COMBINATORIAL AND SPECTRAL PROPERTIES OF GRAPHS AND ASSOCIATION SCHEMES

by

Matt McGinnis

A dissertation submitted to the Faculty of the University of Delaware in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics

Spring 2018

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ABSTRACT

The main topics of this dissertation are related to spectral graph theory, a subtopic of algebraic combinatorics. Algebraic combinatorics is the area of mathematics that implements techniques from linear and abstract algebra to solve problems in combinatorics as well as techniques from combinatorics to study algebraic structures. Spectral graph theory focuses on the study of the eigenvalues associated with various matrices for a graph and how the eigenvalues relate to structural properties of the graph. Properties such as connectedness, diameter, independence number, chromatic number and regularity, among others, are all related to the spectrum of a graph. In this dissertation we will study the spectra of various graphs and incorporate well known techniques in spectral graph theory to gain a better understanding of the structure of these graphs. We focus on three topics (Chapters 2, 3 and 4). The variation of these topics reinforces how diverse and useful spectral techniques in graph theory can be.

In Chapter 1 we cover notation and basic definitions used throughout this dissertation. We also introduce some well known and powerful results relating the structural properties of a graph to its spectrum. This includes a discussion of the properties of graphs that can be established from the spectrum as well as which graphs are determined by their spectrum. Finally, we give definitions and basic results for association schemes and distance-regular graphs since the results of this dissertation are related to these structures.

In Chapter 2 we study the smallest eigenvalues for distance-j graphs in both the Hamming and Johnson association schemes. Our results for distance-j Hamming graphs settle a conjecture proposed by Van Dam and Sotirov in [29]. In fact, we reach a stronger conclusion than the one proposed by Van Dam and Sotirov. Our results for

distance-j Johnson graphs settle a conjecture proposed by Karloff in [46]. Again, we are able to obtain a stronger conslusion than what is presented in Karloff's conjecture.

In Chapter 3 we use the technique of Godsil-McKay switching to construct cospectral mates for graphs formed by taking the union of relations in the Johnson association scheme. Our results offer insight into which graphs in this scheme are not determined by their spectrum. Our work also unifies the switching sets previously found for Johnson graphs in [26] and Kneser graphs in [43]. We also present some open problems related to our work, including a switching set that we would like to see generalized in order to obtain a new infinite family of graphs in the Johnson scheme that are not determined by their spectrum.

In Chapter 4 we examine connectivity properties of distance-regular graphs and graphs related to association schemes. In particular, we prove a result on the minimum number of edges that need to be deleted from a distance-regular graph in order to disconnect it into nonsingleton components. We also prove a result on the edge-connectivity of distance-j twisted Grassmann graphs which supports a conjecture proposed by Godsil in [35]. Finally, we end the chapter by presenting open problems dealing with the connectivity of color classes in association schemes.

Chapter 1

INTRODUCTION

In this chapter we provide an introduction to algebraic combinatorics. We will cover basic definitions and also establish notation. We also provide a small sample of well known results in spectral graph theory and discuss properties of graphs that can be determined from the spectrum. Finally, we give a brief introduction to distance-regular graphs and association schemes. This chapter is a review of material that may be used throughout the rest of this dissertation. I claim none of the results presented in this chapter.

1.1 Algebraic Combinatorics

Discrete mathematics is the study of objects that can be assumed to take distinct values. These objects are considered to be fundamentally discrete as opposed to continuous. Combinatorics is the branch of discrete mathematics that involves the study of finite structures. There are several different areas of study associated with combinatorics including algebraic, arithmetic, enumerative, extremal and geometric combinatorics. In this thesis we will focus on algebraic combinatorics. Algebraic combinatorics involves the use of algebraic structures (e.g. fields, groups, matrices, etc.) associated with combinatorial objects to solve combinatorial problems. Conversely, we may also apply techniques known in combinatorics to solve algebraic problems. For further information regarding algebraic combinatorics we refer the reader to the following sources [9, 36, 39, 54, 66].

The main focus of this dissertation will be on the use of linear algebraic techniques applied to graphs in association schemes. Spectral graph theory is the study of the eigenvalues of different matrices associated with graphs and the relationships between these eigenvalues and the structural properties of the graphs. The theory of association schemes studies the algebra generated by associated matrices. So what exactly does the spectrum tell us about a graph? The spectrum of the adjacency matrix can tell us the number of vertices, the number of edges and the number of closed walks of any length. It can also tell us whether or not a graph is bipartite, regular and, if it is regular, whether or not it is connected and what its girth is. The spectrum of the Laplacian matrix of a graph determines the number of vertices, the number of edges, the number of connected components and spanning trees as well as whether the graph is regular, and if it is regular, its girth. Graph properties such as diameter, chromatic number, independence number, clique number and connectivity are all related to a graph's spectrum too.

Sometimes the spectrum of a graph gives us so much information we can definitively state that there is only one graph up to isomorphism having that exact multiset of eigenvalues. However, in general, the spectrum does not determine the exact structure of a graph. It is possible for two nonisomorphic graphs to have the same spectrum. In fact, many such pairs have been constructed. Because of this, the subject of whether or not a graph is determined by its spectrum has generated a great deal of research recently.

In this dissertation we study the spectra of various families of graphs and how their spectra relate to their structural properties. We will focus on three main topics (Chapters 2–4).

In Chapter 2 we examine the smallest eigenvalues for distance-j graphs in both the Hamming and Johnson association schemes. Graphs in these association schemes are known to behave poorly when using the Goemans-Williamson algorithm to approximate the maximum cut. We explain how the smallest eigenvalue of these graphs plays an important role in determining the performance ratio of the Goemans-Williamson algorithm. The results we present for distance-j Hamming graphs settle a conjecture proposed by Van Dam and Sotirov in [29]. In fact, we reach a stronger conclusion than the one proposed by Van Dam and Sotirov. The results we present for distance-j Johnson graphs settle a conjecture proposed by Karloff in [46]. Again, we are able to obtain a stronger conclusion than what was conjectured by Karloff as well as generalize the hypothesis of his conjecture to obtain results for more graphs in the Johnson scheme.

In Chapter 3 we use the technique of Godsil-McKay switching to construct cospectral mates for graphs formed by taking the union of relations in the Johnson association scheme. The results presented offer insight into which graphs in this scheme are not determined by their spectrum. The work presented in this chapter also unifies the switching sets previously found for Johnson graphs in [26] and Kneser graphs in [43]. At the end of this chapter we also present some open problems related to our work, including a switching set that can potentially be generalized in order to obtain a new infinite family of graphs in the Johnson scheme that are not determined by their spectrum.

In Chapter 4 we examine connectivity properties of distance-regular graphs and graphs related to association schemes. In particular, we prove a result about the minimum number of edges needed to be removed from a distance-regular graph in order to disconnect it into nonsingleton components. We also prove a result on the edge-connectivity of distance-j twisted Grassmann graphs which gives support for a conjecture proposed by Godsil in [35]. Finally, we end the chapter by stating some open problems and partial results dealing with the connectivity of distance-regular graphs and association schemes.

1.2 Definitions, Notations and Basic Graph Theory

Definition 1.2.1. A *graph* Γ is a pair $(V(\Gamma), E(\Gamma))$, where $V(\Gamma)$ is a set of points (called *vertices*) and $E(\Gamma)$ is a set of pairs of vertices (called *edges*). If x and y are vertices in Γ and $\{x, y\} \in E(\Gamma)$, then we say x and y are *adjacent* and write $x \sim y$; otherwise x and y are called *nonadjacent* and we write $x \not\sim y$. Finally, a vertex is said to be *incident* with the edges containing it and vice versa.

Definition 1.2.2. The *order* of a graph is the number of vertices $|V(\Gamma)|$. The *size* of a graph is the number of edges $|E(\Gamma)|$.

To visualize a graph, it is standard to draw the vertices as dots and represent edges as lines joining two vertices. Below we give an example of the path P_3 and the cycle C_6 (see Definition 1.2.12).

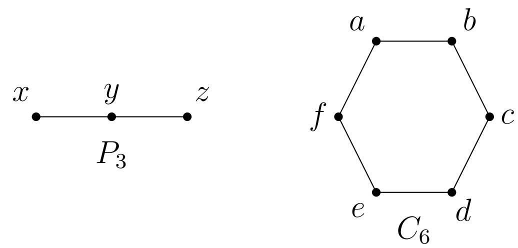


Figure 1.1: The graphs P_3 and C_6

Definition 1.2.3. Let Γ be a graph. A *clique* in Γ is a subset $S \subseteq V(\Gamma)$ such that the vertices of S are pairwise adjacent. The *clique number* of Γ, denoted $\omega(\Gamma)$, is the maximum order of a clique in Γ. An *independent set* (or *coclique*) in Γ is a subset $T \subseteq V(\Gamma)$ such that the vertices of T are pairwise nonadjacent. The *independence number* of Γ, denoted $\alpha(\Gamma)$, is the maximum order of an independent set in Γ.

For example in Figure 1.1, the set $\{x, y\}$ is a clique in P_3 and the set $\{a, b\}$ is a clique in C_6. The set $\{x, z\}$ is an independent set in P_3 and the set $\{a, c, e\}$ is an independent set in C_6. Note that cliques can consist of more than two vertices, however, there are no such cliques in P_3 or C_6. Hence $\omega(P_3) = \omega(C_6) = 2$. As for the independence number, we see $\alpha(P_3) = 2$ while $\alpha(C_6) = 3$.

Definition 1.2.4. A *subgraph* G of Γ is a graph G such that $V(G) = S \subseteq V(\Gamma)$ and $E(G) \subseteq E(\Gamma)$. A subgraph is called *induced* if for every $x, y \in V(G)$ we have $\{x, y\} \in E(G)$ if and only if $\{x, y\} \in E(\Gamma)$. If G is an induced subgraph of Γ, we say that Γ induces G (or that S induces G) and write $G = \Gamma|_S$.

Put less formally, a subgraph is obtained by deleting vertices and edges. An induced subgraph is obtained by deleting vertices. In Figure 1.1 we can see that P_3 is an induced subgraph of C_6.

Definition 1.2.5. Let Γ be a graph, $S, T \subseteq V(\Gamma)$, and H, G be subgraphs of Γ. We define $e(S,T) = \{\{x,y\} \in E(\Gamma) : x \in S \text{ and } y \in T\}$, the set of edges from S to T. We define $e(H,G) = e(V(H), V(G))$.

Definition 1.2.6. A *walk* in a graph Γ is a sequence of vertices in $V(\Gamma)$ such that consecutive vertices are adjacent. The *length* of a walk is the number of edges traversed. A walk is called *closed* if the first and last vertex are the same. A *path* is a walk in which no vertex occurs more than once. A *cycle* is a closed walk with no repeated vertices except for the starting and ending vertex. A graph Γ is *connected* if there is a path from x to y for every $x, y \in V(\Gamma)$. For $x, y \in V(\Gamma)$, the *distance* between x and y, denoted $d_\Gamma(x,y)$, is the smallest length of a path containing x and y. If no such path exists we say $d_\Gamma(x,y) = \infty$. By convention, we say $d_\Gamma(x,x) = 0$. Observe that Γ is connected if $d_\Gamma(x,y)$ is finite for all $x, y \in V(\Gamma)$. Otherwise, Γ is called *disconnected*. If Γ is disconnected and we partition $V(\Gamma)$ into sets $X_1, \cdots, X_k$ such that $\Gamma|_{X_i}$ is connected for each i and $x \nsim y$ for $x \in X_i$ and $y \in X_j$ when $i \neq j$, then we call the subgraphs induced on $X_1, \cdots, X_k$ the *connected components* of Γ.

An example of a path in C_6 from Figure 1.1 would be (a,b,c,d). This path has length 3 and the edges traversed are $\{a,b\}$, $\{b,c\}$ and $\{c,d\}$. The graph, C_6, is itself a cycle of length 6 (so (a,b,c,d,e,f,a) is a cycle). Also, note that in P_3, $d_{P_3}(x,z) = 2$ and in C_6, $d_{C_6}(a,d) = 3$.

Definition 1.2.7. The *diameter*, denoted $\mathrm{diam}(\Gamma)$, of a connected graph Γ is the maximum distance between two vertices of Γ.

In Figure 1.1 one sees that $\mathrm{diam}(P_3) = 2$ and $\mathrm{diam}(C_6) = 3$.

Definition 1.2.8. Let Γ be a graph and $x \in V(\Gamma)$. We define the *neighborhood* of x by $\Gamma(x) = \{v \in V(\Gamma) : v \sim x\}$. The elements of $\Gamma(x)$ are called the *neighbors* of x.

We define the *distance-k neighborhood* of x by $\Gamma_k(x) = \{v \in V(\Gamma) : d_\Gamma(x, v) = k\}$, the set of vertices at distance k from x. The elements of $\Gamma_k(x)$ are called the *distance-k neighbors* of x.

From Figure 1.1 we have $\Gamma(y) = \{x, z\}$ in P_3 and $\Gamma(b) = \{a, c\}$ in C_6. Also note that $\Gamma_2(b) = \{d, f\}$ in C_6.

Definition 1.2.9. Let Γ be a graph. The *complement* of Γ, denoted $\overline{\Gamma}$, is the graph with vertex set $V(\Gamma)$ and for all $x, y \in V(\Gamma)$, $\{x, y\} \in E(\overline{\Gamma})$ if and only if $\{x, y\} \notin E(\Gamma)$.

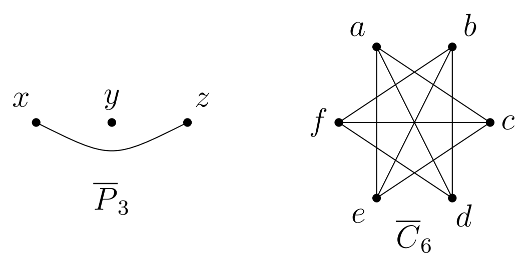


Figure 1.2: The complements of P_3 and C_6

Definition 1.2.10. The *chromatic number* of a graph Γ, denoted $\chi(\Gamma)$, is the minimum number of colors needed to color the vertices of a graph such that no two adjacent vertices share the same color.

Definition 1.2.11. A graph Γ is said to be *bipartite* if $V(\Gamma)$ can be partitioned into sets X and Y such that X and Y are both independent sets (so the only edges are of the form $\{x, y\}$ where $x \in X$ and $y \in Y$).

We note here that a bipartite graph having at least one edge must have chromatic number two.

Definition 1.2.12. The *complete graph* on n vertices, denoted K_n, is the graph on n vertices, all of which are pairwise adjacent. The *complete bipartite graph*, denoted $K_{m,n}$, is the graph with vertices partitioned into independent sets X and Y with $|X| = m$ and $|Y| = n$ such that every vertex of X is adjacent to every vertex of Y. The *cycle* *graph* on n vertices, denoted C_n, is the graph with n vertices whose edges form a cycle of length n. The *path graph* on n vertices, denoted P_n, is the graph on n vertices whose edges form a path of length $n-1$. The *empty graph* on n vertices, denoted $\overline{K}_n$, is the graph on n vertices with no edges.

In Figure 1.1 we provide examples of a path and a cycle. Below we give examples of a complete graph and a complete bipartite graph.

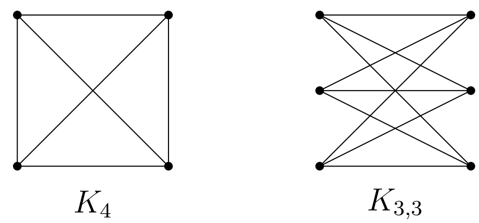
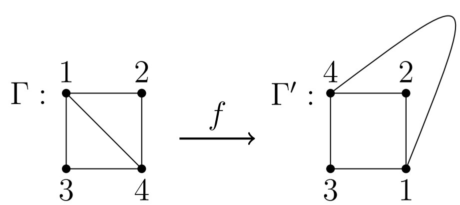


Figure 1.3: The graphs K_4 and $K_{3,3}$

Definition 1.2.13. Two graphs Γ and Γ' are *isomorphic* (which we denote by $\Gamma \cong \Gamma'$) if there exists a bijection $f : V(\Gamma) \to V(\Gamma')$ such that $\{x, y\} \in E(\Gamma)$ if and only if $\{f(x), f(y)\} \in E(\Gamma')$.



$$f := \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 3 & 1 \end{pmatrix}$$

$E(\Gamma)$	$f(E(\Gamma))$	$E(\Gamma')$
$\{1,2\}$	$\{f(1), f(2)\}$	$\{4,2\}$
$\{1,3\}$	$\{f(1), f(3)\}$	$\{4,3\}$
$\{1,4\}$	$\{f(1), f(4)\}$	$\{4,1\}$
$\{2,4\}$	$\{f(2), f(4)\}$	$\{2,1\}$
$\{3,4\}$	$\{f(3), f(4)\}$	$\{3,1\}$

Figure 1.4: Graph isomorphism example

Definition 1.2.14. Let Γ and Γ' be two vertex disjoint graphs. The *graph union* $\Gamma \cup \Gamma'$ is the graph with vertex set $V(\Gamma) \cup V(\Gamma')$ and edge set $E(\Gamma) \cup E(\Gamma')$.

Definition 1.2.15. The *graph Cartesian product* of two graphs Γ and Γ', denoted $\Gamma \,\square\, \Gamma'$, is the graph with vertex set $V(\Gamma) \times V(\Gamma')$ such that for any $u, v \in V(\Gamma)$ and $x, y \in V(\Gamma')$, the vertices (u, x) and (v, y) are adjacent in $\Gamma \,\square\, \Gamma'$ if and only if $u = v$ and $x \sim y$ in Γ' or $u \sim v$ in Γ and $x = y$.

Another way to think about $\Gamma \,\square\, \Gamma'$ is to replace each vertex of Γ with a copy of Γ'. A vertex in a copy of Γ' is adjacent to the same vertex in another copy of Γ' if there was an edge in Γ between the two vertices the two copies of Γ' replaced. We give an example below taking the Cartesian product of K_2 with itself.

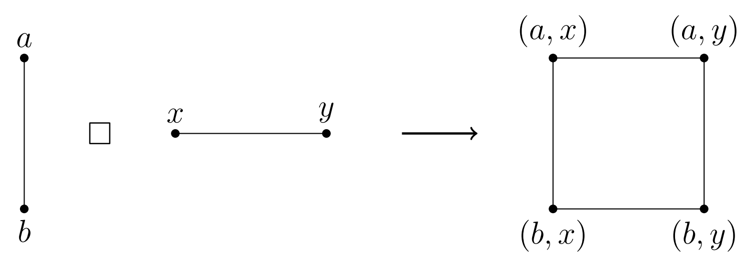


Figure 1.5: Cartesian product $K_2 \,\square\, K_2$

Definition 1.2.16. The n-cube, denoted Q_n, is the graph obtained by taking the Cartesian product of K_2 with itself n times.

Definition 1.2.17. Let Γ be a graph and $v \in V(\Gamma)$. The *degree* (or *valency*) of v, denoted $\deg(v)$, is the number of vertices adjacent to v. That is, $\deg(v) = |\Gamma(v)|$. The *average degree* of Γ, denoted $\overline{d}(\Gamma)$, is the average of the degrees of the vertices in Γ. The *minimum degree* of Γ, denoted $\delta(\Gamma)$, is equal to $\min\limits_{v \in V(\Gamma)} \deg(v)$. The *maximum degree* of Γ, denoted $\Delta(\Gamma)$, is equal to $\max\limits_{v \in V(\Gamma)} \deg(v)$. If every vertex in Γ has the same degree k then we say Γ is *k-regular*. The *degree* (or *valency*) of a k-regular graph is k.

For example, in Figure 1.1, the degree of x in P_3 is one, while the degree of y in P_3 is two. Also note that the graph C_6 is 2-regular.

Definition 1.2.18. The *girth* of a graph Γ is the smallest order of a cycle contained in Γ. If Γ contains no cycles, then we say Γ has infinite girth.

For example, in Figure 1.1, the girth of C_6 is equal to six while P_3 has infinite girth since it contains no cycles.

1.3 Spectra of Graphs

Before we discuss the spectra of graphs, we first review some basic results in linear algebra that will prove to be helpful. For a more thorough review of linear algebra see [4, 48, 53]. Given a matrix M we denote the entry in the i-th row and j-th column of M by $M(i,j)$ or M_{ij}. We refer to the multiset of eigenvalues of a matrix as the *spectrum* of the matrix.

Definition 1.3.1. The identity matrix I_n is the $n \times n$ matrix with diagonal entries equal to one and all other entries equal to 0. The matrix $J_{m,n}$ is the $m \times n$ matrix having all entries equal to one. The matrix J_n is the $n \times n$ matrix having all entries equal to one. The column vector of dimension n having all entries equal to one is denoted by $\mathbf{j}_n$ and the column vector of dimension n having all entries equal to zero is denoted by $\mathbf{0}_n$. When the dimension of these objects is clear we will often write I, J, $\mathbf{j}$ and $\mathbf{0}$.

Definition 1.3.2. For any matrix M the *transpose* of M, denoted $M^\top$, is the matrix with (i,j)-entry equal to $M(j,i)$. A matrix is called *symmetric* if $M = M^\top$.

Proposition 1.3.3. *A real, symmetric, $n \times n$ matrix has n real eigenvalues (including multiplicities), and there is a set of n orthonormal eigenvectors associated with these eigenvalues.*

Proposition 1.3.4. *Let M be a square matrix. The eigenvalues of any polynomial $p(M)$ are $\{p(\lambda) : \lambda$ is an eigenvalue of $M\}$.*

Definition 1.3.5. A real and symmetric matrix M is called *positive semidefinite* if $x^\top Mx \geq 0$ for any real vector x.

Proposition 1.3.6. *A matrix M is positive semidefinite if and only if all of its eigenvalues are non-negative.*

Proposition 1.3.7. *For any real and symmetric matrix M the matrices $M^\top M$ and $MM^\top$ are positive semidefinite. The matrices $M^\top M$ and $MM^\top$ have the same rank and the same nonzero eigenvalues (including multiplicity).*

Definition 1.3.8. Let M be an $n \times n$ matrix. The *trace* of M, denoted $\mathrm{Tr}(M)$, is the sum of the diagonal entries of M.

Proposition 1.3.9. *The sum of the eigenvalues of a matrix M equals $\mathrm{Tr}(M)$.*

The remainder of this section is devoted to covering basic definitions and results in spectral graph theory. We also discuss the spectrum of various families of common graphs. For a more detailed background on spectral graph theory, we refer the reader to [9, 39]. There are several different matrices that can be associated with a graph. For each of these matrices we will assume the rows and columns are indexed by the vertices of the graph.

Definition 1.3.10. Let $\Gamma = (V, E)$ be a graph. The *adjacency matrix* of Γ, denoted $A = A(\Gamma)$, is the matrix with $A(x, y) = 1$ if $x \sim y$ and $A(x, y) = 0$ if $x \not\sim y$. The *degree matrix* of Γ, denoted $D = D(\Gamma)$, is the diagonal matrix with $D(x, x) = \deg(x)$. The *Laplacian matrix* of Γ, denoted $L = L(\Gamma)$, is the matrix $L = D - A$.

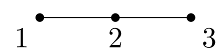
Note that in the above definition we refer to "the" matrices. However, the matrices mentioned above are only unique for a graph up to a permutation of the vertex labelling.

In this dissertation we will mainly focus of the adjacency matrix of a graph with an occasional reference to the Laplacian matrix. A few useful things to recognize are that the adjacency matrix for the complement of a graph Γ is given by $J - I - A(\Gamma)$ and if Γ is k-regular, the Laplacian is given by $kI - A(\Gamma)$.

All of the matrices described in Definition 1.3.10 are real and symmetric matrices. Hence, if the order of Γ is n, these matrices will have n real eigenvalues (including multiplicities). We will refer to the adjacency eigenvalues of a graph Γ simply as the eigenvalues of Γ. When referring to the multiset of eigenvalues of Γ we will simply refer to the spectrum of Γ. We will write the eigenvalues of a graph in descending order as follows $\lambda_1(\Gamma) \geq \lambda_2(\Gamma) \geq \ldots \geq \lambda_n(\Gamma)$, where $\lambda_1(\Gamma)$ is the largest eigenvalue and $\lambda_n(\Gamma)$ is the smallest eigenvalue. When writing the spectrum of a graph we may often use exponents to denote the multiplicities.

As an example, suppose Γ is the path on three vertices as seen below.

Γ :



Then the adjacency matrix is given by

$$A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{array}.$$

and the Laplacian matrix is given by

$$L = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \end{array}$$

The eigenvalues of A are $\sqrt{2}$, 0 and $-\sqrt{2}$ and the eigenvalues of L are 3, 1 and 0.

Proposition 1.3.11. *The spectrum of the complete graph K_n is $\{(n-1)^{(1)}, (-1)^{(n-1)}\}$. The spectrum of the complete bipartite graph $K_{m,n}$ is $\{\sqrt{mn}^{(1)}, 0^{(m+n-2)}, -\sqrt{mn}^{(1)}\}$. The spectrum of the cycle C_n is $\left\{2\cos\left(\frac{2\pi j}{n}\right) : 0 \leq j \leq n-1\right\}$. The spectrum of the path P_n is $\left\{2\cos\left(\frac{\pi j}{n+1}\right) : 1 \leq j \leq n\right\}$.*

Definition 1.3.12. Let M be a matrix with eigenvalues $\lambda_1, \cdots, \lambda_n$. The *spectral radius* of M, denoted $\rho(M)$, is equal to $\max(|\lambda_1|, \cdots, |\lambda_n|)$.

Definition 1.3.13. The spectral radius of a graph Γ, denoted $\rho(\Gamma)$, is equal to $\rho(A(\Gamma))$.

It is clear that $\rho(\Gamma) = \max(\lambda_1(\Gamma), -\lambda_n(\Gamma))$. However, it turns out that $\rho(\Gamma) = \lambda_1(\Gamma)$ which follows as an immediate corollary of the Perron-Frobenius Theorem for irreducible matrices (see [9, Thm. 2.2.1] or [39, Sec. 8.8]).

Corollary 1.3.14. *The spectral radius of a graph is the largest eigenvalue of the graph.*

Proposition 1.3.15. *Let Γ be a connected graph. If Γ is k-regular, then $\rho(\Gamma) = k$. Otherwise we have $\delta(\Gamma) < \overline{d}(\Gamma) < \rho(\Gamma) < \Delta(\Gamma)$.*

Proposition 1.3.16 ([9, Prop. 1.3.3])**.** *Let Γ be a connected graph with diameter d. Then Γ has at least $d+1$ distinct eigenvalues and at least $d+1$ distinct Laplacian eigenvalues.*

The following spectral bounds on the independence number and chromatic number of a graph are due to Hoffman.

Proposition 1.3.17 ([9, Thm. 3.5.2])**.** *If Γ is a connected, k-regular graph on n vertices, then*

$$\alpha(\Gamma) \leq n \cdot \frac{-\lambda_n(\Gamma)}{k - \lambda_n(\Gamma)}.$$

Proposition 1.3.18 ([9, Thm. 3.6.2])**.** *Let Γ be a graph with at least one edge on n vertices and $\chi(\Gamma)$ be its chromatic number, then*

$$\chi(\Gamma) \geq 1 - \frac{\lambda_1}{\lambda_n}.$$

Definition 1.3.19. If A is an $m \times n$ matrix and B is an $s \times t$ matrix, then the *Kronecker product* of A and B, denoted $A \otimes B$ is the $ms \times nt$ block matrix given by

$$\begin{bmatrix} A(1,1)B & \dots & A(1,n)B \\ \vdots & \ddots & \vdots \\ A(m,1)B & \dots & A(m,n)B \end{bmatrix}.$$

Proposition 1.3.20. *Let Γ and Γ' be graphs on n and m vertices, respectively. Provided we have an appropriate ordering of the vertices, the adjacency matrix of $\Gamma \,\Box\, \Gamma'$ is $A(\Gamma \,\Box\, \Gamma') = A(\Gamma) \otimes I_m + I_n \otimes A(\Gamma')$. The eigenvalues of $\Gamma \,\Box\, \Gamma'$ are $\{\lambda_i(\Gamma) + \lambda_j(\Gamma') : 1 \leq i \leq n \text{ and } 1 \leq j \leq m\}$.*

Corollary 1.3.21. *The spectrum of the n-cube Q_n is $\{(n-2i)^{\binom{n}{i}} : 0 \leq i \leq n\}$.*

Proposition 1.3.22. *The spectrum of a disconnected graph is the multiset union of the spectra of its connected components.*

Proposition 1.3.23. *If Γ is a connected, k-regular graph on n vertices with eigenvalues $k = \lambda_1(\Gamma) > \lambda_2(\Gamma) \geq \ldots \geq \lambda_n(\Gamma)$, then its complement $\overline{\Gamma}$ has eigenvalues*

$$n - k - 1 > -1 - \lambda_n(\Gamma) \geq -1 - \lambda_{n-1}(\Gamma) \geq \ldots - 1 - \lambda_3(\Gamma) \geq -1 - \lambda_2(\Gamma).$$

Here we provide a brief introduction to the very useful technique of eigenvalue interlacing. The technique of eigenvalue interlacing has proven to be a very powerful tool in spectral graph theory. For further reading on this technique we refer the reader to [9, Sections 2.3-2.5] or [41].

Definition 1.3.24. Let M be a real and symmetric matrix with rows and columns indexed by $X = \{1, 2, \cdots, n\}$ and let $\mathcal{P} = X_1 \cup \ldots \cup X_m$ be a partition of X. The *characteristic matrix* S of $\mathcal{P}$ is the $n \times m$ matrix with $S(i,j) = 1$ if $i \in X_j$ and $S(i,j) = 0$ otherwise. The *cardinality matrix* K of $\mathcal{P}$ is the $m \times m$ diagonal matrix with $K(i,i) = |X_i|$. Index the rows and columns of M to be consistent with the partition $\mathcal{P}$. So we have

$$M = \begin{bmatrix} M_{1,1} & \ldots & M_{1,m} \\ \vdots & \ddots & \vdots \\ M_{m,1} & \ldots & M_{m,m} \end{bmatrix},$$

where $M_{i,j}$ denotes the submatrix of M with rows indexed by X_i and columns indexed by X_j. The *quotient matrix* of M with respect to $\mathcal{P}$ is the $m \times m$ matrix Q with $Q(i,j) = q_{ij}$ equal to the average row sum of $M_{i,j}$. If the row sum of every row in each $M_{i,j}$ is equal to q_{ij}, then we say the partition $\mathcal{P}$ is *equitable*.

