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Graduate Teaching Assistants' Perception of Student Difficulties and Use in Teaching

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Abstract

Given the important role graduate teaching assistants (TAs) play in undergraduate students' learning, we investigated what TAs identified as students' difficulties from students' written work, their plans to address them, and implementation of their plans in class. Since the difficulties that TAs identified in general matched errors that students made, we analyzed what TAs identified in terms of literature on error handling. We examined levels of specific details of students' work involved in TAs' identifying, planning, and teaching. Our results show that (a) TAs often did not identify the most frequent errors students made, which reflected well-documented difficulties from the literature, (b) the errors TAs identified were mainly procedural in nature, (c) specific details of students' work were mainly included in procedural errors, and (d) the level of specificity of students' work was generally consistent but showed some drops when going from identifying to planning, then to teaching. Our results highlight interesting questions for future research and could be used as resources to design professional development that helps TAs use students' errors in teaching to promote students' learning.

Keywords Graduate teaching assistant · Calculus · Noticing · Teaching · Students' errors

Introduction

The quality of teaching in entry-level undergraduate mathematics courses has been emphasized over decades as necessary for retaining students in Science, Technology, Engineering, and Mathematics (STEM) (Laursen, 2019; Winsløw

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et al., 2021). Recent literature has started investigating how novice university mathematics teachers (NUMTs), especially graduate mathematics teaching assistants (TAs), teach these courses (Park & Rizzolo, 2022; Ellis, 2014; Kim, 2014; Kung & Speer, 2009) and what professional development (PD) opportunities are offered for NUMTs to improve their teaching. A recent review by Winsløw et al. (2021) has shown that many countries and universities in the world consider quality teaching as crucial in NUMTs' academic carrier, and offer various PD opportunities. This literature suggested course content-specific PD that helps teachers deal with specific challenges they face while teaching courses and incorporate research-based general pedagogical approaches to their own course (Brummer et al., 2023; Winsløw et al., 2021).

Investigating novice instructors' and TAs' teaching is a growing field (Speer et al., 2010). Studies have addressed characteristics of their teaching (Park, 2015; Park & Rizzolo, 2022; Güçler, 2013; Kim, 2014; Viirman, 2015) and factors that may impact their teaching such as beliefs (Ellis, 2014), knowledge (Musgrave & Carlson, 2017), research background (Mali et al., 2021), and recognition of their responsibilities to stakeholders (Shultz et al., 2019). Recently, there have been calls for more studies on this topic, especially those that examine NUMTs' teaching (Laursen, 2019; Musgrave & Carlson, 2017).

To address those calls, this study investigates an important aspect of NUMTs' teaching: what they perceive as students' difficulties while evaluating students' written work and how they use them in their teaching. Following suggestions from the literature (Brummer et al., 2023; Winsløw et al., 2021), we conducted our study in a specific context: the first course in a calculus sequence (Calculus I). A group of NUMTs – TAs – are heavily involved in this course especially in the U.S. where the study was conducted (Ellis, 2014). Students' difficulties in this course are well-documented (Bezuidenhout, 2001; Jones, 2017; Sealey, 2014), which allowed us to use related mathematics education literature in our study. Based on our review, this aspect of TAs' teaching has not been studied while extensive research has been conducted at the K-12 level which has emphasized the importance of using students' difficulties and misconceptions as resources for students' learning in teaching (e.g., review by Stahnke et al. (2016)). Given that TAs spend significant time grading students' written work (Kung & Speer, 2009), we are interested in what TAs learn from this experience and how that impacts their teaching. Specifically, our study aims to examine what TAs identified as students' difficulties from the students' work that they evaluated, whether and how the aspects of students' work were reflected in TAs' plans to address those difficulties and their implementation of plans in their class. The following research questions guided the study:

What aspects of students' written work were reflected in TAs' summaries of students' difficulties from their evaluation of students' work?
How do TAs engage with the aspects of students' work reflected in their summaries of students' difficulties in their plan to address those difficulties and implementation of those plans in their class?

Version of Record at: https://doi.org/10.1007/s40753-024-00239-1

To address these questions, we developed a set of tasks aiming to help students move past the difficulties documented in the existing literature, focusing on the main concept of Calculus I – the derivative. To investigate how TAs interact with their students' written work on these tasks and use what they learned in teaching, we adopted the existing literature on teachers' uses of students' errors (Brodie, 2014; Jacobs et al., 2010; Kung, 2010). Our results expand our understanding of TAs' teaching by revealing the process behind going from TAs' learning about students' difficulties by evaluating students' work to using those difficulties in teaching. This addresses recent calls for more studies on NUMTs' teaching, and we consider this study a first step to addressing needs for content-specific PD for TAs because the results provide an understanding of how TAs are currently using students' difficulties. Our goal is to use the results of this study as resources for PD to promote TAs' learning of how to use students' difficulties as resources for their learning of the content, as the existing literature advocates (Li & Schoenfeld, 2019; Smith et al., 1993).

Theoretical Background

This study deals with graduate teaching assistants (TAs) in the U.S. As we view our study as contributing to national and international efforts to understand the population of NUMTs and provide support and PD for them, we review the existing literature for NUMTs focusing on studies that provide insights for studying our population. Additionally, although we asked TAs about difficulties they observed in their students' work, we observed that what TAs identified as difficulties generally matched errors that their students made. To analyze them, we adapted and used teachers' use of students' errors at the elementary and secondary level that provided an analytical lens, which we review here.

Professional Development Opportunities for University Mathematics Teachers

Many professional development (PD) opportunities are provided for NUMTs (Winsløw et al., 2021). A recent review (Winsløw et al., 2021) found that many European universities offer and require general pedagogical PD for NUMTs that is designed based on government frameworks (e.g., Teaching Excellence Framework in England) or on established research (e.g., Scholarship of Teaching and Learning in Germany). Similar PD opportunities are offered nationally in the U.S. through Project NExT (Mathematical Association of America, n.d.) and Academy of Inquiry Based Learning (n.d.) (IBL).

Winsløw et al. (2021) also found that content-specific PD is rarely offered, optional, or up to the participants (e.g., finding their own subject-specific consultant). Thus, NUMTs still had to find their own way to adapt general pedagogical approaches to their subject. The authors argued that the existing mathematics education research specific to the content of mathematics courses should be used to address any deficiencies in NUMTs' knowledge and to provide resources that help NUMTs identify fundamental ideas, gain insights into students' difficulties, and

Version of Record at: https://doi.org/10.1007/s40753-024-00239-1

learn about potential remedies for content-specific areas. Examples of such PD often require participants to have teaching duties to address "specific challenges mathematics teachers face" (p. 65) and use tasks from the research on university mathematics education, which also provide insights for students' responses and potential instructional approaches (e.g., the University of Agder, Norway). Other content-specific PD (German universities) includes content-specific knowledge elements, support measures (e.g., electronic voting), and evaluation instruments for specific course goals (Winsløw et al., 2021).

The existing literature also argued that there is a need for "forming communities and shared resources…to develop teaching" (Winsløw et al., 2021, p. 71), where both mathematics education researchers and teachers are engaged. One such community was Development and Analysis of Teaching in Undergraduate Mathematics (Auckland, New Zealand), in which teachers met regularly to discuss video excerpts of their own teaching based on a mathematics education model (Schoenfeld, 2011). Their 2-year implementation revealed "evidence of improved teaching performance," "key aspects of practice and of undergraduate mathematics," and "theoretical insight into lecturer behaviour in mathematics" (Barton et al., 2015, p. 147). Similarly, in Brummer et al. (2023), PD promoting TAs' adoption of student-centered course materials was initially led by a faculty facilitator. With their iteration of PD, this format shifted towards a community of practice among TAs as the TAs took over the facilitator role in the later meetings, and reallocated time away from solitary lesson prep toward community-based discussions around materials for their students and local contexts.

Our paper focuses on two aspects that are emphasized in the literature review above – PD that is specific to the course that NUMTs teach and its relation to NUMTs' teaching. We followed their suggestion of involving mathematicians and mathematics educators in PD to develop and implement content-specific "material that identifies fundamental or central ideas, provides insight into learning difficulties or obstacles for the students and that shows possible remedies" (Winsløw et al., 2021, p. 74) based on existing mathematics education research. We also followed participating NUMTs' teaching to understand potential relations between what is offered in PD and their teaching. We now review those two aspects: our use of content-specific mathematics education research and what is known about NUMTs' teaching.

Content-Specific Mathematics Education Research Used in PD Materials

Literature regarding students' thinking about Calculus concepts, specifically the derivative, was used in task development for our PD. The purpose of these tasks is helping students to be ready for concepts before they learn them in lectures by solving tasks that are designed to address the difficulties that the existing literature identified. Our literature review focused on three main layers of the derivative: (a) the derivative at a point, (b) the derivative as a function, and (c) the derivative as an operator. In this study, we will collectively refer to these three uses of the derivative and the prerequisite concepts from which the concept of the derivative is built (functions, difference quotients (DQ), and limit) as *layers of the derivative*. We note that layers of the derivative have been previously

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discussed (See Zandieh, 2000) although the layers that we delineate here differ from those outlined in Zandieh (2000). There are extensive bodies of literature on each layer. Since our purpose was to investigate what TAs identify as students' difficulties from reading their students' work on these tasks and how TAs use them in their teaching, we aimed to develop tasks on which students' mistakes would reflect common difficulties. Our review and use of the existing literature serves this purpose and is not intended to provide a comprehensive review on students' thinking about the derivative.

