A STUDY OF THE ORDERED CHINESE RESTAURANT PROCESS AND RANDOM DISSECTIONS

by

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CONTRIBUTIONS

The work appearing in Chapter 2 is heavily based on the work in [45], which was written jointly by the present author and Douglas Rizzolo. The portions of [45] that are used in this dissertation were the contributions made by the present author under the advisement of Rizzolo, the present author's doctoral advisor. Permissions for the use of the joint work are discussed in the appendix.

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ABSTRACT

The ordered Chinese Restaurant Process and polygonal dissections are modern variants of classical objects – namely, the Chinese Restaurant Process and polygonal triangulations. While these classical objects are well-studied in the literature, much less is known about their modern counterparts. The general aim of this thesis is to close this gap by presenting the modern analogues of some classical results.

In the case of the ordered Chinese Restaurant Process, our primary interest lies in the associated up-down chains, a family of Markov processes on integer compositions. We show that, in some scaling limit, these chains converge to a diffusion on the open subsets of $(0, 1)$. This is the analogue of a result of Petrov, in which the limit of the up-down chains associated with the classical Chinese Restaurant Process is identified. Consequently, we construct an ordered analogue to Petrov's diffusion, and by extension, the infinitely-many-neutral-alleles diffusion model of Ethier and Kurtz. We also study the process obtained by projecting the up-down chain to its first coordinate. We state a condition on the initial distribution of the up-down chain that leads to this process having the Markov property and being intertwined with the up-down chain. In particular, these properties hold when the up-down chain is running in stationarity.

Our study of polygonal dissections is focused on describing the maximum vertex degree of a random dissection. We present a concentration inequality for this random variable that is analogous to a result of Gao and Wormald concerning triangulations. As a result, we resolve a conjecture posed in 2012 by Curien and Kortchemski.

Chapter 1

INTRODUCTION

In this thesis, we focus our attention on three objects: (i) the *up-down chains* associated with an ordered variant of the Chinese Restaurant Process, (ii) the corresponding *left-most column process*, and (iii) the *maximum vertex degree* in a random dissection. These objects are investigated in Chapters 2, 3, and 4 respectively. In this chapter, we introduce the settings in which these objects arise, and state the questions that motivate our subsequent study.

1.1 The Chinese Restaurant Process and its relatives

The two-parameter Chinese Restaurant Process (CRP) is a two-parameter family of Markov processes defined as follows. Let (α, θ) satisfy

$$0 \leq \alpha \leq 1,\, \theta + \alpha > 0 \qquad \text{or} \qquad \alpha < 0,\, -\frac{\theta}{\alpha} \in \mathbb{N}. \tag{1.1}$$

The corresponding CRP is the Markov chain $\{\Pi_n^{(\alpha,\theta)}\}_{n\geq 1}$ in which $\Pi_1^{(\alpha,\theta)} = \{\{1\}\}$, and for $n \geq 1$, $\Pi_{n+1}^{(\alpha,\theta)}$ is the random partition of $\{1, 2, \ldots, n+1\}$ constructed from $\Pi_n^{(\alpha,\theta)}$ by

(i) adding the integer $n+1$ to a set in $\Pi_n^{(\alpha,\theta)}$ of size s with probability $\frac{s-\alpha}{n+\theta}$, or

(ii) adding the singleton $\{n+1\}$ to $\Pi_n^{(\alpha,\theta)}$ with probability $\frac{\theta+\alpha N}{n+\theta}$, where N is the number of sets in $\Pi_n^{(\alpha,\theta)}$.

For instance, the partition $\Pi_2^{(\alpha,\theta)}$ will take the values $\{\{1,2\}\}$ and $\{\{1\},\{2\}\}$ with probabilities $\frac{1-\alpha}{1+\theta}$ and $\frac{\theta+\alpha}{1+\theta}$, respectively.

The two-parameter CRP was introduced by Pitman as a way to construct an *exchangeable partition of* $\mathbb{N}$ [40]. This partition, $\Pi_\infty^{(\alpha,\theta)}$, is the common extension of the sequence $\{\Pi_n^{(\alpha,\theta)}\}_{n\geq 1}$ to a random partition of $\mathbb{N}$, and it is exchangeable because its distribution is invariant under any bijection on $\mathbb{N}$ that fixes all but finitely many integers. In the literature, $\Pi_\infty^{(\alpha,\theta)}$ and $\{\Pi_n^{(\alpha,\theta)}\}_{n\geq 1}$ are actually regarded as the same object, allowing the CRP to be interpreted as a two-parameter family of exchangeable partitions. From this perspective, the CRP provides some of the most useful and interesting examples of such objects, serving as guiding examples in [40], exhibiting a number of unique properties [41, 39], and having applications in nonparametric Bayesian inference and machine learning [26, 6, 7, 50]. We note that the theory of exchangeable partitions was initially developed by Kingman to study the geneaology of a sample of alleles from a population [31].