Proposition 1.3.25. *If M is a real and symmetric matrix with $\mathcal{P}$, S, K and Q as in Definition 1.3.24, then we have $S^\top MS = KQ$ and $S^\top S = K$. If the partition is equitable, then we also have $MS = SQ$.*

Proposition 1.3.26. *If M is a real and symmetric matrix with $\mathcal{P}$, S, K, Q and $M_{i,j}$ as in Definition 1.3.24, then for each eigenvector $\mathbf{v}$ of Q with eigenvalue λ, $S\mathbf{v}$ is an eigenvector of M with eigenvalue λ. In addition, we may choose an eigenbasis such that M has two types of eigenvalues:*

(i) the eigenvalues of Q, with eigenvectors constant on X_j for all j, and

(ii) the remaining eigenvalues, with eigenvectors summing to 0 on X_j for each j. These remaining eigenvalues are unchanged if the blocks $M_{i,j}$ are replaced by blocks $M_{i,j} + c_{i,j}J$ for some constants $c_{i,j}$.

Proof. Note that $Q\mathbf{v} = \lambda v$ implies $M(S\mathbf{v}) = S(Q\mathbf{v}) = S(\lambda\mathbf{v}) = \lambda(S\mathbf{v})$. To see that (i) holds we note that the i-th entry of the vector $S\mathbf{v}$ is equal to v_j for all $i \in X_j$. To see that (ii) holds we note that the remaining eigenvectors may be chosen to be orthogonal to those from (i), which implies they sum to zero on each X_j. Adding a multiple of the all ones matrix to a block will not change these eigenvalues since their eigenvectors sum to zero on each X_j. $\square$

Definition 1.3.27. For a real, symmetric matrix M and a nonzero vector $\mathbf{v}$, the *Rayleigh quotient* of $\mathbf{v}$ with respect to M is

$$\mathrm{Ray}(M, \mathbf{v}) = \frac{\mathbf{v}^\top M\mathbf{v}}{\mathbf{v}^\top\mathbf{v}}.$$

Proposition 1.3.28. *If M is a real, symmetric, $n \times n$ matrix and $\mathbf{v}_1, \cdots, \mathbf{v}_n$ is a set of orthonormal eigenvectors corresponding to $\lambda_1 \geq \cdots \geq \lambda_n$ such that $M\mathbf{v}_i = \lambda_i(M)\mathbf{v}_i$ for $1 \leq i \leq n$, then*

(i) $Ray(M, \mathbf{v}) \geq \lambda_i(M)$ if $\mathbf{v} \in span(\mathbf{v}_1, \cdots, \mathbf{v}_i)$, and

(ii) $Ray(M, \mathbf{v}) \leq \lambda_i(M)$ if $\mathbf{v} \in span(\mathbf{v_1}, \cdots, \mathbf{v}_{i-1})^\perp$.

In either case, equality implies $\mathbf{v}$ *is an eigenvector of* M *for the eigenvalue* λ_i.

Definition 1.3.29. Let $\lambda_1 \geq \lambda_2 \geq \ldots \lambda_n$ and $\mu_1 \geq \mu_2 \geq \ldots \mu_m$ be two sequences of real numbers with $m < n$. The sequence $\{\mu_i\}_{1 \leq i \leq m}$ is said to *interlace* the sequence $\{\lambda_i\}_{1 \leq i \leq n}$ if $\lambda_i \geq \mu_i \geq \lambda_{n-m+i}$ for $1 \leq i \leq m$.

Proposition 1.3.30. *If* M *is a real and symmetric matrix with quotient matrix* Q *(with respect to some partition* $\mathcal{P}$*), then the eigenvalues of* Q *interlace the eigenvalues of* M.

Definition 1.3.31. A *principal submatrix* of a square matrix M is a matrix obtained by deleting some rows and the corresponding columns from M.

Proposition 1.3.32. *If* N *is a principal submatrix of a symmetric matrix* M *then the eigenvalues of* N *interlace the eigenvalues of* M.

As we have already seen, the spectrum of a graph can tell us a lot of information about the graph. In fact the following graph properties can be determined just from the spectrum of a graph.

Proposition 1.3.33. *The following properties can be determined from the spectrum of the adjacency and Laplacian for a graph,* Γ*:*

1. *The number of vertices.*
2. *The number of edges.*
3. *Whether or not* G *is regular.*
4. *Whether or not* G *is regular with any fixed girth.*

The following properties can be determined by the spectrum of the adjacency matrix of Γ*:*

1. *The number of closed walks of any fixed length.*

2. Whether or not Γ *is bipartite.*

The following properties can be determined by the spectrum of the Laplacian matrix of Γ*:*

1. The number of connected components.

2. The number of spanning trees.

Definition 1.3.34. If Γ is a graph and every graph having the same spectrum as Γ is isomorphic to Γ, then we say Γ is *determined by spectrum* or *DS* for short.

Some graphs that are determined by spectrum are K_n, P_n, C_n and all regular graphs on less than 10 vertices. However, there are examples of nonisomorphic graphs that have the same spectrum. This will be discussed in greater detail in Section 3.1. Below we give a basic example of a pair of nonisomorphic graphs sharing the same spectrum.

Figure 1.6: Two nonisomorphic graphs with spectrum $\{2^{(1)}, 0^{(3)}, (-2)^{(1)}\}$

1.4 Strongly Regular Graphs, Distance-Regular Graphs and Association Schemes

Here we present basic definitions and notation for association schemes and the structures affiliated with them. For undefined terms and more details, see [8, 9, 28, 36]. Association schemes are one of the most unifying objects in algebraic combinatorics since they relate fields such as coding theory, design theory, algebraic graph theory and finite group theory. Before we present the definitions and basic theory for association schemes, we will try to motivate the idea behind them. We begin by introducing some types of graphs with nice regularity properties.

1.4.1 Strongly regular graphs

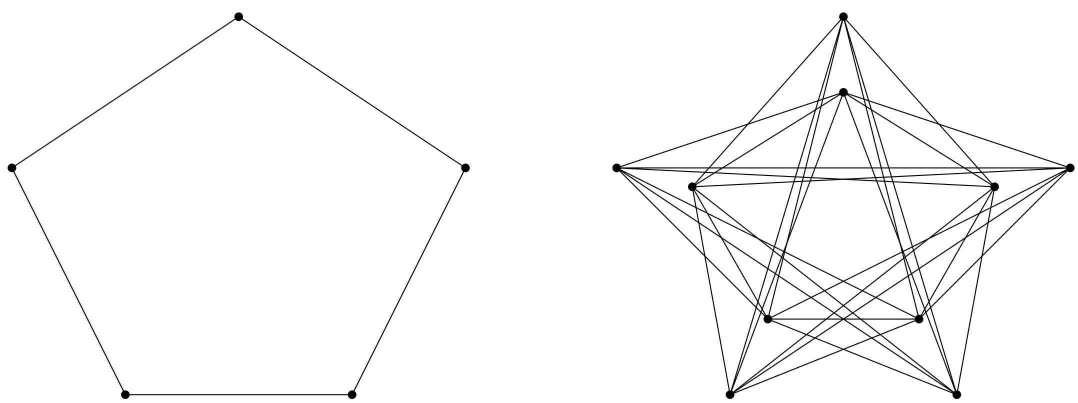
Definition 1.4.1. A simple graph Γ of order v is said to be *strongly regular* with parameters (v, k, λ, μ) whenever

1. Γ is k-regular,

2. for each pair of adjacent vertices x and y in Γ there are λ vertices adjacent to both x and y,

3. for each pair of nonadjacent and distinct vertices x and y in Γ there are μ vertices adjacent to both x and y.

Below we give two examples of strongly regular graphs.



C_5 $\qquad$ $J(5, 2)$

Figure 1.7: The graphs C_5 and $J(5, 2)$

The first graph is C_5 and is strongly regular with parameters $v = 5$, $k = 2$, $\lambda = 0$ and $\mu = 1$. The second is the Johnson graph $J(5, 2)$. This is the graph having as vertices all 2-subsets of a set of cardinality 5 where two subsets are adjacent precisely when their intersection has one element. This graph is strongly regular with parameters $v = 10$, $k = 6$, $\lambda = 3$ and $\mu = 4$. For the general definition of Johnson graphs see Definition 1.4.14.

Proposition 1.4.2 ([9, Sec. 9.1.1])**.** *A graph* Γ *is strongly regular with parameters* (v, k, λ, μ) *if and only if its complement,* $\overline{\Gamma}$*, is strongly regular with parameters* $(v, v-k-1, v-2k+\mu-2, v-2k+\lambda)$*.*

Proposition 1.4.3 ([9, Thm. 9.1.2])**.** *For a simple graph* Γ *of order* v*, not complete or edgeless, with adjacency matrix* A*, the following are equivalent:*

(i) The graph Γ *is strongly regular with parameters* (v, k, λ, μ) *for certain integers* k*,* λ *and* μ*.*

(ii) The adjacency matrix of Γ *satisfies* $A^2 = (\lambda - \mu)A + (k - \mu)I + \mu J$ *for certain real numbers* k*,* λ *and* μ*.*

(iii) The adjacency matrix of Γ *has precisely two distinct eigenvalues other than* k*, the degree of* Γ*.*

Proposition 1.4.4 ([9, Thm. 9.1.3])**.** *Let* Γ *be a strongly regular graph with adjacency matrix* A *and parameters* (v, k, λ, μ)*. Let* r *and* s *(*$r > s$*) be the non-trivial eigenvalues of* A *and let* f *and* g *be their respective multiplicities. Then*

(i) $k(k - \lambda - 1) = \mu(v - k - 1)$*,*

(ii) $rs = \mu - k$*,* $r + s = \lambda - \mu$*,*

(iii) $f, g = \frac{1}{2}\left(v - 1 \mp \frac{(r+s)(v-1)+2k}{r-s}\right)$*.*

1.4.2 Association schemes

The previous definition and results regarding strongly regular graphs will help make some of the definitions and results for associations schemes more clear since strongly regular graphs along with their complements provide one of the most basic examples of an association scheme. This presentation of association schemes follows [9, Chap. 11].

Definition 1.4.5. An association scheme with d classes is a finite set, X, together with $d + 1$ relations, $\mathcal{R}_i$ $(0 \leq i \leq d)$, on X such that

(i) $\{\mathcal{R}_0, \mathcal{R}_1, \ldots, \mathcal{R}_d\}$ is a partition of $X \times X$,

(ii) $\mathcal{R}_0 = \{(x, x) : x \in X\}$,

(iii) if $(x, y) \in \mathcal{R}_i$, then also $(y, x) \in \mathcal{R}_i$, for all $x, y \in X$ and $0 \leq i \leq d$,

(iv) for any $(x, y) \in \mathcal{R}_k$ the number of $z \in X$ such that $(x, z) \in \mathcal{R}_i$ and $(z, x) \in \mathcal{R}_j$ is a constant, p_{ij}^k that only depends on i, j and k (not on x and y).

The numbers p_{ij}^k are called the *intersection numbers* of the association scheme.

Based on Definition 1.4.5 one sees that an association scheme with two classes is the same as a pair of complementary strongly regular graphs.

For the next proposition we introduce the notation $n := |X|$, $n_i := p_{ii}^0$ and $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.

Proposition 1.4.6 ([9, Thm. 11.1.1]). *The intersection numbers of an association scheme satisfy*

1. $p_{0j}^k = \delta_{jk}$, $p_{ij}^0 = \delta_{ij} n_j$, $p_{ij}^k = p_{ji}^k$,
2. $\sum_{i=0}^{d} p_{ij}^k = n_j$, $\sum_{j=0}^{d} n_j = n$,
3. $p_{ij}^k n_k = p_{ik}^j n_j$,
4. $\sum_{\ell=0}^{d} p_{ij}^\ell p_{k\ell}^m = \sum_{\ell=0}^{d} p_{kj}^\ell p_{i\ell}^m$.

Definition 1.4.7. The *intersection matrices, denoted $L_0, \ldots, L_d$, of a d-class association scheme are the matrices such that $L_i(k, j) = p_{ij}^k$ for $0 \leq i, j, k \leq d$.*

From the definition it immediately follows that L_0 is the identity matrix. As an example consider the case when $(X, \mathcal{R}_1)$ corresponds to a strongly regular graph with parameters (v, k, λ, μ). Then the intersection matrices for this scheme are

$$L_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ L_1 = \begin{bmatrix} 0 & k & 0 \\ 1 & \lambda & k - \lambda - 1 \\ 0 & \mu & k - \mu \end{bmatrix} \text{ and } L_2 = \begin{bmatrix} 0 & 0 & v - k - 1 \\ 0 & k - \lambda - 1 & v - 2k + \lambda \\ 1 & k - \mu & v - 2k + \mu - 2 \end{bmatrix}.$$

Next, we introduce the *Bose-Mesner algebra*. This is an algebra generated by the adjacency matrices corresponding to relations in an association scheme. The relations, $\mathcal{R}_i$ $(0 \leq i \leq d)$, of an association scheme are described by their adjacency matrices, A_i, of order n defined by

$$A_i(x,y) = \begin{cases} 1 \text{ whenever } (x,y) \in \mathcal{R}_i, \\ 0 \text{ otherwise.} \end{cases}$$

In other words, A_i is the adjacency matrix of the graph $(X, \mathcal{R}_i)$. All the spectral theory discussed in Section 1.3 applies to these graphs.

In terms of adjacency matrices, the axioms of Definition 1.4.5 become

1. $\sum_{i=0}^{d} A_i = J$,
2. $A_0 = I$,
3. $A_i = A_i^{\top}$, for all $0 \leq i \leq d$,
4. $A_i A_j = \sum_{k=0}^{d} p_{ij}^k A_k$, for all $0 \leq i, j \leq d$.

From (i) we see that the $A_0, A_1, \cdots, A_d$ must be linearly independent and using (ii)-(iv) we find that they generate a commutative $(d+1)$-dimensional algebra, $\mathcal{A}$, of symmetric matrices with constant diagonal. This algebra is known as the *Bose-Mesner algebra*.

Since the A_i commute, they can be diagonalized simultaneously, that is there exists a matrix, S, such that for each $A \in \mathcal{A}$, $S^{-1}AS$ is a diagonal matrix. It follows that $\mathcal{A}$ has a unique basis of minimal idempotents, $E_0, \cdots, E_d$. These idempotents satisfy the following:

(i) $E_i E_j = \delta_{ij} E_i$,

(ii) $\sum_{i=0}^{d} E_i = I$.

The matrix $\frac{1}{n}J$ is a minimal idempotent (it is minimal since $\mathrm{rk}(J) = 1$). Take $E_0 = \frac{1}{n}J$. Let P and $\frac{1}{n}Q$ be the matrices relating our two bases for $\mathcal{A}$ in the following way:

$$A_j = \sum_{i=0}^{d} P(i,j)E_i, \tag{1.1}$$

$$E_j = \frac{1}{n}\sum_{i=0}^{d} Q(i,j)A_i. \tag{1.2}$$

Then it follows that $PQ = QP = nI$ and $A_jE_i = P(i,j)E_i$ for $0 \le i, j \le d$. One quickly sees that the $P(i,j)$ are the eigenvalues of A_j whose eigenvectors are the columns of E_i. Thus $\mu_i = \mathrm{rk}(E_i)$ is the multiplicity of the eigenvalue $P(i,j)$ of A_j (provided $P_{ij} \neq P_{kj}$ for $k \neq i$). We immediately see that $\mu_0 = 1$ and $\sum\limits_{i=0}^{d} \mu_i = n$.

Proposition 1.4.8 ([9, Thm. 11.2.1])**.** *The numbers $P(i,j)$ and $Q(i,j)$, for $0 \le i, j \le d$, satisfy*

(i) $P(i,0) = Q(i,0) = 1,\ P(0,i) = n_i,\ Q(0,i) = \mu_i,$

(ii) $P(i,j)P(i,k) = \sum\limits_{\ell=0}^{d} p_{jk}^{\ell} P_{i\ell},$

(iii) $\mu_i P(i,j) = n_j Q(j,i),\ \sum\limits_{\ell=0}^{d} \mu_\ell P(\ell,j)P(\ell,k) = nn_j\delta_{jk},\ \sum\limits_{\ell=0}^{d} n_\ell Q_{\ell j} Q_{\ell k} = n\mu_j\delta_{jk},$

(iv) $|P(i,j)| \le n_j,\ |Q(i,j)| \le \mu_j.$

If $d = 2$, then $(X, \mathcal{R}_1)$ is strongly regular with parameters (v, k, λ, μ) and the matrices P and Q are

$$P = \begin{bmatrix} 1 & k & v-k-1 \\ 1 & r & -r-1 \\ 1 & s & -s-1 \end{bmatrix} \text{ and } Q = \begin{bmatrix} 1 & f & g \\ 1 & f\frac{r}{k} & g\frac{s}{k} \\ 1 & -f\frac{r+1}{v-k-1} & -g\frac{s+1}{v-k-1} \end{bmatrix}.$$

Proposition 1.4.9. *For $0 \le j \le d$, the intersection matrix L_j has eigenvalues $P(i,j)$ for $0 \le i \le d$.*

1.4.3 Distance-regular graphs

In Section 1.4.5 we presented definitions and results regarding association schemes. In this section we will introduce the concept of distance-regular graphs and provide a brief description of their relation to association schemes.

Definition 1.4.10. Let Γ be a k-regular, connected graph with diameter d. If for each $0 \leq i \leq d$ there exist constants, c_i and b_i, such that for any two vertices, x and y, at distance i, there are exactly c_i neighbors of y in $\Gamma_{i-1}(x)$ and exactly b_i neighbors of y in $\Gamma_{i+1}(x)$, then Γ is said to be *distance-regular*. Note that these parameters do not depend on the choice of x or y, they only depend on $d_\Gamma(x,y)$. The *intersection array* of a distance-regular graph with diameter d is $\{b_0 = k, b_1 \cdots, b_{d-1}; c_1 = 1, c_2, \cdots, c_d\}$.

Note here that the parameters for a distance-regular graph of diameter d are related to an underlying d-class association scheme where the relations are given by $\mathcal{R}_i = \{(x,y) : x, y \in V(\Gamma) \text{ and } d_\Gamma(x,y) = i\}$ for $0 \leq i \leq d$. It follows that $b_i = p^i_{i+1,1}$, $c_i = p^i_{i-1,1}$ and $a_i = k - b_i - c_i = p^i_{i,1}$ (we will often use the notation $a_1 = \lambda$ and $c_2 = \mu$), where k is the valency of the graph. The valencies $p^0_{i,i}$, which we referred to as n_i in Section 1.4.5, are usually denoted by k_i when working with distance-regular graphs. The total number of vertices is usually denoted by v. All results mentioned in Section 1.4.5 hold for the parameters of a distance-regular graph. Next we provide some basic results regarding the parameters of distance-regular graphs.

Suppose Γ is a distance-regular graph with parameters a_i, b_i, c_i and k_i for $0 \leq i \leq d$. It is easy to see

$$b_0 = k,\ b_d = c_0 = 0,\ c_1 = 1. \tag{1.3}$$

Simple counting arguments will verify

$$k_0 = 1,\ k_1 = k,\ k_{i+1} = k_i b_i / c_{i+1} \text{ for } 0 \leq i \leq d-1, \tag{1.4}$$

and

$$v = 1 + k_1 + \ldots + k_d. \tag{1.5}$$

Proposition 1.4.11 ([8, Prop. 5.1.1])**.** *Let* Γ *be a distance-regular graph of diameter* $d \geq 3$*. Then the sequence* $1 = k_0, k_1, \cdots, k_d$ *is unimodal.*

We have the following result regarding the eigenvalues of a distance-regular graph.

Lemma 1.4.12 ([9, Thm. 11.2.2])**.** *The quotient matrix with respect to the distance partition of a distance-regular graph* Γ *with intersection array*

$$\{b_0, b_1, \cdots, b_{d-1}; c_1, c_2, \cdots, c_d\}$$

is the tridiagonal matrix

$$L_1 = \begin{bmatrix} a_0 & b_0 & 0 & \dots & 0 \\ c_1 & a_1 & b_1 & \ddots & \vdots \\ 0 & c_2 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & a_{d-1} & b_{d-1} \\ 0 & \dots & 0 & c_d & a_d \end{bmatrix}.$$

The eigenvalues of Γ *are equal to the eigenvalues of* L_1*.*

The adjacency matrices $A_0, A_1, \dots, A_d$ with entries $A_i(x, y) = 1$ if $d_\Gamma(x, y) = i$ and $A_i(x, y) = 0$ otherwise, satisfy the following relations

$$A_0 = I,\ A_1 = A,$$
$$AA_i = c_{i+1}A_{i+1} + a_iA_i + b_{i-1}A_{i-1} \text{ for } 0 \leq i \leq d-1,$$
$$A_0 + A_1 + \dots + A_d = J.$$

Definition 1.4.13. Let Q be a finite set of cardinality $q \geq 2$. The *Hamming graph*, $H(d, q)$, is the graph whose vertex set consists of the elements of Q^d, where two vertices are adjacent when they differ in precisely one coordinate.

The Hamming graph may also be obtained by taking the Cartesian product (see Definition 1.2.15) of d copies of K_q. The Hamming graph $H(d, q)$ is distance-regular of diameter d with intersection parameters

$$b_i = (q-1)(d-i) \text{ and } c_i = i \text{ for } 0 \leq i \leq d$$

and eigenvalues

$$(q-1)d - qi \text{ with multiplicity } \binom{d}{i}(q-1)^i \text{ for } 0 \leq i \leq d.$$

(see for example [8, Thm. 9.2.1] and [9, Section 12.4.1]). The Hamming graphs will be discussed in more detail in Section 2.1.

Definition 1.4.14. Let X be a finite set with cardinality n. The *Johnson graph* $J(n,d)$ is the graph whose vertex set consists of all d-subsets of X, where two vertices are adjacent when they intersect in precisely $d-1$ elements.

The Johnson graph $J(n,d)$ is distance-regular of diameter $\min(d, n-d)$ with intersection parameters

$$b_i = (d-i)(n-d-i) \text{ and } c_i = i^2 \text{ for } 0 \leq i \leq \min(d, n-d)$$

and eigenvalues

$$(d-i)(n-d-i) - i \text{ with multiplicity } \binom{n}{i} - \binom{n}{i-1} \text{ for } 0 \leq i \leq \min(d, n-d).$$

(see for example [8, Thm. 9.1.2] and [9, Section 12.4.2]). The Johnson graphs will be discussed in more detail in Section 2.2.

Chapter 2

THE SMALLEST EIGENVALUES OF HAMMING GRAPHS AND JOHNSON GRAPHS

In this chapter we determine the smallest and second largest in absolute value eigenvalues for graphs in the Hamming scheme and Johnson scheme for a certain range of parameters. Our results settle a 2016 conjecture of Van Dam and Sotirov [29, Conjecture 8] about the smallest eigenvalues of distance-j Hamming graphs and a 1999 conjecture of Karloff [46, Conjecture 2.12] about the smallest eigenvalues of distance-j Johnson graphs.

The research presented in this chapter represents work done in collaboration with Andries Brouwer, Ferdinand Ihringer and Sebastian Cioabă [7]. We give an outline of the contributions made for the results in Section 2.4. Lemma 2.4.1 was developed by Andries Brouwer and elaborates a condition stated in the conjecture by Van Dam and Sotirov. I have included it here with more details. Lemma 2.4.2 was developed by me with contributions from Ferdinand Ihringer. Lemma 2.4.3 extends the work done by Alon and Sudakov in [2] and was developed by me and Ferdinand Ihringer. Lemmas 2.4.4, 2.4.5 and 2.4.6 were all developed by me, as was Corollary 2.4.7. This work settled the conjecture made by Van Dam and Sotirov. This result was improved with contributions from Andries Brouwer and Ferdinand Ihringer in Theorem 2.4.8. I include more details for the proof of Theorem 2.4.8 in this section.

As far as the contributions for Section 2.5, the conditions of Lemma 2.5.1 were developed by me and Andries Brouwer. The proof is due to Andries Brouwer and I have included more details here. My main contribution was recognizing the gap in the induction hypothesis for our original (incorrect) proof of Theorem 2.5.8. This gap is settled by Lemma 2.5.4, which was developed by me, Andries Brouwer and Ferdinand Ihringer. The proofs of all these facts are given here in extended detail.

Before stating and proving the results we give a brief background on both the Hamming and Johnson schemes and the eigenvalues of their corresponding graphs.

2.1 The Hamming Association Scheme

2.1.1 The graphs in the Hamming scheme

Before we give the definition of the Hamming association scheme we must first define the *Hamming distance* between two codewords. Recall that the Hamming graphs were defined in Section 1.4.3.

Definition 2.1.1. Let Q be a finite set of cardinality $q \geq 2$ and let d be a positive integer. The *Hamming distance*, denoted $d_H(x, y)$, between two points $x = (x_1, ..., x_d)$ and $y = (y_1, ..., y_d)$ in Q^d is given by

$$d_H(x, y) = |\{i : 1 \leq i \leq d \text{ and } x_i \neq y_i\}|.$$

In other words, the Hamming distance between two codewords x and y is given by the number of coordinates in which they differ. The Hamming distance satisfies the axioms of a metric on the set Q^d: reflexivity, symmetry and the triangle inequality.

Definition 2.1.2. Let Q be a finite set of cardinality $q \geq 2$ and let d be a positive integer. The Hamming association scheme $\mathcal{H}(d, q)$ is the association scheme defined on the set Q^d with relations $\mathcal{R}_i$, for $0 \leq i \leq d$, defined as follows:

$$\mathcal{R}_i = \{(x, y) \in Q^d \times Q^d : d_H(x, y) = i\}.$$

We denote by $H(d, q, j)$ $(1 \leq j \leq d)$ the graph representing the distance-j relation in the Hamming scheme $\mathcal{H}(d, q)$. That is the graph with vertex set Q^d for which vertices x and y are adjacent if and only if $d_H(x, y) = j$. Note that the graph $H(d, q, 0)$ is the graph having a single loop at each vertex in Q^d and has adjacency matrix equal to I_{q^d}. The graph $H(d, q, 1)$ is the Hamming graph.

Theorem 2.1.3 ([8, Thm. 9.2.1]). *The Hamming graph, $H(d, q, 1)$, is distance-regular of diameter d with intersection parameters*

$$b_i = (q-1)(d-i) \text{ and } c_i = i \text{ for } 0 \leq i \leq d,$$

and eigenvalues

$$(q-1)d - qi \text{ with multiplicity } \binom{d}{i}(q-1)^i \text{ for } 0 \leq i \leq d.$$

Below we give an example of the graphs (except the trivial graph $H(3, 2, 0)$) making up the Hamming scheme $\mathcal{H}(3, 2)$.

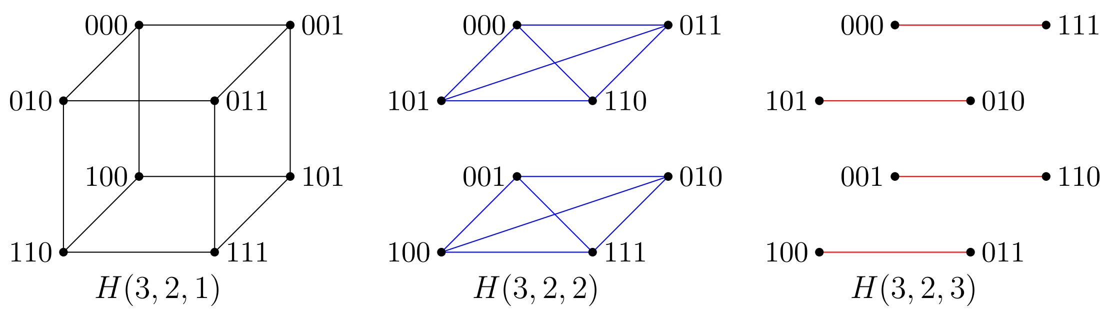


Figure 2.1: Hamming scheme $\mathcal{H}(3, 2)$

2.1.2 Krawtchouk polynomials

The Krawtchouk polynomials $K_j^d(i)$ are a family of orthogonal polynomials arising in the Hamming scheme. These are polynomials of degree j in the variable i. Below we provide basic results for these polynomials that will help us with proofs in Section 2.4.

Definition 2.1.4. The q-ary Krawtchouk polynomials $K_j^d(i)$ (of degree j) are defined as the coefficients of the following polynomial:

$$(1-x)^i(1+(q-1)x)^{d-i} = \sum_{j=0}^{d} K_j^d(i)x^j. \tag{2.1}$$

Proposition 2.1.5 ([30, p. 39] and [31, p. 267])**.** *The following are explicit expressions for the Krawtchouk polynomials:*

$$K_j^d(i) = \sum_{h=0}^{j} (-1)^h (q-1)^{j-h} \binom{i}{h}\binom{d-i}{j-h} \tag{2.2}$$

$$= \sum_{h=0}^{j} (-q)^h (q-1)^{j-h} \binom{i}{h}\binom{d-h}{j-h} \tag{2.3}$$

$$= \sum_{h=0}^{j} (-1)^h q^{j-h} \binom{d-i}{j-h}\binom{d-j+h}{h}. \tag{2.4}$$

Proof. To see that equation (2.2) is true, we first note that the coefficient of x^{j-h} in the expansion of $(1+(q-1)x)^{d-i}$ is $\binom{d-i}{j-h}(q-1)^{j-h}$. Similarly, the coefficient of x^h in the expansion of $(1-x)^i$ is $\binom{i}{h}(-1)^h$. Hence, the coefficient of $x^j = x^{j-h}x^h$ in $(1-x)^i(1+(q-1)x)^{d-i}$ is

$$K_j^d(i) = \sum_{h=0}^{j} (-1)^h (q-1)^{j-h} \binom{i}{h}\binom{d-i}{j-h}.$$

To obtain equation (2.3), write $(1-x)^i = ((1+(q-1)x)-qx)^i$. We now calculate the coefficient of x^j in the expansion of $(1+(q-1)x)^{d-i}((1+(q-1)x)-qx)^i$. Observe that

$$(1+(q-1)x)^{d-i}((1+(q-1)x)-qx)^i = (1+(q-1)x)^{d-i}\left(\sum_{h=0}^{i}\binom{i}{h}(-q)^h x^h (1+(q-1)x)^{i-h}\right)$$

$$= \sum_{h=0}^{i}\binom{i}{h}(-q)^h x^h (1+(q-1)x)^{d-h}.$$

From here we find that the coefficient of x^j in $\sum_{h=0}^{i}\binom{i}{h}(-q)^h x^h(1+(q-1)x)^{d-h}$ is

$$K_j^d(i) = \sum_{h=0}^{i} (-q)^h (q-1)^{j-h} \binom{i}{h}\binom{d-h}{j-h} = \sum_{h=0}^{j} (-q)^h (q-1)^{j-h} \binom{i}{h}\binom{d-h}{j-h}.$$

To obtain equation (2.4), write $(1+(q-1)x)^{d-i} = ((1-x)+qx)^{d-i}$. We now calculate the coefficient of x^j in the expansion of $(1-x)^i((1-x)+qx)^{d-i}$. Observe that

$$(1-x)^i((1-x)+qx)^{d-i} = (1-x)^i \sum_{h=0}^{d-i}\binom{d-i}{h} q^h x^h (1-x)^{d-i-h}$$

$$= \sum_{h=0}^{d-i}\binom{d-i}{h} q^h x^h (1-x)^{d-h}.$$

From here we find that the coefficient of x^j in $\sum_{h=0}^{d-i} \binom{d-i}{h} q^h x^h (1-x)^{d-h}$ is

$$K_j^d(i) = \sum_{h=0}^{d-i} (-1)^{j-h} q^h \binom{d-i}{h} \binom{d-h}{j-h} = \sum_{h=0}^{j} (-1)^h q^{j-h} \binom{d-i}{j-h} \binom{d-j+h}{h}.$$

□

We will mainly use expression (2.2) given for Krawtchouk polynomials throughout the rest of this section and Section 2.4. The Krawtchouk polynomials have the following symmetry.