Regarding the derivative at a point, we focused on students' difficulties about limit that is "inherent in the derivative" but not obvious to students (Hähkiöniemi, 2006, p. 173; Zandieh, 2000). In his case studies, Hähkiöniemi (2006) showed that two of the four participants did not use any limit ideas, or used both algebraic and graphical representations without showing that they understand why the two representations show the same thing. Understanding the limit included in the derivative in both representations and the connection between them has been emphasized in the literature (Asiala et al., 1997; Sánchez-Matamoros et al., 2015).

Regarding the derivative function, we focused on the students' difficulties with its construction from the derivative at several points and the relation between functions and their derivatives. Literature has shown the dominance of symbolic representations in students' thinking (Biza, 2021; Habre & Abboud, 2006; Ryberg, 2018) and suggested using graphs for effective teaching of the derivative (Berry & Nyman, 2003). Existing studies have shown that some students do not recognize the derivative at a point is a specific value (e.g., may identify it as the tangent line) (Park, 2013), which can lead to difficulties in constructing the derivative function based on it, and have trouble relating the idea of the tangent line and its slope with the derived function (Asiala et al., 1997; Biza et al., 2008; Sánchez-Matamoros et al., 2015). Given that characteristics of the original function impact the development of students' understanding of the derivative, studies have also shown students' difficulties identifying relationships among a function and the first and second derivative functions (Baker et al., 2000; Berry & Nyman, 2003).

For the derivative as an operator, we focused on the variables in the operations and their meaning in context. Most Calculus concepts including the derivative are expressed using variables, and connections between variables and Calculus have been emphasized in the literature (Bardini et al., 2014; Trigueros & Ursini, 2003; White & Mitchelmore, 1996). However, studies have shown that undergraduate students often have underdeveloped concepts of variables and lack the ability to use them in the ways needed for calculus (White & Mitchelmore, 1996).

Mathematics Teaching at the University Level

Studies on teaching university mathematics (Biza et al., 2018; Jaworski et al., 2017) cover various topics, including NUMTs' teaching practices and factors that impact teaching such as mathematical knowledge for teaching (Judson & Leingang, 2016;

Musgrave & Carlson, 2017), beliefs (Ellis, 2014), and research background (Mali et al., 2021). We will review studies that examined university mathematics teaching or provided insights for studying TAs in calculus.

First, studies have reported disconnections between mathematics discourse that teachers use and students' ability to understand teachers' discourse. In calculus, teachers often do not explicitly refer to concepts as processes and objects, or connect to point-specific and interval views (Park, 2015; Güçler, 2013, 2016). Those studies suggested being explicit about how those concepts could help students.

Second, studies have examined various facets of NUMTs' classroom instruction, offering potential explanations for their findings. For example, Viirman (2015) reported different routines that seven teachers adopted while teaching "function" and provided potential reasons for such differences, e.g. not being comfortable in the language of teaching potentially limits teachers' adoption of routines connecting mathematical statements with everyday language. Park and Rizzolo (2022) analyzed how various uses of variables were addressed in the classroom teaching of five TAs regarding the derivative. They reported on the limited aspects and applications of variables covered in TAs' instruction. Interviews with the TAs suggested that the limited coverage of variable uses could stem from a lack of knowledge or opportunities to conceptualize them within context. The authors recommended PD programs that offer opportunities for TAs to engage with these concepts before incorporating them into their discussions with students.

Third, studies examined the relation between NUMTs' beliefs and teaching. Some studies have found consistency between beliefs and teaching, while others have not. For example, Kim (2014) showed consistency by observing that international TAs believed in the importance of concept explanations and spent most of class on explaining concepts, whereas U.S. TAs emphasized pattern recognition from solving problems and in class mostly solved real-life problems and emphasized the solution patterns. DeFranco and McGivney-Burelle (2001) showed inconsistency, reporting that 22 GTAs from a mathematics pedagogy course adopted new sets of beliefs (e.g., using problems to uncover or challenge students' misconceptions), but their teaching practices were still teacher-directed with little interactions with students (pp. 586–568). Shultz (2022) observed similar inconsistency by showing that instructors who believe inquiry-oriented instruction is beneficial choose not to use it because of their recognition of professional obligations ("their responsibilities toward…the individual student, mathematics as a discipline, the institution, and society") could work in opposition to those beliefs (p. 227). These studies suggest that PD should provide instructional context for NUMTs to think about implementing strategies reflecting their beliefs.

Fourth, studies pointed out the differences in how TAs and faculty instructors recognize professional obligations. Interviews conducted by Shultz et al. (2019) revealed that both groups showed commitment to integrity of the mathematics, but "GTAs see students as more responsible for their own learning than do professors, and they struggle to recognize an obligation to fostering the social environment for the students" (p. 38). The authors suggested PD that increases GTAs' recognition of students' needs by exposing GTAs to students' struggles with mathematical content, providing chances to hypothesize reasons, then reading the transcript of students'

Version of Record at: https://doi.org/10.1007/s40753-024-00239-1

actual thoughts "to design lessons, grounded in specific mathematical content, which incorporate ways of attending to the environment [and]… adjust to different particular needs" (p. 41).

Our study addresses calls for more studies addressing NUMTs' teaching (Speer et al., 2010) focusing on TAs in particular (Ellis, 2014; Laursen, 2019). We look at a specific aspect of TAs' teaching in relation to students' difficulties with content by examining what TAs conceptualize as students' difficulties from interacting with students' work and their ways to address them in class. This study follows the suggestions from the literature about providing opportunities for TAs to engage with students' struggles before they discuss the ideas with students and plan lessons based on their identifying of students' difficulties (Shultz et al., 2019) with content-specific tasks designed to bring out students' difficulties about the core mathematical concepts in the course that TAs teach (Winsløw et al., 2021).

Use of Students' Errors in Mathematics Classrooms

Use of students' errors in teaching has been emphasized at all levels (e.g., Brodie, 2014; Jacobs et al., 2010; Prediger, 2010) including the university level (Kung, 2010; Nardi, 2008). We take a non-deficit view of errors, consistent with, for example, Smith et al. (1993) who conceptualizes errors that are systematic, common, and reoccurring, as "characteristic of initial phases of learning because students' existing knowledge is inadequate and supports only partial understandings" (p. 123). Systemic errors are produced by "misconceptions" that are a "plausible abstraction" of what students have learned, thus are their "attempts at sense-making" (Li & Schoenfeld, 2019, p. 2), which could be used as resource for their learning. In this study, we conceptualized errors as parts of students' work that deviate from what is mathematically accepted that reflect their difficulties with, in our case, the derivative (See Section "Content-Specific Mathematics Education Research Used in PD Materials"). We did not include what seemed to be nonsystematic errors (i.e., the ones that we could not connect to any layers of the derivative) such as simple arithmetic mistakes. This is consistent with the literature about misconceptions and errors (Barmby et al., 2009; Hansen et al., 2020). That literature classified errors to conceptual and procedural (Mutambara & Bansilal, 2022), which we used to understand the nature of errors that TAs identified and used (further details in Analysis).

Addressing errors in teaching allows teachers to reflect on their students' thinking, emphasize mathematical aspects, and prompt students' mathematical reasoning (Bray, 2011; Ingram et al., 2015; Santagata & Bray, 2016). Studies about teachers' error-handling have been mainly conducted at K-12 levels. They suggested the processes teachers go through such as noticing ("perceiving errors"), interpreting (e.g., "looking for reasons"), and dealing with errors (e.g., planning instructional approaches to help students "overcome the misconception" reflected in errors) (Heinrichs & Kaiser, 2018, p. 84; Jacobs et al., 2010). We found two main features of teachers' high-level engagement with student errors in the literature. The first feature was the level of specificity at which teachers engage with a specific (type of) error made in a problem context (Heinrichs & Kaiser, 2018). For example, Biza

Version of Record at: https://doi.org/10.1007/s40753-024-00239-1

et al. (2018) noted that engaging "with the underpinning mathematical content in the task is essential and strengthens intended pedagogical practice" and differentiated responses that did so from those that did not (p. 74). The second feature was the level at which teachers provide explicit instructional plans to address student errors. For example, Jacobs et al. (2010) explicitly prompted teachers to provide specific instructional approaches (e.g., problems to pose next to the student whose work teachers just engaged with). They used two levels to analyze teachers' approaches; *robust* evidence was when teachers considered the student's strategy and planned next steps that existing literature "has shown likely to further" the students' understandings (p. 189) and *limited* evidence was when teachers assumed students, who adopted different strategies, had similar understandings and proposed next steps not reflecting each student's strategy.

