Well-posedness of a random coefficient damage mechanics model*

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*Dedicated to Professor Robert P. Gilbert on the occasion of his 90th birthday.

Abstract

We study a one-dimensional damage mechanics model in the presence of random materials properties. The model is formulated as a quasilinear partial differential equation of visco-elastic dynamics with a random field coefficient. We prove that in a transformed coordinate system the problem is well-posed as an abstract evolution equation in Banach spaces, and on the probability space it has a strongly measurable and Bochner integrable solution. We also establish the existence of weak solutions in the underlying physical coordinate system. We present numerical examples that demonstrate propagation of uncertainty in the stress–strain relation based on properties of the random damage field.

Keywords: Viscoelasticity; damage mechanics; random coefficient differential equation; mild solutions

AMS Classifications: 35K90; 35Q74; 35R60; 74E35

1. Introduction

1.1. Physical model

Motivated by challenges in the computational study of quasi-brittle media, we analyze a quasilin-ear model that features irreversible damage, viscoelasticity, and randomness due to uncertainty in material properties. In Bažant and Belytschko [1], the authors explored a PDE model based on elastodynamics for quasi-brittle materials. The corresponding elastodynamics with strain softening rheology exhibits strain concentration corresponding to material failure As noted in [1] this model is, in general, ill-posed making it poorly suited for computational studies, particularly when intrinstic uncertainty in material properties is present. If, for instance, the model lacked continuous dependence upon the data, small statistical variations could lead to $O(1)$ variations in the simulation output, ren-dering model predictions uninformative. Indeed, the strain softening leads to a mixed type problem, transitioning between hyperbolic and elliptic regimes. Modeling assumptions as well as numerical simulations suggest that the addition of viscoelasticity may lead to a well-posed problem in suitable function spaces.

In this paper, we make progress towards a well-posedness result, studying the problem in one spatial dimension, including both viscoelasticity and *irreversible* material degradation through continuum damage mechanics. Furthermore, we study the problem in a probabilistic framework that captures random disorder in materials properties. Such stochastic modeling approaches have been studied primarily in engineering literature, e.g. [2,3], where the random disorder is used to model uncertainty in the material damage. We shall not further discuss modeling aspects of the problem and instead focus on analysis of a prototype equation. We refer the reader to [4–7] for more details about continuum damage and fracture models.

In engineering continuum damage mechanics, at each point within the material body, a phenomenological *damage variable* is introduced that captures the density of microvoids and microcracks. Microvoids and microcracks may be present due to imperfect fabrication and are also expected to appear as the material undergoes excess deformation. Under an isotropy assumption the damage variable, $\mathcal{D} = \mathcal{D}(x, t)$ is a scalar valued function taking values in $[0, 1]$. When $\mathcal{D} = 0$, the material is pristine, while $\mathcal{D} = 1$ corresponds to the total failure. Damage can be included in the rheology as

$$\sigma = (1 - \mathcal{D})\sigma^{(\mathrm{L})}(\epsilon)\,, \tag{1}$$

where $\sigma^{(\mathrm{L})}(\epsilon)$ is the linear isotropic stress tensor

$$\epsilon = \tfrac{1}{2}(\nabla u + (\nabla u)^T) \tag{2a}$$

$$\sigma^{(\mathrm{L})}(\epsilon) = 2\mu\epsilon + \lambda \mathrm{Tr}\epsilon, \tag{2b}$$

where μ and λ are the Lamé parameters of the material. As damage accrues there is a loss of stiffness of the material.

To obtain a self-contained model a closure relation is still required for the damage variable. Thermodynamic considerations provide some guidance as to how to define evolution laws for $\mathcal{D}$, see e.g. [6]. However, for the sake of simplicity, we consider a different approach in order to include time irreversibility in our model. We first introduce $\widetilde{\mathcal{D}} = \widetilde{\mathcal{D}}(\epsilon, x)$, a function of the strain, which determines the *instantaneous* damage of the material. It is assumed that $\widetilde{\mathcal{D}}$ takes values in $[0, 1]$ for all admissible strains; the precise form of $\widetilde{\mathcal{D}}$ must still be specified. Irreversibility is introduced by making the damage *history dependent*. Here, we incorporate the history through the definition of a functional

$$\mathcal{D}[\epsilon] = \sup_{s \le t} \widetilde{\mathcal{D}}(\epsilon(x, s), x)\,. \tag{3}$$

Evolution of the displacement field $u(x, t) \in \mathbb{R}^3$ in a three dimensional damage model can then be formulated through the elastodynamics equations with (1) defining the stress field

$$\rho \partial_{tt}^2 u = \mathrm{div}([1 - \mathcal{D}[\epsilon]]\sigma^{(\mathrm{L})})\,. \tag{4}$$

In anticipation of obtaining a well-posed problem viscoelastic terms are added

$$\rho \partial_{tt}^2 u + \eta \partial_t u - \nu \mathrm{div} \partial_t \sigma^{(\mathrm{L})} = \mathrm{div}([1 - \mathcal{D}[\epsilon]]\sigma^{(\mathrm{L})})\,. \tag{5}$$

Due to the nonlinearity of the right-hand side it is not obvious that this formulation yields a well-posed problem.

Disorder in the material properties is modeled by allowing $\widetilde{\mathcal{D}}(\epsilon, x; \omega)$ to be a random field. This leads to a quasilinear PDE with a random coefficient. Understanding the properties of such a random coefficient partial differential equation is the primary focus of this work.

1.2. The random scalar problem

Towards the goal of studying the random coefficient PDE (5), we first reduce the displacement vector field u to the scalar function, and study the problem in one spatial dimension. In the presentation and analysis of the one dimensional model we denote the partial derivatives $u_x = \partial_x u$ and similarly $u_{xx} = \partial_{xx}^2 u$, $u_{xt} = \partial_{xt}^2 u$, etc.

The evolution is then described by the following initial-boundary value problem

$$u_{tt} + \eta u_t - \nu u_{xxt} = \partial_x[(1 - \mathcal{D}[\epsilon])u_x], \quad x \in (0, 1), t > 0, \tag{6a}$$

$$u(0, t) = 0\,, \quad u(1, t) = r(t)\,, \tag{6b}$$

where we have prescribed on the spatial domain $(0,1)$ Dirichlet boundary conditions corresponding to an uniaxial displacement. In this scalar model, the strain is $\epsilon \equiv u_x$, the elasticity constant (Young's modulus) is equal to one, and thus the linear stress is $\sigma^{(\mathrm{L})} = \epsilon$. The stress–strain relation is $\sigma[\epsilon] = (1 - \mathcal{D}[\epsilon])\epsilon$. Here, $r(t)$ denotes a 'loading' function.

In the presence of random disorder in the material strength we define the damage variable as a random field on a probability space $(\Omega, \mathcal{B}, \mathbb{P})$. We adopt the damage model (3), with the instantaneous damage function $\widetilde{\mathcal{D}} = \widetilde{\mathcal{D}}(\epsilon, x; \omega)$, allowing it to be a random field.

Indeed, we assume that the pair $(\widetilde{\mathcal{D}}(\epsilon, x; \omega), \overline{L}_{\mathcal{D}}(\omega))$ is strongly $\mathbb{P}$-measurable pair taking values in the space $C((-\epsilon_{\max}, \epsilon_{\max}) \times [0,1]) \times \mathbb{R}$ and satisfying the following properties. First,

$$0 \le \widetilde{\mathcal{D}}(\epsilon, x; \omega) \le 1, \quad \mathbb{P}-\text{a.s.} \tag{7}$$

for all $|\epsilon| < \epsilon_{\max} \le \infty$ and $x \in [0,1]$.[1] Next, it satisfies a Lipschitz condition over $(-\epsilon_{\max}, \epsilon_{\max}) \times [0,1]$

$$|\widetilde{\mathcal{D}}(\epsilon, x; \omega) - \widetilde{\mathcal{D}}(\epsilon', x; \omega)| \le L_{\mathcal{D}}(x, \omega)|\epsilon - \epsilon'|, \quad \mathbb{P}-\text{a.s.} \tag{8}$$

Lastly, the Lipschitz constant will also be measurable in $C([0,1])$, satisfying

$$0 \le L_{\mathcal{D}}(x, \omega) \le \overline{L}_{\mathcal{D}}(\omega) < \infty, \quad \mathbb{P}-\text{a.s.} \tag{9}$$

The threshold, $\epsilon_{\max}$, is introduced to make it easier to obtain a uniform in ϵ Lipschitz constant in (8). For some models it may be possible to take $\epsilon_{\max} = \infty$.

Given an $\epsilon_{\max} \le \infty$ and an upper bound on the Lipschitz constant, $\overline{L}_{\mathcal{D}}$, we define an admissible class of damage variables by $\mathcal{A}_{\mathcal{D}} \equiv \mathcal{A}_{\mathcal{D}}(\epsilon_{\max}, \overline{L}_{\mathcal{D}})$. In the random case, we assume that strongly $\mathbb{P}$-measurable pair $(\widetilde{\mathcal{D}}, \overline{L}_{\mathcal{D}})$ satisfies $\widetilde{\mathcal{D}}(\cdot, \cdot; \omega) \in \mathcal{A}_{\mathcal{D}}(\epsilon_{\max}, \overline{L}_{\mathcal{D}}(\omega))$ a.s.

The main goal of the present work is to make progress towards showing that (6a) with an instantaneous random damage model in the admissible class, $\mathcal{A}_{\mathcal{D}}$, is well-posed in some sense.

We give two examples of models of $\widetilde{\mathcal{D}}$.

Example 1.1: Let

$$\widetilde{\mathcal{D}}(\epsilon, x; \omega) = \left(1 + \exp\left\{-\frac{\epsilon - \epsilon_\star(x, \omega)}{\Delta\epsilon(x; \omega)}\right\}\right)^{-1} \tag{10}$$

where $\epsilon_\star$ and $\Delta\epsilon$ are strongly $\mathbb{P}$-measurable positive random fields taking values in $C([0,1])$, satisfying, $\mathbb{P}$-a.s.

$$0 < \underline{\epsilon}_\star(\omega) \le \epsilon_\star(x; \omega) \le \overline{\epsilon}_\star(\omega) < \infty, \tag{11a}$$

$$0 < \underline{\Delta\epsilon}(\omega) \le \Delta\epsilon(x; \omega) \le \overline{\Delta\epsilon}(\omega) < \infty. \tag{11b}$$

Here, $\epsilon_\star$ is a peak strain and $\Delta\epsilon$ is the strain softening scale. Obviously, this satisfies (7). Furthermore, (8) and (9) hold

$$L_{\mathcal{D}}(x; \omega) = \frac{1}{4}\frac{1}{\Delta\epsilon(x; \omega)} \le \frac{1}{4}\frac{1}{\underline{\Delta\epsilon}(\omega)} \equiv \overline{L}_{\mathcal{D}}(\omega) < \infty. \tag{12}$$

$\widetilde{\mathcal{D}}$ and $L_{\mathcal{D}}$ inherit strong measurability from $\epsilon_\star$ and $\Delta\epsilon$. This is an example where $\epsilon_{\max} = +\infty$ will be satisfactory.

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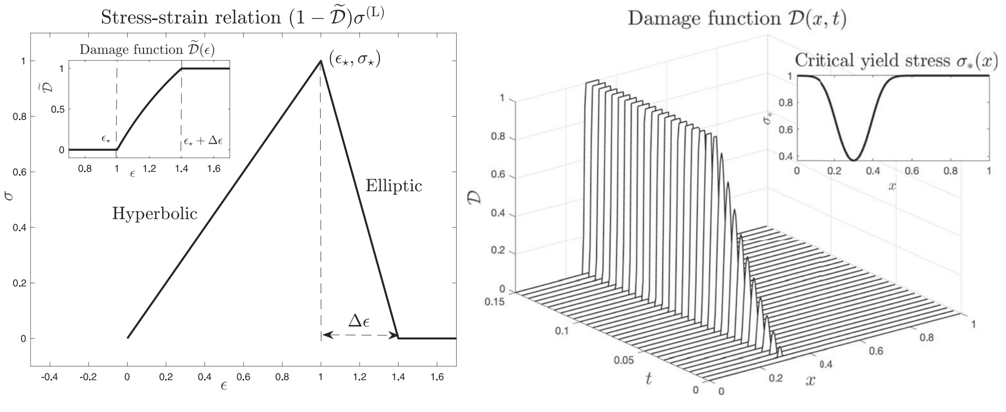

(a) (b)

Figure 1. (a) A sketch of the stress–strain relation (based on (13)) at a fixed point $x \in [0,1]$, also marking the type of the elastodynamics differential equation. The inset depicts the dependence of the damage function $\widetilde{\mathcal{D}}(\epsilon)$ on the strain. (b) The damage function $\mathcal{D}(\epsilon(x,t),x)$ in a consistency test simulation of Section 3 corresponding to the spatial variation of the critical stress $\sigma_\star(x)$ depicted in the inset.