Lemma 2.1.6. *For $q \geq 2$, $d \geq 1$ and $0 \leq i, j \leq d$, the Krawtchouk polynomials satisfy the equation $(q-1)^i \binom{d}{i} K_j^d(i) = (q-1)^j \binom{d}{j} K_i^d(j)$. In particular, the value of the polynomials $K_j^d(i)$ and $K_i^d(j)$ evaluated for integers $0 \leq i, j \leq d$ have the same sign.*

Proof. The statement above is equivalent to

$$(q-1)^{j-i} \frac{\binom{d}{j}}{\binom{d}{i}} K_i^d(j) = K_j^d(i).$$

Using equation (2.2) we observe that

$$\begin{aligned} (q-1)^{j-i} \frac{\binom{d}{j}}{\binom{d}{i}} K_i^d(j) &= (q-1)^{j-i} \frac{\binom{d}{j}}{\binom{d}{i}} \sum_{h=0}^{i} (-1)^h (q-1)^{i-h} \binom{j}{h} \binom{d-j}{i-h} \\ &= \sum_{h=0}^{i} (-1)^h (q-1)^{j-h} \frac{\binom{d}{j}\binom{j}{h}\binom{d-j}{i-h}}{\binom{d}{i}}. \end{aligned}$$

Expanding and simplifying $\binom{d}{j}\binom{j}{h}\binom{d-j}{i-h}/\binom{d}{i}$ we see that it reduces to $\binom{i}{h}\binom{d-i}{j-h}$ and the claim follows by (2.2). □

The following lemma provides another symmetry for the Krawtchouk polynomials.

Lemma 2.1.7. *For $q \geq 2$, $d \geq 1$ and $0 \leq i, j \leq d$, the Krawtchouk polynomials satisfy the equation $K_{d-j}^d(i) = (-1)^{i-j}(q-1)^{d-i-j} K_j^d(d-i)$.*

Proof. We appeal directly to expression (2.2) given for $K_j^d(i)$. Observe that

$$\begin{aligned}(-1)^{i-j}(q-1)^{d-i-j}K_j^d(d-i) &= (-1)^{i-j}(q-1)^{d-i-j}\sum_{h=0}^{j}(-1)^h(q-1)^{j-h}\binom{d-i}{h}\binom{i}{j-h}\\ &= \sum_{h=0}^{j}(-1)^{h+i-j}(q-1)^{d-i-h}\binom{d-i}{h}\binom{i}{j-h}\end{aligned}$$

Making the substitution $h = j - i + \ell$, the previous sum is equal to

$$\sum_{j-i+\ell=0}^{i}(-1)^\ell(q-1)^{d-j-\ell}\binom{d-i}{j-i+\ell}\binom{i}{i-\ell} = \sum_{\ell=0}^{d-j}(-1)^\ell(q-1)^{d-j-\ell}\binom{d-i}{d-j-\ell}\binom{i}{\ell}.$$

Since the right hand side of the above equation is equal to $K_{d-j}^d(i)$ as seen in (2.2), the claim is established. □

The Krawtchouk polynomials satisfy the following three term recurrence.

Lemma 2.1.8. *For $0 \leq i \leq d$, $K_0^d(i) = 1$ and $K_1^d(i) = (q-1)d - qi$. Then for $1 \leq j \leq d-1$ we have $(j+1)K_{j+1}^d(i) = [(d-j)(q-1)+j-qi]K_j^d(i) - (q-1)(d-j+1)K_{j-1}^d(i)$.*

Proof. The fact that $K_0^d(i) = 1$ and $K_1^d(i) = (q-1)d - qi$ may be obtained directly from equation (2.2). To establish the recurrence we begin by differentiating $\sum_{h=0}^{d} K_j^d(i)x^j = (1-x)^i(1+(q-1)x)^{d-i}$ with respect to x to get

$$\sum_{j=1}^{d} jK_j^d(i)x^{j-1} = (1-x)^{i-1}(1+(q-1)x)^{d-i-1}(-qi - d(q-1)(x-1)).$$

Multiplying by $(1-x)(1+(q-1)x)$ we obtain

$$\sum_{j=1}^{d} jK_j^d(i)(1-x)(1+(q-1)x)x^{j-1} = (1-x)^i(1+(q-1)x)^{d-i}(-qi - d(q-1)(x-1)).$$

Writing $-qi - d(q-1)(x-1) = -qi + d(q-1) - d(q-1)x$ and substituting $\sum_{j=0}^{d} K_j^d(i)x^j = (1-x)^i(1+(q-1)x)^{d-i}$, we end up with

$$\begin{gathered}\sum_{j=0}^{d}\left[(-qi + d(q-1))K_j^d(i)x^j - d(q-1)K_j^d(i)x^{j+1}\right]\\ \parallel \\ \sum_{j=0}^{d}\left[(j+1)K_{j+1}^d(i)x^j + (j+1)(q-2)K_{j+1}^d(i)x^{j+1} - (j+1)(q-1)K_{j+1}^d(i)x^{j+2}\right].\end{gathered}$$

Calculating the coefficient of x^j on both sides of the equation gives the desired result. $\square$

The Krawtchouk polynomials also satisfy another three term recurrence.

Lemma 2.1.9. *Let $1 \leq i, j \leq d-1$. Then*

$$(q-1)(d-i)K_j^d(i+1) - (i + (q-1)(d-i) - qj)K_j^d(i) + iK_j^d(i-1) = 0.$$

Proof. Using Lemma 2.1.6 we know that $K_j^d(i) = (q-1)^{j-i}\frac{\binom{d}{j}}{\binom{d}{i}}K_i^d(j)$. Now substituting this into the recurrence established in Lemma 2.1.8 we obtain

$$(j+1)(q-1)^{j-i+1}\frac{\binom{d}{j+1}}{\binom{d}{i}}K_i^d(j+1)$$
$$\|$$
$$[(d-j)(q-1)+j-qi](q-1)^{j-i}\frac{\binom{d}{j}}{\binom{d}{i}}K_i^d(j) - (q-1)(d-j+1)(q-1)^{j-i-1}\frac{\binom{d}{j-1}}{\binom{d}{i}}K_i^d(j-1).$$

Multiplying through by $\frac{\binom{d}{i}}{\binom{d}{j}}$ and dividing by $(q-1)^{j-i}$ we see that this reduces to

$$(q-1)(d-j)K_i^d(j+1) = [(d-j)(q-1)+j-qi]K_i^d(j) - jK_i^d(j-1),$$

from which the conclusion follows. $\square$

It is known that the eigenvalues of the graphs in the Hamming scheme may be expressed in terms of the Krawtchouk polynomials (see [30, Thm. 4.2] or [59, Sec. 2.2–2.4]). Here we give a brief overview of the details for the argument demonstrating that the eigenvalues of the graph $H(d,q,j)$ are given by $K_j^d(i)$ where $0 \leq i \leq d$. Our argument is based on the notes presented by Bill Martin in [59]. To start, we endow the set Q^d with the structure of the abelian group $\mathbb{Z}_q^d$. For $x \in \mathbb{Z}_q^d$, consider the q^d-dimensional vector χ_x with entries indexed by the elements of $\mathbb{Z}_q^d$ and given by $\chi_x(y) = \omega^{x\cdot y}$, where ω is a primitive q-th root of unity and $x \cdot y = (x_1y_1 + \ldots + x_dy_d)$ mod q. Note that each χ_x is a complex character, meaning that $\chi_x(y+z) = \chi_x(y)\chi_x(z)$ for any $y, z \in \mathbb{Z}_q^d$ (see [64]).

Let A_j be the adjacency matrix for $H(d,q,j)$. We will prove that each χ_x (seen as a vector indexed by the vertices of $H(d,q,j)$) is an eigenvector of A_j. Below we compute the y-th entry of the vector $A_j\chi_x$. Suppose $wt(x) = i$ (here wt represents the weight of a vector i.e. the number of nonzero coordinates). We have that

$$\begin{aligned}
(A_j\chi_x)(y) &= \sum_z A_j(y,z)\chi_x(z) \\
&= \sum_{d_H(z,y)=j} \omega^{x\cdot z} \\
&= \sum_{wt(e)=j} \omega^{x\cdot(y+e)} \\
&= \sum_{wt(e)=j} \omega^{x\cdot y + x\cdot e} \\
&= \chi_x(y) \sum_{wt(e)=j} \omega^{x\cdot e}.
\end{aligned}$$

Now we must calculate the value of $\sum_{wt(e)=j} \omega^{x\cdot e}$. We do this by considering the coordinates in which x and e share support. Suppose x and e share support in h coordinates. Since x has weight i, the number of possibilities for where x and e share support is $\binom{i}{h}$. Without loss of generality, suppose x and e share support in the first h coordinates. Note that for each value of h we find

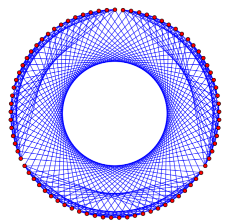
$$\sum_{e_1,\ldots,e_h\neq 0} \omega^{x_1e_1+\ldots+x_he_h} = \left(\sum_{e_1\neq 0}\omega^{x_1e_1}\right)\left(\sum_{e_2\neq 0}\omega^{x_2e_2}\right)\cdots\left(\sum_{e_h\neq 0}\omega^{x_he_h}\right) = (-1)^h.$$

Now, for the remaining $d-i$ coordinates for which x has entries equal to 0, we know $j-h$ of these coordinates must be nonzero in e since $wt(e) = j$. The number of ways this can be achieved is $(q-1)^{j-h}\binom{d-i}{j-h}$. So we see

$$(A_j\chi_x)(y) = \left(\sum_{h=0}(-1)^h(q-1)^{j-h}\binom{i}{h}\binom{d-i}{j-h}\right)\chi_x(y).$$

It follows, by Proposition 2.1.5 that the eigenvalues of $H(d,q,j)$ are given by $K_j^d(i)$ where $0 \leq i \leq d$.

Below we give an example of the first eigenmatrix for the Hamming scheme $\mathcal{H}(4,3)$ as well as a picture of the Hamming graph $H(4,3,1)$.



$$P = \begin{bmatrix} 1 & 8 & 24 & 32 & 16 \\ 1 & 5 & 6 & -4 & -8 \\ 1 & 2 & -3 & -4 & 4 \\ 1 & -1 & -3 & 5 & -2 \\ 1 & -4 & 6 & -4 & 1 \end{bmatrix}$$

Figure 2.2: $H(4, 3, 1)$ with the first eigenmatrix of $\mathcal{H}(4, 3)$

As we mentioned before, the (i, j)-entry of this matrix is $K_j^4(i)$. This means the entries of the first row are $K_j^4(0)$, the valency of each graph in the scheme. Also the entries in the j^{th} column of P give the eigenvalues for the distance-j graph in the scheme. Finally, the second row of P consists of $K_j^4(1)$ for $0 \leq j \leq 4$. These entries will be the ones we focus on in Section 2.4.

2.2 The Johnson Association Scheme

2.2.1 The graphs in the Johnson scheme

In this section we will use the notation $[n] = \{1, \cdots, n\}$ and $\binom{[n]}{d}$ to denote the collection of d-subsets of $[n]$. Recall that the Johnson graphs were defined in Section 1.4.3.

Definition 2.2.1. Let X be a finite set of size n and d be an integer such that $0 \leq d \leq \frac{n}{2}$. The Johnson association scheme, $\mathcal{J}(n, d)$, is the association scheme on the set $\binom{[n]}{d}$ with relations $\mathcal{R}_i$ defined as follows:

$$\mathcal{R}_i = \left\{ (x, y) \in \binom{[n]}{d} \times \binom{[n]}{d} : |x \cap y| = d - i \right\}.$$

We will use the notation $J(n, d, j)$ $(1 \leq j \leq d)$ to denote the graph representing the distance-j relation in the Johnson scheme $\mathcal{J}(n, d)$. That is the graph with vertex set $\binom{[n]}{d}$ for which vertices x and y are adjacent if and only if $|x \cap y| = d - j$. Note that the graph $J(n, d, 0)$ is the graph having a single loop at each vertex in $\binom{[n]}{d}$ and has adjacency matrix equal to $I_{\binom{n}{d}}$. The graph $J(n, d, 1)$ is the Johnson graph.

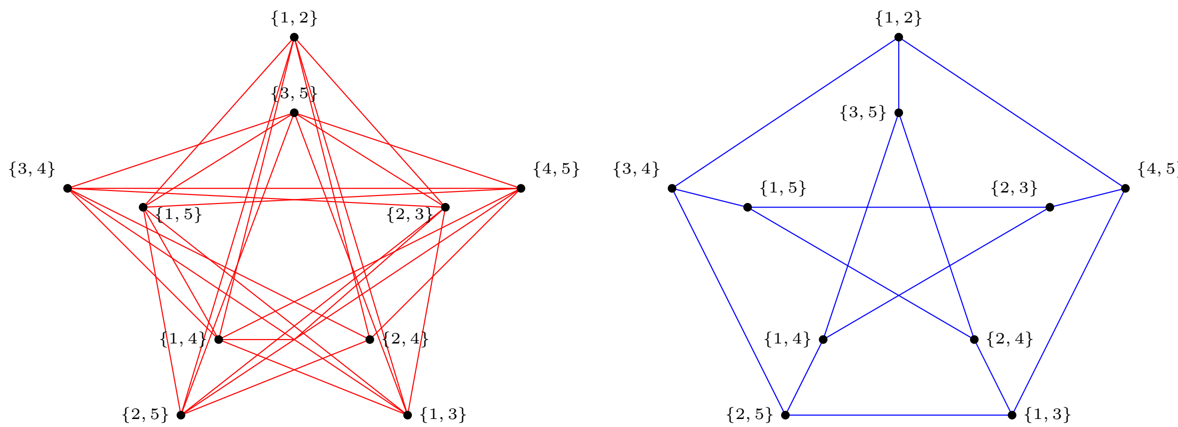
Theorem 2.2.2 ([8, Thm. 9.1.2])**.** *The Johnson graph, $J(n, d, 1)$, is distance-regular of diameter $\min(d, n-d)$ with intersection parameters*

$$b_i = (d-i)(n-d-i) \text{ and } c_i = i^2 \text{ for } 0 \le i \le d,$$

and eigenvalues

$$(d-i)(n-d-i) - i \text{ with multiplicity } \binom{n}{i} - \binom{n}{i-1} \text{ for } 0 \le i \le d.$$

Below we give an example of the graphs making up the Johnson scheme $\mathcal{J}(5, 2)$.



$J(5, 2, 1)$ $J(5, 2, 2)$

Figure 2.3: Johnson scheme $\mathcal{J}(5, 2)$

2.2.2 Eberlein polynomials

The Eberlein polynomials $E_j^{n,d}(i)$ are a family of orthogonal polynomials arising in the Johnson scheme. These are polynomials of degree $2j$ in the variable i. Below we provide basic results for these polynomials that will help us with proofs in Section 2.5. We define the Eberlein polynomials in the following way (see [69]).

Definition 2.2.3. The Eberlein polynomial $E_j^{n,d}(i)$ is defined as the coefficient of the term $x^j y^j$ in the expansion of the following bivariate polynomial:

$$(1-xy)^i(1+x)^{d-i}(1+y)^{n-d-i}. \tag{2.5}$$

Proposition 2.2.4 ([30, p. 48] and [46, Thm. 2.1]). *The following are explicit expressions for the Eberlein polynomials.*

$$E_j^{n,d}(i) = \sum_{h=0}^{j}(-1)^h\binom{i}{h}\binom{d-i}{j-h}\binom{n-d-i}{j-h} \tag{2.6}$$

$$= \sum_{h=0}^{j}(-1)^{j-h}\binom{d-i}{h}\binom{d-h}{j-h}\binom{n-d-i+h}{h} \tag{2.7}$$

$$= \sum_{h=0}^{i}(-1)^{i-h}\binom{i}{h}\binom{d-h}{j}\binom{n-d-i+h}{n-d-j}. \tag{2.8}$$

Proof. To see that equation (2.6) is true, we first note that the coefficient of $(xy)^h$ in the expansion of $(1-xy)^i$ is $(-1)^h\binom{i}{h}$. Similarly, the coefficient of x^{j-h} in the expansion of $(1+x)^{d-i}$ is $\binom{d-i}{j-h}$ and the coefficient of y^{j-h} in the expansion of $(1+y)^{n-d-i}$ is $\binom{n-d-i}{j-h}$. Hence the coefficient of x^jy^j in $(1-xy)^i(1+x)^{d-i}(1+y)^{n-d-i}$ is

$$E_j^{n,d}(i) = \sum_{h=0}^{j}(-1)^h\binom{i}{h}\binom{d-i}{j-h}\binom{n-d-i}{j-h}.$$

To obtain equation (2.7), write $(1+x) = (1-xy) + x(1+y)$. We now calculate the coefficient of x^jy^j in the expansion of $(1-xy)^i((1-xy)+x(1+y)^{d-i})(1+y)^{n-d-i}$. Observe that

$$(1-xy)^i((1-xy)+x(1+y)^{d-i})(1+y)^{n-d-i} = (1-xy)^i(1+y)^{n-d-i}\sum_{h=0}^{d-i}\binom{d-i}{h}x^h(1+y)^h(1-xy)^{d-i-h}$$

$$= \sum_{h=0}^{d-i}\binom{d-i}{h}x^h(1+y)^{n-d-i+h}(1-xy)^{d-h}.$$

From this we find that the coefficient of x^jy^j is

$$E_j^{n,d}(i) = \sum_{h=0}^{d-i}(-1)^{j-h}\binom{d-i}{h}\binom{d-h}{j-h}\binom{n-d-i+h}{h} = \sum_{h=0}^{j}(-1)^{j-h}\binom{d-i}{h}\binom{d-h}{j-h}\binom{n-d-i+h}{h}.$$

To obtain equation (2.8), write $(1-xy) = (1+y) - y(1+x)$. We now calculate the coefficient of x^jy^j in the expansion of $((1+y)-y(1+x))^i(1+x)^{d-i}(1+y)^{n-d-i}$. Observe that

$$((1+y)-y(1+x))^i(1+x)^{d-i}(1+y)^{n-d-i} = (1+x)^{d-i}(1+y)^{n-d-i}\sum_{h=0}^{i}(-1)^{i-h}\binom{i}{h}(1+y)^hy^{i-h}(1+x)^{i-h}$$

$$= \sum_{h=0}^{i}(-1)^{i-h}\binom{i}{h}y^{i-h}(1+x)^{d-h}(1+y)^{n-d-i+h}.$$

From this we find that the coefficient of $x^j y^j$ is

$$E_j^{n,d}(i) = \sum_{h=0}^{i}(-1)^{i-h}\binom{i}{h}\binom{d-h}{j}\binom{n-d-i+h}{j-i+h} = \sum_{h=0}^{i}(-1)^{i-h}\binom{i}{h}\binom{d-h}{j}\binom{n-d-i+h}{n-d-j}.$$

□

We will make use of the following recurrence for Eberlein polynomials in the proof of Theorem 2.5.8.

Lemma 2.2.5. *Let $i, j \geq 1$. Then $E_j^{n+2,d+1}(i) = E_j^{n,d}(i-1) - E_{j-1}^{n,d}(i-1)$.*

Proof. Using $E_j^{n,d}(i) = \sum_{h=0}^{j}(-1)^h\binom{i}{h}\binom{d-i}{j-h}\binom{n-d-i}{j-h}$ the claim follows by applying the identity $\binom{i}{h} = \binom{i-1}{h} + \binom{i-1}{h-1}$. □

It is known that the eigenvalues of the graphs in the Johnson scheme may be expressed in terms of the Eberlein polynomials (see [30, Thm. 4.6] or [38, Sec. 6.5]). Here we give a brief overview of the details for the argument demonstrating that the eigenvalues of the graph $J(n, d, j)$ are given by $E_j^{n,d}(i)$ where $0 \leq i \leq d$. Our argument follows the details presented in Section 4.2 of [30]. To calculate the eigenvalues of graphs in the Johnson scheme, we begin by introducing *inclusion matrices* $W_{s,t}$. Let $W_{s,t}$ denote the $\binom{n}{s} \times \binom{n}{t}$ matrix such that

$$W_{s,t}(S,T) = \begin{cases} 1 \text{ if } S \subseteq T \\ 0 \text{ otherwise.} \end{cases}$$

Proposition 2.2.6 (Vandermonde's Identity)**.** *Let m, n and r be nonnegative integers. Then*

$$\binom{m+n}{r} = \sum_{h=0}^{r}\binom{m}{h}\binom{n}{r-h}.$$

Lemma 2.2.7. *For $0 \leq s, t \leq d$, we have*

$$W_{s,d}W_{t,d}^{\top} = \sum_{r=0}^{\min(s,t)}\binom{n-s-t}{n-d-r}W_{r,s}^{\top}W_{r,t}. \tag{2.9}$$

Proof. Let X be an s-subset of $[n]$ and Y be a t-subset of $[n]$. Then

$$\begin{aligned}
(W_{s,d}W_{t,d}^{\top})(X,Y) &= \sum_{Z\in\binom{[n]}{d}} W_{s,d}(X,Z)W_{t,d}(Y,Z) \\
&= \text{The number of } Z \in \binom{[n]}{d} \text{ such that } X \subseteq Z \text{ and } Y \subseteq Z \\
&= \binom{n-|X\cup Y|}{d-|X\cup Y|}.
\end{aligned}$$

Next we establish that

$$\begin{aligned}
(W_{r,s}^{\top}W_{r,t})(X,Y) &= \sum_{Z\in\binom{[n]}{r}} W_{r,s}(Z,X)W_{r,t}(Z,Y) \\
&= \text{The number of } Z \text{ such that } Z \subseteq X \text{ and } Z \subseteq Y \\
&= \binom{|X\cap Y|}{r}.
\end{aligned}$$

Now, by Vandermonde's identity, observe that

$$\begin{aligned}
\sum_{r=0}^{\min(s,t)} \binom{n-s-t}{n-d-r}(W_{r,s}^{\top}W_{r,t})(X,Y) &= \sum_{r=0}^{\min(s,t)} \binom{n-s-t}{n-d-r}\binom{|X\cap Y|}{r} \\
&= \binom{n-s-t+|X\cap Y|}{n-d} \\
&= \binom{n-|X\cup Y|}{n-d} \\
&= \binom{n-|X\cup Y|}{d-|X\cup Y|}.
\end{aligned}$$

Thus, $W_{s,d}W_{t,d}^{\top} = \sum_{r=0}^{\min(s,t)} \binom{n-s-t}{n-d-r} W_{r,s}^{\top}W_{r,t}$. ☐

Lemma 2.2.8. *Define matrices* $C_h = W_{h,d}^{\top}W_{h,d}$ *for* $0 \le h \le d$. *Then*

$$C_hC_g = \sum_{r=0}^{\min(g,h)} \binom{n-h-g}{n-d-r}\binom{d-r}{h-r}\binom{d-r}{g-r}C_r. \tag{2.10}$$

Proof. Observe that

$$\begin{aligned}
C_h C_g &= (W_{h,d}^\top W_{h,d})(W_{g,d}^\top W_{g,d}) \\
&= W_{h,d}^\top \left(\sum_{r=0}^{\min(g,h)} \binom{n-h-g}{n-d-r} W_{r,h}^\top W_{r,g} \right) W_{g,d} \\
&= \sum_{r=0}^{\min(g,h)} \binom{n-h-g}{n-d-r} (W_{h,d}^\top W_{r,h}^\top)(W_{r,g} W_{g,d}) \\
&= \sum_{r=0}^{\min(g,h)} \binom{n-h-g}{n-d-r} (W_{r,h} W_{h,d})^\top (W_{r,g} W_{g,d}) \\
&= \sum_{r=0}^{\min(g,h)} \binom{n-h-g}{n-d-r} \binom{d-r}{h-r} W_{r,d}^\top \binom{d-r}{g-r} W_{r,d} \\
&= \sum_{r=0}^{\min(g,h)} \binom{n-h-g}{n-d-r} \binom{d-r}{h-r} \binom{d-r}{g-r} W_{r,d}^\top W_{r,d} \\
&= \sum_{r=0}^{\min(g,h)} \binom{n-h-g}{n-d-r} \binom{d-r}{h-r} \binom{d-r}{g-r} C_r.
\end{aligned}$$

□

Next we must find a way to write A_{d-j} as a linear combination of C_h for $0 \leq h \leq d$. This is provided by the following lemma.

Lemma 2.2.9. *For $0 \leq j \leq d$, we have $A_{d-j} = \sum_{h=j}^{d} (-1)^{h-j} \binom{h}{j} C_h$.*

Proof. First, we note that $C_h = \sum_{r=h}^{d} \binom{r}{h} A_{d-r}$. Indeed,

$$A_{d-r}(X,Y) = \begin{cases} 1 \text{ if } |X \cap Y| = r, \\ 0 \text{ otherwise,} \end{cases}$$

so $\left(\sum_{r=h}^{d} \binom{r}{h} A_{d-r} \right)(X,Y) = \binom{|X \cap Y|}{h} = C_h(X,Y)$. Using this, it follows that

$$\sum_{h=j}^{d} (-1)^{h-j} \binom{h}{j} C_h = \sum_{h=j}^{d} (-1)^{h-j} \binom{h}{j} \left[\sum_{r=h}^{d} \binom{r}{h} A_{d-r} \right]. \tag{2.11}$$

We now show that the (X,Y)-entry of the right hand side of equation (2.11) is equal to one if $|X\cap Y| = j$ and 0 otherwise.

(i) Suppose $|X\cap Y| = j$. Then $\left(\sum_{r=h}^{d}\binom{r}{h}A_{d-r}\right)(X,Y) = \binom{j}{h}$. So the right hand side of equation (2.11) is $\sum_{h=j}^{d}(-1)^{h-j}\binom{h}{j}\binom{j}{h} = 1$.

(ii) If $|X\cap Y| < j$, then since $r \geq h \geq j$, it follows that $A_{d-r}(X,Y) = 0$ for all $r \geq j$. Hence the right hand side of equation (2.11) is equal to zero.

(iii) Now suppose $z = |X\cap Y| > j$. Then $\sum_{r=h}^{d}\binom{r}{h}A_{d-r}(X,Y) = \binom{z}{h}$. So it follows that

$$
\begin{aligned}
\sum_{h=j}^{d}(-1)^{h-j}\binom{h}{j}\left[\sum_{r=h}^{d}\binom{r}{h}A_{d-r}(X,Y)\right] &= \sum_{h=j}^{d}(-1)^{h-j}\binom{h}{j}\binom{z}{h}\\
&= \sum_{h=j}^{z}(-1)^{h-j}\binom{h}{j}\binom{z}{h}\\
&= \sum_{h=j}^{z}(-1)^{h-j}\binom{z}{j}\binom{z-j}{h-j}\\
&= \binom{z}{j}\sum_{h=j}^{z}(-1)^{h-j}\binom{z-j}{h-j}\\
&= \binom{z}{j}\sum_{i=0}^{z-j}(-1)^{i}\binom{z-j}{i}\\
&= \binom{z}{j}(1-1)^{z-j}\\
&= 0.
\end{aligned}
$$

□

Since the matrices, C_h and C_g are in the Bose-Mesner algebra for $\mathcal{J}(n,d)$ we can write

$$C_h = \sum_{i=0}^{d}\gamma_{h,i}E_i \text{ and } C_g = \sum_{k=0}^{d}\gamma_{g,k}E_k,$$

where $\gamma_{h,i}, \gamma_{g,k} \in \mathbb{R}$ and $E_0, E_1, \cdots, E_d$ are minimal idempotents. Using $E_iE_k = \delta_{ik}E_k$, we find

$$C_hC_g = \gamma_{h,g}C_g + \sum_{k=0}^{d}\gamma_{g,k}(\gamma_{h,k} - \gamma_{h,g})E_k. \tag{2.12}$$

Comparing the coefficient of C_g in the right hand side of equation (2.10) with the coefficient of C_g in the right hand side of equation (2.12), we find $\gamma_{h,g} = \binom{d-g}{h-g}\binom{n-h-g}{d-h}$. It follows that

$$C_h = \sum_{i=0}^{d} \binom{d-i}{h-i}\binom{n-h-i}{d-h} E_i. \tag{2.13}$$

By Lemma 2.2.9 we know $A_{d-j} = \sum_{h=j}^{d} (-1)^{h-j}\binom{h}{j} C_h$. Substituting the right hand side of equation (2.13) for C_h we find

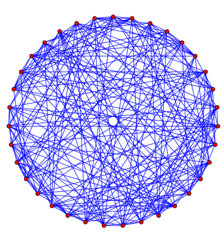
$$\begin{aligned} A_{d-j} &= \sum_{h=j}^{d} (-1)^{h-j}\binom{h}{j}\left[\sum_{i=0}^{d}\binom{d-i}{h-i}\binom{n-h-i}{d-h}E_i\right] \\ &= \sum_{i=0}^{d}\left[\sum_{h=j}^{d}(-1)^{h-j}\binom{h}{j}\binom{d-i}{h-i}\binom{n-h-i}{d-h}\right]E_i. \end{aligned}$$

From Equation (1.1) we know that

$$\sum_{h=j}^{d} (-1)^{h-j}\binom{h}{j}\binom{d-i}{h-i}\binom{n-h-i}{d-h} \text{ for } (0 \le i \le d) \tag{2.14}$$

are the eigenvalues of A_{d-j} corresponding to the i-th eigenspace. But (2.14) is equal to $E_{d-j}^{n,d}(i)$ using expression (2.7). Hence, the eigenvalues of A_j are given by $E_j^{n,d}(i)$ for $0 \le i \le d$. For an alternative proof of this fact, we refer the reader to [38, Sec. 6.4].