Studies have shown that how teachers engage with students' errors is related to their beliefs about errors and their content knowledge. For example, teachers' beliefs that flawed solutions can confuse students limit open discussion about errors in class, and those with weak mathematical knowledge were less likely to identify and understand students' errors and had difficulties "formulating conceptually based questions or explanations in response to errors" (Bray, 2011, p. 31). Moreover, teachers' limited knowledge about key ideas can "undermine the power of focusing on students' errors" which often results in emphasizing correct steps (Santagata & Bray, 2016, p. 563). These studies suggest that handling students' errors requires teachers to do "complex analytic and interpretive work…that draws upon a wealth of knowledge and expertise about students' learning, the mathematics itself, the difficulties students encounter with the particular mathematics, the consequences of different interactional moves and the wider pedagogical goals" (Bray, 2011; Ingram et al., 2015, p. 194), which many NUMTs, including TAs, are not equipped to do (Bray, 2011). This suggests a need for PD that helps teachers "develop their conceptual understanding of key mathematical ideas," which can be aided by literature on students' difficulties (Winsløw et al., 2021) and error handling (Bray, 2011; Brodie, 2014; Ingram et al., 2015; Santagata & Bray, 2016, p. 563).

As mentioned earlier, since what TAs considered as students' difficulties matched types of errors that students made in their work, we built on these studies to examine TAs' engagement with students' errors. We did not explicitly ask them to hypothesize about causes of errors because our intention was not to be invasive of the first-year TAs' teaching. Grading and planning lessons are already included in TAs' general practice, and the literature has shown that years of teaching impact teachers' ability to interpret student misconceptions and common errors (Bray, 2011). Thus, we did not focus on how the first-year TAs interpret students' errors in this paper.

Our study investigating how TAs implement their plan to address students' errors they identified is noteworthy because we have not found studies examining how teachers address what they noticed from their students' written work in their follow-up class or studies about error-handling in class that include teachers' engagement with errors prior to class. Also, we could not find any work that addresses TAs' use of students' errors that they noticed from students' work in their teaching. Given the importance of dealing with students' errors in teaching and that TAs often teach without much background in teaching, this study makes an important initial contribution towards helping TAs learn how to incorporate errors in teaching.

Methods

Research Site and Participants

Our study was conducted in a single-variable calculus course, Calculus I for STEM majors, at a large public U.S. university (the institution). The institution uses a traditional approach to Calculus I and a syllabus created in conjunction with the client STEM departments that reflects what they want their students to learn from the course. At the institution, Calculus I is offered as large-coordinated lectures taught by faculty instructors for 80–100 students with discussion sections taught by TAs for 25–30 students. In the discussion sections, TAs typically solve textbook problems and sometimes review concepts covered in lectures based on questions students have. When we conducted the study, three lecture sections paired with nine discussion sections were offered. Three faculty members, including the authors, taught the lecture sections, and the five TAs (Mathematics PhD students – Amy, Lia, Kay, Dan, and Eddy, pseudonyms) taught the discussion sections. We invited all TAs to participate in our study. We explained the purpose of this study was to gain an understanding of the role of the TAs in undergraduate mathematics education, and that participating in our study would involve having the research team observe and video-record their classroom teaching and having them attend four PD sessions. They all volunteered to participate. We decided not to share specific goals for class observation explicitly because we wanted to observe whether and how they implement their plans to address students' difficulties on their own after their identifying and planning. We note that Lia and Eddy taught the students in the PD facilitator's sections. We acknowledge the potential for bias given that PD was conducted by their instructor rather than a neutral party. Any bias introduced might have resulted in TAs being more inclined to excel in the PD and related assignments. To mitigate this potential bias, we opted not to disclose the specific goals of our study to prevent behavior being influenced by those goals.

All five TAs were first years, and only two of them Lia and Eddy had previous teaching experience before joining the mathematics PhD program, as instructors of 20–30 student classes during their Master's degree studies and as temporary faculty at another institution. The TAs all taught discussion sections for Calculus I in the previous semester, which we observed on the topics for which we later developed the tasks. During this time, TAs' discussion sections were in a lecture format, meaning that they solved problems from the textbook while asking students some closed-ended questions with no explicit mention or use of students' work. TAs followed a coordinated schedule and instructors' directions on what to cover. Prior to our study and before starting teaching at the institution, the TAs had participated in a one-day training session, in which they received general instruction about their responsibilities (e.g., grading, being prepared for class, attire). Additionally, they each did a 5-min teaching demonstration on a calculus topic and received feedback from a faculty member who led the training. They received no specific training on knowledge for teaching mathematics, such as how to understand students' difficulties. This study represents initial efforts to improve TAs' knowledge for teaching mathematics by exposing TAs to students' difficulties about the content *before* they teach students.

Version of Record at: https://doi.org/10.1007/s40753-024-00239-1

Data Collection

Problem Sets and Guidelines

This study was conducted by a research team consisting of two faculty instructors and two doctoral students (not participating TAs). First, we developed a set of eight calculus tasks about the derivative, the main topic for Calculus I, addressing common students' difficulties identified in existing studies (Section "Use of Students' Errors in Mathematics Classrooms", Table 1) (Appendix).

We used online sources such as The Better File Cabinet Home Page (n.d.), which is a "searchable data base of calculus problems," CLEAR Calculus (n.d.), which is "a research-based effort to make calculus conceptually accessible to more students" and provides instructor resources, and the most widely-used calculus textbooks in the U.S. (Bressoud et al., 2013). The purpose of these tasks was preparing students to learn new concepts by making potentially difficult mathematical aspects explicit. This was communicated with the TAs in PD.

Additionally, for each task, we developed a written sample solution and guidelines for TAs (4–6 specific guidelines given as bullet points), for a total of 41 guidelines in all (Appendix). We had four main principles when developing the guidelines:

- Clarifying the purpose of tasks (e.g., "Clarify that part 4 is related to the slope at a point $x = c$ and part 5 is related to how the function changes its behavior around those points" for Pre-lab 4.3),
- Emphasizing crucial or easily-missed points of solutions (e.g., "Be sure students are graphing the sine function with x in radians" for Pre-lab 3.3),
- Providing potential students' difficulties with problems (e.g., "Students might be confused about finding the ordered pairs corresponding to values c and d from parts 3 and 7 above" for Pre-lab 4.3), and
- Providing extra explanations to solutions (e.g., "Emphasize the difference between a relation and a function: a function is a type of relation, but not all relations are functions for Pre-lab 3.5)

Students' Written Work

To collect students' written work on the tasks that we developed, we invited all the students who attended the TAs' discussion sessions to participate in our study. We explained that the purpose of this study was to improve calculus education and that participating in our study would involve having the research team collect copies of their written work that they to complete as part of the course, and for us to video-record their classes focusing on their TAs' teaching. They all volunteered to participate.

For each task, we asked the students to complete it as homework, and returned collected students' work to their TAs. On average, 91 students submitted their responses per task, totaling 727 submissions. We, the research team, analyzed students' work focusing on the errors that they made and categorized them into the main layers of the derivative after observing that what the TAs submitted as difficulties mostly reflected errors their students made (Table 1, pp. 13–14).

Version of Record at: https://doi.org/10.1007/s40753-024-00239-1

Table 1 Purpose for tasks and anticipated difficulties or common difficulties

Layer	Purpose for Tasks (Task Number)	Anticipated Difficulties or Common Difficulties
Derivative at a point	• Making the limit evident by asking to compute numerical values for the DQ for different values of the variable included in the limit (e.g., h in $\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$), and to interpret them as the slopes of the secant lines whose limit is the slope of the tangent line (2.7, 2.8)	• Overlooking the connection between numerical computation and the symbolic representation of the limit of the DQ (e.g., treating 'h' solely as a numerical value rather than as a variable • Challenges with identifying the object to which the limit is applied (e.g., taking the limit of the original function instead of the DQ)
Derivative function	• Addressing the construction of the derivative function, which involves exchanging the slope of the tangent line with the derivative at a point (3.1, 3.3, 3.5–2) • Examining the meaning of the derivative functions in relation to the original function (4.3)	• Drawing tangent lines as the y values for the derivative graph instead of determining their slopes • Using differentiation rules to find the equation for the derivative function instead of utilizing the derivative graph they had constructed • Mirroring textbook explanations regarding the connection between a function and its first derivative (e.g., if $f(x)$ increases, then $f'(x) \geq 0$) but not being able to extend similar statements to describe the relationship between $f'(x)$ and $f''(x)$
Derivative as Operator	• Eliciting students' use of variables in decomposing a function in preparation for the chain rule (3.4) • Eliciting any differences in students' use of variables regarding different types of differentiation (3.5, 3.9)	• Difficulties in selecting independent variables (e.g., consistently assuming 'x' was the independent variable, or omitting variables from function or derivative notation without explicit acknowledgment)

Version of Record at: https://doi.org/10.1007/s40753-024-00239-1

	A	B	C	D	E	F	G	H
1	Pre-labs	Student Difficulty 1	Possible Plan for 1	Student Difficulty 2	Possible Plan for 2	Student Difficulty 3	Possible Plan for 3	
2	Prelab for 2.7							
3	Prelab for 2.8							

Fig. 1 Spreadsheet Shared with TAs

TAs' Interaction with Students' Work

We asked each TA to evaluate the students' work on each task from his/her own class and, in a Google Spreadsheet shared with the researchers only (Fig. 1), submit what they noticed as students' difficulties, and explain how they planned to address those difficulties before they solve it in class. They had 1–2 days for submissions. The PD about the task happened before the TAs evaluated students' written work. During the meeting the verbal prompts "what are the students' difficulties that you identified while evaluating?" and "what's your plan to address those difficulties?" were given to the TAs to complete after they had evaluated students' work.