Example 1.2: A simple linear damage model (Figure 1) that has these properties and is used in numerical experiments in Section 3 is given by

$$\widetilde{\mathcal{D}}(\epsilon,x;\omega)=\begin{cases}0 & \epsilon\le\epsilon_\star(x;\omega),\\ \dfrac{\epsilon_\star(x;\omega)+\Delta\epsilon(x;\omega)}{\Delta\epsilon(x;\omega)}\left(1-\dfrac{\epsilon_\star(x;\omega)}{\epsilon}\right) & \epsilon_\star(x;\omega)<\epsilon\le\epsilon_\star(x;\omega)+\Delta\epsilon(x;\omega),\\ 1 & \epsilon>\epsilon_\star(x;\omega)+\Delta\epsilon(x;\omega),\end{cases} \tag{13}$$

where $\epsilon_\star(x;\omega)$ and $\Delta\epsilon(x;\omega)$ are given positive random fields satisfying (11a). A Lipschitz constant bound can be obtained

$$\overline{L}_{\mathcal{D}}(\omega)=\frac{\overline{\epsilon}_\star(\omega)+\overline{\Delta\epsilon}(\omega)}{\underline{\epsilon}_\star(\omega)\underline{\Delta\epsilon}(\omega)}\,. \tag{14}$$

This is another example where we may take $\epsilon_{\max}=+\infty$. In computations in Section 3 we have worked in the stress coordinate instead of strain, and we only randomized the peak stress which corresponds to $\epsilon_\star$.

Overview of main results. The main aim of this work is to establish well-posedness results for solution to (6a), subject to suitable assumptions on the nonlinearity. This, in turn, will help justify numerical simulations on this type of visco-elastic random models. The main results are contained in Theorem 2.11, Theorem 2.21, and Corollary 2.24.

To summarize: For the model (6a), given sufficiently regular initial conditions that are compatible with the boundary conditions, a transformed problem, (15), is almost surely well-posed and strongly $\mathbb{P}$-measurable. The solutions to the transformed problem ensure that the original system (6a) has, almost surely, weak solutions.

Note that this is a result that holds almost surely allowing for randomness in the damage model, and thus accounting for uncertainty in the material properties. The moments of the solution and its time of existence are also bounded. While we were unable to obtain a full well-posedness result in the original variables, (u,u_t), obtaining only existence of the solutions, numerical experiments in these coordinates suggest that sufficient regularity is present for well-posedness.

The analysis is built upon prior work used in [8,9] in which a clever change of variables maps (6a) into a semilinear parabolic equation. A key novelty of this work is that, in contrast to these earlier works on viscoelastic materials, we include a history dependent nonlinearity that precludes spontaneous 'healing' of the material; once broken, it cannot unbreak.

We also propose a numerical method based on finite differences in space and the Newmark's integration scheme combined with Monte Carlo sampling, that allows us to explore the behavior of (6a) numerically. This reveals that there are notable differences, on average, in the evolution of the material depending on the nature of the randomness (e.g. the spatial correlation length).

2. Existence and uniqueness of solutions

To make progress, we first reformulate (6a) as a semilinear partial differential equation with a random coefficient in the definition of its right-hand side. More precisely after a transformation of unknowns (u, u_t) we study the first-order system in the new coordinates $z = (p, q)$ (see (16a))

$$\frac{\mathrm{d}z}{\mathrm{d}t} + Az = \mathbf{f}[z, r] + \mathbf{g}(t), \quad t > 0, \quad \text{with the initial condition } z(0) = z_0. \tag{15}$$

Introducing the random pair $(\widetilde{\mathcal{D}}, \overline{L}_{\mathcal{D}})$ on a probability space $(\Omega, \mathcal{B}, \mathbb{P})$, as discussed in Section 1.2, we arrive at an abstract evolution equation with a random right-hand side $\mathbf{f}[\mathbf{z}, \mathbf{r}; \omega]$. This random evolution equation will be studied path-wise, i.e. for almost all realizations ω.

2.1. Transformation to semi-linear parabolic system

The aforementioned change of coordinates is based on [9] and was also used previously in multi-dimensional setting, e.g. in [10,11] to study long-time behavior of nonlinear viscoelasticity with non-monotone stress–strain function. We formally write

$$p(x,t) = \int_0^x u_t(y,t) - r'(t)y \,\mathrm{d}y - \int_0^1 \int_0^x u_t(y,t) - r'(t)y \,\mathrm{d}y \,\mathrm{d}x \tag{16a}$$

$$q(x,t) = \nu(u_x(x,t) - r(t)) - p(x,t). \tag{16b}$$

For suitably regular p and q, subject to $\int_0^1 p(x,t)\,\mathrm{d}x = \int_0^1 q(x,t)\,\mathrm{d}x = 0$ and

$$u_x = \nu^{-1}(p+q) + r(t)\,. \tag{17}$$

Thus, u can be recovered by integration while u_t can be recovered by differentiation. In addition, p satisfies the Neumann boundary conditions

$$p_x(0,t) = u_t(0,t) = 0\,, \tag{18a}$$

$$p_x(1,t) = u_t(1,t) - r'(t) = 0\,. \tag{18b}$$

No boundary conditions are imposed on q.

One can check that the new coordinate $z = (p, q)$ satisfies the semilinear Equation (15). The linear term is (with the identity operator denoted I)

$$A = \begin{pmatrix} \eta I - \nu\partial_{xx} & 0 \\ 0 & I \end{pmatrix}, \tag{19}$$

the nonlinear term becomes

$$\begin{aligned}\mathbf{f}[z,r] &= \begin{pmatrix} \sigma[\nu^{-1}(p+q)+r] - \int_0^1 \sigma[\nu^{-1}(p+q)+r]\,\mathrm{d}x \\ q + \eta p - \sigma[\nu^{-1}(p+q)+r] + \int_0^1 \sigma[\nu^{-1}(p+q)+r]\,\mathrm{d}x \end{pmatrix} \\ &= \begin{pmatrix} f_1[z,r] \\ q + \eta p - f_1[z,r] \end{pmatrix}, \end{aligned} \tag{20}$$

and the forcing term is

$$\mathbf{g}(t) = (\eta r'(t) + r''(t)) \begin{pmatrix} \frac{1}{6} - \frac{x^2}{2} \\ -\frac{1}{6} + \frac{x^2}{2} \end{pmatrix} = \begin{pmatrix} g_1(x,t) \\ -g_1(x,t) \end{pmatrix}. \tag{21}$$

The analysis of (15) proceeds along the well-established lines developed for semilinear PDEs (e.g. [12–14]) with one non-standard feature arising from the history dependence of $\mathcal{D}$. To accommodate randomness in the problem, we treat the solutions as of (15) as random variables on the probability space $(\Omega, \mathcal{B}, \mathbb{P})$ with values in a space-time Banach space. For a given realization of $(\widetilde{\mathcal{D}}, \overline{L}_{\mathcal{D}})$ we analyse the solutions to the problem (15) as functions $z : [0, T] \to X$ with values in the Banach space

$$X = \left\{ (p, q) \in L^2(0,1) \times C([0,1]) \mid \int_0^1 p(x,t)\,\mathrm{d}x = \int_0^1 q(x,t)\,\mathrm{d}x = 0 \right\}. \tag{22}$$

The space X is equipped with the usual sum norm

$$\|z\|_X = \|z_1\|_{L^2} + \|z_2\|_C \,. \tag{23}$$

The domain of the operator A, (19),

$$D(A) = \{(p,q) \in X \mid p \in H^2(0,1), p_x(0) = 0, p_x(1) = 0,\ q \in C([0,1])\} = H^2_b \times C_a \,.$$

Due to the diagonal structure of the operator $A = \mathrm{diag}\{A_1, I\}$ we have the fractional power operator $A^\alpha = \{A_1^\alpha, I\}$ with the domain $D(A^\alpha) = D(A_1^\alpha) \times C_a$ with $0 < \alpha < 1$. The operator A induces an analytic semigroup, and we make use of its properties. For any $\theta \in (0, 1]$

$$\|A^\theta \mathrm{e}^{-tA}\| \le a_\theta t^{-\theta} \,, \tag{24}$$

$$\|(\mathrm{e}^{-tA} - I)A^{-\theta}\| \le b_\theta t^\theta \,. \tag{25}$$

In the case of our problem, the constant a_θ is uniform over $\eta > 0$ and $\nu > 0$, see [14, (2.128), (2.129)] for details.

Recall that $D(A_1^\alpha)$ can be defined as the closure of $D(A_1)$ with respect to the norm

$$\|A_1^\alpha f\|^2_{L^2} \equiv \|f\|^2_{H^{2\alpha}} = \sum_{k=1}^{\infty} \lambda_k^{2\alpha} |\langle f, \varphi_k \rangle|^2 \tag{26}$$

where a standard calculation reveals that

$$\lambda_k = \eta + \nu(k\pi)^2, \quad \varphi_k(x) = \begin{cases} 1 & k = 0, \\ \sqrt{2}\cos(k\pi x) & k > 0. \end{cases} \tag{27}$$

We denote the spaces $D(A_1^\alpha) = X_1^\alpha \subset H^{2\alpha}(0,1)$. These are Sobolev type spaces, and for $\alpha > 1/4$, $H^{2\alpha}(0,1) \hookrightarrow C([0,1])$. For $\alpha > 3/4$, $H^{2\alpha}(0,1) \hookrightarrow C^1([0,1])$ and these functions will satisfy the Neumann boundary conditions. We will always identify elements of $H^{2\alpha}$, for $\alpha > 1/4$, with their continuous versions.

We define $X^\alpha = D(A^\alpha) = X_1^\alpha \times X_2$ equipped with the sum norm $\|z\|_{X^\alpha} = \|z_1\|_{H^{2\alpha}} + \|z_2\|_C$. The space $C([0,T]; X^\alpha)$ will also be used with the norm

$$\|f\|_{X_T^\alpha} = \sup_{t \le T} \|f(t)\|_{X^\alpha} \,. \tag{28}$$

We will also make use of the spaces $H_T^{2\alpha} = C([0,T]; H^{2\alpha})$ and $C_T = C([0,T]; C([0,1])) \equiv C([0,T] \times [0,1])$, both equipped with the sup norms; the spatial norm is denoted $\|f\|_C = \sup_x |f(x)|$, and the space-time norm is $\|\cdot\|_{C_T}$. We will also need to control $r(t) \in C^k([0,T])$; this will use the sup norm $|r|_{C_T^k} \equiv \sum_{\ell=0}^{k} \sup_{t\in[0,T]} |\partial^\ell r(t)|$ for $k \geq 0$.

Remark 2.1: We note that the spaces $C([0,T];C([0,1]))$ and $C([0,T]\times[0,1])$ are isometric, and will use them interchangeably. See Lemma A.3 for a proof.

We are now in the position to define notions of solutions to (15).

Definition 2.2 (Mild solution): Given $z_0 \in X^\alpha$ the function $z \in C([0,T];X^\alpha)$ is a mild solution of (15) if it solves the integral equation

$$z(t) = \mathrm{e}^{-tA} z_0 + \int_0^t \mathrm{e}^{-(t-s)A}(\mathbf{f}[z,r](s) + \mathbf{g}(s))\,\mathrm{d}s, \quad 0 \leq t \leq T, \tag{29}$$

with equality holding in the sense of X^α.

Definition 2.3 (Strong solution): Given $z_0 \in X^\alpha$, the function

$$z \in C([0,T];X) \cap C([0,T];X^\alpha) \cap C^1((0,T];X) \cap C((0,T];X^1) \tag{30}$$

is a strong solution if it satisfies (15) with equality holding in the sense of X for $0 < t \leq T$ and $z(0) = z_0$.

Finally, we use strong solutions of (15) with $\alpha > 1/4$ to introduce a weak solution.

Definition 2.4 (Weak solution): A function

$$u \in C([0,T];C^1([0,1])) \cap C^1((0,T];H^1([0,1])) \cap C^2((0,T];H^{-1}) \tag{31}$$

is a weak solution of (6a) provided:

(i) for all $0 < t \leq T$

$$\langle u_{tt},\varphi\rangle = -\eta\,\langle u_t,\varphi\rangle - \nu\,\langle u_{tx},\varphi_x\rangle - \langle\sigma,\varphi_x\rangle \quad \text{for all } \varphi \in H_0^1([0,1]), \tag{32}$$

(ii) the initial conditions $u(0) = u_0$, $u_t(0) = u_1$, and the boundary conditions (6b).

For all of these notions of solutions, when we permit $(\widetilde{\mathcal{D}}, \overline{L}_{\mathcal{D}})$ to be a strongly $\mathbb{P}$-measurable pair, a.s. in the admissible class, $\mathcal{A}_{\mathcal{D}}(\epsilon_{\max}, \overline{L}_{\mathcal{D}}(\cdot))$, we will obtain strongly measurable $z(\omega)$ taking values in the above function spaces.

In order to prove local in time existence we have to first establish boundedness and Lipschitz continuity of $\mathbf{f} : C([0,T];X^\alpha) \to C([0,T];X)$. Observe that $\mathbf{f}$ is examined as a mapping over the time-space function space because of the history dependence. Before studying the mapping $\mathbf{f}$ we need to establish properties of the random damage function and the stress.

2.2. Random damage function

We first derive estimates on the damage function $\mathcal{D}$ related to a deterministic instantaneous damage $\widetilde{\mathcal{D}}$ in the admissible class $\mathcal{A}_{\mathcal{D}}(\epsilon_{\max}, \overline{L}_{\mathcal{D}})$ for deterministic $0 \leq \overline{L}_{\mathcal{D}} < \infty$ and $0 < \epsilon_{\max} \leq \infty$. When $(\widetilde{\mathcal{D}}, L_{\mathcal{D}})$ is a strongly $\mathbb{P}$-measurable pair the properties will be inherited $\mathbb{P}$-almost surely. In the sequel, we use the notation $\mathcal{D}[\epsilon]$ to emphasize that $\mathcal{D}$ is viewed as a mapping between Banach spaces.

