

A High Order Lagrangian Scheme for Flow Through Unsaturated Porous Media

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ABSTRACT. A new high order Lagrangian method uses moving basis functions to represent the moisture content in an unsaturated porous material. The basis functions are localized elliptical Gaussians that can move, spread, elongate and rotate under the action of the local velocity, velocity deviations, and diffusivity with corrections for local material properties. The velocity and its deviations are calculated from the local moisture potential and its derivatives which can be obtained from published experiments. This numerical technique is naturally adaptive in the sense that computational effort is expended only where there is moisture and nowhere else, and this method is capable of capturing infiltration instabilities in the wetting front observed by experimentalists and predicted by linear stability analysis. This method borrows many ideas from high Reynolds number vortex methods and other applications of Lagrangian schemes to nonlinear partial differential equations.

1. Introduction

The numerical treatment of laboratory and geophysical flows through unsaturated porous media without hysteresis poses many numerical challenges including strong nonlinearities in the media properties and small scale features over unbounded domains. Lagrangian methods have been used successfully for basic linear convection-diffusion problems as well as a small collection of nonlinear problems, and the basic description of the motion of moisture through unsaturated media without hysteresis is also conducive to this approach. Lagrangian methods express a field as a linear combination of localized basis functions that move and evolve with the flow. As is the case with many unsaturated flow problems, these methods generally perform well when there is a Lagrangian structure in the problem and the field being simulated is limited to a small subset of the entire domain. In fact, if one wishes to numerically approximate the solution to any partial differential equation with the general form

$$(1) \quad \frac{Dw}{Dt} = f(w),$$

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where $\frac{D}{Dt} = \partial_t + (\vec{u} \cdot \nabla)$ one should consider a Lagrangian method. When the field w is localized in small regions of a large domain, one can say that Lagrangian methods are naturally adaptive in the sense that the use of moving basis functions has the advantage of concentrating computational resources where w is dominant and not elsewhere. In this paper, we develop an accurate Lagrangian method to compute moisture evolution in porous media. To capture the governing partial differential equation to a high spatial accuracy, we meld diverse ideas such as diffusion velocities and core spreading to generate a novel scheme employing deforming basis functions.

While the author is unaware of any attempt to use Lagrangian schemes for unsaturated transport phenomena, these methods have played an important role in the numerical approximation of simple convection-diffusion problems. For linear convection-diffusion problems, the use of particle methods to track the motion of passive tracers is ubiquitous in the literature (for examples and further discussion, see [18, 20, 25, 33]). Particle tracking schemes for transport problems arose as an alternative to finite difference or finite element schemes which have large numerical diffusion owing to large grid Peclet numbers. That is, if h is the characteristic size of the computational element (either a grid cell or a finite element), the grid Peclet number would be defined at Vh/ν where V is the characteristic flow velocity and ν is the characteristic diffusion coefficient. Of course, it is possible, even likely, that in most situations, the spatial domain is large while ν is small. Thus, one finds that the numerical diffusion plays a larger role in the problem than hydrodynamic diffusion in the porous matrix. One proposed solution, among several options, is to use particle tracking methods. Particle tracking schemes are Lagrangian methods that approximate the concentration of passive tracers with a large number number of discrete particles. The particles move with the flow velocity, and concentration information can be computed by counting the number of particles in a given area. The problem is necessarily linear, and the flow is computed independently of the passive field. In the geosciences, diffusive effects are typically captured using either constant time or constant displacement random walks where in a random disturbance is added to the particle trajectories. (A comparative study can be found in [33].) While random walks are a convergent means of capturing diffusive transport that is relatively simple to develop, investigators of nonlinear phenomena tend to avoid them in favor of either exchange or core spreading methods because random walks are slow to converge [6, 28].

