (\varphi_1, \varphi_2)\big) - \tau_\epsilon \mathcal{B}_\epsilon\big((w_\epsilon, v_\epsilon); (\varphi_1, \varphi_2)\big) = 0 \qquad \text{for all} \quad (\varphi_1, \varphi_2) \in X(D)$$

where the sesquilinear forms in $X(D) \times X(D)$ are

$$\mathcal{A}_\epsilon\big((w_\epsilon, v_\epsilon); (\varphi_1, \varphi_2)\big) := \int\limits_D A_\epsilon \nabla w_\epsilon \cdot \nabla \overline{\varphi}_1 - 2A_\epsilon \nabla w_\epsilon \cdot \nabla \overline{\varphi}_2 + \nabla v_\epsilon \cdot \nabla \overline{\varphi}_2 \, dx$$
$$+\eta \int\limits_D n_\epsilon w_\epsilon \overline{\varphi}_1 + v_\epsilon \overline{\varphi}_2 - 2n_\epsilon w_\epsilon \overline{\varphi}_2 \, dx$$
$$\mathcal{B}_\epsilon\big((w_\epsilon, v_\epsilon); (\varphi_1, \varphi_2)\big) := \int\limits_D n_\epsilon w_\epsilon \overline{\varphi}_1 + v_\epsilon \overline{\varphi}_2 - 2n_\epsilon w_\epsilon \overline{\varphi}_2 \, dx$$

and the eigenvalue parameter $\tau_\epsilon = k_\epsilon^2 - \eta$. For the case where $A_\epsilon < I$ and $n_\epsilon < 1$ we can choose the parameter η independent of ϵ such that $\mathcal{A}_\epsilon(\cdot\,;\cdot)$ is uniformly coercive with respect to ϵ. This linear eigenvalue problem can be studied with standard techniques to avoid the limitations in the non-linear analysis.

Chapter 5

HOMOGENIZATION APPROACH TO THE TRANSMISSION EIGENVALUE PROBLEM FOR PERIODIC MEDIA

The theory of homogenization is used to study differential equations with rapidly oscillating coefficients. The rapid oscillations of the coefficients model the constitutive parameters of a composite anisotropic material with a fine microstructure. The purpose of designing composite materials is to use two or more materials so that the product will inherit the properties of the different components. We consider the interior transmission problem associated with the scattering by an inhomogeneous (possibly anisotropic) periodic media. We use a homogenization approach to arrive at the homogenized interior transmission problem and show that it is an approximation of the original problem in some weak sense. Then we include the bulk and boundary correctors to obtain strong convergence. We also show that the real transmission eigenvalues of the periodic media converge to the real transmission eigenvalues of the homogenized problem. Finally we show how to use the first transmission eigenvalue of the periodic media, which is measurable from scattering data, to obtain information about constant effective parameters of the periodic media. We include in this investigation the cases for both isotropic and anisotropic materials, since the transmission eigenvalue problem takes different forms. This part of the thesis is published as the article F. Cakoni, H. Haddar and I. Harris "Homogenization approach for the transmission eigenvalue problem for periodic media and application to the inverse problem" (Accepted) *Inverse Problems and Imaging.*

5.1 The Scattering Problem for Highly Oscillating Media

We consider the scattering problem for an inhomogeneous (possibly anisotropic) periodic media in the frequency domain. The governing equations have rapidly oscillating periodic coefficients which typically model the wave propagation through composite anisotropic materials with a fine microstructure. Such composite materials are at the foundation of many contemporary engineering designs and are used to produce materials with special properties by combining in a particular structure (usually in periodic patterns) different materials. Mathematically it is desirable to understand these special properties, in particular the macrostructure behavior of the composite materials which is achievable by using a homogenization approach [7], [74]. The homogenization for the direct scattering problem for a periodic media is studied in [26]. Our concern here is with homogenization of the transmission eigenvalue problem corresponding to media with rapidly oscillating periodic coefficients. It has already been shown in [20] and Chapter 2 Theorem 2.2.1 that transmission eigenvalues can be determined for the scattering data and they provide information about the material properties of the scattering media. In this study the goal is to understand what kind of information the real transmission eigenvalues hold for periodic materials.

More precisely, let $D \subset \mathbb{R}^d$ be a bounded simply connected open set with Lipschitz boundary ∂D. Assume we have a symmetric coefficient matrix $A(x) \in L^\infty\left(D, \mathbb{R}^{d\times d}\right)$ that is positive definite and $n(x) \in L^\infty(D)$ such that $n > 0$. When we study the convergence using boundary correctors we will need to augment the regularity assumptions on the coefficients and boundary. Let $\epsilon > 0$ be very small in comparison to the size of D and let $Y = (0,\, 1)^d$ be the rescaled unit periodic cell. Furthermore, we assume that both $A(y)$ and $n(y)$ are periodic in $y = x/\epsilon$ with period

Y (here $x \in D$ is refer to as the slow variable where $y = x/\epsilon \in Y$ is referred to as the fast periodic variable). Let us introduce the following notations:

$$\inf_{y \in Y} \inf_{|\xi|=1} \overline{\xi} \cdot A(y)\xi = A_{min} > 0 \quad \text{and} \quad \sup_{y \in Y} \sup_{|\xi|=1} \overline{\xi} \cdot A(y)\xi = A_{max} < \infty \tag{5.1}$$

$$\inf_{y \in Y} n(y) = n_{min} > 0 \quad \text{and} \quad \sup_{y \in Y} n(y) = n_{max} < \infty. \tag{5.2}$$

We recall the corresponding interior transmission eigenvalue problem for the anisotropic media ($d = 2$ in electromagnetic scattering and $d = 3$ in acoustic scattering): find (w_ϵ, v_ϵ) satisfying

$$\nabla \cdot A\left(x/\epsilon\right) \nabla w_\epsilon + k_\epsilon^2 n\left(x/\epsilon\right) w_\epsilon = 0 \quad \text{in} \quad D \tag{5.3}$$

$$\Delta v_\epsilon + k_\epsilon^2 v_\epsilon = 0 \quad \text{in} \quad D \tag{5.4}$$

$$w_\epsilon = v_\epsilon \quad \text{on} \quad \partial D \tag{5.5}$$

$$\frac{\partial w_\epsilon}{\partial \nu_{A_\epsilon}} = \frac{\partial v_\epsilon}{\partial \nu} \quad \text{on} \quad \partial D \tag{5.6}$$

where $\frac{\partial w}{\partial \nu_A} = \nu \cdot A \nabla w$. Note that the spaces for the solution (w_ϵ, v_ϵ) will become precise later on since they depends on whether $A = I$ or not.

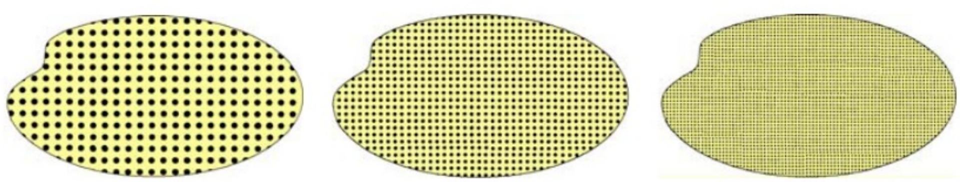

Figure 5.1: A periodic domain for three different values of ϵ.

Definition 5.1.1. *The values $k_\epsilon \in \mathbb{C}$ for which (5.3)-(5.6) has a nontrivial solution are called transmission eigenvalues. The corresponding nonzero solutions (w_ϵ, v_ϵ) are* *called eigenfunctions.*

It is known that assuming roughly that $A-I$ or/and $n-1$ do not change sign in D and are bounded away from zero real transmission eigenvalues exists [22],[29]. However the transmission eigenvalue problem is non-selfajoint and this causes complications in the analysis particularly concerning the existence of the eigenvalues. In this study we are interested in the behavior of eigenvalues k_ϵ^2 and eigenfunctions (w_ϵ, v_ϵ) in the limiting case as $\epsilon \to 0$. In particular, we are interested in the limit of the *real transmission eigenvalues* since they are proven to exists and can be determined from scattering data. For fixed $\epsilon > 0$, we denote by $k_{\epsilon,j} > 0$ the j-th real transmission eigenvalue corresponding to the transmission eigenvalue problem with A_ϵ and n_ϵ, and will investigate $\lim_{\epsilon\to 0} k_{\epsilon,j}$ for fixed $j \in \mathbb{N}$. Note that under our assumptions the set of transmission eigenvalues is discrete, hence it is possible to order the real eigenvalues in increasing order.

We are interested in developing the asymptotic theory of (5.3)-(5.6) as the period size $\epsilon \to 0$. To this end we need to define the space

$$H^1_\#(Y) := \{u \in H^1(Y) \,|\, u(y) \text{ is } Y\text{-periodic}\}$$

and consider the subspace of Y-periodic H^1-functions of mean zero, i.e.

$$\widehat{H}^1_\#(Y) := \left\{ u \in H^1_\#(Y) \,|\, \int_Y u(y)\,dy = 0 \right\}.$$

One expects (as our convergence analysis will confirm) that the homogenized or limiting transmission eigenvalue problem will be

$$\nabla_x \cdot A_h \nabla_x w + k^2 n_h w = 0 \quad \text{in} \quad D \tag{5.7}$$

$$\Delta_x v + k^2 v = 0 \quad \text{in} \quad D \tag{5.8}$$

$$w = v \quad \text{on} \quad \partial D \tag{5.9}$$

$$\frac{\partial w}{\partial \nu_{A_h}} = \frac{\partial v}{\partial \nu} \quad \text{on} \quad \partial D, \tag{5.10}$$

where

$$A_h = \frac{1}{|Y|} \int_Y \Big(A(y) - A(y) \nabla_y \vec{\psi}(y) \Big)\, dy \qquad \text{and} \qquad n_h = \frac{1}{|Y|} \int_Y n(y)\, dy. \tag{5.11}$$

The so-called cell function $\psi_i(y) \in \widehat{H}^1_{\#}(Y)$ is the unique solution to

$$\nabla_y \cdot A \nabla_y \psi_i = \nabla_y \cdot A e_i \ \text{ in } \ Y, \tag{5.12}$$

where e_i is the i-th standard basis vector in $\mathbb{R}^d$. We recall that it is well known that the homogenized (otherwise known as effective) anisotropic constitutive parameter of the periodic medium A_h satisfies the following estimates [2]

$$\left(\frac{1}{|Y|} \int\limits_Y A^{-1}(y) dy \right)^{-1} \xi \cdot \overline{\xi} \leq A_h \xi \cdot \overline{\xi} \leq \left(\frac{1}{|Y|} \int\limits_Y A(y) dy \right) \xi \cdot \overline{\xi} \quad \xi \in \mathbb{C}^d \tag{5.13}$$

hence (5.1) and (5.2) is also satisfied by A_h and n_h.

The question now is whether the eigenvalues k_ϵ and corresponding eigenfunctions v_ϵ, w_ϵ of (5.3)-(5.6) converge to eigenvalues and eigenfunctions of (5.7)-(5.10).

For the Dirichlet and Neumann eigenvalue problem for periodic structures the question of convergence is studied in details. In particular for these problems, the convergence is proven in [52] and [53] and the rate of convergence with explicit first order correction involving the boundary layer is studied in [49], [51], [65], [66] and [75]. Given the peculiarities of the transmission eigenvalue problem such as non-selfadjointness and the lack of ellipticity, the above approaches cannot be applied in a straightforward manner. Furthermore the transmission eigenvalue problem exhibits different properties in the case when $A \neq I$ or $A = I$, hence each of these cases need to be studied separately [27]. We remark that the existence of transmission eigenvalues in general settings is proven in [59], [60] and [73], and the existence of an infinite set of real transmission eigenvalues along with monotonicity properties are proven in [22] and [29]. In the next section we justify the formal asymptotic for the resolvent corresponding to the transmission eigenvalue problem using the two scale convergent approach developed in [1]. This is followed by the proof of convergence results for a subset of real transmission eigenvalues. The last section is dedicated to some preliminary numerical examples where we investigate convergence properties of the first transmission eigenvalue and demonstrate the feasibility of using the first real transmission eigenvalue to determine the effective material properties A_h and n_h.

5.2 Convergence of the interior transmission problem

We start with the study of the convergence of the solutions to the interior transmission problem for highly oscillating media to the solution of the corresponding homogenized interior transmission problem. The approach to study the interior transmission problem depends on whether $A(y) \neq I$ for all $y \in Y$ or $A(y) = I$.

5.2.1 The case of $A(y) \neq I$

We assume that $A_{min} > 1$ or $0 < A_{max} < 1$ in addition to (5.1) and (5.2). For f_ϵ and g_ϵ in $L^2(D)$ strongly convergent to f and g, respectively, as $\epsilon \to 0$ we consider the interior transmission problem of finding $(w_\epsilon, v_\epsilon) \in H^1(D) \times H^1(D)$ such that

$$\nabla \cdot A\left(x/\epsilon\right) \nabla w_\epsilon + k^2 n\left(x/\epsilon\right) w_\epsilon = f_\epsilon \quad \text{in} \quad D \tag{5.14}$$

$$\Delta v_\epsilon + k^2 v_\epsilon = g_\epsilon \quad \text{in} \quad D \tag{5.15}$$

$$w_\epsilon = v_\epsilon \quad \text{on} \quad \partial D \tag{5.16}$$

$$\frac{\partial w_\epsilon}{\partial \nu_{A_\epsilon}} = \frac{\partial v_\epsilon}{\partial \nu} \quad \text{on} \quad \partial D. \tag{5.17}$$

The following result is known (see [11] for the proof).

Lemma 5.2.1. *Assume that $A_{min} > 1$ or $0 < A_{max} < 1$ then the problem (5.14)-(5.17) satisfies the Fredholm alternative. In particular it has a unique solution $(w_\epsilon, v_\epsilon) \in H^1(D) \times H^1(D)$ provided k is not a transmission eigenvalue.*

The following lemma is proven in [22] and [27].

Lemma 5.2.2. *Assume that $A_{min} > 1$ or $0 < A_{max} < 1$ and either $n = 1$ or if $n \neq 1$ and $\int_Y (n(y) - 1)\, dy \neq 0$ then the set of transmission eigenvalues $k \in \mathbb{C}$ is at most discrete with $+\infty$ as the only accumulation point.*

Note that (5.13) implies that $A_h - I$ is positive definite if $A_{min} > 1$ and $I - A_h$ is positive definite if $A_{max} < 1$.