Lemma 2.5: *Given* $\epsilon_{\max} > 0$ *consider all* $\epsilon \in C([0,T];C([0,1]))$ *such that* $\|\epsilon\|_{C_T} < \epsilon_{\max}$. *Then for any* $\widetilde{\mathcal{D}} \in \mathcal{A}_{\mathcal{D}}(\epsilon_{\max}, \overline{L}_{\mathcal{D}})$ *we have* $\mathcal{D}[\epsilon] \in C([0,T];C([0,1]))$ *and* $0 \le \mathcal{D}[\epsilon](x,t) \le 1$ *for any* $t \le T$ *and* $x \in [0,1]$.

If ϵ' *satisfies the same assumptions, then for any* $t \le T$ *we have*

$$\|\mathcal{D}[\epsilon](t) - \mathcal{D}[\epsilon'](t)\|_C \le \overline{L}_{\mathcal{D}} \|\epsilon - \epsilon'\|_{C_t}.$$

Note that in in the above result we allow $t \le T$ defining the norm to vary

$$\|\epsilon\|_{C_t} = \sup_{s \le t} \|\epsilon(s)\|_C.$$

Proof: Using the assumed continuity of ϵ and $\widetilde{\mathcal{D}}$, $\widetilde{\mathcal{D}}(\epsilon(\cdot,\cdot),\cdot) \in C([0,T];C([0,1])$. Now, $\widetilde{\mathcal{D}}(\epsilon(\cdot,\cdot),\cdot) \in C([0,T];C([0,1])) = C([0,1];C([0,T]))$. Hence, for each x, the following is well-defined:

$$\mathcal{D}[\epsilon](x,t) = \sup_{s \le t} \widetilde{\mathcal{D}}(\epsilon(x,s),x).$$

By Lemma A.1, for each x, $\mathcal{D}[\epsilon](x,\cdot) \in C([0,T])$ and for $t \le T$ and $x \in [0,1]$

$$\mathcal{D}[\epsilon](x,t) = \sup_{s \le t} \widetilde{\mathcal{D}}(\epsilon(x,s),x) \in [0,1].$$

Next, we check that this is continuous in x

$$|\mathcal{D}[\epsilon](x,t) - \mathcal{D}[\epsilon](x',t)| \le \sup_{s \le t} |\widetilde{\mathcal{D}}(\epsilon(x,s),x) - \widetilde{\mathcal{D}}(\epsilon(x',s),x')| \tag{33}$$

Since $\widetilde{\mathcal{D}}(\epsilon(\cdot,\cdot),\cdot) \in C([0,T];C([0,1])) = C([0,T] \times [0,1])$ is uniformly continuous in its arguments, for any $\varepsilon > 0$, there exists $\delta > 0$, such that if $|x - x'| + |t - t'| < \delta$, then

$$|\widetilde{\mathcal{D}}(\epsilon(x,t),x) - \widetilde{\mathcal{D}}(\epsilon(x',t'),x')| < \varepsilon.$$

In (33), $t = t' = s$, so for any $\varepsilon > 0$, if $|x - x'| < \delta$ for the corresponding δ,

$$|\mathcal{D}[\epsilon](x,t) - \mathcal{D}[\epsilon](x',t)| \le \varepsilon,$$

and we even have

$$\sup_{t \le T} |\mathcal{D}[\epsilon](x,t) - \mathcal{D}[\epsilon](x',t)| \le \varepsilon.$$

Thus, $x \mapsto \mathcal{D}[\epsilon](x,\cdot)$ is a $C([0,1];C([0,T])) = C([0,T];C([0,1]))$ mapping. We conclude $\mathcal{D}[\epsilon] \in C([0,T];C([0,1]))$.

Next, we show continuity of the functional $\epsilon \mapsto \mathcal{D}[\epsilon]$. At any fixed $x \in [0,1]$ and $t \in [0,T]$

$$\begin{aligned} |\mathcal{D}[\epsilon](x,t) - \mathcal{D}[\epsilon'](x,t)| &\le \sup_{s \le t} |\widetilde{\mathcal{D}}(\epsilon(x,s),x) - \widetilde{\mathcal{D}}(\epsilon'(x,s),x)| \\ &\le \overline{L}_{\mathcal{D}} \sup_{s \le t} |\epsilon(x,s) - \epsilon'(x,s)| \le \overline{L}_{\mathcal{D}} \|\epsilon - \epsilon'\|_{C_t} \end{aligned}$$

Taking the supremum over x yields the result. ■

2.3. *Estimates on the stress function*

Establishing properties of the damage function allows us to address the properties of the stress function $\sigma[\epsilon] = (1 - \mathcal{D}[\epsilon])\epsilon$.

Lemma 2.6: *Given* $\epsilon_{\max} > 0$ *consider all* $\epsilon \in C([0,T]; C([0,1]))$ *such that* $\|\epsilon\|_{C_T} < \epsilon_{\max}$, *and assume* $\widetilde{\mathcal{D}} \in \mathcal{A}_{\mathcal{D}}(\epsilon_{\max}, \overline{L}_{\mathcal{D}})$. *Then* $\sigma[\epsilon] \in C([0,T]; C([0,1]))$ *and for* $t \leq T$

$$\|\sigma[\epsilon](t)\|_C \leq \|\epsilon(t)\|_C ,$$

If ϵ' *satisfies the same assumptions, then*

$$\|\sigma[\epsilon](t) - \sigma[\epsilon'](t)\|_C \leq (1 + \overline{L}_{\mathcal{D}}(\|\epsilon\|_{C_T} + \|\epsilon'\|_{C_T}))\|\epsilon - \epsilon'\|_{C_t}$$

Proof: From Lemma 2.5 we have $\mathcal{D}[\epsilon] \in [0,1]$ and so is $1 - \mathcal{D}[\epsilon]$, hence $\|\sigma[\epsilon](t)\|_C \leq \|\epsilon(t)\|_C$ follows immediately. Next, since $C([0,T]; C([0,1]))$ is an algebra, then $(1 - \mathcal{D}[\epsilon)]\epsilon$ is in it. To obtain the Lipschitz estimate we have for any $x \in [0,1]$ and $t \in [0,T]$

$$\begin{aligned} |\sigma[\epsilon](x,t) - \sigma[\epsilon'](x,t)| &\leq |(1 - \mathcal{D}[\epsilon](x,t))(\epsilon(x,t) - \epsilon'(x,t))| \\ &\quad + |(\mathcal{D}[\epsilon](x,t) - \mathcal{D}[\epsilon'](x,t))\epsilon'(x,t)| \\ &\leq |1 - \mathcal{D}[\epsilon](x,t)||\epsilon(x,t) - \epsilon'(x,t)| \\ &\quad + |\epsilon'(x,t)||\mathcal{D}[\epsilon](x,t) - \mathcal{D}[\epsilon'](x,t)| , \end{aligned}$$

and using Lemma 2.5 we have

$$\begin{aligned} \|\sigma[\epsilon](t) - \sigma[\epsilon'](t)\|_C &\leq \|\epsilon(t) - \epsilon'(t)\|_C + \|\epsilon'\|_{C_T}\|\mathcal{D}[\epsilon](t) - \mathcal{D}[\epsilon'](t)\|_C \\ &\leq (1 + \overline{L}_{\mathcal{D}}(\|\epsilon\|_{C_T} + \|\epsilon'\|_{C_T}))\|\epsilon - \epsilon'\|_{C_t} , \end{aligned}$$

and the result follows. ■

2.4. *Estimates on the nonlinearity and forcing terms*

In this section we have that $z \in C([0,T]; X^\alpha)$, so that, after embedding, $p, q \in C([0,T]; C([0,1]))$ and we also assume $r \in C([0,T])$. Consequently, the strain is

$$\epsilon = \nu^{-1}(p + q) + r \in C([0,T]; C([0,1])) , \tag{34}$$

and we have, for any $t \in [0,T]$, $\nu \leq \nu_0 < \infty$, the bound

$$\|\epsilon(t)\|_C \leq c(\alpha, \nu_0)\nu^{-1}(\|z(t)\|_{X^\alpha} + |r(t)|) \tag{35}$$

We are now ready to prove Lipschitz results on the nonlinearity $\mathbf{f}$.

Proposition 2.7: *For any* $\alpha > 1/4$, *let* $z \in X^\alpha_T$, *and* $r \in C([0,T])$. *Assume* $\epsilon_{\max}$ *is sufficiently large that*

$$\nu^{-1}\max\{c_\alpha, 1\}\|z\|_{X^\alpha_T} + |r|_{C_T} < \epsilon_{\max} ,$$

where c_α *is the Sobolev embedding constant of* $H^{2\alpha} \hookrightarrow C$. *Also assume* $\widetilde{\mathcal{D}} \in \mathcal{A}_{\mathcal{D}}(\epsilon_{\max}, \overline{L}_{\mathcal{D}})$. *Then* $\mathbf{f}[z,r] \in X_T$ *and there exists a constant c > 0 such that for any* $t \leq T$

$$\|\mathbf{f}[z,r](t)\|_X \leq c((\nu^{-1} + \eta + 1)\|z(t)\|_{X^\alpha} + |r(t)|)$$

Furthermore, let $z, z' \in X^\alpha_T$, $r \in C([0,T])$, *and*

$$\epsilon_{\max} > \nu^{-1}\max\{c_\alpha, 1\}\max\{\|z\|_{X^\alpha_T}, \|z'\|_{X^\alpha_T}\} + |r|_{C_T} .$$

Then there exists a constant $\tilde{c} > 0$ *such that for* $t \leq T$

$$\begin{aligned}\|\mathbf{f}[z,r](t)-\mathbf{f}[z,r'](t)\|_X &\le \tilde{c}[1+\eta+\nu^{-1}(1+\nu^{-1}\bar{L}_{\mathcal{D}}(\|z\|_{X_T^\alpha}+\|z'\|_{X_T^\alpha}+|r|_{C_T}))]\\ &\quad\times\|z-z'\|_{X_t^\alpha}\end{aligned}$$

The constants depend only upon α, ν_0, and η_0.

Remark 2.8: As we are also interested in the small regularization limit, $\nu \to 0$ and $\eta \to 0$, assuming $\nu \le \nu_0 < \infty$ and $\eta \le \eta_0 < \infty$, the estimates can be written as

$$\begin{aligned}\|\mathbf{f}[z,r](t)\|_X &\le c\nu^{-1}(\|z\|_{X_t^\alpha}+|r|_{L_t^\infty})\\ \|\mathbf{f}[z,r](t)-\mathbf{f}[z,r'](t)\|_X &\le c'\nu^{-2}(c''+\bar{L}_{\mathcal{D}}(\|z\|_{X_T^\alpha}+\|z'\|_{X_T^\alpha}+|r|_{C_T}))\|z-z'\|_{X_t^\alpha}\end{aligned}$$

where the constants, c, c', and c'' depend upon α, η_0, ν_0, but not on the other quantities.

Proof: Letting $\epsilon(t)=\nu^{-1}(p(t)+q(t))+r(t)$, by the embedding of $p(t)\in H^{2\alpha}\hookrightarrow C([0,1])$, we have that $\epsilon\in C([0,T];C([0,1]))$, so the stress $\sigma[\epsilon]\in C([0,T];C([0,1]))$ too. Consequently, $\int_0^1\sigma[\epsilon](y,t)dy\in C([0,T])$, and all of the components of $\mathbf{f}$, (20), are in $C([0,T];C([0,1]))$. It is also clear that $\int_0^1 f_1(r,z)dy=0$. Since p and q are also mean zero, we have $\int_0^1 f_2(r,z)dy=0$. We conclude that $\mathbf{f}(r,z)\in C([0,T];C([0,1]))\hookrightarrow C([0,T];X)\equiv X_T$.

Next, we denote $\bar{\sigma}[\epsilon](t)=\int_0^1\sigma(\epsilon(x,t))\,dx$, and we have for any $t\le T$

$$\begin{aligned}\|\mathbf{f}[z,r](t)\|_X &\le \|\sigma[\epsilon](t)-\bar{\sigma}[\epsilon](t)\|_{L^2}+\|\sigma[\epsilon](t)-\bar{\sigma}[\epsilon](t)\|_C+\|q(t)\|_C+\eta\|p(t)\|_C\\ &\le 4\|\sigma[\epsilon](t)\|_C+\|q(t)\|_C+\eta c_\alpha\|p(t)\|_{H^{2\alpha}}\,.\end{aligned}$$

Next, using Lemma 2.6 and (35)

$$\|\sigma[\epsilon](t)\|_C\le\|\epsilon(t)\|_C\le c\nu^{-1}(\|z(t)\|_{X^\alpha}+|r(t)|)$$

Combining these estimates completes the bound on $\mathbf{f}[z,r](t)$.