To develop the method for flows through unsaturated media, it will be necessary to borrow ideas from the vortex method community. One of the first documented uses of a Lagrangian computational method for a nonlinear problem can be attributed to Rosenhead's manual calculation of the interface in an inviscid mixing layer in 1932 [27]. In this case, each of the eight computational elements were point vortices moving with the velocity induced by all the other vortices in the domain evaluated with a periodic Biot-Savart integral. In 1973, Chorin incorporated viscosity in a modern, computerized implementation of a similar algorithm [4]. In this case, he added stochastic noise (random walks) to the positions of each computational element with the appropriate variance to capture a spatially constant diffusivity. Since this time, "vortex methods" have become a standard approach in many engineering applications. While these schemes have developed and matured over time including superior techniques for capturing diffusion and faster summation algorithms, the underlying concept of using basis functions that move and

evolve with the flow has remained unchanged. There are several reviews, collections and expository works on vortex methods for incompressible flows including [5, 11, 21, 26].

Since the velocity field governing moisture movement in unsaturated soils is compressible, it is essential to introduce a fictitious velocity field, closely related to diffusion velocities and similar constructs used by other investigators. For instance, Ogami & Cheer determined have used moving Gaussians to approximate inviscid and viscous Burgers equations [23]. Similar to the work presented in this article, the computation elements the flow velocity modified by nonlinear interactions from the underlying partial differential equations. Also, Lions & Mas-Gallic have successfully applied Lagrangian methods to one-dimensional linear and nonlinear diffusion problems [22]. Indeed, the use of diffusion velocities is a well-established procedure for a variety of compressible and incompressible problems [3, 19]. In this paper, we use a similar methodology by capturing spatially varying and nonlinear diffusive flux through a diffusion velocity term.

2. Flow Through Unsaturated Media

Moisture in unsaturated media is subject to a variety of forces including wetting, double layer effects, capillarity and gravity, but in this paper, we shall begin, as most investigators do, with a model described in Philip's review paper on infiltration [24]. We expect the flow velocity will be described by

$$(2) \quad \vec{v} = K(w)\nabla\Phi,$$

where \vec{v} is the flow velocity in the unsaturated media, w is the volumetric water content, $K(w)$ is the permeability and Φ is the total flow potential. In addition, we assume that the total potential can be decomposed into the moisture potential and the gravitational potential,

$$(3) \quad \Phi = \Psi(w) + z.$$

With this velocity field, we require that mass be conserved:

$$(4) \quad \frac{\partial w}{\partial t} = \nabla \cdot (K\nabla\Phi),$$

$$(5) \quad \frac{\partial w}{\partial t} = \nabla \cdot (E\nabla w) + K' \frac{\partial w}{\partial z} \quad \text{where } E = K\Psi'.$$

Thus, given material properties in the functions K and E , (for example, see Fig. 1), one obtains a general infiltration model.

To distill a Lagrangian structure from the model, let us define the diffusion velocity \vec{u} as

$$(6) \quad \vec{u} = \begin{bmatrix} -E' \frac{\partial w}{\partial x} \\ -E' \frac{\partial w}{\partial y} \\ -E' \frac{\partial w}{\partial z} - K' \end{bmatrix}.$$

Then, (5) can be rewritten as

$$(7) \quad \frac{Dw}{Dt} = \frac{\partial w}{\partial t} + \vec{u} \cdot \nabla w = E\nabla^2 w.$$

Thus, we see that in the Lagrangian frame traveling with diffusion velocity \vec{u} , the moisture content diffuses with a spatially varying diffusivity E .

3. A Lagrangian Scheme

Having determined a suitable velocity field, not to be confused with the actual fluid flow velocity, we can consider developing a scheme capable of capturing (7). In this article, we shall restrict our attention to 2-dimensional flows for computational and analytical simplicity, but extending these ideas to one more dimension uses the same ideas. To begin, we can express an approximate moisture content as a linear combination of basis functions:

$$(8) \quad \widehat{w}(\vec{x}, t) = \sum_{i=1}^N m_i f(\vec{x} - \vec{x}_i, \vec{p}_i)$$

where m_i is the total moisture content and \vec{x}_i is the position of i^{th} element. Also, we allow for individual variations in the shapes of individual elements by including a vector of additional parameters \vec{p}_i . In traditional vortex methods, only the position of the computational element evolves with time, but there is no reason why one should accept this a priori restriction. Thus, we allow all parameters, m_i , \vec{x}_i and \vec{p}_i to evolve with time.

