A FINITE ELEMENT METHOD FOR APPROXIMATING ELECTROMAGNETIC SCATTERING FROM A CONDUCTING OBJECT

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Abstract. We provide an error analysis of a fully discrete finite element – Fourier series method for approximating Maxwell's equations. The problem is to approximate the electromagnetic field scattered by a bounded, inhomogeneous and anisotropic body. The method is to truncate the domain of the calculation using a series solution of the field away from this domain. We first prove a decomposition for the Poincaré-Steklov operator on this boundary into an isomorphism and a compact perturbation. This is proved using a novel argument in which the scattering problem is viewed as a perturbation of the free space problem. Using this decomposition, and edge elements to discretize the interior problem, we prove an optimal error estimate for the overall problem.

1. Introduction. Motivated by the problem of computing the interaction of microwave radiation with biological tissue, we shall analyze a finite element method for approximating Maxwell's equations in an infinite domain. We suppose that there is a bounded anisotropic, conducting object, called the scatterer, illuminated by a time-harmonic microwave source. The microwave source produces an incident electromagnetic field that interacts with the scatterer and produces a scattered time harmonic electromagnetic field. It is the scattered field that we wish to approximate.

From the mathematical point of view, the problem can be reduced to that of approximating the total electric field $\boldsymbol{E} = \boldsymbol{E}(\boldsymbol{x})$, $\boldsymbol{x} \in \mathbb{R}^3$, which satisfies Maxwell's equations in all space:

$$\nabla \times \mu_r^{-1} \nabla \times \boldsymbol{E} \;-\; k^2 \epsilon_r \boldsymbol{E} \;=\; 0\,, \quad \text{in } \mathbb{R}^3 \tag{1.1}$$

where μ_r and ϵ_r are respectively the relative permeability tensor (real 3×3 matrix) and the relative permittivity tensor (complex 3×3 matrix) describing the electromagnetic properties of the scatterer. The constant k is the wave number of the material in the background medium. Recall that the magnetic field $\boldsymbol{H}$ can be computed from the electric field using

$$\boldsymbol{H} \;=\; \frac{1}{ik}\,\mu_r^{-1} \nabla \times \boldsymbol{E}.$$

The boundedness of the scatterer implies the existence of a radius $a > 0$ such that

$$\mu_r(\boldsymbol{x}) \;=\; \epsilon_r(\boldsymbol{x}) \;=\; I \quad \text{if } |\boldsymbol{x}| > a$$

where I is the identity matrix.

The incident field $\boldsymbol{E}^i$ is assumed known and to satisfy Maxwell's equations in the background medium

$$\nabla \times \nabla \times \boldsymbol{E}^i \;-\; k^2 \boldsymbol{E}^i \;=\; 0 \quad \text{in } \mathbb{R}^3,$$

so $\boldsymbol{E}^i$ is a wave propagating in the background homogeneous medium. We have in mind the plane wave

$$\boldsymbol{E}^i \;=\; \boldsymbol{p} \exp(ik\boldsymbol{x} \cdot \boldsymbol{d}) \tag{1.2}$$

where $|\boldsymbol{d}| = |\boldsymbol{p}| = 1$ and $\boldsymbol{p}$ is orthogonal to $\boldsymbol{d}$. Other incident fields can be incorporated into the theory including for example the field due to a point source provide the source is outside the sphere of radius a enclosing the scatterer.

The total field $\boldsymbol{E}$ consists of the known incident field $\boldsymbol{E}^i$ and the scattered field denoted $\boldsymbol{E}^s$ so that

$$\boldsymbol{E} \;=\; \boldsymbol{E}^i \;+\; \boldsymbol{E}^s \quad \text{in } \mathbb{R}^3. \tag{1.3}$$

Finally the scattered field is assumed to be "outgoing" so that it satisfies the Silver-Müller radiation condition

$$\lim_{r \to \infty} \left[(\nabla \times \boldsymbol{E}^s) \times \boldsymbol{x} - ikr\boldsymbol{E}^s \right] \;=\; 0 \tag{1.4}$$

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where $r = |\boldsymbol{x}|$.

Under suitable assumptions on the coefficients in the model described shortly, the scattering problem (1.1)-(1.4) has a unique solution (for a general discussion of problems of this type see [5]). The theory presented here will produce an alternative proof of this fact.

There are a variety of ways to approximate the solution of (1.1)-(1.4). One way is by integral equations based on a volume formulation (see for example [5, 22]). This handles the infinite domain precisely, but requires the evaluation of singular integrals and the approximate inversion of a large dense system (of course using a suitable iterative method).

An alternative method, which is the focus of this paper, is to use a finite element method. To do this the infinite spatial domain must be truncated by introducing an artificial boundary containing the scatterer in it's interior. A suitable boundary condition must then be formulated on the artificial boundary to mimic the scattering behavior of the infinite part of the domain. Using an appropriate Calderon operator (or Dirichlet-to-Neumann map) we proposed and analyzed such a scheme (see [27] and the correction in [28]). That method however supposed that the Calderon operator is computed exactly. In a related work Demkowicz and Pal [9] analyzed the same method this time discretizing the Calderon operator but not the interior problem. In this paper we shall use a domain decomposition approach to analyze a fully discrete problem (discretizing with finite elements in the interior and discretizing the Calderon operator). The analysis of the fully discrete problem is desirable to illuminate potential stability conflicts.

Of course the use of the Steklov-Poincaré operator and its application to computing the Dirichlet to Neumann map has a long history. In particular we draw attention to the following work related to using a special function expansion in the exterior of a ball as we shall do here [17, 23, 19, 7, 8, 18]. In our paper we shall present an a priori analysis of this method applied to Maxwell's equations.

The approach we adopt here is a natural extension of our work on the Helmholtz equation [26]. This proves convergence of the Keller-Givoli Dirichlet-to-Neumann map absorbing boundary condition [24, 13] which is presented in detail by Ihlenburg [21]. In our paper in two dimensions, and later in the paper of Grote and Keller [15] in three dimensions, stability is proved without requiring a relationship between the number of modes on the artificial boundary and the mesh size of the finite element method. Unfortunately we have been unable to prove this for Maxwell's equations, and instead must require that the mesh size be sufficiently small compared to the number of modes used on the boundary (see Theorem 5.1).

A similar approach to obtaining artificial boundary conditions for Maxwell's equations has been proposed by Grote and Keller [16] using the special function expansion of the solution that we use in this paper. They showed how to implement the method efficiently for the time dependent problem, but did not provide error estimates. For more recent results in this direction (again for the time domain problem) see [14]. Our results show that, for the time-harmonic problem, the use of truncated spherical-harmonic expansions on the artificial boundary produces a well-posed discrete problem (under the conditions of Theorem 5.1) that converges to the exact solution.

The scattering problem is decomposed into two parts, one on the bounded domain inside the artificial boundary and the other on it's infinite complement. Matching is done on the artificial boundary using a Poincaré-Steklov problem. As a result the method is said to be a non-overlapping scheme. An alternative scheme proposed in [20] is to use an overlapping method. This introduces a coupling between some interior points and points on the artificial boundary. Our method decouples the two problems (effectively only coupling the nodes on the artificial boundary). Nevertheless, the method of [20] allows a very general artificial boundary, whereas the method we describe here is restricted to a spherical outer boundary (more general boundaries are possible, for example ellipsoidal boundaries, at the expense of working with suitable basis functions in more general coordinate systems).

There are many other possible methods for approximating this scattering problem. For example the interior finite element method can be coupled to a boundary element method that effectively computes the Calderon operator (and allows a rather general artificial boundary), see for example [22]. Finally we note that it is possible to use the perfectly matched layer of Bérénger [2] to terminate the finite element region. The analysis of the perfectly matched layer for Maxwell's equations has yet to be done.

The layout of the paper is as follows. In the next section, § 2, we describe in detail the assumptions on ϵ_r and μ_r, and we formulate a domain decomposition scheme for the continuous problem. Then we show that the domain decomposed problem possess a unique solution. The approach is novel in that we view the scattering problem as a compact perturbation of the free space problem in a suitable sense. We also show that despite the fact that we do not explicitly handle the divergence condition, the solution is unique.

Section § 3 is devoted to describing the finite dimensional discrete problem based on using the edge finite elements of Nédélec [31] in the interior and a Fourier space on the surface of the sphere.

In Section § 4 we analyze the interior finite element problem and derive an error estimate for the interior scheme.

Finally in Section § 5 we analyze the overall discrete problem, prove that it possesses a unique solution and derive an error estimate. The analysis of this problem is complicated by the fact that we have been unable to write the boundary Poincaré-Steklov operator as a compact perturbation of a coercive operator (see also [8]). Thus we have to adopt a more general splitting writing the operator as an invertible operator plus a compact perturbation. This is possible because of the very special boundary space that we use on the artificial boundary.

Throughout the paper we shall denote by $\|\cdot\|$ the $\big(L^2(\Omega_R)\big)^3$ norm. For other normed spaces $\mathcal{X}$ we denote the norm by $\|\cdot\|_{\mathcal{X}}$.

2. The Truncated Problem. Before we show how to reduce the scattering problem to a problem posed on a bounded domain, we shall make explicit the assumptions on the coefficients ϵ_r and μ_r. Later, in the section on numerical analysis, we shall further restrict the class of coefficients to enable us to prove error estimates.

Our assumptions are essentially the same as those in [20] but modified to allow for matrix functions of position. We suppose that $\mathbb{R}^3$ can be decomposed into $N+1$ disjoint open sets with non-empty interior and Lipschitz smooth boundaries. We denote these domains $\Omega_0, \ldots, \Omega_N$. We assume that Ω_0 is unbounded and that the remaining sub-domains are bounded. The coefficients μ_r and ϵ_r satisfy the following conditions:

- On Ω_0, we have $\epsilon_r = \mu_r = I$ (the scatterer is bounded).
- On each Ω_n, $1 \le n \le N$, the coefficients ϵ_r and μ_r are uniformly bounded, Lipschitz continuous matrix functions of position. Furthermore $\Re(\epsilon_r)$ and μ_r are real symmetric uniformly positive definite matrix functions of position.
- On at least one sub-domain (say Ω_J, for some J with $1 \le J \le N$), $\Im(\epsilon_r)$ is strictly uniformly positive definite. On every domain, either $\Im(\epsilon_r)$ is strictly positive definite or $\Im(\epsilon_r) = 0$.

The assumptions are not very restrictive since they allow for piecewise smooth media. At least one sub-domain is assumed to be absorbing, and this allows us to prove uniqueness of certain interior problems used in the method. Obviously this assumption is satisfied for most biological media, but it could be dropped at the cost of needing to pick the auxiliary boundary carefully.

2.1. Domain Decomposition. We introduce an artificial boundary Γ_R that is the surface of the ball of radius R. We assume $R > a$ so that the scatterer is contained in the interior of the ball. We denote by Ω_R the ball of radius R.

Inside Ω_R, the electric field satisfies the Maxwell system

$$\nabla \times \mu_r^{-1} \nabla \times \boldsymbol{E} \;-\; k^2 \epsilon_r \boldsymbol{E} \;=\; 0 \quad \text{in } \Omega_R\,.$$

Outside Ω_R, in $\mathbb{R}^3 \setminus \overline{\Omega_R}$, the scattered field $\boldsymbol{E}^s$ satisfies the following constant coefficient Maxwell system together with the Silver-Müller radiation condition

$$\nabla \times \nabla \times \boldsymbol{E}^s \;-\; k^2 \boldsymbol{E}^s = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{\Omega_R}\,,$$
$$\lim_{r\to\infty} \big[(\nabla \times \boldsymbol{E}^s) \times \boldsymbol{x} - ikr\boldsymbol{E}^s\big] = 0\,.$$

Across the artificial boundary, these problems are linked by enforcing the continuity of the tangential components of the electric and magnetic fields:

$$\hat{\boldsymbol{x}} \times \frac{1}{ik} \nabla \times \boldsymbol{E}\Big|_i = \hat{\boldsymbol{x}} \times \frac{1}{ik} \nabla \times \boldsymbol{E}^s \;+\; \hat{\boldsymbol{x}} \times \frac{1}{ik} \nabla \times \boldsymbol{E}^i \quad \text{on } \Gamma_R\,, \tag{2.1}$$

$$\hat{\boldsymbol{x}} \times \boldsymbol{E}\big|_i = \hat{\boldsymbol{x}} \times \boldsymbol{E}^s \;+\; \hat{\boldsymbol{x}} \times \boldsymbol{E}^i \quad \text{on } \Gamma_R\,. \tag{2.2}$$

Next we want to explicitly decouple the two fields and pose the problem as an operator equation on Γ_R. We introduce the interior and exterior Calderon operators denoted G_i and G_e, respectively (see [4]). Proceeding formally, suppose that $\boldsymbol{\lambda}$ is a suitably smooth (to be made precise shortly) tangential vector field on Γ_R, then we define the exterior Calderon operator G_e by

$$G_e\boldsymbol{\lambda} \;=\; \hat{\boldsymbol{x}} \times \boldsymbol{u}(\boldsymbol{\lambda})|_{\Gamma_R}$$

where $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{\lambda})$ is the solution of

$$\nabla \times \nabla \times \boldsymbol{u} \;-\; k^2\boldsymbol{u} = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{\Omega_R}\,, \tag{2.3}$$

$$\hat{\boldsymbol{x}} \times \frac{1}{ik}\nabla \times \boldsymbol{u} = \boldsymbol{\lambda} \quad \text{on } \Gamma_R \tag{2.4}$$

$$\lim_{r\to\infty}\big[(\nabla \times \boldsymbol{u}) \times \boldsymbol{x} - ikr\boldsymbol{u}\big] = 0\,. \tag{2.5}$$

The interior Calderon operator is defined in a similar way by

$$G_i\boldsymbol{\lambda} \;=\; \hat{\boldsymbol{x}} \times \boldsymbol{w}(\boldsymbol{\lambda})|_{\Gamma_R}$$

where $\boldsymbol{w} = \boldsymbol{w}(\boldsymbol{\lambda})$ is the solution of

$$\nabla \times \mu_r^{-1}\nabla \times \boldsymbol{w} - k^2\epsilon_r\boldsymbol{w} = 0 \quad \text{in } \Omega_R\,, \tag{2.6}$$

$$\hat{\boldsymbol{x}} \times \frac{1}{ik}\nabla \times \boldsymbol{w} = \boldsymbol{\lambda} \quad \text{on } \Gamma_R\,. \tag{2.7}$$

Using the two Calderon operators we see that if $\boldsymbol{\lambda} = \hat{\boldsymbol{x}} \times \frac{1}{ik}\nabla \times \boldsymbol{E}$ on Γ_R, then using the boundary relations (2.1)-(2.2) we have

$$G_i\boldsymbol{\lambda} \;-\; G_e\left(\boldsymbol{\lambda} - \hat{\boldsymbol{x}} \times \frac{1}{ik}\nabla \times \boldsymbol{E}^i\right) \;=\; \hat{\boldsymbol{x}} \times \boldsymbol{E}^i|_{\Gamma_R}.$$

Shortly we shall analyze the mapping properties of G_e and G_i, but in order to state precisely the operator equation resulting from the above equality we recall the space of tangential vector fields with well defined divergences by

$$H^s(\mathrm{Div};\Gamma_R) \;=\; \left\{\boldsymbol{u} \in (H^s(\Gamma_R))^3 \mid \boldsymbol{u}\cdot\hat{\boldsymbol{x}} = 0 \text{ a.e.}, \nabla_{\Gamma_R}\cdot\boldsymbol{u} \in H^s(\Gamma_R)\right\}$$

for $s \in \mathbb{R}$ and where $\nabla_{\Gamma_R}\cdot$ is the surface divergence on Γ_R. We shall be particularly interested in $H^{-1/2}(\mathrm{Div};\Gamma_R)$. For later use we also recall that

$$H(\mathrm{curl};\Omega_R) = \left\{\boldsymbol{u} \in (L^2(\Omega_R))^3 \mid \nabla \times \boldsymbol{u} \in (L^2(\Omega_R))^3\right\},$$
$$H_0(\mathrm{curl};\Omega_R) = \{\boldsymbol{u} \in H(\mathrm{curl};\Omega_R) \mid \hat{\boldsymbol{x}} \times \boldsymbol{u} = 0 \text{ on } \Gamma_R\},$$

with norm $\|\cdot\|_{H(\mathrm{curl};\Omega_R)}$.