An alternative description of the CRP comes from the setting of *partition structures*. A partition of a positive integer n is a tuple consisting of positive integers that are arranged into non-increasing order and sum to n. A partition structure is a sequence $\mu_1, \mu_2, \ldots$ of probability distributions on the partitions of $1, 2, \ldots$, respectively, satisfying the following consistency condition: if a random partition of $n+1$ is drawn according to μ_{n+1} and one of its components is chosen according to a size-biased pick, then the partition of n obtained by decrementing that component by 1 and rearranging has distribution μ_n. These objects were introduced by Kingman in an effort to describe the frequencies of alleles in a sample from a population [29]. A major result in this area, due to Kingman, establishes a correspondence between partition structures and exchangeable partitions [30, 31]. Under this correspondence, the CRP is identified with the *Ewens-Pitman partition structures*, Pitman's two-parameter extension of the Ewens sampling formula [14, 40]. Through this connection, the CRP has arisen in the context of population genetics [14], random permutations and matrices [39, 35], and stable subordinators [37].

Yet another point of view on the CRP comes from Kingman's *paintbox correspondence*, a de Finetti-type result that describes an exchangeable partition of $\mathbb{N}$ as a mixture of partitions obtained via Kingman's paintbox construction [31]. This result identifies an exchangeable partition with a mixing measure, a probability distribution on the closure of the Kingman simplex

$$\overline{\nabla}_\infty = \Big\{\mathbf{x} = (x_1, x_2, \dots) \ : \ x_1 \geq x_2 \geq \dots \geq 0, \ \sum_i x_i \leq 1\Big\}$$

that describes the limiting frequencies of the sets in the exchangeable partition. In the case of the CRP, the mixing measures form the two-parameter family of Poisson-Dirichlet distributions, Pitman and Yor's extension of Kingman's one-parameter Poisson-Dirichlet distribution [28, 44]. Aside from giving rise to the Poisson-Dirichlet Process (or Pitman-Yor Process), which has had utility in statistical settings, this distribution establishes a surprising connection between the CRP and the study of prime numbers [25, 5].

One recent development on the CRP was made by Petrov, who computed the scaling limit of some Markov chains associated to the CRP and identified the limiting process as a two-parameter extension of the infinitely-many-neutral-alleles diffusion model of Ethier and Kurtz [38, 12]. These Markov chains are the up-down chains associated to the CRP and are discussed in Section 2.1 (some related chains are discussed below). Other work has focused on establishing ordered analogues of the relevant theory – this is the study of composition structures, exchangeable compositions, ordered variants of the CRP, Poisson-Dirichlet interval partitions, and their applications [11, 23, 22, 43, 42, 16, 27]. Of particular interest was the problem of constructing ordered analogues to Petrov's diffusions, which was addressed in [16, 17, 18, 48, 49]. One question that remained open is whether any of these diffusions arise as limits of up-down chains associated to an ordered variant of the CRP (see [46] for a conjecture). Chapter 2 sheds light on this question.

We will study the ordered variant of the CRP that is defined as follows. Let (α, θ) satisfy

$$0 \leq \alpha \leq 1, \ \theta \geq 0, \ \theta + \alpha > 0. \tag{1.2}$$

The corresponding ordered CRP (oCRP) is the Markov chain $\{\mathbf{C}_n^{(\alpha,\theta)}\}_{n\geq 1}$ in which $\mathbf{C}_1^{(\alpha,\theta)} = (\{1\})$, and for $n \geq 1$, $\mathbf{C}_{n+1}^{(\alpha,\theta)}$ is the random composition (i.e. ordered partition) of $\{1, 2, \dots, n+1\}$ constructed from $\mathbf{C}_n^{(\alpha,\theta)}$ by

(i) adding the integer $n+1$ to a set in $\mathbf{C}_n^{(\alpha,\theta)}$ of size s with probability $\frac{s-\alpha}{n+\theta}$,

(ii) inserting the singleton $\{n+1\}$ into a new component created to the left of the first component of $\mathbf{C}_n^{(\alpha,\theta)}$ with probability $\frac{\theta}{n+\theta}$, or

(iii) inserting the singleton $\{n+1\}$ into a new component created to the right of the k^{th} component of $\mathbf{C}_n^{(\alpha,\theta)}$ with probability $\frac{\alpha}{n+\theta}$, where k ranges over all components of $\mathbf{C}_n^{(\alpha,\theta)}$.