Below is an example of the first eigenmatrix for the Johnson scheme $\mathcal{J}(7,3)$ as well as a picture of the Johnson graph $J(7,3,1)$.



$$P = \begin{bmatrix} 1 & 12 & 18 & 4 \\ 1 & 5 & -3 & -3 \\ 1 & 0 & -3 & 2 \\ 1 & -3 & 3 & -1 \end{bmatrix}$$

Figure 2.4: $J(7,3,1)$ with the first eigenmatrix of $\mathcal{J}(7,3)$

As we mentioned before, the (i,j)-entry of this matrix is $E_j^{7,3}(i)$. This means the entries of the first row are $E_j^{7,3}(0)$, the valency of each graph in the scheme. Also the entries in the j^{th} column of P give the eigenvalues for the distance-j graph in the scheme. Finally, the second row of P consists of $E_j^{7,3}(1)$ for $0 \leq j \leq 3$. These entries will be the ones we focus on in Section 2.5.

2.3 The Max-Cut Problem

The *Max-Cut problem* is a well studied problem in graph theory, optimization and computer science. The problem can be stated as follows: given a graph $\Gamma = (V, E)$ determine $mc(\Gamma) = \max_{S \subset V} e(S, V \setminus S)$, which is the maximum number of edges of a bipartite subgraph of Γ.

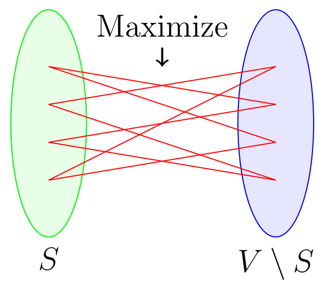


Figure 2.5: Max-Cut visualization

The Max-Cut problem is known to be NP-Hard (see [3, 34, 47]). So it is important to develop efficient algorithms to find good approximate solutions.

Using the following greedy algorithm we may obtain a cut of size at least $\frac{1}{2}|E|$. Take an arbitrary partition $V = S \cup S^c$. Now pick an arbitrary vertex v in either side of the partition. Suppose that the vertex we pick is in S. Of the edges incident with v, some may be incident with vertices in S and others may be incident with vertices in S^c. Now, if v has more neighbors in S than in S^c, then we can increase the size of our cut by moving v to S^c. This process can be repeated for all vertices until there is no such vertex whose movement will increase the size of the cut.

The same vertex may be moved several times. However, the process must terminate since each move must increase the size of the cut. Since the size of the cut can be at most $|E|$ we know the number of movements must be finite. Once we are done we know each vertex has at least half of its incident edges joining it to a vertex in the complementary set. Therefore, the cut obtained by this algorithm must have size at least $\frac{1}{2}|E|$.

This is a good start to approximating $mc(\Gamma)$. However, due to an algorithm proposed by Goemans and Williamson [40], we now know that we can do much better. We give a brief overview of the work done by Goemans and Williamson below.

Let $\Gamma = (V, E)$ be a graph with $|V| = n$ and let (S, S^c) denote a cut. By assigning a variable $x_i = +1$ to each vertex $i \in S$ and $x_i = -1$ to each vertex $v \in S^c$ solving the Max-Cut problem is equivalent to the following integer program:

$$\max_{x_i \in \{-1,1\}} \sum_{\{i,j\} \in E} \frac{1 - x_i x_j}{2}, \text{ for } 1 \le i \le n. \tag{2.15}$$

In fact, if we can find a solution to this problem, we can actually construct a maximum cut based on the solution. As we mentioned before, this problem is known to be NP-hard. However, we can relax it to the following semidefinite program:

$$\max_{\|v_i\|^2 = 1} \sum_{\{i,j\} \in E} \frac{1 - v_i^t v_j}{2}, \tag{2.16}$$

where each $v_i \in \mathbb{R}^k$ for $1 \le i \le n$ (if $|V(\Gamma)| = n$, we take $k \le n$). This is a semidefinite programming problem (see [68] for an overview of semidefinite programming) which can be solved with exponentially small error in polynomial time. Suppose we have a solution $\{x_i\}_{1 \le i \le n}$ to (2.15). Note the vectors $v_i = (x_i, 0, \cdots, 0)$ (for $1 \le i \le n$) form a feasible solution to this semidefinite program. Therefore, the optimal value z^* of (2.16) is at least as large as the size of the max-cut of Γ.

Given a solution $v_1, \cdots, v_n$ of the semidefinite program (2.16), Goemans and Williamson suggested the following procedure for constructing a cut in Γ. Choose a random unit vector $\mathbf{r} \in \mathbb{R}^k$ and define $S = \{i \mid \mathbf{r}^t v_i \le 0\}$ and $S^c = \{i \mid \mathbf{r}^t v_i > 0\}$. This produces a cut (S, S^c) of Γ. Let W denote the size of a random cut produced in this way and let $\mathbf{E}[W]$ be its expected value. Using the linearity of the expected value, we find that $\mathbf{E}[W]$ is the sum over all $\{i, j\} \in E$ of the probabilities that vertices i and j

lie in opposite sides of the cut. The probability vertices i and j lie on opposite sides of the cut is $\arccos(v_i^t v_j)/\pi$. Hence

$$\mathbf{E}[W] = \sum_{\{i,j\}\in E} \frac{\arccos(v_i^t v_j)}{\pi}.$$

Recall the optimal value z^* of the semidefinite program is equal to

$$z^* = \sum_{\{i,j\}\in E} \frac{1 - v_i^t v_j}{2}.$$

Therefore the ratio between $\mathbf{E}[W]$ and z^* satisfies

$$\frac{\mathbf{E}[W]}{z^*} = \frac{\sum_{\{i,j\}\in E} \arccos(v_i^t v_j)/\pi}{\sum_{\{i,j\}\in E}(1 - v_i^t v_j)/2} \geq \min_{\{i,j\}\in E} \frac{\arccos(v_i^t v_j)/\pi}{(1 - v_i^t v_j)/2}.$$

Denote $\alpha = \frac{2}{\pi} \min\limits_{0<\theta\leq\pi} \frac{\theta}{1-\cos(\theta)}$. Let $f(\theta) = \frac{\theta}{1-\cos(\theta)}$ for $0 < \theta \leq \pi$. We note $f'(\theta) = \frac{1-\cos(\theta)-\theta\sin(\theta)}{(1-\cos(\theta))^2}$. We quickly see that our function is minimized at the nonzero root of $\cos(\theta) + \theta\sin(\theta) - 1 = 0$. So we find $\alpha \in (0.87856, 0.87857)$. Thus, $\mathbf{E}[W] \geq \alpha \cdot z^*$. Since z^* is at least the size of $mc(\Gamma)$, we conclude that $\alpha \leq \frac{\mathbf{E}[W]}{mc(\Gamma)}$. Therefore, the Goemans-Williamson algorithm provides an α-approximation for $mc(\Gamma)$. Moreover, by the discussion above, the expected size of the cut produced by this algorithm is no better than $\alpha \cdot mc(\Gamma)$ if $z^* = mc(\Gamma)$.

If the value of the semidefinite program happens to be a large fraction of the total number of edges of Γ, Goemans and Williamson [40] show that their performance ratio of the algorithm is even better. Indeed, put $h(t) = \arccos(1 - 2t)/\pi$ and let t_0 be the value of t for which $h(t)/t$ attains its minimum on the interval $(0, 1]$. Then $t_0 \approx 0.84458$. Define $A = z^*/|E|$. Goemans and Williamson proved that if $A \geq t_0$, then $\mathbf{E}[W] \geq \frac{h(a)}{A} z^* \geq \frac{h(a)}{A} mc(\Gamma)$. As mentioned before, the expected size of the cut produced by the algorithm is no better than $\frac{h(A)}{A} mc(\Gamma)$ if $z^* = mc(\Gamma)$.

The previous arguments guarantee that the performance ratio of the Goemans-Williamson algorithm is at least α or at least $\frac{h(A)}{A}$ when the solution of the semidefinite program is a large fraction of the edge set of Γ.

In 1999, Karloff [46] showed that for an infinite family of graphs the performance ratio of the Goemans-Williamson algorithm is exactly α. Karloff worked with graphs that were troublesome for the Goemans-Williamson algorithm. The graphs Karloff focused on were the graphs $J(2d, d, j)$ in the Johnson scheme with $j \geq \frac{5d}{6}$. The smallest eigenvalues of these graphs are important for solving the semidefinite program relaxation of the Max-Cut problem as well as demonstrating that the solution to the semidefinite program is equal to the max-cut of these graphs. In his work he makes the following conjecture.

Conjecture 2.3.1 ([46, Conjecture 2.12])**.** *Let $\frac{d}{2} < j \leq d$. Then the smallest eigenvalue of the graph $J(2d, d, j)$ is $E_j^{2d,d}(1)$.*

Conjecture 2.3.1 was proven by Karloff [46] for $d \geq j \geq \frac{5d}{6}$. We prove Conjecture 2.3.1 along with a more general result about the smallest eigenvalues of graphs in the Johnson scheme in Section 2.5.

A year later, Alon and Sudakov [2] were able to extend Karloff's result using the following relation between (2.16) and the smallest eigenvalue of the adjacency matrix of a graph.

Proposition 2.3.2 ([2, Prop. 3.1])**.** *Let $\Gamma = (V, E)$ be a graph with vertex set $V = \{1, 2, \cdots, n\}$ and adjacency matrix $A = (a_{ij})$ with $1 \leq i, j \leq n$, and let $\lambda_1 \geq \ldots \geq \lambda_n$ be the eigenvalues of A. Then*

$$\sum_{i<j} a_{ij} \frac{1 - v_i^\top v_j}{2} \leq \frac{1}{2}|E| - \frac{1}{4}\lambda_n \cdot n,$$

for any $k > 0$ and any set $v_1, \cdots, v_n$ of unit vectors in $\mathbb{R}^k$.

Alon and Sudakov [2] were able to show that the performance ratio of the Goemans-Williamson algorithm is not only tight for the worst case, but also for graphs in which the size of the maximum cut is a large fraction of the number of edges. More precisely, they show that for infinitely many graphs $\frac{\mathbf{E}[W]}{z^*} = \frac{\mathbf{E}[W]}{\text{Max-Cut}(G)} = \frac{h(A)}{A}$.

Alon and Sudakov [2] worked with graphs in the binary Hamming scheme. Again, the smallest eigenvalue of these graphs are important for finding an optimal solution to (2.16) as well as demonstrating that the max-cut for these graphs is equal to z^*. Alon and Sudakov were able to show that for $q = 2$ and d large enough with j/d fixed, if $\frac{d}{2} < j \leq d$, then $K_j^d(1)$ is the smallest eigenvalue of $H(d, 2, j)$.

More recently, semidefinite programming techniques were adapted for the Max-k-Cut problem by Van Dam and Sotirov [29]. The Max-k-Cut problem is the problem of partitioning the vertex set of a graph into k subsets such that the edges between these subsets is maximized. For $k = 2$, this problem is the Max-Cut problem. In [29], Van Dam and Sotirov provide the following semidefinite program for the Max-k-Cut problem.

Van Dam and Sotirov (2016): SDP relaxation of Max-k-Cut problem

$$\begin{aligned} \max \quad & \frac{1}{2} tr(LY), \\ s.t. \quad & \text{diag}(Y) = j_n, \\ & kY - J_n \succeq 0,\ Y \geq 0, \end{aligned} \tag{2.17}$$

where L is the Laplacian matrix for the graph. Relaxing the constraint $\text{diag}(Y) = \mathbf{j}_n$ to $\text{tr}(Y) = n$ and removing the nonnegativity constraints, Van Dam and Sotirov developed the following semidefinite program:

$$\begin{aligned} \max \quad & \frac{1}{2} tr(LY), \\ s.t. \quad & \text{tr}(Y) = n, \\ & kY - J_n \succeq 0. \end{aligned} \tag{2.18}$$

Now (2.18) can be solved to get the following eigenvalue bound for the Max-k-Cut problem.

Theorem 2.3.3 ([29, Thm. 3])**.** *Let $\Gamma = (V, E)$ be a graph with $|V| = n$ and let L be the Laplacian matrix of Γ. Then*

$$\frac{n(k-1)}{2k} \lambda_{\max}(L) \geq mc_k(\Gamma) \tag{2.19}$$

where $mc_k(\Gamma) = \max_{V_1 \cup \cdots \cup V_k = V} e(V_1, \cdots, V_k)$.

In their work, Van Dam and Sotirov investigate when the solutions to (2.17) and the eigenvalue bound (2.19) coincide. They are able to prove these two bounds are equal for certain graphs in the Hamming scheme with $k \leq q$. In addition, they also demonstrate that the eigenvalue bound is tight for $k = q$. It is not surprising that in their work, the smallest eigenvalue of the graphs $H(d, q, j)$ plays a role in the analysis of these bounds. Their work with the smallest eigenvalues of these graphs led them to make the following conjecture.

Conjecture 2.3.4 ([29, Conjecture 8])**.** *Let $q \geq 2$ and $j \geq d - \frac{d-1}{q}$ where j is even when $q = 2$. Then the smallest eigenvalue of $H(d, q, j)$ is $K_j^d(1)$.*

As we mentioned before, Alon and Sudakov [2] proved this result for graphs in the binary Hamming scheme for d large enough with j/d fixed. Unbeknownst to Van Dam and Sotirov, the binary case was solved completely in 2013 by Dumer and Kapralova [32] in their work regarding spherically punctured biorthogonal codes.

Proposition 2.3.5 ([32, Cor. 10])**.** *Let $q = 2$, then for all integers $1 \leq j \leq d-1$ and $1 \leq i \leq d-1$, $|K_j^d(i)| \leq |K_j^d(1)|$ except when $d = 2j$, in which case the maximum is achieved at $i = 2$.*

In Section 2.4 we prove Conjecture 2.3.4 as well as a stronger result for $q \geq 3$ in Theorem 2.4.8.

2.4 Distance-j Hamming Graphs

The occurrence of $d - \frac{d-1}{q}$ in Conjecture 2.3.4 is explained by the following lemma. When we refer to $K_j^d(1)$ or $K_j^d(2)$ in the following lemma, it is assumed that $d \geq 1$ or $d \geq 2$.

Lemma 2.4.1. *Let $q \geq 2$ and $0 \leq j \leq d$.*

(i) $K_j^d(1) < 0$ *if and only if* $j \geq d - \frac{d-1}{q}$.

(ii) $K_j^d(2) = K_j^d(1)$ *if and only if* $j = 0$ *or* $j = d - \frac{d-1}{q}$.

(iii) $K_j^d(2) > K_j^d(1)$ *if and only if* $j > d - \frac{d-1}{q}$.

(iv) $K_j^d(2) = \frac{-1}{q-1} K_j^d(1)$ *if and only if* $j = (d-1)(1-\frac{1}{q})$ *or* $j = d$.

(v) Let $d - \frac{d-1}{q} \leq j \leq d$. *Then* $|K_j^d(2)| \leq |K_j^d(1)|$.

Proof. In the following proofs we appeal directly to the expression (2.2) given for the Krawtchouk polynomial $K_j^d(i)$ and the various lemmas established in Section 2.1.2.

(i) By Lemma 2.1.6 we know $K_j^d(i)$ and $K_i^d(j)$ have the same sign when evaluated at integers $0 \leq i, j \leq d$. Hence $K_j^d(1)$ has the same sign as $K_1^d(j)$. Now the claim follows from the fact $K_1^d(j) = (q-1)d - qj$.

(ii) By Lemma 2.1.6 we know $K_j^d(i) = \binom{d}{j}(q-1)^{j-i} K_i^d(j) / \binom{d}{i}$. So an equivalent claim to $K_j^d(2) = K_j^d(1)$ is $K_2^d(j) = \frac{1}{2}(q-1)(d-1)K_1^d(j)$. We know

$$\begin{aligned} K_1^d(j) &= (q-1)(d-j) - j, \\ K_2^d(j) &= \frac{(q-1)^2(d-j)(d-j-1)}{2} - (q-1)j(d-j) + \frac{j(j-1)}{2}. \end{aligned}$$

So showing our claim now becomes a matter of determining when

$$(q-1)^2(d-j)(d-1) - (q-1)(d-1)j = (q-1)^2(d-j)(d-j-1) - 2(q-1)j(d-j) + j(j-1).$$

Combining like terms and moving everything to the right hand side of the equation we are left with the following:

$$(q-1)^2 j(d-j) + (q-1)j(d-2j+1) - j(j-1) = 0.$$

This left hand side is quadratic in j and has roots $j = 0$ and $j = d - \frac{d-1}{q}$. This establishes the claim.

(iii) Again, we use the fact that $K_j^d(i)$ and $K_i^d(j)$ have the same sign. Considering the argument of part (ii) and noting that $K_2^d(j)$ has positive leading coefficient $\frac{q^2}{2}$ in the variable j, the claim follows immediately.

(iv) Using Lemma 2.1.6 we see the claim is equivalent to $K_2^d(j) = -\frac{1}{2}(d-1)K_1^d(j)$. The expression $K_2^d(j) + \frac{1}{2}(d-1)K_1^d(j)$ is quadratic in j with roots $j = d$ and $j = (d-1)\left(1 - \frac{1}{q}\right)$.

(v) Using Lemma 2.1.6 we see an equivalent claim is $|K_2^d(j)| \leq \frac{1}{2}(q-1)(d-1)|K_1^d(j)|$. Since for $d - \frac{d-1}{q} \leq j \leq d$ we have $K_1^d(j) < 0$, this gives the pair of conditions $K_2^d(j) - \frac{1}{2}(q-1)(d-1)K_1^d(j) \geq 0$ and $-K_2^d(j) - \frac{1}{2}(q-1)(d-1)K_1^d(j) \geq 0$.

In (ii) we demonstrated the expression $K_2^d(j) - \frac{1}{2}(q-1)(d-1)K_1^d(j)$ is quadratic in j and equals zero when $j = 0$ or $j = d - \frac{d-1}{q}$. Knowing this, it quickly follows that the inequality $K_2^d(j) - \frac{1}{2}(q-1)(d-1)K_1^d(j) \geq 0$ holds for $j \geq d - \frac{d-1}{q}$.

For the latter it suffices to solve $-K_2^d(j) - \frac{1}{2}(d-1)K_1^d(j) \geq 0$.

In (iv) we demonstrated that $-K_2^d(j) - \frac{1}{2}(d-1)K_1^d(j)$ is quadratic in j and equals zero when $j = d$ or $j = (d-1)\left(1 - \frac{1}{q}\right)$. Knowing this, it quickly follows that the inequality $-K_2^d(j) - \frac{1}{2}(q-1)(d-1)K_1^d(j) \geq 0$ holds for $d - \frac{d-1}{q} - 1 \leq j \leq d$. Therefore the claim holds for $d - \frac{d-1}{q} \leq j \leq d$.

□

Lemma 2.4.2. *For $q \geq 2$ and $0 \leq i, j \leq d$, $|K_j^d(i)| \leq (q-1)^{d-i}\binom{d}{j}$.*

Proof. Since $\binom{d-i}{j-h} = 0$ unless $j - h \leq d - i$, an application of the triangle inequality along with Vandermonde's identity yields

$$\begin{aligned}
|K_j^d(i)| &= \left|\sum_{h=0}^{i}(-1)^h(q-1)^{j-h}\binom{i}{h}\binom{d-i}{j-h}\right| \\
&\leq \sum_{h=i+j-d}^{i}(q-1)^{j-h}\binom{i}{h}\binom{d-i}{j-h} \\
&\leq (q-1)^{d-i}\sum_{h=i+j-d}^{i}\binom{i}{h}\binom{d-i}{j-h} \\
&= (q-1)^{d-i}\binom{d}{j}.
\end{aligned}$$

□

Lemma 2.4.3. *Let $1 < i < d$ and $d - \frac{d-1}{q} \leq j \leq d$. If $qj \leq 2(q-1)(d-i)$, then $|K_j^d(i+1)| \leq \max(|K_j^d(i-1)|,\ |K_j^d(i)|)$.*

Proof. Put $a = (q-1)(d-i)$. From Proposition 2.1.9 one has $aK_j(i+1) - (i - qj + a)K_j(i) + iK_j(i-1) = 0$. If $|K_j(i-1)| \le M$ and $|K_j(i)| \le M$, then

$$a|K_j(i+1)| \le |i - qj + a|M + iM,$$

and the conclusion follows if $i + |i - qj + a| \le a$. Now

$$qj - i - a \ge d(q-1) + 1 - i - d(q-1) + (q-1)i > (q-2)i \ge 0.$$

So we need $i - i + qj - a \le a$, which reduces to $qj \le 2a$. But this is one of the hypotheses. Therefore the claim is true. □

Lemma 2.4.4. *If $1 \le i \le d$, then $|K_d^d(i)| \le |K_d^d(1)|$.*

Proof. Using Lemma 2.1.7 and letting $j = 0$, we find $K_d^d(i) = (-1)^i(q-1)^{d-i}$. Since $1 \le i \le d$, we reach the conclusion $|K_d^d(i)| = (q-1)^{d-i} \le (q-1)^{d-1} = |K_d^d(1)|$. □

Lemma 2.4.5. *Let $d \ge 29$, $q \ge 3$ and $1 \le i \le d$. If $qj = d(q-1)+1$, then $|K_j^d(i)| \le |K_j^d(1)|$, unless $(q,d,i,j) = (3,4,3,3)$. If $(q,d,j) = (3,4,3)$ then $K_j^d(0) = 32$, $K_j^d(i) = -4$ for $i = 1,2,4$, and $K_j^d(3) = 5$.*

Proof. Based on Lemma 2.4.1 and Lemma 2.4.3 if $d - i \ge \frac{d-2}{2} + \frac{1}{2(q-1)}$ then the claim follows by induction on i. So assume $d - i < \frac{d-2}{2} + \frac{1}{2(q-1)}$. Now

$$\begin{aligned} K_j^d(1) &= (q-1)^{j-1}\left(q - 1 - \frac{qj}{d}\right)\binom{d}{j} \\ &= (q-1)^{d-\frac{d-1}{q}-1}(q-1-(q-1)-1/d)\binom{d}{j} \\ &= -\frac{(q-1)^{d-\frac{d-1}{q}-1}}{d}\binom{d}{j}. \end{aligned}$$

Hence $|K_j^d(1)| = \frac{(q-1)^{d-\frac{d-1}{q}-1}}{d}\binom{d}{j}$. By Lemma 2.4.2 we have $|K_j^d(i)| \le (q-1)^{\left(\frac{d-2}{2}+\frac{1}{2(q-1)}\right)}\binom{d}{j}$. Thus we want $(q-1)^{d-\frac{d-1}{q}-1} \ge d(q-1)^{\left(\frac{d-2}{2}+\frac{1}{2(q-1)}\right)}$. Dividing the left hand side by the $(q-1)^{\left(\frac{d-2}{2}+\frac{1}{2(q-1)}\right)}$ term we see it is equivalent to $(q-1)^{\left(\frac{d}{6}+\frac{1}{12}\right)} \ge d$. This holds for $q \ge 3$ and $d \ge 29$. □

Before we state and prove the following lemma, we argue that the inequality $K_j^d(i) - K_j^d(1) \geq 0$ is equivalent to

$$q\binom{d-1}{j} - \binom{d}{j} \leq \sum_{h=0}^{i} (-1)^h (q-1)^{1-h} \binom{i}{h} \binom{d-i}{j-h}. \tag{2.20}$$

Indeed, we see

$$K_j^d(i) - K_j^d(1) = \sum_{h=0}^{i} (-1)^h (q-1)^{j-h} \binom{i}{h} \binom{d-i}{j-h} - (q-1)^j \binom{d-1}{j} + (q-1)^{j-1} \binom{d-1}{j-1}$$

and dividing by $(q-1)^{j-1}$ gives the equivalent statement

$$\sum_{h=0}^{i} (-1)^h (q-1)^{1-h} \binom{i}{h} \binom{d-i}{j-h} - (q-1)\binom{d-1}{j} + \binom{d-1}{j-1} \geq 0.$$

Noting that $\binom{d-1}{j} + \binom{d-1}{j-1} = \binom{d}{j}$ we see that (2.20) is equivalent to

$$\sum_{h=0}^{i} (-1)^h (q-1)^{1-h} \binom{i}{h} \binom{d-i}{j-h} - q\binom{d-1}{j} + \binom{d}{j} \geq 0.$$

We now prove inequality (2.20).

Lemma 2.4.6. *Let $d \geq 2$, $q \geq 3$ and $d - \frac{d-1}{q} \leq j \leq d$. Then*

$$q\binom{d-1}{j} - \binom{d}{j} \leq \sum_{h=0}^{i} (-1)^h (q-1)^{1-h} \binom{i}{h} \binom{d-i}{j-h}.$$

Proof. If $j = d$, then the claim follows from Lemma 2.4.4. If $qj = d(q-1)+1$, then the claim follows from Lemma 2.4.5. Since we know the claim holds for the case $j = d$, we may assume $d - \frac{d-1}{q} \leq j \leq d-1$. This gives the condition $q \leq d-1$. The cases $2 \leq d \leq 29$ with $2 \leq q \leq d-1$ were verified by computer by Andries Brouwer. For the rest of the claim we will use induction on d. Since the case $d = 29$ has already been verified, we assume $d > 29$ and

$$q\binom{(d-1)-1}{j} - \binom{d-1}{j} \leq \sum_{h=0}^{i} (-1)^h (q-1)^{1-h} \binom{i}{h} \binom{d-i-1}{j-h}$$

and

$$q\binom{(d-1)-1}{j-1} - \binom{d-1}{j-1} \leq \sum_{h=0}^{i} (-1)^h (q-1)^{1-h} \binom{i}{h} \binom{d-i-1}{j-h-1}.$$

Adding the left hand sides of the above inequalities yields

$$q\left[\binom{(d-1)-1}{j-1}+\binom{(d-1)-1}{j}\right]-\left[\binom{d-1}{j-1}+\binom{d-1}{j}\right]=q\binom{d-1}{j}-\binom{d}{j}.$$

Adding the right hand sides of the above inequalities yields

$$\sum_{h=0}^{i}(-1)^h(q-1)^{1-h}\binom{i}{h}\left[\binom{d-i-1}{j-h-1}+\binom{d-i-1}{j-h}\right]=\sum_{h=0}^{i}(-1)^h(q-1)^{1-h}\binom{i}{h}\binom{d-i}{j-h}.$$

Therefore, we see

$$q\binom{d-1}{j}-\binom{d}{j}\leq\sum_{h=0}^{i}(-1)^h(q-1)^{1-h}\binom{i}{h}\binom{d-i}{j-h}.$$

□

As mentioned before Lemma 2.4.6, the inequality $K_j^d(i)-K_j^d(1)\geq 0$ is equivalent to

$$q\binom{d-1}{j}-\binom{d}{j}\leq\sum_{h=0}^{i}(-1)^h(q-1)^{1-h}\binom{i}{h}\binom{d-i}{j-h}.$$

Conjecture 2.3.4 now follows as a corollary of Lemma 2.4.6 and Proposition 2.3.5.

Corollary 2.4.7. *Let $q\geq 2$ and $d-\frac{d-1}{q}\leq j\leq d$, then $K_j(1)\leq K_j(i)$ for all i, $0\leq i\leq d$.*

Given $q\geq 2$, $d\geq 2$ and $d-\frac{d-1}{q}\leq j\leq d$ it turns out that $K_j^d(1)$ is not only the smallest eigenvalue of $H(d,q,j)$ but also the second largest in absolute value with one exception. The following stronger result was developed by me with the help of Andries Brouwer and Ferdinand Ihringer.

Theorem 2.4.8. *Let $q\geq 3$ and $d-\frac{d-1}{q}\leq j\leq d$.*

(i) One has $K_j(1)\leq K_j(i)$ for all i, $0\leq i\leq d$.

(ii) One has $|K_j(i)|\leq|K_j(1)|$ for all $i\geq 1$, unless $(q,d,i,j)=(3,4,3,3)$.

If $(q,d,j)=(3,4,3)$, then $K_j(0)=32$, $K_j(i)=-4$ for $i=1,2,4$, and $K_j(3)=5$.

Proof. Since $K_j(1) < 0$ (and $K_j(0)$ is the largest eigenvalue), part (i) follows from part (ii).

The case $i = 2$ was handled in Lemma 2.4.1, so we may assume $i \geq 3$ and the case $j = d$ was taken care of in Lemma 2.4.4, so we may assume $j \leq d - 1$.

For $j = d - 1$ we have $K_{d-1}^d(i) = \sum_{h=0}^{i}(-1)^h(q-1)^{d-h-1}\binom{i}{h}\binom{d-i}{d-h-1}$. Noting $\binom{d-i}{d-h-1} = 0$ for $h < i - 1$, one sees that

$$\begin{aligned} K_{d-1}^d(i) &= (-1)^{i-1}(q-1)^{d-i}i + (-1)^i(q-1)^{d-i-1}(d-i) \\ &= (-1)^{i-1}(q-1)^{d-i-1}[(q-1)i - (d-i)] \\ &= (-1)^{i-1}(q-1)^{d-i-1}(qi-d). \end{aligned}$$

Since $d - \frac{d-1}{q} \leq d-1$ it follows that $q \leq d-1$. So we have $|K_{d-1}^d(1)| = (q-1)^{d-2}(d-q)$. To prove (ii) in this case, it suffices to show that $qi - d \leq (q-1)^{i-1}(d-q)$ for $i \geq 1$. Since $d \geq q+1$ we have $d - q \geq 1$ and $qi - d \leq q(i-1) - 1$. So it suffices to have $q(i-1) - 1 \leq (q-1)^{i-1}$. This is true unless $(q, i) = (3, 3)$. When $(q, i) = (3, 3)$, the inequality $qi - d \leq (q-1)^{i-1}(d-q)$ still holds, unless $d = 4$.

Thus, we may now assume $d - \frac{d-1}{q} \leq j \leq d-2$. From this we obtain the bound $3 \leq q \leq (d-1)/2$.