To analyze (5.14)-(5.17) we define the variational space

$$X(D) := \left\{(w, v) : \; w, v \in H^1(D) \,|\, w - v \in H^1_0(D)\right\}$$

equipped with the $H^1(D) \times H^1(D)$ norm and assume that k is not a transmission eigenvalue for all $\epsilon > 0$ small enough. Let $(w_\epsilon, v_\epsilon) \in X(D)$ be the solution of (5.14)-(5.17) for $\epsilon \geq 0$ small enough (for $\epsilon = 0$ we take the interior transmission problem with the homogenized coefficients A_h and n_h) and assume that (w_ϵ, v_ϵ) is bounded in the $X(D)$-norm with respect to $\epsilon > 0$ (this assumption will be discussed later in the paper). This solution satisfies the variational problem

$$\int\limits_D A_\epsilon \nabla w_\epsilon \cdot \nabla \overline{\varphi}_1 - \nabla v_\epsilon \cdot \nabla \overline{\varphi}_2 - k^2 (n_\epsilon w_\epsilon \overline{\varphi}_1 - v_\epsilon \overline{\varphi}_2)\, dx = \int\limits_D g_\epsilon \overline{\varphi}_2 \, dx - \int\limits_D f_\epsilon \overline{\varphi}_1 \, dx \quad (5.18)$$

or all $(\varphi_1, \varphi_2) \in X(D)$. Hence we have that there is a $(w, v) \in X(D)$ such that a subsequence $(w_\epsilon, v_\epsilon) \rightharpoonup (w, v)$ in $X(D)$ (strongly in $L^2(D) \times L^2(D)$). We now show that (w, v) solves the homogenized interior transmission problem. We adopt the formal two-scale convergence framework: we say that a sequence α_ϵ of $L^2(D)$ two-scale converges to $\alpha \in L^2(D \times Y)$ if

$$\int_D \alpha_\epsilon \varphi(x) \phi(x/\epsilon) dx \to \frac{1}{|Y|} \int_D \int_Y \alpha(x, y) \varphi(x) \phi(y) dy dx$$

for all $\varphi \in L^2(D)$ and $\phi \in C_\#(Y)$ (the space of Y-periodic continuous functions). From [1, Proposition 1.14] there exists w_1 and $v_1 \in L^2(D, H^1_\#(Y))$ such that (up to a subsequence), ∇w_ϵ and ∇v_ϵ respectively two-scale converge to $\nabla_x w(x) + \nabla_y w_1(x, y)$ and $\nabla_x v(x) + \nabla_y v_1(x, y)$. Let θ_1 and θ_2 in $C_0^\infty(D)$, ϕ_1 and ϕ_2 in $C_\#^\infty(Y)$ (Y-periodic C^∞ functions) and $(\psi_1, \psi_2) \in X(D)$. Applying (5.18) to $(\varphi_1, \varphi_2) \in X(D)$ such that

$\varphi_i(x) = \psi_i(x) + \epsilon\theta_i(x)\phi_i(x/\epsilon)$, $i = 1, 2$ then taking the two-scale limit implies

$$\int_D \int_Y A(y)(\nabla w(x) + \nabla_y w_1(x,y)) \cdot (\nabla \psi_1(x) + \theta_1(x)\nabla\phi_1(y))dydx$$
$$- \int_D \int_Y (\nabla v(x) + \nabla_y v_1(x,y)) \cdot (\nabla \psi_2(x) + \theta_2(x)\nabla\phi_2(y))dydx$$
$$- k^2 \int_D \int_Y n(y)w(x)\psi_1(x) - v(x)\psi_2(x)dydx = |Y| \int_D g(x)\psi_2(x) - f(x)\psi_1(x)\, dx. \tag{5.19}$$

Taking $\psi_1 = \psi_2 = 0$ one easily deduces

$$w_1(x,y) = -\vec{\psi}(y) \cdot \nabla w(x) + \overline{w}_1(x) \text{ and } v_1(x,y) = \overline{v}_1(x). \tag{5.20}$$

Then considering again (5.19) with $\theta_1 = \theta_2 = 0$ implies that $(w, v) \in X(D)$ satisfies

$$\int_D A_h \nabla w \cdot \nabla\psi_1 - \nabla v \cdot \nabla\psi_2 - k^2(n_h w\psi_1 - v\psi_2)\, dx = \int_D g\psi_2\, dx - \int_D f\psi_1\, dx \tag{5.21}$$

which is the variational formulation of the homogenized problem

$$\nabla \cdot A_h \nabla w + k^2 n_h w = f \quad \text{in} \quad D \tag{5.22}$$
$$\Delta v + k^2 v = g \quad \text{in} \quad D \tag{5.23}$$
$$w = v \text{ and } \frac{\partial w}{\partial \nu_{A_h}} = \frac{\partial v}{\partial \nu} \quad \text{on} \quad \partial D. \tag{5.24}$$

The above analysis was based on the assumption that the sequence that solves (5.14)-(5.17) is bounded with respect to $\epsilon > 0$. Now we wish to show that any sequence that solves (5.14)-(5.17) is indeed bounded independently of ϵ which will guaranty that there is a weakly convergent subsequence whose limit solves (5.22)-(5.24).

Theorem 5.2.1. *Assume that either $A_{min} > 1$ or $0 < A_{max} < 1$ and that k^2 is not a transmission eigenvalue for $\epsilon \geq 0$ small enough. Then for (w_ϵ, v_ϵ) solving (5.14)-(5.17) there exists $C > 0$ independent of (f_ϵ, g_ϵ) and ϵ such that*

$$||w_\epsilon||_{H^1(D)} + ||v_\epsilon||_{H^1(D)} \leq C \left(||f_\epsilon||_{L^2(D)} + ||g_\epsilon||_{L^2(D)} \right).$$

Proof. We will prove that (5.14)-(5.17) satisfies the Fredholm property following the $\mathbb{T}$-coercivity approach in [11]. To this end we recall the variational formulation (5.18) equivalent to (5.14)-(5.17). Let us first assume that $A_{min} > 1$, which means that $A_\epsilon - I$ is positive definite in D uniformly with respect to $\epsilon > 0$, and define the bounded sesquilinear forms in $X(D) \times X(D)$

$$a_\epsilon\big((w_\epsilon, v_\epsilon); (\varphi_1, \varphi_2)\big) := \int\limits_D A_\epsilon \nabla w_\epsilon \cdot \nabla \overline{\varphi}_1 + A_{min} w_\epsilon \overline{\varphi}_1 \, dx - \int\limits_D \nabla v_\epsilon \cdot \nabla \overline{\varphi}_2 + v_\epsilon \overline{\varphi}_2 \, dx$$

$$b_\epsilon\big((w_\epsilon, v_\epsilon); (\varphi_1, \varphi_2)\big) := -\int\limits_D (k^2 n_\epsilon + A_{min}) w_\epsilon \overline{\varphi}_1 \; - (k^2+1) v_\epsilon \overline{\varphi}_2 \, dx$$

Then (5.18) can be written as

$$a_\epsilon\big((w_\epsilon, v_\epsilon); (\varphi_1, \varphi_2)\big) + b_\epsilon\big((w_\epsilon, v_\epsilon); (\varphi_1, \varphi_2)\big) = F_\epsilon(\varphi_1, \varphi_2)$$

where $F_\epsilon(\varphi_1, \varphi_2)$ is the bounded conjugate linear functional on $X(D)$ that is bounded independently of ϵ, defined by the right hand side of (5.18). Let us define by $\mathbb{A}_\epsilon : X(D) \to X(D)$ and $\mathbb{B}_\epsilon : X(D) \to X(D)$ the bounded linear operators defined by $a_\epsilon\big((w_\epsilon, v_\epsilon)\big)$ and $b_\epsilon\big((w_\epsilon, v_\epsilon)\big)$ by means of the Riesz representation theorem. It is clear that $\mathbb{B}_\epsilon$ is compact. We next show that $\mathbb{A}_\epsilon$ is invertible with bounded inverse uniformly with respect to $\epsilon > 0$. To this end we consider the isomorphism $\mathbb{T}(w,v) = (w, -v+2w) : X(D) \mapsto X(D)$ (it is easy to check that $\mathbb{T} = \mathbb{T}^{-1}$). Using this we now show that $a_\epsilon\big((w_\epsilon, v_\epsilon); \mathbb{T}(\varphi_1, \varphi_2)\big)$ is coercive in $X(D)$. Therefore we have that

$$\begin{aligned} \big|a_\epsilon\big((w_\epsilon, v_\epsilon); \mathbb{T}(w_\epsilon, v_\epsilon)\big)\big| \;&\geq\; \int\limits_D A_\epsilon \nabla w_\epsilon \cdot \nabla \overline{w}_\epsilon + A_{min}|w_\epsilon|^2\, dx + \int\limits_D |\nabla v_\epsilon|^2 + |v_\epsilon|^2\, dx \\ &-\; 2\left| \int\limits_D \nabla v_\epsilon \cdot \nabla \overline{w}_\epsilon + v_\epsilon \overline{w}_\epsilon\, dx \right| \end{aligned}$$

But we can estimate

$$\left| 2\int\limits_D \nabla v_\epsilon \cdot \nabla \overline{w}_\epsilon + v_\epsilon \overline{w}_\epsilon\, dx \right| \leq \frac{1}{\delta}||w_\epsilon||^2_{H^1(D)} + \delta ||v_\epsilon||^2_{H^1(D)} \qquad \text{for any} \quad \delta > 0.$$

Hence we obtain that

$$\big|a_\epsilon\big((w_\epsilon, v_\epsilon); \mathbb{T}(w_\epsilon, v_\epsilon)\big)\big| \;\geq\; \left(A_{min} - \frac{1}{\delta}\right) ||w_\epsilon||^2_{H^1(D)} + (1-\delta)||v_\epsilon||^2_{H^1(D)}$$

So for any $\delta \in \left(\frac{1}{A_{min}}, 1\right)$ we have that there is a constant $\alpha > 0$ independent of ϵ such that

$$\big|a_\epsilon\big((w_\epsilon, v_\epsilon); \mathbb{T}(w_\epsilon, v_\epsilon)\big)\big| \geq \alpha \left(||w_\epsilon||^2_{H^1(D)} + ||v_\epsilon||^2_{H^1(D)} \right).$$

Next we assume that $A_{max} < 1$ which means that $I - A_\epsilon$ is positive definite in D uniformly with respect to $\epsilon > 0$. Similarly we define

$$\begin{aligned} a_\epsilon\big((w_\epsilon, v_\epsilon); (\varphi_1, \varphi_2)\big) &:= \int\limits_D A_\epsilon \nabla w_\epsilon \cdot \nabla \overline{\varphi_1} + A_{max} w_\epsilon \overline{\varphi_1}\, dx - \int\limits_D \nabla v_\epsilon \cdot \nabla \overline{\varphi_2} + v_\epsilon \overline{\varphi_2}\, dx \\ b_\epsilon\big((w_\epsilon, v_\epsilon); (\varphi_1, \varphi_2)\big) &:= -\int\limits_D (k^2 n_\epsilon + A_{max}) w_\epsilon \overline{\varphi_1} \; - (k^2+1) v_\epsilon \overline{\varphi_2}\, dx \end{aligned}$$

and the corresponding bounded linear operators $\mathbb{A}_\epsilon : X(D) \to X(D)$ and $\mathbb{B}_\epsilon : X(D) \to X(D)$. To show that $\mathbb{A}_\epsilon$ is invertible we now consider the isomorphism $\mathbb{T}(w,v) = (w-2v,-v) : X(D) \mapsto X(D)$ (it is easy to check that $\mathbb{T} = \mathbb{T}^{-1}$). We then have that

$$\begin{aligned} \Big|a_\epsilon\big((w_\epsilon,v_\epsilon);\mathbb{T}(w,v)\big)\Big| \;&\geq\; \int\limits_D A_\epsilon \nabla w_\epsilon \cdot \nabla \overline{w}_\epsilon + A_{max}|w_\epsilon|^2\,dx + \int\limits_D |\nabla v_\epsilon|^2 + |v_\epsilon|^2\,dx \\ &-\; 2\left|\int\limits_D A_\epsilon \nabla w_\epsilon \cdot \nabla \overline{v}_\epsilon + A_{max} w_\epsilon \overline{v}_\epsilon\,dx\right|. \end{aligned}$$

Using that A_ϵ is symmetric positive definite we have that for any $\delta > 0$

$$\left|2\int\limits_D A_\epsilon \nabla w_\epsilon \cdot \nabla \overline{v}_\epsilon\,dx\right| \leq \delta \int\limits_D A_\epsilon \nabla w_\epsilon \cdot \nabla \overline{w}_\epsilon\,dx + \frac{A_{max}}{\delta}\int\limits_D |\nabla v|_\epsilon^2\,dx.$$

We also use that for any $\mu > 0$

$$\left|2\int\limits_D A_{max} w_\epsilon \overline{v}_\epsilon\,dx\right| \leq \frac{A_{max}^2}{\mu}||w_\epsilon||^2_{L^2(D)} + \mu||v_\epsilon||^2_{L^2(D)}.$$

Form the above estimates we see that

$$\begin{aligned} \Big|a_\epsilon\big((w_\epsilon,v_\epsilon);\mathbb{T}(w_\epsilon,v_\epsilon)\big)\Big| \;&\geq\; A_{min}\,(1-\delta)\,||\nabla w_\epsilon||^2_{L^2(D)} + \left(1-\frac{A_{max}}{\delta}\right)||\nabla v_\epsilon||^2_{L^2(D)} \\ &+\; A_{max}\left(1-\frac{A_{max}}{\mu}\right)||w_\epsilon||^2_{L^2(D)} + (1-\mu)||v_\epsilon||^2_{L^2(D)}. \end{aligned}$$

So for any $\mu\,,\delta \in (A_{max},1)$. Hence $\mathbb{A}_\epsilon^{-1} : X(D) \mapsto X(D)$ exists for all $\epsilon > 0$ with $||\mathbb{A}_\epsilon^{-1}||_{\mathcal{L}(X(D))}$ bounded independently of ϵ.

In both cases, the above analysis also proves that the Fredholm alternative can be applied to the operator $(\mathbb{A}_\epsilon + \mathbb{B}_\epsilon)$ and equivalently to (5.14)-(5.17). Therefore if k^2 is not a transmission eigenvalue for $\epsilon \geq 0$ we have that there is a constant C_ϵ that does not depend on (f_ϵ, g_ϵ) but possibly on $\epsilon > 0$ such that the unique solution (w_ϵ, v_ϵ) of (5.14)-(5.17) satisfies

$$||w_\epsilon||_{L^2(D)} + ||v_\epsilon||_{L^2(D)} \leq C_\epsilon \left(||f_\epsilon||_{L^2(D)} + ||g_\epsilon||_{L^2(D)}\right).$$

The above analysis shows that if $(w_\epsilon, v_\epsilon) \in X(D)$ solves (5.14)-(5.17) then

$$(\mathbb{I} + \mathbb{K}_\epsilon)(w_\epsilon, v_\epsilon) = (\alpha_\epsilon, \beta_\epsilon),$$

where $\mathbb{K}_\epsilon$ is compact such that

$$||\mathbb{K}_\epsilon(w_\epsilon, v_\epsilon)||_{X(D)} \leq M_1 \left(||w_\epsilon||_{L^2(D)} + ||v_\epsilon||_{L^2(D)}\right), \tag{5.25}$$

and $(\alpha_\epsilon, \beta_\epsilon) \in X(D)$ is such that

$$||\alpha_\epsilon||_{H^1(D)} + ||\beta_\epsilon||_{H^1(D)} \leq M_2 \left(||f_\epsilon||_{L^2(D)} + ||g_\epsilon||_{L^2(D)}\right) \tag{5.26}$$

with M_1 and M_2 independent of ϵ (Note that (5.25) holds for $\mathbb{K}_\epsilon = \mathbb{A}_\epsilon^{-1}\mathbb{B}_\epsilon$ since obviously $\|\mathbb{B}_\epsilon(w_\epsilon, v_\epsilon)\|_{X(D)}$ is bounded by the $L^2(D) \times L^2(D)$ norm of (w_ϵ, v_ϵ) and $\|\mathbb{A}_\epsilon^{-1}\|_{\mathcal{L}(X(D))}$ is uniformly bounded with respect to ϵ).