Now we can state the problem we wish to solve precisely. Given $\boldsymbol{f} \in H^{-1/2}(\mathrm{Div};\Gamma_R)$ we wish to find $\boldsymbol{\lambda} \in H^{-1/2}(\mathrm{Div};\Gamma_R)$ such that

$$(G_i - G_e)\boldsymbol{\lambda} \;=\; \boldsymbol{f}\,. \tag{2.8}$$

As we have seen, in applications to the scattering problem

$$\boldsymbol{f} \;=\; \hat{\boldsymbol{x}} \times \boldsymbol{E}^i \;-\; G_e\left(\hat{\boldsymbol{x}} \times \frac{1}{ik}\nabla \times \boldsymbol{E}^i\right). \tag{2.9}$$

Once we have computed $\boldsymbol{\lambda}$ via (2.8), we can compute $\boldsymbol{E}$ on Ω_R by solving (2.6)-(2.7). Similarly we can compute $\boldsymbol{E}^s$ in $\mathbb{R}^3 \setminus \overline{\Omega_R}$ by solving (2.3)-(2.5) with $\boldsymbol{\lambda}$ replaced by $\boldsymbol{\lambda} - \hat{\boldsymbol{x}} \times (1/ik)\nabla \times \boldsymbol{E}^i$.

The remainder of this section is devoted to showing that the boundary operator equation (2.8) is well defined and has a unique solution.

2.2. Analysis of the Domain Decomposed Problem. This part is devoted to establishing the following theorem concerning the continuous Calderon operators for the coupled problem. It is the cornerstone of our later analysis of the numerical method.

THEOREM 2.1. *Suppose the coefficients ϵ_r and μ_r satisfy the conditions outlined at the beginning of this section and that Ω_R is chosen such that k is not an eigenvalue for the interior magnetic Maxwell eigenvalue problem when $\epsilon_r = \mu_r = 1$. Then, for any s,*

$$G_i - G_e \;=\; T + K$$

where T is a bounded invertible operator from $H^s(\mathrm{Div};\Gamma_R)$ onto $H^s(\mathrm{Div};\Gamma_R)$ and K is a compact perturbation.

The outline of the proof (which is proved after some preliminary results later in this section) is as follows. First we establish the theorem in the case when $\epsilon_r = \mu_r = 1$. We denote by $\tilde{G}_i$ the interior Calderon operator in this case and use a suitable series solution to establish the result. Then we show how the result can be extended to the general case.

In order to perform the series based analysis let us recall some standard facts about special function solutions of Maxwell's equations (see for example [5]). Following [5], let $\{Y_n^m(\hat{\boldsymbol{x}})\}_{m=-n}^{n}$ denote an orthonormal sequence of spherical harmonics of order n on the unit sphere. Using these spherical harmonics, basis functions for tangential fields on any sphere centered at the origin are

$$\boldsymbol{U}_n^m \;=\; \frac{1}{\sqrt{n(n+1)}}\nabla_{\Gamma_1} Y_n^m\,, \quad \text{and} \quad \boldsymbol{V}_n^m \;=\; \hat{\boldsymbol{x}} \times \boldsymbol{U}_n^m\,, \tag{2.10}$$

for $-n \le m \le n$ and $n \in \mathbb{N}$. Here ∇_{Γ_1} represents the surface gradient of scalar functions on the unit sphere. For a smooth function ϕ we have that for any sphere of radius R

$$\nabla\phi|_{\Gamma_R} \;=\; \frac{1}{R}\nabla_{\Gamma_1}\phi \;+\; \frac{\partial\phi}{\partial r}\hat{\boldsymbol{x}}\,,$$

where $\boldsymbol{x} = r\hat{\boldsymbol{x}}$.

Using the tangential basis functions, a tangential field $\boldsymbol{\lambda} \in H^s(\mathrm{Div};\Gamma_R)$ can be written as

$$\boldsymbol{\lambda} \;=\; \sum_{n=1}^{\infty}\sum_{m=-n}^{n}\left(a_n^m\,\boldsymbol{U}_n^m \;+\; b_n^m\,\boldsymbol{V}_n^m\right) \tag{2.11}$$

and the $H^s(\mathrm{Div};\Gamma_R)$ norm is given by

$$\|\boldsymbol{\lambda}\|^2_{H^s(\mathrm{Div};\Gamma_R)} \;\equiv\; \sum_{n=1}^{\infty}\sum_{m=-n}^{n}\left[n^{2(1+s)}|a_n^m|^2 + n^{2s}|b_n^m|^2\right].$$

For a more detailed discussion see [27].

Corresponding to the surface basis functions, there are standard radiating vector basis functions for the scattered field

$$\boldsymbol{M}_n^m(\boldsymbol{x}) = \nabla \times \{\boldsymbol{x} h_n^{(1)}(kr) Y_n^m(\hat{\boldsymbol{x}})\}\,, \tag{2.12}$$

$$\boldsymbol{N}_n^m(\boldsymbol{x}) = \frac{1}{ik}\nabla \times \boldsymbol{M}_n^m(\boldsymbol{x})\,. \tag{2.13}$$

For the interior field we need basis functions that are bounded at $r = 0$ and these are

$$\hat{\boldsymbol{M}}_n^m(\boldsymbol{x}) = \nabla \times \{\boldsymbol{x} j_n(kr) Y_n^m(\hat{\boldsymbol{x}})\}\,, \tag{2.14}$$

$$\hat{\boldsymbol{N}}_n^m(\boldsymbol{x}) = \frac{1}{ik}\nabla \times \hat{\boldsymbol{M}}_n^m(\boldsymbol{x})\,. \tag{2.15}$$

Here, $h_n^{(1)}$ and j_n denote the spherical Hankel function and Bessel functions, respectively. These basis functions are discussed in many books (again see for example [5]). The volume fields and boundary basis (see (2.10)) are related as follows (see (6.64) and (6.65) of [5]). On the sphere of radius R

$$\hat{\boldsymbol{x}} \times \boldsymbol{N}_n^m(\boldsymbol{x}) = \frac{1}{ikR}\,\tilde{h}_n(kR)\sqrt{n(n+1)}\,\boldsymbol{V}_n^m(\hat{\boldsymbol{x}})\,, \tag{2.16}$$

$$\hat{\boldsymbol{x}} \times \boldsymbol{M}_n^m(\boldsymbol{x}) = h_n^{(1)}(kR)\sqrt{n(n+1)}\,\boldsymbol{U}_n^m(\hat{\boldsymbol{x}}), \tag{2.17}$$

where $\tilde{h}_n(z) = h_n^{(1)}(z) + z{h_n^{(1)}}'(z)$ with similar expressions for $\hat{\boldsymbol{x}} \times \hat{\boldsymbol{N}}_n^m$ and $\hat{\boldsymbol{x}} \times \hat{\boldsymbol{M}}_n^m$.

We have the following lemma summarizing the basic mapping properties of G_e:

Lemma 2.2. *If $\boldsymbol{\lambda} \in H^s(\mathrm{Div};\Gamma_R)$ is given by (2.11) then the solution $\boldsymbol{u}$ of (2.3)-(2.5) is given by*

$$\begin{aligned}\boldsymbol{u} = &\sum_{n=1}^{\infty}\sum_{m=-n}^{n} \frac{ikR\,b_n^m}{h_n^{(1)}(kR) + kR\,(h_n^{(1)})'(kR)}\,\frac{\boldsymbol{M}_n^m}{\sqrt{n(n+1)}}\\ &-\sum_{n=1}^{\infty}\sum_{m=-n}^{n} \frac{a_n^m}{h_n^{(1)}(kR)}\,\frac{\boldsymbol{N}_n^m}{\sqrt{n(n+1)}}\,.\end{aligned} \tag{2.18}$$

and the exterior Calderon operator has the representation

$$G_e\boldsymbol{\lambda} \;=\; \sum_{n=1}^{\infty}\sum_{m=-n}^{n}\left[\frac{b_n^m}{\delta_n}\boldsymbol{U}_n^m - \delta_n a_n^m \boldsymbol{V}_n^m\right]. \tag{2.19}$$

where

$$\delta_n \;=\; \frac{1}{ikR}\left(1 + kR\,\frac{({h_n^{(1)}}')(kR)}{h_n^{(1)}(kR)}\right).$$

In particular, $G_e : H^s(\mathrm{Div};\Gamma_R) \to H^s(\mathrm{Div};\Gamma_R)$ and G_e is invertible.

Proof. The representation of $\boldsymbol{u}$ in (2.18) in terms of $\boldsymbol{M}_n^m$ and $\boldsymbol{N}_n^m$ is proved in [5]. The representation of $\boldsymbol{u}$ and $G_e\boldsymbol{\lambda}$ in terms of the coefficients of $\boldsymbol{\lambda}$ then follows using the relationship surface and volume basis functions in (2.16)-(2.17). Finally the mapping properties follow from the definition of the norm on $H^s(\mathrm{Div};\Gamma_R)$ and the fact (see [27]) that there are positive constants c_1 and c_2 such that

$$c_1 n \;\le\; |\delta_n| \;\le\; c_2 n$$

for $n = 1, 2, \ldots$. □

Now let $\tilde{G}_i$ denote the interior Calderon operator in the case when $\epsilon_r = \mu_r = 1$. We have the following lemma:

Lemma 2.3. *Suppose R is chosen such that $j_n(kR) \neq 0$ for all n and, with*

$$\tilde{\delta}_n \;=\; \frac{1}{ikR}\left(1 + kR\,\frac{(j_n)'(kR)}{j_n(kR)}\right),$$

then $\tilde{\delta}_n \neq 0$. When $\epsilon_r = \mu_r = 1$ we have

$$\tilde{G}_i\boldsymbol{\lambda} \;=\; \sum_{n=1}^{\infty}\sum_{m=-n}^{n}\left[\frac{b_n^m}{\tilde{\delta}_n}\,\boldsymbol{U}_n^m \;-\; \tilde{\delta}_n a_n^m\,\boldsymbol{V}_n^m\right].$$

The operator $\tilde{G}_i : H^s(\mathrm{Div};\Gamma_R) \to H^s(\mathrm{Div};\Gamma_R)$ is bounded linear operator for any s.

Remark: The conditions in δ_n imply a restriction on R which can be checked a-priori from a knowledge of the spherical Bessel functions. In fact this restriction can be avoided entirely by using a more complex interior problem having an absorbing ball at the center, but this complicates the analysis.

Proof. The proof is the same as for Lemma 2.2 provided the conditions on $\tilde{\delta}_n$ are satisfied. □

Theorem 2.4. *Under the conditions on Γ_R in Lemma 2.3, if $\epsilon_r = \mu_r = 1$, then*

$$G_e - \tilde{G}_i \;=\; T + K_1$$

where $T : H^s(\mathrm{Div};\Gamma_R) \to H^s(\mathrm{Div};\Gamma_R)$ is bounded and invertible and K_1 is compact for any s.

Proof. By Lemma 2.2 and Lemma 2.3, if $\boldsymbol{\lambda}$ is given by (2.11) then

$$(G_e - \tilde{G}_i)\boldsymbol{\lambda} \;=\; \sum_{n=1}^{\infty}\sum_{m=-n}^{n}\left[b_n^m\left(\frac{1}{\delta_n} - \frac{1}{\tilde{\delta}_n}\right)\boldsymbol{U}_n^m \;-\; a_n^m\left(\delta_n - \tilde{\delta}_n\right)\boldsymbol{V}_n^m\right].$$

Using the following recurrence asymptotic relations for spherical Hankel and Bessel functions (see [5]):

$$h_n^{(1)}(z) = \frac{(2n-1)!!}{iz^{n+1}}\left(1 + \mathcal{O}(\frac{1}{n})\right), \quad n \to \infty \tag{2.20}$$

$$(h_n^{(1)})'(z) = -\frac{n+1}{z}h_n^{(1)}(z) \;+\; h_{n-1}^{(1)}(z) \tag{2.21}$$

$$(h_n^{(1)})'(z) = -(n+1)\frac{(2n-1)!!}{iz^{n+2}}\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right) \tag{2.22}$$

$$j_n(z) = \frac{z^n}{(2n+1)!!}\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right), \tag{2.23}$$

$$(j_n)'(z) = n\,\frac{z^{n-1}}{(2n+1)!!}\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right), \tag{2.24}$$

and the Wronskian identity

$$h_n^{(1)'}(z)\, j_n(z) \;-\; h_n^{(1)}(z)\, j_n'(z) \;=\; \frac{i}{z^2}$$

we can derive the following estimates

$$\frac{1}{\delta_n} \;-\; \frac{1}{\tilde{\delta}_n} \;=\; -\frac{2ikR}{n}\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right)$$

and

$$\delta_n \;-\; \tilde{\delta}_n \;=\; \frac{2i}{kR}n\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right).$$

Hence if we define the operator T by

$$T\boldsymbol{\lambda} \;=\; -\sum_{n=1}^{\infty}\sum_{m-n}^{n}\left[\frac{2ikR}{n}\, b_n^m\,\boldsymbol{U}_n^m \;+\; \frac{2i}{kR}\, n\, a_n^m\,\boldsymbol{V}_n^m\right] \tag{2.25}$$

we have derived the desired decomposition. □

The next step is to extend the above result to the case of a general medium. To do this we state the following regularity result proved in the appendix. In this result we choose $a < \rho < R$ so that the scatterer is still contained in the interior of the ball of radius ρ. Let A denote the annulus $\{\boldsymbol{x} \;:\; \rho < r < R\}$ having boundaries Γ_R and Γ_ρ.