For example, the composition $\mathbf{C}_2^{(\alpha,\theta)}$ will take the values $(\{1,2\})$, $(\{1\},\{2\})$, and $(\{2\},\{1\})$ with probabilities $\frac{1-\alpha}{1+\theta}$, $\frac{\alpha}{1+\theta}$, and $\frac{\theta}{1+\theta}$, respectively. We note that this oCRP is the left-to-right reversal of the one in [43] but is consistent with the order structure in [17, 18]. Moreover, the condition (1.2) is necessary – it is the only variant of (1.1) in which the above process is well-defined.

The up-down chains associated to the oCRP are a three-parameter family $\{\mathbf{X}_n^{(\alpha,\theta)}\}_{n\geq 1}$ in which (α,θ) satisfies (1.2) and $\mathbf{X}_n^{(\alpha,\theta)}$ is a Markov chain on the compositions of the integer n (i.e. rearrangements of partitions of n). As with any up-down chain, each transition in $\mathbf{X}_n^{(\alpha,\theta)}$ consists of a random enlargement – an up-step – followed by a random reduction – a down-step. For our chains, an up-step from the composition σ is performed by

(i) increasing the j^{th} component of σ by 1 with probability $\frac{\sigma_j-\alpha}{n+\theta}$, where j ranges over all components of σ,

(ii) inserting a 1 to the left of the first component of σ with probability $\frac{\theta}{n+\theta}$, or

(iii) inserting a 1 (immediately) to the right of the k^{th} component of σ with probability $\frac{\alpha}{n+\theta}$, where k ranges over all components of σ.

A down-step from the composition τ is performed by selecting a component of τ according to a size-biased pick, decrementing that component by 1, and deleting that component if its new value is zero. It should be clear that the up-step dynamic models the evolution of the sizes of the components in the oCRP. It is for this reason that we say these up-down chains are associated to the oCRP.

After considering the limiting behavior of the up-down chains in Chapter 2, we proceed to study the left-most column process. This is the process obtained by projecting an up-down chain onto its first coordinate, and is considered in Chapter 3. Here, we are interested in general properties (e.g. the Markov property), but the underlying motivation is to obtain a better understanding of the main result of Chapter 2. Although we make substantial progress on the immediate goal, further study is necessary (e.g. a convergence result) to meet the broader aim.

1.2 Polygonal Triangulations and Dissections

A triangulation of a convex polygon is a union of that polygon and some of its diagonals in which the diagonals partition the polygon into triangular regions and do not intersect. These objects (and their variants) have been studied extensively in the literature, appearing in combinatorics, computational geometry, probability, and physics [51, 34, 24, 3, 32, 2].

Viewing a triangulation as a graph, it is natural to wonder about its graph-theoretical properties. Some work in this direction include [10, 19, 20, 21, 33]. Of particular note is [20], in which Gao and Wormald offer a comprehensive description of the maximum vertex degree in a uniformly drawn dissection of the n-gon by means of a concentration inequality and limiting distribution.

In the case of dissections (where the regions are not necessarily triangular), much less is known about the maximum vertex degree. Concentration inequalities have been given by Bernasconi et. al. and Curien and Kortchemski, but these results are believed to be suboptimal [4, 9]. The latter authors conjectured that the optimal result is given by

$$\mathbb{P}\left(\left|\Delta_n - (\log_b(n) + \log_b \log_b(n))\right| > c \log_b \log_b(n)\right) \longrightarrow 0 \tag{1.3}$$

as $n \to \infty$, where $b = 1 + \sqrt{2}$ and c is an arbitrary positive number. This follows from the heuristic that the vertex degrees of large random dissections behave like independent random variables whose limiting behavior can be described [9]. This conjecture partly motivated the study in Chapter 4. Our other motivation is the work of Gao and Wormald, whose results we aimed to replicate in the dissection setting. Although we make no progress on the description of the limiting distribution, we do obtain appropriate analogues of their other results. This includes a concentration inequality that resolves the above conjecture.