Next we need to show that C_ϵ is bounded independently of ϵ. Assume to the contrary that C_ϵ is not bounded as $\epsilon \to 0$. If this is true we can find a subsequence such that

$$||w_\epsilon||_{L^2(D)} + ||v_\epsilon||_{L^2(D)} \geq \gamma_\epsilon \left(||f_\epsilon||_{L^2(D)} + ||g_\epsilon||_{L^2(D)}\right)$$

where the sequence $\gamma_\epsilon \longrightarrow \infty$ as $\epsilon \to 0$. So we define the sequence $(\tilde{w}_\epsilon, \tilde{v}_\epsilon) \in X(D)$

$$\tilde{w}_\epsilon := \frac{w_\epsilon}{||w_\epsilon||_{L^2(D)} + ||v_\epsilon||_{L^2(D)}} \quad \text{and} \quad \tilde{v}_\epsilon := \frac{v_\epsilon}{||w_\epsilon||_{L^2(D)} + ||v_\epsilon||_{L^2(D)}}.$$

Note that $||\tilde{w}_\epsilon||_{L^2(D)} + ||\tilde{v}_\epsilon||_{L^2(D)} = 1$ and we have that $(\tilde{w}_\epsilon, \tilde{v}_\epsilon)$ solves (5.14)-(5.17) with $(\tilde{f}_\epsilon, \tilde{g}_\epsilon) \in L^2(D) \times L^2(D)$ given by

$$\tilde{f}_\epsilon := \frac{f_\epsilon}{||w_\epsilon||_{L^2(D)} + ||v_\epsilon||_{L^2(D)}} \quad \text{and} \quad \tilde{g}_\epsilon := \frac{g_\epsilon}{||w_\epsilon||_{L^2(D)} + ||v_\epsilon||_{L^2(D)}}.$$

Notice we have that $||\tilde{f}_\epsilon||_{L^2(D)} + ||\tilde{g}_\epsilon||_{L^2(D)} \leq \frac{1}{\gamma_\epsilon} \longrightarrow 0$ as $\epsilon \to 0$. Now from (5.25) and (5.26) we have that $(\mathbb{I} + \mathbb{K}_\epsilon)(\tilde{w}_\epsilon, \tilde{v}_\epsilon) = (\tilde{\alpha}_\epsilon, \tilde{\beta}_\epsilon)$. Therefore for all ϵ sufficiently small

$$\begin{aligned} ||\tilde{w}_\epsilon||_{H^1(D)} + ||\tilde{v}_\epsilon||_{H^1(D)} &\leq ||\mathbb{K}_\epsilon(\tilde{w}_\epsilon, \tilde{v}_\epsilon)||_{X(D)} + ||(\tilde{\alpha}_\epsilon, \tilde{\beta}_\epsilon)||_{X(D)}, \\ &\leq M_1 \left(||\tilde{w}_\epsilon||_{L^2(D)} + ||\tilde{v}_\epsilon||_{L^2(D)}\right) + M_2 \left(||\tilde{f}_\epsilon||_{L^2(D)} + ||\tilde{g}_\epsilon||_{L^2(D)}\right), \\ &\leq M_1 + M_2. \end{aligned}$$

Since M_1 and M_2 are independent of ϵ we have that $(\tilde{w}_\epsilon, \tilde{v}_\epsilon)$ is a bounded sequence in $X(D)$ and therefore has a subsequence that converges to $(\tilde{w}, \tilde{v})$ weakly in $X(D)$ (strongly in $L^2(D) \times L^2(D)$). Also we have that $(\tilde{w}, \tilde{v})$ solves (5.22)-(5.24) with $(f, g) = (0, 0)$. Since k^2 is not a transmission eigenvalue for $\epsilon = 0$ we have that $(\tilde{w}, \tilde{v}) = (0, 0)$ which contradicts the fact that $||\tilde{w}||_{L^2(D)} + ||\tilde{v}||_{L^2(D)} = 1$ which proves the claim. □

Notice that Theorem 5.2.1 gives that any sequence (w_ϵ, v_ϵ) that solves (5.14)-(5.17) is bounded in $X(D)$ since f_ϵ and g_ϵ are assumed to converge strongly in $L^2(D)$. We can now prove the following convergence result based on the above analysis.

Theorem 5.2.2. *Assume that either $A_{min} > 1$ or $A_{max} < 1$ and that k is not a transmission eigenvalue for $\epsilon \geq 0$ small enough. Then we have that (w_ϵ, v_ϵ) solving (5.14)-(5.17) converges weakly in $X(D)$ $\big($strongly in $L^2(D) \times L^2(D)\big)$ to (w, v) that is a solution of (5.22)-(5.24). If we assume in addition that $w \in H^2(D)$ then, v_ϵ strongly converges to v in $H^1(D)$ and $w_\epsilon(x) - w(x) - \epsilon w_1(x, x/\epsilon)$ strongly converges to 0 in $H^1(D)$ where $w_1(x, y) := -\vec{\psi}(y) \cdot \nabla w(x)$.*

Proof. The first part of the theorem is a direct consequence of the above analysis and the uniqueness of solutions to (5.22)-(5.24). The corrector type result is obtained using the $\mathbb{T}$-coercivity property as follows. We first observe that, due to the strong convergence of the right hand side of the variational formulation of interior transmission problem, we have that

$$(a_\epsilon + b_\epsilon)\big((w_\epsilon, v_\epsilon); \mathbb{T}(w_\epsilon, v_\epsilon)\big) \to F(\mathbb{T}(w, v)) = (a + b)\big((w, v); \mathbb{T}(w, v)\big)$$

as $\epsilon \to 0$ where a and b have similar expressions as a_ϵ and b_ϵ with A_ϵ and n_ϵ respectively replaced by A_h and n_h and F has the same expression as F_ϵ with f_ϵ and g_ϵ respectively replaced with f and g. The L^2 strong convergence implies that

$$b_\epsilon\big((w_\epsilon, v_\epsilon); \mathbb{T}(w_\epsilon, v_\epsilon)\big) \to b\big((w, v); \mathbb{T}(w, v)\big).$$

We therefore end up with,

$$a_\epsilon\big((w_\epsilon, v_\epsilon); \mathbb{T}(w_\epsilon, v_\epsilon)\big) \to a\big((w, v); \mathbb{T}(w, v)\big) \tag{5.27}$$

as $\epsilon \to 0$. Let us set $w_1^\epsilon(x) := w_1(x, x/\epsilon)$. From the expression of w_1 one has (see for instance [65])

$$\epsilon^{1/2}\|w_1^\epsilon\|_{H^{1/2}(\partial D)} \le C$$

for some constant C independent of ϵ. Therefore we can construct a lifting function $v_1^\epsilon \in H^1(D)$ such that $v_1^\epsilon = w_1^\epsilon$ on ∂D and

$$\epsilon\|v_1^\epsilon\|_{H^1(D)} \to 0 \text{ as } \epsilon \to 0. \tag{5.28}$$

Now, taking as test functions $\varphi_1 = \tilde{w}_\epsilon$ and $\varphi_2 = \tilde{v}_\epsilon$ where $\tilde{w}_\epsilon(x) := w(x)+\epsilon w_1(x, x/\epsilon)$ and $\tilde{v}_\epsilon(x) := v(x) + \epsilon v_1^\epsilon(x)$, one has

$$(a_\epsilon + b_\epsilon)\big((w_\epsilon, v_\epsilon); \mathbb{T}(\tilde{w}_\epsilon, \tilde{v}_\epsilon)\big) \to F(\mathbb{T}(w, v)).$$

Using the two-scale convergence of the sequences w_ϵ and v_ϵ together with the form (and regularity) of w_1 as well as (5.28), we easily see that

$$b_\epsilon\big((w_\epsilon, v_\epsilon); \mathbb{T}(\tilde{w}_\epsilon, \tilde{v}_\epsilon)\big) \to b\big((w, v); \mathbb{T}(w, v)\big)$$

while

$$a_\epsilon\big((w_\epsilon, v_\epsilon); \mathbb{T}(\tilde{w}_\epsilon, \tilde{v}_\epsilon)\big) \to L(w, w_1, v)$$

with

$$\begin{aligned} L(w, w_1, v) = &\frac{1}{|Y|}\int\limits_D\int\limits_Y A(y)(\nabla w(x) + \nabla_y w_1(x, y)) \cdot (\nabla\overline{w}(x) + \nabla_y\overline{w}_1(x, y))dydx \\ &+ \int\limits_D |\nabla v(x)|^2 + A_{min}|w(x)|^2 + |v(x)|^2 - 2\nabla\overline{w}(x)\nabla v(x) - 2\overline{w}(x)v(x)dx \end{aligned}$$

in the case $A_{min} > 1$ and

$$\begin{aligned} L(w, w_1, v) = \frac{1}{|Y|} \int_D \int_Y & A(y)(\nabla w(x) + \nabla_y w_1(x,y)) \cdot (\nabla \overline{w}(x) + \nabla_y \overline{w}_1(x,y)) dy dx \\ & - 2\frac{1}{|Y|} \int_D \int_Y A(y)(\nabla w(x) + \nabla_y w_1(x,y)) \cdot \nabla \overline{v}(x) dy dx \\ & \quad + \int_D |\nabla v(x)|^2 + A_{min}|w(x)|^2 + |v(x)|^2 - 2A_{\min}\overline{v}(x) w(x) dx \end{aligned}$$

in the case $A_{max} < 1$. Hence we can conclude that

$$F(\mathbb{T}(w,v)) = L(w, w_1, v) + b\big((w,v); \mathbb{T}(w,v)\big)$$

and therefore

$$a\big((w,v); \mathbb{T}(w,v)\big) = L(w, w_1, v). \tag{5.29}$$

Using (5.27) and (5.29) and the T-coercivity, we can apply similar arguments as in [1, Theorem 2.6] to obtain the result. Indeed, the T-coercivity shows that it is sufficient to prove that

$$a_\epsilon\big((w_\epsilon - \tilde{w}_\epsilon, v_\epsilon - v); \mathbb{T}(w_\epsilon - \tilde{w}_\epsilon, v_\epsilon - v)\big) \to 0. \tag{5.30}$$

Now, using the two-scale convergence of the sequences v_ϵ and w_ϵ, we observe that each of the quantities

$$a_\epsilon\big((w_\epsilon, v_\epsilon); \mathbb{T}(\tilde{w}_\epsilon, v)\big),\ a_\epsilon\big((\tilde{w}_\epsilon, v); \mathbb{T}(w_\epsilon, v_\epsilon)\big) \text{ and } a_\epsilon\big((\tilde{w}_\epsilon, v); \mathbb{T}(\tilde{w}_\epsilon, v)\big)$$

converges to $L(w, w_1, v)$

Finally, using (5.27) we can conclude that

$$a_\epsilon\big((w_\epsilon - \tilde{w}_\epsilon, v_\epsilon - v); \mathbb{T}(w_\epsilon - \tilde{w}_\epsilon, v_\epsilon - v)\big) \to a\big((w, v); \mathbb{T}(w, v)\big) - L(w, w_1, v)$$

and then the result is a direct consequence of (5.30). □

Remark 5.2.1. *The same convergence analysis holds for the interior transmission problem formulated as follows*

$$\begin{aligned} \nabla \cdot A\,(x/\epsilon)\,\nabla w_\epsilon + k^2 n\,(x/\epsilon)\,w_\epsilon = 0 \quad & in \quad D \\ \Delta v_\epsilon + k^2 v_\epsilon = 0 \quad & in \quad D \\ w_\epsilon - v_\epsilon = f_\epsilon \quad & on \quad \partial D \\ \frac{\partial w_\epsilon}{\partial \nu_{A_\epsilon}} - \frac{\partial v_\epsilon}{\partial \nu} = g_\epsilon \quad & on \quad \partial D, \end{aligned}$$

where $(f_\epsilon, g_\epsilon) \in H^{1/2}(\partial D) \times H^{-1/2}(\partial D)$ *is strongly convergent to* (f, g) *and* $\epsilon \to 0$. *This formulation is closer to what arises from the scattering theory from periodic media [14]. From the Introduction we see that* $f_\epsilon = u^i$ *and* $g_\epsilon = \frac{\partial u^i}{\partial \nu}$

5.2.2 Boundary Corrector and Convergence Rate for $A(y) \neq I$

To establish a convergence rate for the interior transmission problem we construct boundary correctors for problem (5.14)-(5.17) for $f_\epsilon = f$ and $g_\epsilon = g$ (i.e. independent of ϵ). The boundary correctors are needed since the boundary ∂D disrupts the periodicity or the material and causes the gradient of w_ϵ to be highly oscillatory . Now we define $\mathbf{u}_\epsilon := A\,(x/\epsilon)\,\nabla w_\epsilon$, where we assume that

$$\mathbf{u}_\epsilon(x) \quad = \quad \mathbf{u}^{(0)}(x, y) + \epsilon \mathbf{u}^{(1)}(x, y) + \epsilon^2 \mathbf{u}^{(2)}(x, y) + \cdots$$

therefore we have for a fixed f and g in $L^2(D)$ that

$$\nabla \cdot \mathbf{u}_\epsilon + k^2 n\left(x/\epsilon\right) w_\epsilon = f \quad \text{in} \quad D. \tag{5.31}$$

We then us the chain rule to rewrite the partial differential equations for w_ϵ and $\mathbf{u}_\epsilon$. To do so we use the definition of the multi-scale gradient operator given by $\nabla = \nabla_x + \frac{1}{\epsilon}\nabla_y$. We now have that by using the multi-scale gradient operator and our asymptotic expansions collecting similar powers of ϵ we have that

$$\nabla_y \cdot \mathbf{u}^{(0)} = 0 \tag{5.32}$$

$$\mathbf{u}^{(0)} = A_\epsilon \nabla_x w + A_\epsilon \nabla_y w_1 \tag{5.33}$$

$$\nabla_x \cdot \mathbf{u}^{(0)} + \nabla_y \cdot \mathbf{u}^{(1)} = -k^2 n_\epsilon w + f \tag{5.34}$$

It has been shown that $w_1(x,y) = -\psi(y) \cdot \nabla_x w(x)$ therefore

$$\mathbf{u}^{(0)} = \left(A_\epsilon - A_\epsilon \nabla_y \vec{\psi}\right)\nabla_x w,$$

where the components of $\vec{\psi}$ are defined by (5.12). Taking the average of $\mathbf{u}^{(0)}$ over the cell Y gives $\mathbf{u}_h^{(0)} = A_h \nabla_x w$. Using this and (5.32) we can obtain that $\mathbf{u}^{(1)}$, to this end let $q(x,y) \in \left(H^1_\sharp(Y)\right)^d$ be the solution to

$$\nabla_y \times q(x,y) = \mathbf{u}^{(0)} - \mathbf{u}_h^{(0)}$$

which exists since $\mathbf{u}^{(0)} - \mathbf{u}_h^{(0)}$ is divergence free in the y variable where in $\mathbb{R}^2$ the $\nabla \times$ is the rot of the scalar function q. Then let $\mathbf{u}^{(1)} = \nabla_x \times q(x,y)$, which gives that (5.34) is satisfied by taking an average over Y. For the convergence we have seen that one does not need to construct v_1 in the expansions of v_ϵ since it satisfies the

Helmholtz equation. Due to the boundary condition (5.6) we will need to correct at the Neumann boundary condition for v_ϵ. We now need the boundary correctors given by

$$\nabla \cdot A\left(x/\epsilon\right)\nabla\varphi_\epsilon + k^2 n\left(x/\epsilon\right)\varphi_\epsilon = 0 \quad \text{in} \quad D \quad (5.35)$$

$$\Delta\phi_\epsilon + k^2\phi_\epsilon = 0 \quad \text{in} \quad D \quad (5.36)$$

$$\varphi_\epsilon - \phi_\epsilon = -w_1 \;\; \text{and} \;\; \frac{\partial\varphi_\epsilon}{\partial\nu_{A_\epsilon}} - \frac{\partial\phi_\epsilon}{\partial\nu} = \left(\frac{\mathbf{u}_h^{(0)} - \mathbf{u}^{(0)}}{\epsilon} - \mathbf{u}^{(1)}\right)\cdot\nu \quad \text{on} \quad \partial D \quad (5.37)$$

Notice that the Neumann boundary condition is given by $\frac{\partial\varphi_\epsilon}{\partial\nu_{A_\epsilon}} - \frac{\partial\phi_\epsilon}{\partial\nu} = \big(\nabla \times q\big)\cdot\nu$, where the full curl is given by $\nabla\times = \nabla_x \times + \frac{1}{\epsilon}\nabla_y\times$. We now define the preliminary error function as

$$z_\epsilon^w = w_\epsilon - w - \epsilon w_1 \qquad \text{and} \qquad z_\epsilon^v = v_\epsilon - v$$

$$\eta_\epsilon^w = A_\epsilon\nabla w_\epsilon - \mathbf{u}^{(0)} - \epsilon\mathbf{u}^{(1)} \qquad \text{and} \qquad \eta_\epsilon^v = \nabla v_\epsilon - \nabla v.$$

From straight forward calculations using (5.32)-(5.34) and using the multi-scale gradient we obtain that

$$\begin{aligned} A_\epsilon\nabla z_\epsilon^w - \eta_\epsilon^w &= \mathbf{u}^{(0)} - A_\epsilon\nabla w - \epsilon\left(A_\epsilon\nabla w_1 - \mathbf{u}^{(1)}\right) \\ &= \mathbf{u}^{(0)} - A_\epsilon\nabla_x w - \epsilon\left(A_\epsilon\nabla_x w_1 + \frac{1}{\epsilon}A_\epsilon\nabla_y w_1 - \mathbf{u}^{(1)}\right) \\ &= \epsilon\left(\mathbf{u}^{(1)} - A_\epsilon\nabla_x w_1\right). \end{aligned}$$

From the definition of $\mathbf{u}^{(1)}$ and w_1 we have the following estimate

$$\|A_\epsilon\nabla z_\epsilon^w - \eta_\epsilon^w\|_{L^2(D)} \leq C\epsilon||w||_{H^2(D)} \qquad (5.38)$$

since $q(x,y)$ can be chosen such that

$$\sup_{y\in Y}|\mathbf{u}^{(1)}| \leq C\left(\sum_{i,j}\left|\frac{\partial^2 w}{\partial x_i \partial x_j}\right| + |w|\right)$$

where the constant C is independent of ϵ and w.