THEOREM 2.5. *Assume that ρ is chosen so that k is not a Maxwell eigenvalue for the annulus A (i.e. the following interior problem possesses a unique solution). Let the operator $L : \boldsymbol{\gamma} \to \hat{\boldsymbol{x}} \times \boldsymbol{E}|_{\Gamma_R}$ be defined by*

$$\begin{aligned}
\nabla \times \boldsymbol{E} \;-\; ik\boldsymbol{H} = 0 \quad & \text{in } A\,,\\
\nabla \times \boldsymbol{H} \;+\; ik\boldsymbol{E} = 0 \quad & \text{in } A\,,\\
\hat{\boldsymbol{x}} \times \boldsymbol{E} = \boldsymbol{\gamma} \quad & \text{on } \Gamma_\rho\,,\\
\hat{\boldsymbol{x}} \times \boldsymbol{H} = 0 \quad & \text{on } \Gamma_R\,.
\end{aligned}$$

Then L is bounded from $H^{-1/2}(\mathrm{Div};\Gamma_\rho)$ into $H^s(\mathrm{Div};\Gamma_R)$ for any s.

Now we consider the interior problem with general coefficients. Let us define the following sesquilinear forms (sometimes also denoting appropriate duality pairings):

$$(\boldsymbol{u},\boldsymbol{v}) = \int_{\Omega_R} \boldsymbol{u}\cdot\overline{\boldsymbol{v}}\,dV, \text{ and } \langle \boldsymbol{u},\boldsymbol{v}\rangle = \int_{\Gamma_R} \boldsymbol{u}\cdot\overline{\boldsymbol{v}}\,dA,$$

where the overline denotes complex conjugation.

Given $\boldsymbol{\lambda}\in H^{-1/2}(\mathrm{Div};\Gamma_R)$ we recall that we can define the operator

$$G_i : H^{-1/2}(\mathrm{Div};\Gamma_R) \to H^{-1/2}(\mathrm{Div};\Gamma_R)$$

by $G_i\boldsymbol{\lambda} = \hat{\boldsymbol{x}}\times \boldsymbol{w}|_{\Gamma_R}$ where $\boldsymbol{w}\in H(\mathrm{curl};\Omega_R)$ satisfies (2.6) and (2.7). To obtain a variational formulation of this problem suitable for later finite element discretization we can multiply (2.6) by a test function $\boldsymbol{\phi}\in H(\mathrm{curl};\Omega_R)$ and integrate by parts (using (2.7) for the boundary term) to obtain

$$\left(\mu_r^{-1}\nabla\times\boldsymbol{u},\nabla\times\boldsymbol{\phi}\right) \;-\; k^2\left(\epsilon_r\boldsymbol{u},\boldsymbol{\phi}\right) \;+\; ik\langle\boldsymbol{\lambda},\boldsymbol{\phi}\rangle \;=\; 0 \tag{2.26}$$

for all $\boldsymbol{\phi}\in H(\mathrm{curl};\Omega_R)$. In order to show that $G_i\boldsymbol{\lambda}$ is well defined and maps $H^{-1/2}(\mathrm{Div};\Gamma_R)$ into $H^{-1/2}(\mathrm{Div};\Gamma_R)$ it suffices to show that a unique solution of (2.26) in $H(\mathrm{curl};\Omega_R)$ exists. As usual for problems of this type we do this in two steps. First we show uniqueness and then use the Fredholm alternative to obtain existence.

LEMMA 2.6. *Problem (2.26) has at most one solution.*

Proof. By linearity it suffices to consider the case when $\boldsymbol{\lambda}=0$ and choose $\boldsymbol{\phi}=\boldsymbol{u}$ in (2.26). Then

$$(\mu_r^{-1}\nabla\times\boldsymbol{u},\nabla\times\boldsymbol{u}) \;-\; k^2(\epsilon_r\boldsymbol{u},\boldsymbol{u}) \;=\; 0$$

and hence (since μ_r is real symmetric) $\Im(\epsilon_r\boldsymbol{u},\boldsymbol{u})=0$. This implies that $\boldsymbol{u}=0$ in every sub-domain in which $\Im(\epsilon_r)\neq 0$ (at least one such sub-domain exists). Now using the unique continuation result of [32] in the same way as in the proof of uniqueness in [20] we conclude $\boldsymbol{u}=0$ in Ω_R. □

Next we prove that $\boldsymbol{u}$ exists using the Fredholm alternative and this proves the existence of $G_i : H^{-1/2}(\mathrm{Div};\Gamma_R) \to H^{-1/2}(\mathrm{Div};\Gamma_R)$. To do this we first have to prove a compact embedding result as follows. Let use define the space

$$X \;=\; \left\{\boldsymbol{v}\in H(\mathrm{curl};\Omega_R)\mid \left(\epsilon_r\boldsymbol{v},\nabla q\right)=0 \text{ for all } q\in H^1(\Omega_R)\right\}. \tag{2.27}$$

THEOREM 2.7. *The space X defined in (2.27) is compactly embedded in $\left(L^2(\Omega_R)\right)^3$.*

Remark: This result is well-known if ϵ_r is a real valued matrix function of position (under the conditions we have given on the functions). For a presentation of the proof of this result and a discussion of the literature see [30].

Proof. Our proof is a slight generalization of that given in [20] for the case of scalar ϵ_r. Let $\{\boldsymbol{u}_k\}\subset X$ be a bounded sequence. Note that

$$(\epsilon_r\boldsymbol{u}_k,\nabla q)=0 \quad \text{for all } q\in H^1(\Omega_R)\,.$$

Now let $\boldsymbol{\phi}_k\in H_0(\mathrm{curl};\Omega_R)$ satisfy

$$\begin{aligned}\nabla\times\boldsymbol{\phi}_k = \epsilon_r\boldsymbol{u}_k \quad &\text{in } \Omega_R\,,\\ \nabla\cdot\boldsymbol{\phi}_k = 0 \quad &\text{in } \Omega_R\,.\end{aligned}$$

Since Ω_R is a sphere, this problem has a unique solution and because the boundary of the sphere is smooth

$$\boldsymbol{\phi}_k\in\left(H^1(\Omega_R)\right)^3 \quad \text{with} \quad \|\boldsymbol{\phi}_k\|_{H^1(\Omega_R)} \;\le\; C\|\epsilon_r\boldsymbol{u}_k\| \;\le\; C\|\boldsymbol{u}_k\|$$

where $C>0$ is independent of k. Since the set $\{\boldsymbol{\phi}_k\}$ is bounded in $\left(H^1(\Omega_R)\right)^3$, the standard compact embedding of $H^1(\Omega_R)$ into $L^2(\Omega_R)$ proves the existence of a subsequence (still denoted $\{\boldsymbol{\phi}_k\}$) converging weakly to a function $\boldsymbol{\phi}\in\left(H^1(\Omega_R)\right)^3$ and strongly in $\left(L^2(\Omega_R)\right)^3$. Now let $\boldsymbol{u}_{l,k}=\boldsymbol{u}_l-\boldsymbol{u}_k$ and $\boldsymbol{\phi}_{l,k}=\boldsymbol{\phi}_l-\boldsymbol{\phi}_k$, then

$$\left(\epsilon_r\boldsymbol{u}_{l,k},\boldsymbol{u}_{l,k}\right) \;=\; \left(\nabla\times\boldsymbol{\phi}_{l,k},\boldsymbol{u}_{l,k}\right) \;=\; \left(\boldsymbol{\phi}_{l,k},\nabla\times\boldsymbol{u}_{l,k}\right),$$

where we have used the boundary condition on $\boldsymbol{\phi}_{l,k}$ to simplify the above result. Since $\nabla \times \boldsymbol{u}_{l,k}$ is bounded in $\left(L^2(\Omega_R)\right)^3$ and $\{\boldsymbol{\phi}_k\}$ is a Cauchy sequence in $\left(L^2(\Omega_R)\right)^3$, we conclude that $\{\boldsymbol{u}_k\}$ is a Cauchy sequence in $\left(L^2(\Omega_R)\right)^3$ and hence convergent. ∎

Now suppose that $\boldsymbol{u} \in X$ and $\nabla \times \boldsymbol{u} = 0$ in Ω_R. Since $\boldsymbol{u}$ is curl free, there is a function $p \in H^1(\Omega_R)/\mathbb{R}$ such that $\boldsymbol{u} = \nabla p$ and since $\boldsymbol{u} \in X$, we have $(\epsilon_r \nabla p, \nabla p) = 0$. The positive definiteness of the real part of ϵ_r now implies $p = 0$. Combining this uniqueness result with the above compactness result implies the following corollary:

COROLLARY 2.8. *There exists a constant $C > 0$ such that for all $\boldsymbol{u} \in X$*

$$\|\boldsymbol{u}\| \;\leq\; C\|\nabla \times \boldsymbol{u}\|\,.$$

Using these results we can prove the promised existence result.

THEOREM 2.9. *There exists a unique solution $\boldsymbol{u}$ to the interior boundary value problem (2.26) and hence $G_i : H^{-1/2}(\mathrm{Div};\Gamma_R) \to H^{-1/2}(\mathrm{Div};\Gamma_R)$ is well defined and bounded.*

Proof. For given $\boldsymbol{\lambda} \in H^{-1/2}(\mathrm{Div};\Gamma_R)$ we define $p \in H^1_0(\Omega_R)$ by

$$k^2\big(\epsilon_r \nabla p, \nabla q\big) \;=\; ik\langle \boldsymbol{\lambda}, \nabla q\rangle \quad \text{for all } q \in H^1(\Omega_R)\,. \tag{2.28}$$

For this problem, existence and uniqueness follow from the Lax-Milgram Lemma since $\Re(\epsilon_r)$ is uniformly positive definite. Now we make the ansatz

$$\boldsymbol{u} \;=\; \boldsymbol{z} \;+\; \nabla p\,, \tag{2.29}$$

where $\boldsymbol{z} \in H(\mathrm{curl};\Omega_R)$ satisfies

$$\big(\mu_r^{-1}\nabla \times \boldsymbol{z}, \nabla \times \boldsymbol{\phi}\big) \;-\; k^2\big(\epsilon_r \boldsymbol{z}, \boldsymbol{\phi}\big) \;=\; -ik\langle \boldsymbol{\lambda}, \boldsymbol{\phi}\rangle \;+\; k^2\big(\epsilon_r \nabla p, \boldsymbol{\phi}\big) \tag{2.30}$$

for all $\boldsymbol{\phi} \in H(\mathrm{curl};\Omega_R)$. By choosing $\boldsymbol{\phi} = \nabla q$ for an arbitrary $q \in H^1_0(\Omega_R)$ we see that $\big(\epsilon_r \boldsymbol{z}, \nabla q\big) = 0$ and thus $\boldsymbol{z} \in X$.

Since $H(\mathrm{curl};\Omega_R)$ is the direct sum of X and $\nabla H^1(\Omega_R)$ we can rewrite (2.30) as the problem of finding $\boldsymbol{z} \in X$ such that

$$\big(\mu_r^{-1}\nabla \times \boldsymbol{z}, \nabla \times \boldsymbol{\phi}\big) \;-\; k^2\big(\epsilon_r \boldsymbol{z}, \boldsymbol{\phi}\big) \;=\; -ik\langle \boldsymbol{\lambda}, \boldsymbol{\phi}\rangle \;+\; k^2\big(\epsilon_r \nabla p, \boldsymbol{\phi}\big) \tag{2.31}$$

for all $\boldsymbol{\phi} \in X$. By Corollary 2.8, the first term on the left hand side of (2.31) defines a bounded and coercive sesquilinear form on X. Hence we can define the operator $B : \left(L^2(\Omega_R)\right)^3 \to X$ by

$$\big(\mu_r^{-1}\nabla \times B\boldsymbol{z}, \nabla \times \boldsymbol{\phi}\big) \;=\; \big(\epsilon_r \boldsymbol{z}, \boldsymbol{\phi}\big) \tag{2.32}$$

for all $\boldsymbol{\phi} \in X$. The operator B restricted to X is compact since it is continuous from $\left(L^2(\Omega_R)\right)^3$ into X, and X is compactly embedded in $\left(L^2(\Omega_R)\right)^3$ (see Theorem 2.7). Hence if we define $\boldsymbol{F} \in X$ by

$$\big(\mu_r^{-1}\nabla \times \boldsymbol{F}, \nabla \times \boldsymbol{\phi}\big) \;=\; -ik\langle \boldsymbol{\lambda}, \boldsymbol{\phi}\rangle \;+\; k^2\big(\epsilon_r \nabla p, \boldsymbol{\phi}\big) \tag{2.33}$$

for all $\boldsymbol{\phi} \in X$, the original problem is equivalent to finding $z \in X$ such that

$$(I - k^2 B)\boldsymbol{z} \;=\; \boldsymbol{F}$$

and existence of a solution to this problem (and hence to the original problem) follows from the Fredholm alternative and the uniqueness result proved in Lemma 2.6. ∎

Now that we have verified that the existence of the operator G_i we can prove Theorem 2.1.