If $qj \leq 2(q-1)(d-i+1)$, then we can apply Lemma 2.4.3 (and induction on i) along with Lemma 2.4.1 to conclude that $|K_j^d(i)| \leq \max(|K_j^d(1)|, |K_j^d(2)|) \leq |K_j^d(1)|$, and we are done. So, assume $qj > 2(q-1)(d-i+1)$.

One has $|K_j^d(1)| = (q-1)^{j-1}(\frac{qj}{d} - q + 1)\binom{d}{j}$. Lemma 2.4.2 tells us $|K_j^d(i)| \leq (q-1)^{d-i}\binom{d}{j}$. Hence $|K_j(i)| \leq |K_j(1)|$ when $d \leq (q-1)^{i+j-d-1}(qj - (q-1)d)$.

It remains to figure out when the inequality $d \leq (q-1)^{i+j-d-1}(qj - (q-1)d)$ holds. We will take advantage of the following information:

(i) $qj - (q-1)d \geq 1$ implies $(q-1)^{i+j-d-1} \leq (q-1)^{i+j-d-1}(qj - (q-1)d)$.

(ii) $qj > 2(q-1)(d-i+1)$ and $q \geq 3$ imply $d - i + 1 < \frac{qj}{2(q-1)} \leq \frac{3j}{4}$.

(iii) $j \geq d - \frac{d-1}{q}$ and $q \geq 3$ imply $j \geq d - \frac{d-1}{3} > \frac{2d}{3}$.

Now, (ii) and (iii) imply $i+j-d-1 = j-(d-i+1) \geq \frac{d}{6}$. Combining this with (i) and the fact $q \geq 3$ we find it suffices to show $d^6 \leq 2^d$. This holds for $d \geq 30$ and the finitely many remaining cases were checked by computer by Andries Brouwer. $\square$

2.5 Distance-j Johnson Graphs

We use the notation $e := n-d$ to make our formulas shorter and nicer. The occurrence of $de \leq j(n-1)$ in Theorem 2.5.8 is explained by the following lemma. When we refer to $E_j^{n,d}(1)$ or $E_j^{n,d}(2)$ in the following lemma, it is assumed that $d \geq 1$ and $d \geq 2$.

Lemma 2.5.1. *Let $j > 0$. Then*

(i) $E_j^{n,d}(1) = 0$ *if and only if* $j = de/n$.

(ii) $E_j^{n,d}(1) < 0$ *if and only if* $j > de/n$,

(iii) $E_j^{n,d}(1) = E_j^{n,d}(2)$ *if and only if* $j(n-1) = de$.

(iv) $E_j^{n,d}(1) < E_j^{n,d}(2)$ *if and only if* $j(n-1) > de$.

Proof. In the following we use expression (2.6) given for the Eberlein polynomials $E_j^{n,d}(i)$.

(i) We have $E_j^{n,d}(1) = (1 - \frac{jn}{de})\binom{d}{j}\binom{e}{j}$. Hence $E_j^{n,d}(1) = 0$ if and only if $1 - \frac{jn}{de} = 0$.

(ii) We have $E_j^{n,d}(1) = (1 - \frac{jn}{de})\binom{d}{j}\binom{e}{j}$. Hence $E_j^{n,d}(1) < 0$ if and only if $1 - \frac{jn}{de} < 0$.

(iii) & (iv) Using expression (2.6) for the Eberlein polynomial $E_j^{n,d}(i)$ we find

$$\begin{aligned} E_j^{n,d}(1) &= \sum_{h=0}^{j}(-1)^h\binom{1}{h}\binom{d-1}{j-h}\binom{e-1}{j-h} \\ &= \binom{d-1}{j}\binom{e-1}{j} - \binom{d-1}{j-1}\binom{e-1}{j-1} \\ &= \left(\frac{(d-j)(e-j)}{de} - \frac{j^2}{de}\right)\binom{d}{j}\binom{e}{j} \\ &= \left(1 - \frac{j(e+d)}{de}\right)\binom{d}{j}\binom{e}{j} \\ &= \left(1 - \frac{jn}{de}\right)\binom{d}{j}\binom{e}{j} \end{aligned}$$

and

$$\begin{aligned} E_j^{n,d}(2) &= \sum_{h=0}^{j}(-1)^h\binom{2}{h}\binom{d-2}{j-h}\binom{e-2}{j-h} \\ &= \binom{d-2}{j}\binom{e-2}{j} - 2\binom{d-2}{j-1}\binom{e-2}{j-1} + \binom{d-2}{j-2}\binom{e-2}{j-2}. \end{aligned}$$

For now, assume $j > 1$ (we will consider the case when $j = 1$ at the end). Dividing both $E_j^{n,d}(1)$ and $E_j^{n,d}(2)$ by $\binom{d-2}{j-2}\binom{e-2}{j-2}$ and multiplying by $j^2(j-1)^2$ we find that $E_j^{n,d}(1) \leq E_j^{n,d}(2)$ is equivalent to

$$\begin{aligned} (de-jn)(d-1)(e-1) &\leq (d-j)(d-j-1)(e-1)(e-j-1) - 2(d-j)(e-j)j^2 + (j-1)^2j^2 \\ &= (de-nj)(d-1)(e-1) - (de-nj)(n-2)j - (n-2)j^2. \end{aligned}$$

So we now see the previous inequality is equivalent to

$$(n-2)j^2 \leq (nj-de)(n-2)j.$$

Since $i = 2$ only occurs for $d \geq 2$ and $n \geq 4$ we may divide both sides by a factor of $(n-2)j$ and realize $E_j^{n,d}(1) \leq E_j^{n,d}(2)$ if and only if $j(n-1) \geq de$.

Finally, if $j = 1$, then we have

$$E_1^{n,d}(i) = (d-i)(e-i) - i = de - i(n-i+1).$$

So $E_1^{n,d}(1) = de - n$ and $E_1^{n,d}(2) = de - 2n + 2$. It follows that $E_1^{n,d}(1) \leq E_1^{n,d}(2)$ if and only if $n \leq 2$. However, this is false since $d \geq 2$ and $n \geq 4$.

□

Before we dive into the main results of this section we briefly describe the hypergeometric distribution, as it will prove useful in the lemmas that follow. Consider an urn with N balls in it, M of which are white and the remaining $N - M$ black. We draw n balls uniformly and at random from the urn without replacement. Let X be the random variable representing the number of white balls drawn. Then X is said to have a *hypergeometric distribution* with parameters N, M and n.

The probability mass function for this distribution is

$$\mathbf{Pr}[X = m] = \frac{\binom{M}{m}\binom{N-M}{n-m}}{\binom{N}{n}}.$$

The expected value is given by

$$\mathbf{E}[X] = n\frac{M}{N}.$$

The variance is given by

$$\mathrm{Var}[X] = n\frac{M(N-M)(N-n)}{N^2(N-1)}.$$

The following lemma is due to Chvátal [16]. An excellent review and explanation of this result is provided by Skala in [65].

Lemma 2.5.2. *Let X be a hypergeometric random variable with parameters N, M and n and $t \geq 0$. Then*

$$\boldsymbol{Pr}[|X - \boldsymbol{E}[X]| \geq tn] \leq 2\exp(-2t^2 n).$$

The next couple of paragraphs and the following lemma will help us take care of a gap in the induction hypothesis that will appear in the proof of Theorem 2.5.8.

For any k-regular graph Γ with v vertices and adjacency matrix A, the sum of the squares of the eigenvalues is the trace of A^2, which is vk. We apply this to $J(n, d, j)$, and find $vk_j = \sum_{i=0}^{d} m_i E_j(i)^2$, where $v = \binom{n}{d}$ is the number of vertices of

$J(n,d)$, $k_j = \binom{d}{j}\binom{e}{j}$ is the valency of $J(n,d,j)$ (with $e := n-d$), and $m_i = \binom{n}{i} - \binom{n}{i-1}$ is the multiplicity of the i-th eigenvalue for $0 \le i \le d$ (cf. [8, 9.1.2]). It follows that $E_j(i)^2 \le vk_j/m_i$.

We need to estimate k_j close to its maximum value. To do this we use the tail inequality for the hypergeometric distribution given in Lemma 2.5.2.

Lemma 2.5.3. *Let $I = (\frac{de}{n} - \sqrt{d}, \frac{de}{n} + \sqrt{d})$. Then $\sum_{j \in I} k_j > \frac{8}{11}v$.*

Proof. Let X be the random variable taking value j with probability $\frac{k_j}{v} = \frac{\binom{d}{j}\binom{n-d}{j}}{\binom{n}{d}} = \frac{\binom{n-d}{j}\binom{d}{d-j}}{\binom{n}{d}}$. The distribution of this random variable is hypergeometric with parameters n, e and d. Hence $\mathbf{E}[X] = \frac{de}{n}$. Now, based on Lemma 2.5.2, we know

$$\mathbf{Pr}\left[\left|j - \frac{de}{n}\right| \ge td\right] = \mathbf{Pr}[|X - \mathbf{E}[X]| \ge td] \le 2\exp(-2t^2 d).$$

Allowing $t = d^{-1/2}$ we find

$$\mathbf{Pr}\left[\left|j - \frac{de}{n}\right| \ge \sqrt{d}\right] = 1 - \sum_{j \in I} \frac{k_j}{v} \le 2\exp(-2).$$

From this it follows that

$$\frac{8}{11}v < (1 - 2\exp(-2))v \le \sum_{j \in I} k_j.$$

□

Lemma 2.5.4. *Let $j_0 = \frac{de}{n}$ and $j_0 \le j < j_0 + \frac{3}{2}$. If $\frac{de}{n-1} \le j < d$ and $i \ge 3$, then $|E_j(i)| \le |E_j(1)|$.*

Proof. We will make use of the following observations that hold for d large enough.

(i) Since $\frac{de}{n-1} \le j \le d-1$ and $n = d+e$, we find $de \le (d-1)(n-1) = (d-1)(d+e-1)$. Hence $de \le de - e + d^2 - 2d + 1$. This gives us the condition $e \le (d-1)^2$.

(ii) Consider the function $f(e) = \frac{n^3}{d^2e^2} = \frac{(e+d)^2}{d^2e^2}$. We see that $f'(e) = \frac{(e+d)^2(e-2d)}{d^2e^3}$. It follows that $f(e)$ is decreasing for $e < 2d$ and increasing for $e > 2d$. Hence, on the interval $d \leq e \leq (d-1)^2$, $f(e)$ attains its maximum at $e = (d-1)^2$ for $d \geq 3$. From this it follows that

$$\frac{n}{j_0^2} = \frac{n^3}{d^2e^2} = \frac{(e+d)^3}{d^2e^2} \leq \frac{(d^2-d+1)^3}{d^2(d-1)^4} < 1 + \frac{3}{2d}$$

for $d \geq 10$.

(iii) We show that $k_{j-1}/k_j < 3$ if $d \geq 10$. First note $k_{j-1}/k_j = c_j/b_{j-1} = j^2/(d-j+1)(e-j+1)$. Hence $k_{j-1}/k_j < 3$ is equivalent to $j^2 < 3(d-j+1)(e-j+1)$. Expanding and moving everything to the same side we see that we wish to show $de - n(j-1) + (j-1)^2 - \frac{j^2}{3} > 0$. Let $g(j) = de - n(j-1) + (j-1)^2 - \frac{j^2}{3}$. Then $g'(j) = \frac{4j}{3} - (n+2) < 0$, since $j \leq d \leq \frac{n}{2}$. Since $g(j)$ is decreasing it suffices to prove the inequality for $j = j_0 + \frac{3}{2}$. Since $de = nj_0$, we find

$$de - n(j-1) + (j-1)^2 - \frac{j^2}{3} = -\frac{1}{2}n + \frac{2}{3}j_0^2 - \frac{1}{2}.$$

Multiplying by 2 we see our claim reduces to $\frac{4}{3}j_0^2 > n+1$. Now, noting $j_0^2 \geq \frac{d^2}{4}$ we find by (ii) that

$$\frac{n+1}{j_0^2} < 1 + \frac{3}{2d} + \frac{1}{j_0^2} < \frac{4}{3}$$

for $d \geq 10$.

(iv) We show that $v/k_j < \frac{1}{6}(n-5)$ for $n \geq 42$. Lemma 2.5.3 states $\sum_{|\ell-j_0|<\sqrt{d}} k_\ell > \frac{8}{11}v$. Let $k_{j_1} = \max_{\ell\,:\,|\ell-j_0|<\sqrt{d}} k_j$. We argue that $\lfloor j_0 \rfloor \leq j_1 \leq \lceil j_0 \rceil$. Recall that the k_j are unimodal (see 1.4.11) and $\frac{k_{j-1}}{k_j} = \frac{j^2}{(d-j+1)(e-j+1)}$. From this it follows that $k_{j-1} < k_j$ if $j < \frac{de-1}{n+2} + 1$ and $k_j < k_{j-1}$ if $j > \frac{de-1}{n+2} + 1$. Next we see, since $\frac{de}{n} < \frac{de-1}{n+2} + 1 < \frac{de}{n} + 1$, it follows that if $j \leq \lfloor j_0 \rfloor$, then $k_{j-1} < k_j$ and if $j \geq \lceil j_0 \rceil$, then $k_j < k_{j-1}$. Hence $\lfloor j_0 \rfloor \leq j_1 \leq \lceil j_0 \rceil$. Now, it follows that $2\sqrt{d}k_{j_1} > \frac{8}{11}v$, that is $\frac{v}{k_{j_1}} < \frac{11}{4}\sqrt{d}$. Since $j_0 \leq j \leq j_0 + \frac{3}{2}$ we know j differs from j_1 by at most 2 and that $j \geq j_1$. Now, using $\frac{k_{j-2}}{k_{j-1}} = \frac{(j-1)^2}{(d-j+2)(e-j+2)}$ and $\frac{k_{j-1}}{k_j} = \frac{j^2}{(d-j+1)(e-j+1)}$, we see that $\frac{k_{j-2}}{k_{j-1}} \leq \frac{k_{j-1}}{k_j} < 3$, by (iii). It follows that $\frac{k_{j_1}}{k_j} \leq \frac{k_{j-2}}{k_j} < 9$. Thus, $\frac{v}{k_j} < \frac{99}{4}\sqrt{d}$.

Our aim is to show $\frac{v}{k_j} < \frac{1}{6}(n-5)$. Since $n \geq 2d$ implies $n \geq \sqrt{2n}\sqrt{d}$ we find $\frac{v}{k_j} < \frac{99\sqrt{n}}{4\sqrt{2}}$. Now, $\frac{99\sqrt{n}}{4\sqrt{2}} \leq \frac{1}{6}(n-5)$ for $n \geq 11037$. The finitely many remaining cases with $42 \leq n \leq 11036$ were checked by computer.

(v) Finally, we show that $E_j(i)^2 \leq E_j(1)^2$ if $i \geq 3$. In the discussion at the beginning of this section we saw that $E_j(i)^2 \leq vk_j/m_i$, where $m_i \geq m_3 = \frac{1}{6}n(n-1)(n-5)$ (for $n \geq 20$). On the other hand, $E_j(1) = \binom{d}{j}\binom{e}{j}(1 - \frac{j}{j_0}) = k_j\left(\frac{j_0 - j}{j_0}\right)$. Note $j - j_0 \geq \frac{de}{n-1} - \frac{de}{n} = \frac{j_0}{n-1}$. So we find $E_j(i)^2 \leq E_j(1)^2$ when $\frac{6vk_j}{n(n-1)(n-5)} \leq k_j^2\left(\frac{1}{n-1}\right)^2$. This is equivalent to $v/k_j \leq \frac{1}{6}\frac{n(n-5)}{n-1}$, which follows from (iv).

Earlier, we had the conditions $d \geq 10$ (or $n \geq 42$), but if $d \leq 9$, then $n \leq 73$, and these cases were checked by computer.

□

Lemma 2.5.5. *Let $(j-1)(n+1) \geq de$. Then $E_j^{n,d}(0) + |E_{j-1}^{n,d}(1)| + |E_j^{n,d}(1)| \leq E_{j-1}^{n,d}(0)$.*

Proof. Appealing directly to equation (2.6) for $E_j^{n,d}(i)$ we see

$$E_j^{n,d}(0) = \binom{d}{j}\binom{e}{j} \text{ and } E_j^{n,d}(1) = \binom{d}{j}\binom{e}{j}\left(1 - \frac{jn}{de}\right).$$

Noting $jn > de$ we see the desired inequality is equivalent to

$$\binom{d}{j}\binom{e}{j} + \binom{d}{j-1}\binom{e}{j-1}\left|1 - \frac{(j-1)n}{de}\right| + \binom{d}{j}\binom{e}{j}\left(\frac{jn}{de} - 1\right) \leq \binom{d}{j-1}\binom{e}{j-1}.$$

Dividing through by $\binom{d}{j-1}\binom{e}{j-1}$ we get

$$\frac{(d-j+1)}{j} \cdot \frac{e-j+1}{j} + \left|1 - \frac{(j-1)n}{de}\right| + \frac{d-j+1}{j} \cdot \frac{e-j+1}{j}\left(\frac{jn}{de} - 1\right) \leq 1.$$

Simplifying we end up with

$$\frac{jn}{de} \cdot \frac{d-j+1}{j} \cdot \frac{e-j+1}{j} + \left|1 - \frac{(j-1)n}{de}\right| \leq 1.$$

We next consider the following two cases:

1. If $(j-1)n \leq de$, then we have to show

$$\frac{jn}{de} \cdot \frac{d-j+1}{j} \cdot \frac{e-j+1}{j} \leq \frac{(j-1)n}{de}.$$

With some simplification, we can see this inequality is equivalent to $de \leq (n+1)(j-1)$, which was our hypothesis.

2. If $(j-1)n \geq de$, then we have to show

$$\frac{jn}{de} \cdot \frac{d-j+1}{j} \cdot \frac{e-j+1}{j} + \frac{(j-1)n}{de} \leq 2.$$

This reduces to $(d-j+1)(e-j+1)+j(j-1) \leq \frac{2jde}{n}$. Expanding out the left hand side and factoring out a $(j-1)$ term we end up with $de-(n-2j+1)(j-1) \leq \frac{2jde}{n}$. Simplifying, we find an equivalent inequality is $de(n-2j) \leq n(j-1)(n-2j+1)$. Since

$$de(n-2j) \leq n(j-1)(n-2j) < n(j-1)(n-2j+1),$$

the claim is true.

□

In this section we investigate when $|E_j^{n,d}(1)|$ is largest among $|E_j^{n,d}(i)|$ for $1 \leq i \leq d$, and moreover when $E_j(1) < 0$ implies $E_j(1)$ is the smallest among the $E_j(i)$ for $0 \leq i \leq d$. We prove in Theorem 2.5.8 that this is the case if and only if $j \geq \frac{de}{n-1}$.

Lemma 2.5.6. (Lovász [55]) *The eigenvalues of the Kneser graph are*

$$E_d^{n,d}(i) = (-1)^i \binom{n-d-i}{d-i} = (-1)^i \binom{n-d-i}{n-2d}.$$

Proof. Using equation (2.7) given above for $E_j^{n,d}(i)$ we know

$$E_d^{n,d}(i) = \sum_{h=0}^{j} (-1)^{d-h} \binom{d-i}{h} \binom{n-d-i+h}{h}.$$

Using the identity $\binom{n+h}{h} = (-1)^h \binom{-n-1}{h}$ and Vandermonde's identity we find

$$\begin{aligned}
\sum_{h=0}^{j} (-1)^{d-h} \binom{d-i}{h} \binom{n-d-i+h}{h} &= (-1)^d \sum_{h=0}^{j} \binom{d-i}{h} (-1)^h \binom{n-d-i+h}{h} \\
&= (-1)^d \sum_{h=0}^{j} \binom{d-i}{h} \binom{d+i-n-1}{h} \\
&= (-1)^d \sum_{h=0}^{j} \binom{d-i}{d-i-h} \binom{d+i-n-1}{h} \\
&= (-1)^d \binom{2d-n-1}{d-i}.
\end{aligned}$$

Again, using the identity $\binom{n+h}{h} = (-1)^h \binom{-n-1}{h}$, we have $(-1)^{d-i} \binom{-(n-2d)-1}{d-i} = \binom{n-d-i}{d-i}$. It follows that $(-1)^d \binom{2d-n-1}{d-i} = (-1)^i \binom{n-d-i}{d-i}$, and the claim is proven. □

Now we know the eigenvalues of the Kneser graph, so the case $j = d$ follows immediately.

Lemma 2.5.7. *Let $d \geq 1$. The smallest eigenvalue of the Kneser graph $K(n, d)$, and second largest in absolute value, is $E_d^{n,d}(1)$.* □

The following theorem is the main result of this section. It implies Conjecture 2.3.1.

Theorem 2.5.8. *Let $j > 0$. Then $E_j(1)$ is the smallest eigenvalue of $J(n, d, j)$ if and only if $j(n-1) \geq de$. In this case, $E_j(1)$ is also the second largest in absolute value among the eigenvalues of $J(n, d, j)$.*

Proof. By Lemma 2.5.1, if $E_j^{n,d}(1)$ is the smallest eigenvalue, then $j(n-1) \geq de$, so $E_j^{n,d}(1) < 0$.

Next, we show that if $j(n-1) \geq de$, then $|E_j^{n,d}(i)| \leq |E_j^{n,d}(1)|$, for $1 \leq i \leq d$. If $j = d$, then the statement follows by Lemma 2.5.7. If $\frac{de}{n-1} \leq j < \frac{de}{n-3}$, then since $n \geq 2d$ and $d \geq 3$, $j_0 < j < j_0 + \frac{3}{2}$, where $j_0 = \frac{de}{n}$. In this case, the fact that $E_j^{n,d}(1)$ is the smallest eigenvalue, and second largest in absolute value, of $J(n, d, j)$ holds by Lemma 2.5.4.

The remainder of the proof will proceed using induction on d. Since the cases for $j = d$ and $\frac{de}{n-1} \leq j < \frac{de}{n-3}$ have already been established, we may assume $\frac{de}{n-3} \leq j \leq d-1$. This implies $d \geq 5$. The cases when $d = 5$, $n \leq 21$ and $\frac{5(n-5)}{n-1} \leq j \leq 4$ were checked by computer by Andries Brouwer. So assume $d \geq 5$ and

$$|E_j^{n,d}(i)| \leq |E_j^{n,d}(1)| \text{ for } j \geq \frac{de}{n-1}$$

and

$$|E_{j-1}^{n,d}(i)| \leq |E_{j-1}^{n,d}(1)| \text{ for } j \geq \frac{de}{n-1} + 1.$$

We aim to show that if $j(n-1) \geq (d+1)(e+1)$, then $|E_j^{n+2,d+1}(i)| \leq |E_j^{n+2,d+1}(1)|$. By Lemma 2.2.5 we find this is equivalent to

$$\left|E_j^{n,d}(i-1) - E_{j-1}^{n,d}(i-1)\right| \leq \left|E_j^{n,d}(0) - E_{j-1}^{n,d}(0)\right|.$$

Using the triangle inequality we see it suffices to show

$$E_j^{n,d}(0) + \left|E_j^{n,d}(i-1)\right| + \left|E_{j-1}^{n,d}(i-1)\right| \leq E_{j-1}^{n,d}(0).$$

Now $j(n-1) \geq (d+1)(e+1)$ implies $j(n-1) \geq de$ and $(j-1)(n-1) \geq de$. So by our induction hypothesis, or trivially if $i = 2$, we find

$$E_j^{n,d}(0) + \left|E_j^{n,d}(i-1)\right| + \left|E_{j-1}^{n,d}(i-1)\right| \leq E_j^{n,d}(0) + \left|E_j^{n,d}(1)\right| + \left|E_{j-1}^{n,d}(1)\right| \leq E_{j-1}^{n,d}(0),$$

where the last inequality holds by Lemma 2.5.5. □

2.6 Open Problems

In this section we discuss some open problems and potential areas of future research regarding the eigenvalues of graphs in the Hamming and Johnson association schemes. As we saw in both Sections 2.1.2 and 2.4 the graph $H(d, q, j)$ can have several repeated eigenvalues. Recall the first eigenmatrix of the graph $H(4, 3, 3)$ given by

$$P = \begin{bmatrix} 1 & 8 & 24 & 32 & 16 \\ 1 & 5 & 6 & -4 & -8 \\ 1 & 2 & -3 & -4 & -4 \\ 1 & -1 & -3 & 5 & -2 \\ 1 & -4 & -6 & -4 & -1 \end{bmatrix}$$

.

One sees that the graph $H(4,3,3)$ has exactly 3 distinct eigenvalues. Hence it is strongly regular and has parameters $v = 81$, $k = 32$, $\lambda = 13$ and $\mu = 12$. The spectrum is $\{32^{(1)}, 5^{(32),(-4)^{48}}\}$.

In the Hamming scheme $\mathcal{H}(7,2)$ the first eigenmatrix is

$$\begin{bmatrix} 1 & 7 & 21 & 35 & 35 & 21 & 7 & 1 \\ 1 & 5 & 9 & 5 & -5 & -9 & -5 & -1 \\ 1 & 3 & 1 & -5 & -5 & 1 & 3 & 1 \\ 1 & 1 & -3 & -3 & 3 & 3 & -1 & -1 \\ 1 & -1 & -3 & 3 & 3 & -3 & -1 & 1 \\ 1 & -3 & 1 & 5 & -5 & -1 & 3 & -1 \\ 1 & -5 & 9 & -5 & -5 & 9 & -5 & 1 \\ 1 & -7 & 21 & -35 & 35 & -21 & 7 & -1 \end{bmatrix}.$$

We see above that the graph $H(7,2,4)$ has two connected components, both isomorphic to the graph Γ on 64 binary vectors of length 7 with even weight where two vectors are adjacent when they differ in 4 coordinates. The graph Γ is strongly regular with parameters $v = 64$, $k = 35$, $\lambda = 18$ and $\mu = 20$ and has spectrum $\{35^{(1)}, 3^{(35)}, (-5)^{(28)}\}$. We believe an interesting area for future research involves classifying the graphs in both the Hamming and Johnson scheme with few distinct eigenvalues. That being said, we end this chapter with the following conjecture attributed to Andries Brouwer (see [7] for more information).

Conjecture 2.6.1. *If $H(d,q,j)$ is connected, then it has more than $\frac{d}{2}$ distinct eigenvalues.*

Chapter 3

COSPECTRAL MATES FOR GRAPHS IN THE JOHNSON ASSOCIATION SCHEME

In this chapter we construct cospectral mates for graphs formed by taking unions of relations in the Johnson association scheme. Our work extends the work done by Van Dam, Haemers, Koolen and Spence in [26], and the work done by Haemers and Ramezani in [43]. The research presented represents joint work with Sebastian Cioabă, Willem Haemers and Travis Johnston from [18].

A major area of study in spectral graph theory is classifying which graphs are determined by their spectrum (see [24, 25] for example). In 1973, Schwenk [63] proved the important result that almost all trees are not determined by spectrum. In general, the problem of determining whether a graph is determined by spectrum or not is nontrivial. Haemers has proposed the following conjecture.

Conjecture 3.0.1 ([42, Haemers])**.** *Almost all graphs are determined by spectrum.*

In Section 3.1 we provide an overview of a well known technique for constructing cospectral graphs due to Godsil and McKay [37]. In Section 3.2 we discuss previous work done regarding the spectral characterization of graphs in the Johnson association scheme. Here we give an outline of the contributions made for the results in Section 3.3. The switching set described in Theorem 3.3.1 was found using Travis Johnston's code for $J(10, 5, \{0, 1, 2\})$. I generalized this switching set for graphs of the form $J(n, 2k+1, S)$ with $k \geq 2$ and $S = \{0, 1, \cdots, k\}$. I also proved Lemma 3.3.3 as well as Theorem 3.3.2.

As far as contributions for Section 3.4, the switching set described in Theorem 3.4.1 was found using Travis Johnston's code for $J(9, 4, \{0, 1\})$. I generalized this switching set for graphs of the form $J(4k+1, 2k, S)$ with $k \geq 2$ and $S = \{0, 1, \cdots, k-1\}$. Willem Haemers made the suggestion that our switching sets for these graphs could be generalized to give us Theorem 3.4.1. The proofs of Theorem 3.4.1, Lemma 3.4.3 and Theorem 3.4.2 are all due to me.

The work in Section 3.5 represents my exploration of how the switching sets described in Theorem 3.4.1 are related to switching sets previously found by Van Dam, Haemers, Koolen and Spence in [26] and Haemers and Ramezani in [43]. I also describe my attempts to further generalize the switching sets found in Theorem 3.4.1.

At the end of this chapter, we report on our computational results (using code written by Travis Johnston) obtained while searching for switching sets in various classes and unions of classes of the Johnson scheme including the Kneser graphs $K(9, 3)$ and $K(10, 3)$ which are the smallest examples of Kneser graphs for which it is not known whether they are determined by spectrum. We also present some open problems and further directions for research.

3.1 Godsil-McKay Switching and Cospectral Graphs

As we saw in Chapter 1 the spectrum of the adjacency matrix of a graph tells us a lot about the structure. However, in general, the spectrum does not determine the graph, meaning there exist nonisomorphic graphs with the same spectrum. Graphs are called *cospectral* if they have the same spectrum. Recall a graph Γ is said to be *determined by spectrum* if any graph cospectral to Γ must be isomorphic to Γ. Two non-isomorphic graphs that are cospectral are called *cospectral mates.*

The main tool we will make use of in this chapter is due to Godsil and McKay [37]. Given a graph Γ with a certain regular structure the following theorem allows us to construct a graph with the same spectrum by swapping edges and non-edges in a specific way.

Theorem 3.1.1 (Godsil-McKay [37]). *Let $\Gamma = (V, E)$ be a graph and $V = C_1 \cup C_2 \cup \cdots \cup C_t \cup D$ a partition of the vertex set of Γ satisfying the following conditions for $1 \leq i, j \leq t$:*

(i) Any two vertices in C_i have the same number of neighbors in C_j.

(ii) For each C_i every vertex in D is adjacent to $0, |C_i|/2$ or $|C_i|$ vertices in C_i.

Construct a new graph Γ' as follows. For every vertex u not in C_i with $|C_i|/2$ neighbors in C_i, delete the $|C_i|/2$ edges between u and C_i and join u to the other $|C_i|/2$ vertices in C_i. Then Γ and Γ' have the same spectrum.

Proof. We demonstrate that Γ and Γ' are cospectral by showing that their adjacency matrices are similar. For any positive integer m, define $Q_m = \frac{2}{m}J_m - I_m$ where J_m is the $m \times m$ all ones matrix and I_m is the $m \times m$ identity matrix. Then Q_m has the following properties

(1) $Q_m^2 = I_m$.