Similarly we have that since $\mathbf{u}^{(1)} = \nabla_x \times q(x,y)$ that

$$\begin{aligned}
\nabla \cdot \eta_\epsilon^w &= \nabla \cdot A_\epsilon \nabla w_\epsilon - \nabla \cdot \mathbf{u}^{(0)} - \epsilon \nabla \cdot \mathbf{u}^{(1)} \\
&= -k^2 n_\epsilon w_\epsilon + f - \nabla_x \cdot \mathbf{u}^{(0)} - \nabla_y \cdot \mathbf{u}^{(1)} \\
&= -k^2 n_\epsilon (w_\epsilon - w) \\
&= -k^2 n_\epsilon z_\epsilon^w + \epsilon k^2 n_\epsilon w_1 \qquad (5.39)
\end{aligned}$$

and it is clear that

$$\nabla z_\epsilon^v - \eta_\epsilon^v = 0 \quad \text{and} \quad \nabla \cdot \eta_\epsilon^v = -k^2 z_\epsilon^v.$$

Theorem 5.2.3. *Let the pair (w_ϵ, v_ϵ) be the solution to (5.14)-(5.17) with (w,v) being the solutions to (5.22)-(5.24) such that $w \in H^2(D)$, and the boundary correctors $(\varphi_\epsilon, \phi_\epsilon)$ are solutions to (5.35)-(5.37). If $A_\epsilon - I$ is positive (or negative) definite and k^2 is not a transmission eigenvalue for $\epsilon \geq 0$ sufficiently small then we have that*

$$\|w_\epsilon - w - \epsilon(w_1 + \varphi_\epsilon)\|_{H^1(D)} + \|v_\epsilon - v - \epsilon\phi_\epsilon\|_{H^1(D)} \leq C\epsilon ||w||_{H^2(D)}$$

where the constants C is independent of ϵ and w

Proof. To prove the result we will use a duality argument, to this end let $\ell(\cdot) \in X(D)'$ (the dual space of $X(D)$) where we let $(\alpha_\epsilon, \beta_\epsilon) \in X(D)$ be the solutions of the variational problem

$$\int_D A_\epsilon \nabla\alpha_\epsilon \cdot \nabla\varphi_1 - \nabla\beta_\epsilon \cdot \nabla\varphi_2 - k^2(n_\epsilon\alpha_\epsilon\varphi_1 - \beta_\epsilon\varphi_2)\,dx = \ell(\varphi_1, \varphi_2).$$

Notice that the error functions $z_\epsilon^w - \epsilon\varphi_\epsilon$ and $z_\epsilon^v - \epsilon\phi_\epsilon$ have the same trace on the boundary ∂D. Therefore we have that

$$\begin{aligned}\ell(z_\epsilon^w - \epsilon\varphi_\epsilon, z_\epsilon^v - \epsilon\phi_\epsilon) = &\\ &\int_D A_\epsilon \nabla\alpha_\epsilon \cdot \nabla z_\epsilon^w - \nabla\beta_\epsilon \cdot \nabla z_\epsilon^v - k^2(n_\epsilon\alpha_\epsilon z_\epsilon^w - \beta_\epsilon z_\epsilon^v)\,dx\\ &\quad -\epsilon\int_D A_\epsilon \nabla\alpha_\epsilon \cdot \nabla\varphi_\epsilon - \nabla\beta_\epsilon \cdot \nabla\phi_\epsilon - k^2(n_\epsilon\alpha_\epsilon\varphi_\epsilon - \beta_\epsilon\phi_\epsilon)\,dx.\end{aligned}$$

Now by using integration by parts on the second integral and using the Neumann condition in (5.37) we have that

$$\begin{aligned}\ell(z_\epsilon^w - \epsilon\varphi_\epsilon, z_\epsilon^v - \epsilon\phi_\epsilon) = &\\ &\int_D A_\epsilon \nabla\alpha_\epsilon \cdot \nabla z_\epsilon^w - \nabla\beta_\epsilon \cdot \nabla z_\epsilon^v - k^2(n_\epsilon\alpha_\epsilon z_\epsilon^w - \beta_\epsilon z_\epsilon^v)\,dx\\ &-\int_{\partial D} \alpha_\epsilon \left(\mathbf{u}_h^{(0)} - \mathbf{u}^{(0)} - \epsilon\mathbf{u}^{(1)}\right)\cdot\nu\,ds.\end{aligned}$$

Notice that since A_ϵ is symmetric and that $\nabla z_\epsilon^v - \eta_\epsilon^v = 0$ we obtain

$$
\begin{aligned}
\ell(z_\epsilon^w - \epsilon\varphi_\epsilon, z_\epsilon^v - \epsilon\phi_\epsilon) = \\
\int_D (A_\epsilon \nabla z_\epsilon^w - \eta_\epsilon^w) \cdot \nabla\alpha_\epsilon + \eta_\epsilon^w \cdot \nabla\alpha_\epsilon - \eta_\epsilon^v \cdot \nabla\beta_\epsilon - k^2(n_\epsilon\alpha_\epsilon z_\epsilon^w - \beta_\epsilon z_\epsilon^v)\, dx \\
- \int_{\partial D} \alpha_\epsilon \left(\mathbf{u}_h^{(0)} - \mathbf{u}^{(0)} - \epsilon\mathbf{u}^{(1)}\right) \cdot \nu \, ds. \qquad (5.40)
\end{aligned}
$$

Applying the Divergence theorem and using (5.39) gives that

$$
\begin{aligned}
\int_D \eta_\epsilon^w \cdot \nabla\alpha_\epsilon \, dx &= -\int_D \alpha_\epsilon \nabla\cdot\eta_\epsilon^w \, dx + \int_{\partial D} \alpha_\epsilon \eta_\epsilon^w \cdot \nu \, ds \\
&= \int_D \alpha_\epsilon (k^2 n_\epsilon z_\epsilon^w - \epsilon k^2 n_\epsilon w_1)\, dx \\
&+ \int_{\partial D} \alpha_\epsilon \big(A_\epsilon \nabla w_\epsilon - \mathbf{u}^{(0)} - \epsilon\mathbf{u}^{(1)}\big) \cdot \nu \, ds \qquad (5.41)
\end{aligned}
$$

and since $\nabla \cdot \eta_\epsilon^v = -k^2 z_\epsilon^v$

$$
\begin{aligned}
-\int_D \eta_\epsilon^v \cdot \nabla\beta_\epsilon \, dx &= \int_D \beta_\epsilon \nabla\cdot\eta_\epsilon^v \, dx - \int_{\partial D} \beta_\epsilon \eta_\epsilon^v \cdot \nu \, ds \\
&= -\int_D k^2 \beta_\epsilon z_\epsilon^v \, dx - \int_{\partial D} \alpha_\epsilon \big(\nabla v_\epsilon - \nabla v\big) \cdot \nu \, ds. \qquad (5.42)
\end{aligned}
$$

Now by substituting (5.41) and (5.42) into (5.40) and noticing the cancelation of the boundary integral by the boundary conditions(5.6) and (5.24) along with the cancelations of the volume integrals gives that

$$
\ell(z_\epsilon^w - \epsilon\varphi_\epsilon, z_\epsilon^v - \epsilon\phi_\epsilon) = \int_D (A_\epsilon \nabla z_\epsilon^w - \eta_\epsilon^w) \cdot \nabla\alpha_\epsilon - \epsilon k^2 n_\epsilon w_1 \alpha_\epsilon \, dx.
$$

From the definition of w_1 we have the estimate

$$\|n_\epsilon w_1\|_{L^2(D)} \leq C n_{max} \|w\|_{H^2(D)}$$

therefore by (5.38) we can now conclude that

$$|\ell(z_\epsilon^w - \epsilon\varphi_\epsilon, z_\epsilon^v - \epsilon\phi_\epsilon)| \leq C\epsilon \|w\|_{H^2(D)} \|\alpha_\epsilon\|_{H^1(D)}.$$

Since k^2 is not a transmission eigenvalue for ϵ sufficiently small we have that there is a constant C independent of ϵ such that

$$||\alpha_\epsilon||_{H^1(D)} + ||\beta_\epsilon||_{H^1(D)} \leq C||\ell||_{X(D)'}.$$

The result follows by dividing by $||\ell||_{X(D)'}$ and taking the supremum over $X(D)'$. $\square$

Notice the if k^2 is not a transmission eigenvalue for $\epsilon \geq 0$ sufficiently small then we have that (5.35)-(5.37) is well posed and the boundary correctors satisfy

$$\|\varphi_\epsilon\|_{H^1(D)} + \|\phi_\epsilon\|_{H^1(D)} \leq C\Big(||w_1||_{H^{1/2}(\partial D)} + ||(\nabla \times q) \cdot \nu||_{H^{-1/2}(\partial D)}\Big)$$

where the constant C is independent of ϵ, w_1 and q. It can be shown using interpolation just as in [65] and [66] that

$$||w_1||_{H^{1/2}(\partial D)} \leq C\epsilon^{-1/2}||w||_{H^2(D)}.$$

Similarly as in [26] using integration by parts and a duality argument interpolation yields that $(\nabla \times q) \cdot \nu$ is bounded by the $H^2(D)$ of w in $H^{-1}(\partial D)$ whereas in $L^2(\partial D)$

$(\nabla \times q) \cdot \nu$ is bounded by $\epsilon^{-1}||w||_{H^2(D)}$ giving that

$$||(\nabla \times q) \cdot \nu||_{H^{-1/2}(\partial D)} \leq C\epsilon^{-1/2}||w||_{H^2(D)}.$$

Therefore we have the following result.

Corollary 5.2.1. *Let the pair (w_ϵ, v_ϵ) be the solution to (5.14)-(5.17) with (w, v) being the solutions to (5.22)-(5.24) such that $w \in H^2(D)$. If $A_\epsilon - I$ is positive (or negative) definite and k^2 is not a transmission eigenvalue for $\epsilon \geq 0$ sufficiently small then we have that*

$$\|w_\epsilon - w - \epsilon w_1\|_{H^1(D)} + \|v_\epsilon - v\|_{H^1(D)} \leq C\epsilon^{1/2}||w||_{H^2(D)}$$

where the constants C is independent of ϵ.

Remark 5.2.2. *Exactly in the same way as in Appendix 2 of [26] using duality just as in Theorem 5.2.3 we have that the boundary correctors φ_ϵ and ϕ_ϵ are bounded in $L^2(D)$ by $\|w\|_{H^2(D)}$ therefore we have that*

$$\|w_\epsilon - w\|_{L^2(D)} + \|v_\epsilon - v\|_{L^2(D)} \leq C\epsilon||w||_{H^2(D)}.$$

5.2.3 The case of $A(y) = I$

Here we now assume that either $n_{min} > 1$ or $0 < n_{max} < 1$. For the case where $A_\epsilon \equiv I$ the interior transmission problem becomes: find $(w_\epsilon, v_\epsilon) \in L^2(D) \times L^2(D)$

such that

$$\Delta w_\epsilon + k^2 n\left(x/\epsilon\right) w_\epsilon = 0 \quad \text{in} \quad D \tag{5.43}$$

$$\Delta v_\epsilon + k^2 v_\epsilon = 0 \quad \text{in} \quad D \tag{5.44}$$

$$w_\epsilon - v_\epsilon = f_\epsilon \quad \text{on} \quad \partial D \tag{5.45}$$

$$\frac{\partial w_\epsilon}{\partial \nu} - \frac{\partial v_\epsilon}{\partial \nu} = g_\epsilon \quad \text{on} \quad \partial D \tag{5.46}$$

for the boundary data $(f_\epsilon, g_\epsilon) \in H^{3/2}(\partial D) \times H^{1/2}(\partial D)$ converging strongly to $(f, g) \in H^{3/2}(\partial D) \times H^{1/2}(\partial D)$ as $\epsilon \to 0$. Just as in the case for anisotropic media we require that k^2 is not a transmission eigenvalue for $\epsilon \geq 0$ small enough. We formulate the interior transmission problem for the difference $U_\epsilon := w_\epsilon - v_\epsilon \in H^2(D)$. Using the interior transmission problem one can show that this U_ϵ satisfies

$$0 \;=\; \left(\Delta + k^2 n_\epsilon\right) \frac{1}{n_\epsilon - 1} \left(\Delta + k^2\right) U_\epsilon \;\; \text{in} \;\; D \tag{5.47}$$

where

$$v_\epsilon \;=\; -\frac{1}{k^2(n_\epsilon - 1)} \left(\Delta U_\epsilon + k^2 n_\epsilon U_\epsilon\right) \quad \text{in} \;\; D \tag{5.48}$$

$$w_\epsilon \;=\; -\frac{1}{k^2(n_\epsilon - 1)} \left(\Delta U_\epsilon + k^2 U_\epsilon\right) \quad \text{in} \;\; D \tag{5.49}$$

Theorem 5.2.4. *Assume that either $(n_{min} - 1) > 0$ or $(n_{max} - 1) < 0$ and $U_\epsilon \in H^2(D)$ is a bounded sequence, then there is a subsequence such that $U_\epsilon \rightharpoonup U$ in $H^2(D)$ and $(w_\epsilon, v_\epsilon) \rightharpoonup (w, v)$ in $L^2(D) \times L^2(D)$ $\big($strongly in $L^2_{loc}(D) \times L^2_{loc}(D)\big)$. Moreover* *we have that the limit U satisfies*

$$\left(\Delta + k^2 n_h\right) \frac{1}{n_h - 1} \left(\Delta + k^2\right) U = 0 \qquad in \qquad D, \tag{5.50}$$

$$U = f \qquad and \qquad \frac{\partial U}{\partial \nu} = g \qquad on \qquad \partial D, \tag{5.51}$$

$U = w - v$, and (w, v) satisfy

$$\Delta v + k^2 v = 0 \qquad and \qquad \Delta w + k^2 n_h w = 0 \qquad in \quad D, \tag{5.52}$$

$$w - v = f \qquad and \qquad \frac{\partial w}{\partial \nu} - \frac{\partial v}{\partial \nu} = g \qquad on \quad \partial D. \tag{5.53}$$