The order was set this way because we wanted the tasks and PD to be blended with their course. Specifically, (a) each of our tasks were designed to prepare for a certain section in the textbook and we wanted to collect students' work after they completed the previous section and before they started the section for the task, (b) we wanted the TAs to build a common understanding of the task and how to approach it during PD, and then (c) read and evaluate their own students work before they solve the task with students. The classes where the TAs solved those tasks were video-recorded and transcribed, and analyzed to examine how and to what extent the TAs used difficulties that they noticed from students' work.

Professional Development

We conducted four 70-min PD sessions. Each covered two of the 8 tasks we developed beforehand (since the lectures were scheduled three times a week generally covering one textbook section per class, but the discussion sections were scheduled only twice a week, TAs often had to cover two sections in one class). For each task, two TAs prepared their solutions to the task but only one presented. Two TAs did a demo lesson once and the others did two over the PD. During each 70-min meeting, the facilitator led a short discussion about layers of the derivative (Section "Content-Specific Mathematics Education Research Used in PD Materials") and the purpose of the task regarding layers and their relations. Then one TA did a 15–20 min demo lesson solving one task. The presenting TA explained what their solution addressed regarding the layers of the derivative and the TAs discussed which part of the demo lesson might be hard for students, how to improve it, and different ways to solve the task. Then, they repeated the process for the other task. For example, about Pre-lab 2.7, the presenting TA (Amy) explicitly mentioned that her focus on addressing “the process of finding the derivative at a point” as “the limit on the difference quotient,” and “secant lines … getting closer and closer to the tangent line” can be more explicitly discussed with how h changes. Lia added that the solution should indicate that the slopes of secant lines (represented discretely) converge to the slope of the tangent line and how the behavior of secant lines represents that. There was limited mention of “students” in their discussion when discussing which part might be hard for students. For example, in her solution of Pre-lab 2.8, the presenting TA (Kay) used the definition of the derivative involving h because she thought factoring would be harder for students in the other form of definition.

The purpose of PD was to prepare a mathematically strong lesson together by (a) discussing ways to solve the tasks that they would cover regarding layers of the derivative and students' difficulties about them (Section "Content-Specific Mathematics Education Research Used in PD Materials") and (b) having TAs to realize their own misconceptions or confusions about the tasks and discuss those with their peers and the PD facilitator. The purpose (a) was also addressed by the written guidelines for the TAs to use in support of the PD, which we distributed at the end of PD.

Since PD happened before the TAs had chances to look at their students' work on the task, it did not specifically address errors that their students made. Asking the TAs to identify students' difficulties from their work and plan to address them was our effort to understand their current use of what they learned from students' work in class, which would inform our future PD where we also incorporate TAs' engagement with students' work in their preparation of lessons in addition to preparing mathematically strong lessons.

Analysis

We analyzed three sets of data: (a) TAs' submission of students' difficulties and plans to address them, (b) TAs' classes solving those tasks, and (c) their students' work on tasks. We decided to add part (c) after observing that the TAs' submissions generally matched errors in students' work. Our analysis involves three terms:

- “Error”: Individual student error that the research team identified (excluding arithmetic mistakes)
- “Error type”: Types of errors that the research team identified by grouping similar individual errors (called “codes” in the paper, examples in Table 2)
- “Error types the TAs identified”: the error types that the researchers identified from TAs' submissions. They were a subset of all error types the researchers identified from their students' work.

Table 2 Examples of codes and corresponding difficulties TAs submitted

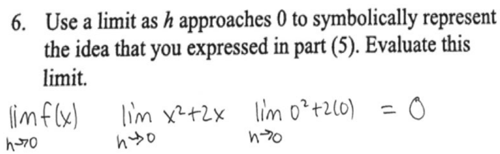
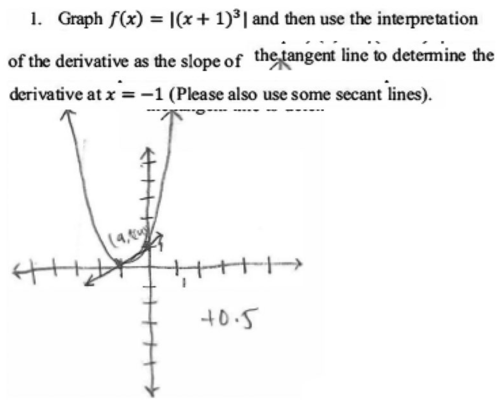
Layer	Error Type Description	Nature	TA's identifying	Student Examples
Limit-of-DQ	Students take the limit of the original function [No difference quotient seen]	How to	"The students do not under-stand that they are taking the limit of the slope of the secant line. They were taking the limit as h goes to 0 of $f(x)$ in many cases" (Amy)	 (2.7)
	Students incorrectly draw secant lines to show the limit process when asked	How to	"Either drawing one tangent line incorrectly or one or two secant lines and not seeing the trend of the slopes of the secant lines going to zero" (Kay)	 (2.8)

Table 2 (continued)

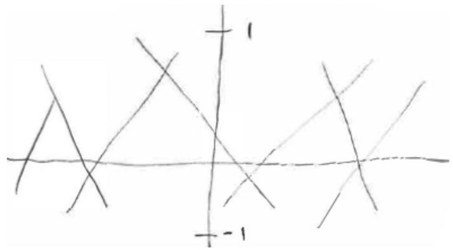
Layer	Error Type Description	Nature	TA's identifying	Student Examples
	When asked to graph the derivative students re-graph tangent lines (with or without the original function)	How to	"Students had a lot of problems plotting the slopes of the tangent lines. Many plotted the tangent lines." (Amy)	**4. Plot these estimated slopes as y-values at the same x-values from part (2) on a new (x, y)-plane.**  (3.3)

Table 2 (continued)

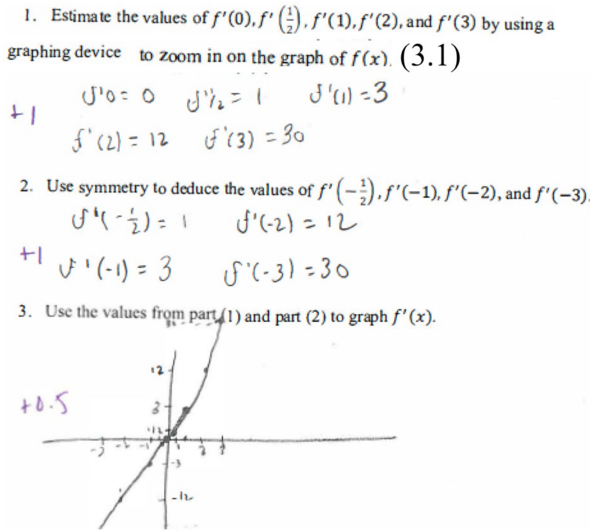
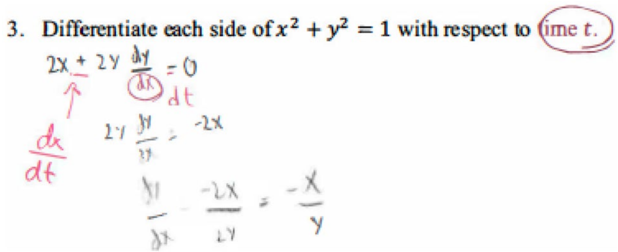
Layer	Error Type Description	Nature	TA's identifying	Student Examples
Derivative as-function	Students have difficulty with translating the slope of tangent lines to the coordinates (x, y) for the derivative function graph or vice versa	Meaning or Relation	"Not making the connection between Qs 1 & 2 and plotting x^3 instead of its derivative" (Kay)	

Table 2 (continued)

Layer	Error Type Description	Nature	TA's identifying	Student Examples
Derivative-as-operation	Students take the derivative with respect to a variable which is not the independent variable	How to	"Some students' answer for number 1 and 2 do not make sense. Some use implicit differentiations. They had issues in distinguishing taking derivatives in terms of t and x" (Dan)	 (3.9)

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For example, consider an error type "when computing $f(x+h)$ or $f(a+h)$, students use a numerical example for 'h' rather than the letter h".[1] Different numerical values for h would produce different errors, but regardless of the values they used, these errors would still be categorized under this error type. A priori, there could be differences between what TAs considered as students' difficulties and what we considered as error types. However, during our analysis, the difficulties that the TAs identified easily matched with the codes, or small collections of codes, we developed, which is further explained below.

Analysis of Students' Written Work

For analysis, we expanded our research team to include three advanced undergraduate students as research assistants. For each task, a faculty member and two research assistants analyzed 50–60 students' work individually first and then together regarding the types of errors that students made about the main layers of the derivative (Section "Content-Specific Mathematics Education Research Used in PD Materials") through open coding of our data (Strauss & Corbin, 1990). Since students made errors regarding other prerequisite layers such as functions and DQs, we also added those error types as our codes. Then, those three members came up with common codes (error types) and descriptions to apply to the rest of the data. It should be noted that our error types were detailed to catch all TAs' submissions (Table 1). Then, a research assistant finished the rest of the coding based on the developed codes and met with a faculty member to discuss unclear descriptions or new cases to be added. Once coding ended, we randomly picked 20% of students' work on each task to code again, which all matched the original coding. Thus, our codes are the types of errors we found in students' written work.