Proof. Using Theorem 2.4 and the definition of $\tilde{G}_i$ and G_i we can write

$$G_e\boldsymbol{\lambda} - G_i\boldsymbol{\lambda} \;=\; (G_e - \tilde{G}_i)\boldsymbol{\lambda} \;+\; (\tilde{G}_i - G_i)\boldsymbol{\lambda} \;=\; T\boldsymbol{\lambda} \;+\; K_1\boldsymbol{\lambda} \;+\; \hat{\boldsymbol{x}} \times (\tilde{\boldsymbol{u}} - \boldsymbol{u})$$

where $\boldsymbol{u}$ and $\tilde{\boldsymbol{u}}$ solve (2.26) and (2.26) corresponding to $\epsilon_r = \mu_r = 1$, respectively. Now if we define $\boldsymbol{w} = \tilde{\boldsymbol{u}} - \boldsymbol{u}$ then $\boldsymbol{w} \in H(\mathrm{curl};\Omega_R)$ satisfies

$$\left(\mu_r^{-1}\nabla\times\boldsymbol{w},\nabla\times\boldsymbol{\phi}\right) \;-\; k^2\left(\epsilon_r\boldsymbol{w},\boldsymbol{\phi}\right) \;=\; \left((1-\mu_r^{-1})\nabla\times\tilde{\boldsymbol{u}},\nabla\times\boldsymbol{\phi}\right) \;+\; k^2\left((\epsilon_r-1)\tilde{\boldsymbol{u}},\boldsymbol{\phi}\right)$$

for all $\boldsymbol{\phi} \in H(\mathrm{curl};\Omega_R)$. Using exactly the same argument as in the previous theorem (but with a different right hand side $\boldsymbol{F}$) we can see that $\boldsymbol{w}$ is the unique solution of the above variational problem. Now let us choose $\rho < R$ such that Γ_ρ contains the support of $(1-\mu_r)$ and $(\epsilon_r - 1)$. Then $\hat{\boldsymbol{x}} \times \boldsymbol{w}|_{\Gamma_\rho} \in H^{-1/2}(\mathrm{Div};\Gamma_\rho)$ is bounded in terms of the curl norm of $\tilde{\boldsymbol{u}}$ and hence in terms of the $H^{-1/2}(\mathrm{Div};\Gamma_R)$ norm of $\boldsymbol{\lambda}$. Then using Theorem 2.5, we conclude $\hat{\boldsymbol{x}} \times \boldsymbol{w} \in H^l(\mathrm{Div};\Gamma_R)$ for any l. Hence $\tilde{G}_i - G_i$ is a compact map from $H^{-1/2}(\mathrm{Div};\Gamma_R)$ into $H^{-1/2}(\mathrm{Div};\Gamma_R)$. We have thus proved the main theorem of this section (Theorem 2.1) since

$$G_e - G_i \;=\; T \;+\; (K_1 + \tilde{G}_i - G_i)\,.$$

and $K = K_1 + \tilde{G}_i - G_i$ is compact. □

Theorem 2.1 can be used to prove the existence of a weak solution of the original scattering problem (of course this is already known from our previous work [27]). We start by indicating a proof of uniqueness of the solution of (2.8).

LEMMA 2.10. *Problem (2.8) has at most one solution.*

Proof. By linearity it suffices to consider the case $\boldsymbol{f} = 0$. For a given solution $\boldsymbol{\lambda}$ we define $\boldsymbol{u}_1$ to satisfy (2.3)-(2.5) and define $\boldsymbol{u}_2$ to satisfy (2.6)-(2.7). Existence and uniqueness of $\boldsymbol{u}_1$ is classical (see for example [5]) and we have proved existence and uniqueness of a weak solution of (2.6)-(2.7) in Theorem 2.9.

Then equation (2.8) ensures that

$$\boldsymbol{u} \;=\; \begin{cases} \boldsymbol{u}_1 & \text{in } \mathbb{R}^3 \setminus \overline{\Omega_R}\,, \\ \boldsymbol{u}_2 & \text{in } \Omega_R\,, \end{cases}$$

is a solution of the Maxwell system (1.1)-(1.4) with vanishing incident field. Classical uniqueness arguments for the solution of the Maxwell system then show that $\boldsymbol{u} = 0$ outside the scatterer, and Theorem 2.6 proves uniqueness inside. Thus $\boldsymbol{u} = 0$ and hence $\boldsymbol{\lambda} = 0$. □

Hence, using Theorem 2.1 and the above uniqueness lemma, by the application of the Fredholm alternative we can prove the following result:

THEOREM 2.11. *For every $\boldsymbol{f} \in H^{-1/2}(\mathrm{Div};\Gamma_R)$ there exists a unique solution $\boldsymbol{\lambda} \in H^{-1/2}(\mathrm{Div};\Gamma_R)$ to (2.8).*

3. The finite dimensional problem. In this section we describe the discrete problem related to (2.8). The idea is to seek an approximation of $\boldsymbol{\lambda}$ on Γ_R using the space S_N defined as follows

$$S_N \;=\; \left\{ \boldsymbol{u} \in H^{-1/2}(\mathrm{Div};\Gamma_R) \mid \boldsymbol{u} = \sum_{n=1}^{N}\sum_{|m|\le n}\left[\alpha_{n,m}\boldsymbol{U}_n^m + \beta_{n,m}\boldsymbol{V}_n^m\right] \text{ with } \alpha_{n,m},\beta_{n,m} \in \mathbb{C} \right\}.$$

In other words we seek to approximate $\boldsymbol{\lambda}$ by a finite Fourier series. For future reference we define $P_N : H^{-1/2}(\mathrm{Div};\Gamma_R) \to S_N$ to be the orthogonal projection in the $H^{-1/2}(\mathrm{Div};\Gamma_R)$ inner product. Due to the orthogonality properties of the basis functions, this is nothing more than the truncation operator. Of course, for any $\boldsymbol{\lambda} \in H^{-1/2}(\mathrm{Div};\Gamma_R)$

$$P_N\boldsymbol{\lambda} \;\longrightarrow\; \boldsymbol{\lambda} \text{ in } H^{-1/2}(\mathrm{Div};\Gamma_R) \text{ as } N \to \infty\,.$$

We also have the following error estimate

$$\|(I-P_N)\boldsymbol{\lambda}\|_{H^{-1/2}(\mathrm{Div};\Gamma_R)} \;\le\; N^{-\sigma}\|\boldsymbol{\lambda}\|_{H^{\sigma}(\mathrm{Div};\Gamma_R)} \tag{3.1}$$

for any $\sigma \ge -1/2$. This is seen by the estimate

$$\|(I-P_N)\boldsymbol{\lambda}\|^2_{H^{-1/2}(\mathrm{Div};\Gamma_R)} = \sum_{n>N}\sum_{m=-n}^{n}\left[n\,|a_n^m|^2 + \frac{1}{n}\,|b_n^m|^2\right]$$

$$
\begin{aligned}
&= \sum_{n>N} n^{-1-2\sigma} \sum_{m=-n}^{n} \left[n^{2+2\sigma} \, |a_n^m|^2 + n^{2\sigma} \, |b_n^m|^2 \right] \\
&\le \frac{1}{N^{1+2\sigma}} \sum_{n>N} \sum_{m=-n}^{n} \left[n^{2+2\sigma} \, |a_n^m|^2 + n^{2\sigma} \, |b_n^m|^2 \right] \\
&\le \frac{1}{N^{1+2\sigma}} \|\boldsymbol{\lambda}\|^2_{H^\sigma(\mathrm{Div};\Gamma_R)}
\end{aligned}
$$

where $\boldsymbol{\lambda} = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \left(a_n^m \, \boldsymbol{U}_n^m + b_n^m \, \boldsymbol{V}_n^m \right)$.

Furthermore, since T from (2.25) is a diagonal operator when restricted to S_N, it is easy to see that T and P_N commute:

$$
P_N T \; = \; T P_N \, .
$$

For any function $\boldsymbol{\lambda} \in S_N$, the function $G_e \boldsymbol{\lambda}_N$ is easy to calculate using the truncation of (2.19).

We also note that S_N satisfies the inverse estimate

$$
\|\boldsymbol{\lambda}_N\|_{H^{1/2}(\mathrm{Div};\Gamma_R)} \; \le \; N \|\boldsymbol{\lambda}_N\|_{H^{-1/2}(\mathrm{Div};\Gamma_R)} \, . \tag{3.2}
$$

This is again seen by using the series representation of $\boldsymbol{\lambda}_N$:

$$
\begin{aligned}
\|\boldsymbol{\lambda}\|^2_{H^{1/2}(\mathrm{Div};\Gamma_R)} &= \sum_{n=1}^{N} \sum_{m=-n}^{n} \left[n^3 \, |a_n^m|^2 + n \, |b_n^m|^2 \right] \; \le \; N^2 \sum_{n=1}^{N} \sum_{m=-n}^{n} \left[n \, |a_n^m|^2 + \frac{1}{n} \, |b_n^m|^2 \right] \\
&= N^2 \|\boldsymbol{\lambda}_N\|_{H^{-1/2}(\mathrm{Div};\Gamma_R)} \, .
\end{aligned}
$$

The interior operator G_i also need to be discretized. For this we propose to use the finite element method using the edge elements of Nédélec [31]. We will provide details of these elements later. At this stage we shall simply assume that we have a suitable finite element space $V_h \subset H(\mathrm{curl};\Omega_R)$. Then for any function $\boldsymbol{\lambda} \in H^{-1/2}(\mathrm{Div};\Gamma_R)$ we can define $G_{i,h}\boldsymbol{\lambda} = \hat{\boldsymbol{x}} \times \boldsymbol{u}_h$ where $\boldsymbol{u}_h \in V_h$ satisfies the discrete analogue of (2.26):

$$
\left(\mu_r^{-1} \nabla \times \boldsymbol{u}_h, \nabla \times \boldsymbol{\phi}_h \right) \; - \; k^2 \left(\epsilon_r \boldsymbol{u}_h, \boldsymbol{\phi}_h \right) \; + \; ik \langle \boldsymbol{\lambda}, \boldsymbol{\phi}_h \rangle \; = \; 0 \quad \text{for all } \boldsymbol{\phi}_h \in V_h \, . \tag{3.3}
$$

In § 4 we shall show that this problem has a unique solution, and derive some error estimates.

Now that we have a discrete analogue of G_i, we define the discrete analogue of (2.8) to be the problem of finding $\boldsymbol{\lambda}_{N,h} \in S_N$ such that

$$
(P_N G_{i,h} - G_e) \boldsymbol{\lambda}_{N,h} \; = \; P_N \boldsymbol{f} . \tag{3.4}
$$

The remainder of the paper is devoted to showing that this problem has a unique solution that converges at an optimal rate to the exact solution.

4. Analysis of the Interior Finite Element Problem. Here we shall detail the construction of the finite element space V_h described previously and prove an error estimate for the interior finite element problem. To construct V_h we shall use the lowest order edge elements of Nédélec [31] as modified by Dubois [11] to allow for the curved outer boundary (and curved interfaces at discontinuities of the coefficients ϵ and μ).

Let Ω_R be covered by a tetrahedral mesh (allowing curvilinear tetrahedra near the outer boundary Γ_R and the boundaries of the sub-domains Ω_n, $n = 1, 2, \cdots, N$) of regular, quasi-uniform finite elements with a maximum diameter h. We denote by τ_h such a mesh. For a precise description of the mesh and the notion of regularity in this case see [11].

Following [11], we define the reference tetrahedron $\hat{K}$:

$$
\hat{K} \; = \; \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid 1 - x_1 - x_2 - x_3 \ge 0 \, , \; x_1 \ge 0 \, , \; x_2 \ge 0 \, , \; x_3 \ge 0 \right\} .
$$

Now if $K \in \tau_h$ is an element, there is an invertible map F_K

$$
F_K : \hat{K} \; \longrightarrow \; K \, .
$$

This map can be chosen affine if K is a tetrahedron. For curvilinear elements the map can be constructed as shown in [11]. The Jacobian matrix for F_K is denoted DF_K.

To describe V_h we next define the set

$$\mathcal{R}_1 \;=\; \left\{\hat{\boldsymbol{u}} : \hat{K} \to \mathbb{C}^3 \,|\, \hat{\boldsymbol{u}}(\hat{\boldsymbol{x}}) = \boldsymbol{a} + \hat{\boldsymbol{x}} \times \boldsymbol{b} \text{ for some } \boldsymbol{a}, \boldsymbol{b} \in \mathbb{C}^3\right\},$$

and using this set, for each $K \in \tau_h$ we define

$$\mathcal{R}_1(K) \;=\; \left\{\boldsymbol{u} : K \to \mathbb{C}^3 \mid \boldsymbol{u}(F_K(\hat{\boldsymbol{x}})) = DF_K^{-T}(\hat{\boldsymbol{x}})\hat{\boldsymbol{u}}(\hat{\boldsymbol{x}}) \text{ for some } \hat{\boldsymbol{u}} \in \mathcal{R}_1 \text{ and all } \hat{\boldsymbol{x}} \in \hat{K}\right\}.$$

Then the edge element space V_h is defined by

$$V_h \;=\; \{\boldsymbol{u}_h \in H(\text{curl};\Omega_R) \mid \boldsymbol{u}_h|_K \in \mathcal{R}_1(K) \text{ for all } K \in \tau_h\}.$$

The degrees of freedom for this space are the moments of the tangential components of the field along the edges in the mesh. If e is a generic edge in the mesh and $\boldsymbol{\tau}_e$ is the unit tangent to this edge we define , for any sufficiently smooth function $\boldsymbol{u}$,

$$\Sigma(\boldsymbol{u}) \;=\; \left\{\int_e \boldsymbol{\tau}_e \cdot \boldsymbol{u}\, ds \text{ for all } e \in \tau_h\right\}. \tag{4.1}$$

Now let

$$PH^1(\text{curl};\Omega_R) = \Big\{\boldsymbol{u} \in \big(L^2(\Omega_R)\big)^3 \mid \boldsymbol{u}|_{\Omega_n\cap\Omega_R} \in \big(H^1(\Omega_n \cap \Omega_R)\big)^3 \text{ and}$$
$$\nabla \times \boldsymbol{u}|_{\Omega_n\cap\Omega_R} \in \big(H^1(\Omega_n \cap \Omega_R)\big)^3,\ n = 0, 1, \dots, N\Big\}$$

where the domains Ω_n, $n = 0, 1, \dots, N$ were introduced in § 2 and we have assumed that the interfaces between the domains Ω_n, $n = 0, \dots, N$ lies along faces of the mesh. The norm on this space is

$$\|\boldsymbol{u}\|^2_{PH^1(\text{curl};\Omega_R)} \;=\; \|\boldsymbol{u}\|^2 \;+\; \sum_{n=0}^{N} \left[\|\boldsymbol{u}\|^2_{H^1(\Omega_n\cap\Omega_R)} + \|\nabla \times \boldsymbol{u}\|^2_{H^1(\Omega_n\cap\Omega_R)}\right].$$

Using Dubois arguments we can show that the interpolation operator $\pi_h : PH^1(\text{curl};\Omega_R) \to V_h$ corresponding to the above degrees of freedom (4.1) is well defined and the following estimate holds:

$$\|\boldsymbol{u} - \pi_h\boldsymbol{u}\| \;+\; \|\nabla \times (\boldsymbol{u} - \pi_h\boldsymbol{u})\| \;\le\; Ch\,\|\boldsymbol{u}\|_{PH^1(\text{curl};\Omega_R)}. \tag{4.2}$$

Of course the interpolation operator is well defined for much less regular function (see [1]) but we wish to prove optimal error estimates for which the above smoothness is sufficient.