Chapter 2

DIFFUSIONS ARISING FROM THE ORDERED CHINESE RESTAURANT PROCESS

2.1 Introduction

Recall that $\{\mathbf{X}_n^{(\alpha,\theta)}\}_{n\geq 1}$ is the family of up-down chains associated to the oCRP (see section 1.1). Letting `dec` be the map that reorders a tuple's components into non-increasing order, the processes $\{\mathtt{dec}(\mathbf{X}_n^{(\alpha,\theta)})\}_{n\geq 1}$ are the up-down chains associated with the classical Chinese Restaurant Process. In [38], Petrov showed that these chains converge (in an appropriate sense) to a Fellerian diffusion on the closure of the Kingman simplex (see section 1.1). This diffusion provides a two-parameter extension of the Ethier and Kurtz's infinitely-many-neutral-alleles diffusion model, and as a result of Petrov's algebraic approach, it could be described explicitly by specifying its generator on a core described by symmetric functions.

Results. In this chapter, we obtain results for the ordered up-down chains analogous to those of Petrov for the classical up-down chains. Our main result, Theorem 2.1.1 below, establishes the convergence of these processes (in an appropriate sense) to a Fellerian diffusion on $\mathcal{U}$, the set of open subsets of $(0,1)$ equipped with the Hausdorff metric on the complements of sets. In Proposition 2.7.1, we offer an explicit description of this diffusion by specifying its generator on a core described by *quasisymmetric* functions. In Corollary 2.1.1, we show that our diffusions are natural ordered analogues of the diffusions constructed by Petrov (and consequently, Ethier and Kurtz).

To state our results, we will need an inclusion map ι that sends $\sigma = (\sigma_1, \ldots, \sigma_l)$, a composition of n, to the open set given by

$$\iota(\sigma) = \left(0, \frac{\sigma_1}{n}\right) \cup \left(\frac{\sigma_1}{n}, \frac{\sigma_1+\sigma_2}{n}\right) \cup \ldots \cup \left(1 - \frac{\sigma_l}{n}, 1\right).$$

Theorem 2.1.1. *Let (α, θ) be parameters satisfying $0 \le \alpha < 1$, $\theta \ge 0$, and $\alpha + \theta > 0$. There is a Feller diffusion $\mathbf{U}^{(\alpha,\theta)}$ on $\mathcal{U}$ such that the path convergence*

$$\left(\iota(\mathbf{X}_n^{(\alpha,\theta)}(\lfloor n^2 t \rfloor))\right)_{t\ge 0} \longrightarrow_d \left(\mathbf{U}^{(\alpha,\theta)}(t)\right)_{t\ge 0}$$

holds in distribution on the Skorokhod space $D([0,\infty),\mathcal{U})$ whenever the initial distributions converge, $\iota(\mathbf{X}_n^{(\alpha,\theta)}(0)) \to_d \mathbf{U}^{(\alpha,\theta)}(0)$. Here, $\lfloor a \rfloor$ is the integer part of a.

Recall that every open set in $\mathcal{U}$ can be written uniquely as a union of disjoint open intervals. Therefore, we can define a map `decLengths` $: \mathcal{U} \to \overline{\nabla}_\infty$ that takes an open set to the sequence containing the lengths of these intervals in non-increasing order. By noting that this map is continuous (we equip $\overline{\nabla}_\infty$ with the supremum norm), we obtain the following corollary.

Corollary 2.1.1. The process $(\texttt{decLengths}(\mathbf{U}^{(\alpha,\theta)}(t)))_{t\ge 0}$ is exactly the (α,θ) diffusion constructed by Petrov.

Techniques. Our proof of Theorem 2.1.1 follows the spirit of [38]. Namely, we use combinatorial and algebraic techniques to identify convenient bases in which to express the transition operators of the up-down chains in a pseudo-triangular form. From this, it will be easy to compute the limit of the generators of the up-down chains. The convergence of processes will then follow from the convergence of generators by some standard arguments.

Outline. This chapter is organized as follows. In Section 2.2, we introduce a graph structure and some operations on the space of compositions. In Section 2.3, we obtain an explicit description of the up-down chains. In Section 2.4, we introduce the algebra of quasisymmetric functions and establish its connection with the graph of compositions. In Section 2.5, we obtain explicit formulas for the transition operators of the up-down chains in terms of quasisymmetric functions. In Section 2.6, we address metric properties of $\mathcal{U}$ and $C(\mathcal{U})$, and identify a useful homomorphism from the algebra of quasisymmetric functions into $C(\mathcal{U})$. In Section 2.7, the convergence results are obtained.