(2) If X is an $m \times n$ matrix with constant row sums and constant column sums, then $Q_m X Q_n = X$.

(3) If $\mathbf{x}$ is a vector with $2m$ entries, m of which are zero and m of which are one, then $Q_{2m}\mathbf{x} = \mathbf{j}_{2m} - \mathbf{x}$, where $\mathbf{j}_{2m}$ is the all ones vector of length $2m$.

If the vertices of Γ are labelled in an order consistent with the partition $C_1 \cup C_2 \cup \cdots \cup C_t \cup D$, then the adjacency matrix of Γ has the form

$$\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1t} & D_1 \\ C_{12}^\top & C_{22} & \cdots & C_{2t} & D_2 \\ \vdots & \vdots & & \vdots & \vdots \\ C_{1t}^\top & C_{2t}^\top & \cdots & C_{tt} & D_t \\ D_1^\top & D_2^\top & \cdots & D_t^\top & D \end{bmatrix}.$$

The required properties of our partition ensure that each C_i and each C_{ij} has constant row sums and constant column sums, and that each column of each D_i has either 0, $|C_i|/2$ or $|C_i|$ ones. Let $Q = \operatorname{diag}(Q_{|C_1|}, Q_{|C_2|}, \cdots, Q_{|C_t|}, I_{|D|})$. Then it follows that $QA_\Gamma Q = A_{\Gamma'}$ where A_Γ and $A_{\Gamma'}$ are the adjacency matrices of Γ and Γ', respectively. Since $Q^2 = I$, this proves that Γ and Γ' are cospectral. □

We note here that Γ and Γ' will have cospectral complements as well since switching in the complement of G with respect to the same partition generates a graph isomorphic to the complement of Γ'.

The operation above that changes Γ into Γ' is called Godsil-McKay switching or GM-switching for short. The partition $V = C_1 \cup C_2 \cup \ldots \cup C_t \cup D$ satisfying (i) and (ii) is known as a *switching partition*. For our purposes we will only need $t = 1$. Hence, from here forward we will simply use C to denote C_1 and refer to C as a *switching set*.

The following provides a visual example for the process of GM-switching. Let Γ be the graph of C_8 with a new vertex v adjoined to exactly half of the vertices in C_8. From this, we form a new graph Γ' through GM-switching. Here we can take C_8 to be our switching set, C, and v to be the only vertex in D. Since v is adjacent to exactly half of the vertices in C_8, we simply swap edges with non-edges.

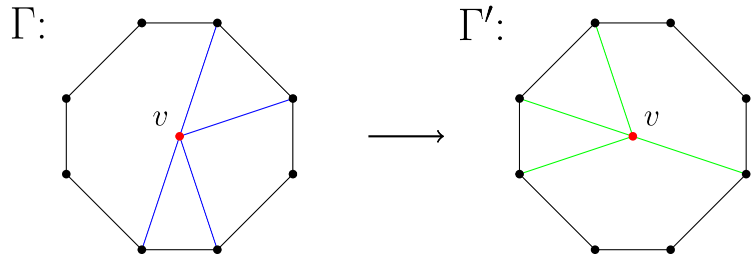


Figure 3.1: GM-switching produces a nonisomorphic graph

In the previous figure we saw how GM-switching may be applied to obtain a pair of nonisomorphic graphs with the same spectrum. However, the process of GM-switching does not always produce a nonisomorphic cospectral mate. It is clear that any set of two vertices satisfies the conditions of being a switching set. However, switching with respect to such a set will always produce a graph that is isomorphic to the original graph. Thus it is only beneficial to search for switching sets of order at least four. The problem of finding sufficient conditions for a graph obtained from GM-switching to be nonisomorphic with the original graph has gained attention recently. Abiad, Brouwer and Haemers [1] give a sufficient condition for being isomorphic after switching. Below, we provide another example of GM-switching applied to a graph resulting in an isomorphic copy of the original graph.

Again, let Γ be the graph of C_8 with a new vertex v adjoined to exactly half of the vertices in C_8. From this, we form a new graph Γ' through GM-switching. Here we can take C_8 to be our switching set, C, and v to be the only vertex in D. Since v is adjacent to exactly half of the vertices in C_8, we simply swap edges with non-edges. However, this time, we see that Γ' has the exact same structure as Γ.

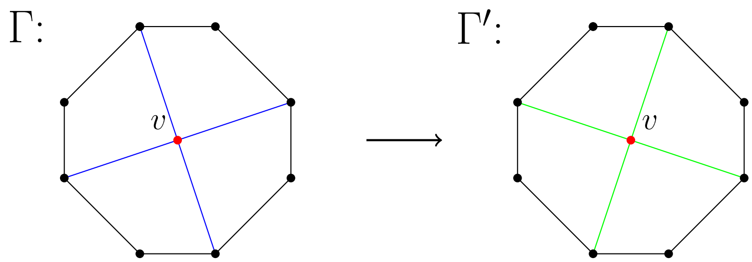
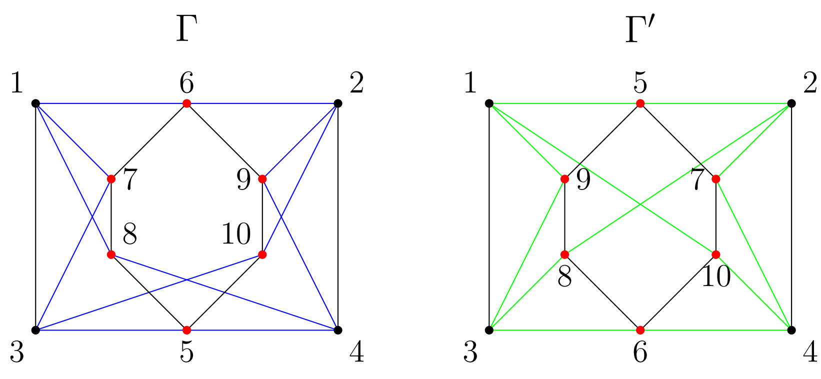


Figure 3.2: GM-switching produces an isomorphic graph

Because of the properties required for the switching partition, GM-switching is very useful when working with graphs with some regular structure. Below we provide an example of how *GM*-switching may be applied to obtain one of the four pairs of cospectral regular graphs on 10 vertices. Our switching set $C = \{1, 2, 3, 4\}$ is a union of two edges and $D = \{5, 6, 7, 8, 9, 10\}$ consists of the remaining vertices.



Spectrum: $4,\ 1\ ,\ -1^{(4)},\ \pm\sqrt{5},\ \frac{1}{2}(1 \pm \sqrt{17})$

Figure 3.3: GM-switching on a regular graph

3.2 Spectral Characterization of Graphs in the Johnson Scheme

In this section we provide a brief review of the work that has been done concerning the spectral characterization of graphs in the Johnson association scheme. More generally, we consider this problem for the union of classes in the Johnson association scheme.

Let $n \geq m \geq 2$ be two integers and S a subset of $\{0, 1, \ldots, m-1\}$. The graph $J(n, m, S)$ has as vertices the m-subsets of the n-set $[n] = \{1, \ldots, n\}$ and two m-subsets A and B are adjacent if $|A \cap B| \in S$. Using this notation, the Johnson graph $J(n, m, 1)$ (as seen in Chapter 2) is the graph $J(n, m, \{m-1\})$. From now on we will just denote this graph as $J(n, m)$. We will also denote the Kneser graph, $J(n, m, \{0\})$, by $K(n, m)$.

It is known that $J(n, 2)$ is determined by its spectrum precisely when $n \neq 8$ (see [15, 23, 44]). The graph $J(8, 2)$ is strongly regular with parameters $v = 28$, $k = 12$, $\lambda = 6$ and $\mu = 4$. There are exactly 3 cospectral mates for $J(8, 2)$. These graphs are known as the Chang graphs and can be described using GM-switching [15]. Each of these graphs may be obtained by switching with respect to one of the following sets [14, p. 52]:

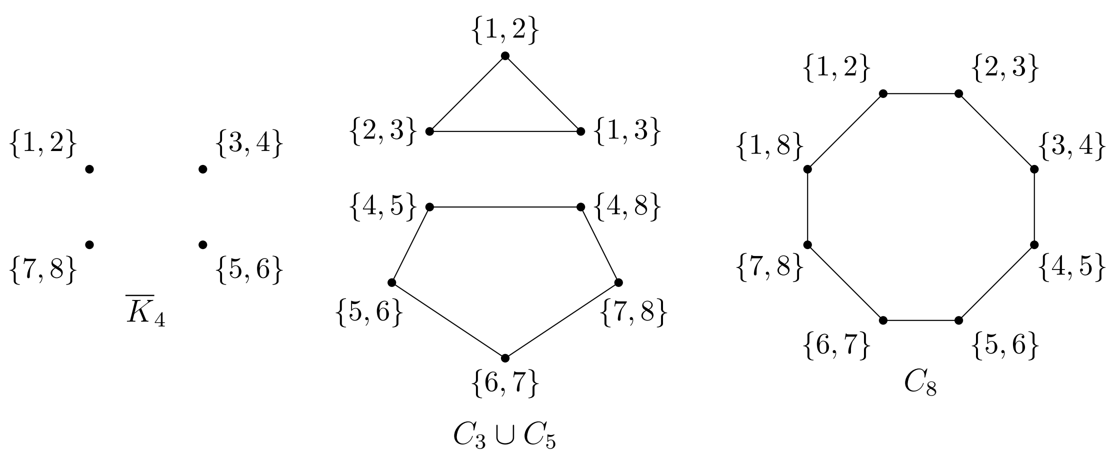


Figure 3.4: Switching sets for the Chang graphs from $J(8, 2)$

Huang and Liu [45] show that the odd graphs $K(2m + 1, m)$ for $m \geq 2$ are determined by spectrum. Currently the graphs $J(n, 2)$ with $n \neq 8$ (along with their complements) and the odd graphs $K(2m+1, m)$ with $m \geq 2$ (along with their complements) are the only nontrivial graphs in the Johnson scheme known to be determined by their spectra.

There has also been significant work done in constructing cospectral mates for graphs in the Johnson scheme. Van Dam, Haemers, Koolen and Spence [26] have shown that for $3 \leq m \leq n-3$, the Johnson graphs, $J(n, m)$, are not determined by their spectrum. The key to their result is GM-switching. The switching partition is constructed as follows. Fix a set Y of four elements of $[n]$. Let D be the set of m-tuples that do not contain precisely three elements of Y. For each $(m-3)$-subset T on $[n] \setminus Y$, let C_T be the set of (four) m-tuples containing T and precisely three elements of Y.

The union of the C_T is an equitable partition of $V \setminus D$, with quotient graph $J(n-4, m-3)$. Now, there are three possibilities for the adjacency between a vertex $x \in D$ and a set C_T.

(i) If x intersects Y in at most one element, then x has no neighbors in C_T.

(ii) If x intersects Y in two elements, then x has either two or zero neighbors in C_T (depending on whether x contains T or not).

(iii) If x intersects Y in four elements, then x has either zero or four neighbors in C_T.

The following is an illustration of a switching set in $J(7, 3)$. Let $Y = \{1, 2, 3, 4\}$. Note that $m = 3$ (we take a 0-tuple of Y to construct a switching set), so our switching partition consists of just one switching set C containing the vertices $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 3, 4\}$ and $\{2, 3, 4\}$.

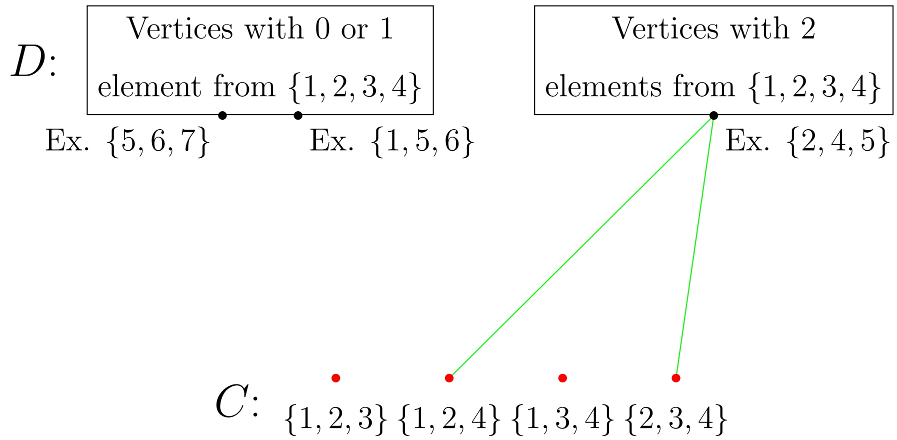


Figure 3.5: Example of GM-switching on $J(7,3)$

The collection of C_T described above forms a switching partition as described in Theorem 3.1.1. After switching with respect to this partition the graph we end up with is not isomorphic to $J(n,m)$. In fact the resulting graph is not even distance-regular. Indeed, let x be a vertex in $V \setminus D$, and let y be a vertex in D with one point in Y, which is also in x, and containing all $m-3$ points of x in $[n] \setminus Y$ (these exist since $n \geq m+3$). Then x and y are nonadjacent and they have precisely two common neighbors (namely the two vertices intersecting x and y in their common point of Y, containing the point of Y which is not contained in x, the $m-3$ points of x in $[n] \setminus Y$, and one of the two points of $y \setminus x$ in $[n] \setminus Y$). Since the Johnson graph has $c_2 = 4$ (not 2), the cospectral graph is not distance-regular.

Haemers and Ramezani [43] have shown that for $k \geq 3$, the Kneser graphs, $K(n,m)$, with $n = 3m-1$ or $2n = 6m - 3 + \sqrt{8m^2+1}$, as well as the mod-2 Kneser graph $J(n,m,S)$ (where S is the set of even numbers in $\{0, 1, \ldots, m-1\}$) are not determined by spectrum. The cospectral mates for both of these families of graphs were obtained using GM-switching. Here we provide a brief description of how switching works for $K(3m-1, m)$.

Let ℓ, m and n be positive integers such that $\ell < m < n/2$. Define

$$C = \{v \in V(K(n,m)) : \{1, \cdots, m-\ell\} \subset v\}.$$

Then C is a switching set in $K(n, m)$ if n, m and ℓ satisfy the following equation

$$\binom{n-m+\ell}{\ell} = 2\binom{n-2m+\ell}{\ell}.$$

Indeed, C induces an independent set of order $h = \binom{n-m+\ell}{\ell}$. Now, let x be a vertex outside of V_ℓ. There are two possibilities

(i) If x is disjoint from $\{1, \cdots, m-\ell\}$, then x has $\binom{n-2m-\ell}{\ell} = \frac{h}{2}$ neighbors in C.

(ii) If x has nonempty intersection with $\{1, \cdots, m-\ell\}$, then x is nonadjacent to all vertices of C.

The following is an illustration of a switching set in $K(8, 3)$. In this instance, C consists of all vertices containing the elements 1 and 2. It follows that any vertices containing a 1 or 2 will be nonadjacent to every vertex of C and any vertex disjoint from $\{1, 2\}$ will be adjacent to exactly half the vertices of C.

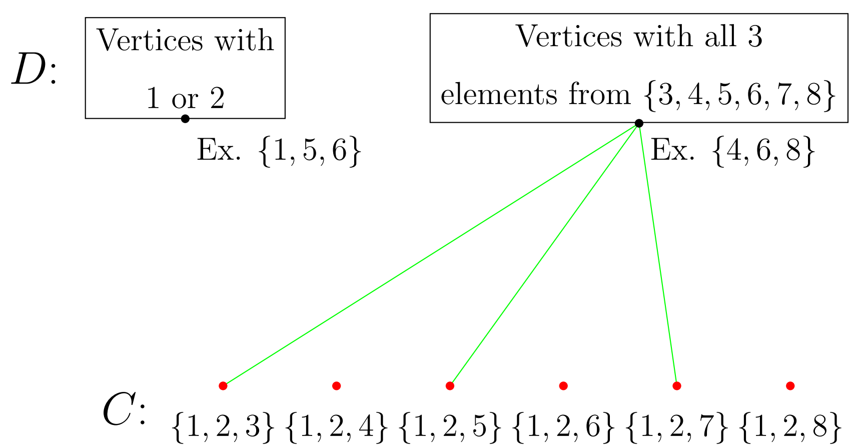


Figure 3.6: Example of GM-switching on $K(8, 3)$

To demonstrate that the graph obtained after switching with respect to C is not isomorphic to $K(n, m)$, the authors demonstrate that $n \geq 3m-1$ implying that $K(n, m)$ has diameter two. Next, consider the vertices $v = \{1, \cdots, m\}$ and $u = \{1, \cdots, m-\ell-1, m-\ell+1, \cdots, m+1\}$. Then $v \in C$, $u \notin C$ and $u \not\sim v$. The authors then show that, after switching, u and v have no common neighbors. Hence the diameter of the cospectral mate is at least three.

Until now, the Johnson graphs $J(8, 2)$, $J(n, m)$ (with $3 \leq m \leq n-3$), the Kneser graphs $K(n, m)$ (with $n = 3m - 1$ or $2n = 6m - 3 + \sqrt{8m^2 + 1}$) and the Modulo-2 Kneser graphs $J(n, m, S)$ with $3 \leq m \leq n - 3$ along with their complements were the only graphs in the Johnson scheme known to have cospectral mates.

In Section 3.3, we construct cospectral mates for the graphs $J(n, 2k + 1, S)$ for $k \geq 2$, $n \geq 4k + 2$ and $S = \{0, 1, \cdots, k\}$. And in Section 3.4 we construct cospectral mates for the graphs $J(3m - 2k - 1, m, S)$ for $k \geq 0$, $m \geq \max(k + 2, 3)$ and $S = \{0, 1, \cdots, k\}$. This provides two new infinite families of graphs corresponding to unions of classes in the Johnson scheme that are not determined by spectrum.

3.3 $J(n, 2k + 1, S)$ where $S = \{0, 1, \ldots, k\}$

In this section we use GM-switching to construct cospectral mates for the graphs $J(n, 2k + 1, S)$ with $k \geq 2$ and $S = \{0, 1, \ldots, k\}$. The following theorem describes the switching set used to construct the cospectral mates.

Theorem 3.3.1. *Let $k \geq 2$, $n \geq 4k + 2$ and $S = \{0, 1, \ldots, k\}$. If $\Gamma = J(n, 2k + 1, S)$ and $C = \{c \in V(\Gamma) : c \subset \{1, \ldots, 2k + 2\}$ and $|c| = 2k + 1\}$, then C is a switching set in Γ.*

Proof. First, note that C is an independent set of size $\binom{2k+2}{2k+1} = 2k + 2$ since the intersection of any two vertices in C has cardinality $2k$. Let $R = [2k + 2]$ and $u \in V(\Gamma) \setminus C$. We now consider the following cases:

(i) If $0 \leq |u \cap R| \leq k$, then for $c \in C$, $|u \cap c| \leq k$. Hence u will be adjacent to every vertex in C.

(ii) If $|u \cap R| = k + 1$, then there are $\binom{(2k+2)-(k+1)}{k} = \binom{k+1}{k} = k + 1$ vertices in C that intersect u in $k + 1$ elements, implying they will not be adjacent to u. The remaining $k + 1$ vertices in C must be adjacent to u as their intersection with u

will have cardinality k. Hence u will be adjacent to exactly half of the vertices in C.

(iii) If $k+2 \leq |u \cap R| \leq 2k$, then $|R \setminus u| \leq (2k+2)-(k+2) = k$. So for $c \in C$, $|u \cap c| \geq k+1$. Hence u will have no neighbors in C.

□

Below we give an example of what this switching set looks like in the graph $J(11, 5, \{0, 1, 2\})$. In this case we have $n = 11$ and $k = 2$. Hence our switching set, C, will consist of all 5-subsets of the set $R = \{1, 2, 3, 4, 5, 6\}$. Vertices intersecting R in less than 3 elements will be adjacent to every vertex in C. Vertices intersecting R in 3 elements will be adjacent to exactly half the vertices in C. Finally, vertices intersecting R in at least four elements will be adjacent to no vertices in C.

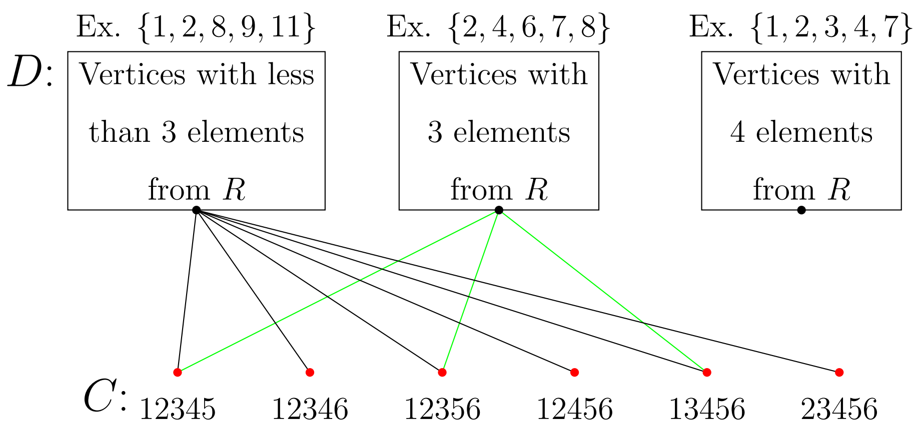


Figure 3.7: Example of GM-switching on $J(11, 5, \{0, 1, 2\})$

As we mentioned in Section 3.1, the process of GM-switching does not always result in a graph that is not isomorphic with the original graph. Thus, the remainder of this section will be devoted to establishing the following theorem.

Theorem 3.3.2. *Let $k \geq 2$, $S = \{0, 1, \ldots, k\}$ and C be the switching set described in Theorem 3.3.1. If $\Gamma = J_S(n, 2k+1)$ and Γ' is the graph obtained by switching with respect to C, then Γ and Γ' are not isomorphic.*

To prove Theorem 3.3.2 we define the *common neighbor count* $\lambda_\Gamma(x,y)$ of two vertices, x and y in Γ, as the number of vertices that are neighbors of both x and y. The *common neighbor pattern* of a vertex, x in Γ, is the multi-set of all possible values of $\lambda_\Gamma(x,y)$ where y runs through the vertex set of Γ.

Consider the following vertices

$$c_0 = [2k+1] \text{ and } v = \{2k+2, \ldots, 4k+2\}$$

in Γ and Γ'.

Lemma 3.3.3. *If Γ and Γ' are isomorphic, then $\lambda_\Gamma(c_0,v) = \lambda_{\Gamma'}(c_0,v)$.*

Proof. As Γ is vertex-transitive, all vertices of Γ will have the same common neighbor pattern. If Γ and Γ' are isomorphic, then all vertices of Γ' will have the same common neighbor pattern as well. In particular, v will have the same common neighbor pattern before and after switching. Based on the way switching is defined it follows that $\lambda_\Gamma(u,v) = \lambda_{\Gamma'}(u,v)$ for all u not in C. This implies

$$\{\lambda_\Gamma(c_0,v), \ldots, \lambda_\Gamma(c_{2k+1},v)\} = \{\lambda_{\Gamma'}(c_0,v), \ldots, \lambda_{\Gamma'}(c_{2k+1},v)\},$$

where c_i is the vertex in C that does not contain the element i for $1 \le i \le 2k+1$.

Moreover, the permutation $(1, 2, \ldots, 2k+1)$ of $[n]$ induces an automorphism of Γ that fixes c_0 and v and cyclically shifts $c_1, c_2, ..., c_{2k+1}$. It follows that

$$\lambda_\Gamma(c_1,v) = \cdots = \lambda_\Gamma(c_{2k+1},v).$$

This permutation remains an automorphism after switching, thus

$$\lambda_{\Gamma'}(c_1,v) = \cdots = \lambda_{\Gamma'}(c_{2k+1},v).$$

Therefore, if Γ and Γ' are isomorphic $\lambda_\Gamma(c_0,v) = \lambda_{\Gamma'}(c_0,v)$. □

Proof of Theorem 3.3.2. By Lemma 3.3.3, it is sufficient to show $\lambda_\Gamma(c_0,v) \neq \lambda_{\Gamma'}(c_0,v)$. Before switching c_0 is adjacent to vertices of the form

$$\binom{[2k+1]}{m} \cup \{2k+2\} \cup \binom{[n] \setminus [2k+2]}{k}.$$

Of these vertices, there are exactly $\binom{2k+1}{k}\left(\sum_{i=0}^{m-1}\binom{2k}{i}\binom{n-(4k+2)}{k-i}\right)$ adjacent to v. This accounts for the number of common neighbors of c_0 and v lost during switching. After switching c_0 becomes adjacent to vertices of the form

$$\binom{[2k+1]}{k+1} \cup \binom{[n] \setminus [2k+2]}{k}.$$

Of these vertices, there are exactly $\binom{2k+1}{k+1}\left(\sum_{i=0}^{m}\binom{2k}{i}\binom{n-(4k+2)}{k-i}\right)$ adjacent to v. As $n \geq 4k+2$, it follows that $\lambda_{\Gamma'}(c_0, v) = \lambda_{\Gamma}(c_0, v) + \binom{2k+1}{k+1}\binom{2k}{k}$. Therefore, Γ and Γ' are nonisomorphic. □

Corollary 3.3.4. *For $k \geq 2$, $n \geq 4k+2$ and $S = \{0, 1, \ldots, k\}$, the graphs $J(n, 2k+1, S)$ are not determined by spectrum.*

3.4 $J(3m-2k-1, m, S)$ where $S = \{0, \ldots, k\}$

In this section we use GM-switching to construct cospectral mates for the graphs $J(3m-2k-1, m, S)$ with $k \geq 2$ and $S = \{0, 1, \ldots, k\}$. The following theorem describes the switching set used to construct the cospectral mates.

Theorem 3.4.1. *Let $k \geq 0$, $m \geq k+2$ and $S = \{0, 1, \ldots, k\}$. If $\Gamma = J_S(3m-2k-1, m, S)$ and $C = \{c \in V(\Gamma) : [m-1] \subset c\}$, then C is a switching set in Γ.*

Proof. First, note that C is an independent set. Let $R = [3m-2k-1] \setminus [m-1]$. Then $|R| = 2(m-k)$.

Now, every $c \in C$ has the form $[m-1] \cup \{r\}$ for some $r \in R$. So $|C| = 2(m-k)$. Let $u \in V(\Gamma) \setminus C$. We consider the following cases:

(i) If $2 \leq |u \cap R| \leq m-k-1$, then $|u \cap [m-1]| \geq m-(m-k-1) = k+1$. Hence u has no neighbors in C.

(ii) If $|u \cap R| = m-k$, then $|u \cap [m-1]| = k$. So there will be $m-k$ vertices in C sharing exactly k elements with u and $m-k$ vertices in C sharing exactly $k+1$ elements with u. Hence, u will be adjacent to half the vertices in C.

(iii) If $m-k+1 \leq |u \cap R| \leq m$, then $|u \cap [m-1]| \leq m-(m-k+1) = k-1$. Hence u will be adjacent to each vertex in C.

□

Below we give an example of what this switching set looks like in the graph $J(9,4,\{0,1\})$. In this case we have $n=9$, $m=4$ and $k=1$. Hence our switching set, C, will consist of all 4-subsets containing $\{1,2,3\}$. Vertices intersecting $R=\{4,5,6,7,8\}$ in two elements will be adjacent to no vertices in C. Vertices intersecting R in three elements will be adjacent to exactly half the vertices in C. Finally, vertices intersecting R in four elements will be adjacent to no vertices in C.

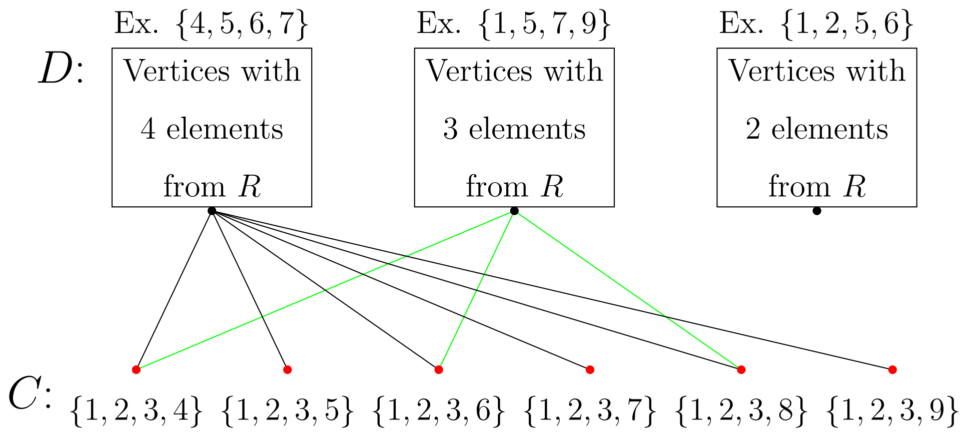


Figure 3.8: Example of GM-switching on $J(9,4,\{0,1\})$

As in Section 3.3, the remainder of this section will be devoted to proving that the graphs obtained by switching with respect to the set described in Theorem 3.4.1 are not isomorphic with $J(3m-2k-1,m,S)$.

Theorem 3.4.2. *Let $k \geq 0$, $m \geq \max(k+2,3)$, $S=\{0,1,\ldots,k\}$ and C be the switching set described in Theorem 3.4.1. If $\Gamma = J(3m-2k-1,m,S)$ and Γ' is the graph obtained by switching with respect to C, then Γ and Γ' are not isomorphic.*

Consider the vertices

$$c_0 = [m],\ c_1 = [m-1] \cup \{m+1\} \text{ and } w = [m-2] \cup \{m, m+1\}$$

in Γ and Γ'.

Lemma 3.4.3. *If Γ and Γ' are isomorphic, then $\lambda_\Gamma(c_0, w) = \lambda_{\Gamma'}(c_0, w)$.*

Proof. As Γ is vertex-transitive, all vertices of Γ will have the same common neighbor pattern. So if Γ and Γ' are isomorphic, then all vertices of Γ' will have the same common neighbor pattern. In particular, w will have the same common neighbor pattern before and after switching. Based on the way switching is defined it follows that $\lambda_\Gamma(u, w) = \lambda_{\Gamma'}(u, w)$ for all u not in C.

Hence,

$$\{\lambda_\Gamma(c_0, w), \ldots, \lambda_\Gamma(c_{2m-2k-1}, w)\} = \{\lambda_{\Gamma'}(c_0, w), \ldots, \lambda_{\Gamma'}(c_{2m-2k-1}, w)\},$$

where $c_i = [m-1] \cup \{m+i\}$ for $0 \le i \le 2m-2k-1$.