As is the case for finite difference, finite element and most other numerical schemes, the design of a Lagrangian scheme evolves hand-in-hand with efforts to understand its convergence properties. As soon as one can describe the accuracy of a method, one immediately attempts to improve the accuracy using the convergence description as a guide. The typical consistency and stability description that is common to finite difference methods is not applicable to Lagrangian schemes because there is no underlying difference equation. That is, it is not possible to substitute an exact solution into the difference equation to measure a residual. While many investigators use the terms “consistency” and “stability” in convergence proofs for vortex methods, the terms do not have the same meaning as for finite difference methods [1, 12, 13]. Similarly, there is no variational principle for Lagrangian methods as are commonly used in finite element method convergence analyses. However, a different type of residual for Lagrangian methods can be defined by substituting the approximate system of ordinary differential equations into the exact partial differential equation. The other component of the convergence process corresponding to a stability condition would be related to the well-posedness of the exact partial differential equation. This second aspect is beyond the scope of this paper, but the full technique is applied to the vorticity equations in [28]. Here, we shall focus on understanding the accuracy of Lagrangian methods for (7) and using this knowledge to develop an effective scheme.

To design the method and gauge its spatial accuracy, we consider a residual operator for (7):

$$(9) \quad R[h] = \partial_t h + (\vec{u} \cdot \nabla)h - E(w)\nabla^2 h$$

where \vec{u} is the diffusion velocity corresponding to the exact solution w . In this sense, the residual is linearized about the convective term, even though the latter term has a nonlinearity. Thus, we can see that

$$(10) \quad R[w] = 0,$$

if w is an exact solution to (7). However, the residual of the numerical approximation,

$$(11) \quad R[\hat{w}] = \sum_{i=1}^N m_i R[f(\vec{x} - \vec{x}_i, \vec{p}_i)],$$

is not necessarily zero, and its size allows one to measure the spatial accuracy of the method. Of course, the residual is governed by the choice of basis function, and some basis functions are better than others.

3.1. Example: Gaussian basis functions. As a simple example for the general procedure, one might choose Gaussian basis functions:

$$(12) \quad f(\vec{x} - \vec{x}_i, \vec{p}_i) = \frac{m_i}{4\pi\sigma_i^2} \exp\left(-\frac{|\vec{x} - \vec{x}_i|^2}{4\sigma_i^2}\right).$$

In this example, \vec{p}_i is a vector with two elements, m_i and σ_i^2 which are the volume and area of the basis function, respectively. To simplify the notation, we shall define

$$(13) \quad X = x - x_i,$$

$$(14) \quad Y = y - y_i,$$

and drop all indices with the exception of \vec{x}_i in order to distinguish it from \vec{x} . When necessary, the velocity field will be expressed in individual components as $\vec{u} = \begin{bmatrix} u^{(1)} \\ u^{(2)} \end{bmatrix}$. Now, we compute $R[f]$ and gather together terms of the same order:

$$(15) \quad R[f] = \frac{m}{4\pi\sigma^2} \exp\left(-\frac{X^2 + Y^2}{4\sigma^2}\right) \times \left\{ \left(\frac{\frac{d}{dt}(m)}{m} - \frac{\frac{d}{dt}(\sigma^2)}{\sigma^2} + \frac{E(w)}{\sigma^2} \right) + \left(\frac{\frac{d}{dt}(x_i)}{2\sigma^2} - \frac{u^{(1)}}{2\sigma^2} \right) X + \left(\frac{\frac{d}{dt}(y_i)}{2\sigma^2} - \frac{u^{(2)}}{2\sigma^2} \right) Y + \left(\frac{\frac{d}{dt}(\sigma^2)}{4\sigma^2} - \frac{E(w)}{4\sigma^2} \right) (X^2 + Y^2) \right\}.$$