Notation. The following will be used throughout this chapter. For a topological space X, we denote by $C(X)$ the space of continuous functions from X to $\mathbb{R}$ equipped with the supremum norm. Finite topological spaces will always be equipped with the discrete topology. A monotone map is a map that is strictly increasing. Any sum or product over an empty index set will be regarded as a zero or one, respectively. The set of positive integers will be denoted by $\mathbb{N}$. The subset of positive integers $\{1, ..., k\}$ will be denoted by $[k]$, and $[0]$ will denote the empty subset of $\mathbb{N}$. The falling factorial will be denoted using *factorial exponents* – that is, $x^{\downarrow b} = x(x-1) \cdot \ldots \cdot (x-b+1)$ for a real number x and non-negative integer b , and $0^{\downarrow 0} = 1$ by convention. We note here the following properties, which hold whenever b is positive:

$$(x+1)^{\downarrow b} = x^{\downarrow b} + b\, x^{\downarrow (b-1)}, \qquad x\, x^{\downarrow (b-1)} = x^{\downarrow b} + (b-1)\, x^{\downarrow (b-1)}. \tag{2.1}$$

2.2 The Graph of Compositions

In this section, we introduce the graph of compositions and some combinatorial identities on this graph. These identities will be useful for analyzing the transition operator of the up-down chains.

Compositions provide the state spaces for our up-down chains. They are defined as follows.

Definition 2.2.1. For $n \geq 1$, a *composition* of n is a tuple $\sigma = (\sigma_1, ..., \sigma_k)$ of positive integers that sum to n. The composition of $n = 0$ is the empty tuple, which we denote by $\varnothing$. If σ is a composition of n with k components, we say it has *size* $|\sigma| = n$ and *length* $\ell(\sigma) = k$. We denote the set of all compositions of n by $\mathcal{C}_n$ and their union by $\mathcal{C} = \cup_{n \geq 0} \mathcal{C}_n$.

We can associate a unique diagram of boxes to every composition, similarly to how a Young diagram can be associated to a partition of an integer. The diagram for a composition σ will contain $|\sigma|$ boxes arranged into $\ell(\sigma)$ columns with σ_j boxes in the j^{th}

column, see Figure 2.1. The diagram corresponding to $\varnothing$ contains no boxes. Throughout this work, we think of a composition both as a tuple and as its corresponding diagram.

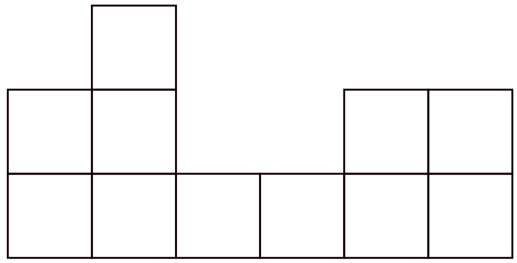

Figure 2.1: The composition diagram of $\tau = (2,3,1,1,2,2)$ has 11 boxes arranged into 6 columns with τ_j boxes in the j^{th} column

We will need the following operations on $\sigma = (\sigma_1, \sigma_2, \dots \sigma_{\ell(\sigma)}) \in \mathcal{C}$. We define

$$\sigma_k^l = \begin{cases} (\sigma_k, \dots, \sigma_l), & 1 \leq k \leq l \leq \ell(\sigma), \\ \varnothing, & \text{else.} \end{cases}$$

For $k \in [\ell(\sigma)]$, we define the *stacking* operation by

$$\sigma + \square_k = (\sigma_1^{k-1}, \sigma_k + 1, \sigma_{k+1}^{\ell(\sigma)}),$$

which can be thought diagrammatically as the composition obtained by *stacking* a box on top of the k^{th} column of σ. For $s \in [\ell(\sigma)+1]$, we define the *insertion* operation by

$$\sigma \oplus \square_s = (\sigma_1^{s-1}, 1, \sigma_s^{\ell(\sigma)}),$$

which can be thought of diagrammatically as the composition obtained by *inserting* a one-box column into σ that becomes the s^{th} column. For $k \in [\ell(\sigma)]$, we define

$$\sigma / \square_k = (\sigma_1^{k-1}, 1, \sigma_{k+1}^{\ell(\sigma)}),$$

which can be thought of diagrammatically as the composition obtained by replacing the k^{th} column of σ with a single box.

We also introduce operations $-$ and $\ominus$ inverse to $+$ and $\oplus$, respectively, so that $\tau - \Box_r = \sigma$ whenever $\tau = \sigma + \Box_r$ and $\tau \ominus \Box_s = \sigma$ whenever $\tau = \sigma \oplus \Box_s$. The number of ways to obtain τ from σ by stacking or inserting a box will be denoted by

$$\kappa(\sigma, \tau) = |\{r : \tau = \sigma + \Box_r\}| + |\{s : \tau = \sigma \oplus \Box_s\}|.$$

When $\kappa(\sigma, \tau) > 0$, we write $\sigma \nearrow \tau$. We write $\sigma \nearrow \tau^+$ or $\sigma \nearrow \tau^\oplus$ when τ can be obtained from σ using the stacking or inserting operation, respectively.