Proof. Since U_ϵ is a bounded sequence in $H^2(D)$, from (5.48) and (5.49) we have that (w_ϵ, v_ϵ) is a bounded sequence in $L^2(D) \times L^2(D)$. Therefore we have that there is a subsequence still denoted by (w_ϵ, v_ϵ) that is weakly convergent in $L^2(D) \times L^2(D)$. So we have that for all $\varphi \in \mathcal{C}_0^\infty(D)$, there is a $v \in L^2(D)$ such that:

$$0 = \int_D v_\epsilon(\Delta\varphi + k^2\varphi)\, dx \xrightarrow{\epsilon \to 0} \int_D v(\Delta\varphi + k^2\varphi)\, dx.$$

This gives that $\Delta v + k^2 v = 0$ in the distributional sense. By interior elliptic regularity (see e.g. [81]) for all $\Omega \subset \overline{\Omega} \subset D$ and all $\epsilon > 0$ we have

$$||v_\epsilon||_{H^1(\Omega)} \leq C$$

for some constant independent of ϵ which implies (using an increasing sequence of domains Ω_n that converges to D and a diagonal extraction process of the subsequence) that a subsequence v_ϵ converges to v strongly in $L^2_{loc}(D)$. Next since $w_\epsilon = U_\epsilon + v_\epsilon$ and U_ϵ is bounded in $H^2(D)$, we have that w_ϵ converges to some w weakly in $L^2(D)$

and strongly in $L^2_{loc}(D)$. Now using the strong convergence we have that for all $\varphi \in \mathcal{C}_0^\infty(D)$ such that $\overline{\text{supp}(\varphi)} \subset D$ we obtain that

$$0 = \int\limits_D w_\epsilon(\Delta\varphi + k^2 n_\epsilon \varphi)\, dx \overset{\epsilon\to 0}{\longrightarrow} \int\limits_D w(\Delta\varphi + k^2 n_h \varphi)\, dx,$$

which gives that $\Delta w + k^2 n_h w = 0$ in the distributional sense. Now, the fact that $-k^2(n_\epsilon - 1)w_\epsilon = \Delta U_\epsilon + k^2 U_\epsilon$, the weak convergence of U_ϵ to U in $H^2(D)$ and the local strong convergence of w_ϵ to the above w imply that the limit U satisfies $(\Delta + k^2 n_h)\, \frac{1}{n_h - 1}\, (\Delta + k^2)\, U = 0$ in D and $U = w - v$. Finally, integration by parts formulas together with (5.45) and (5.46) guarantee that $U := w - v$ satisfies the boundary conditions (5.52) and (5.53) which ends the proof. □

The above result that connects w_ϵ, v_ϵ and U_ϵ with the respective limits requires that U_ϵ is a bounded sequence. Next we show that this is the case for every solution to the interior transmission problem. To this end, since $(f_\epsilon, g_\epsilon) \in H^{3/2}(\partial D) \times H^{1/2}(\partial D)$ there is a lifting function $\phi_\epsilon \in H^2(D)$ such that $\phi_\epsilon\big|_{\partial D} = f_\epsilon$ and $\frac{\partial \phi_\epsilon}{\partial \nu}\big|_{\partial D} = g_\epsilon$ and

$$||\phi_\epsilon||_{H^2(D)} \leq C\left(||f_\epsilon||_{H^{3/2}(\partial D)} + ||g_\epsilon||_{H^{1/2}(\partial D)}\right) \tag{5.54}$$

where the constant C is independent of ϵ and $\phi_\epsilon \to \phi$ strongly in $H^2(D)$ where $\phi\big|_{\partial D} = f$ and $\frac{\partial \phi}{\partial \nu}\big|_{\partial D} = g$. Now following [22] and [27] we define the bounded sesquilinear forms on $H_0^2(D) \times H_0^2(D)$:

$$\mathcal{A}_\epsilon(u,\varphi) = \int_D \frac{1}{n_\epsilon - 1}\left(\Delta u + k^2 u\right)\left(\Delta\overline{\varphi} + k^2\overline{\varphi}\right) + k^4 u\overline{\varphi}\,dx, \tag{5.55}$$

$$\widehat{\mathcal{A}}_\epsilon(u,\varphi) = \int_D \frac{n_\epsilon}{1 - n_\epsilon}\left(\Delta u + k^2 u\right)\left(\Delta\overline{\varphi} + k^2\overline{\varphi}\right) + \Delta u \Delta\overline{\varphi}\,dx, \tag{5.56}$$

$$\mathcal{B}(u,\varphi) = \int_D \nabla u \cdot \nabla\overline{\varphi}\,dx. \tag{5.57}$$

With the help of the lifting function ϕ_ϵ, we have that $u_\epsilon \in H_0^2(D)$ where $U_\epsilon = u_\epsilon + \phi_\epsilon$ and that u_ϵ solve the variational problems

$$\mathcal{A}_\epsilon(u_\epsilon,\varphi) - k^2\mathcal{B}(u_\epsilon,\varphi) = L_\epsilon(\varphi) \tag{5.58}$$

$$\widehat{\mathcal{A}}_\epsilon(u_\epsilon,\varphi) - k^2\mathcal{B}(u_\epsilon,\varphi) = \widehat{L}_\epsilon(\varphi) \tag{5.59}$$

where the conjugate linear functionals are defined as follows

$$L_\epsilon(\varphi) = k^2\mathcal{B}(\phi_\epsilon,\varphi) - \mathcal{A}_\epsilon(\phi_\epsilon,\varphi) \quad \text{and} \quad \widehat{L}_\epsilon(\varphi) = k^2\mathcal{B}(\phi_\epsilon,\varphi) - \widehat{\mathcal{A}}_\epsilon(\phi_\epsilon,\varphi).$$

Let $\mathbb{A}_\epsilon : H_0^2(D) \to H_0^2(D)$, $\widehat{\mathbb{A}}_\epsilon : H_0^2(D) \to H_0^2(D)$ and $\mathbb{B} : H_0^2(D) \to H_0^2(D)$ be bounded linear operators defined by the sesquilinear forms (5.55), (5.56) and (5.57) by means of Riesz representation theorem. Obviously $\mathbb{B}$ is a compact operator and it does not depend on ϵ, and furthermore $\|\mathbb{B}(u_\epsilon)\|_{H^2(D)}$ is bounded by $\|u_\epsilon\|_{H^1(D)}$. In [27] it is shown that $\mathcal{A}_\epsilon(\cdot,\cdot)$ is coercive when $\frac{1}{n_\epsilon - 1} \geq \alpha > 0$ for all $\epsilon > 0$ (which is satisfied if $n_{min} > 1$) whereas $\widehat{\mathcal{A}}_\epsilon(\cdot,\cdot)$ is coercive when $\frac{n_\epsilon}{1-n_\epsilon} \geq \alpha > 0$ for all $\epsilon > 0$ (which is satisfied if $0 < n_{max} < 1$) and furthermore the coercivity constant depends only on D and α. Hence $\mathbb{A}_\epsilon^{-1}$ exists if $n_{min} > 1$ and $\widehat{\mathbb{A}}_\epsilon^{-1}$ exists if $0 < n_{max} < 1$ and their norm is uniformly bounded with respect to ϵ.

Theorem 5.2.5. *Assume that either $n_{min} > 1$ or $0 < n_{max} < 1$, and that k is not a transmission eigenvalue for $\epsilon \geq 0$ small enough. If $U_\epsilon \in H^2(D)$ is a solution to (5.47) such that $U_\epsilon = f_\epsilon$ and $\frac{\partial U_\epsilon}{\partial \nu} = g_\epsilon$ on ∂D, then there is a constant $C > 0$ independent of $\epsilon \geq 0$ and (f_ϵ, g_ϵ) such that:*

$$||U_\epsilon||_{H^2(D)} \leq C \left(||f_\epsilon||_{H^{3/2}(\partial D)} + ||g_\epsilon||_{H^{1/2}(\partial D)} \right).$$

Proof. First recall that $U_\epsilon = u_\epsilon + \phi_\epsilon$ where $u_\epsilon \in H_0^2(D)$ satisfies either (5.58) or (5.59) and $\phi_\epsilon \in H^2(D)$ satisfies (5.54). Therefore it is sufficient to prove the result for u_ϵ. From the discussion above we know that u_ϵ satisfies

$$(\mathbb{I} - k^2 \mathbb{K}_\epsilon)(u_\epsilon) = \alpha_\epsilon \tag{5.60}$$

where $\mathbb{K}_\epsilon = \mathbb{A}_\epsilon^{-1}\mathbb{B}$ and $\alpha_\epsilon \in H_0^2(D)$ is the Riesz representation of L_ϵ if $n_{min} > 1$, and $\mathbb{K}_\epsilon = \widehat{\mathbb{A}}_\epsilon^{-1}\mathbb{B}$ and $\alpha_\epsilon \in H_0^2(D)$ is the Riesz representation of $\widehat{L}_\epsilon$ if $0 < n_{max} < 1$. In both cases

$$\|\mathbb{K}_\epsilon(u_\epsilon)\|_{H^2(D)} \leq M_1 \|u_\epsilon\|_{H^1(D)}$$

and

$$\|\alpha_\epsilon\|_{H^2(D)} \leq M_2 \left(||f_\epsilon||_{H^{3/2}(\partial D)} + ||g_\epsilon||_{H^{1/2}(\partial D)} \right)$$

with M_1 and M_2 independent of $\epsilon > 0$. Now since k^2 is not a transmission eigenvalue for $\epsilon \geq 0$ (small enough), the Fredholm alternative applied to (5.60) guaranties the existence of a constant C_ϵ independent of f_ϵ, g_ϵ such that

$$||u_\epsilon||_{H^2(D)} \leq C_\epsilon \left(||f_\epsilon||_{H^{3/2}(\partial D)} + ||g_\epsilon||_{H^{1/2}(\partial D)} \right).$$

In the same way as in Theorem 5.2.1, we can now show that C_ϵ is bounded independently of ϵ. Indeed, to the contrary assume that C_ϵ is not bounded as $\epsilon \to 0$. Then we can find a subsequence u_ϵ such that

$$||u_\epsilon||_{H^1(D)} \geq \gamma_\epsilon \left(||f_\epsilon||_{H^{3/2}(\partial D)} + ||g_\epsilon||_{H^{1/2}(\partial D)}\right)$$

and $\gamma_\epsilon \to \infty$ as $\epsilon \to 0$. Let us define the sequences $\tilde{u}_\epsilon := \frac{u_\epsilon}{||u_\epsilon||_{H^1(D)}}$, $\tilde{f}_\epsilon := \frac{f_\epsilon}{||u_\epsilon||_{H^1(D)}}$ and $\tilde{g}_\epsilon := \frac{g_\epsilon}{||u_\epsilon||_{H^1(D)}}$. Hence we have that $(\tilde{f}_\epsilon, \tilde{g}_\epsilon) \to (0,0)$ as $\epsilon \to 0$ and $(\mathbb{I} - k^2\mathbb{K}_\epsilon)(\tilde{u}_\epsilon) = \tilde{\alpha}_\epsilon$. Hence

$$\begin{aligned} ||\tilde{u}_\epsilon||_{H^2(D)} &\leq k^2||\mathbb{K}_\epsilon(\tilde{u}_\epsilon)||_{H^2(D)} + ||\tilde{\alpha}_\epsilon||_{H^2(D)}, \\ &\leq M_1||\tilde{u}_\epsilon||_{H^1(D)} + M_2\left(||\tilde{f}_\epsilon||_{H^{-3/2}(\partial D)} + ||\tilde{g}_\epsilon||_{H^{1/2}(\partial D)}\right) \leq M_1 + M_2. \end{aligned}$$

Hence $\tilde{u}_\epsilon$ is bounded and therefore has a weak limit in $H^2_0(D)$, which from Theorem 5.2.4 is a solution to the homogenized equation (5.50) with zero boundary data. This implies that $\tilde{u} = 0$ since k^2 is not a transmission eigenvalue for $\epsilon = 0$ which contradicts the fact that $||\tilde{u}||_{H^1(D)} = 1$, proving the result. □

We can now state the convergence result for the interior transmission problem.

Theorem 5.2.6. *Assume that either $n_{min} > 1$ or $0 < n_{max} < 1$ and k is not a transmission eigenvalue for $\epsilon \geq 0$ small enough. Let $(w_\epsilon, v_\epsilon) \in L^2(D) \times L^2(D)$ be such that $U_\epsilon = w_\epsilon - v_\epsilon \in H^2(D)$ is a sequence of solutions to (5.47) with $(f_\epsilon, g_\epsilon) \in H^{3/2}(\partial D) \times H^{1/2}(\partial D)$ converging strongly to $(f,g) \in H^{3/2}(\partial D) \times H^{1/2}(\partial D)$ as $\epsilon \to 0$. Then $U_\epsilon \rightharpoonup U$ in $H^2(D)$ and $(w_\epsilon, v_\epsilon) \rightharpoonup (w,v)$ in $L^2(D) \times L^2(D)$ $\big($strongly in*

$L^2_{loc}(D) \times L^2_{loc}(D)\big)$, *where the limit* U *satisfies*

$$\left(\Delta + k^2 n_h\right) \frac{1}{n_h - 1} \left(\Delta + k^2\right) U = 0 \quad \text{in} \quad D \tag{5.61}$$

$$U = f \quad \text{and} \quad \frac{\partial U}{\partial \nu} = g \quad \text{on} \quad \partial D, \tag{5.62}$$

$U = w - v$, *and* (w, v) *satisfy*

$$\Delta v + k^2 v = 0 \qquad \text{and} \qquad \Delta w + k^2 n_h w = 0 \quad \text{in} \quad D \tag{5.63}$$

$$w - v = f \qquad \text{and} \qquad \frac{\partial w}{\partial \nu} - \frac{\partial v}{\partial \nu} = g \qquad \text{on} \quad \partial D \tag{5.64}$$

Proof. The result follows from combining Theorem 5.2.4 and Theorem 5.2.5 and the uniqueness of solution for (5.61)-(5.62). □

5.3 Convergence of the Transmission Eigenvalues

Using the convergence analysis for the solution of the interior transmission problem, we now prove the convergence of a sequence of real transmission eigenvalues of the periodic media, namely of those who are bounded with respect to the small parameter ϵ. The following lemmas provide conditions for the existence of real transmission eigenvalues that are bounded in ϵ.

Lemma 5.3.1. *The following holds:*

1. *Assume that* $A_\epsilon = I$ *for all* $\epsilon > 0$ *and either* $n_{min} > 1$ *or* $0 < n_{max} < 1$. *There exists an infinite sequence of real transmission eigenvalues* k^j_ϵ, $j \in \mathbb{N}$ *of (5.3)-(5.6) accumulating at* $+\infty$ *such that*

   $$k^j(n_{max}, D) \leq k^j_\epsilon < k^j(n_{min}, D) \qquad \text{if} \quad n_{min} > 1$$
   $$k^j(n_{min}, D) \leq k^j_\epsilon < k^j(n_{max}, D) \qquad \text{if} \quad 0 < n_{max} < 1$$

   where $k^j(n, D)$ *denotes an eigenvalue of (5.3)-(5.6) with* $A_\epsilon = I$ *and* $n_\epsilon = n$.

2. *Assume that* $n_\epsilon = 1$ *for all* $\epsilon > 0$ *and either* $A_{min} > 1$ *or* $0 < A_{max} < 1$. *There exists an infinite sequence of real transmission eigenvalues* k_ϵ^j, $j \in \mathbb{N}$ *of (5.3)-(5.6) accumulating at* $+\infty$ *such that*

$$\begin{array}{ll} k^j(a_{max}, D) \le k_\epsilon^j \le k^j(a_{min}, D) & \text{if} \quad a_{min} > 1 \\ k^j(a_{min}, D) \le k_\epsilon^j \le k^j(a_{max}, D) & \text{if} \quad 0 < a_{max} < 1 \end{array}$$

where $k^j(a, D)$ *denotes an eigenvalue of (5.3)-(5.6) with* $A_\epsilon = aI$ *and* $n_\epsilon = 1$.