Ideally, one would like to be able to cancel the coefficients of each power of X and Y , but we quickly see that the coefficients at all orders vary in space. At the simplest level, one could approximate $E(w)$ and \vec{u} with their values at the center of the computational element, $E[w(\vec{x}_i)]$ and $\vec{u}(\vec{x}_i)$ respectively, and throw away all the corrections. Remembering that the residual is localized in space, it would be fine to do this to terms at the highest order $O(X^2 + Y^2)$, but truncating at a lower order spoils the higher order terms. If we will linearize the $E(w)$ in the $O(1)$ term, as well as \vec{u} in the $O(X)$ and $O(Y)$ terms, and discard the quadratic remainders,

we are left with the following modified system

$$(16) \quad R[f] = \frac{m}{4\pi\sigma^2} \exp\left(-\frac{X^2 + Y^2}{4\sigma^2}\right) \times \left\{ \left(\frac{\frac{d}{dt}(m)}{m} - \frac{\frac{d}{dt}(\sigma^2)}{\sigma^2} + \frac{E(w_i)}{\sigma^2} \right) + \left(\frac{\frac{d}{dt}(x_i)}{2\sigma^2} - \frac{u_i^{(1)}}{2\sigma^2} + \frac{E'(w_i)}{\sigma^2}(\partial_x w_i) \right) X + \left(\frac{\frac{d}{dt}(y_i)}{2\sigma^2} - \frac{u_i^{(2)}}{2\sigma^2} + \frac{E'(w_i)}{\sigma^2}(\partial_y w_i) \right) Y + \left(\frac{\frac{d}{dt}(\sigma^2)}{4\sigma^2} - \frac{E(w_i)}{4\sigma^2} \right) (X^2 + Y^2) + O(X^2) + O(Y^2) + O(XY) \right\}$$

where $w_i = w(\vec{x}_i)$, $u_i^{(1)} = u^{(1)}(\vec{x}_i)$ and $u_i^{(2)} = u^{(2)}(\vec{x}_i)$. In this expansion, there are four equations and four unknown functions. The reason why the first $E(w)$ and \vec{u} in (15) were not expanded further is that it would result in unmatched terms in $O(XY)$. (This subsection describes *an example* of a type of basis function, but this is *not an endorsement* for using Gaussians for this problem.)

From (16), we can recover the following system of ordinary differential equations.

$$(17) \quad \frac{d}{dt}(m) = 0,$$

$$(18) \quad \frac{d}{dt}(\vec{x}_i) = \vec{u}_i - 2E'(w_i)\nabla w_i,$$

$$(19) \quad \frac{d}{dt}(\sigma^2) = E(w_i).$$

If $E(w)$ is a constant, this system reduces to a simple core spreading method. By themselves, core spreading methods are inconsistent for arbitrary but finite time, but they can be corrected (for further discussion on this issue, see [10, 16, 17, 15, 28]).

We see that the Lagrangian system described here does not capture all the localized terms to second order, and we are motivated to find a basis function with more internal degrees of freedom (i.e. a larger vector \vec{p}) so that the basis function will be consistent at second order.

3.2. Elliptical Gaussian Basis Functions. The residual has exposed the weaknesses of Gaussian elements for (7), so we shall introduce another self-similar basis function with more internal degrees of freedom in the hopes that we can reduce the residual further. To address the problem, we examine elliptical Gaussian basis functions:

$$(20) \quad f(\vec{x} - \vec{x}_i, \vec{p}) = \frac{m_i}{4\pi\sigma_i^2} \times \exp\left\{-\frac{[c_i(x - x_i) + s_i(y - y_i)]^2/a_i^2 + [-s_i(x - x_i) + c_i(y - y_i)]^2 a_i^2}{4\sigma_i^2}\right\},$$

where $\vec{p} = (m_i, \sigma_i^2, a_i^2, \theta_i)^T$ corresponds to the volume, area, aspect ratio and orientation of the elliptical Gaussian, and $c_i = \cos(\theta_i)$ and $s_i = \sin(\theta_i)$ (see Fig. 1 for a schematic). By introducing these two additional degrees of freedom into each basis

function, we can capture the quadratic terms in the residual and generate a high spatial order Lagrangian method.