Once we finished coding, we recorded how frequent error types were observed by recording (a) the number of codes we identified for each task and each TA's class, and (b) the number of errors corresponding to such codes.

To understand the general distribution of the error types in students' work, we labelled natures of codes. Consistent with recent literature on students' errors, we classified them as conceptual and procedural (Mutambara & Bansilal, 2022), and further classified conceptual errors to meaning-or-relationships and missing-components:

- How to compute or graph (How-to),
- Meaning of mathematical objects or their relationships (Meaning-or-Relationships), and
- Missing mathematical components in their solutions (Missing-Component)

through open coding (Strauss & Corbin, 1990) of all codes we developed (Table 2). With this classification, we were interested in identifying what aspect of students' work was incorrect. Specifically, we aimed to identify whether their work

[1] We note that this is a type of error in evaluating a function, not in taking the limit as h approaches 0.

was incorrect regarding carrying out a procedure, meaning of mathematical objects and relating them, or missing a component, not to further interpret students' thought processes that may have led to the particular part of their solution being incorrect. Based on this classification, we wanted to look for patterns in where TAs identified difficulties from students' work. This is in contrast to other studies that aim to understand the underlying causes of errors. Our goal was to identify what aspects of students' work TAs considered when they identify student difficulties.

Analysis of What TAs Identified as Difficulties in Terms of Error Types

As mentioned above, once we observed that generally students' difficulties in TAs' submission matched types of errors their students made, we identified error types that their submissions addressed (Table 1). In most cases, there was a one-to-one correspondence, but when what TAs submitted as one difficulty addressed multiple error types, we recorded all such error types as what their submission addressed.

Comparison between Error Types that TAs and Research Team Identified

Once we identified the error types that TAs identified, in each task for each TA, we compared them with the error types that we identified. To show the proportion of (a) the error types each TA identified for a task among (b) the error types the research team identified for the task in that TA's class, we computed the percentage by dividing the number of (a) by the number of (b). Then, to show the percentage of (c) the errors that correspond to the error types each TA identified for a task among (d) the errors that the research team identified for the task in that TA's class, we computed the percentage by dividing the number (c) by the number of (d). We then computed average percentages from the eight tables of those percentages (one for each task). To understand the distribution of the natures of the error types that TAs identified, we also computed the average percentage for each of How-to, Meaning-or-Relationships, and Missing-component error types that they identified in the same way.

Analysis of TAs' Submissions – Levels

For each error type that TAs identified, the two faculty researchers developed a framework (descriptions) for their identifying, planning and teaching (Tables 3, 4 and 5) using different levels (See 2.4), and applied it to data individually, and our

Table 3 Levels of TAs' identifying

Level	Description
Robust	Identifying error types with specifics of students' work
Limited	Stating a general category of error types regarding layers of the derivative and/or their nature without specifics of student work (Category), or Mentioning they saw students' work with examples (Saw)
No-Data	Not filling out the Google Spreadsheet or mentioning students' error type in class

Table 4 Levels of TAs' planning

Levels	General Description	Specific Categories	Description
Robust	Incorporating specifics of students' work with extra/modified problems, or alternative approaches to address the error types	Extra Details/Example/ Representation	Providing extra details/representations that are not considered as typical solutions and/or suggesting extra/modified problems
		Direct Aspect	Comparing students' answers to correct one and explaining why they were wrong
		Counter Example	Asking for counter examples to students' approaches
Limited	Considering students' work, but not explicitly addressing the specifics of their work or only addressing partial aspects	Limited Details/ Example/ Representation	Providing extra details/examples/representations that addressed limited mathematical aspects of students' errors
		Correct Answer	Providing a correct answer to the part of the task where students made errors
		Correct Statement	Providing the correct statement that students' solution violated
		Revisit Task	Reading the task to help students understand what is asked
No-Data	Not filling out Google Spreadsheet	N/A	N/A

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Table 5 Levels of TAs' use of students' difficulties in teaching

	General Description	Specific Category	Description
Robust	See above	Direct Aspect	See above
		Extra Details/Example/ Representation	
Limited	Mentioning error types or the mathematical aspects that they are regarding without explicitly addressing the specifics of students' work or only addressing partial aspects	Limited Details/ Example/ Representation	See above
		Correct Answer	
		Correct Statement	
		Wrong vs. Right	Pointing out what was wrong vs. right in part of students' solution and/or providing what could have been right without further explanation
		Should/Should Not	Explaining what students should or should not do
No-Use	TAs identified error types and planned to address them, but there was no evidence of using them in class	N/A	N/A

coding matched. We developed specific categories within each level through open coding (Strauss & Corbin, 1990) of the TAs' identifying, planning, and teaching. Thus, they are the results of our analysis, but we present them here to make the differences in levels explicit. In the results, we will focus on the levels.

TAs' attending to the specifics of students' work (Robust) is differentiated from when TAs gave a general category without the specifics (Limited) (Table 3). When we refer to the "specifics" of students' work, we are not necessarily indicating TAs' inclusion of particular students' examples. Instead, we are focusing on specific mathematical aspects within students' responses that are included. For example, Amy's submission of "students do not understand that they are taking the limit of the slope of the secant line. They were taking the limit as *h* goes to 0 of *f(x)* in many cases" was.

coded as 'Robust' because it explicitly pointed out the incorrect object to which the student applied the limit. In contrast, Lia's submission of "students did not understand how to calculate the difference quotient limit" was coded as 'Limited' because it only included that the error type was regarding how to compute the limit of the difference quotient without mentioning what students did ("Category" in Table 3).

When in class TAs mentioned the error types that they noticed that they did not initially submit, we included them as part of their identifying, assuming they had first identified students' error types before mentioning them, and coded them using the three levels depending on what they said in class. There was one occasion where a TA simply mentioned examples of students' wrong responses in class without discussing their layers or nature (i.e., "I saw some of you put…"). We categorized this as "saw" under limited identifying to differentiate it from No-data. In this paper, we considered TAs' engagement with an error type from identifying, to planning, to teaching as a case. For example, if a TA identified a code at the Robust level, and his/her planning and teaching addressed it at the Robust level, the triplet (Identifying-Robust, Planning-Robust, Teaching-Robust) would be considered as one case.

Our scales for TAs' plans reflect the levels of specifics of error types that the plans address and the level of explicitness of their instructional approach (see Section "Use of Students' Errors in Mathematics Classrooms") (Table 4).

We analyzed the TAs' teaching, first by identifying and classifying the parts of the recorded class where they solved the tasks from PD according to the error types that they identified and planned to address. Then, we coded each part using a three-level system that was modified from the planning levels: Robust, Limited, and No-Use (Table 5). No-Use cases include when TAs identified error types, but there was no evidence of using them in class. The specific categories in planning also apply to the levels for TAs' use of the error types in teaching, with additional categories that we identified in our analysis of teaching. The reliability of coding for planning and teaching was assessed using the same method as the one employed for coding identifying.

Version of Record at: https://doi.org/10.1007/s40753-024-00239-1

Table 6 Number of error types that each TA identified

	Amy	Lia	Kay	Dan	Eddy	Sum
Number of Error Types Identified	17	7	14	12	8	58

Results

This section reports our analysis of what TAs identified as difficulties from their students' work, their planning and teaching addressing those difficulties. Recall from the Methods section that the difficulties identified by the TAs matched the error types, which allowed us to compare the error types that were captured in the TAs' submissions to those that were not. Since the error types that the TAs' submissions captured were the ones that were addressed in their planning and teaching, our results will mainly discuss the TAs' identification, and then planning and teaching in relation to their identification.

Identifying Errors

To have a general idea of the error types the TAs identified, we investigated them from three aspects. First, we compared them with the error types we identified from their students' work regarding how frequent those error types appeared in their students work. We also examined the nature of those error types and levels of specificity of students' work reflected in the TAs' submissions. Then, we will discuss specifics of those error types in the context of students' difficulties that were identified in the existing literature (Section "Content-Specific Mathematics Education Research Used in PD Materials").

TAs' Identifying – Frequencies of Error Types

The total number of the error types each TA identified varied from 7 to 17 (Table 6, if the same error type was addressed on multiple tasks, it was counted each time it was identified). Each TAs' submission captured 7% to 25% of the error types that we identified from their students' work on average across the tasks.

Given that a large percentage of errors were not captured in TAs' identifying, we sought to learn about which error types were and were not identified by the TAs. First, we looked at the *frequent error types*, which we operationalize as when they were observed in 3 or more students' work in one task in each TA's class, and found that the error types that TAs identified were not in general the frequent ones.[2] The average percentage of frequent errors that were included in each TAs' submission ranged from 8 to 44%. In fact, of the 39 combinations of 5 TAs and 8 tasks (one TA did not submit the scan of their students' work for one task), there were only 2 combinations

[2] We chose the frequency of 3 because 3 was about 10% of the submitted work from each TA's class for each problem.

Table 7 Levels of TAs' identifying and the nature of error types identified

	How-to	Meaning	Missing comp	Total
Robust	27	7	3	37
Limited	12	8	1	21
Total	39	15	4	58

where the TAs identified all the frequent error types for the task. Moreover, in 24 combinations among those 39 combinations (62%), the most frequent error types (whose frequencies ranged from 6 to 16) were not identified by TAs. This shows that the TAs often did not identify types of errors made by a relatively large number of students in their class as difficulties while evaluating their work.