Let $\mathcal{P}_1$ denote the standard space of polynomials of total degree at most one in three variables. Dubois shows that the standard piecewise linear finite element space defined by

$$S_h \;=\; \left\{p_h \in H^1(\Omega_R)/\mathbb{R} \mid p_h|_K\big(F_K(\hat{\boldsymbol{x}})\big) = \hat{p}(\hat{\boldsymbol{x}}) \text{ for some } \hat{p} \in \mathcal{P}_1 \text{ and all } \hat{\boldsymbol{x}} \in \hat{K},\ K \in \tau_h\right\}$$

is such that $\nabla S_h \subset V_h$.

We can thus define the space of discrete divergence free fields to be

$$X_h \;=\; \{\boldsymbol{u}_h \in V_h \mid (\epsilon_r\boldsymbol{u}_h, \nabla p_h) = 0 \text{ for all } p_h \in S_h\}.$$

Now suppose we have a sequence of refinements of the mesh indexed by mesh sizes $h_1 > h_2 > \dots$. We assume $h_n \to 0$ as $n \to \infty$ and set

$$\Lambda \;=\; \{h_n \mid n = 1, 2, \dots\}.$$

We want to show convergence of G_{i,h_n} to G_i as n increases. In order to prove this, we proceed as in [10] using a discrete compactness argument. First we give the discrete compactness result in for the Dubois space. It is a generalization to variable ϵ of the original result of Kikuchi [25].

THEOREM 4.1 (Discrete Compactness). *Suppose* $\{\boldsymbol{u}_n\}_{n=1}^{\infty} \subset H(\mathrm{curl};\Omega_R)$ *is a bounded sequence such that for each* n *there is an* $m = m(n)$ *such that* $\boldsymbol{u}_n \in X_{h_m}$ *and* $h_m \to 0$ *as* $n \to 0$. *Then there is a subsequence, also denoted by* $\{\boldsymbol{u}_n\}_{n=1}^{\infty}$ *which converges weakly in* $H(\mathrm{curl};\Omega_R)$ *to a function* $\boldsymbol{u} \in X$, *i.e.,*

$$\big(\epsilon \boldsymbol{u}, \nabla p\big) \;=\; 0 \quad \textit{for all } p \in H^1(\Omega_R)\,,$$

and $\boldsymbol{u}_n \to \boldsymbol{u}$ *strongly in* $\big(L^2(\Omega_R)\big)^3$.

Proof. For a polyhedral Lipschitz domain with $\epsilon = \mu = 1$ and standard Nédélec elements this result is proved by Kikuchi [25]. This is extended to allow for variable real ϵ in [3]. We outline these proofs here checking that they can be modified to allow for our case.

First we prove discrete compactness when $\epsilon = 1$ for the Dubois space. For each n we define $\boldsymbol{u}^{(n)} \in X$ by

$$\begin{aligned} \nabla \times \boldsymbol{u}^{(n)} &= \nabla \times \boldsymbol{u}_n \text{ in } \Omega,\\ \nabla \cdot \boldsymbol{u}^{(n)} &= 0 \text{ in } \Omega,\\ \boldsymbol{u}^{(n)} \cdot \hat{\boldsymbol{x}} &= 0 \text{ on } \Gamma_R. \end{aligned}$$

Hence by Theorem 2.8 we know that there is a subsequence of $\{\boldsymbol{u}^{(n)}\}$ (still denoted by $\{\boldsymbol{u}^{(n)}\}$) and a function $\boldsymbol{u} \in X$ such that $\boldsymbol{u}^{(n)} \to \boldsymbol{u}$ strongly in $(L^2(\Omega))^3$ (and weakly in $H(\mathrm{curl};\Omega_R)$). But since Ω_R is smooth, $\boldsymbol{u}^{(n)} \in (W^{1,t}(\Omega_R))^3$ for any $t \geq 2$ [12] . In this case $\pi_{h_m}\boldsymbol{u}^{(n)}$ is defined (see [1]) and using the same arguments as in [12] we have the error estimate

$$\|\boldsymbol{u}^{(n)} - \pi_{h_m}\boldsymbol{u}^{(n)}\| \leq C h_m \|\nabla \times \boldsymbol{u}_n\|_{L^t(\Omega)}.$$

Since the mesh is regular and quasi-uniform, standard inverse estimates show that

$$\|\nabla \times \boldsymbol{u}_n\|_{L^t(\Omega)} \leq C h^{3/t-3/2} \|\nabla \times \boldsymbol{u}_n\|.$$

Thus, for $2 < t < 3$,

$$\|\boldsymbol{u}^{(n)} - \pi_{h_m}\boldsymbol{u}^{(n)}\| \leq C h_m^{3/t-1/2} \|\nabla \times \boldsymbol{u}_n\|.$$

Now we may write, using the Helmholtz decomposition,

$$\boldsymbol{u}_n = \boldsymbol{u}^{(n)} + \nabla p^{(n)}$$

for some $p^{(n)} \in H^1(\Omega_R)/\mathbb{R}$. For the Dubois space we also have the standard commuting property that, provided the interpolant is well defined, $\pi_{h_m}\nabla p^{(n)} = \nabla p_n$ for some $p_n \in S_{h_m}$. Hence

$$\boldsymbol{u}_n = \pi_{h_m}\boldsymbol{u}^{(n)} + \nabla p_n$$

and using the fact that $\boldsymbol{u} \in X$ and $\boldsymbol{u}_n \in X_{h_m}$ we may write

$$\begin{aligned} (\boldsymbol{u} - \boldsymbol{u}_n, \boldsymbol{u} - \boldsymbol{u}_n) &= (\boldsymbol{u} - \boldsymbol{u}_n, \boldsymbol{u} - \pi_{h_m}\boldsymbol{u}^n)\\ &= (\boldsymbol{u} - \boldsymbol{u}_n, \boldsymbol{u} - \boldsymbol{u}^{(n)}) + (\boldsymbol{u} - \boldsymbol{u}_n, \boldsymbol{u}^{(n)} - \pi_{h_m}\boldsymbol{u}^{(n)}). \end{aligned}$$

Hence

$$\|\boldsymbol{u} - \boldsymbol{u}_n\| \leq \|\boldsymbol{u} - \boldsymbol{u}^{(n)}\| + \|\boldsymbol{u}^{(n)} - \pi_{h_m}\boldsymbol{u}^{(n)}\| \to 0 \text{ as } n \to \infty.$$

Hence the discrete compactness result is proved when $\epsilon = 1$.

Now we follow [3] allowing for the fact that ϵ is a complex valued matrix function of position. From Lemma 4.1 of [27] we can define a bounded operator $P^{\epsilon} : H(\mathrm{curl};\Omega_R) \to \nabla H^1(\Omega_R)/\mathbb{R}$ by requiring, for given $\boldsymbol{u} \in H(\mathrm{curl};\Omega_R)$ that $P^{\epsilon}\boldsymbol{u} \in \nabla H^1(\Omega_R)/\mathbb{R}$ satisfy

$$(\epsilon P^{\epsilon}\boldsymbol{u}, \nabla q) = (\epsilon \boldsymbol{u}, \nabla q) \quad \forall q \in H^1(\Omega_R)/\mathbb{R}.$$

Thus the corresponding projection operator when $\epsilon = 1$ is P^1. In the discrete case we can also define $P_h^\epsilon : H(\text{curl};\Omega_R) \to \nabla S_h$ by requiring $P_h^\epsilon \boldsymbol{u} \in \nabla S_h$ satisfy

$$(\epsilon P_h^\epsilon \boldsymbol{u}_h, \nabla q_h) = (\epsilon \boldsymbol{u}, \nabla q_h) \quad \forall q_h \in S_h.$$

Consider the sequence $\{(I - P_{h_m}^1)\boldsymbol{u}_n\}$ where $\{\boldsymbol{u}_n\}$ is the sequence in the statement of the theorem for general ϵ. Since $P_{h_m}^1(I - P_{h_m}^1) = 0$ this sequence is discrete divergence free with $\epsilon = 1$ and bounded in $H(\text{curl};\Omega_R)$. The discrete compactness property for $\epsilon = 1$ proved above shows that there is a subsequence, still denoted $\{(I - P_{h_m}^1)\boldsymbol{u}_n\}$, and a function $\tilde{\boldsymbol{u}} \in H(\text{curl};\Omega_R)$ such that

$$\begin{aligned}(I - P_{h_m}^1)\boldsymbol{u}_n &\to \tilde{\boldsymbol{u}} \text{ strongly in } (L^2(\Omega_R))^3 \text{ as } n \to \infty,\\ \nabla \cdot \tilde{\boldsymbol{u}} &= 0 \text{ in } \Omega_R,\\ \tilde{\boldsymbol{u}} \cdot \hat{\boldsymbol{x}} &= 0 \text{ on } \Gamma_R.\end{aligned}$$

Now let $\boldsymbol{u} = (I - P^\epsilon)\tilde{\boldsymbol{u}}$. We claim that $\boldsymbol{u}_n \to \boldsymbol{u}$ strongly in $(L^2(\Omega_R))^3$ as $n \to \infty$. Expanding terms

$$(\epsilon(\boldsymbol{u}_n - \boldsymbol{u}), (\boldsymbol{u}_n - \boldsymbol{u})) = (\epsilon(\boldsymbol{u}_n - \boldsymbol{u}), (P_{h_m}^1 \boldsymbol{u}_n + (I - P_{h_m}^1)\boldsymbol{u}_n - \boldsymbol{u})).$$

But since $\boldsymbol{u} \in X$ and $\boldsymbol{u}_n \in X_{h_m}$

$$(\epsilon(\boldsymbol{u}_n - \boldsymbol{u}), \nabla \xi_{h_m}) = 0 \qquad \forall \xi_{h_m} \in S_{h_m}$$

so that, recalling that $P_{h_m}^1 \boldsymbol{u}_n \in \nabla S_{h_m}$, we can see that

$$\begin{aligned}(\epsilon(\boldsymbol{u}_n - \boldsymbol{u}), (\boldsymbol{u}_n - \boldsymbol{u})) &= (\epsilon(\boldsymbol{u}_n - \boldsymbol{u}), ((I - P_{h_m}^1)\boldsymbol{u}_n - \boldsymbol{u} - \nabla \xi_{h_m}))\\ &= (\epsilon(\boldsymbol{u}_n - \boldsymbol{u}), ((I - P_{h_m}^1)\boldsymbol{u}_n - (I - P^\epsilon)\tilde{\boldsymbol{u}} - \nabla \xi_{h_m})).\end{aligned}$$

Hence

$$\|\boldsymbol{u}_n - \boldsymbol{u}\| \le \|(I - P_{h_m}^1)\boldsymbol{u}_n - \tilde{\boldsymbol{u}}\| + \inf_{\xi_{h_m} \in S_{h_m}} \|P^\epsilon \tilde{\boldsymbol{u}} - \nabla \xi_{h_m}\| \to 0$$

as $n \to \infty$ since ∇S_{h_m} is dense in $\nabla H^1(\Omega_R)/\mathbb{R}$. This verifies that the arguments of [3] carry over to complex ϵ and completes the proof of the result. □

Using this result as in [10] we have the following result which generalizes Lemma 2.8 to the finite element context:

Corollary 4.2. *Provided h_1 is sufficiently small, there is a constant C such that*

$$\|\boldsymbol{w}\| \;\le\; C\,\|\nabla \times \boldsymbol{w}\| \quad \textit{for all } \boldsymbol{w} \in W \;=\; X \cup \left\{ \bigcup_{h \le h_1} X_h \right\},$$

and W is a precompact set in $(L^2(\Omega_R))^3$.

Now we show that $G_{i,h}$ is well defined by showing that (3.3) has a solution. We proceed along the same lines as in the proof of Theorem 2.9 with $H(\text{curl};\Omega_R)$ and $H^1(\Omega_R)$ replaced by V_h and S_h, respectively.

Let $p_h \in S_h$ satisfy the discrete analogue of (2.28)

$$k^2\big(\epsilon_r \nabla p_h, \nabla \xi_h\big) \;=\; ik\,\langle \boldsymbol{\lambda}, \nabla \xi_h \rangle \quad \text{for all } \xi_h \in S_h\,. \tag{4.3}$$

Since the real part of ϵ_r is positive definite, and the average value of p_h is zero, this problem has a solution by the Lax-Milgram lemma.