Similar to (15) but with elliptical Gaussian basis functions (20), we shall compute the residual in a coordinate system rotated by θ to simplify the computation:

$$(21) \quad R[f] = \frac{m}{4\pi\sigma^2} \exp\left(-\frac{X^2/a^2 + Y^2a^2}{4\sigma^2}\right) \times \left\{ \left[\frac{\frac{d}{dt}(m)}{m} - \frac{\frac{d}{dt}(\sigma^2)}{\sigma^2} + \frac{E(w)}{2\sigma^2} (a^2 + a^{-2}) \right] + \left(\frac{\frac{d}{dt}(x_i)}{2a^2\sigma^2} - \frac{u_1}{2a^2\sigma^2} \right) X + \left(\frac{\frac{d}{dt}(y_i)a^2}{2\sigma^2} - \frac{u_2a^2}{2\sigma^2} \right) Y + \left(\frac{\frac{d}{dt}(\sigma^2)}{4a^2\sigma^2} + \frac{\frac{d}{dt}(a^2)}{4a^4\sigma^2} - \frac{E(w)}{4\sigma^2} \right) X^2 + \left(\frac{a^2 \frac{d}{dt}(\sigma^2)}{4\sigma^2} + \frac{\frac{d}{dt}(a^2)}{4\sigma^2} - \frac{E(w)}{4\sigma^2} \right) Y^2 - \left(\frac{(a^{-2} - a^2) \frac{d}{dt}(\theta)}{2\sigma^2} \right) XY \right\}.$$

This time, we can expand both $E(w)$ in a second order Taylor series and similarly linearize \vec{u} :

$$(22) \quad \vec{u}(\vec{x}) = \vec{u}_i + D\vec{u}(\vec{x} - \vec{x}_i) + O(|\vec{x} - \vec{x}_i|^2)$$

where $D\vec{u} = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}$ is the matrix of partial derivatives so that $d_{11} = \partial_x u^{(1)}(\vec{x}_i)$, $d_{12} = \partial_y u^{(1)}(\vec{x}_i)$, $d_{21} = \partial_x u^{(2)}(\vec{x}_i)$ and $d_{22} = \partial_y u^{(2)}(\vec{x}_i)$.

$$(23) \quad R[f] = \frac{m}{4\pi\sigma^2} \exp\left(-\frac{X^2/a^2 + Y^2a^2}{4\sigma^2}\right) \times \left\{ \left[\frac{\frac{d}{dt}(m)}{m} - \frac{\frac{d}{dt}(\sigma^2)}{\sigma^2} + \frac{E(w_i)}{2\sigma^2} (a^2 + a^{-2}) \right] + \left[\frac{\frac{d}{dt}(x_i)}{2a^2\sigma^2} - \frac{u_{1,i}}{2a^2\sigma^2} + \frac{E'(w_i)}{2\sigma^2} (\partial_x w_i) (a^2 + a^{-2}) \right] X + \left[\frac{\frac{d}{dt}(y_i)a^2}{2\sigma^2} - \frac{u_{2,i}a^2}{2\sigma^2} + \frac{E'(w_i)}{2\sigma^2} (\partial_y w_i) (a^2 + a^{-2}) \right] Y + \left[\frac{\frac{d}{dt}(\sigma^2)}{4a^2\sigma^2} + \frac{\frac{d}{dt}(a^2)}{4a^4\sigma^2} - \frac{d_{11}}{2\sigma^2 a^2} + \frac{E''(w_i)(a^{-2} + a^2)}{4\sigma^2} (\partial_x w_i)^2 - \frac{E(w_i)}{4\sigma^2} \right] X^2 + \left[\frac{a^2 \frac{d}{dt}(\sigma^2)}{4\sigma^2} + \frac{\frac{d}{dt}(a^2)}{4\sigma^2} - \frac{d_{22}a^2}{2\sigma^2} + \frac{E''(w_i)(a^{-2} + a^2)}{4\sigma^2} (\partial_y w_i)^2 - \frac{E(w_i)}{4\sigma^2} \right] Y^2 - \left[\frac{(a^{-2} - a^2) \frac{d}{dt}(\theta)}{2\sigma^2} + \frac{d_{12}a^{-2} + d_{21}a^2}{2\sigma^2} - \frac{E''(w_i)(a^{-2} + a^2)}{2\sigma^2} (\partial_x w_i) (\partial_y w_i) \right] XY + \text{H.O.T.} \right\},$$