The following proposition records basic properties of these operations that we will frequently need.

Proposition 2.2.1. Let $\sigma \in \mathcal{C}$, $r, r' \in [\ell(\sigma)]$, and $s, s' \in [\ell(\sigma) + 1]$. The following properties hold:

(i) $\sigma + \Box_r \neq \sigma \oplus \Box_s$.

(ii) $\sigma + \Box_r = \sigma + \Box_{r'}$ if and only if $r = r'$.

(iii) $\sigma \oplus \Box_s = \sigma \oplus \Box_{s'}$ if and only if $\sigma_u = 1$ for $\min(s, s') \leq u < \max(s, s')$, which holds if and only if $(\sigma \oplus \Box_s)_u = 1$ for $\min(s, s') \leq u \leq \max(s, s')$.

(iv) $\kappa(\sigma, \sigma + \Box_r) = 1$.

(v) $\kappa(\sigma, \sigma \oplus \Box_s)$ = length of the longest sequence of one-box columns in $\sigma \oplus \Box_s$ containing the box in column s.

(vi) There exists a unique $c \in [\ell(\sigma) + 1]$ such that $\sigma \oplus \Box_s = \sigma \oplus \Box_c$ and either $c = 1$ or $\sigma_{c-1} \neq 1$.

(vii) For every u with $\sigma_u = 1$, there exists a unique $c \in [\ell(\sigma)]$ such that $\sigma \ominus \Box_u = \sigma \ominus \Box_c$ and either $c = 1$ or $\sigma_{c-1} \neq 1$.

Proof. To obtain (i), observe that the two compositions differ in length. For (ii) and (iii), a direct computation will verify the claim. The property in (iv) then follows from directly (i) and (ii).

For (v)-(vi), we consider the following equivalence relation on $[\ell(\sigma)+1]$: $u \sim u'$ whenever $\sigma \oplus \Box_u = \sigma \oplus \Box_{u'}$. Using (iii), it can be verified that the resulting equivalence classes are intervals of integers, and that $u-1 \sim u$ if and only if $\sigma_{u-1} = 1$. Therefore, the minimum of the class containing s is the unique c in (vi). It also follows from (iii) that every sequence of one-box columns in $\sigma \oplus \Box_s$ corresponds to a subinterval of an equivalence class. Accordingly, the length of the longest such sequence containing column s is the length of the longest interval containing s and lying in some equivalence class. Since our equivalence classes are intervals themselves, this is exactly the size of the class containing s. Applying now (i), we see that this quantity coincides with $\kappa(\sigma, \sigma \oplus \Box_s)$, establishing (v).

The statement in (vii) can be obtained in a manner similar to (vi).

□

The graph of compositions is the directed multi-graph whose vertices are the elements of $\mathcal{C}$ and that contains $\kappa(\sigma, \tau)$ directed edges from σ to τ. On this graph, moving along an edge in the forward direction corresponds to either stacking or inserting a box, while moving in the reverse direction corresponds to the inverse operation, either $-$ or $\ominus$. Accordingly, a path can be viewed as both a construction and a deconstruction, providing a way of adding boxes to the smaller composition to obtain the larger one and vice versa. Under the deconstruction interpretation, a path decomposes into two parts: (1) a *box selection*, which identifies the boxes to be removed, and (2) an *order of removal*, which specifies when to remove each box. In what follows, we make use of this decomposition to count the number of paths between compositions. We denote by $g(\sigma, \tau)$ the number of paths from σ to τ and set $g(\tau) = g(\varnothing, \tau)$.

Fix compositions σ and $\tau \neq \varnothing$ with $g(\sigma, \tau) > 0$. Since the operations $-$ and $\ominus$ remove boxes from the top of a column, identifying which boxes to remove from a column is equivalent to providing the number of boxes to remove from that column. As a result, a box selection can be described as a tuple b whose r^{th} component indicates the number of boxes to remove from the r^{th} column of τ. For the removal of these boxes to result in σ, the tuple must satisfy $\tau - b \equiv \sigma$, where the relation $x \equiv y$ means that the tuples x and y are equal after removing all zero-valued components from each. Therefore, every box selection associated with a path from σ to τ can be identified as an element in

$$B_{\sigma,\tau} = \{b \in \mathbb{Z}_{\geq 0}^{\ell(\tau)} : \tau - b \equiv \sigma\}.$$