The permutation $(m+2, \ldots, 3m-2k-1)$ of $[3m-2k-1]$ induces an automorphism of Γ that fixes c_0, c_1 and w and cyclically shifts $c_2, \ldots, c_{2m-2k-1}$.

It follows that

$$\lambda_\Gamma(c_2, w) = \cdots = \lambda_\Gamma(c_{2m-2k-1}, w).$$

This permutation remains an automorphism after switching, thus

$$\lambda_{\Gamma'}(c_2, w) = \cdots = \lambda_{\Gamma'}(c_{2m-2k-1}, w).$$

Hence, if Γ and Γ' are isomorphic $\{\lambda_\Gamma(c_0, w), \lambda_\Gamma(c_1, w)\} = \{\lambda_{\Gamma'}(c_0, w), \lambda_{\Gamma'}(c_0, w)\}$. Observing $\lambda_\Gamma(c_0, w) = \lambda_\Gamma(c_1, w)$ gives the desired result. □

Proof of Theorem 3.4.2. By Lemma 3.4.3, it is sufficient to show $\lambda_\Gamma(c_0, w) \neq \lambda_{\Gamma'}(c_0, w)$. Before switching c_0 is adjacent to vertices of the form

$$\binom{[m-1]}{k} \cup \binom{[3m-2k-1] \setminus c_0}{m-k}.$$

Of these vertices, there are exactly $\binom{m-2}{k}\binom{2(m-k-1)}{m-k} + \binom{m-2}{k-1}\binom{2m-2k-1}{m-k}$ adjacent to w. This accounts for the number of common neighbors of c_0 and w deleted during switching. After switching c_0 becomes adjacent to vertices of the form

$$\binom{[m-1]}{k} \cup \{m\} \cup \binom{[3m-2k-1] \setminus c_0}{m-k-1}.$$

Of these vertices, there are exactly $\binom{m-2}{k-1}\binom{2(m-k-1)}{m-k-1}$ adjacent to w. As $m \geq k+2$, it follows that $\binom{m-2}{m}\binom{2(m-k-1)}{m-k} + \binom{m-2}{k-1}\binom{2m-2k-1}{m-k} > \binom{m-2}{k-1}\binom{2(m-k-1)}{m-k-1}$. Hence, $\lambda_\Gamma(c_0, w) > \lambda_{\Gamma'}(c_0, w)$. Therefore, Γ and Γ' are not isomorphic. □

Corollary 3.4.4. *For $k \geq 0$, $m \geq \max(k+2, 3)$, $S = \{0, 1, ..., k\}$, the graphs $J(3m-2k-1, m, S)$ are not determined by spectrum.*

3.5 Relation to Previous Work

In this section we discuss how the switching sets found in Theorem 3.4.1 are related to the switching sets found for Kneser graphs in [43] as well as the switching partition found for Johnson graphs in [26]. We also describe our attempts to generalize the switching sets we found in ways similar to what was done in both of these papers.

3.5.1 Relation to Kneser graphs and Johnson graphs

We begin by noting that the switching sets found in Theorem 3.4.1 include the switching sets found for the Kneser graphs $K(3m-1, m)$ in [43]. Indeed, taking $k = 0$ in Theorem 3.4.1, we obtain the switching sets found for the graphs $K(3m-1, m)$.

In addition to this generalization, one may also notice that the switching sets described in Theorem 3.4.1 are a generalization of the switching sets found for the Johnson graphs $J(n, 3)$ in [26]. Indeed, a switching set for $J(n, 3)$ can be obtained by taking the 4 vertices being 3 element subsets of a set of size 4. We note here $J(n, m, S)$ is isomorphic with $J(n, n-m, S+n-2m)$ and with the complement of $J(n, m, M \setminus S)$, where $M = \{0, 1, ..., m-1\}$. Using these facts, it follows that the complement of $J(n, 3, \{2\})$ is $J(n, 3, \{0, 1\})$ and $J(n, 3, \{0, 1\})$ is isomorphic to $J(n, n-3, \{n-6, n-5\})$. Thus, this family of graphs can be described in Theorems 3.4.1 and 3.4.2 by letting $k = m-2$.

What is more, the switching sets for $J(n, 3)$ were extended for $m > 3$ by using the more general form of switching described in Theorem 3.1.1. Recall, a switching partition can be constructed for $J(n, m)$, $n-3 \geq m \geq 3$ by fixing a set Y of size 4 and letting D be the set of all m-sets that do not contain precisely 3 elements from

Y. Then for each $(m-3)$-subset I of $[n] \setminus Y$, we allow C_I to be the set of four m-sets containing I and precisely 3 elements from Y.

This leads to the question of whether or not we can make a similar generalization in Theorems 3.4.1 and 3.4.2. In an attempt to extend our results for $n \neq 3m-2k-1$ we construct similar sets C_I in the following way. For each $(m-1)$-subset I of $[n-2(m-k)]$ take C_I to be the set of $2(m-k)$ vertices containing i and one element from $[n] \setminus [n-2(m-k)]$. When $k = m-2$ we obtain switching sets for the complements of the Johnson graphs $J(n,m)$, $n-3 \leq m \leq 3$.

Now, consider the graph $J(3m-2k-1, m, S)$ described in Theorems 3.4.1 and 3.4.2 and take v to be a vertex containing $[k] \cup \{m\}$ and $m-k-1$ elements from $[n] \setminus [n-2(m-k)]$. Consider the switching set C_A formed by taking the $2(m-k)$ vertices containing $[m-1]$ and one element from $[n] \setminus [n-2(m-k)]$. It is easily seen that v will have $m-k+1$ neighbors in C_A. As $m-k < m-k+1 < 2(m-k)$ it follows that v cannot be in D. Thus, the only option is for v to be in some other C_I which can only occur if $k = m-2$.

3.5.2 Vertices containing $[m-2]$ as a possible switching set

In [43] it was shown that taking C to be the set of vertices containing $[m-2]$ as a subset is a switching set for the graphs $K(n,m)$ satisfying $2n = 6m-3+\sqrt{8m^2+1}$. It is reasonable to try to generalize this in a way similar to what we have done for $K(3m-1,m)$.

Consider the graph $J(n,m,S)$ where $S = \{0, ..., k\}$ and let C be the set of vertices containing $[m-2]$ as a subset. Suppose u is a vertex not in C. We consider the following cases:

(i) If u has $k+1$ or more elements in $[m-2]$, then u is adjacent to no vertices in C.

(ii) If u has $k-2$ or less elements in $[m-2]$, then u is adjacent to all vertices in C.

(iii) If u has $k-1$ elements in $[m-2]$, then u is adjacent to $\binom{n-m+2}{2} - \binom{m-k+1}{2}$ vertices in C.

(iv) If u has k elements in $[m-2]$, then u is adjacent to $\binom{n-m+2}{2} - (n-2m+k+2)(m-k) - \binom{m-k}{2}$ vertices in C.

So our restrictions on n, m and k will come from (iii) and (iv). Note that $\binom{n-m+2}{2} - \binom{m-k+1}{2} \neq |C|$ as $k < m-1$. If $\binom{n-m+2}{2} - \binom{m-k+1}{2} = 0$, then $n = 2m-k-1$, but evaluating (iv) with $n = 2m-k-1$ gives 0 so this would not give a desirable switching set.

If $\binom{n-m+2}{2} - \binom{m-k+1}{2} = (1/2)|C|$, then solving for n we obtain

$$n = (1/2)(\sqrt{8m^2 - 16mk + 8m + 8k^2 - 8k + 1} + 2m - 3).$$

If (iv) is equal to $|C|$, then we find $n = (1/2)(3m-k-3)$. Setting these equal gives no solutions.

If (iv) is equal to 0, then we find $n = 2m-k-1$ or $n = 2m-k-2$, which will make (iii) equal to 0 or $k-m$, respectively.

Finally, if (iv) is equal to $(1/2)|C|$, then (iii) and (iv) are equal and we find $n = 2m-k+1$. Solving for m we find $m = (1/2)(2k - \sqrt{33} + 3)$ or $m = (1/2)(2k + \sqrt{33} + 3)$, neither of which are integers.

One thing to note is when $k = 0$, (iii) is not a possibility and we need (iv) equal to $(1/2)|C|$. Solving for n in this case we obtain $2n = 6m - 3 + \sqrt{8m^2+1}$, the same parameters found previously for Kneser graphs.

3.6 Computational Results

In this section, we report on a list of classes and union of classes in the Johnson scheme we have checked by computer for switching sets. We used two pieces of code to search for switching sets. This code was written by Travis Johnston during his time in the Computer and Information Sciences Department at the University of Delaware.

The first code made use of GPUs (general purpose Graphical Processing Units) to exhaustively search a graph for (small) switching sets. The key feature of GPUs is that they allow for massive parallel computation. In our case, each independent thread examines one induced subgraph of size $2i$ for $i = 2, 3, 4, 5, \ldots$ specified by the user. Because graphs in the Johnson scheme are vertex-transitive we were able to reduce the necessary computation and only examine subgraphs that included vertex 1. While the GPU dramatically speeds up the computation, if either the Johnson graph is large or the size of the subgraphs being examined is large then the computation is still prohibitively long. Focusing mainly on Kneser graphs, we were able to eliminate the possibility of switching sets of size 8 in $K(9,3)$, $K(10,3)$, $K(11,3)$, $K(12,3)$ and $K(10,4)$ as well as switching sets of size 10 in $K(9,3)$ and $K(10,3)$. Our computations extend the computations of Haemers and Ramezani [43] which did not find any switching sets of size 4 or 6 in the Kneser graphs $K(9,3)$ nor $K(10,3)$. At the present time, these are the smallest graphs in the Johnson scheme whose spectral characterization is not known.

The second code employed a technique similar to backtracking and searched only for switching sets of size 4 (an independent set, an induced matching, an induced cycle, and a complete graph), and switching sets of size 6 restricted to independent sets, induced matchings, and an induced 6-cycle. Because of the restrictions on the type of switching sets that were searched for (and the relatively small size) we were able to explore larger graphs in the Johnson scheme. Both codes are available on github at https://github.com/jtjohnston/computational_combinatorics/tree/master/GM-switching. Below we provide tables outlining our search for switching sets for graphs in the Johnson scheme. The notation for these tables is described as follows.

- 0b indicates that no switching sets were found using backtracking technique.
- 0eX indicates that no switching sets were found of size 4, 6, ..., X using the exhaustive search on GPU.
- 1(DS) indicates that these graphs have already been proven to be DS.
- 1(NDS) indicates that these graphs have already been proven to not be DS.
- 1+ indicates we found a new switching set and the graph after switching is non-isomorphic.
- 1- indicates we found a new switching set but the graph after switching is isomorphic.

n	$S=\{0\}$	$S=\{1\}$	$S=\{2\}$	$S=\{0,1\}$	$S=\{0,2\}$	$S=\{1,2\}$
6	1(DS)	1(NDS)	1(NDS)	1(NDS)	1(NDS)	1(DS)
7	1(DS)	1(NDS)	1(NDS)	1(NDS)	1(NDS)	1(DS)
8	1(NDS)	1(NDS)	1(NDS)	1(NDS)	1(NDS)	1(NDS)
9	0e10	1(NDS)	1(NDS)	1(NDS)	1(NDS)	0e10
10	0e10	1(NDS)	1(NDS)	1(NDS)	1(NDS)	0e8
11	0e8	1(NDS)	1(NDS)	1(NDS)	1(NDS)	0e6
12	0e8	1(NDS)	1(NDS)	1(NDS)	1(NDS)	0b
13	0b	1(NDS)	1(NDS)	1(NDS)	1(NDS)	0b
14	0b	1(NDS)	1(NDS)	1(NDS)	1(NDS)	0b
15	0b	1(NDS)	1(NDS)	1(NDS)	1(NDS)	0b

Table 3.1: Switching sets checked for in $J(n, m, S)$ with $m = 3$

n	$S=\{0\}$	$S=\{1\}$	$S=\{2\}$	$S=\{3\}$	$S=\{0,1\}$	$S=\{0,2\}$	$S=\{1,2\}$
8	1(DS)	0e8	1-	1(NDS)	0e8	1(NDS)	0e8
9	1(DS)	0e8	0e8	1(NDS)	1+	1(NDS)	0e8
10	0e8	0b	0b	1(NDS)	0b	1(NDS)	0b
11	1(NDS)	0b	0b	1(NDS)	0b	1(NDS)	0b
12	0e6	0b	0b	1(NDS)	0b	1(NDS)	0b

Table 3.2: Switching sets checked for in $J(n, m, S)$ with $m = 4$

n	$S=\{0\}$	$S=\{1\}$	$S=\{2\}$	$S=\{3\}$	$S=\{4\}$
10	1(DS)	0e6	0e6	0e6	1(NDS)
11	1(DS)	0b	0b	0b	1(NDS)

Table 3.3: Switching sets checked for in $J(n, m, S)$ with $m = 5$

3.7 Open Problems

In this section we review some open problems dealing with the spectral characterization for graphs constructed from classes in the Johnson association scheme. As mentioned before, the smallest open case as of now is $K(9,3)$. Regarding the tables above, we were only able to find two new graphs with potential switching sets. For $J(9,4,\{0,1\})$ we were able to generalize the switching set we found in Theorem 3.4.1.

Our computer search for switching sets found switching sets of size order four for the graph $J(8,4,\{2\})$. None of the switching sets of order four we found produced nonisomorphic cospectral mates. Focusing more closely on this graph, we extended our computations to look for switching sets of size 8. We found two different switching sets of order 8 that produced nonisomorphic cospectral mates. One such switching set is

$$\{\{1,2,3,4\},\{1,2,5,6\},\{1,2,3,5\},\{1,2,4,6\},\{3,4,7,8\},\{3,5,7,8\},\{4,6,7,8\},\{5,6,7,8\}\}$$

which is the union of two 4-cycles. The second is a 6-regular graph on 8 vertices

$$\{\{1,2,3,4\},\{1,2,3,5\},\{1,4,6,7\},\{1,5,6,7\},\{2,3,4,8\},\{2,3,5,8\},\{4,6,7,8\},\{5,6,7,8\}\}.$$

We explored both of these switching sets but were unable to establish a generalization that resulted in an infinite family of graphs with cospectral mates. For this reason, we believe it is worth exploring possible generalizations of these switching sets in order to generate more infinite families of graphs in the Johnson scheme with cospectral mates.

Chapter 4

CONNECTIVITY OF SOME GRAPHS IN ASSOCIATION SCHEMES

4.1 Some Conjectures of Brouwer and Godsil

In this chapter we study the connectivity of graphs in association schemes. In Section 4.2 we give a brief background of the work that has been done dealing with the connectivity of distance-regular graphs and graphs related to association schemes. Our work provides support for conjectures regarding the connectivity of graphs in association schemes. Below we provide definitions for the edge-connectivity and vertex-connectivity of a graph Γ. More details concerning the connectivity of graphs will be discussed later in this chapter.

Definition 4.1.1. The *edge-connectivity* of a graph Γ is the minimum number of edges that need to be deleted from Γ in order to disconnect Γ.

Definition 4.1.2. The *vertex-connectivity* of a graph Γ is the minimum number of vertices that need to be deleted from Γ in order to disconnect Γ.

The first conjecture we present is due to Godsil regarding the edge-connectivity of color classes in association schemes.

Conjecture 4.1.3 (Godsil, [35])**.** *The edge-connectivity of any graph corresponding to a connected basis relation in any symmetric association scheme is equal to the degree of the graph.*

The second stronger conjecture is due to Brouwer, who makes the same claim for the vertex-connectivity of color classes in association schemes.

Conjecture 4.1.4 (Brouwer, [6])**.** *The vertex-connectivity of any graph corresponding to a connected basis relation in any symmetric association scheme is equal to the degree of the graph.*

The final conjecture we discuss is again due to Brouwer, who proposed the following condition on the connectivity of strongly regular graphs.

Conjecture 4.1.5 (Brouwer, [6])**.** *The minimum number of vertices needed to be removed from a strongly regular graph in order to disconnect it into non-singleton components is* $2k - \lambda - 2$*.*

Note that the bound $2k - \lambda - 2$ appearing in Conjecture 4.1.5 is the order of the neighborhood of an edge. In Section 4.3 we consider the edge version of Conjecture 4.1.5 for distance-regular graphs of diameter at least three. The main contributions made in Section 4.3 include several lemmas and the main result presented at the end. The main theorem states that the minimum number of edges needed to be removed from a distance-regular graph to disconnect it into nonsingleton components is $2k - 2$ where k is the degree of regularity.

In Section 4.5 we verify that the edge-connectivity of the distance-j twisted Grassmann graphs is equal to their degree, k_j. This provides support for Conjecture 4.1.3. As far as the contributions for Section 4.5, the main result provides support for a Conjecture 4.1.3. Specifically, we prove that the minimum number of edges needed to be removed to disconnect a distance-j graph of the twisted Grassmann graph is equal to k_j, the degree of the graph.

4.2 Connectivity of Graphs in Association Schemes

In order to motivate the work done in this chapter we provide a brief survey of previous results dealing with the connectivity of graphs in association schemes.

In 1972, Plesník [62] proved that the edge-connectivity of a strongly regular graph is equal to its degree. In 1985, Brouwer and Mesner [13] proved that the vertex-connectivity of a strongly regular graph Γ is equal to its degree and the only disconnecting sets of minimum order are the neighborhoods of its vertices. In 2005, Brouwer and Haemers [10] proved that the edge-connectivity of a distance-regular graph is equal to its degree. Brouwer and Mesner's result for vertex-connectivity of strongly regular graphs was extended in 2009 by Brouwer and Koolen [12] who proved that the vertex-connectivity of a distance-regular graph of valency at least 3 is equal to its degree and the only disconnecting sets of minimum order are the neighborhoods of its vertices.

Regarding Conjecture 4.1.5, Cioabă, Kim and Koolen [20] proved that it is false in general by showing that the triangular graphs $T(n)$ $(n \geq 6)$ and the symplectic graphs $Sp(2r, q)$ $(r \geq 2)$ are counterexamples. In this paper they also show that Conjecture 4.1.5 is true for many families of graphs including graphs from copolar spaces and Δ-spaces, conference graphs, the generalized quadrangles $GQ(q, q)$, the lattice graphs and some Latin square graphs. The authors conjecture that Conjecture 4.1.5 is true for all strongly regular graphs having $k \geq 2\lambda + 1$. Cioabă, Koolen and Li [21] proved that Conjecture 4.1.5 is true for all strongly regular graphs with $\max(\lambda, \mu) \leq k/4$. They also prove other families of strongly regular graphs satisfy Conjecture 4.1.5, including block graphs of Steiner 2-designs and many Latin square graphs.

In [35], Godsil proved that if Γ is a connected k-regular graph corresponding to a basis relation in an association scheme, then the edge-connectivity of Γ is at least $\frac{k}{2}\frac{|X|}{|X|-1}$. In 2006, Evdokimov and Ponomarenko proved Conjecture 4.1.4 for specific families of graphs in association schemes (see [33] for definitions and details).

More recently, Kodalen and Martin [49] proved that the deletion of the neighborhood of any vertex leaves behind at most one non-singleton component. Note that in the case of distance-regular graphs a stronger result was proven by Cioabă and Koolen [19]. We say two vertices $x,\ y \in X$ are *twins* in $\Gamma = (X, \mathcal{R}_j \cup \mathcal{R}_j^\top)$ if they have identical neighborhoods: $\Gamma(x) = \Gamma(y)$. Kodalen and Martin also characterize twins in polynomial association schemes and show that, in the absence of twins, the deletion of any vertex and its neighbors in Γ results in a connected graph.

Much more can be said about the connectivity of vertex-transitive and edge-transitive graphs. For a good overview of the work done for these graphs we refer the reader to [39, Sec. 3.3–4]. Mader [57] and Watkins [67] independently obtained the following two results on connectivity in 1970. The vertex-connectivity of an edge-transitive graph is equal to its minimum degree. A vertex-transitive graph of degree k has vertex-connectivity at least $\frac{2}{3}(k+1)$. Further, in 1971, Mader [58] proved that any vertex-transitive graph has edge-connectivity equal to its degree.

In the following sections we present our contributions to problems dealing with the connectivity of graphs in association schemes.

4.3 Disconnecting Distance-Regular Graphs into Nonsingleton Components by Removing Edges

In this section we consider the problem of determining the number of edges needed to be removed from a distance-regular graph in order to disconnect it into non-singleton components. This problem was considered by Cioabă, Koolen and Li for strongly regular graphs where the authors proved the following result [21].

Theorem 4.3.1. *Let Γ be a connected (v, k, λ, μ)-SRG. If A is a subset of vertices with $2 \leq A \leq v/2$, then*

$$|e(A, A^c)| \geq 2k - 2.$$

Equality happens if and only if A consists of two adjacent vertices or A induces a triangle in $K_{2,2,2}$ or A induces a triangle in the line graph of $K_{3,3}$.

Our main result is given in Theorem 4.3.16, where we generalize Theorem 4.3.1 for distance-regular graphs. The majority of the work preceding this theorem consists of lemmas that will ultimately lead us to the main result. The following lemma gives a well known bound on the size of an edge-cut in a graph.

Lemma 4.3.2 ([9, Cor. 4.8.4] or [60]). *Let* Γ *be a graph on* v *vertices and* $A \subset V(\Gamma)$*. Then*

$$|e(A, A^c)| \geq \mu_2 |A| \left(1 - \frac{|A|}{v}\right),$$

where μ_2 *is the second smallest eigenvalue of the Laplace matrix of* Γ*.*

For the remainder of this section suppose $T = e(A, A^c)$ is an edge cut in the graph Γ, where A is a connected component of Γ minus T of order at most $v/2$. Suppose for now that Γ is a distance-regular graph of diameter at least three and degree at least four. The following lemma due to Cioabă, Koolen and Li [21] helps us verify that if there exists an edge cut of Γ of size less than $2k-2$, then the smallest connected component (say A) left after the removal of these edges has to have size at least k.

Lemma 4.3.3. *Let* Γ *be a distance-regular graph of valency* $k \geq 4$*. If* $A \subset V$ *with* $3 \leq |A| \leq k-1$*, then* $|e(A, A^c)| \geq 3k-6$*.*

Indeed, since $|e(A, A^c)| \geq 3k - 6 \geq 2k - 2$ for $k \geq 4$, we find $|A| \geq k$ whenever $|e(A, A^c)| < 2k - 2$. This will prove useful in the proof of the following lemma.

Lemma 4.3.4. *If* A *is a connected component of a distance-regular graph* ($k \geq 4$) *minus a disconnecting set of edges* T *then there exists a vertex of* A *that is incident with at most one edge of* T*.*

Proof. If every vertex of A is incident with at least two edges of T, then the subgraph induced by A has maximum degree at most $k-2$. Hence $e(A) \leq |A|(k-2)/2$, where $e(A)$ is the number of edges with both endpoints in A. It follows that

$$\begin{aligned} e(A, A^c) &\geq k|A| - 2e(A) \\ &\geq k|A| - (k-2)|A| \\ &= 2|A|. \end{aligned}$$

By Lemma 4.3.3 it follows that $e(A, A^c) \geq 2k$. □

Lemma 4.3.5 ([10, Lemma 4.3])**.** *Let T be a disconnecting set of edges of a distance regular graph Γ, and let A be the vertex set of a component of Γ minus T. Fix a vertex $a \in A$ and let t_i be the number of edges in T that join $\Gamma_{i-1}(a)$ and $\Gamma_i(a)$. Then $|A \cap \Gamma_i(a)| \geq \left(1 - \sum_{j=1}^{i} \frac{t_j}{c_j k_j}\right) k_i$, so that*

$$|A| \geq v - \sum_i \frac{t_i}{c_i k_i}(k_i + \ldots + k_d).$$

Using Lemma 4.3.5 along with the fact that $(k_i + \ldots + k_d)/k_i \geq (k_{i+1} + \ldots + k_d)/k_{i+1}$ it follows that if there exists a vertex of A incident with at most one edge of T, then

$$\begin{aligned} |A| &\geq v - \frac{1}{k}(v-1) - \frac{|T|-1}{\mu k_2}(v-k-1) \\ &> v\left(1 - \frac{1}{k} - \frac{|T|-1}{\mu k_2}\right). \end{aligned}$$

Similarly, if there exists a vertex of A incident with no edges of T, then

$$|A| > v\left(1 - \frac{|T|}{\mu k_2}\right). \tag{4.1}$$

For the rest of this section we may use the notation $a = |A|$.

Lemma 4.3.6. *If Γ is a distance-regular graph of valency 3 and A is a set of vertices such that $2 \leq |A| \leq v/2$, then $e(A, A^c) \geq 2k - 2$.*

Proof. There are 10 cubic distance-regular graphs of diameter at least 3 (see Figure 4.1 below).

d	v	Name	Intersection Array
3	8	3-Cube	$\{3, 2, 1; 1, 2, 3\}$
3	14	Heawood	$\{3, 2, 2; 1, 1, 3\}$
4	18	Pappus	$\{3, 2, 2, 1; 1, 1, 2, 3\}$
4	28	Coxeter	$\{3, 2, 2, 1; 1, 1, 1, 2\}$
4	30	Tutte's 8-cage	$\{3, 2, 2, 2; 1, 1, 1, 3\}$
5	20	Dodecahedron	$\{3, 2, 1, 1, 1; 1, 1, 1, 2, 3\}$
5	20	Desargues	$\{3, 2, 2, 1, 1; 1, 1, 2, 2, 3\}$
6	126	Tutte's 12-cage	$\{3, 2, 2, 2, 2, 2; 1, 1, 1, 1, 1, 3\}$
7	102	Biggs-Smith	$\{3, 2, 2, 2, 1, 1, 1; 1, 1, 1, 1, 1, 1, 3\}$
8	90	Foster	$\{3, 2, 2, 2, 2, 1, 1, 1; 1, 1, 1, 1, 2, 2, 2, 3\}$

Table 4.1: Cubic distance-regular graphs of diameter at least three

All of these graphs have $b_1 = 2$. For the sake of contradiction, suppose $e(A, A^c) \leq 3$. From Lemma 4.3.4 and the results of Lemma 4.3.5, it follows that

$$|A| > v\left(1 - \frac{1}{k} - \frac{2k-4}{\mu k_2}\right) = v(1 - 1/3 - 2/6) = v/3.$$

Using Lemma 4.3.2 we have that

$$e(A, A^c) \geq (k - \theta_1)a\left(1 - \frac{a}{v}\right).$$

Taking the derivative of the right hand side with respect to a, we find that right hand side of the inequality is increasing for $a < v/2$. As $|A| > v/3$ we see that $e(A, A^c) > \frac{2v}{9}(3 - \theta_1)$. So it follows that $3 > \frac{2v}{9}(3 - \theta_1)$. This was checked and found to be false for each cubic distance-regular graph of diameter at least three. □

Lemma 4.3.7. *The minimum number of edges that need to be removed from the icosahedron to disconnect it into components of order 2 or greater is 8.*

Proof. The icosahedron has 12 vertices with intersection array $\{5, 2, 1; 1, 2, 5\}$ and second largest eigenvalue $\sqrt{5}$. From Lemma 4.3.3 it follows that $|A| \geq 5$. Using Lemma 4.3.2 we have that

$$e(A, A^c) \geq (k - \theta_1)a\left(1 - \frac{a}{v}\right),$$

where $a = |A|$. Taking the derivative of the right hand side with respect to a, we find that the right hand side of the inequality is increasing for $a < v/2 = 6$. Using this in Lemma 4.3.2 we have

$$e(A, A^c) \geq 5(5 - \sqrt{5})\left(1 - \frac{5}{12}\right) > 8.$$

□

The following theorem and three propositions can be found in [8]. These will come in handy when dealing with some cases in the proof of Lemma 4.3.15. For more information of the geometric structures mentioned in this section we refer the reader to [8, Chapter 4].

Definition 4.3.8. The line graph $L(\Gamma)$ of a graph Γ is the graph with vertex set equal to $E(\Gamma)$ where two elements of $E(\Gamma)$ are adjacent if and only if they are incident with the same vertex.

Definition 4.3.9. For each vertex γ of a graph Γ let us write $\gamma^\perp = \{\gamma\} \cup \Gamma(\gamma)$. If A is a set of vertices, we let $A^\perp = \bigcap_{\gamma \in A} \gamma^\perp$ be the set of all vertices at distance at most one from each vertex in A. *Singular lines* of Γ are sets L of the form $\{\gamma, \delta\}^{\perp\perp}$ where γ and δ are adjacent vertices.

Theorem 4.3.10 ([8, Thm. 4.2.16])**.** *Let Δ be a connected graph such that its line graph $\Gamma = L(\Delta)$ is distance-regular. Then we have one of the following:*

(i) $\Delta \cong K_{1,m}$ *and* $\Gamma \cong K_m$*, a complete graph;*

(ii) $\Delta \cong K_m$ *and* $\Gamma \cong T(m)$*, a triangular graph;*

(iii) $\Delta \cong K_{m,m}$ *and* $\Gamma \cong K_m \square K_m$*;*

(iv) Δ *is a polygon and* $\Gamma \cong \Delta$*;*

(v) Δ *is regular with diameter* $d \geq 3$ *and girth* $g = 2d$*, i.e.,* Δ *is a regular generalized* $2d$*-gon of order* $(1, s)$*, say, where* $d \in \{3, 4, 6\}$*, and* Γ *is a generalized* $2d$*-gon of order* $(s, 1)$*; in other words,* Δ *and* Γ *are the incidence graph and the flag graph of a regular generalized* d*-gon of order* (s, s)*;*

(vi) Δ *is regular with diameter* $d \geq 2$ *and girth* $g = 2d + 1$*, i.e.,* Δ *is a Moore graph and* $d = 2$*;* Δ *has* 10*,* 50 *or perhaps* 3250 *vertices.*

Proposition 4.3.11 ([8, Prop. 4.3.2])**.** *If, in a distance-regular graph, one of the following conditions*

(i) $\mu = 1$ *or* $\lambda = 1$*,*

(ii) $\mu = 2$ *and* $k < \frac{1}{2}\lambda(\lambda + 3)$*,*

(iii) $b_{d-1} = 1$ *and* $a_d = \lambda + 1$*,*

(iv) $b_i > 1$ *and* $c_i = b_1$

holds, then all singular lines have size $\lambda+2$*. In case (iii) we additionally have* $(\lambda+2)|k_d$*.*

Proposition 4.3.12 ([8, Prop. 4.3.3])**.** *If in a distance-regular graph* Γ *all singular lines have size* $\lambda + 2$*, then*

$$(\lambda + 1)|k,\ (\lambda + 1)(\lambda + 2)|vk$$

and

$$a_2 \geq \lambda\mu.$$

If $\mu = 1$*, then also*

$$\frac{vk}{(\lambda + 1)(\lambda + 2)} \geq 1 + \frac{\lambda + 2}{\lambda + 1}b_1 + \frac{\lambda + 2}{\lambda + 1}b_1^2.$$

Proposition 4.3.13 ([8, Prop. 4.3.4])**.** *If all singular lines in* Γ *have size* $\lambda + 2$*, and* $k = 2(\lambda + 1)$*, then* Γ *is a line graph (and hence satisfies the conclusion of Theorem 4.3.10).*

The following lemma provides a useful bound on the smallest and second largest eigenvalue of a distance-regular graph in terms of the parameter b_1.