where H.O.T. are higher order terms in X and Y .

Setting to zero each coefficient for powers of X and Y and performing some routine algebra, one arrives at the following system of ordinary differential equations.

$$(24) \quad \frac{d}{dt}(m) = m(d_{11} + d_{22}) - \frac{m}{2} E''(w_i)(a^{-2} + a^2) \left[a^2 (\partial_x w_i)^2 + a^{-2} (\partial_y w_i)^2 \right],$$

$$(25) \quad \frac{d}{dt}(\vec{x}_i) = \vec{u}_i - E'(w_i) \nabla w(\vec{x}_i)(a^{-2} + a^2),$$

$$(26) \quad \frac{d}{dt}(\sigma^2) = \sigma^2(d_{11} + d_{22}) + \frac{1}{2} E(w_i)(a^{-2} + a^2) - \frac{1}{2} \sigma^2 E''(w_i)(a^{-2} + a^2) \left[a^2 (\partial_x w_i)^2 + a^{-2} (\partial_y w_i)^2 \right],$$

$$(27) \quad \frac{d}{dt}(a^2) = a^2(d_{11} - d_{22}) + \frac{E(w_i)}{2\sigma^2}(1 - a^4) - \frac{1}{2} a^2 E''(w_i)(a^{-2} + a^2) \left[a^2 (\partial_x w_i)^2 - a^{-2} (\partial_y w_i)^2 \right],$$

$$(28) \quad \frac{d}{dt}(\theta) = - \frac{d_{12} a^{-2} + d_{21} a^2 - E''(w_i)(a^{-2} + a^2) (\partial_x w_i) (\partial_y w_i)}{a^{-2} - a^2}.$$

The first evolution equation, together with the third, captures the effects of compressibility. The second equation describes the motion of computational elements under the influence of the derived diffusion velocity plus corrections from the spatially varying flux. The third equation defines the evolution of the core size with contributions from compressibility and diffusion. We can see that diffusion is enhanced in elongated elements which is what one would expect because elongation increases the gradient of the moisture field. In the fourth equation, the aspect ratio grows if there is an applied straining field either from the velocity field or from variations in the flux. The last equation for the evolution of the orientation can be rewritten as

$$(29) \quad \frac{d}{dt}(\theta) = \frac{1}{2}(d_{21} - d_{12}) - \frac{1}{2} [d_{12} + d_{21} - 2E''(w_i) (\partial_x w_i) (\partial_y w_i)] \frac{a^{-2} + a^2}{a^{-2} - a^2},$$

so that one can see that computational elements rotate under the influence of local rotations in the velocity field and through straining if the computational element is not aligned with the principle axes of the local straining field. The singularity as $a \rightarrow 1$ can be treated directly in any standard integration scheme because the orientation becomes undefined if the element becomes axisymmetric [30]. Finally, the evolution equations (24-29) can be reoriented back to standard coordinates by a suitable transformation of $D\vec{u}$.