An order of removal must specify when to remove each box in a box selection. However, since a box can only be removed from the top of a column, specifying which box is the k^{th} box to be removed is equivalent to specifying the k^{th} column to remove from. One way, then, to describe an order of removal for N boxes in τ is with a map $c : [N] \to [\ell(\tau)]$ that sends k to the column location of the k^{th} box to be removed. An alternative is to provide the tuple of preimages $(c^{-1}\{1\}, c^{-1}\{2\}, ..., c^{-1}\{\ell(\tau)\})$. Using the latter, it follows that an order of removal for the box selection $b \in B_{\sigma,\tau}$ is described by an ordered partition of $[\sum b_r]$ whose parts have sizes given by b. Ignoring the positions where $b_r = 0$, these objects can be identified as compositions of $[\sum b_r]$, and it follows that there are exactly $(\sum b_r)!/\prod b_r!$ of them. The number of paths from σ to τ is then given by

$$g(\sigma, \tau) = \sum_{b \in B_{\sigma,\tau}} \frac{(|\tau| - |\sigma|)!}{\prod_r b_r!}. \tag{2.2}$$

We remark that, although we initially placed conditions on σ and τ, the above identity holds for all compositions. The remaining cases $g(\sigma, \tau) = 0$ and $\tau = \varnothing$ can be verified directly since $B_{\sigma,\tau}$ is either empty or the singleton $\{\varnothing\}$. In addition, we have the special case

$$g(\tau) = \frac{|\tau|!}{\prod_r \tau_r!},$$

which follows from setting $\sigma = \varnothing$ and observing that $B_{\varnothing,\tau} = \{\tau\}$.

2.3 The Up and Down Kernels

In this section, we obtain an explicit description of the up-down chains by factorizing their transition kernels into an up-step kernel and down-step kernel. This factorization naturally leads to a factorization for the transition operators.

To begin, we give a diagrammatical description of the up-steps and down-steps that govern the transitions in the up-down chains. Let (α, θ) satisfy $\theta \geq 0$, $0 \leq \alpha < 1$, and $\alpha + \theta > 0$. Given a composition σ, we perform an (α, θ) *up-step* by stacking a box to the i^{th} column of σ with probability proportional to $\sigma_i - \alpha$, inserting a one-box column that becomes the left-most column with probability proportional to θ, and inserting a box to the right of any column of σ with probability proportional to α. We perform a *down-step* from a non-empty composition τ by removing a box from τ uniformly at random (when this box belongs to a one-box column, we delete the entire column, and when this box lies below other boxes, the other boxes should 'fall' into place).

Let $p^{\uparrow}_{(\alpha,\theta)}(\sigma, \cdot)$ and $p^{\downarrow}(\tau, \cdot)$ be the distributions of the compositions resulting from an (α, θ) up-step from σ and a down-step from τ, respectively. Then, $p^{\uparrow}_{(\alpha,\theta)}$ is a transition kernel from $\mathcal{C}_n$ to $\mathcal{C}_{n+1}$, which we call the up-step kernel, and $p^{\downarrow}$ is a transition kernel from $\mathcal{C}_{n+1}$ to $\mathcal{C}_n$, which we call the down-step kernel. For each $n \geq 0$, the transition kernel $T_n^{(\alpha,\theta)}$ of the up-down chain $(\mathbf{X}_n^{(\alpha,\theta)}(k))_{k \geq 0}$ is a kernel on $\mathcal{C}_n$ and it admits the factorization

$$T_n^{(\alpha,\theta)}(\sigma, \sigma') = \sum_{\tau \in \mathcal{C}_{n+1}} p^{\uparrow}_{(\alpha,\theta)}(\sigma, \tau) p^{\downarrow}(\tau, \sigma').$$

It follows from Proposition 2.2.1 and the above descriptions that the up- and down-step kernels are given by

$$p^{\uparrow}_{(\alpha,\theta)}(\sigma, \tau) = \begin{cases} \frac{\sigma_i - \alpha}{|\sigma| + \theta}, & \tau = \sigma + \square_i, \\ \frac{\theta + \alpha(\kappa(\sigma,\tau) - 1)}{|\sigma| + \theta}, & \tau = \sigma \oplus \square_1, \\ \frac{\alpha}{|\sigma| + \theta} \kappa(\sigma, \tau), & \tau = \sigma \oplus \square_j \neq \sigma \oplus \square_1, \\ 0, & \text{else}, \end{cases}$$

and

$$p^{\downarrow}(\tau, \sigma) = \begin{cases} \frac{\tau_k}{|\tau|} \kappa(\sigma, \tau), & \tau \in \{\sigma + \square_k, \sigma \oplus \square_k\}, \\ 0, & \text{else}. \end{cases}$$

Notice that $p^{\downarrow}$ is well-defined since Parts (i), (ii), and (iii) of Proposition 2.2.1 imply that if $\tau \in \{\sigma + \square_k, \sigma \oplus \square_k\}$ and $\tau \in \{\sigma + \square_{k'}, \sigma \oplus \square_{k'}\}$, then $\tau_k = \tau_{k'}$.