To get the Lipschitz bound, for any $t\in[0,T]$,

$$\begin{aligned}\|\mathbf{f}[z,r](t)-\mathbf{f}[z',r](t))\|_X &\le \|f_1[z,r](t)-f_1[z',r](t)\|_{L^2}+\|f_2[z,r](t)-f_2[z',r](t)\|_C\\ &\le 2\|f_1[z,r](t)-f_1[z',r](t)\|_C+\eta\|p(t)-p'(t)\|_C+\|q(t)-q'(t)\|_C\\ &\le 4\|\sigma[\epsilon](t)-\sigma[\epsilon'](t)\|_C+C_\alpha\eta\|p(t)-p'(t)\|_{H^{2\alpha}}+\|q(t)-q'(t)\|_C\\ &\le 4\|\sigma[\epsilon](t)-\sigma[\epsilon'](t)\|_C+\max\{c_\alpha\eta,1\}\|z(t)-z'(t)\|_{X^\alpha}\end{aligned}$$

Next, using (35)

$$\|\epsilon(t)\|_C\le\nu^{-1}\max\{c_\alpha,1\}\|z\|_{X_T^\alpha}+|r|_{C_T}$$

and

$$\begin{aligned}\|\epsilon(t)-\epsilon'(t)\|_C &\le \nu^{-1}(\|p(t)-p'(t)\|_C+\|q(t)-q'(t)\|_C)\\ &\le \nu^{-1}\max\{c_\alpha,1\}\|z(t)-z'(t)\|_{X^\alpha}\end{aligned}$$

Thus, using Lemma 2.6, we get

$$\begin{aligned}\|\sigma[\epsilon](t)-\sigma[\epsilon'](t)\|_C &\le c(1+\bar{L}_D\nu^{-1}\max\{c_\alpha,1\}(\|z\|_{X_T^\alpha}+\|z'\|_{X_T^\alpha}+|r|_{C_T}))\\ &\quad\times\nu^{-1}\|z-z'\|_{X_t^\alpha}.\end{aligned}$$

Combining these estimates we have the result. ■

Proposition 2.9: *For* $r \in C^2([0,T])$, $\mathbf{g} \in X_T$, *and*

$$\|g(t)\|_X \leq c(|r'(t)| + |r''(t)|)$$

Proof: Recall the definition of **g**, (21). The proof is immediate since the spatial components are polynomials in x. ■

2.5. Well-posedness of the deterministic evolution

We first establish existence and continuity with respect to data for a deterministic instantaneous damage function $\widetilde{\mathcal{D}}$ and then generalize them for a random $\widetilde{\mathcal{D}}$.

In formulating and proving the results we impose several conditions on the initial conditions and the damage function $\widetilde{\mathcal{D}}$.

Assumption 2.10: Assumptions on the data:

(A1) Given $Z_0 > 0$, assume $z_0 \in X^\alpha$ with $\alpha \in (1/4, 1)$ satisfies $\|z_0\|_{X^\alpha} \leq \frac{1}{2} Z_0/a_0$ where the constant a_0 is from (24).

(A2) The boundary condition satisfies $r \in C^2(0, T_r)$, and there exists $\epsilon_{\max}$ such that

$$\nu^{-1} \max\{c_\alpha, 1\} Z_0 + |r|_{C_{T_r}} < \epsilon_{\max}\,. \tag{36}$$

(A3) The instantaneous damage function $\widetilde{\mathcal{D}} \in \mathcal{A}_{\mathcal{D}}(\epsilon_{\max}, \overline{L}_{\mathcal{D}})$.

(A4) The viscosity parameters $\nu \leq \nu_0 < \infty$, and $\eta \leq \eta_0 < \infty$, for some positive ν_0, η_0.

2.5.1. Mild solutions

First, we prove the existence of a mild solution.

Theorem 2.11 (Mild Solutions): *Assume* (A1)–(A4) *then there exist positive constants c and* c', *independent of* ν *and* $\overline{L}_{\mathcal{D}}$, *such that for*

$$T_e = c \min\left\{ \frac{\nu Z_0}{Z_0 + |r|_{C_{T_r}}}, \frac{\nu^2}{c' + \overline{L}_{\mathcal{D}}(Z_0 + |r|_{C_{T_r}})} \right\}^{\frac{1}{1-\alpha}} > 0\,, \tag{37}$$

there exists a unique mild solution to the transformed problem (15), $z \in C([0, T_e]; X^\alpha)$. *The constants depend only upon* α, ν_0, *and* η_0.

Remark 2.12: Because $\epsilon_{\max}$ may be finite, in the uniaxial strain setting, if $r(t) \nearrow \infty$, we cannot have a global in time solution. Thus, we introduce T_r, which may be thought of as a maximum simulation time. If $\epsilon_{\max} = \infty$, we can take $T_r = \infty$.

Remark 2.13: We take some care in arriving at the time of existence in (37), as it will be essential to track the dependence upon $\overline{L}_{\mathcal{D}}$ when it is a random variable.

Remark 2.14: The time of existence that we derive in (37) is a *local* time of existence. We do not address the maximal time of existence in this work.

Proof: The proof follows the standard strategy based on Banach fix point theorem by showing that the right-hand side (29) is a contraction on a suitably chosen set. In the sequel we fix a time $0 < T_r < \infty$

(1) For any $T \in (0, T_r]$ we define the set

$$\mathcal{K}(T, Z_0) = \left\{ z \in X_T^\alpha \mid \|z\|_{X_T^\alpha} \le Z_0 \right\} . \tag{38}$$

Using (35), for any $z \in \mathcal{K}(T, Z_0)$, the corresponding strain satisfies

$$\|\epsilon(t)\|_C \le \|\epsilon\|_{C_T} \le \nu^{-1} \max\{c_\alpha, 1\} Z_0 + |r|_{C_{T_r}} < \epsilon_{\max} ,$$

hence $\widetilde{\mathcal{D}}$ is well defined on $\mathcal{K}(T, Z_0)$. On this set we define the mapping

$$\mathcal{J}[z](t) = \mathrm{e}^{-tA} z_0 + \int_0^t \mathrm{e}^{-(t-s)A} \mathbf{f}[z, r](s) \, \mathrm{d}s + \int_0^t \mathrm{e}^{-(t-s)A} \mathbf{g}(s) \, \mathrm{d}s \tag{39}$$

(2) First, we verify that for $\theta \in [\alpha, 1)$ the mapping $\mathcal{J}$ maps $\mathcal{K}(T, Z_0)$ into itself. Indeed,

$$\begin{aligned}
\|\mathcal{J}[z](t)\|_{X^\theta} &\le \|\mathrm{e}^{-tA} z_0\|_{X^\theta} + \int_0^t \|\mathrm{e}^{-(t-s)A} \mathbf{f}[z, r](s)\|_{X^\theta} \, \mathrm{d}s \\
&\quad + \int_0^t \|\mathrm{e}^{-(t-s)A} \mathbf{g}(s)\|_{X^\theta} \, \mathrm{d}s \\
&\le \|A^{\theta-\alpha} \mathrm{e}^{-tA}\| \|A^\alpha z_0\|_X + \int_0^t \|A^\theta \mathrm{e}^{-(t-s)A}\|_X \|\mathbf{f}[z, r](s)\|_X \, \mathrm{d}s \\
&\quad + \int_0^t \|A^\theta \mathrm{e}^{-(t-s)A}\|_X \|\mathbf{g}(s)\|_X \, \mathrm{d}s
\end{aligned}$$

By Proposition 2.7 and Remark 2.8 we have

$$\|\mathbf{f}[z, r](s)\|_X \le c\nu^{-1} (\|z(s)\|_{X^\alpha} + |r(s)|) \le c\nu^{-1} (Z_0 + |r|_{C_{T_r}}) ,$$

and by Proposition 2.9 $\|\mathbf{g}(s)\| \le c|r|_{C_T^2}$. Thus, for $t > 0$,

$$\begin{aligned}
\|\mathcal{J}[z](t)\|_{X^\theta} &\le a_{\theta-\alpha} t^{-(\theta-\alpha)} \|z_0\|_{X^\alpha} + \int_0^t c(Z_0 + |r|_{C_T}) a_\theta (t-s)^{-\theta} \, \mathrm{d}s \\
&\quad + \int_0^t c|r|_{C_{T_r}^2} \, a_\theta (t-s)^{-\theta} \, \mathrm{d}s \\
&\le a_{\theta-\alpha} t^{-(\theta-\alpha)} \|z_0\|_{X^\alpha} + \frac{a_\theta}{1-\theta} c(Z_0 + |r|_{C_{T_r}^2}) t^{1-\theta} .
\end{aligned} \tag{40}$$

In the case that $\theta = \alpha$ we can take this down to $t = 0$, and for small enough T, $\|\mathcal{J}[z]\|_{X_T^\alpha} \le Z_0$

$$\frac{a_\alpha}{1-\alpha} T^{1-\alpha} c\nu^{-1} (Z_0 + |r|_{C_{T_r}^2}) \le \frac{1}{2} Z_0 . \tag{41}$$

Lastly, since the semigroup e^{-tA} preserves the mean zero property, and the components of $\mathbf{f}[z, r]$ and $\mathbf{g}$ are mean zero, the components of $\mathcal{J}[z]$ will be mean zero too.

(3) Next, we show that the solution is continuous in time. For $0 < s \le t \le T$

$$\begin{aligned}
\mathcal{J}[z](t) &= \mathrm{e}^{-(t-s)A} \mathrm{e}^{-sA} z_0 + \int_s^t \mathrm{e}^{-(t-\tau)A} \mathbf{f}[z, r](\tau) \, \mathrm{d}\tau \\
&\quad + \mathrm{e}^{-(t-s)A} \int_0^s \mathrm{e}^{-(s-\tau)A} \mathbf{f}[z, r](\tau) \, \mathrm{d}\tau
\end{aligned}$$

$$+\int_s^t \mathrm{e}^{-(t-\tau)A}\mathbf{g}(\tau)\,\mathrm{d}\tau + \mathrm{e}^{-(t-s)A}\int_0^s \mathrm{e}^{-(s-\tau)A}\mathbf{g}(\tau)\,\mathrm{d}\tau$$

so that

$$\begin{aligned}\mathcal{J}[z](t) - \mathcal{J}[z](s) &= (\mathrm{e}^{-(t-s)A} - I)\mathcal{J}[z](s) + \int_s^t \mathrm{e}^{-(t-\tau)A}\mathbf{f}[z,r](\tau)\,\mathrm{d}\tau \\ &\quad + \int_s^t \mathrm{e}^{-(t-\tau)A}\mathbf{g}(\tau)\,\mathrm{d}\tau\,. \end{aligned} \tag{42}$$

Let $\gamma > 0$ be such that $\theta = \alpha + \gamma < 1$, then, using our previous computations and the properties of the semigroup

$$\begin{aligned}\|\mathcal{J}[z](t) - \mathcal{J}[z](s)\|_{X^\alpha} &\le \|(\mathrm{e}^{-(t-s)A} - I)A^{-\gamma}\|\,\|\mathcal{J}[z](s)\|_{X^\theta} \\ &\quad + \int_s^t \|A^\alpha \mathrm{e}^{-(t-\tau)A}\|(\|\mathbf{f}[z,r](\tau)\|_{X_T} + \|\mathbf{g}(\tau)\|_X)\,\mathrm{d}\tau \\ &\le b_\gamma (t-s)^\gamma \|\mathcal{J}[z](s)\|_{X^\theta} + c a_\alpha (Z_0 + |r|_{C_T^2})\int_s^t (t-\tau)^{-\alpha}\,\mathrm{d}\tau \\ &\le b_\gamma (a_\gamma s^{-\gamma}\|z_0\|_{X^\alpha} + c a_\theta (1-\theta)^{-1}(Z_0 + |r|_{C_T^2})s^{1-\theta})(t-s)^\gamma \\ &\quad + c a_\alpha (Z_0 + |r|_{C_T^2})(1-\alpha)(t-s)^{1-\alpha} \\ &\lesssim s^{-\gamma}(t-s)^\gamma\,, \end{aligned} \tag{43}$$

and this obviously vanishes as $t \searrow s$.

For continuity at $t = 0$, we observe that the term $\mathrm{e}^{-tA}z_0$ is clearly continuous, thus all what needs to be handled are the integral terms. A direct calculation shows

$$\begin{aligned}\left\|\int_0^t \mathrm{e}^{-(t-s)A}\mathbf{f}[z,r](s)\,\mathrm{d}s\right\|_{X^\alpha} &\le a_\alpha c(Z_0 + |r|_{C_T})\int_0^t (t-s)^{-\alpha}\,\mathrm{d}s \\ &\lesssim t^{1-\alpha}\,, \end{aligned} \tag{44}$$

which also vanishes as $t \searrow 0$. Analogously, we have

$$\left\|\int_0^t \mathrm{e}^{-(t-s)A}\mathbf{g}(s)\,\mathrm{d}s\right\|_{X^\alpha} \le a_\alpha c|r|_{C_T^2}\int_0^t (t-s)^{-\alpha}\,\mathrm{d}s \lesssim t^{1-\alpha}\,. \tag{45}$$

Thus, provided $T \le \min\{T_r, T_1\}$, we have $\mathcal{J}[z] \in C([0,T]; X^\alpha) = X_T^\alpha$. Moreover, (43) implies that the solution will be locally Hölder continuous with the Hölder exponent γ for all $0 < s < t \le T$.