Here j *counts the eigenvalue in the sequence under consideration which may not necessarily be the* j*-th transmission eigenvalue. In particular the first transmission eigenvalue satisfies the above estimates.*

Proof. The detailed proof of the above statements can be found in [22]. We remark that the statements are not proven for all real transmission eigenvalues. For example in the case of first statement, from the proofs in [22], real transmission eigenvalues are roots of $\lambda_j(\tau, n_\epsilon, D) - \tau = 0$, where λ_j, $j = 1 \ldots$, are eigenvalues of some auxiliary selfadjoint eigenvalue problem satisfying the Rayleigh quotient. The latter implies lower and upper bounds for λ_j in terms of n_{min} and n_{max}, and these bounds are also satisfied by the transmission eigenvalues that are the smallest root of each $\lambda_j(\tau, n_\epsilon, D) - \tau = 0$. Same argument applies to the second statement also. In particular the estimates hold for the first transmission eigenvalue. □

The existence results and estimates on real transmission eigenvalues are more restrictive for the case when both $A_\epsilon \ne I$ and $n_\epsilon \ne 1$. The following result is proven in [29] (see also [14]).

Lemma 5.3.2. *The following holds:*

1. *Assume that either* $a_{min} > 1$ *and* $0 < n_{max} < 1$ *or* $0 < a_{max} < 1$ *and* $n_{min} > 1$. *There exists a infinite sequence of real transmission eigenvalues* k_ϵ^j, $j \in \mathbb{N}$ *of* (5.3)-(5.6) *accumulating at* $+\infty$ *satisfying*

$$k^j(a_{max}, n_{min}, D) \leq k_\epsilon^j < k^j(a_{min}, n_{max}, D) \quad \text{if} \quad a_{min} > 1,\ 0 < n_{max} < 1$$
$$k^j(a_{min}, n_{max}, D) \leq k_\epsilon^j < k^j(a_{max}, n_{min}, D) \quad \text{if} \quad 0 < a_{max} < 1,\ n_{min} > 1$$

   where $k^j(a, n, D)$ *denotes an eigenvalue of* (5.3)-(5.6) *with* $A_\epsilon = aI$ *and* $n_\epsilon = n$.

2. *Assume that* $a_{min} > 1$ *and* $n_{min} > 1$ *or* $0 < a_{max} < 1$ *and* $0 < n_{max} < 1$. *There exists finitely many real transmission eigenvalues* k_ϵ^j, $j = 1 \cdots p$ *of* (5.3)-(5.6) *provided that* n_{max} *is small enough. In addition they satisfy*

$$0 < k_\epsilon^j < k^j(a_{min}/2, D) \quad \text{if} \quad a_{min} > 1,\ n_{min} > 1$$
$$0 < k_\epsilon^j < k^j(a_{max}/2, D) \quad \text{if} \quad 0 < a_{max} < 1,\ 0 < n_{max} < 1$$

   where $k^j(a, D)$ *denotes an eigenvalue of* (5.3)-(5.6) *with* $A_\epsilon = aI$ *and* $n_\epsilon = 1$.

Here j *counts the eigenvalue in the sequence under consideration which may not necessarily be the* j*-th transmission eigenvalue. In particular the first transmission eigenvalue satisfies the above estimates.*

Proof. The estimates follow by the same argument as in the proof of Lemma 5.3.1 combined with the existence proofs in [29]. In particular, the estimates can be obtained by modifying the proof of Theorem 2.6 and Theorem 2.10 in [29] in a similar way as in the proof of Corollary 2.6 in [22]. □

5.3.1 The case of $A(y) \neq I$

We assume that $A_{min} > 1$ or $A_{max} < 1$ in addition to (5.1) and (5.2) and let k_ϵ be one of the transmission eigenvalues described in Lemma 5.3.1 and Lemma 5.3.2. In particular $\{k_\epsilon\}$ is bounded and hence there is a positive number $k \in \mathbb{R}$ such that $k_\epsilon \to k$ as $\epsilon \to 0$. Let (w_ϵ, v_ϵ) be a corresponding pair of eigenfunctions normalized such that $||w_\epsilon||_{L^2(D)} + ||v_\epsilon||_{L^2(D)} = 1$. Notice from Section 5.2 that the transmission eigenfunctions satisfy

$$\mathcal{A}_\epsilon\left((w_\epsilon, v_\epsilon); (\varphi_1, \varphi_2)\right) = 0 \quad \text{for all} \;\; (\varphi_1, \varphi_2) \in X(D)$$

where the sesquilinear form $\mathcal{A}_\epsilon(\cdot;\cdot)$ is given by

$$\mathcal{A}_\epsilon\big((w_\epsilon, v_\epsilon); (\varphi_1, \varphi_2)\big) := \int_D A_\epsilon \nabla w_\epsilon \cdot \nabla \overline{\varphi}_1 - \nabla v_\epsilon \cdot \nabla \overline{\varphi}_2 - k_\epsilon^2 (n_\epsilon w_\epsilon \overline{\varphi}_1 - v_\epsilon \overline{\varphi}_2)\, dx.$$

Obviously if $\mathbb{T} : X(D) \mapsto X(D)$ is a continuous bijection then we have that the pair of the eigenfunction (w_ϵ, v_ϵ) satisfies

$$\mathcal{A}_\epsilon\big((w_\epsilon, v_\epsilon); \mathbb{T}(w_\epsilon, v_\epsilon)\big) = 0. \tag{5.65}$$

We will use (5.65) to prove that the sequence (w_ϵ, v_ϵ) is bounded in $X(D)$. To do so we must control the norm of the gradients of the functions in the sequence. Indeed, assuming that $A_{min} > 1$ and letting $\mathbb{T}(w, v) = (w, -v + 2w)$ gives that

$$\int_D A_\epsilon \nabla w_\epsilon \cdot \nabla \overline{w}_\epsilon + |\nabla v_\epsilon|^2 - 2\nabla v_\epsilon \cdot \nabla \overline{w}_\epsilon \, dx = k_\epsilon^2 \int_D n_\epsilon |w_\epsilon|^2 + |v_\epsilon|^2 - 2 v_\epsilon \overline{w}_\epsilon \, dx, \tag{5.66}$$

which by using Young's inequality gives that $||\nabla w_\epsilon||^2_{L^2(D)} + ||\nabla v_\epsilon||^2_{L^2(D)}$ is bounded independently of $\epsilon > 0$. Similarly in the case when $0 < A_{max} < 1$ we obtain the result using $\mathbb{T}(w, v) = (w - 2v, -v)$.

Therefore, in both cases we have that (w_ϵ, v_ϵ) is a bounded sequence in $X(D)$. This implies that there is a subsequence, still denoted by (w_ϵ, v_ϵ), that converges weakly (strongly in $L^2(D) \times L^2(D)$ to some (w, v) in $X(D)$). The L^2-strong limit implies that $||w||_{L^2(D)} + ||v||_{L^2(D)} = 1$ hence $(w, v) \neq (0, 0)$. Using similar argument as at the beginning of Section 5.2 we have that k is a transmission eigenvalue, with (w, v) in $X(D)$ the corresponding transmission eigenfunctions, for the homogenized transmission eigenvalue problem

$$\nabla \cdot A_h \nabla w + k^2 n_h w = 0 \quad \text{and} \quad \Delta v + k^2 v = 0 \ \text{ in } \ D, \tag{5.67}$$

$$w = v \quad \text{and} \quad \frac{\partial w}{\partial \nu_{A_h}} = \frac{\partial v}{\partial \nu} \ \text{ on } \ \partial D. \tag{5.68}$$

Hence we have proven the following result for the transmission eigenvalue problem.

Theorem 5.3.1. *Assume that either $A_{min} > 1$ or $0 < A_{max} < 1$ and let k_ϵ be a sequence of transmission eigenvalues for (5.3)-(5.6) with corresponding eigenfunctions (w_ϵ, v_ϵ). Then, if k_ϵ is bounded with respect to ϵ, then there is a subsequence of $\{(w_\epsilon, v_\epsilon), k_\epsilon\} \in X(D) \times \mathbb{R}$ such that $(w_\epsilon, v_\epsilon) \rightharpoonup (w, v)$ in $X(D)$ $\Big($strongly in $L^2(D) \times L^2(D)\Big)$ and $k_\epsilon \to k$ as $\epsilon \to 0$, where $\{(w, v), k\} \in X(D) \times \mathbb{R}$ is an eigenpair for (5.67)-(5.68).*

5.3.2 The case of $A(y) = I$

In this case we assume that either $n_{min} > 1$ or $0 < n_{max} < 1$. Let k_ϵ be an eigenvalue of (5.3)-(5.6) with corresponding eigenfunctions $(w_\epsilon, v_\epsilon) \in L^2(D) \times L^2(D)$ such that $u_\epsilon = w_\epsilon - v_\epsilon \in H_0^2(D)$. As discussed in Section 5.2.3, (w_ϵ, v_ϵ) are distributional solutions to:

$$\Delta v_\epsilon + k_\epsilon^2 v_\epsilon = 0 \ \text{ and } \ \Delta w_\epsilon + k_\epsilon^2 n_\epsilon w_\epsilon = 0 \ \text{ in } \ D, \tag{5.69}$$

whereas $u_\epsilon \in H_0^2(D)$ solves

$$0 = \left(\Delta + k^2 n_\epsilon\right) \frac{1}{n_\epsilon - 1} \left(\Delta + k^2\right) u_\epsilon \quad \text{in } D, \tag{5.70}$$

which in the variational form reads

$$\int_D \frac{1}{n_\epsilon - 1} \left(\Delta u_\epsilon + k_\epsilon^2 u_\epsilon\right) \left(\Delta \overline{\varphi} + k_\epsilon^2 n_\epsilon \overline{\varphi}\right) \, dx = 0 \text{ for all } \varphi \in H_0^2(D). \tag{5.71}$$

We recall that w_ϵ, v_ϵ and u_ϵ are related by

$$v_\epsilon = -\frac{1}{k^2(n_\epsilon - 1)} \left(\Delta u_\epsilon + k^2 n_\epsilon u_\epsilon\right) \quad \text{in } D \tag{5.72}$$

$$w_\epsilon = -\frac{1}{k^2(n_\epsilon - 1)} \left(\Delta u_\epsilon + k^2 u_\epsilon\right) \quad \text{in } D. \tag{5.73}$$

Without loss of generality we consider the first real transmission eigenvalue $k_\epsilon := k_\epsilon^1$ and set $\tau_\epsilon := (k_\epsilon)^2$. Since the corresponding eigenfunctions are nontrivial we can take $||u_\epsilon||_{H^1(D)} = 1$, and in addition we have the existence of a limit point τ for the set $\{\tau_\epsilon\}_{\epsilon > 0}$. Similarly to the previous case we wish to show that the normalized sequence u_ϵ is bounded in $H_0^2(D)$. We start with the case when $n_{min} > 1$ and let $\frac{1}{n_{max} - 1} = \alpha > 0$. Taking $\varphi = u_\epsilon$ in (5.71) implies

$$\int_D \frac{1}{n_\epsilon - 1} |\Delta u_\epsilon|^2 + \frac{2\tau_\epsilon}{n_\epsilon - 1} \Re(\Delta u_\epsilon \overline{u_\epsilon}) + \frac{\tau_\epsilon^2 n_\epsilon}{n_\epsilon - 1} |u_\epsilon|^2 \, dx = 0$$

Therefore, making use of Lemma 5.3.1 part 1, we have that:

$$\left| \int_D \frac{2\tau_\epsilon}{n_\epsilon - 1} (\Delta u_\epsilon) \overline{u_\epsilon} \, dx \right| \leq \frac{2\tau(n_{min}, D)}{n_{min} - 1} \left| \int_D (\Delta u_\epsilon) \overline{u_\epsilon} \, dx \right| \leq \frac{2\tau(n_{min}, D)}{n_{min} - 1} \int_D |\nabla u_\epsilon|^2 \, dx.$$

Which gives that:

$$\alpha||\Delta u_\epsilon||^2_{L^2(D)} \leq \frac{\tau(n_{min}, D)^2 n_{max}}{n_{min}-1}||u_\epsilon||^2_{L^2(D)} + \frac{2\tau(n_{min}, D)}{n_{min}-1}||\nabla u_\epsilon||^2_{L^2(D)}.$$

Now since $||u_\epsilon||_{H^1(D)} = 1$ and using that $||\Delta \cdot ||_{L^2(D)}$ is an equivalent norm on $H^2_0(D)$ we have that u_ϵ is a bounded sequence. By the construction of (w_ϵ, v_ϵ) we have that this is a bounded sequence in $L^2(D) \times L^2(D)$. Note that a similar argument holds if $0 < n_{max} < 1$, by multiplying the variational form by -1. Now by similar argument as in the proof of Theorem 5.2.4 we can now conclude the following result.

Theorem 5.3.2. *Assume that $A_\epsilon \equiv I$ for all $\epsilon > 0$ and either $n_{min} > 1$ or $n_{max} < 1$, and furthermore let k_ϵ be a transmission eigenvalue for (5.3)-(5.6) with corresponding eigenfunctions (w_ϵ, v_ϵ). Then, if k_ϵ is bounded with respect to ϵ, there is a subsequence of $\{(w_\epsilon, v_\epsilon), k_\epsilon\} \in (L^2(D)\times L^2(D))\times\mathbb{R}_+$ such that $(w_\epsilon, v_\epsilon) \rightharpoonup (w, v)$ in $L^2(D)\times L^2(D)$ and $k_\epsilon \to k$ as $\epsilon \to 0$, where $\{(w, v), k\} \in (L^2(D) \times L^2(D)) \times \mathbb{R}_+$ is an eigenpair corresponding to*

$$\Delta v + k^2 v = 0 \;\; \text{and} \;\; \Delta w + k^2 n_h w = 0 \;\; \text{in} \;\; D, \quad w - v \in H^2_0(D).$$

The proofs of both Theorem 5.3.1 and Theorem 5.3.2 simply depend on the boundedness of the sequence of any real transmission eigenvalue in terms of ϵ, therefore the proofs hold for all the eigenvalues that satisfy bounds stated in Lemma 5.3.1 and Lemma 5.3.2.

Remark 5.3.1. The transmission eigenvalues of the limiting problem (5.67)-(5.68) satisfy the same type of estimates as in Lemma 5.3.1 and Lemma 5.3.2. Furthermore, from the proof of Theorem 5.3.1 and Theorem 5.3.2 one can see that the limit k of the sequence $\{k_\epsilon\}$, where each k_ϵ is the first transmission eigenvalue of (5.3)-(5.6), is the first transmission eigenvalue of (5.67)-(5.68).

5.3.3 Notes on the Convergence Rate

In this section we provide a partial result on the convergence rate for the transmission eigenvalues by appealing to Osborn's theorem (see [69]) on the perturbation of the spectrum of a compact operator. Our study is by no means complete but we introduce the main ideas for the special case when the periodicity in only in $A(y) \neq I$ while $n(y)$ is a constant different from one. For sake of presentation we consider the case where $A_{min} > 1$ and $n > 1$, the same results can be proven for the case when $0 < A_{max} < 1$ and $0 < n < 1$. As we will see the convergence rate of the transmission eigenvalues is related to the convergence rate for the resolvent (i.e. the interior transmission problem) which was studied in Section 5.2.2. To obtain a corrector for the transmission eigenvalues one needs to analyze the limit of the boundary correctors, which is still an open problem in general. We start by stating the main result in [69] for the general analytic framework. To this end, suppose that H is a Hilbert space with $\mathbf{T}_\epsilon : H \to H$ is a sequence of compact linear operators such that $\mathbf{T}_\epsilon \to \mathbf{T}_0$ in norm (the adjoint also converge in norm). Let λ_0 be a non-zero eigenvalues corresponding to $\mathbf{T}_0$ with algebraic multiplicity m, now let $\mathbf{E}$ be the spectral projection onto the m dimensional generalized eigenspace of $\mathbf{T}_0$.