TAs' Identifying – Nature of Error Types and Levels of Specific Students' Work

Before we further investigated the natures of the error types TAs identified, it should be noted that in the researchers' coding of all student data, how-to error types were most dominant, but there were also significant numbers of error types of other natures. Specifically, how-to error types ranged from 54% ~ 60% of all codes, Meaning-and-Relationship error types ranged from 26 ~ 32%, and Missing-component error types ranged from 9% ~ 17% (average percentages over 8 tasks for each TA) and corresponding numbers of errors showed a similar distribution.

In the TAs' submissions, we found that they identified how-to error types more frequently than the other two error types. All TAs identified "how-to" error types (12% ~ 34%) much more frequently than "meaning-or-relationships" (0% ~ 17%) or "missing-component" (0% ~ 20%) error types in comparison to what we identified, and some TAs almost exclusively identified How-to error types.

Of 58 error types, the TAs identified 37 at the Robust level (64%) and 21 at the limited level (36%). When we further categorized the levels according to the nature of error types (Table 7), we see that the error types TAs identified at the robust level were primarily about How-to, and Meaning-and-relation error types were less likely to be identified at the robust level. Among the 39 how-to error types, the TAs identified 27 (about 70%) at the robust level. In comparison, among the 15 meaning-or-relation error types, they identified less than half (7 among 15) at the robust level.

Our observation that (a) a large number of error types, especially the frequent ones, were not captured in their submission, (b) how-to error types were dominant in their identification, and (c) high levels of specific aspects of students' work was mainly included in how-to error types motivated us to further investigate the error types the TAs identified in comparison to those they did not.

Error Types Captured in TAs' Submission

To understand the content that the error types the TAs identified address, we first connected those error types to the difficulties about the derivative identified in the literature (Section "Content-Specific Mathematics Education Research Used in PD

Version of Record at: https://doi.org/10.1007/s40753-024-00239-1

Materials"). We then separated those difficulties that were addressed in multiple TAs submissions (Table 8) and in only one TA's submissions (Table 9). To make a comparison, we also looked at the error types that we observed in more than 3 TAs' classes, but were not included in any of their submissions (Table 10). These separations allowed us to observe what the TAs identified in the mathematical context of our study. The details are as follows.

Some error types TAs identified relate to prerequisite materials such as functions and DQs. For those layers, the TAs included various error types reflecting misuse of function properties in algebraic manipulations or graphing (Table 8). For example, TAs include students' difficulties decomposing a function (e.g., Pre-lab 3.4, Appendix), using algebraic properties of functions correctly (e.g., Pre-lab 3.5), and graphing given functions (e.g., Pre-lab 3.1). Although TAs did not include a frequent error type regarding composite functions (Table 10), it can be considered similar to those that they have already identified. Those error types that most TAs included were dominantly how-to error types.

Less prominent in their identification regarding functions and DQs were students' difficulties with use of variables and ordered pairs (Table 9). We identified error types reflecting such difficulties in multiple TAs' classes such as using numerical values instead of letter h when evaluating a function (Pre-lab 2.7), not including variables while decomposing functions (Pre-lab 3.4), making mistakes transitioning between an ordered pair as a point on the graph of the function and function notations (Pre-labs 2.7 & 3.9), and students' misuse of the DQ as a specific number when writing the DQ as a function of a letter h (Pre-lab 2.7). However, only one TA included each of those error types. These less prominent error types were mostly about meaning or relationships, or missing components.

Regarding the limit layer, our tasks were designed to address the limit not being obvious in students' thinking (Pre-lab 2.7). All TAs included a variety of error types reflecting this difficulty such as missing or incorrectly reflecting the limit in their computation of the derivative, or not being able to provide the limit definition of the derivative (Table 8), which seem to mainly address what students should have done for correct solutions. Less obvious in their submissions were students' difficulties applying the limit to the right object, which addresses what students actually did in their solutions. For example, although students applying the limit on a wrong object (e.g., the original function not on the DQ) was identified in all 5 sections, only one TA included it in her submission (Table 9).

Another difficulty that was less obvious in their submissions was regarding the meaning of the derivative at a point as a quantity. Those error types include representing the derivative as a tangent line, not as its slope, or providing incorrect justification of different slopes based on the original function. Also, students' difficulties with drawing tangent lines or estimating its slope were identified in all sections (See Limit layer in Table 10), but it was not included in any TAs' submission. The importance of tangent line in analytic contexts where the derivative is discussed was emphasized in the literature (Shultz et al., 2019) and being able to estimate its slope as a number is one of the crucial abilities in the construction of the derivative of a function that is given as a graph (Stewart, 2016).

Version of Record at: https://doi.org/10.1007/s40753-024-00239-1

Table 8 Difficulties addressed in multiple TAs' identification

Layer	Topics Students Have Difficulty with	Description of Error Types
Function	Composition of functions	Students could decompose a function into two functions, not into three
		Functions students write do not compose into the original function
	Properties or graph of functions	Students misuse properties of functions (e.g., symmetry, square root)
		Students provide incorrect graph for equations
DQ	Algebraic mistakes in computing DQ	Students do not use absolute values in the limit of the DQ
		Students make algebraic mistakes setting up, plugging in or simplifying the DQ
		Students give a wrong number as the limit of DQ or slope of secant lines
Limit	Limit not obvious in the derivative	Students do not complete the limit process or has work missing
		Students have no (or wrong) limit symbol
		Students incorrectly draw secant lines to show the limit
		Students confuse Average Rate of Change and Instantaneous Rate of Change
		Students do not provide the limit definition of the derivative
		Students use the derivative rules to obtain the derivative, not the limit definition
	Computation mistakes in limit of the DQ	Students make algebraic or computational error when computing the derivative
Derivative as a function	Connection between the derivative at a point and the derivative as a function	Students use the derivative rules to obtain the derivative not examining the graph of the derivative
		Students do not represent the derivative as a function with collection of derivative values at several points
		Students have difficulty with translating the slope of tangent lines to the coordinates (x, y) for the derivative function graph or vice versa
		Students do not correctly graph derivative
	Meaning of the derivative function	Students express an incorrect relationship between the function behavior and the first (or second) derivative from reading or drawing the graph of the derivative
		Students confuse dx/dt and dy/dt (horizontal vs. vertical)
Derivative as an operator	Variables in implicit differentiation	Students take the derivative with respect to a variable which is not the independent variable
	Computation of the derivative as an operator	Students do not take the derivative of the 'inside function' when applying the chain rule

Version of Record at: https://doi.org/10.1007/s40753-024-00239-1

Table 9 Difficulties addressed in only one of the TAs' identification

Layer	Topics Students Have Difficulty with	Description of Error Types
Function	Independent variables not explicit	Students do not include independent variables when writing function
		Students do not use independent variables when decomposing a given function
DQ	h as a variable and h as a number	When computing $f(x+h)$ or $f(a+h)$, students use a numerical example for 'h' rather than the letter h
	DQ as a function of h	Students do not recognize the difference between the DQ as a function of h and a specific value of the DQ
Limit on DQ	Object on which the limit was applied to	Students take the limit of the original function [No difference quotient seen]
	Relation between the derivative at a point and the function	When asked to graph the derivative, students graph tangent lines
		Students explain that 2 slopes found at the same x value because the slopes are symmetric to the x-axis
		When asked for the point on the curve where a specific slope is achieved (e.g. where $f'(x) = 0$), students give a correct x-value and the slope
		Unit missing for a value of DR

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Table 10 Repeated frequent error types that none of the TAs identified

Layer	Topics Students Have Difficulty with	Description of Error Types
Function	Composite function	Students could not find more than one way to decompose the given function into two functions
Limit	Slope of the tangent line as the derivative at a point	Students incorrectly approximate the slope of the tangent line from the graph
		Students graph tangent lines inaccurately
	Relation between the derivative and the original function	Students write general limit definition of the derivative but does not adjust for the function and/or a point
Derivative as a function	Relation between the representations of the derivative function	Students use the limit definition of the derivative to find the equation of the derivative instead of examining the graph of the derivative

Regarding the derivative as a function layer, the TAs' submissions addressed various students' difficulties with graphically constructing the derivative as a function, which was reflected in frequent error types in multiple sections (Table 8). These error types included "not representing the derivative as a function with collection of the derivative values at a several points" or using different methods (e.g., the differentiation rules) (Pre-labs 3.1. and 3.3). The TAs also included incorrect relations between the original function and its first and second derivative functions that students' answer included (Pre-lab 4.3) (Table 8). Note that those are addressing meaning and relationships, but at the same time are part of a well-established procedure of constructing the derivative graph from the textbook (Stewart, 2016). What was less obvious in TAs' submissions was students' transitioning between graphical and algebraic representations of the derivative. Specifically, in all 5 sections, students' alternative methods (e.g., limit definition) to find the equation of the derivative function instead of utilizing the graph of the derivative function that they had just constructed were identified, but not included in any TAs' submissions.

Regarding the derivative as an operator, two of the TAs included the incorrect variables that students adopted in the implicit differentiation (e.g., "wrong variable") in their submissions. The other three did not include those in their submissions although those error types were frequently identified in all TAs' classes and in multiple tasks.