(4) To prove that $\mathcal{J}$ is a contraction we again make use of Proposition 2.7 and Remark 2.8 to get

$$\begin{aligned}&\|\mathcal{J}[z](t) - \mathcal{J}[z'](t)\|_{X^\alpha} \\ &\le \int_0^t \|A^\alpha \mathrm{e}^{-(t-s)A}\|\,\|\mathbf{f}[z,r](s) - \mathbf{f}[z',r](s)\|_X\,\mathrm{d}s \\ &\le \left(\int_0^t a_\alpha (t-s)^{-\alpha}\,\mathrm{d}s\right) c\nu^{-2}(c' + \overline{L}_{\mathcal{D}}(\|z\|_{X_T^\alpha} + \|z'\|_{X_T^\alpha}) + |r|_{C_T})\|z - z'\|_{X_s^\alpha}\end{aligned}$$

$$
\begin{aligned}
&\le \frac{ca_\alpha}{1-\alpha}\nu^{-2}T^{1-\alpha}(c' + \overline{L}_{\mathcal{D}}(Z_0 + |r|_{C_{T_r}}))\|z - z'\|_{X_T^\alpha} \\
&\le c(c' + \overline{L}_{\mathcal{D}}(Z_0 + |r|_{C_{T_r}}))\nu^{-2}T^{1-\alpha}\|z - z'\|_{X_T^\alpha}\,.
\end{aligned}
$$

Thus by choosing T sufficiently small to have

$$
c\nu^{-2}(c' + \overline{L}_{\mathcal{D}}(Z_0 + |r|_{C_T}))T^{1-\alpha} < 1 \tag{46}
$$

we obtain the required bound for $\mathcal{J}$ to be a contraction.

(5) Choosing the constant c properly for a time of existence T_e in (37), it will satisfy $T_e \le T_r$, (41), and (46). Setting $T = T_e$ shows that $\mathcal{J}$ is a contraction on $\mathcal{K}(T_e, Z_0)$, and thus we obtain a unique mild solution.

■

Proposition 2.15 (Continuous dependence on data): *Assume* (A1)–(A4) *then there exist positive constants c and c′, independent of ν and $\overline{L}_{\mathcal{D}}$, such that if*

$$
T_c = c \min\left\{\frac{\nu Z_0}{Z_0 + |r|_{C_{T_r}}}, \frac{\nu^2}{c' + \overline{L}_{\mathcal{D}}(Z_0 + |r|_{C_{T_r}})}\right\}^{\frac{1}{1-\alpha}}, \tag{47}
$$

then the mild solutions, $z, z' \in X_{T_c}^\alpha$ with initial conditions z_0, z_0', respectively, satisfy the bound

$$
\|z - z'\|_{X_{T_c}^\alpha} \le c''\|z_0 - z_0'\|_{X^\alpha}
$$

The constants depend only upon α, ν_0, and η_0.

Proof: The proof follows from a direct calculation. By Theorem 2.11, z, z' have a common time of existence $T > 0$. Then using Proposition 2.7 and Remark 2.8

$$
\begin{aligned}
\|z(t) - z'(t))\|_{X^\alpha} &\le \|A^\alpha \mathrm{e}^{-tA}(z_0 - z_0')\| \\
&\quad + \int_0^t \|A^\alpha \mathrm{e}^{-(t-s)A}\|\,\|\mathbf{f}[z, r]](s) - \mathbf{f}[z', r](s)\|\,\mathrm{d}s \\
&\le a_0\|z_0 - z_0'\|_{X^\alpha} \\
&\quad + \int_0^t a_\alpha(t-s)^{-\alpha}c\nu^{-2}(c' + \overline{L}_{\mathcal{D}}(\|z\|_{X_T^\alpha} + \|z'\|_{X_T^\alpha} + |r|_{C_T}))\|z - z'\|_{X_s^\alpha}\,\mathrm{d}s \\
&\le c\|z_0 - z_0'\|_{X^\alpha} + c\nu^{-2}(c' + \overline{L}_{\mathcal{D}}(Z_0 + |r|_{C_T}))\int_0^t (t-s)^{-\alpha}\|z - z'\|_{X_s^\alpha}\,\mathrm{d}s
\end{aligned}
$$

Consequently,

$$
\|z - z'\|_{X_t^\alpha} \le c\|z_0 - z_0'\|_{X^\alpha} + c\nu^{-2}(c' + \overline{L}_{\mathcal{D}}(Z_0 + |r|_{C_T}))\frac{t^{1-\alpha}}{1-\alpha}\|z - z'\|_{X_t^\alpha}\,,
$$

and, for $t \le T$ with T sufficiently small,

$$
\begin{aligned}
&0 < \left(1 - c\nu^{-2}(c' + \overline{L}_{\mathcal{D}}(Z_0 + |r|_{C_T}))\frac{T^{1-\alpha}}{1-\alpha}\right)\|z - z'\|_{X_t^\alpha} \le c\|z_0 - z_0'\|_{X^\alpha}\,, \\
&\frac{1}{2}\|z - z'\|_{X_t^\alpha} \le c\|z_0 - z_0'\|_{X^\alpha}\,.
\end{aligned}
$$

This sets the constant in (47) to determine T_c, along with the value of T_e from (37); $T_c \leq T_e$. ■

To eventually address the random damage field problem, we now show that the solutions depend continuously upon the damage field variable.

Proposition 2.16 (Continuous dependence on damage): *Assume that* (A1)–(A4) *hold then there exist positive constants c and c', independent of ν and $\overline{L}_{\mathcal{D}}$, such that if*

$$T_d = c \min\left\{ \frac{\nu Z_0}{Z_0 + |r|_{C_{T_r}}}, \frac{\nu^2}{c' + \overline{L}_{\mathcal{D}}(Z_0 + |r|_{C_{T_r}})} \right\}^{\frac{1}{1-\alpha}} \tag{48}$$

then the two mild solutions z and z' corresponding to damage fields $\mathcal{D}$ and $\mathcal{D}'$ exist in $X^{\alpha}_{T_d}$ and

$$\|z - z'\|_{X^{\alpha}_{T_d}} \leq c'' T_d^{1-\alpha} \nu^{-2} (Z_0 + |r|_{C_{T_r}}) \|\widetilde{\mathcal{D}} - \widetilde{\mathcal{D}}'\|_C \,.$$

The constants depend only upon α, ν_0, and η_0.

Proof: We may presume that we have a time of existence $T = T_c$ as in Proposition 2.15, so that the solutions exist in the common space X^{α}_T.

Letting ϵ and ϵ' denote the strain fields corresponding to z and z' then, using Lemma 2.6 and (35), since

$$|\sigma[\epsilon](x,t) - \sigma'[\epsilon'](x,t)| \leq |\sigma[\epsilon](x,t) - \sigma[\epsilon'](x,t)| + |(\mathcal{D}[\epsilon'](x,t) - \mathcal{D}'[\epsilon'](x,t))\epsilon'(x,t)| \,,$$

we obtain

$$\begin{aligned}
\|\sigma[\epsilon](t) - \sigma'[\epsilon'](t)\|_C &\leq (1 + \overline{L}_{\mathcal{D}}(\|\epsilon\|_{C_T} + \|\epsilon'\|_{C_T}))\|\epsilon - \epsilon'\|_{C_t} \\
&\quad + \|\epsilon'\|_{C_T} \sup_{s \leq t} \sup_{x} |\widetilde{\mathcal{D}}(\epsilon'(x,s),x) - \widetilde{\mathcal{D}}'(\epsilon'(x,s),x)| \\
&\leq c\nu^{-2}(c' + \overline{L}_{\mathcal{D}}(\|z\|_{X^{\alpha}_T} + \|z'\|_{X^{\alpha}_T} + |r|_{C_T}))\|z - z'\|_{X^{\alpha}_t} \\
&\quad + c''\nu^{-1}(\|z'\|_{X^{\alpha}_T} + |r|_{C_T})\|\widetilde{\mathcal{D}} - \widetilde{\mathcal{D}}'\|_C.
\end{aligned}$$

Following the steps of the proof of Proposition 2.7 along with Remark 2.8 we have

$$\begin{aligned}
\|\mathbf{f}[z,r](t) - \mathbf{f}'[z',r](t)\| &\leq c\|\sigma[\epsilon](t) - \sigma'[\epsilon'](t)\|_C + c\|z(t) - z'(t)\|_{X^{\alpha}} \\
&\leq c\nu^{-2}(c' + \overline{L}_{\mathcal{D}}(Z_0 + |r|_{C_T}))\|z - z'\|_{X^{\alpha}_t} + c''\nu^{-1}(Z_0 + |r|_{C_T})\|\widetilde{\mathcal{D}} - \widetilde{\mathcal{D}}'\|_C
\end{aligned}$$

with constants c' and c'' which are again independent of the damage fields, the forcing, and the solutions.

Now, taking differences of the mild forms of the solutions

$$\begin{aligned}
\|z(t) - z'(t)\|_{X^{\alpha}} &\leq \int_0^t \|A^{\alpha} \mathrm{e}^{-(t-s)A}\| \|\mathbf{f}[z,r](s) - \mathbf{f}'[z',r](s)\| \,\mathrm{d}s \\
&\leq \int_0^t a_{\alpha}(t-s)^{-\alpha} c(\nu^{-2}(c' + \overline{L}_{\mathcal{D}}(Z_0 + |r|_{C_T}))\|z - z'\|_{X^{\alpha}_s} \\
&\quad + \nu^{-1}(Z_0 + |r|_{C_T})\|\widetilde{\mathcal{D}} - \widetilde{\mathcal{D}}'\|_C) \,\mathrm{d}s \\
&\leq cT^{1-\alpha}\nu^{-2}(c' + \overline{L}_{\mathcal{D}}(Z_0 + |r|_{C_T}))\|z - z'\|_{X^{\alpha}_T} + (Z_0 + |r|_{C_T})\|\widetilde{\mathcal{D}} - \widetilde{\mathcal{D}}'\|_C) \,.
\end{aligned}$$

Then, if T is sufficiently small,

$$\begin{aligned} &0 < \left(1 - cT^{1-\alpha}\nu^{-2}(c' + \overline{L}_{\mathcal{D}}(Z_0 + |r|_{C_T}))\right) \|z - z'\|_{X_T^\alpha} \le Z_0 T^{1-\alpha} \|\widetilde{\mathcal{D}} - \widetilde{\mathcal{D}}'\|_C, \\ &\frac{1}{2}\|z - z'\|_{X_T^\alpha} \le cT^{1-\alpha}\nu^{-2}(Z_0 + |r|_{C_T})\|\widetilde{\mathcal{D}} - \widetilde{\mathcal{D}}'\|_C\,. \end{aligned}$$

This determines the constant in (48), and $0 < T_d \le T_c \le T_e \le T_r$. ■

2.5.2. Strong solutions

To obtain strong solutions, we must introduce the weighted (in time) Hölder spaces; see [14]. For a generic Banach space X, the space $\mathcal{F}^{\beta,\gamma}((0,T];X)$ with $0 < \gamma < \beta \le 1$ satisfies the properties: for $F \in \mathcal{F}^{\beta,\gamma}((0,T];X)$

(1) For $\beta < 1$, the limit $\lim_{t\to 0^+} t^{1-\beta}F(t)$ exists.

(2) Weighted Hölder continuity holds in the following sense

$$\sup_{0\le s<t\le T} \frac{s^{1-\beta+\gamma}\|F(t) - F(s)\|_X}{(t-s)^\gamma} < \infty. \tag{49}$$

(3) The scalar function

$$h_F(t) = \sup_{0\le s<t} \frac{s^{1-\beta+\gamma}\|F(t) - F(s)\|_X}{(t-s)^\gamma} \tag{50}$$

satisfies $\lim_{t\to 0^+} h_F(t) = 0$.

(4) The space is $\mathcal{F}^{\beta,\gamma}((0,T];X)$ equipped with the norm

$$\|F\|_{\mathcal{F}^{\beta,\gamma}} = \sup_{0\le t\le T} t^{1-\beta}\|F(t)\|_X + \sup_{0\le s<t\le T} \frac{s^{1-\beta+\gamma}\|F(t) - F(s)\|_X}{(t-s)^\gamma} \tag{51}$$

is a Banach space.

Our result on strong solutions is based on the following theorem from [14].

Theorem 2.17 (Theorems 3.4 and 3.5 of [14]): *Given the Cauchy problem*

$$\frac{\mathrm{d}U}{\mathrm{d}t} + AU = F(t), \quad U(0) = U_0\,,$$

if $F \in \mathcal{F}^{\alpha,\gamma}((0,T];X)$ *and* $U_0 \in D(A^\alpha)$, *then there exists a solution*

$$U \in C([0,T];X) \cap C([0,T];D(A^\alpha)) \cap C((0,T];D(A)) \cap C^1((0,T];X).$$

First, we need a few regularity results.

Proposition 2.18: *For* $r \in C^2([0,T])$ *and any* $0 < \gamma < \alpha \le 1$, $\mathbf{g} \in \mathcal{F}^{\alpha,\gamma}((0,T];X)$.