Theorem 4.3.14 ([8, Thm. 4.4.3])**.** *Let Γ be a distance-regular graph of diameter at least three with eigenvalues $k = \theta_0 > \theta_1 > \cdots > \theta_d$ and put $b^+ = \frac{b_1}{\theta_1+1}$, $b^- = \frac{b_1}{\theta_d+1}$. Then $b^+ > 0$, $b^- < -1$, and we have:*

(i) Each neighborhood $\Gamma(\gamma)$ of a vertex γ is a graph with smallest eigenvalue at least $-1-b^+$ and second largest eigenvalue at most $-1-b^-$ (here the second largest eigenvalue is taken to be the valency λ in the case $\Gamma(\gamma)$ is disconnected).

(ii) If $b^+ < 1$, then either $\mu = 1$ or Γ is the icosahedron.

(iii) If $b^- > -2$, then either $\lambda = 0$ or Γ is the icosahedron.

Lemma 4.3.15. *Suppose Γ is a distance-regular graph of degree $k \geq 4$. If A is a subset of vertices with $2 \leq |A| \leq v/2$, then $e(A, A^c) \geq 2k-2$.*

Proof. By Theorem 4.3.14 (ii) we have: either Γ is the icosahedron, $\mu = 1$ or $\theta_1 \leq b_1 - 1$. If $3 \leq |A| \leq k-1$, then by Lemma 4.3.3

$$e(A, A^c) \geq 3k-6 \geq 2k-2,$$

as $k \geq 4$.

Now, if $k \leq |A| \leq v/2$ Lemma 4.3.2 tells us

$$e(A, A^c) \geq (k-\theta_1)\left(\frac{a(v-a)}{v}\right) \geq \frac{(k-\theta_1)(k)}{2}.$$

If $\mu > 1$, then $\theta_1 \leq b_1 - 1$ and it follows that $k - \theta_1 \geq k - b_1 + 1 = \lambda + 2$. Hence $e(A, A^c) \geq \frac{(\lambda+2)(k))}{2} \geq 2k-2$, for $\lambda \geq 2$. Now we only need to verify the following three cases:

i) $\lambda = 0$

We consider two possibilities. First, suppose A contains a vertex incident with no edges of T. Then as a result of Lemma 4.3.5 we have $|A| > v\left(1 - \frac{2k-3}{\mu k_2}\right) =$

$v\left(1-\frac{2k-3}{k(k-1)}\right) \geq v/2$ for $k \geq 4$.

Otherwise, every vertex in A is incident with an edge of T. Hence $|A| \leq 2k-3$. Using the results of Lemma 4.3.5 we also have $|A| > v\left(1-\frac{1}{k}-\frac{2k-4}{k(k-1)}\right) \geq v/2$ for $k \geq 5$. Now, if $k=4$, we have that $5 \geq |A| > \frac{5v}{12}$ so $v < 12$. Based on the results of [11] there is one such graph satisfying these conditions. Specifically, $K_{5,5}$ minus a matching.

Using Lemma 4.3.2, we find that $e(A, A^c) \geq (4-\theta_1)a\left(1-\frac{a}{v}\right) \geq 3(4)(1-2/5) > 7$.

ii) $\lambda = 1$

Since $\lambda = 1$ we have that $b_1 = k-2$. So it follows that $\mu k_2 = k(k-2)$. We consider two separate cases. First, suppose there is a vertex in A incident with no vertices in T. Then it follows from the results of Lemma 4.3.5 that $|A| > v\left(1-\frac{2k-3}{k(k-2)}\right) \geq v/2$ for $k \geq 5$. By Propositions 4.3.11 and 4.3.12 we have that $(\lambda+1)|k$, so the only case left to consider is when $(k, \lambda) = (4, 1)$. This case will be handled later on.

Otherwise, every vertex in A is incident with an edge of T. Using the results of Lemma 4.3.5 we have $|A| > v\left(1-\frac{1}{k}-\frac{2k-4}{k(k-2)}\right) \geq v/2$ for $k \geq 6$. By Propositions 4.3.11 and 4.3.12 we have that $(\lambda+1)|k$, so the only case left to consider is when $(k, \lambda) = (4, 1)$. As mentioned, this case will be handled later.

iii) $\mu = 1$

If $\mu = 1$ then it follows as a result of Lemma 4.3.5 that

$$|A| > v\left(1-\frac{1}{k}-\frac{2k-4}{k_2}\right) \geq v/2$$

if $k_2 \geq 4k$. By Propositions 4.3.11 and 4.3.12 we have that $(\lambda+1)|k$ from which it follows $(\lambda+1)|b_1$. Since $k_2 = b_1 k$, we have $b_1 \leq 3$. This leaves us with the cases $(k, \lambda) \in \{(3,0), (4,0), (4,1), (6,2)\}$.

The cases $(k,\lambda) = (3,0)$ and $(k,\lambda) = (4,0)$ have already been taken care of.

Suppose $(k,\lambda) = (4,1)$. Since $\lambda = 1$ we also have $b_1 = 2$, so $b_1 k = 8$. From the results of Lemma 4.3.5 it follows that $|A| > v\left(1 - \frac{1}{4} - \frac{4}{8}\right) = \frac{v}{4}$. Using Lemma 4.3.2 we have that

$$e(A, A^c) \geq (k - \theta_1)a\left(1 - \frac{a}{v}\right).$$

Taking the derivative of the right hand side with respect to a, we find that right hand side of the inequality is increasing for $a < v/2$. Hence $e(A, A^c) > \frac{3v}{16}(4 - \theta_1)$.

In [11] Brouwer and Koolen classify all the possibly intersection arrays of distance-regular graphs of valency 4. Of these intersection arrays there are 4 such that $\lambda = 1$ and the diameter is at least 3. Specifically, the graphs are one of the following:

1. The line graph of the Petersen Graph $\{4, 2, 1; 1, 1, 4\}$

   We see that

$$e(A, A^c) > \frac{(3)(15)}{16}(4 - 2) > 5.$$

2. The flag graph of $PG(2,2)$ $\{4, 2, 2; 1, 1, 2\}$

   We see that

$$e(A, A^c) > \frac{(3)(21)}{16}(4 - (1 + \sqrt{2})) > 6.$$

3. The flag graph of $GQ(2,2)$ $\{4, 2, 2, 2; 1, 1, 1, 2\}$

   We see that

$$e(A, A^c) > \frac{(3)(45)}{16}(4 - 3) > 8.$$

4. The flag graph of $GH(2,2)$ $\{4, 2, 2, 2, 2, 2; 1, 1, 1, 1, 1, 2\}$

   We see that

$$e(A, A^c) > \frac{(3)(189)}{16}(4 - (1 + \sqrt{6})) > 19.$$

In each case we have verified that $e(A, A^c) \geq 6 = 2k - 2$.

Suppose $(k, \lambda) = (6, 2)$. Since $\mu = 1$, by Proposition 4.3.11 all singular lines have size $\lambda + 2$. Since $k = 2(\lambda + 1)$ it follows from Proposition 4.3.13 that Γ is a line graph and satisfies the conditions of Theorem 4.3.10. According to Theorem 4.3.10 we know Γ must be a generalized $2d$-gon of order $(3, 1)$ where $d \in \{3, 4, 6\}$.

1. $d = 3$

   If $d = 3$, then we have $v = 52$ and $\theta_1 = 2 + \sqrt{3}$. From the results of Lemma 4.3.5 we know that

   $$|A| > v\left(1 - \frac{1}{k} - \frac{2k-4}{\mu k_2}\right) = 52\left(1 - \frac{1}{6} - \frac{8}{18}\right) > 20.$$

   Applying this in Lemma 4.3.2 we find that

   $$e(A, A^c) \geq (6 - (2 + \sqrt{3}))(21)\left(1 - \frac{21}{56}\right) > 29.$$

2. $d = 4$

   If $d = 4$, then we have $v = 160$ and $\theta_1 = 2 + \sqrt{6}$. From the results of Lemma 4.3.5 we know that

   $$|A| > v\left(1 - \frac{1}{k} - \frac{2k-4}{\mu k_2}\right) = 160\left(1 - \frac{1}{6} - \frac{8}{18}\right) > 62.$$

   Applying this in Lemma 4.3.2 we find that

   $$e(A, A^c) \geq (6 - (2 + \sqrt{6}))(62)\left(1 - \frac{62}{160}\right) > 58.$$

3. $d = 6$

   If $d = 6$, then we have $v = 1456$ and $\theta_1 = 5$. From the results of Lemma 4.3.5 we know that

   $$|A| > v\left(1 - \frac{1}{k} - \frac{2k-4}{\mu k_2}\right) = 1456\left(1 - \frac{1}{6} - \frac{8}{18}\right) > 566.$$

   Applying this in Lemma 4.3.2 we find that

   $$e(A, A^c) \geq (6 - 5)(566)\left(1 - \frac{566}{1456}\right) > 345.$$

In each case we have verified that $e(A, A^c) > 2k - 2 = 10$.

□

Theorem 4.3.16. *If Γ is a distance-regular graph of degree $k \geq 2$, then the minimum number of edges needed to be removed to disconnect Γ into nonsingleton components is $2k - 2$.*

Proof. We know that if Γ is strongly regular, then the conclusion is true based on Theorem 4.3.1. So we will assume $d \geq 3$. Now, if Γ is cubic then the conclusion holds by Lemma 4.3.6. So we may now assume $k \geq 4$. But this case was taken care of in Lemma 4.3.15. Therefore, the minimum number of edges needed to be removed to disconnect Γ into nonsingleton components is $2k - 2$. □

4.4 The Grassmann Graphs and Twisted Grassmann Graphs

In this section we briefly describe the Grassmann and twisted Grassmann graphs. We give the definitions of each graph, note that they are distance-regular and give the parameters as well as the spectrum for each.

Definition 4.4.1. Let $\mathbb{F}$ be a field, and let V be an n-dimensional vector space over $\mathbb{F}$. The Grassmann graph of the d-subspaces of V has as vertex set $\begin{bmatrix} V \\ d \end{bmatrix}$, the collection of linear subspaces of V of dimension d. Two vertices A and B are adjacent whenever $\dim(A \cap B) = d - 1$. We will denote this graph by $Gr(n, d, q)$.

Let $\begin{bmatrix} n \\ m \end{bmatrix}_q$ be the q-ary Gaussian binomial coefficient:

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{(q^n - 1) \dots (q^{n-m+1})}{(q^m - 1) \dots (q - 1)}.$$

The following lemma provides some well known facts about the dimensions of vector spaces over a finite field of order q.

Lemma 4.4.2. *Let V be as in Definition 4.4.1. Suppose $0 \leq i,\ j \leq n$. Then*

(i) The number of m-spaces of V is $\begin{bmatrix} n \\ m \end{bmatrix}_q$.

(ii) *If X is a j-space of V, then there are precisely $q^{ij} \left[{n-j \atop i} \right]_q$ i-spaces Y in V such that $X \cap Y = \mathbf{0}$.*

(iii) *If X is a j-space of V, then there are precisely $q^{(i-m)(j-m)} \left[{n-j \atop i-m} \right]_q \left[{j \atop m} \right]_q$ i-spaces Y in V such that $X \cap Y$ is an m-space.*

Using the results of Lemma 4.4.2 one can establish the following proposition, which describes the parameters as well as the spectrum of the Grassmann graphs.

Proposition 4.4.3 ([8, Thm. 9.3.3]). *Let $Gr(n, d, q)$ be the Grassmann graph of the d-subspaces of V where V is an n-dimensional vector space over $\mathbb{F}_q$. Then $Gr(n, d, q)$ has diameter $\min(d,\ n-d)$ and is distance-transitive with intersection array given by*

$$b_i = q^{2i+1} \begin{bmatrix} d-i \\ 1 \end{bmatrix}_q \begin{bmatrix} n-d-i \\ 1 \end{bmatrix},\ c_i = \begin{bmatrix} i \\ 1 \end{bmatrix}^2 \text{ where } 0 \le i \le d.$$

The graph $Gr(n, d)$ has eigenvalues and multiplicities given by

$$\theta_i = q^{i+1} \begin{bmatrix} d-j \\ 1 \end{bmatrix} \begin{bmatrix} n-d-i \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix}$$

and

$$f_i = \begin{bmatrix} n \\ i \end{bmatrix} - \begin{bmatrix} n \\ i-1 \end{bmatrix}.$$

Proofs of Lemma 4.4.2 and Proposition 4.4.3 may be found in [8] (9.3.2 and 9.3.3).

In [27] Van Dam and Koolen construct a new family of distance-regular graphs with unbounded diameter. These graphs have the same parameters, and hence the same spectrum, as the Grassmann graphs $Gr(2d+1, d, q)$. The construction of these graphs is outlined in the following proposition.

Proposition 4.4.4 ([27]). *Let V be a $(2d+1)$-dimensional vector space over the field $\mathbb{F}_q$, and let H be a hyperplane in V. Consider the graph, ${}^*Gr(d, q)$, whose vertices are all*

(i) *the $(d+1)$-dimensional subspaces of V that are not contained in H, and* *(ii) the $(d-1)$-dimensional subspaces of H.*

Adjacency is defined as follows:

(1) Two vertices from (i) are adjacent if they intersect in a d-dimensional subspace.

(2) A vertex from (i) is adjacent to a vertex from (ii) if the vertex from (ii) is a subspace of the vertex from (i).

(3) Two vertices from (ii) are adjacent if they intersect in a $(d-2)$-dimensional subspace.

Then ${}^*Gr(d,q)$ is distance-regular, with the same parameters as the Grassmann graph $Gr(2d+1,d,q)$. Moreover, ${}^*Gr(d,q)$ is not vertex-transitive, and hence not isomorphic to $Gr(2d+1,d,q)$.

In [61] it was shown that ${}^*Gr(d,q)$ may be obtained from $Gr(2d+1,d,q)$ through GM-switching. Indeed, let H be a fixed hyperplane of V and partition the vertices of ${}^*Gr(d,q)$ into the sets

$$\begin{aligned} \mathcal{A} &= \left\{ W \in \begin{bmatrix} V \\ d+1 \end{bmatrix}_q \;:\; W \not\subset H \right\}, \\ \mathcal{B} &= \begin{bmatrix} H \\ d-1 \end{bmatrix}_q . \end{aligned}$$

Now let

$$\begin{aligned} C_U &= \{W \in \mathcal{A} \;:\; W \cap H = U\} \quad \left(U \in \begin{bmatrix} H \\ d \end{bmatrix} \right), \\ D &= \begin{bmatrix} H \\ d+1 \end{bmatrix}, \\ \mathcal{C} &= \left\{ C_U \cup C_{\sigma(U)} \;:\; U \in \begin{bmatrix} H \\ d \end{bmatrix} \right\}, \end{aligned}$$

where σ is a polarity of H. Then $\mathcal{A} = \bigcup_{U \in \left[\begin{smallmatrix} H \\ d \end{smallmatrix}\right]} C_U$ (disjoint).

Proposition 4.4.5 ([61, Thm. 3]). *The graph obtained by applying GM-switching to the Grassmann graph $Gr(2d+1, d, q)$ with respect to the partition $\mathcal{C} \cup D$ is isomorphic to the twisted Grassmann graph $^*Gr(d, q)$.*

To end this subsection, we note that the twisted Grassmann graphs are currently the only known family, with unbounded diameter, of distance-regular graphs that are not vertex-transitive. A proposition in Section 4.5 will make it clear why we decided to focus on the edge-connectivity of distance-j graphs of $^*Gr(d, q)$.

4.5 Edge-Connectivity of Distance-j Twisted Grassmann Graphs

In this section we examine the edge-connectivity of the distance-j twisted Grassmann graphs. Our main result is given in Theorem 4.5.5, where we prove that the edge-connectivity of the distance-j graphs of the twisted Grassmann graphs is equal to their degree k_j.

To make it more clear why we focus on the twisted Grassmann graphs, we will need the following lemma ([39, Lemma 3.3.3]).

Lemma 4.5.1. *If Γ is a connected vertex-transitive graph, then its edge-connectivity is equal to its degree.*

Proposition 4.5.2. *Let Γ be a vertex-transitive distance-regular graph. If the distance-j graph Γ_j of Γ is connected, then the edge-connectivity of Γ_j is equal to its degree k_j.*

Proof. Since the distance graph of a vertex-transitive graph is also vertex-transitive, the result follows directly from Lemma 4.5.1 □

The twisted Grassmann graphs are the only known family of distance-regular graphs of unbounded diameter that are not vertex-transitive. This means the edge connectivity of their distance-j graphs does not follow immediately from the previous proposition. We will now verify that the edge-connectivity of these graphs is equal to their degree.

The following lemma relating the second largest eigenvalue in absolute value of a regular graph is due to Krivelevich and Sudakov [52] (see [17] for a stronger result).

Lemma 4.5.3. *Let Γ be a k-regular graph on n vertices and $\theta = \max_{2 \le i \le n} |\theta_i(\Gamma)|$. If $k - \theta \ge 2$, then Γ is k-edge-connected.*

Lemma 4.5.4 ([7, Prop. 5.4])**.** *Let $G_j^{n,d}(i) = \sum_{h=0}^{j} (-1)^{j-h} q^{hi + \binom{j-h}{2}} \begin{bmatrix} d-i \\ h \end{bmatrix} \begin{bmatrix} d-h \\ j-h \end{bmatrix} \begin{bmatrix} n-d-i+h \\ h \end{bmatrix}$ denote the i-th eigenvalue of the distance-j graph of the Grassmann graph $Gr(n, d, q)$. Then*

(i) $G_j^{n,d}(1) < 0$ if and only if $j = d$. The eigenvalue $G_j^{n,d}(1)$ is never zero.

(ii) Let $i \ge 1$. Then $|G_j^{n,d}(i)| \le |G_j^{n,d}(1)|$.

Theorem 4.5.5. *Let $q \ge 2$, $d \ge 2$ and $1 \le j \le d$. The edge-connectivity of the distance-j graph of the twisted Grassmann graph ${}^*Gr(d, q)$ is equal to the degree.*

Proof. Since it was shown that the edge-connectivity of any distance-regular graph is equal to its degree (see [10]) we know the result holds for $j = 1$.

So assume $j \ge 2$. Note ${}^*Gr(d, q)$ has the same intersection array and is cospectral with the Grassmann graph $Gr(2d + 1, d, q)$. It follows that the distance-j graphs of ${}^*Gr(d, q)$ and $Gr(2d + 1, d, q)$ are cospectral. Since $G_j^{n,d}(0) = k_j$ it suffices to show $G_j^{n,d}(0) - G_j^{n,d}(i) \ge 2$ for $1 \le i \le d$, by Lemma 4.5.3.

Using Lemma 4.5.4 we find that for $j < d$

$$\begin{aligned} G_j^{n,d}(0) - G_j^{n,d}(i) &\ge G_j^{n,d}(0) - G_j^{n,d}(1) \\ &= q^{j^2} \begin{bmatrix} d \\ j \end{bmatrix} \begin{bmatrix} d+1 \\ j \end{bmatrix} - q^{j^2} \begin{bmatrix} d-1 \\ j \end{bmatrix} \begin{bmatrix} d+1 \\ j \end{bmatrix} + q^{j(j-1)} \begin{bmatrix} d \\ j \end{bmatrix} \begin{bmatrix} d \\ j-1 \end{bmatrix} \\ &\ge q^{j(j-1)} \begin{bmatrix} d \\ j \end{bmatrix} \begin{bmatrix} d \\ j-1 \end{bmatrix} \\ &\ge 2. \end{aligned}$$

For $j = d$ we find

$$\begin{aligned} G_j^{n,d}(0) - G_j^{n,d}(i) &\ge G_j^{n,d}(0) - |G_j^{n,d}(1)| \\ &= q^{j^2} \begin{bmatrix} d \\ j \end{bmatrix} \begin{bmatrix} d+1 \\ j \end{bmatrix} + q^{j^2} \begin{bmatrix} d-1 \\ j \end{bmatrix} \begin{bmatrix} d+1 \\ j \end{bmatrix} - q^{j(j-1)} \begin{bmatrix} d \\ j \end{bmatrix} \begin{bmatrix} d \\ j-1 \end{bmatrix} \\ &= q^{d^2} \begin{bmatrix} d+1 \\ d \end{bmatrix} - q^{d(d-1)} \begin{bmatrix} d \\ d-1 \end{bmatrix} \\ &\ge 2. \end{aligned}$$

Therefore, it follows that the edge-connectivity of the distance-j twisted Grassmann graphs is equal to their degree. □

4.6 Open Problems

In this section we review some open problems dealing with the connectivity of graphs in association schemes that we believe would be interesting and merit further research.

4.6.1 Conjecture 4.1.3 and Conjecture 4.1.4

The results of Proposition 4.5.2 support Conjecture 4.1.3 since many known distance-regular graphs are vertex-transitive. However, this does not take care of the conjecture completely. For this reason we feel that Conjecture 4.1.3 should be explored further.

4.6.2 Vertex-connectivity of graphs in association schemes

In Section 4.1 we mentioned Conjecture 4.1.4. Some progress has been made towards establishing this conjecture. As mentioned in Section 4.2 we have the following proposition regarding the vertex-connectivity of vertex-transitive graphs (see [39, Thm. 3.4.2]).

Proposition 4.6.1. *A vertex-transitive graph of degree k has vertex-connectivity at least $\frac{2}{3}(k+1)$.*

We also have the following proposition due to Watkins (see [67]).

Proposition 4.6.2. *The vertex-connectivity of a connected edge-transitive graph is equal to its minimum degree.*

Proposition 4.6.3. *Let Γ be a distance-transitive graph. If the distance-j graph, Γ_j, is connected, then the vertex-connectivity of Γ_j equals k_j, its degree.*

Proof. Let Γ be a distance-transitive graph. We show that the distance-j graph Γ_j must be edge-transitive. So suppose $x \sim y$ and $u \sim v$ in Γ_j, then $d_\Gamma(x, y) = d_\Gamma(u, v) = j$.

Since Γ is distance-transitive, there exists $g \in Aut(\Gamma)$ such that $g(x) = u$ and $g(y) = v$. We need to show $g \in Aut(\Gamma_j)$. Indeed, $a \sim b$ in Γ_j is equivalent to $d_\Gamma(a, b) = j$. Since $g \in Aut(\Gamma)$, we know $d_\Gamma(g(a), g(b)) = j$, which means $g(a) \sim g(b)$ in Γ_j. Hence, Γ_j is edge-transitive. The result now follows from Proposition 4.6.2. □

This verifies that the conjecture is true for the distance-j graphs of distance-transitive graphs. However, there are several families of distance-regular graphs which are not distance-transitive. Currently, the families known with unbounded diameter are the Doob graphs, twisted Grassmann graphs, Hemmeter graphs, Ustimenko graphs and quadratic forms graphs. Determining the vertex-connectivity of any of these families of graphs would be a great contribution to resolving this conjecture.

4.6.3 Disconnecting distance-regular graphs into nonsingleton components by removing vertices

In this subsection we investigate the following problem concerning the number of vertices required to be removed from a distance-regular graph with diameter at least three in order to disconnect it into nonsingleton components.

Problem 4.6.4 ([28, Problem 41])**.** *Determine whether one needs to remove at least $2k - a_1 - 2$ vertices in order to disconnect a distance-regular graph with diameter at least three such that each resulting component has at least two vertices.*

The work presented in this subsection represents work done in collaboration with Jack Koolen. We give examples of vertex sets of order $2k-a_1-2$ that do not correspond to the vertex-neighborhood of an edge in distance-regular graphs with diameter three whose removal leaves behind nonsingleton components. Before we construct these examples we will need to cover a few basic definitions.

Definition 4.6.5. A distance-regular graph with intersection array

$$\{k, \mu, 1; 1, \mu, k\}$$

is called a *Taylor graph*.

For a Taylor graph $k = k_2$, $k_3 = 1$ and $v = 2k+2$. Hence it is an antipodal double cover of the complete graph K_{k+1}.

Proposition 4.6.6 ([8, Thm. 1.5.3])**.** *A graph Γ is a Taylor graph if and only if for some (and then for every) $\gamma \in \Gamma$ the local graph $\Gamma(\gamma)$ is strongly regular and satisfies the conditions $k(\Gamma(\gamma)) = 2\mu(\Gamma(\gamma))$.*

We introduce a family of graphs called the *symplectic graphs* that will be used to construct an infinite family of Taylor graphs with a vertex set of order $2k - a_1 - 2$ that does not correspond to the vertex-neighborhood of an edge and whose removal leaves behind non-singleton components.

Let q be a prime power and $r \geq 2$ be an integer. If x is a non-zero (column) vector of $\mathbb{F}_q^{2r}$, denote by $[x]$ the 1-dimensional vector subspace of $\mathbb{F}_q^{2r}$ that is spanned by x and denote by x^t the row vector that is the transpose of x. Let M be the $2r \times 2r$ block diagonal matrix whose diagonal blocks are $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

The *symplectic graph*, $Sp(2r, q)$, over $\mathbb{F}_q$ is the complement of the orthogonality graph of the unique non-degenerate symplectic form over $\mathbb{F}_q^{2r}$. More precisely, its vertex set is formed by the 1-dimensional subspaces $[x]$ of $\mathbb{F}_q^{2r}$ with $[x] \sim [y]$ if and only if $x^t My \neq 0$. This graph is called *symplectic* as the function $f(x, y) = x^t My$ is known as a symplectic form. The symplectic graph $Sp(2r, q)$ is a strongly regular graph with parameters $v = \frac{q^{2r}-1}{q-1}$, $k = q^{2r-1}$, $\lambda = q^{2r-2}(q-1)$ and $\mu = q^{2r-2}(q-1)$.

First, we observe that in order to construct a Taylor graph with the symplectic graph $Sp(2r, q)$ as the local graph, we need to satisfy $k(Sp(2r, q)) = 2\mu(Sp(2r, q))$. Hence we need

$$q^{2r-1} = 2q^{2r-2}(q-1).$$

This condition reduces to $q = 2(q-1)$ and we see we require $q = 2$. Note that the graph $Sp(2r, 2)$ is strongly regular with parameters $v = 2^{2r} - 1$, $k = 2^{2r-1}$, $\lambda = 2^{2r-2}$ and $\mu = 2^{2r-2}$.

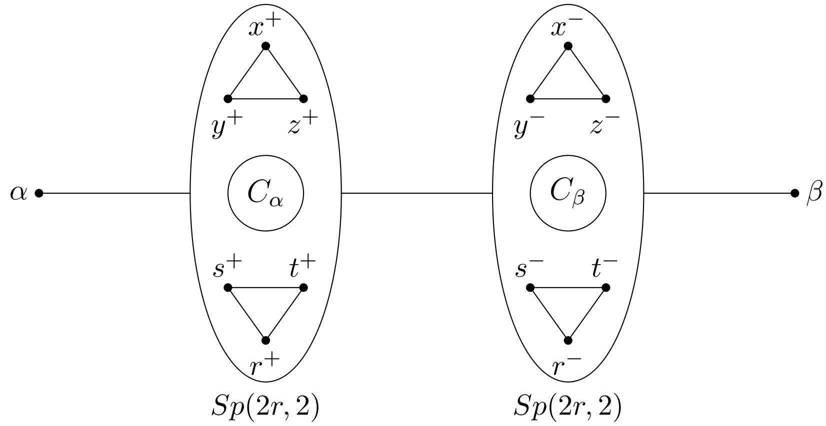


Figure 4.1: Taylor graph with local graph $Sp(2r, 2)$

Now, let Γ be the Taylor graph having $Sp(2r, 2)$ as its local subgraph. This graph is regular of degree $k = 2^{2r} - 1$ and has $a_1 = 2^{2r-1}$. For more information on the structure of Taylor graphs see Section 1.5 of [8]. For our argument we refer to Figure 4.1 above and note that there are no edges between the triangles $\{x^+, y^+, z^+\}$ and $\{x^-, y^-, z^-\}$ and there are no edges between the triangles $\{r^+, s^+, t^+\}$ and $\{r^-, s^-, t^-\}$. Also, each vertex in the triangle $\{x^+, y^+, z^+\}$ is adjacent to every vertex of the triangle $\{r^-, s^-, t^-\}$ and each vertex in the triangle $\{x^-, y^-, z^-\}$ is adjacent to every vertex of the triangle $\{r^+, s^+, t^+\}$.

In [20] it was shown that the graphs $Sp(2r, 2)$ have separating sets of vertices of order $2^{2r-1} + 2^{2r-2} - 3$ whose removal leaves two disjoint triangles. We denote these separating sets by C_α and C_β in Figure 4.1. It follows that the set $\{\alpha, \beta\} \cup C_\alpha \cup C_\beta$ is a separating set in Γ of order $2^{2r} + 2^{2r-1} - 4$ whose removal disconnects the graph into two components of order six. Since $k = 2^{2r} - 1$ and $a_1 = 2^{2r-1}$ in Γ we find $2k - a_1 - 2 = 2^{2r+1} - 2^{2r-1} - 4 = 2^{2r} + 2^{2r-1} - 4$. So we have found nontrivial separating sets of vertices in Taylor graphs whose removal leaves nonsingleton components.

An answer to Problem 4.6.4 is still unknown at this time. It would be interesting to see if examples exist for distance-regular graphs of any diameter. As mentioned before, Conjecture 4.1.5 is believed to be true for strongly regular graphs with $k \geq 2\lambda+1$

(see [20]). Suppose Γ is a distance-regular graph that has a pair of distinct vertices x and y such that the number of common neighbors of x and y is about half the degree of Γ. In [50] the authors show that if the diameter is at least three, then such a graph, besides a finite number of exceptions, is a Taylor graph, bipartite with diameter three or a line graph. Since an answer to Problem 4.6.4 is still unknown, we believe that Problem 4.6.4 is worth further exploration.

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