In summary, we observed that TAs identified the how-to error types dominantly, and the meaning-and-relation or missing-comments error types that are mostly considered parts of performing well-established procedures (e.g., constructing the graph of the derivative). We also observed that TAs included what students did not do or did wrong, but often did not include what about their responses made it incorrect. This is related back to our distinction between levels of identification between robust and limited. We consider this distinction important because being able to identify the aspects of students' work that makes their answer incorrect beyond just them being incorrect is a starting point of understanding where students come from to make plans to address them from students' perspective instead of only addressing them from the perspective of mathematical correctness.

Planning

All error types the TAs identified in their submission (56) were used in their planning: 24 plans were at the robust level and 32 were at the limited level. The detailed planning approaches were presented in Analysis (3.3.4, Table 4). The most common planning categories at the robust level were providing extra examples, representations, and details addressing the error types they identified (20). The most common categories at the limited level were providing a correct answer or statement to the part of the task where the TAs had identified error types (for example, "making the relation between a function behavior and the sign of its first and second derivatives clear" for the wrong relationship stated by students) (27). However, there were four plans to provide extra examples, representations, and details addressing the error types they identified, which included limited specific details of students' work. For example, to address the error type, "Students have difficulty with translating the slope of tangent lines to the coordinates (x, y) for the derivative function graph or vice versa", Lia planned to emphasize the slopes as y values on a new plane under the original function graph for the matching x values, while explaining how the new y values correspond to the slopes of tangent lines of the original function at the Robust level. In comparison, Eddy planned to provide a table of x values with the corresponding values of the derivative, which we considered limited because the difficulties might have been about interpreting the slope of the tangent line as the y value of the derivative function. However, we consider plans for providing limited representations/examples/details to be much closer to using students' errors as resource for learning because they directly address what students' error informed about their misconceptions, and could be easily scaffolded to the robust level with PD.

Our comparison between the TAs' identifying and planning suggested the importance of the specific aspects of students' work included in identifying in devising instructional approaches that directly addressed students' difficulties, thus using them as resources in teaching. Specifically, when comparing TAs' planning and identifying, we found most of the robust plans were paired with robust identifying. Among 24 robust plans, 22 were paired with robust identifying, whereas only 2 were paired with limited identifying. Those two TAs' plans both included using the vertical line test (Robust Planning – extra representations) to address student difficulty with "identifying two functions from an equation of a circle" that they identified (limited identifying-category). Similarly, more than half of limited plans were paired with TAs' limited identifying (18/32, 56%). However, there were significant drops in levels from identifying to planning; 14 limited plans were paired with robust identifying. Most of these 14 cases include providing correct answers or statements to the part of the task where TAs identified the error types (13).

Teaching

Among the 58 error types that the TAs identified, 41 error types were used in their teaching at the Robust level (23/41, 56%) or the Limited level (18/41, 44%). The detailed approaches we observed in the TAs' teaching are presented in Analysis (3.3.4, Table 5). Similar to planning, the most frequent categories for Robust teaching were providing extra details, examples, and representations (22). The most frequent categories for limited teaching were providing correct answers or correct statements to address errors (10).

Generally, the levels of specificity of students' work involved in the TAs' teaching were the same as the levels of specificity of students' work involved in their identifying: Robust-Identifying paired with Robust-Teaching (20) and Limited-Identifying paired with Limited-Teaching (12). However, we also observed significant level drops between identifying and teaching. Most of Robust-Identifying and Limited-Teaching cases involved How-to error types (4 of 6) which the TAs addressed by providing the correct answers or statement or saying what students should or should not do. Also, there were 16 error types the TAs identified but did not use during teaching, and most of them were also How-to error types (14 of 16).

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These 41 error types that were used in teaching show close ties between teaching and planning. There were 20 pairs of Robust planning-Robust teaching, and 16 Limited planning-Limited teaching. In fact, the subcategories for planning and teaching were the same for 33 error types. There were 16 error types that TAs planned to address but did not use in teaching. Those plans mainly include providing the correct answer or statements (11 of 16) and addressed How-to error types (13 of 16). Their solving of the task looked like what one would expect of teachers who did not see their students' work.

Discussion and Conclusions

This study contributes to the field of undergraduate mathematics teaching by responding to the call for more studies by examining TAs use of students' work in their teaching. We incorporated suggestions from the recent literature of providing content-specific PD opportunities for university mathematics teachers, addressing the specific challenges that they face (Winsløw et al., 2021) including the ones stemming from disconnections between their teaching and students' ability to understand it (Park & Rizzolo, 2022; Güçler, 2013, 2016). Specifically, we investigated what TAs identify as students' difficulties from students' written work (identifying), and how they use those difficulties in their plan to address them (planning) and implementation (teaching). Since the difficulties the TAs identified matched types of errors that their students made in their work (error types), we developed our framework based on well-established research about teachers' use of students' errors in teaching (Bray, 2011; Brodie, 2014; Ingram et al., 2015; Santagata & Bray, 2016) focusing on the level of specifics of students' work and the levels of explicit instructional approaches that were reflected in TAs' engagement of students' errors (Biza et al., 2018; Heinrichs & Kaiser, 2018; Jacobs et al., 2010). Our analysis of TAs' identifying, planning, and teaching together provided several important observations.

Overall, our analysis shows that most of the difficulties that TAs identified were identified at the robust level, and that robust identification tended to result in robust planning and teaching. This shows that TAs attend to the specifics of students' responses and use the difficulties they observe to try to help their students overcome those difficulties. Thus, they have a strong foundation for learning more about using student difficulties as resources for learning. Our analysis also highlighted several important areas needing further investigation to understand how to best support TAs.

Our results suggest that identifying is a key stage in the process of TAs using student difficulties as a resource for learning. In particular, while there were drops in the level of engagement with student's difficulties from identifying to planning and then to teaching, overall TAs typically addressed student difficulties in teaching at the same level at which they identified the difficulties. In our analysis of what TAs identified, we found that they generally did not identify what the research team found to be the most frequent error types made by their students. Additionally, we found that TAs identified how-to error types at a higher rate than meaning-and-relationship or missing-component error types, which could be related to previous findings from the literature that domestic TAs in the U.S. view problem solving via pattern matching as an important part of Calculus I (Kim,

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2014). Based on our results, we believe the issue of what TAs identify as student difficulties from student work is a rich area for future investigation. In particular, it would be helpful to understand the reasons TAs identify the error types that they do. There are several possibilities for this that would have relevant practical consequences for PD. For example, are TAs not identifying frequent error types because they did not notice that they were frequent? If this is the case, then PD emphasizing how to identify frequent errors could be helpful. Alternatively, if TAs did not identify frequent errors because they are of a type or nature that TAs do not consider important to address, it might be more helpful to PD to focus on the learning goals for the course and the difficulties students have meeting those goals to reach consensus about what is important to address.

We also found that, in the difficulties TAs did identify, the specific details that TAs included were mainly about what students should have done to proceed to correct solutions rather than the approaches that students actually used in their work. For example, errors due to applying the limit on a wrong object (e.g., the original function not on the DQ) were identified in all 5 sections, but included in only one TA's identifying while other TAs' identifying pointed out what students should have done such as providing the limit definition and adjusting it to the original function as their first step. Based on this, we think it would be helpful to understand why TAs choose the instructional approaches that they do. For example, in terms of the characterization of instructional approaches in Smith et al. (1993), TAs primarily adopt the provide-the-right-answer approach, rather than the using student difficulties as a resource for learning approach advocated for (Li & Schoenfeld, 2019). It would be interesting to understand why TAs are adopting these approaches. For example, are they actively choosing these approaches because these are the approaches that they thought worked well for them as students? Or is it because they have not been exposed to other approaches? Answering these questions would be relevant for designing PD because it would impact how PD could productively address teaching practices to help TAs use the errors as a resource for learning (Smith et al., 1993) and to view student errors as a "plausible application" of their existing knowledge (Li & Schoenfeld, 2019).

Finally, while our analysis did show that levels were generally consistent going from identifying to planning to teaching, there were noticeable drops in levels of specifics of students' work reflected in the process of identifying, then planning, and then teaching. It would be interesting to understand how TAs decide which difficulties merit discussion in class.

Our results and the answers to the questions posed here would help for designing PD. In our view such PD promoting using students' difficulties as resources for their learning could be administered concurrently with TAs' teaching, similar to the setting of the current study, and would benefit TAs with different levels of teaching experience. Identifying students' difficulties together from students' work would particularly benefit beginning TAs who have little experience predicting and explaining the types of difficulties students have. Once TAs can predict and hypothesize about the difficulties their students may have (Park & Rizzolo, 2023), planning the lesson together based on guidelines from the mathematics education literature would help them navigate various instructional methods to use those as resources for students' learning.

Version of Record at: https://doi.org/10.1007/s40753-024-00239-1

Appendix

Problem Set

Pre-lab for Section 2.7: Derivatives and Rates of Change

1. Graph the function $f(x) = x^2 + 2x$.
2. Consider a number $h > 0$. For example, h can be 0.5 or 1. Plot $(-1 + h)$ on the x axis on your graph from Problem 1. Then what is the value of $f(x)$ when $x = (-1 + h)$?
3. Plot a point $(-1 + h, f(-1 + h))$ on the graph. Draw the secant line between this point and $(-1, -1)$. What is the slope of this secant line in terms of h?
4. Choose another value for h, and plot another point $(-1 + h, f(-1 + h))$ closer to $(-1, -1)$ on the curve. Draw another secant line between this point and $(-1, 1)$. What is the slope of this secant line in terms of h?
5. How does the slope of the secant line between $(-1 + h, f(-1 + h))$ and $(-1, -1)$ change as h approaches 0?
6. Use a limit as h approaches 0 to symbolically represent the idea that you expressed in part (5). Evaluate this limit.
7. Write an equation for the tangent line to $f(x)$ at $(-1, -1)$. Graph the tangent line on your graph of $f(x)$.