Theorem 5.3.3. *Let $\phi_1, \cdots, \phi_m$ be the normalized basis for $\mathcal{R}(\mathbf{E})$. Then there is a set of eigenvalues $\{\lambda_{\epsilon,j}\}_{j=1}^m$ and a constant C such that*

$$\left|\lambda_0 - \frac{1}{m}\sum_{j=1}^m \lambda_{\epsilon,j} - \frac{1}{m}\sum_{j=1}^m \big((\mathbf{T}_\epsilon - \mathbf{T}_0)\phi_j, \phi_j\big)\right| \leq C \left\|(\mathbf{T}_\epsilon - \mathbf{T}_0)\,\big|_{\mathcal{R}(\mathbf{E})}\right\| \left\|(\mathbf{T}^*_\epsilon - \mathbf{T}^*_0)\,\big|_{\mathcal{R}(\mathbf{E})}\right\|.$$

Consider the pair $(w_\epsilon, v_\epsilon) \in H^1(D) \times H^1(D)$ solving

$$\begin{aligned} \nabla \cdot A_\epsilon / n \nabla w_\epsilon - w_\epsilon = f/n \quad \text{and} \quad \Delta v_\epsilon - v_\epsilon = g \quad \text{in} \quad D \\ w_\epsilon = v_\epsilon \quad \text{and} \quad \frac{\partial w_\epsilon}{\partial \nu_{A_\epsilon}} = \frac{\partial v_\epsilon}{\partial \nu} \quad \text{on} \quad \partial D \end{aligned}$$

with $(f, g) \in L^2(D) \times L^2(D)$, which has been proven to exist uniquely in [17]. A priori estimate proved in [17] gives that there exists $C > 0$ independent of (f, g) and ϵ with

$$||w_\epsilon||_{H^1(D)} + ||v_\epsilon||_{H^1(D)} \leq C \left(||f||_{L^2(D)} + ||g||_{L^2(D)} \right).$$

Now we define the operator $\mathbf{T}_\epsilon : L^2(D) \times L^2(D) \to L^2(D) \times L^2(D)$ such that $\mathbf{T}_\epsilon(f, g) = (w_\epsilon, v_\epsilon)$. Notice that $k_\epsilon^2 = \tau_\epsilon - 1$ is a transmission eigenvalue corresponding to (5.3)-(5.6) for $n(y) = n$ (i.e. constant) provided that $\mathbf{T}_\epsilon(w_\epsilon, v_\epsilon) = -\frac{1}{\tau_\epsilon}(w_\epsilon, v_\epsilon)$. We have that the operator $\mathbf{T}_\epsilon$ is compact on $L^2(D) \times L^2(D)$ due to the compact embedding of $H^1(D) \times H^1(D)$ in $L^2(D) \times L^2(D)$. To use Theorem 5.3.3 to obtain a convergence rate we still need to prove that $\mathbf{T}_\epsilon \to \mathbf{T}_0$ in norm. We now assume that we have that following elliptic regularity estimate:

Assumption 5.3.1. *Let $(w, v) \in H^1(D) \times H^1(D)$ be the solutions of*

$$\begin{aligned} \nabla \cdot A_h \nabla w - nw = f \quad \textit{and} \quad \Delta v - v = g \quad \textit{in} \quad D \\ w = v \quad \textit{and} \quad \frac{\partial w}{\partial \nu_{A_h}} = \frac{\partial v}{\partial \nu} \quad \textit{on} \quad \partial D \end{aligned}$$

for $(f, g) \in L^2(D) \times L^2(D)$ and where A_h is the constant homogenized matrix. Then $(w, v) \in H^2(D) \times H^2(D)$ and satisfy the a priori estimate

$$||w||_{H^2(D)} + ||v||_{H^2(D)} \leq C \left(||f||_{L^2(D)} + ||g||_{L^2(D)} \right).$$

If ∂D is $C^{2,\beta}$ smooth, this regularity assumption is satisfied when $A_h = a_h I$. It can be proven by using standard regularity results for the difference $u = w - v$ which satisfies $\Delta u = \frac{n}{a_h} w - v + f - g$ in D. Indeed we have that $u \in H_0^1(D)$ and $\Delta u \in L^2(D)$ therefore we have that $u \in H^2(D) \cap H_0^1(D)$, moreover we can conclude that

$$||u||_{H^2(D)} \leq C\left(||f||_{L^2(D)} + ||g||_{L^2(D)}\right)$$

by standard elliptic regularity results. Now since $(1 - a_h)\frac{\partial w}{\partial \nu} = \frac{\partial u}{\partial \nu} \in H^{1/2}(\partial D)$ we conclude that $w \in H^2(D)$ and satisfies

$$||w||_{H^2(D)} \leq C\left(||f||_{L^2(D)} + ||g||_{L^2(D)}\right).$$

The function v satisfies the same estimates by the triangle inequality. For the case of a constant anisotropic matrix, one possible way to prove the regularity assumption is to use an integral equation approach exactly as it is done in [38]. After a change of variables in the equation for w and moving the source terms to boundary terms by means of a volume potential which leads to $H^2(D)$ regularity. Then the result follows from the mapping properties of the integral operators (see e.g. [63]).

Notice that by Remark 5.2.2 we have that

$$\begin{aligned} \left\|(\mathbf{T}_\epsilon - \mathbf{T}_0)(f,g)\right\|_{L^2(D)\times L^2(D)} &\leq C\epsilon ||w||_{H^2(D)} \\ &\leq C\epsilon\left(||f||_{L^2(D)} + ||g||_{L^2(D)}\right). \end{aligned}$$

This gives that $\mathbf{T}_\epsilon \to \mathbf{T}_0$ in norm and therefore we may apply Theorem 5.3.3 to the transmission eigenvalue problem. We also conclude that since $\mathcal{R}(\mathbf{E})$ is finite dimensional that

$$\frac{1}{m}\sum_{j=1}^{m}\big((\mathbf{T}_\epsilon - \mathbf{T}_0)\phi_j, \phi_j\big) \leq C\epsilon$$

where ϕ_j are the eigenfunctions for $\mathbf{T}_0$ with eigenvalue τ. Notice that we also have that $\|\mathbf{T}_\epsilon - \mathbf{T}_0\| = \|\mathbf{T}^*_\epsilon - \mathbf{T}^*_0\|$ and therefore $\|\mathbf{T}^*_\epsilon - \mathbf{T}^*_0\| \leq C\epsilon$. Therefore by appealing to Theorem 5.3.3 we have the following result.

Theorem 5.3.4. *Assume that either $A_{min} > 1$ and $n > 1$ or $0 < A_{max} < 1$ and $0 < n < 1$. Let $\tau_\epsilon = k_\epsilon^2 + 1$ where k_ϵ is a transmission eigenvalue corresponding to (5.3)-(5.6) and $\tau = k^2 + 1$ where k is a transmission eigenvalue corresponding to (5.67)-(5.68) with algebraic multiplicity m. Then there is a set of eigenvalues $\big\{k_{\epsilon,j}\big\}_{j=1}^{m}$ such that $\left|\frac{1}{\tau} - \frac{1}{m}\sum\limits_{j=1}^{m}\frac{1}{\tau_{\epsilon,j}}\right| = \mathcal{O}(\epsilon)$.*

5.4 Numerical Experiments

We start this section with a preliminary numerical investigate the convergence of the first transmission eigenvalue. To this end, we consider different $A(y)$ and $n(y)$ and investigate the behavior of the first transmission eigenvalue $k_1(\epsilon)$ on ϵ. More specifically, we investigate the convergence rate of $k_1(\epsilon)$ to the first eigenvalue for the homogenized problem $k_h^{(1)}$. The first transmission eigenvalue for the periodic media and homogenized problem is computed using a mixed finite element method with an eigenvalues searching technique described in [77] and [79]. In addition we show numerical example of determining first few real transmission eigenvalues from the far field scattering data. This section is concluded with some examples of using the first real transmission eigenvalue corresponding to the periodic media to obtain information about the effective material properties of the homogenized problem.

5.4.1 Numerical Tests for the Order of Convergence

We consider the case where the domain $D = B_R$ with $R = 2$ and for the first example assume that the periodic media is isotropic, i.e. $A_\epsilon = I$, with refractive index

$$n_\epsilon = \sin^2(2\pi x_1/\epsilon) + \cos^2(2\pi x_2/\epsilon) + 2.$$

Obviously $n_h = 3$. If the domain is a ball of radius two in $\mathbb{R}^2$ separation of variables gives that the roots of

$$d_0(k) = \mathrm{J}_0\left(2k\sqrt{n_h}\right)\mathrm{J}_1(2k) - \sqrt{n_h}\mathrm{J}_1\left(2k\sqrt{n_h}\right)\mathrm{J}_0(2k)$$

are the transmission eigenvalues. Using the secant method we see that $k_h^{(1)} \approx 2.0820$. The values of the first transmission eigenvalue for the periodic media for different values of ϵ are shown in Table 5.1.

ϵ	1/3	1/4	1/5	1/6	1/7
$k_1(\epsilon)$	2.0842	2.0834	2.0829	2.0828	2.0824

Table 5.1: First TEV for various ϵ with $A_\epsilon = I$ and $n_\epsilon \neq 1$

To find the convergence rate we assume that the error satisfies that

$$|k_1(\epsilon) - k_h^{(1)}| = C\epsilon^p \quad \text{which gives} \quad \log\left(|k_1(\epsilon) - k_h^{(1)}|\right) = \log(C) + p\log(\epsilon)$$

for some constant C independent of ϵ. Using the `polyfit` command in Matlab we can find a p that approximately satisfies the above equality. The calculations give that in this case $p = 2.1486$ (see Figure 5.2).

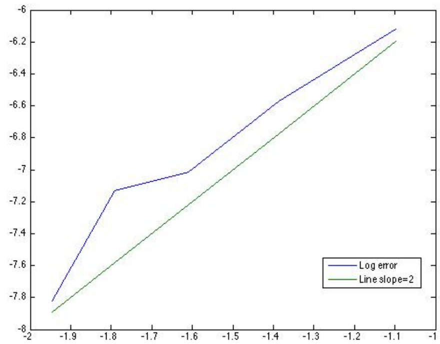

Figure 5.2: Here is a `Log-Log` plot that compares the log $\left|k_1(\epsilon) - k_h^{(1)}\right|$ to the line with slope 2.

In the next example we keep the same domain D and take the periodic constitutive parameters of the media

$$n_\epsilon = \sin^2(2\pi x_1/\epsilon) + \cos^2(2\pi x_2/\epsilon) + 2 \;\text{ and }\; A_\epsilon = \frac{1}{3}\begin{pmatrix} \sin^2(x_2/\epsilon)+1 & 0 \\ 0 & \cos^2(x_1/\epsilon)+1 \end{pmatrix}$$

Notice that $\nabla_y \cdot A_\epsilon e_i = 0$ which gives that $A_h = \frac{1}{2}I$ and $n_h = 3$. In this case the first zero of

$$d_0(k) = \mathrm{J}_0\left(2k\sqrt{\frac{n_h}{A_h}}\right)\mathrm{J}_1(2k) - \sqrt{n_h A_h}\,\mathrm{J}_1\left(2k\sqrt{\frac{n_h}{A_h}}\right)\mathrm{J}_0(2k)$$

is the first transmission eigenvalue $k_h^{(1)}$ for the homogenized problem which turns out to be $k_h^{(1)} = 1.0582$. Similarly we use `polyfit` in Matlab to find a p such that $\log\left(|k_1(\epsilon) - k_h^{(1)}|\right) = \log(C) + p\log(\epsilon)$. In this case we calculate that $p = 1.4421$. The results are shown in Table 5.2 and Figure 5.3.

ϵ	1	1/2	1/3	1/4	1/5	1/6
$k_1(\epsilon)$	1.0592	1.0591	1.0587	1.0586	1.0584	1.0583

Table 5.2: First TEV for various ϵ with $A_\epsilon \neq I$ and $n_\epsilon \neq 1$

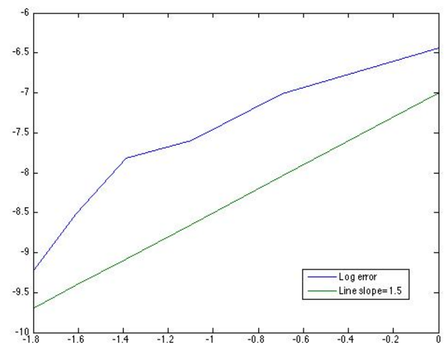

Figure 5.3: Here is a `Log-Log` plot that compares the log $\left|k_1(\epsilon) - k_h^{(1)}\right|$ to the line with slope

In these two examples the convergence rate seems to be better than of order ϵ. Notice that the boundary correction in these both cases does not appear since there is no boundary correction if $A = I$ and in the second example we have

$$\nabla_y \cdot A_\epsilon e_i = 0 \Longrightarrow \psi(y) = 0$$

which yield no boundary correction. This is the reason that the homogenized problem is a good approximation of the periodic media. We now wish to investigate the numerical convergence rate when $\psi(y) \neq 0$. Hence take $n_\epsilon = \sin^2(2\pi x_1/\epsilon) + 2$ and

$\tilde{A}_\epsilon = T A_\epsilon T^\top$ where

$$T = \begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix} \quad \text{and} \quad A_\epsilon = \frac{1}{3}\begin{pmatrix} \sin^2(2\pi x_2/\epsilon) + 1 & 0 \\ 0 & \cos^2(2\pi x_1/\epsilon) + 1 \end{pmatrix}$$

with $\phi = 1$ radian. We now compute the first transmission eigenvalue with coefficients n_ϵ and $\tilde{A}_\epsilon$. Since now $\psi(y) \neq 0$ we can not compute analytically A_h (one need to dove the cell problem numerically in order to compute A_h) and hence we do not have a value for the first transmission eigenvalue of the homogenized problem. In this case, in order to obtain an idea about the order of the convergence of the first transmission eigenvalue we define the relative error as:

$$\text{R.E.} = \frac{|k_1(\epsilon) - k_1(\epsilon/2)|}{k_1(\epsilon/2)}$$

and find the convergence rate for the relative error is a similar manner as discussed above. The Table 5.3 and Figure 5.4 show the computed first transmission eigenvalue for various epsilon in the square $D := [0, 2] \times [0, 2]$ and the circular domain $D := B_R$ of radius $R = 1$.