Now we make the ansatz

$$\boldsymbol{u}_h \;=\; \boldsymbol{z}_h \;+\; \nabla p_h \tag{4.4}$$

where $\boldsymbol{z}_h \in X_h$ satisfies

$$\big(\mu_r^{-1} \nabla \times \boldsymbol{z}_h, \nabla \times \boldsymbol{\phi}_h\big) \;-\; k^2\big(\epsilon_r \boldsymbol{z}_h, \boldsymbol{\phi}_h\big) \;=\; -ik\langle \boldsymbol{\lambda}, \boldsymbol{\phi} \rangle \;+\; k^2\big(\epsilon_r \nabla p_h, \boldsymbol{\phi}_h\big) \tag{4.5}$$

for all $\boldsymbol{\phi}_h \in X_h$. To convert this to an operator equation we define $B_h : \left(L^2(\Omega_R)\right)^3 \to X_h$ by defining $B_h\boldsymbol{f} = \boldsymbol{w}_h \in X_h$ which satisfies

$$\left(\mu_r^{-1}\nabla\times\boldsymbol{w}_h, \nabla\times\boldsymbol{\phi}_h\right) \;=\; \left(\epsilon_r\boldsymbol{f}, \boldsymbol{\phi}_h\right) \quad \text{for all } \boldsymbol{\phi}_h \in X_h\,.$$

Note that B_h is actually a bounded map from $\left(L^2(\Omega_R)\right)^3$ into X_h. We also define $\boldsymbol{F}_h \in X_h$ as the solution of

$$\left(\mu_r^{-1}\nabla\times\boldsymbol{F}_h, \nabla\times\boldsymbol{\phi}_h\right) \;=\; -ik\langle\boldsymbol{\lambda}, \boldsymbol{\phi}\rangle \;+\; k^2\left(\epsilon_r\nabla p_h, \boldsymbol{\phi}_h\right) \quad \text{for all } \boldsymbol{\phi}_h \in X_h\,. \tag{4.6}$$

By Corollary 4.2 and the Lax-Milgram lemma these problems have a unique solution. Thus we consider the operator equation of finding $\boldsymbol{v} \in \left(L^2(\Omega_R)\right)^3$ such that

$$(I - k^2 B_h)\boldsymbol{v} \;=\; \boldsymbol{F}_h\,. \tag{4.7}$$

Note that if we can uniquely solve this problem for a given h, then

$$\boldsymbol{v} \;=\; k^2 B_h\boldsymbol{v} \;+\; \boldsymbol{F}_h \;\in\; X_h$$

so that $\boldsymbol{v} \in X_h$. In addition

$$\left(\mu_r^{-1}\nabla\times(\boldsymbol{v} - k^2 B_h\boldsymbol{v}), \nabla\times\boldsymbol{\phi}_h\right) \;=\; \left(\mu_r^{-1}\nabla\times\boldsymbol{F}_h, \nabla\times\boldsymbol{\phi}_h\right) \quad \text{for all } \boldsymbol{\phi}_h \in X_h\,.$$

Now using the definition of B_h and $\boldsymbol{F}_h$ we see that $\boldsymbol{v}$ satisfies (4.5) and so in fact $\boldsymbol{z}_h = \boldsymbol{v}$. We have the following result:

THEOREM 4.3. *The collection of operators $\{B_h\}_{h\in\Lambda}$ converges point-wise to the operator B from § 2.2.in $\left(L^2(\Omega_R)\right)^3$. In addition the set of operators $\{B_h\}_{h\in\Lambda}$ is collectively compact when considered as maps from $\left(L^2(\Omega_R)\right)^3$ to $\left(L^2(\Omega_R)\right)^3$.*

Proof. This proof parallels the proof of a similar result in [10]. We prove point-wise convergence first. Rewriting the definition of $B\boldsymbol{f}$ as a mixed problem we see that $B\boldsymbol{f} \in H(\text{curl};\Omega_R)$ and $q \in H^1(\Omega_R)$ satisfies

$$\left(\mu_r^{-1}\nabla\times B\boldsymbol{f}, \nabla\times\boldsymbol{\phi}\right) \;+\; \left(\epsilon_r\boldsymbol{\phi}, \nabla q\right) = \left(\epsilon_r\boldsymbol{f}, \boldsymbol{\phi}\right) \quad \text{for all } \boldsymbol{\phi} \in H(\text{curl};\Omega_R)\,, \tag{4.8}$$

$$\left(\epsilon_r B\boldsymbol{f}, \nabla\xi\right) = 0 \quad \text{for all } \xi \in H^1(\Omega_R)\,. \tag{4.9}$$

The second equation ensures $B\boldsymbol{f} \in X \subset W$. Similarly we can see that $B_h\boldsymbol{f} \in V_h$ and $q_h \in S_h$ to satisfy

$$\left(\mu_r^{-1}\nabla\times B_h\boldsymbol{f}, \nabla\times\boldsymbol{\phi}_h\right) \;+\; \left(\epsilon_r\boldsymbol{\phi}_h, \nabla q_h\right) = \left(\epsilon_r\boldsymbol{f}, \boldsymbol{\phi}_h\right) \quad \text{for all } \boldsymbol{\phi}_h \in V_h\,, \tag{4.10}$$

$$\left(\epsilon_r B_h\boldsymbol{f}, \nabla\xi_h\right) = 0 \quad \text{for all } \xi_h \in S_h\,. \tag{4.11}$$

Again the last equation ensures $B_h\boldsymbol{f} \in X_h \subset W$.

Now Corollary (4.2) shows that the bilinear form $\left(\mu_r^{-1}\nabla\times\cdot, \nabla\times\cdot\right)$ is coercive on X_h and the fact that $\nabla S_h \subset V_h$ can be used to verify the Babuska-Brezzi condition (see [10]). Thus we know that

$$\|B\boldsymbol{f} - B_h\boldsymbol{f}\|_{H(\text{curl};\Omega_R)} \;\le\; C\left\{\inf_{\boldsymbol{\chi}_h\in V_h}\|B\boldsymbol{f} - \boldsymbol{\chi}_h\|_{H(\text{curl};\Omega_R)} \;+\; \inf_{\xi_h\in S_h}\|\nabla(q - \xi_h)\|\right\}\,. \tag{4.12}$$

Then the density of V_h in $H(\text{curl};\Omega_R)$ and of S_h in $H^1(\Omega_R)/\mathbb{R}$ completes the proof (actually we have proved pointwise convergence in $H(\text{curl};\Omega_R)$ which is more than sufficient.

Next we show that the set of operators is collectively compact. Let $U \subset \left(L^2(\Omega_R)\right)^3$ be a bounded set. Then, if $\boldsymbol{u} \in U$, we know that $B_h\boldsymbol{u} \in X_h$ satisfies

$$\left(\mu_r^{-1}\nabla\times B_h\boldsymbol{u}, \nabla\times\boldsymbol{\phi}_h\right) \;=\; \left(\epsilon_r\boldsymbol{u}, \boldsymbol{\phi}_h\right) \quad \text{for all } \boldsymbol{\phi}_h \in X_h\,.$$

It follows that $\|\nabla\times B_h\boldsymbol{u}\| \le C\|\boldsymbol{u}\|$. But using the discrete Friedrichs inequality in Corollary 4.2 we have

$$\|B_h\boldsymbol{u}\| \;+\; \|\nabla\times B_h\boldsymbol{u}\| \;\le\; C\,\|\boldsymbol{u}\|\,.$$

Thus $B_h(U) \subset W$. Since W is compactly contained in $\left(L^2(\Omega_R)\right)^3$ we can extract a convergent subsequence from $B_h(U)$. Thus $B_h(U)$ is pre-compact in $\left(L^2(\Omega_R)\right)^3$ as required. ◻

Now that we have written the finite element problem as an operator equation, we can prove the basic existence and convergence theorem for $G_{i,h}$:

THEOREM 4.4. *For sufficiently small h the operator $G_{i,h}$ is well defined and*

$$\|(G_i - G_{i,h})\boldsymbol{\lambda}\|_{H^{-1/2}(\mathrm{Div};\Gamma_R)} \to 0$$

as $h \to 0$.

Proof. First we show that $\boldsymbol{z}_h$ (see (4.4)) is well defined. Using the collective compactness and pointwise convergence of B_h, we know that provided h is small enough $(I - k^2 B_h)$ is invertible with uniformly bounded inverse as a map from $\left(L^2(\Omega_R)\right)^3$ into itself (see [29]). Hence $\boldsymbol{z}_h$ and p_h in (4.4) are well defined. Furthermore, the following error estimate holds:

$$\|\boldsymbol{z} - \boldsymbol{z}_h\| \;\leq\; C\Big(\|(B_h - B)B\boldsymbol{z}\| \;+\; \|\boldsymbol{F} - \boldsymbol{F}_h\| \;+\; \|(B_h - B)\boldsymbol{F}\|\Big)$$

We estimate the first term on the right hand side. Taking $\boldsymbol{\phi} = \nabla q$ in (4.8) we see that since $B\boldsymbol{z} \in X$ we can conclude that $q = 0$. Hence using (4.12) with $\xi_h = 0$ we have

$$\|(B - B_h)B\boldsymbol{z}\| \;\leq\; C \inf_{\boldsymbol{\chi}_h \in V_h} \|B^2\boldsymbol{z} - \boldsymbol{\chi}_h\|_{H(\mathrm{curl};\Omega_R)}\,.$$

Now we estimate $\|\boldsymbol{F} - \boldsymbol{F}_h\|$. The function $\boldsymbol{F}$ is defined by (2.33) and using the decomposition that any function $\boldsymbol{\phi} \in H(\mathrm{curl};\Omega_R)$ can be written $\boldsymbol{\phi} = \tilde{\boldsymbol{\phi}} + \nabla q$ for some $\tilde{\boldsymbol{\phi}} \in X$ and $q \in H^1(\Omega_R)$ we conclude that (2.33) actually holds for any $\boldsymbol{\phi} \in H(\mathrm{curl};\Omega_R)$. Similarly (4.6) actually holds for any $\boldsymbol{\phi}_h \in V_h$. We can now use (2.33) and (4.6) to write

$$\begin{aligned}
\|\boldsymbol{F} - \boldsymbol{F}_h\|^2_{H(\mathrm{curl};\Omega_R)} &\leq C\left(\mu_r^{-1}\nabla\times(\boldsymbol{F} - \boldsymbol{F}_h), \nabla\times(\boldsymbol{F} - \boldsymbol{\eta}_h)\right) \\
&\quad + \left(\mu_r^{-1}\nabla\times(\boldsymbol{F} - \boldsymbol{F}_h), \nabla\times(\boldsymbol{\eta}_h - \boldsymbol{F}_h)\right) \\
&= C\Big[\left(\mu_r^{-1}\nabla\times(\boldsymbol{F} - \boldsymbol{F}_h), \nabla\times(\boldsymbol{F} - \boldsymbol{\eta}_h)\right) \;+\; k^2\left(\epsilon_r\nabla(p - p_h), \boldsymbol{\eta}_h - \boldsymbol{F}_h\right)\Big] \\
&= C\Big[\left(\mu_r^{-1}\nabla\times(\boldsymbol{F} - \boldsymbol{F}_h), \nabla\times(\boldsymbol{F} - \boldsymbol{\eta}_h)\right) \;+\; k^2\left(\epsilon_r\nabla(p - p_h), \boldsymbol{F} - \boldsymbol{F}_h\right) \\
&\quad + k^2\left(\epsilon_r\nabla(p - p_h), \boldsymbol{\eta}_h - \boldsymbol{F}\right)\Big]
\end{aligned}$$

for any $\boldsymbol{\eta}_h \in V_h$. Now using the Cauchy-Schwarz inequality and (4.2)

$$\begin{aligned}
\|\boldsymbol{F} - \boldsymbol{F}_h\|^2_{H(\mathrm{curl};\Omega_R)} &\leq C\Big[\|\boldsymbol{F} - \boldsymbol{F}_h\|_{H(\mathrm{curl};\Omega_R)}\,\|\boldsymbol{F} - \boldsymbol{\eta}_h\|_{H(\mathrm{curl};\Omega_R)} \;+\; \|\nabla(p - p_h)\|\,\|\boldsymbol{F} - \boldsymbol{F}_h\| \\
&\quad + \|\nabla(p - p_h)\|\|\boldsymbol{F} - \boldsymbol{\eta}_h\|\Big]\,.
\end{aligned}$$

But since p satisfies (2.28) and p_h satisfies (4.3) we know that standard estimates for coercive elliptic problems give

$$\|\nabla(p - p_h)\| \;\leq\; C\|p - \phi_h\|_{H^1(\Omega_R)} \tag{4.13}$$

for any $\phi_h \in S_h$. Thus the arithmetic geometric mean inequality gives

$$\|\boldsymbol{F} - \boldsymbol{F}_h\| \;\leq\; \|\boldsymbol{F} - \boldsymbol{F}_h\|_{H(\mathrm{curl};\Omega_R)} \;\leq\; C\Big(\|\boldsymbol{F} - \boldsymbol{\eta}_h\|_{H(\mathrm{curl};\Omega_R)} \;+\; \|p - \phi_h\|_{H^1(\Omega_R)}\Big)\,.$$

Finally, using the fact that $\boldsymbol{F}$ is divergence free we can again use (4.12) to show that

$$\|(B_h - B)\boldsymbol{F}\| \leq C\|B\boldsymbol{F} - \boldsymbol{\psi}_h\|_{H(\mathrm{curl};\Omega_R)}$$

for all $\boldsymbol{\psi}_h \in V_h$.