Pre-lab for Section 2.8: The Derivative as a Function

1. Graph $f(x) = |(x + 1)^3|$ and then use the interpretation of the derivative as the slope of the tangent line to determine the derivative at $x = -1$ (Please also use some secant lines).
2. Determine $f'(x)$ at $x = -1$ for $f(x) = |(x + 1)^3|$ using the definition of the derivative at a certain point.

3.Pre-lab for Section 3.1: Derivatives of Polynomials and Exponential Functions

Let $f(x) = x^3$.

1. Estimate the values of $f'(0), f'\left(\frac{1}{2}\right), f'(1), f'(2)$, and $f'(3)$ by using a graphing device to zoom in on the graph of $f(x)$.
2. Use symmetry to deduce the values of $f'\left(-\frac{1}{2}\right), f'(-1), f'(-2)$, and $f'(-3)$.
3. Use the values from part (1) and part (2) to graph $f'(x)$.
4. Guess the formula for $f'(x)$.
5. Use the definition of derivative to prove that your guess in part (4) is correct.

Pre-lab for Section 3.3: Derivatives of Trigonometric Functions

1. Graph $f(x) = \sin(x)$ on $[-2\pi, 2\pi]$. Label and scale the x-axis and y-axis.
2. Sketch tangent lines along the curve of $f(x)$ at $x = -2\pi, -\frac{3\pi}{2}, -\pi, -\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, and 2π.
3. Estimate the slopes of these tangent lines.
4. Plot these estimated slopes as y-values at the same x-values from part (2) on a new (x, y)-plane. Label and scale the x-axis and y-axis.
5. Use your answers from part (4) to make a conjecture for an equation of $f'(x)$. Explain your thinking. (Hint: the derivative of *sine* is another trigonometric function.)
6. Let $g(x) = \cos x$. Use a similar process from part (1) to part (5), and make a conjecture for an equation $g'(x)$. Explain your thinking.

Pre-lab for Section 3.4: The Chain Rule

1. $f(x) = \sqrt{x^2 + 1}$
   a) Write $f(x)$ as a composition of two functions.
   b) Find another way to write $f(x)$ as a composition of two functions.
   c) Write $f(x)$ as a composition of three functions.
2. $g(x) = (2x^2 - x)^{\frac{5}{2}}$
   a) Write $g(x)$ as a composition of two functions.
   b) Find another way to write $g(x)$ as a composition of two functions.
   c) Write $g(x)$ as a composition of three functions.

3. $h(x) = \frac{1}{2^{sin\pi x}}$
   a) Write $h(x)$ as a composition of two functions.
   b) Find another way to write $h(x)$ as a composition of two functions.
   c) Write $h(x)$ as a composition of three functions.

Pre-lab for Section 3.5: Implicit Differentiation

1. Consider the relation $x^2 + y^2 = 1$.
   a) Find two functions $y(x)$ that solve this equation. Find the domain for each function.
   b) Graph $x^2 + y^2 = 1$ on a xy plane. Label each of the functions you found in part (a) on the graph.
   c) Differentiate the two functions from part (a) with respect to x.
   d) Find the slope(s) of the line(s) tangent to the curve at $x = \frac{1}{2}$. Draw the tangent line(s) on the graph.

2. You should have found the two different slopes for the tangent lines for the relation $x^2 + y^2 = 1$ at $x = \frac{1}{2}$. Given an interpretation of why this is happening in terms of the graph.

Pre-lab for Section 3.9: Related Rates

Suppose a particle is moving along a circle described by the equation $x^2 + y^2 = 1$. Here x and y depend on time t. In other words, x can be written as $x(t)$, and y can be written as $y(t)$. The unit length is one meter, and time t is measured in seconds.

1. Which derivative expresses the rate (in m/s) at which x is changing (the horizontal velocity of the particle)?
2. Which derivative expresses the rate (in m/s) at which y is changing (the vertical velocity of the particle)?
3. Differentiate each side of $x^2 + y^2 = 1$ with respect to time t.
4. Suppose the particle is moving through the point $(\frac{1}{2}, \frac{\sqrt{3}}{2})$ with horizontal velocity 4 m/s. What is the vertical velocity of the particle?

Pre-lab for Section 4.3: How the derivatives affect the shape of a graph

Let $f(x) = 4x^3 - 12x^2 + 8x$.

1. Find the x and y intercepts of f.
2. Differentiate f and simplify.
3. Find all points $(c, f(c))$ where $f'(c) = 0$.
4. Graph f on a graphing calculator. Use the graph of f to give an interpretation of your answer in part (3).
5. For each c such that $f'(c) = 0$: How does the graph of f behave on an interval around c? Explain why you think this is happening in terms of how the derivative of f behaves around c.
6. Find the second derivative of f and simplify.
7. Find all points $(d, f(d))$ where $f''(d) = 0$.
8. Use the graph of f in part (4) to give an interpretation of the results that you found in part (7).
9. For each d such that $f''(d) = 0$: How does the graph of f behave on an interval around d? Explain why you think this is happening in terms of how the ***first*** and ***second*** derivatives of f behave around d.
10. Plot all intercepts and points you found in parts (3) and (7) for $f(x)$. Be sure to label the points and scale your graph. Then, sketch a graph of $f(x)$, paying special attention to what you have learned about the behavior of $f(x)$ at and around c and d. Compare your sketch with the graph on the calculator.

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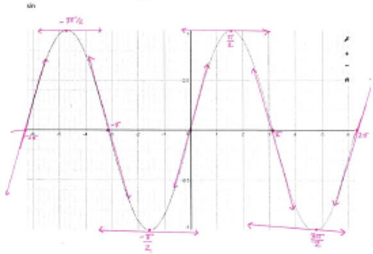
Example of a Sample Solution and Guidelines

Pre-lab 3.3 Derivatives of Trigonometric Functions

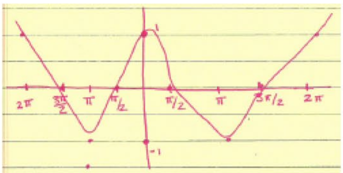
Sample Solutions: Pre-lab

1. Graph and tangent lines



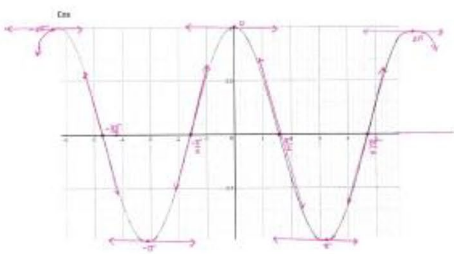
3. Estimated slopes:
$f'(-2\pi) = 1$
$f'\left(-\frac{3\pi}{2}\right) = 0$
$f'(-\pi) = -1$
$f'\left(-\frac{\pi}{2}\right) = 0$
$f'(0) = 1$
$f'\left(\frac{\pi}{2}\right) = 0$
$f'(\pi) = -1$
$f'\left(\frac{3\pi}{2}\right) = 0$
$f'(2\pi) = 1$

4.



5. cosine, because $\cos(0) = 1$ and has the same shape.

6. Graph for $g(x)$ and tangent lines



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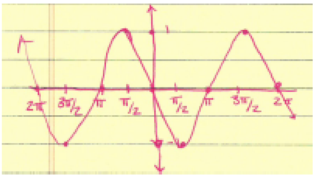
$f'(-2\pi) = 0$
$f'\left(-\frac{3\pi}{2}\right) = -1$
$f'(-\pi) = 0$
$f'\left(-\frac{\pi}{2}\right) = 1$
$f'(0) = 0$
$f'\left(\frac{\pi}{2}\right) = -1$
$f'(\pi) = 0$
$f'\left(\frac{3\pi}{2}\right) = 1$
$f'(2\pi) = 0$



$-\sin(x)$, it has the same shape and 0's as sine, but the non-zeroes $(\frac{\pi}{2}, \frac{3\pi}{2})$ are flipped.

Guidelines

1. In part 1, be sure students are graphing the sine function with x in radians.
2. Help students notice the relationships between tangent line slopes as they sketch and estimate in parts 2 and 3, e.g., the slope at $x = -\pi$ should be equal the magnitude but opposite the sign as the slope at $x = -2\pi$.
3. For parts 4 & 5, remind students that the dependent variable of the derivative function is the slopes of the original function. This should help them realize that the slopes they estimated in part 3 correspond to the y-values of the cosine function.
4. Make sure to check students' labels and scales on the x- and y- axes.
5. For part 6, make sure students are not just guessing that $g'(x) = \sin x$. Show them why this doesn't make sense and help them use what they know about slopes and derivatives to arrive at $g'(x) = -\sin x$.

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Declarations

Conflict of Interests There is no conflict of interest.

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