Proof: Recalling the definition of $\mathbf{g}$ in (21), and by the assumed regularity in time, it is in fact Lipschitz in time. First, it is immediately clear based on the regularity of $\mathbf{g}$ in space and time that for any such α

$$\lim_{t\to 0^+} t^{1-\alpha}\mathbf{g}(t)$$

exists. Next, since the $\mathbf{g}$ is Lipschitz, for $0 \le s < t \le T$:

$$s^{1-\alpha+\gamma}\frac{\|\mathbf{g}(t)-\mathbf{g}(s)\|_X}{(t-s)^\gamma} \lesssim s^{1-\alpha+\gamma}(t-s)^{1-\gamma}$$

so its supremum over this set is finite; (49) holds. Finally,

$$\begin{aligned} h_g(t) &= \sup_{0\le s<t} s^{1-\alpha+\gamma}\frac{\|\mathbf{g}(t)-\mathbf{g}(s)\|_X}{(t-s)^\gamma} \\ &\lesssim s^{1-\alpha+\gamma}(t-s)^{1-\gamma} \lesssim t^{1-\alpha+\gamma}(t-s)^{1-\gamma} \end{aligned}$$

and this vanishes as $t \to 0^+$, so (50) holds. Thus $\mathbf{g}$ is in the Hölder space. ■

Corollary 2.19: *If z is a mild solution of the transformed problem, then for any $\gamma \in (0, 1-\alpha)$, z is in the weighted Hölder space $\mathcal{F}^{\alpha,\gamma}((0,T];X^\alpha)$*

Proof: For a mild solution to the equation, $z \in X^\alpha_T$, so $t^{1-\alpha}z(t) \to 0$ in X^α as $t \to 0^+$. Next, since $z = \mathcal{J}[z]$, a consequence of (43) is that we have a weighted Hölder bound, and for $0 \le s < t \le T$

$$\frac{s^{1-\alpha+\gamma}\|z(t)-z(s)\|_{X^\alpha}}{(t-s)^\gamma} \le cs^{1-\alpha},$$

so (49) and (50) hold. ■

Proposition 2.20: *If z is a mild solution of the transformed problem, then for any $\gamma \in (0,1-\alpha)$, $\mathbf{f}[z,r]$ is in the weighted Hölder space $\mathcal{F}^{\alpha,\gamma}((0,T];X)$*

Proof: A consequence of Proposition 2.7, for all $t \in (0,T]$, $\mathbf{f}[r,z] \in X_T$, so $t^{1-\alpha}\mathbf{f}[r,z](t) \to 0$ as $t \to 0^+$ in the X. Next, per the analysis in the proof of Proposition 2.7, for $0 < s < t \le T$

$$\|\mathbf{f}[z,r](t)-\mathbf{f}[z,r](s)\|_X \le 4\|\sigma[\epsilon](t)-\sigma[\epsilon](s)\|_C + \max\{1, c_\alpha\eta\}\|z(t)-z(s)\|_{X^\alpha}$$

By Corollary 2.19, the second term on the last line has the necessary property

$$\|z(t)-z(s)\|_{X^\alpha} \le cs^{-\gamma}(t-s)^\gamma .$$

For any $x \in [0,1]$

$$\begin{aligned} |\sigma[\epsilon](x,t)-\sigma[\epsilon](x,s)| &\le |1-\mathcal{D}[\epsilon](x,t)||\epsilon(x,t)-\epsilon(x,s)| \\ &\quad |\epsilon(x,s)||\mathcal{D}[\epsilon](x,t)-\mathcal{D}[\epsilon](x,s)| \end{aligned}$$

Since z is a mild solution, we know that $\epsilon \in C([0,T];C([0,1]))$, and Lemma 2.5 ensures $0 \le 1 - \mathcal{D}(\epsilon)(x,t) \le 1$. Combining this with (35) we have

$$|\sigma[\epsilon](x,t)-\sigma[\epsilon](x,s)| \lesssim |\epsilon(x,t)-\epsilon(x,s)| + |\mathcal{D}[\epsilon](x,t)-\mathcal{D}[\epsilon](x,s)| .$$

Next, note that by the admissibility of $\widetilde{\mathcal{D}}$ and the regularity of r,

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$$
\begin{aligned}
|\widetilde{\mathcal{D}}(\epsilon(x,t),x) - \widetilde{\mathcal{D}}(\epsilon(x,s),x)| &\le \overline{L}_{\mathcal{D}}|\epsilon(x,t) - \epsilon(x,s)| \\
&\le c(\|z(t) - z(s)\|_{X^\alpha} + |r(t) - r(s)|) \le cs^{-\gamma}(t-s)^\gamma
\end{aligned}
$$

Then, by Lemma A.2, we have

$$|\mathcal{D}[\epsilon](x,t) - \mathcal{D}[\epsilon](x,s)| \le cs^{-\gamma}(t-s)^\gamma .$$

Thus, we are assured that

$$\|\sigma[\epsilon](t) - \sigma[\epsilon](s)\|_C \le cs^{-\gamma}(t-s)^\gamma$$

and

$$\|\mathbf{f}[z,r](t) - \mathbf{f}[z,r](s)\|_X \le cs^{-\gamma}(t-s)^\gamma .$$

Consequently, (49) and (50) also hold which concludes the proof. ■

We can now state and prove our result on strong solutions of (15).

Theorem 2.21 (Strong Solutions): *If z is a mild solution, then it is also a strong solution with*

$$z \in C([0,T];X) \cap C([0,T];X^\alpha) \cap C^1((0,T];X) \cap C((0,T];X^1)$$

***Proof*:** For the problem (15), if we substitute the mild solution into the right-hand side, we have by Proposition 2.20 that $\mathbf{f}[z,r] \in \mathcal{F}^{\alpha,\gamma}((0,T];X)$. Furthermore, $\mathbf{g}$ also belongs to this space by Proposition 2.18. Thus, interpreting $\mathbf{f}[z,r] + \mathbf{g}$ as F in Theorem 2.17 we have the result. ■

2.5.3. Weak solutions

Assuming we have obtained a strong solution to (15) we have the following result.

Proposition 2.22: *If $z = (p,q)$ is a strong solution of (15) with $\alpha > 1/4$, let*

$$u(x,t) = \int_0^x \nu^{-1}(p(y,t) + q(y,t)) + r(t)\,\mathrm{d}y . \tag{52}$$

Then, u is a weak solution of (6a), satisfying the boundary conditions (6b), with the initial conditions $u(0) = u_0$, $u_t(0)) = u_1$ given by

$$u_0(x) = \int_0^x \nu^{-1}(p_0(y) + q_0(y)) + r(0)\,\mathrm{d}y \in C^1([0,1]) , \tag{53a}$$

$$u_1(x) = \partial_x p_0(x) + r'(0)x \in H^{2\alpha-1}([0,1]) . \tag{53b}$$

***Proof*:** By the regularity of p and q, for $0 < t \le T$, using that we have a strong solution,

$$u_t = \int_0^x \nu^{-1}(p_t + q_t) + r'(t)\,\mathrm{d}y = \int_0^x (\nu^{-1}(\nu p_{xx}) + r'(t))\,\mathrm{d}y = p_x + r'(t)x$$

with equality holding in the sense of H^1. Next, we have

$$u_{tt} = p_{tx} + r''(t)x$$

in the distributional sense; that is for all $\varphi \in H_0^1([0,1])$,

$$\langle u_{tt}, \varphi \rangle = -\left\langle p_t - r''\left(\frac{1}{6} - \frac{x^2}{2}\right), \varphi_x \right\rangle$$

Indeed,

$$\langle u_{tt}, \varphi \rangle = -\int_0^1 \varphi_x \left(p_t - r''\left(\tfrac{1}{6} - \tfrac{x^2}{2}\right)\right)\,\mathrm{d}x$$

$$
\begin{aligned}
&= -\int_0^1 \varphi_x \left(-\eta p + \nu p_{xx} + \sigma - \bar{\sigma} - \eta r' \left(\tfrac{1}{6} - \tfrac{x^2}{2}\right)\right) \mathrm{d}x \\
&= \int_0^1 \varphi_x \eta \left(p + r'(t) \left(\tfrac{1}{6} - \tfrac{x^2}{2}\right)\right) \mathrm{d}x + \int_0^1 \varphi_{xx} \nu (p_x + r'(t)x)\, \mathrm{d}x \\
&\quad - \int_0^1 \varphi_x (\sigma - \bar{\sigma})\, \mathrm{d}x \\
&= -\int_0^1 \varphi \eta u_t \,\mathrm{d}x - \int_0^1 \nu \varphi_x u_{tx}\, \mathrm{d}x - \int_0^1 \varphi_x (\sigma - \bar{\sigma})\, \mathrm{d}x\,,
\end{aligned}
$$

where we denoted $\bar{\sigma} = \int_0^1 \sigma \, dx$. Thus (32) holds. Note that in the above computations, the stress is well defined as $u_x \equiv \epsilon \in C([0,T]; C([0,1])) \hookrightarrow C([0,T]; L^2(0,1)$. The displacement u will satisfy the boundary conditions, as, using (52), $u(0,t) = 0$, while

$$
u(1,t) = \nu^{-1} \int_0^1 p(y,t) + q(y,t)\, \mathrm{d}y + r(t) = r(t)
$$

since p and q are mean zero.

Since $p_0 \in H_a^{2\alpha} \hookrightarrow C_a$ and $q_0 \in C_a$, substituting into (53a) and (53b), we infer that $u_0 \in C^1$ and $u_1 \in H^{2\alpha-1}$. ■

As we will eventually start with data in the displacement velocity coordinates, we will also benefit from the following lemma

Lemma 2.23: *Assume* $u_0 \in C^1([0,1])$ *and* $u_1 \in H_0^{2\alpha-1}([0,1])$ *for* $\alpha > \frac{1}{4}$, $u_0(0) = 0$ *and* $u_0(1) = r(0)$. *Then*

$$
p_0(x) = \int_0^x (u_1(y) - r'(0)y)\, \mathrm{d}y - \int_0^1 \int_0^x (u_1(y) - r'(0)y)\, \mathrm{d}y\, \mathrm{d}x \in H_a^{2\alpha}([0,1]) \tag{54a}
$$

$$
q_0(x) = \nu(\partial_x u_0(x) - r(0)) - p_0(x) \in C_a([0,1]) \tag{54b}
$$

Proof: Using the Fourier sine series representation of u_1, it can be shown that p_0 is well-defined, and it is in $H_a^{2\alpha}$, since $\alpha > 1/4$, we thus have $p_0 \in C_a$. Then, by the assumption on u_0, we have $q_0 \in C_a$. ■

Corollary 2.24 (Existence of weak solutions): *Under the assumptions of Lemma 2.23, there exists a weak solution to* (6a) *satisfying boundary conditions* (6b).

Proof: By the lemma, the data, (p_0, q_0), are sufficiently regular so that Theorem 2.11 ensures that a mild solution exists. Theorem 2.21 ensures it is also a strong solution in (p,q) space. From Proposition 2.22 we thus have that there is a corresponding weak solution. ■

2.6. Well-posedness of the evolution with random damage

In this section we generalize the previous results for a deterministic case (a fixed realization of $\mathcal{D}$) to the case of the random damage field.

Theorem 2.25: *Assume* (A1)–(A4), *and furthermore let* $(\widetilde{\mathcal{D}}, \overline{L}_{\mathcal{D}})$ *be a strongly measurable pair, almost surely satisfying* $\widetilde{\mathcal{D}}(\cdot,\cdot;\omega) \in \mathcal{A}_{\mathcal{D}}(\epsilon_{\max}, \overline{L}_{\mathcal{D}}(\omega))$.

Then there exists a strongly $\mathbb{P}$*-measurable mild solution* $\omega \mapsto z(\omega) \in X^{\alpha}_{T(\omega)}$ *with* $T(\omega) > 0$ *almost surely. Furthermore,* $z(\omega)$ *are strong solutions for almost all* ω*, and they induce corresponding weak solutions of* (6a).

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Proof: Since α, Z_0, and r are fixed, we can use (48) from Proposition 2.16 to assume that, for a given value of $\overline{L}_{\mathcal{D}}(\omega) < \infty$,

$$T(\omega) = \left\{ \frac{cv^2}{c' + \overline{L}_{\mathcal{D}}(\omega)} \right\}^{\frac{1}{1-\alpha}} \in (0, \infty), \quad \mathbb{P} - \text{a.s.}, \tag{55}$$

where the positive constants c and c' are independent of the realization ω.

Since $(\widetilde{\mathcal{D}}, \overline{L}_{\mathcal{D}})$ is strongly $\mathbb{P}$-measurable, there exist simple functions

$$\widetilde{\mathcal{D}}^{(n)} = \sum_{i=1}^{N_n} \widetilde{\mathcal{D}}^{(n,i)} 1_{A^{(n,i)}}(\omega), \quad \overline{L}_{\mathcal{D}}^{(n)} = \sum_{i=1}^{N_n} \overline{L}_{\mathcal{D}}^{(n,i)} 1_{A^{(n,i)}}(\omega), \quad N_n < \infty,$$

converging, almost surely, in $C((-\epsilon_{\max}, \epsilon_{\max}) \times [0,1]) \times \mathbb{R}$; see, for instance, corollaries in [15, Corollary 1.1.7, 1.1.21]. These simple functions may be constructed from elements of the image, $\{(\widetilde{\mathcal{D}}(\cdot,\cdot,\omega), \overline{L}_{\mathcal{D}}(\omega)) | \omega \in \Omega\}$; see [16, Lemma A.1.4]. Consequently, the $\widetilde{\mathcal{D}}^{(n,i)} \in \mathcal{A}_{\mathcal{D}}(\epsilon_{\max}, \overline{L}_{\mathcal{D}}^{(n,i)})$.