While this article makes no attempt to review and analyze various methods of capturing diffusive terms with Lagrangian methods, it is necessary to discuss core spreading and spatial resolution. Since the cores of computational elements can and will grow over time, strictly integrating (24-29) will result in a method that remains spatially accurate only for a short time (see [16]). The problem can be resolved for arbitrary times by introducing a *spatial refinement* procedure wherein an element that has grown too wide ($\sigma_i > l$) is replaced with a configuration of thinner elements that approximate the wide element to a controllably small error (see [28, 30] for details on spatial refinement). This guarantees a minimum spatial

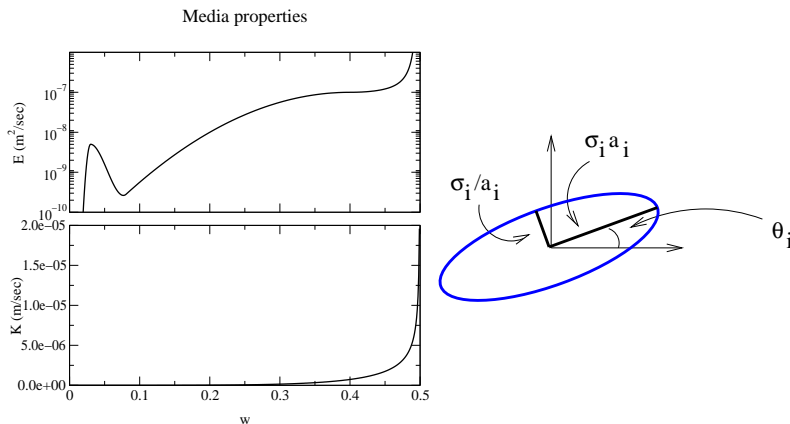


FIGURE 1. The physical properties, $K(w)$ and $E(w)$, of the porous media to be studied (left) based on [24], and a schematic diagram of the parameters for an elliptical Gaussian basis function (right).

resolution of l . There is also a corresponding procedure for merging together similar elements that nearly overlap (see [29, 31] for details).

We can summarize a resulting Lagrangian method as follows:

- (1) Express initial conditions as a collection of basis functions.
- (2) Integrate (24-29) for a small amount of time.
- (3) If any elements have core widths exceeding l , replace them with a configuration of thinner elements.
- (4) Merge any overlapping, similar elements to reduce the total number of computational elements, if such opportunities arise.
- (5) Return to (2) to advance in time.

To solve (24-29), we integrate particle positions with a fourth-order ‘‘Adams family’’ multistep method, but all other terms are computed with fourth order Runge-Kutta.¹

4. Infiltration

The infiltration of water into soils is an important problem that has received experimental, theoretical and some numerical treatment. Several decades ago, it was assumed that the progress of a wetting front into unsaturated soils was essentially one-dimensional. Indeed, treatment of the one-dimensional problem is by no means simple (see [2, 32] for analysis and review of various approaches.) However, Hill & Parlange and others have shown that under certain conditions, the wetting front can exhibit instabilities [14]. This area has received substantial experimental and analytical attention (see [8, 7, 9] for reviews and discussion), but numerical treatment has been limited to linear stability analysis with good reason. The full problem is highly nonlinear and the domain is unbounded in all directions. Fortunately, Lagrangian methods are naturally adaptive so that the boundary conditions at infinity do not pose a problem.

¹This implementation is a substantial modification of the BlobFlow (TM) open source project (see <http://www.math.udel.edu/~rossi/BlobFlow>) and is available from the author upon request.