We have thus proved that

$$\|\boldsymbol{z}-\boldsymbol{z}_h\| \le C\ \{\|B^2\boldsymbol{z}-\boldsymbol{\chi}_h\|_{H(\mathrm{curl};\Omega_R)} + \|p-\psi_h\|_{H^1(\Omega_R)} + \|\boldsymbol{F}-\boldsymbol{\eta}_h\|_{H(\mathrm{curl};\Omega_R)}$$
$$+\|B\boldsymbol{F}-\boldsymbol{\psi}_h|_{H(\mathrm{curl};\Omega_R)}\}\,. \tag{4.14}$$

It remains to derive an error estimate in the $H(\mathrm{curl};\Omega_R)$ norm. For this we define $\boldsymbol{e}=\boldsymbol{z}-\boldsymbol{z}_h$ and

$$a(\boldsymbol{u},\boldsymbol{v}) \;=\; \big(\mu_r^{-1}\nabla\times\boldsymbol{u},\nabla\times\boldsymbol{v}\big) \;-\; k^2(\epsilon_r\boldsymbol{u},\boldsymbol{v})\,.$$

Using (2.29) and (4.4) we can see that

$$a(\boldsymbol{e},\boldsymbol{\phi}_h) \;=\; k^2\big(\epsilon_r\nabla(p-p_h),\boldsymbol{\phi}_h\big) \quad \text{for all } \boldsymbol{\phi}_h\in V_h\,.$$

Using this "orthogonality" relation we have that

$$a(\boldsymbol{e},\boldsymbol{e}) \;=\; a(\boldsymbol{e},\boldsymbol{z}-\boldsymbol{\tau}_h) \;+\; a(\boldsymbol{e},\boldsymbol{\tau}_h-\boldsymbol{z}_h) \;=\; a(\boldsymbol{e},\boldsymbol{z}-\boldsymbol{\tau}_h) \;+\; k^2\big(\epsilon_r\nabla(p-p_h),\boldsymbol{\tau}_h-\boldsymbol{z}_h\big)$$

for any $\boldsymbol{\tau}_h\in V_h$. Thus adding $(k^2+1)(\epsilon_r\boldsymbol{e},\boldsymbol{e})$ to both sides of this equation we have that

$$\|\boldsymbol{e}\|_{H(\mathrm{curl};\Omega_R)} \;\le\; C\Big[\big|\big(\epsilon_r\nabla(p-p_h),\boldsymbol{\tau}_h-\boldsymbol{z}_h\big)\big| \;+\; |a(\boldsymbol{e},\boldsymbol{z}-\boldsymbol{\tau}_h)| \;+\; |(\epsilon_r\boldsymbol{e},\boldsymbol{e})|\Big]\,. \tag{4.15}$$

We have already proved above an estimate for $\|\boldsymbol{e}\|$ (see (4.14)) and for $\|\nabla(p-p_h)\|$. Hence using the arithmetic geometric mean inequality and collecting terms in (4.15) we have shown that

$$\|\boldsymbol{e}\|_{H(\mathrm{curl};\Omega_R)} \;\le\; C\Big\{\|\nabla(p-p_h)\|\big[\|\boldsymbol{z}-\boldsymbol{\tau}_h\|+\|\boldsymbol{e}\|\big] \;+\; \|\boldsymbol{e}\|^2 \;+\; \|\boldsymbol{z}-\boldsymbol{\tau}_h\|^2_{H(\mathrm{curl};\Omega_R)}\Big\}\,.$$

But using (4.14)

$$\|\boldsymbol{e}\|_{H(\mathrm{curl};\Omega_R)} \;\le C\,\{\|\boldsymbol{z}-\boldsymbol{\tau}_h\|_{H(\mathrm{curl};\Omega_R)} + \|B^2\boldsymbol{z}-\boldsymbol{\chi}_h\|_{H(\mathrm{curl};\Omega_R)} + \|p-\phi_h\|_{H^1(\Omega_R)}$$
$$+\|\boldsymbol{F}-\boldsymbol{\eta}_h\|_{H(\mathrm{curl};\Omega_R)} + \|B\boldsymbol{F}-\boldsymbol{\psi}_h\|_{H(\mathrm{curl};\Omega_R)},\}\,. \tag{4.16}$$

for any $\boldsymbol{\tau}_h$, $\boldsymbol{\chi}_h$, $\boldsymbol{\eta}_h$, $\boldsymbol{\psi}_h\in V_h$ and $\phi_h\in S_h$. This proves convergence of $\boldsymbol{z}_h$ to $\boldsymbol{z}$ by using the density of the relevant finite element spaces in $H(\mathrm{curl};\Omega_R)$ and $H^1(\Omega_R)/\mathbb{R}$.

To obtain an estimate for $\boldsymbol{u}-\boldsymbol{u}_h$ we use (4.4), (2.29) and (4.13) to write

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{H(\mathrm{curl};\Omega_R)} \;\le\; C\Big[\|\boldsymbol{e}\|_{H(\mathrm{curl};\Omega_R)} \;+\; \|\nabla(p-p_h)\|\Big]. \tag{4.17}$$

Again p_h converges to p in $H^1(\Omega_R)$ and using the trace theorem for $H(\mathrm{curl};\Omega_R)$ we have

$$\|(G_i-G_{i,h})\boldsymbol{\lambda}\|_{H^{-1/2}(\mathrm{Div};\Gamma_R)} \;\le\; C\,\|\boldsymbol{u}-\boldsymbol{u}_h\|_{H(\mathrm{curl};\Omega_R)} \tag{4.18}$$

where $\boldsymbol{u}$ satisfies (2.7) and $\boldsymbol{u}_h$ satisfies (3.3) so that we have proved the desired result. □

The above estimate shows that the $G_{i,h}$ converges to G_i with very general assumptions on the smoothness of the data to the problem but with no rate of convergence. To obtain error estimates for the coupled problem we actually need an order estimate for the convergence rate. Hence for the remainder of the paper, we shall assume that the coefficients ϵ_r and μ_r are sufficiently smooth that the following a priori estimates hold.

1. For every $p\in H^1(\Omega_R)/\mathbb{R}$ satisfying $\nabla\cdot\epsilon_r\nabla p=0$ in Ω_R we have

$$\|\nabla p\|_{PH^1(\mathrm{curl};\Omega_R)} \;\le\; C\left\|\frac{\partial p}{\partial\hat{\boldsymbol{x}}}\right\|_{H^{1/2}(\Gamma_R)}\,. \tag{4.19}$$

   This is a typical elliptic regularity estimate for p for smooth data.

2. Suppose $\boldsymbol{f}\in\big(L^2(\Omega_R)\big)^3$ is such that $\nabla\cdot(\mu_r\boldsymbol{f})=0$ in Ω_R. Let $\boldsymbol{u}\in H(\mathrm{curl};\Omega_R)$ satisfy

$$\nabla\times\boldsymbol{u}=\mu_r\boldsymbol{f} \quad \text{in } \Omega_R\,,$$
$$\nabla\cdot(\epsilon_r\boldsymbol{u})=0 \quad \text{in } \Omega_R\,,$$
$$\hat{\boldsymbol{x}}\cdot\boldsymbol{u}=0 \quad \text{on } \Gamma_R\,,$$

   then $\boldsymbol{u}\in PH^1(\mathrm{curl};\Omega_R)$ and

$$\|\boldsymbol{u}\|_{PH^1(\mathrm{curl};\Omega_R)} \;\le\; C\|\boldsymbol{f}\|\,. \tag{4.20}$$

3. Let $\boldsymbol{f} \in \left(L^2(\Omega_R)\right)^3$ and $\boldsymbol{g} \in H^{1/2}(\mathrm{Div};\Gamma_R)$ satisfy the compatibility condition that

$$\left(\epsilon_r \boldsymbol{f}, \nabla\xi\right) \; + \; \langle \boldsymbol{g}, \nabla\xi\rangle \;=\; 0 \quad \text{for all } \xi \in H^1(\Omega_R)\,. \tag{4.21}$$

Let $\boldsymbol{v} \in H(\mathrm{curl};\Omega_R)$ satisfy

$$\begin{aligned} \nabla\times\boldsymbol{v} &= \epsilon_r \boldsymbol{f} \quad \text{in } \Omega_R\,,\\ \nabla\cdot(\mu_r\boldsymbol{v}) &= 0 \quad \text{in } \Omega_R\,,\\ \hat{\boldsymbol{x}}\times\boldsymbol{v} &= \boldsymbol{g} \quad \text{on } \Gamma_R\,, \end{aligned}$$

then $\boldsymbol{v} \in PH^1(\mathrm{curl};\Omega_R)$ and

$$\|\boldsymbol{v}\|_{PH^1(\mathrm{curl};\Omega_R)} \;\leq\; C\big[\|\boldsymbol{f}\| \;+\; \|\boldsymbol{g}\|_{H^{1/2}(\mathrm{Div};\Gamma_R)}\big]\,. \tag{4.22}$$

Obviously these assumptions rule out rough boundaries between the domains Ω_n where ϵ and μ are smooth. For a discussion of regularity of Maxwell's equations in the presence of piece-wise smooth functions with smooth interfaces see Weber [33] and for the case of piece-wise constant coefficients with non-smooth interfaces see Costabel et al. [6].

Using (4.20) and (4.22), if $\boldsymbol{f}$ and $\boldsymbol{g}$ satisfy the compatibility condition (4.21) and if $\boldsymbol{u} \in X$ satisfies

$$\begin{aligned} \nabla\times\mu_r^{-1}\nabla\times\boldsymbol{u} &= \epsilon_r\boldsymbol{f} \quad \text{in } \Omega_R\,,\\ \hat{\boldsymbol{x}}\times\nabla\times\boldsymbol{u} &= \boldsymbol{g} \quad \text{on } \Gamma_R\,, \end{aligned}$$

then

$$\|\boldsymbol{u}\|_{PH^1(\mathrm{curl};\Omega_R)} \;\leq\; C\big[\|\boldsymbol{f}\| \;+\; \|\boldsymbol{g}\|_{H^{1/2}(\mathrm{Div};\Gamma_R)}\big]\,.$$

The goal of the remainder of this section is to prove the following error estimate:

Theorem 4.5. *Assume that (4.19)–(4.22) hold. Then if $\boldsymbol{\lambda} \in H^{1/2}(\mathrm{Div};\Gamma_R)$ there exists a constant C such that*

$$\|(G_i - G_{i,h})\boldsymbol{\lambda}\|_{H^{-1/2}(\mathrm{Div};\Gamma_R)} \;\leq\; C\,h\,\|\boldsymbol{\lambda}\|_{H^{1/2}(\mathrm{Div};\Gamma_R)}\,, \quad h\in\Lambda\,.$$

Before starting the proof of this theorem, note that if the functions p and $B\boldsymbol{f}$ are smooth we also have an error estimate for B_h as the next theorem shows (this is proved from (4.12) and the approximation properties of V_h and S_h).

Theorem 4.6. *If $B\boldsymbol{f} \in PH^1(\mathrm{curl};\Omega_R)$ and $p \in PH^2(\Omega_R)$ then*

$$\|B\boldsymbol{f} - B_h\boldsymbol{f}\|_{H(\mathrm{curl};\Omega_R)} \;\leq\; Ch\Big(\|B\boldsymbol{f}\|_{PH^1(\mathrm{curl};\Omega_R)} \;+\; \|p\|_{PH^2(\Omega_R)}\Big)\,.$$

Proof. [of Theorem 4.5.] We can simply use estimate (4.16) followed by (4.17) and (4.18). Choosing $\boldsymbol{\chi}_h = \pi_h B^2\boldsymbol{z}$ and using (4.2) we have that

$$\|B^2\boldsymbol{z} - \boldsymbol{\chi}_h\| \;\leq\; Ch\,\|B^2\boldsymbol{z}\|_{PH^1(\mathrm{curl};\Omega_R)}\,.$$

Now using a priori estimate (4.22) with $\boldsymbol{f} = B\boldsymbol{z}$ and $\boldsymbol{g} = 0$ we have

$$\|B^2\boldsymbol{z} - \boldsymbol{\chi}_h\| \;\leq\; Ch\,\|B\boldsymbol{z}\|\,.$$

Again using (4.22) and the well-posedness of the equation for $\boldsymbol{z}$ in $H(\mathrm{curl};\Omega_R)$ we have

$$\|B^2\boldsymbol{z} - \boldsymbol{\chi}_h\| \;\leq\; Ch\,\|\boldsymbol{z}\|_{H(\mathrm{curl};\Omega_R)} \;\leq\; Ch\,\|\boldsymbol{F}\|_{H(\mathrm{curl};\Omega_R)}\,.$$

Using the definition of $\boldsymbol{F}$ we have

$$\|\boldsymbol{F}\|_{H(\mathrm{curl};\Omega_R)} \leq C\left(\|\boldsymbol{\lambda}\|_{H^{-1/2}(\mathrm{Div};\Gamma_R)} + \|\nabla p\|\right).$$

Next we estimate $p-\phi_h$. Note that $p\in H^1(\Omega_R)/\mathbb{R}$ is defined by (2.28) and so satisfies

$$\nabla\cdot(\epsilon_r\nabla p)=0 \quad \text{in } \Omega_R\,,$$
$$\frac{\partial p}{\partial\hat{\boldsymbol{x}}}=-\frac{i}{k}\nabla_{\Gamma_R}\cdot\boldsymbol{\lambda} \quad \text{on } \Gamma_R\,.$$

Using (4.19) and choosing ϕ_h to be the $H^1(\Omega_R)/\mathbb{R}$ projection of p we have

$$\|p-\phi_h\|_{H^1(\Omega_R)}\le Ch\|p\|_{PH^2(\Omega_R)}\le Ch\|\boldsymbol{\lambda}\|_{Hpdiv}.$$

Now we estimate $\|\boldsymbol{F}-\boldsymbol{\eta}_h\|$ by choosing $\boldsymbol{\eta}_h=\pi_h\boldsymbol{F}$. Then

$$\|\boldsymbol{F}-\boldsymbol{\eta}_h\|_{H(\mathrm{curl};\Omega_R)}\le\ Ch\|\boldsymbol{F}\|_{PH^1(\mathrm{curl};\Omega_R)}.$$

Using (4.22) we have that

$$\|\boldsymbol{F}\|_{PH^1(\mathrm{curl};\Omega_R)}\ \le\ c\Big(\|\nabla p\|\ +\ \|\boldsymbol{\lambda}\|_{H^{1/2}(\mathrm{Div};\Gamma_R)}\Big)$$

and thus

$$\|\boldsymbol{F}-\boldsymbol{\eta}_h\|\ \le\ Ch\Big(\|\nabla p\|\ +\ \|\boldsymbol{\lambda}\|_{H^{1/2}(\mathrm{Div};\Gamma_R)}\Big)\le\ Ch\|\boldsymbol{\lambda}\|_{H^{1/2}(\mathrm{Div};\Gamma_R)}\,.$$

It remains only to estimate $\boldsymbol{z}-\boldsymbol{\tau}_h$ and $B\boldsymbol{F}-\boldsymbol{\psi}_h$. We choose $\boldsymbol{\tau}_h=\pi_h\boldsymbol{z}$ and $\boldsymbol{\psi}_h=\pi_hB\boldsymbol{F}$ and proceed as for the other estimates to show that

$$\|\boldsymbol{z}-\pi_h\boldsymbol{z}\|_{H(\mathrm{curl};\Omega_R)}\le Ch\|\boldsymbol{z}\|_{PH^1(\mathrm{curl};\Omega_R)}\le Ch\|\boldsymbol{\lambda}\|_{H^{1/2}(\mathrm{Div};\Gamma_R)},$$
$$\|B\boldsymbol{F}-\pi_hB\boldsymbol{F}\|_{H(\mathrm{curl};\Omega_R)}\le Ch\|B\boldsymbol{F}\|_{PH^1(\mathrm{curl};\Omega_R)}\le Ch\|\boldsymbol{\lambda}\|_{H^{1/2}(\mathrm{Div};\Gamma_R)}.$$

Combining all the estimates in (4.16), (4.17) and (4.18) proves the theorem. ☐

5. Error Estimates for the Fully Discrete Problem. Here we shall analyze the fully discrete problem. We shall prove the following basic theorem:

THEOREM 5.1. *Assume that (4.19)-(4.22) hold. Then there is a* $\delta>0$ *such that for* N *sufficiently large and* $hN<\delta$ *there is unique solution* $\boldsymbol{\lambda}_{N,h}\in S_N$ *of (3.4), furthermore*

$$\|\boldsymbol{\lambda}_{N,h}-\boldsymbol{\lambda}\|_{H^{-1/2}(\mathrm{Div};\Gamma_R)}\le c\left(h\|\boldsymbol{f}\|_{H^{1/2}(\mathrm{Div};\Gamma_R)}+(h+1/N)\|\boldsymbol{\lambda}\|_{H^{1/2}(\mathrm{Div};\Gamma_R)}\right).$$

Remark: We can obtain a higher power of N in this estimate (at the expense of a higher norm of $\boldsymbol{\lambda}$).