For each n, let $\omega \in A^{(n,i)}$ for some i. We then have a corresponding mild solution $z^{(n,i)}$ associated with with $\widetilde{\mathcal{D}}^{(n,i)}$. This gives us the simple functions

$$z^{(n)}(\omega) = \sum_{i=1}^{N_n} z^{(n,i)} 1_{A^{(n,i)}}(\omega)\,. \tag{56}$$

These solutions have the corresponding Lipschitz bounds $\overline{L}_{\mathcal{D}}^{(n,i)}$ and the times of existence $T^{(n,i)}$ from (55). Hence,

$$\begin{aligned} T^{(n)}(\omega) = \sum_{i=1}^{N_n} T^{(n,i)} 1_{A_{n,i}}(\omega) &= \sum_{i=1}^{N_n} \left\{ \frac{cv^2}{c' + \overline{L}_{\mathcal{D}}^{(n,i)}} \right\}^{\frac{1}{1-\alpha}} 1_{A_{n,i}}(\omega) \\ &= \left\{ \frac{cv^2}{c' + \overline{L}_{\mathcal{D}}^{(n)}(\omega)} \right\}^{\frac{1}{1-\alpha}}. \end{aligned} \tag{57}$$

By the choice of these times of existence

$$\|z^{(n,i)}\|_{X^{\alpha}_{T(n,i)}} \le Z_0\,.$$

As we have almost sure convergence of $\overline{L}_{\mathcal{D}}^{(n)} \to \overline{L}_{\mathcal{D}} \in [0, \infty)$, we get almost sure convergence of $T^{(n)} \to T \in (0, \infty)$. Thus there exists an $N(\omega)$ such that for all $n \ge N(\omega)$, $T^{(n)}(\omega) \ge T(\omega)/2$. Let $T'(\omega) = T(\omega)/2$, a common time of existence for the $z^{(n)}(\omega)$ provided $n \ge N(\omega)$. We can then apply Proposition 2.16 with $T_d = T'$ such that for all $n, m \ge N(\omega)$

$$\|z^{(n)}(\omega) - z^{(m)}(\omega)\|_{X^{\alpha}_{T'(\omega)}} \le cT'(\omega)^{1-\alpha} \|\tilde{D}^{(n)}(\omega) - \tilde{D}^{(m)}(\omega)\|_C.$$

As $T'(\omega) \in (0, \infty)$ almost surely and the $\tilde{D}^{(n)}(\omega)$ are Cauchy, the sequence $z^{(n)}(\omega)$ is also Cauchy, almost surely. The limit, $z(\omega)$, is strongly $\mathbb{P}$-measurable, as it is the limit of simple (hence strongly $\mathbb{P}$-measurable) functions; see corollaries in [15, Corollary 1.1.9, 1.1.23].

Lastly, since z is almost surely a mild solution, by Theorem 2.21, it is also a strong solution. By Proposition 2.22, this implies the existence of corresponding weak solution of (6a), which will also be strongly measurable. ■

2.7. Moment bounds

Here, we provide estimates on moments of derived solutions in the case of a random damage field $\mathcal{D}$.

Corollary 2.26: *Under the same assumptions as in Theorem 2.25, for any $m > 0$, there exist constants c and c' such that*

$$\mathbb{E}[T^m] \le \left\{ \frac{c\nu^2}{c' + \mathbb{E}[\bar{L}_{\mathcal{D}}]} \right\}^{\frac{m}{1-\alpha}} < \infty .$$

Proof: This result follows from (55) in the proof of Theorem 2.25, and an application of Jensen's inequality. ■

Remark 2.27: Corollary 2.26 reveals how our local time of existence scales with respect to the viscoelastic regularization and the Lipschitz constant. Per Remark 2.14, this is only a result upon the local time of existence achievable via a Banach space fixed point argument; we have not addressed what the maximal time of existence is.

Corollary 2.28: *Under the same assumptions as Theorem 2.25, z is Bochner integrable and we have the following bound on the m-th moment*

$$\mathbb{E}[\|z\|^m_{X^\alpha_T}] \le c(1 + \mathbb{E}[T^{1-\alpha}])^m < \infty ,$$

where T is the local time of existence obtained from Theorem 2.25.

Proof: From the theorem, z is strongly $\mathbb{P}$-measurable, and in our proof of Theorem 2.25, we infer that there exists a constant c independent of the realizations such that almost surely

$$\|z(\omega) - z^{(n)}(\omega)\|_{X^\alpha_{T(\omega)}} \le cT(\omega)^{1-\alpha}\|\widetilde{\mathcal{D}}(\omega) - \widetilde{\mathcal{D}}(\omega)^{(n)}\|_C \le 2cT(\omega)^{1-\alpha} < \infty ,$$

where $z^{(n)}$ are the simple functions from (56). By their construction, almost surely,

$$\|z^{(n)}(\omega)\|_{X^\alpha_{T(\omega)}} \le \sum_{i=1}^{n} \|z^{(n,i)}\|_{X^\alpha_{T(\omega)}} 1_{A^{(n,i)}}(\omega) \le \sum_{i=1}^{n} Z_0 1_{A^{(n,i)}}(\omega)$$

and the right most quantity is obviously integrable. Then, by the dominated convergence theorem, since $\|z - z_n\|_{X^\alpha_T} \to 0$ a.s., we conclude z is a Bochner integrable random variable.

Using (40) from the proof of Theorem 2.11, since any mild solution is a fixed point, hence almost surely $\mathcal{J}[z] = z$, and

$$\|z(\omega)\|_{X^\alpha_T} \le Z_0 + \frac{a_\alpha}{1-\alpha} c(Z_0 + |r|_{C_T})T(\omega)^{1-\alpha} .$$

The moment bound then follows from Jensen's inequality. ■

Corollary 2.29: *For $\alpha \ge 1/2$, Given (u, v) obtained from the strong solution (p, q) through Corollary* 2.24

$$\mathbb{E}[\|u\|^m_{W^{1,\infty}_T} + \|u_t\|^m_{L^2_T}] \le C(1 + \mathbb{E}[T^{1-\alpha}])^m < \infty \tag{58}$$

where T is the local time of existence obtained from Theorem 2.25.

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Proof: Using Proposition 2.22, since $p_x \in L^2 \subset H^{2\alpha}$ for all $t \in [0, T]$, we have almost surely

$$\|v(t)\|_{L^2} \lesssim \|p(t)\|_{H^{2\alpha}} + |r'(t)|\,,$$

hence $\|v\|_{L^2_T} \lesssim \|z\|_{X^\alpha_T} + |r|_{C^1_T}$. Next, since $p \in H^{2\alpha}([0,1]) \hookrightarrow C([0,1])$ and $q \in C([0,1])$,

$$\begin{aligned}\|u(t)\|_C &\lesssim \|p(t)\|_C + \|q(t)\|_C + |r|_{C_T}\\ &\lesssim \|p\|_{H^{2\alpha}_T} + \|q\|_{C_T} + |r|_{C_T}\\ &\lesssim \|z\|_{X^\alpha_T} + |r|_{C_T}\,,\end{aligned}$$

and an analogous computation holds for $\|u_x(t)\|_C$. ■

3. Numerical experiments

While there remains an analytic gap in our result, regarding the uniqueness of the solution in the (u, u_t) variables and its continuous dependence upon the data, numerical experiments suggest that the problem has greater regularity than has thus far been obtained.

Using the piece-wise linear damage model, (13) and Figure 1, with a random peak stress[2] $\sigma_\star$, we perform a classical second-order centered difference approximation of the viscoelastic term, and a midpoint flux approximation of the damage augmented stress $\sigma(\epsilon) = (1 - \mathcal{D}[\epsilon])\epsilon$. We then integrate the spatially discretized system

$$\ddot{\boldsymbol{u}} + \eta\dot{\boldsymbol{u}} = \nu A_L\dot{\boldsymbol{u}} + A_N(\boldsymbol{u})\boldsymbol{u} + \boldsymbol{g}(t)\,. \tag{59}$$

The matrix A_L corresponds to discretization of the operator ∂_{xx} and $A_N(\boldsymbol{u})$ corresponds to discretization of $\partial_x[(1 - \mathcal{D})\partial_x u]$. The system (59) is integrated with the Newmark-β scheme, [17,18]. To address the history dependence in the nonlinearity, we rely on a splitting, holding $A_N(\boldsymbol{u})$ constant across a Newmark scheme update, and then updating it where damage has been accrued.

The self-consistency of this method is demonstrated in Figure 2 in a test with non-random damage $\mathcal{D}$. In this convergence testing the Rayleigh damping parameters were chosen $\eta = \nu = 0.1$, the expansion speed $c = 0.5$, the maximum time $T_{\max} = 1$, and the stress softening $\Delta\sigma = 0.1$. Furthermore, the deterministic critical stress field $\sigma_\star$ was set to $\sigma_\star = \exp(-\exp[-100(x - 0.3)^2])$, depicted in Figure 1).

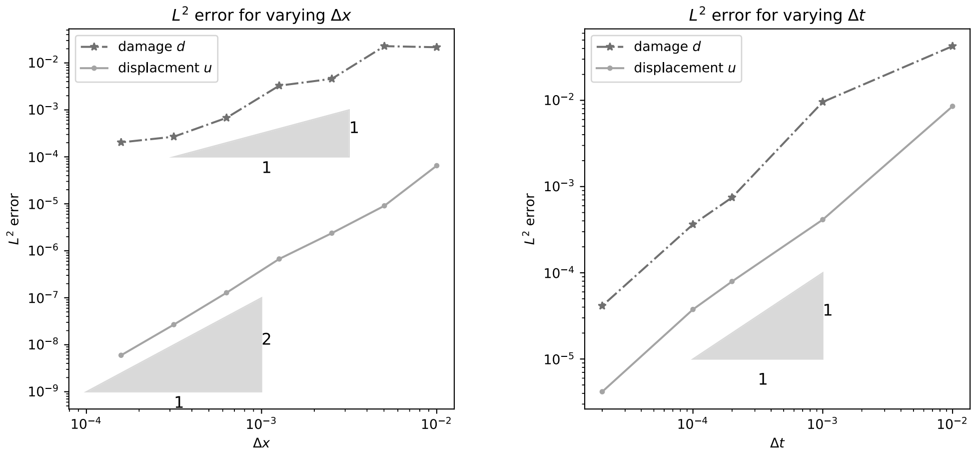
4. Stress–strain characteristics for random yield stress fields

In the presence of random disorder the damage variable $\widetilde{\mathcal{D}}$ becomes a random field due to uncertainty of the value for the critical (peak) stress $\sigma_\star$. Thus the peak stress is defined as a random field $\sigma_\star(x, \omega)$. We present an example which demonstrates the dependence of a global quantity of interest, that mimics experimentally obtained stress–strain curve, on the spatial correlation length of the random disorder. For different choices of the correlation length ℓ we investigate the mean and variance of the global stress $\sigma_{\text{total}}(t; \omega) = \int_0^1 \sigma(u_x(x, t); \omega)\,\mathrm{d}x$ and its dependence on the boundary displacement $u(1, t) \equiv ct$.

The computational results used $N_{\text{smpl}} = 1000$ independent samples from the log-normal random yield stress fields. The log-normal field was obtained from a Gaussian field (using an exponential transformation), see, e.g. [19, Chapter 9], with the constant mean $\mu = -2$ and the covariance operator

$$\mathcal{C} = \left(\partial_{xx} + \ell^{-2} I\right)^{-1}\,. \tag{60}$$

The parameter ℓ controls the mean spatial correlation length.



(a) (b)

Figure 2. Convergence of the discretization scheme for the approximation of the displacement $u(x,t)$ and the damage variable $\mathcal{D}(u_x(x,t))$. (a) Convergence of the L_2 spatial discretization error. (b) Convergence of the L_2 time discretization error.

We ran numerical simulations by solving Equation (59) to a final stopping time $T_{\max} = 2$. The Rayleigh damping parameters were $\eta = \nu = 0.01$, the expansion speed $c = 0.075$, and the stress-softening $\Delta\sigma = 0.1$. At each time step we collected the total cumulative stress

$$\sigma_{\text{total}}(t;\omega) = \int_0^1 \sigma(x,t;\omega)\,\mathrm{d}x\,, \tag{61}$$

and plotted it against the total strain for the system. Since the boundary conditions were $u(0,t) = 0, \quad u(1,t) = ct$, the total strain is given by

$$\epsilon_{\text{total}}(t) = \int_0^1 u_x(x,t)\,\mathrm{d}x = ct\,. \tag{62}$$

Since for all simulations c is fixed at the same value, for simplicity we plot the total mean stress as a function of t instead of the total mean strain. The results for various values of ℓ are summarized in Figure 3. From each ensemble of samples we estimated the mean and standard deviation for the stress–strain curve. These moments were computed point-wise for each value of t. The results can be seen in Figure 4.