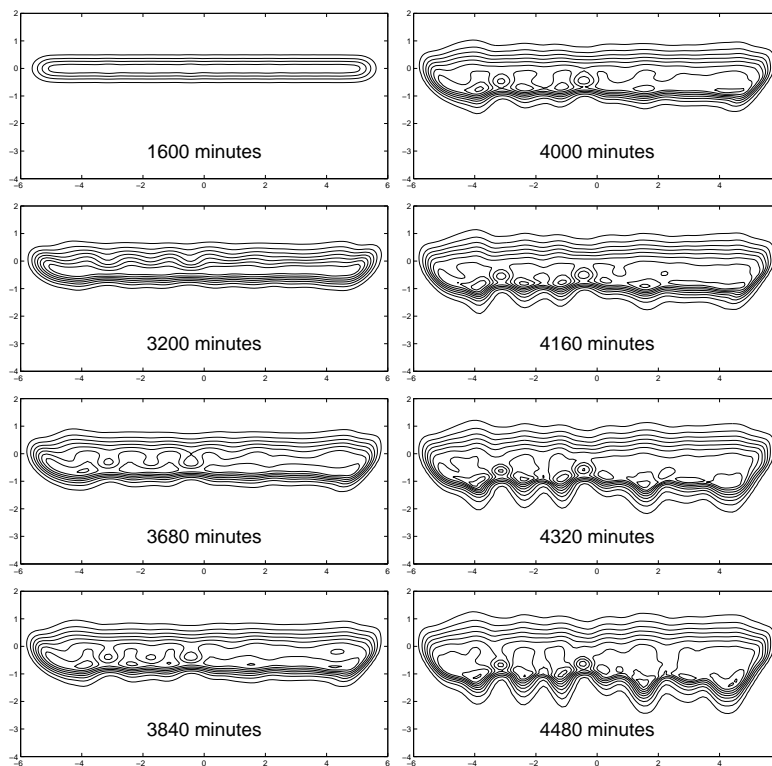


FIGURE 2. Sample computations of a steady infiltration problem. The moisture content contour intervals are 0.05. In these calculations, we see that water oozes into the porous matrix as a one-dimensional wetting front until it reaches a large enough volumetric content to trigger the instability.

For specific material properties, we use a continuously differentiable interpolating function from the data for Yolo light clay in [24] and scale the permeability K to correspond to a fine sand or gravel while keeping E identical to the published values which amounts to changing the ratio of gravitational potential to moisture potential (see Fig. 1). Moisture is added uniformly to the domain along the y -axis on the interval $[-6, 6]$ at an infiltration rate of approximately 1.7×10^{-6} m² per second. For this configuration, this would amount to about half a centimeter per hour of rainfall or colloquially “a light drizzle.”

To resolve the flow, the minimum spatial resolution, l , is fixed at 0.2 meters. Halving the spatial resolution does not visibly change the solution. The solution is integrated in time with an interval of 2 minutes. Again, temporal refinement yielded no visible change in the solution. In short, the computations are fully resolved. Some merging of computational elements is permitted as long as the induced error is less than 0.001 every 160 minutes which has the affect of adding a very small amount of noise to the system. Initially, this noise is damped out by moisture diffusion, but as the water content increases, these and other numerical errors excite a wetting front instability. Typical calculations beginning with a

few dozen elements and ending with several thousand, spanning over 48 hours of simulated time, requires about an hour of CPU time on a 450 mhz Pentium III running Linux. While the material properties for various fingering instability experiments have not been measured or published, we believe that the material properties correspond loosely in general trend and magnitude to sand or gravel. In Fig. 2, we see that the computations yield excellent qualitative agreement with the published analyses and experiments [7, 9].

5. Conclusions and future work

We have developed a new Lagrangian method by using elliptical Gaussian elements to capture all terms in the residual of a unsaturated flow model to second order. The resulting method retains higher spatial accuracy than those using rigid basis functions, and effectively captures the behavior of the wetting front in a steady infiltration experiment. While this method is both practical and flexible, there are several avenues for future work. First, this technique is easily extensible to three dimensions using ellipsoidal Gaussian elements. Second, this method permits elements to attain arbitrarily large aspect ratios and at the same time diminish into nothingness (see equations (24-29) for E'' large and positive). Finding a way to treat this situation would improve the efficiency of the scheme. Third, the method requires $O(N^2)$ computations, but this could be reduced substantially because the basis functions and their gradients are localized.

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