Proof. Note first that by operating on (2.8) by P_N and using the fact that G_e and P_N commute we have

$$P_NG_i\boldsymbol{\lambda}-G_eP_N\boldsymbol{\lambda}=P_N\boldsymbol{f} \tag{5.1}$$

Let us define $\boldsymbol{e}_{N,h}=\boldsymbol{\lambda}_{N,h}-P_N\boldsymbol{\lambda}$. Then using (5.1) and (3.4) we have

$$\begin{aligned}(P_NG_{i,h}-P_NG_e)\boldsymbol{e}_{N,h}&=P_N(G_i-G_e)\boldsymbol{e}_{N,h}+P_N(G_{i,h}-G_i)\boldsymbol{e}_{N,h}\\&=P_N(G_i-G_{i,h})P_N\boldsymbol{\lambda}-P_NG_i(P_N\boldsymbol{\lambda}-\boldsymbol{\lambda}),\end{aligned}$$

where we have used the fact that

$$P_NG_e(P_N\boldsymbol{\lambda}-\boldsymbol{\lambda})=G_eP_N(P_N\boldsymbol{\lambda}-\boldsymbol{\lambda})=0.$$

Using Theorem 2.1 we have the decomposition

$$G_i-G_e=T+K$$

where $T,K:H^{-1/2}(\mathrm{Div};\Gamma_R)\to H^{-1/2}(\mathrm{Div};\Gamma_R)$, T is an isomorphism and K is compact. Using the fact that P_N and T commute we obtain our fundamental error equation:

$$T\boldsymbol{e}_{N,h}+P_NK\boldsymbol{e}_{N,h}+P_N(G_{i,h}-G_i)\boldsymbol{e}_{N,h}=P_N(G_i-G_{i,h})P_N\boldsymbol{\lambda}-P_NG_i(P_N\boldsymbol{\lambda}-\boldsymbol{\lambda}).$$

First we need to show that this equation has a solution. Since P_N is the orthogonal projection for $H^{-1/2}(\mathrm{Div};\Gamma_R)$ into S_N and K is compact in this space we know that $P_N K \to K$ in the operator norm of $H^{-1/2}(\mathrm{Div};\Gamma_R)$. For the other term on the left hand side above we can use the error estimate for the finite element solution in Theorem 4.5, together with the inverse estimate (3.2) to show that

$$\begin{aligned}\|P_N(G_{i,h}-G_i)\boldsymbol{e}_{N,h}\|_{H^{-1/2}(\mathrm{Div};\Gamma_R))} &\le \|(G_{i,h}-G_i)\boldsymbol{e}_{N,h}\|_{H^{-1/2}(\mathrm{Div};\Gamma_R)}\\ &\le Ch\|\boldsymbol{e}_{N,h}\|_{H^{1/2}(\mathrm{Div};\Gamma_R)}\\ &\le CNh\|\boldsymbol{e}_{N,h}\|_{H^{-1/2}(\mathrm{Div};\Gamma_R)}.\end{aligned}$$

Thus for sufficiently large N and small Nh the operator $T+P_NK+P_N(G_{i,h}-G_i)$ is invertible with uniformly bounded inverse. This implies that $\boldsymbol{e}_{N,h}$ and hence $\boldsymbol{\lambda}_{N,h}$ is well defined and we can obtain an error estimate simply by estimating the right hand side using Theorem (4.5) and the estimate (3.1):

$$\begin{aligned}\|P_N(G_i-G_{i,h})P_N\boldsymbol{\lambda}\|_{H^{-1/2}(\mathrm{Div};\Gamma_R)} &\le Ch\|P_N\boldsymbol{\lambda}\|_{H^{1/2}(\mathrm{Div};\Gamma_R)},\\ \|P_NG_i(P_N\boldsymbol{\lambda}-\boldsymbol{\lambda})\|_{H^{-1/2}(\mathrm{Div};\Gamma_R)} &\le C\|P_N\boldsymbol{\lambda}-\boldsymbol{\lambda}\|_{H^{-1/2}(\mathrm{Div};\Gamma_R)} \le \frac{C}{N}\|\boldsymbol{\lambda}\|_{H^{-1/2}(\mathrm{Div};\Gamma_R)}.\end{aligned}$$

Putting these estimates together we obtain the estimate

$$\|\boldsymbol{e}_{N,h}\|_{H^{-1/2}(\mathrm{Div};\Gamma_R)} \le C\left(h\|\boldsymbol{f}\|_{H^{1/2}(\mathrm{Div};\Gamma_R)} + (h+1/N)\|\boldsymbol{\lambda}\|_{H^{1/2}(\mathrm{Div};\Gamma_R)}\right).$$

The use of the triangle equality then proves the estimate of the theorem. □

Our final result gives an error estimate for the field near and in the scatterer. It follows from the previous result.

COROLLARY 5.2. *Assume that (4.19)–(4.22) hold. Let $\boldsymbol{E}$ satisfy (1.1)–(1.3). Define $\boldsymbol{E}_h \in V_h$ to satisfy*

$$(\mu_r^{-1}\nabla\times\boldsymbol{E}_h,\nabla\times\boldsymbol{\phi}_h) - k^2(\epsilon_r\boldsymbol{E}_h,\boldsymbol{\phi}) + ik\langle\boldsymbol{\lambda}_{h,N},\boldsymbol{\phi}_h\rangle = 0 \quad \forall\boldsymbol{\phi}_h\in V_h$$

where $\boldsymbol{\lambda}_{N,h}\in S_N$ satisfies (3.4). Then $\boldsymbol{E}_h$ is well defined and

$$\|\boldsymbol{E}-\boldsymbol{E}_h\|_{H(\mathrm{curl};\Omega_R)} \le C\left(h\|\boldsymbol{f}\|_{H^{1/2}(\mathrm{Div};\Gamma_R)} + (h+1/N)\|\boldsymbol{\lambda}\|_{H^{1/2}(\mathrm{Div};\Gamma_R)}\right).$$

6. Conclusion.

We have proved convergence of the combined spectral – finite element scheme, under the stability condition that hN is sufficiently small. It would be interesting to determine if this condition is necessary. Of more practical importance is to generalize the class of exterior boundaries allowed (here only a sphere) to cope with scatterers of a more general geometry. A numerical test of the algorithm is clearly desirable.

7. Appendix.

THEOREM 2.6. *Assume that ρ is chosen so that k is not a Maxwell eigenvalue for the annulus A (i.e. the following interior problem possesses a unique solution). Let the operator $L:\boldsymbol{\gamma}\to\hat{\boldsymbol{x}}\times\boldsymbol{E}|_{\Gamma_R}$ be defined by*

$$\begin{aligned}\nabla\times\boldsymbol{E} \;-\; ik\boldsymbol{H} &= 0 \quad \text{in } A,\\ \nabla\times\boldsymbol{H} \;+\; ik\boldsymbol{E} &= 0 \quad \text{in } A,\\ \hat{\boldsymbol{x}}\times\boldsymbol{E} &= \boldsymbol{\gamma} \quad \text{on } \Gamma_\rho,\\ \hat{\boldsymbol{x}}\times\boldsymbol{H} &= 0 \quad \text{on } \Gamma_R.\end{aligned}$$

Then L is bounded from $H^{-1/2}(\mathrm{Div};\Gamma_\rho)$ into $H^s(\mathrm{Div};\Gamma_R)$ for any s.

Proof. The tangential fields $\boldsymbol{U}_n^m$ and $\boldsymbol{V}_n^m$ are defined in (2.10). Using the vector basis functions defined in (2.13) and (2.15) we know that

$$\boldsymbol{E} \;=\; \sum_{n=1}^{\infty}\sum_{m=-n}^{n}\left[\left(\alpha_n^m\boldsymbol{M}_m^n + \beta_n^m\boldsymbol{N}_n^m\right) \;+\; \left(\hat{\alpha}_n^m\hat{\boldsymbol{M}}_n^m + \hat{\beta}_n^m\hat{\boldsymbol{N}}_n^m\right)\right].$$

for suitable constants $\{\alpha_m^m, \hat\alpha_n^m, \beta_n^m, \hat\beta_n^m\}$. Then using the relationships between the boundary and volume basis in (2.16)-(2.17) with similar relationships for the interior fields we obtain that

$$\hat{\boldsymbol{x}} \times \boldsymbol{E} \;=\; \sum_{n=1}^{\infty}\sum_{m=-n}^{n} \left[\alpha_n^m h_n^{(1)}(kr)\boldsymbol{U}_n^m + \frac{\tilde h_n(kr)}{ikr}\beta_n^m \boldsymbol{V}_n^m\right] \;+\; \left[\hat\alpha_n^m j_n(kr)\hat{\boldsymbol{U}}_n^m + \frac{\tilde j_n(kr)}{ikr}\hat\beta_n^m \hat{\boldsymbol{V}}_n^m\right],$$

where $r = \rho$ or $r = R$ depending on which boundary is under consideration. Furthermore, since $\boldsymbol{H} = (1/ik)\nabla \times \boldsymbol{E}$, we have

$$\boldsymbol{H} \;=\; \sum_{n=1}^{\infty}\sum_{m=-n}^{n} \left[\left(\alpha_n^m \boldsymbol{N}_m^n - \beta_n^m \boldsymbol{M}_n^m\right) \;+\; \left(\hat\alpha_n^m \hat{\boldsymbol{N}}_n^m - \hat\beta_n^m \hat{\boldsymbol{M}}_n^m\right)\right]$$

and hence

$$\hat{\boldsymbol{x}} \times \boldsymbol{H} \;=\; \sum_{n=1}^{\infty}\sum_{m=-n}^{n} \left[\alpha_n^m h_n^{(1)}(kr)\boldsymbol{U}_n^m + \frac{\tilde h_n(kr)}{ikr}\beta_n^m \boldsymbol{V}_n^m\right] \;+\; \left[\hat\alpha_n^m j_n(kr)\hat{\boldsymbol{U}}_n^m \frac{\tilde j_n(kr)}{ikr}\hat\beta_n^m \hat{\boldsymbol{V}}_n^m\right]$$

where $r = R$ or $r = \rho$ depending on if we are at the inner or outer boundary of the annular region.

We determine the coefficients α_n^m, β_n^m, $\hat\alpha_n^m$ and $\hat\beta_n^m$, $m = -n, \ldots, n$ and $n = 1, 2, \ldots$ from the boundary conditions. Suppose

$$\boldsymbol{\lambda} \;=\; \sum_{n=1}^{\infty}\sum_{m=-n}^{n} \left[a_n^m \boldsymbol{U}_n^m \;+\; b_n^m \boldsymbol{V}_n^m\right].$$

Then using the boundary condition on $r = \rho$ we have

$$\begin{aligned} \alpha_n^m h_n^{(1)}(k\rho) \;+\; \hat\alpha_n^m j_n(k\rho) &= a_n^m/\sqrt{n(n+1)}\,, \\ \beta_n^m \tilde h_n(k\rho) \;+\; \hat\beta_n^m \tilde j_n(k\rho) &= ik\,\rho b_n^m/\sqrt{n(n+1)}\,. \end{aligned}$$

On the boundary $r = R$ we have the vanishing tangential component of the magnetic field. Hence

$$\begin{aligned} \alpha_n^m \tilde h_n(kR) \;+\; \hat\alpha_n^m \tilde j_n(kR) &= 0\,, \\ \beta_n^m h_n^{(1)}(kR) \;+\; \hat\beta_n^m j_n(k\rho) &= 0\,. \end{aligned}$$

These two systems can be solved for the unknown coefficients to yield

$$\begin{aligned} \alpha_n^m &= \frac{1}{D_1}\frac{a_n^m\,\tilde j_n(kR)}{\sqrt{n(n+1)}}\,, \\ \hat\alpha_n^m &= -\frac{1}{D_1}\frac{a_n^m\,\tilde h_n(kR)}{\sqrt{n(n+1)}}\,, \\ \beta_n^m &= \frac{1}{D_2}\frac{ik\rho\,b_n^m\,j_n(kR)}{\sqrt{n(n+1)}}\,, \\ \hat\beta_n^m &= -\frac{1}{D_2}\frac{ik\rho\,b_n^m\,h_n^{(1)}(kR)}{\sqrt{n(n+1)}}\,, \end{aligned}$$

where

$$\begin{aligned} D_1 &= h_n^{(1)}(k\rho)\,\tilde j_n(kR) \;-\; \tilde h_n(kR)\,j_n(k\rho)\,, \\ D_2 &= \tilde h_n(k\rho)\,j_n(kR) \;-\; \tilde j_n(k\rho)\,h_n^{(1)}(kR)\,. \end{aligned}$$

Now using the asymptotic estimates (2.20)-(2.24) we can easily show that

$$L\boldsymbol{\lambda} = \sum_{n=1}^{\infty}\sum_{m=n}^{n}\left[a_n^m\,h_n(kR) + \hat a_n^m\,j_n(kR)\right]\boldsymbol{U}_n^m \;+$$

$$
\begin{aligned}
&+ \sum_{n=1}^{\infty} \sum_{m=n}^{n} \left[b_n^m \, \tilde{h}_n(kR) + \hat{b}_n^m \, \tilde{j}_n(kR) \right] \boldsymbol{V}_n^m \\
&= \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \left[\left(\frac{\rho}{R} \right)^n (1+\lambda_n) \, a_n^m \, \boldsymbol{U}_n^m \; + \; \left(\frac{\rho}{R} \right)^{n+2} (1+\mu_n) \, b_n^m \, \boldsymbol{V}_n^m \right],
\end{aligned}
$$

where $\lambda_n, \mu_n = \mathcal{O}(1/n)$. □

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