ϵ	1	1/2	1/4	1/8	Convergence Rate
Circle $k_1(\epsilon)$	2.460	2.453	2.472	2.518	1.32
Square $k_1(\epsilon)$	2.201	2.213	2.230	2.273	0.917

Table 5.3: First TEV for various ϵ with $\tilde{A}_\epsilon$ and n_ϵ with convergence rate

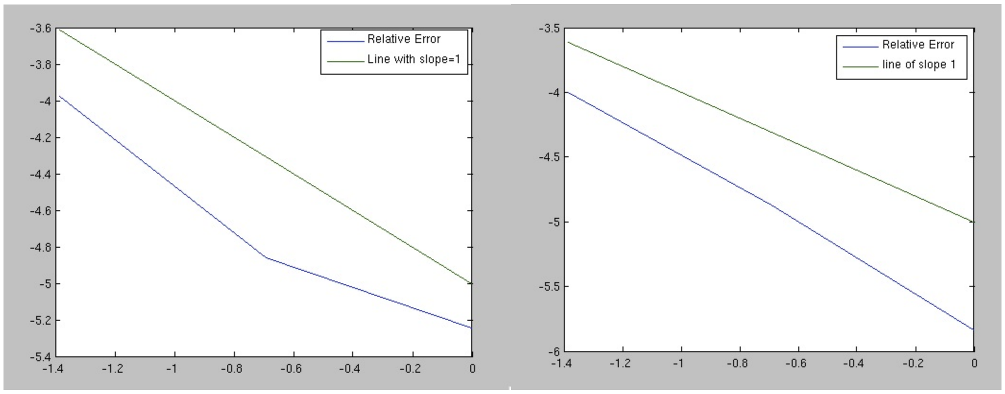

Figure 5.4: Convergence graph for relative error when $\psi(y) \neq 0$ compare to the line with slope one. On the left we have the `Log-Log` plot for the square and on the right for the disk.

The above results seem to suggest that the relative error is first order ϵ. In order to obtain the optimal convergence rate it is necessary to include the boundary correction.

5.4.2 Transmission Eigenvalues and the Determination of Effective Material Properties

For the given inhomogeneous media, the corresponding transmission eigenvalues are closely related to the so-called non-scattering frequencies, i.e. the values of k for which there exists an incident wave doesn't scatter (see the Introduction). The far field operator in $\mathbb{R}^2$ can be seen as $F : L^2(0, 2\pi) \mapsto L^2(0, 2\pi)$ is defined by

$$(Fg)(\theta) := \int_0^{2\pi} u_\epsilon^\infty(\theta, \phi) g(\phi)\, d\phi.$$

Now we would like to investigate how the first transmission eigenvalue determined from the far field operator depends on the parameter ϵ. Here to find the transmission eigenvalues from the far field data, we follow the approach in Chapter 2. To this end, let $\Phi_\infty(\cdot,\cdot)$ be the far field pattern for the fundamental solution to the Helmholtz equation. If $g_{z,\delta}$ is the Tikhonov regularized solution of the far field equation, i.e. the unique minimizer of the functional:

$$\|F_\epsilon g - \Phi_\infty(\cdot, z)\|^2_{L^2(0,\,2\pi)} + \alpha \|g\|^2_{L^2(0,\,2\pi)}$$

with the regularization parameter $\alpha := \alpha(\delta) \to 0$ as the noise level $\delta \to 0$, then at a transmission eigenvalue $\|v_{g_{z,\delta}}\|_{L^2(D)} \to \infty$ as $\delta \to 0$ for almost every $z \in D$, whereas otherwise bounded, where $v_g(x) := \displaystyle\int_0^{2\pi} g(\phi) e^{ik(x_1 \cos\phi + x_2 \sin\phi)}\, d\phi$. To compute the simulated data we use a FEM method to approximate the far field pattern corresponding to the scattering problem. Using the approximated u^∞_ϵ we then solve: $F_\epsilon g = \Phi_\infty(\cdot, z)$ for 25 random values of $z \in D$ where the regularization parameter is chosen based on Morozov's discrepancy principle. The transmission eigenvalues will appear as spikes in the plot of $||g_z||_{L^2[0,2\pi]}$ versus k. In our example we choose the domain $D := B_R$ to be the ball of radius $R = 1$ and the material properties given by

$$n_\epsilon = n(x/\epsilon) = \sin^2(2\pi x_1/\epsilon) + 2 \;\text{ and }\; A_\epsilon = \frac{1}{3}\begin{pmatrix} \sin^2(x_2/\epsilon) + 1 & 0 \\ 0 & \cos^2(x_1/\epsilon) + 1 \end{pmatrix}.$$

The effective material properties are $A_h = \frac{1}{2} I$ and $n_h = \frac{3}{2}$ and the corresponding first transmission eigenvalue is $k_h^{(1)} = 2.5340$. The computed transmission eigenvalue for this configuration for the choices of $\epsilon = 1$ and $\epsilon = 0.1$ are shown in Figure 5.5

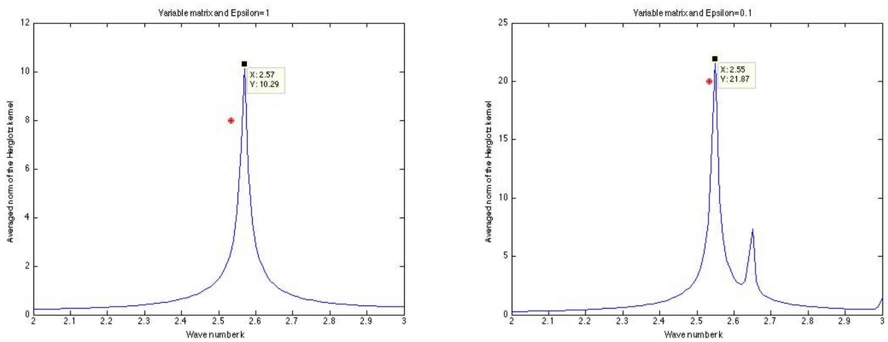

Figure 5.5: On the left $\epsilon = 1$ and on the right is $\epsilon = 0.1$. The red dot indicates $k_h^{(1)}$ whereas the peak indicates $k_1(\epsilon)$.

The measured first transmission can be used to obtain information about the effective material properties A_h and n_h. If $A_\epsilon = I$, it is known that $k_h^{(1)}$ uniquely determines n_h and also the transmission eigenvalue depend continuously on n_h [15], [41], [47]. From the scattering data we measure $k_1(\epsilon)$ which for epsilon small enough is close to $k_h^{(1)}$. Hence having available $k_1(\epsilon)$ we find a constant n such that the first transmission eigenvalue of the homogeneous media with refractive index n has $k_1(\epsilon)$ as the first transmission eigenvalue. Then by continuity this constant n is close to n_h. In Table 5.4 we show the calculations for the ball of radius D, $A_\epsilon = I$ and $n_\epsilon = n(x/\epsilon) = \sin^2(2\pi x_1/\epsilon) + 2$.

ϵ	$k_{\epsilon,1}$	n_h	reconstructed n_h
0.1	5.046	2.5	2.5188

Table 5.4: Reconstruction of n_h.

Similarly, we can obtain information about the contact matrix A_h [21], [25]. In particular, in the case when $n_\epsilon = 1$, from the first transmission eigenvalue $k_h^{(1)}$ we can determine a constant α which is in the middle of the smallest and the largest eigenvalues (in fact earlier numerical example suggest that this constant is roughly the arithmetic average of the eigenvalues of A_h). As an example we again consider the ball $D := B_R$ of radius $R = 1$, $n_\epsilon = 1$ and

$$A_\epsilon = \frac{1}{3}\begin{pmatrix} \sin^2(2\pi x_2/\epsilon) + 1 & 0 \\ 0 & \cos^2(2\pi x_1/\epsilon) + 1 \end{pmatrix}$$

Then having the measured $k_1(\epsilon)$, we find the constant a such that the first eigenvalue of the homogeneous media with $A = aI$ and $n = 1$ is equal to $k_1(\epsilon)$. The calculation are shown in Table 5.5.

ϵ	$k_1(\epsilon)$	A_h	reconstructed A_h
0.1	7.349	$0.5I$	$0.4851I$

Table 5.5: Reconstruction of affective material property from FFE in unit disk

In the above both examples we see that the measured first transmission eigenvalue corresponding to the periodic highly oscillating media can accurately determine the effective isotropic material properties $A_h = a_h I$ or n_h. Next we consider an example where A_h is constant matrix. We again take the ball $D := B_R$ of radius $R = 1$ and n_ϵ and

$$A_\epsilon = \frac{1}{3} T \begin{pmatrix} \sin^2(2\pi x_2/\epsilon) + 1 & 0 \\ 0 & \cos^2(2\pi x_1/\epsilon) + 1 \end{pmatrix} T^\top \tag{5.74}$$

where

$$T = \begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix} \qquad \text{with } \phi = 1 \text{ radian.}$$

In this case it becomes non-trivial to compute A_h (one needs to solve the cell PDE problem). However the constant a found as in the above example is in between (roughly the average) of the smallest and the largest eigenvalue of A_h. The results are shown in Table 5.6

ϵ	$k_1(\epsilon)$	reconstructed a
0.1	7.5499	$0.4921I$

Table 5.6: Reconstruction for the unit disk and A_ϵ given by (5.74).

Furthermore, if both $A_\epsilon \neq I$ and $n \neq 1$ we use a similar method as the above to obtain information about A_h/n_h [29]. Here we look for a constant α such that the first eigenvalue of

$$\begin{aligned} \Delta w + \alpha k^2 w = 0 \qquad &\text{and} \qquad \Delta v + k^2 v = 0 \quad \text{in} \quad D \\ w = v \qquad &\text{and} \qquad \frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} \quad \text{on} \quad \partial D \end{aligned}$$

coincide with $k_1(\epsilon)$ (note that here we incorrectly drop the jump in the normal derivative), where we take

$$n_\epsilon = \sin^2(2\pi x_1/\epsilon) + 2 \;\text{ and }\; A_\epsilon = \frac{1}{3}\begin{pmatrix} \sin^2(2\pi x_2/\epsilon) + 1 & 0 \\ 0 & \cos^2(2\pi x_1/\epsilon) + 1 \end{pmatrix}$$

giving that the ratio $\frac{n_h}{a_h} = 5$. The reconstruction is shown in Table 5.7.

ϵ	$k_1(\epsilon)$	reconstructed $\frac{n_h}{a_h}$
0.1	2.5415	4.788

Table 5.7: Reconstruction of the ratio $\frac{n_h}{a_h} = 5$ of effective material property for the unit disk D.

In all the examples so far we have considered smooth coefficients A_ϵ and n_ϵ. Hence, our next example concerns a checker board patterned media where the coefficients take different values in the white and black squares. Again here we let the period for the coefficients by $Y = [0,1]^2$. The white and black squares are assumed to cover that same area of a unit cell. See Figure 5.6 for the definition of the coefficients. In this case we have that $n_h = 7/2$ and A_h is shown in [80] to be a scalar matrix, i.e. $A_h = a_h I$ where a_h can be computed numerically.

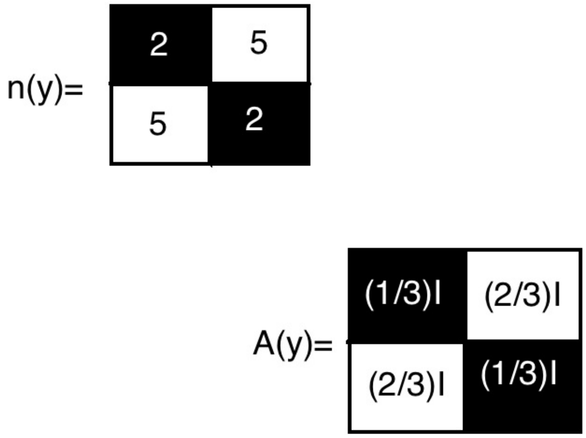

Figure 5.6: Definition of checker board coefficients.

See Table 5.8 for a comparison between the first transmission eigenvalue of the homogenized media and periodic media.

$k_1(n(y))$	$k_1(n_h)$	$k_1(A(y))$	$k_1(A_h)$	$k_1(n(y), A(y))$	$k_1(n_h, A_h)$
1.0930	1.0757	1.9027	1.896	0.7673	0.7139

Table 5.8: Media with checkerboard pattern in $[-3, 3]^2$

Next we use the first transmission eigenvalue for the actual media to determine the effective material properties. The result are shown in Table 5.9

$A(y) = I$, $n(y)$	reconstructed $n_h = 3.4123$ (exact $n_h = 3.5$)
$A(y)$, $n(y) = 1$	reconstructed $a_h = 0.4472$
$A(y)$, $n(y)$	reconstructed $n_h/a_h = 7.4704$ which gives $a_h = 0.4685$

Table 5.9: Reconstructed of effective material properties for the checkerboard.

Lastly consider the case of a media with periodically spaced voids (subregions with $n_\epsilon = 1$ and $A_\epsilon = I$). Our analysis does not cover these type of material property but we never the less consider an example of this type. In particular, we consider an example of isotropic media with refractive index $A(y) = I$ and

$$n(y) = \begin{cases} 1 & \text{if} \quad (y_1 - 0.5)^2 + (y_2 - 0.5)^2 < 0.25^2 \\ 5 & \text{if} \quad (y_1 - 0.5)^2 + (y_2 - 0.5)^2 \geq 0.25^2 \end{cases}$$

which gives that $n_h = 5 - \frac{\pi}{4}$, and an example of anisotropic case with the same $n(y)$ and

$$A(y) = \begin{cases} I & \text{if} \quad (y_1 - 0.5)^2 + (y_2 - 0.5)^2 < 0.25^2 \\ 0.5I & \text{if} \quad (y_1 - 0.5)^2 + (y_2 - 0.5)^2 \geq 0.25^2 \end{cases}$$

where the period is $Y = [0,1]^2$ and the is domain $D = [-3,3]^2$. See Table 5.10 for the comparison of the first transmission eigenvalue for the homogenized media and the actual periodic media.

$k_1(n(y))$	$k_1(n_h)$	$k_1(n(y), A(y))$	$k_1(n_h, A_h)$
0.8745	0.8781	0.7599	0.7231

Table 5.10: Media with periodic voids in $[-3, 3]^2$

In Table 5.11 we show reconstructed effective material properties based on the first transmission eigenvalue. Note that a_h is between the smallest and the largest eigenvalues of A_h.

$A(y) = I$, $n(y)$	reconstructed $n_h = 4.2678$ (exact $n_h = 4.2146$)
$A(y)$, $n(y)$	reconstructed $n_h/a_h = 5.0550$ which gives $a_h = 0.8337$

Table 5.11: Reconstructed of effective material properties for periodic voids

Chapter 6

OUTLOOK AND OPEN PROBLEMS

In this thesis we have investigated a variety of questions pertaining to non-destructive testing of anisotropic materials. We have developed reconstruction methods for an anisotropic background, studied the transmission eigenvalue problem for defective and periodic media as well as considered the inverse spectral problem of using the transmission eigenvalues for parameter identification. There are still many interesting open question and future research opportunities pertaining to non-destructive testing inspired by the work in this thesis. Here are some of the open problems

1. Construct a numerical algorithm using the asymptotic expansion given in Theorem 4.3.9 in conjunction with the MUSIC algorithm to reconstruct the material parameters for small volume defects.

2. In our study of the inverse spectral problem of using the transmission eigenvalues to determine material properties of the anisotropic media we only use the first eigenvalue to obtain partial information about the coefficients. It remains an open problem on how to use higher eigenvalues to obtain more information about A and n.

3. Further investigation of the transmission eigenvalue problem for highly oscillating materials. Our study initiated this area and there are still many open questions. One such question is on how can the results in Chapter 5 be extended for periodic materials with voids. In our approach the operator $\mathbb{A}_\epsilon$ given by the sesquilinear forms defined in Theorem 5.2.1 does not have an inverse that is bounded independently with respect to ϵ in the case when the periodic cell contains voids, which is essential to proving the convergence results. It is

also desirable to use the transmission eigenvalues to obtain information about the microstructure of the material where our results only provide information on the macrostructure of the material.

4. Studying the same problem's discussed in this thesis for Maxwell's Equations:

   (a) Proving existence and discreetness of transmission eigenvalues for voids in an anisotropic material.

   (b) Developing the factorization and generalized linear sampling methods for penetrable defects.

   (c) Study the transmission eigenvalue problem for wave propagation in a highly oscillating material by electromagnetic waves.

   (d) Derive a MUSIC algorithm for small volume defects for electromagnetic waves governed by Maxwell's equation for an anisotropic material.

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