5. Discussion

We have made progress in providing a rigorous basis for the use of damage mechanics type models within an uncertainty quantification framework by transforming the problem into one for which there is a proper well-posedness theory. But there are several outstanding analytical challenges. First, uniqueness and continuous dependence upon the data within the original displacement-velocity coordinates needs to be established. Ideally, this would be achieved without any additional regularization to the problem beyond the viscoelasticic terms we included. Second, it would be desirable to assess the maximal time of existence and explore the potential for global in time existence. The extension of the results to the multi-dimensional setting and full visco-elastic dynamics poses additional analytical challenges. In the mechanical engineering community various dynamical models with random material properties have been used in simulations of damage and failure. In order to improve

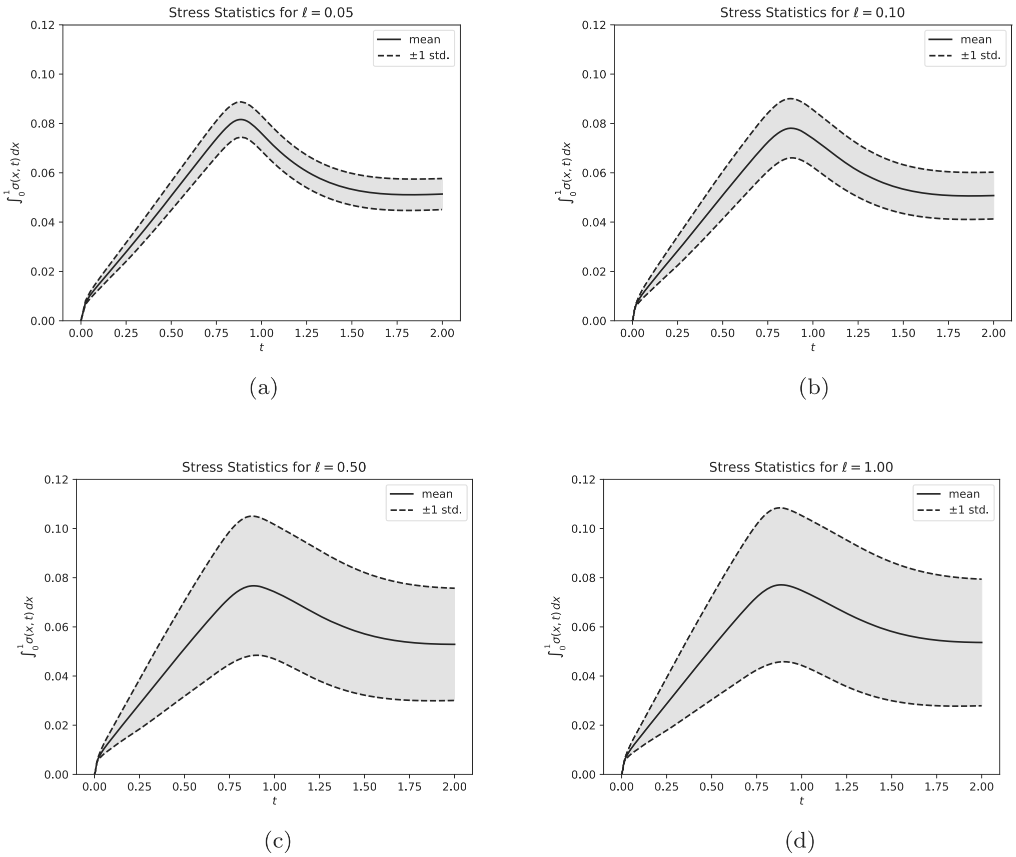


Figure 3. Global mean stress–strain curve in uni-axial loading together with a one-standard deviation uncertainty interval (confidence intervals of the estimators are below resolution of the graph). (a) $\ell = 0.05$. (b) $\ell = 0.1$. (c) $\ell = 0.5$. (d) $\ell = 1.0$.

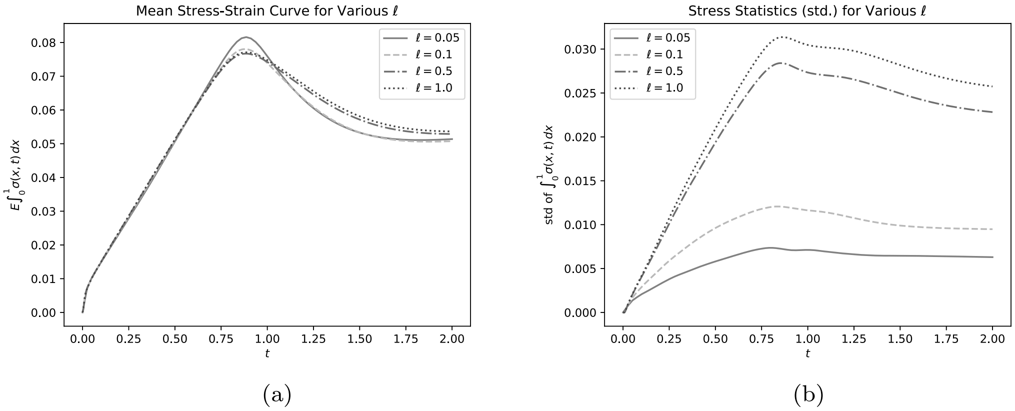


Figure 4. Comparison of the moments for the global stress–strain curves for random (critical) yield stress fields $\sigma_\star$ with different correlation lengths ℓ (confidence intervals of the estimators are below resolution of the graph). (a) Mean for different ℓ. (b) Standard deviation for different ℓ.

numerical results of such simulations it would be useful to explore, based on some results presented here, the almost sure well-posedness of such models.

Our computational examples illustrate a key challenge, and opportunity, for uncertainty quantification of the mechanical failure of materials. In particular, we see the dependence upon the correlation length in the material parameters, through the damage model, in the inferred stress–strain uncertainty profiles. Indeed, the experiments demonstrate that the stress–strain curve becomes highly uncertain as the correlation length (relative to the size of the specimen) increases. This suggests that using approximation approaches based on polynomial chaos expansions may become computationally infeasible as the dimension of the stochastic space will have to rapidly increase with time. Therefore development of efficient Monte Carlo sampling methods that will allow us to extend similar numerical experiments to two and three-dimensional models is a desirable next step. At the same time, the simulations reveal that uncertainty of the stress–strain curves carry a signal of the correlation length of the material properties. Thus, if one can overcome computational bottlenecks, it may be possible to perform a statistical inversion via, for instance, Bayesian Markov chain Monte Carlo.

Notes

1. Recall that if $X(\omega)$ is strongly $\mathbb{P}$-measurable, it is $\mathbb{P}$-a.s. the point-wise limit of simple functions, [15].
2. Since in our model the elastic constant is equal to one, the critical strain $\epsilon_\star$ in (13) and the critical peak stress $\sigma_\star$ are interchangeable.

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Auxiliary calculations

Classical analysis results

For the convenience of the reader we recall certain results from classical analysis of continuous functions.

Lemma A.1: *Given* $f \in C([0,T])$, $0 < T < \infty$, *and defining the function*

$$F(t) = \sup_{s \le t} f(s) \,,$$

we have $F \in C([0,T])$ *with* $\|F\|_C \le \|f\|_C$.

Proof: We immediately have $|F(t)| \le \sup_{s\le t} |f(s)| \le \|f\|_C$, so F is a bounded function. Next we establish continuity. Fix $t \in [0,T]$, and let $\epsilon > 0$, either $f(t) = F(t)$ or $f(t) < F(t)$.

In the case that $f(t) < F(t)$, by the continuity of f over the compact set $[0,t]$, there exists $t_0 \in [0,t]$ such that $F(t) = F(t_0) = f(t_0)$, and, we may assume that t_0 is the largest such value in the interval $[0,t]$. By monotonicity of F, for $t' \in [t_0, t]$

$$F(t_0) \le F(t') \le F(t) = F(t_0),$$

Hence, F is constant on the interval $[t_0,t]$. Next, by the continuity of f, there exists a $\delta > 0$, such that for $|t - t'| \le \delta$

$$|f(t) - f(t')| \le \frac{1}{2}(F(t) - f(t)) \,,$$

and we may take $\delta < (t - t_0)/2$, then for all $s \in [t, t+\delta]$

$$f(s) \le \frac{1}{2}(F(t) + f(t)) < F(t) = F(t_0) \,.$$

Thus, in the interval $|t - t'| < \delta$, $|F(t) - F(t')| = 0 < \epsilon$.

In the case that $f(t) = F(t)$, pick δ such that $|f(t) - f(t')| \le \epsilon$ for $|t - t'| \le \delta$. Then for $t - \delta \le t' \le t$, by monotonicity of F,

$$F(t) - \epsilon = f(t) - \epsilon \le f(t') \le F(t') \le F(t) \,.$$

Analogously, for $t \le t' \le t + \delta$

$$F(t) \le F(t') = \sup_{s \le t'} f(s) = \sup_{s\in[t,t']} f(s) \le \sup_{s\in[t,t']} f(s) f(t) + \epsilon = F(t) + \epsilon \,.$$

And in this case $|F(t) - F(t')| \le \epsilon$ for $|t - t'| \le \delta$. ■

Lemma A.2: *Given* $f \in C^{0,\gamma}([0,T])$ *with* $0 < T < \infty$ *and* $0 < \gamma \le 1$ *define*

$$F(t) = \sup_{s \le t} f(s) \,,$$

then $F \in C^{0,\gamma}([0,T])$. *Furthermore, if*

$$\sup_{t \ne s} \frac{|f(t) - f(s)|}{|t - s|^{\gamma}} = L < \infty \,,$$

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then

$$\sup_{t\neq s}\frac{|F(t)-F(s)|}{|t-s|^{\gamma}}\leq L\,.$$

Proof: Per Lemma A.1, we immediately know that $F(t)$ is a continuous function. Next, fix $0\leq t_1\leq t_2\leq T$. If $F(t_1)=F(t_2)$, there is nothing to prove. Hence, assume $F(t_1)\neq F(t_2)$ and let

$$\begin{aligned} t_1' &= \sup\{\tau\geq t_1 \mid F(\tau)=F(t_1)\} \\ t_2' &= \inf\{\tau\leq t_2 \mid F(\tau)=F(t_2)\,. \end{aligned}$$

Since t_1' is the largest such value with $F(t_1)=F(t_1')$, we must have that $F(t_1')=f(t_1')$, and analogously, $F(t_2')=f(t_2')$. Thus, if C_1 is the Hölder coefficient of f at t_1', then

$$|F(t_1)-F(t_2)|=|F(t_1')-F(t_2')|=|f(t_1')-f(t_2')|\leq C_1|t_2'-t_1'|^{\gamma}\leq C_1|t_2-t_1|^{\gamma}\,.$$

Thus, for all t, there exists a constant $c>0$, such that

$$|F(t)-F(s)|\leq c|t-s|^{\gamma}\,.$$

Furthermore, if f has a global Hölder constant, such that

$$|f(t)-f(s)|\leq L|t-s|^{\gamma}\,,$$

then the same constant holds for F. ■

Lemma A.3: *The following function spaces of real valued functions are isometric:*

$$C([a,b]\times[c,d])\cong C([a,b];C([c,d]))\cong C([c,d];C([a,b])) \tag{A1}$$

Proof: The proof is based on the compactness of the spaces and the implied uniform continuity. We will show the isometry $C([a,b]\times[c,d])\cong C([a,b];C([c,d]))$, as the third relation can then be deduced in the same way.

Let Φ be the mapping defined on $C([a,b]\times[c,d])$ as $\Phi(f)(x)(y)=f(x,y)$. First, observe

$$\|\Phi(f)\|_C=\sup_x\|\Phi(f)(x)\|_C=\sup_x\left\{\sup_y|\Phi(f)(x)(y)|\right\}=\sup_x\{\sup_y|f(x,y)|\}=\|f\|_C$$

Since f is jointly continuous and $[a,b]\times[c,d]$ is compact, it is uniformly continuous. Thus, there exists $\delta>0$ such that for $|x-x'|\leq\delta$ and $|y-y'|\leq\delta$

$$|\Phi(f)(x)(y)-\Phi(f)(x')(y')|=|f(x,y)-f(x',y')|\leq\epsilon\,.$$

Thus, for each $x\in[a,b]$, $\Phi(f)(x)\in C([c,d])$. Additionally,

$$\|\Phi(f)(x)-\Phi(f)(x')\|_\infty=\sup_y|f(x,y)-f(x',y)|\leq\epsilon$$

provided $|x-x'|\leq\delta$. Consequently, $\Phi(f)\in C([a,b];C([c,d]))$.

Next, let Ψ be defined on $C([a,b];C([c,d]))$ as $\Psi(f)(x,y)=f(x)(y)$. Again,

$$\|\Psi(f)\|_C=\sup_{x,y}|f(x)(y)|=\sup_x\left\{\sup_y|f(x,y)|\right\}=\sup_x\|f(x)\|_C=\|f\|_C$$

Fix $\epsilon>0$ and $(x,y)\in[a,b]\times[c,d]$. There exists $\delta>0$ such that for $|x-x'|\leq\delta$

$$\|\Psi(f)(x)-\Psi(f)(x')\|_C\leq\epsilon/2$$

Also, since $\Psi(f)(x)(\cdot)=f(x)(\cdot)\in C([c,d])$, it is uniformly continuous in y, so there exists $\delta_x>0$ such that for $|y-y'|\leq\delta_x$,

$$|\Psi(f)(x)(y)-\Psi(f)(x)(y')|\leq\epsilon/2\,.$$

Thus, for $|x-x'|\leq\delta$ and $|y-y'|\leq\delta_x$, and

$$\begin{aligned}|\Psi(f)(x)(y)-\Psi(f)(x')(y')| &\le |\Psi(f)(x)(y)-\Psi(f)(x)(y')| \\ &\quad + |\Psi(f)(x)(y')-\Psi(f)(x')(y')| \le \epsilon\end{aligned}$$

Thus, $\Psi(f)$ is jointly continuous.

Clearly, Φ and Ψ are inverses of one another, since, for $f \in C([a,b]\times[c,d])$

$$\Psi(\Phi(f))(x,y) = \Phi(f)(x)(y) = f(x,y)\,,$$

and for $f \in C([a,b];C([c,d]))$

$$\Phi(\Psi(f))(x)(y) = \Psi(f)(x,y) = f(x)(y)\,.$$

■