2.4 The Algebra of Quasisymmetric Functions

In this section, we introduce the algebra of quasisymmetric functions and establish its connection to the graph of compositions. In particular, we show that quasisymmetric functions can be easily expressed in terms of the path-counting function g (Proposition 2.4.1). This result inspires our later choice to write the transition operators in terms of quasisymmetric functions.

To begin, we define

$$\mathcal{I}_k = \{i \colon [k] \to \mathbb{N} \,\big|\, i \text{ is monotone}\}, \qquad k \geq 0,$$

and

$$\mathcal{I}_{k,l} = \{i \in \mathcal{I}_k : \text{range } i \subset [\,l\,]\}, \qquad k, l \geq 0.$$

Note that when $k = 0$, these sets are singletons containing the empty function.

Every composition σ has an associated quasisymmetric *monomial* in the formal variables $y_1, y_2, \ldots$ defined by

$$m_\sigma = \sum_{i \in \mathcal{I}_{\ell(\sigma)}} \prod_{r=1}^{\ell(\sigma)} y_{i_r}^{\sigma_r}.$$

Note the special case $m_\varnothing \equiv 1$. The collection $\{m_\sigma\}_{\sigma \in \mathcal{C}}$ is known to be a linear basis for Λ, the real algebra of quasisymmetric functions in the formal variables $\{y_k\}$. This algebra admits a filtration by the finite-dimensional spaces

$$\Lambda_k = \text{span}\, \{m_\sigma\}_{|\sigma| \leq k}, \qquad k \geq 0.$$

Every quasisymmetric function $q \in \Lambda$ has a natural identification as a function on $\mathcal{C}_n$, denoted by q_n, which is formally obtained by setting the variables $y_{n+1}, y_{n+2}, \ldots$ equal to 0 and treating the resulting formal sum as a polynomial in n variables. For monomials, this is given by

$$(m_\sigma)_n(\tau) = \sum_{i \in \mathcal{I}_{\ell(\sigma), \ell(\tau)}} \prod_{r=1}^{\ell(\sigma)} \tau_{i_r}^{\sigma_r}, \qquad \tau \in \mathcal{C}_n.$$

It will often be more convenient to work with a variant of the monomials, obtained by replacing the exponents of a monomial by factorial powers:

$$m^*_\sigma = \sum_{i \in \mathcal{I}_{\ell(\sigma)}} \prod_{r=1}^{\ell(\sigma)} y_{i_r}^{\downarrow \sigma_r}.$$

Again, we have the special case $m^*_\varnothing \equiv 1$. Moreover, since the homogeneous component of largest degree in m^*_σ is m_σ, the collection $\{m^*_\sigma\}_{\sigma \in \mathcal{C}}$ is also a linear basis for Λ. Still, the primary reason we consider these functions is because they arise naturally in the identity below.

Proposition 2.4.1. For all compositions σ and τ, the following identity holds:

$$(m^*_\sigma)_{|\tau|}(\tau) = \frac{g(\sigma,\tau)|\tau|^{\downarrow|\sigma|}}{g(\tau)}.$$

Proof. Let us first assume that $\sigma, \tau \neq \varnothing$ and $g(\sigma,\tau) > 0$. In this case, we construct a bijection between the collection of box selections $B_{\sigma,\tau}$ and the collection of monotone maps

$$\mathcal{I}_{\sigma,\tau} = \{i \in \mathcal{I}_{\ell(\sigma),\ell(\tau)} : \sigma_r \leq \tau_{i_r} \text{ for all } r\}.$$

To begin, fix a box selection b in $B_{\sigma,\tau}$ and place an order of removal on b. This defines a deconstruction of τ into σ from which the columns in σ can be identified as descendants of the columns in τ. Let i be the map associated with this identification – that is, i sends (the position of) a column in σ to (the position of) its ancestor in τ. Since deconstructing a composition preserves the order of columns, this map must be monotone. In addition, since the r^{th} column of σ is formed by removing b_{i_r} boxes from the i_r^{th} column of τ, the identity $\sigma_r = \tau_{i_r} - b_{i_r}$ must hold. From this, it follows that $i \in \mathcal{I}_{\sigma,\tau}$, and we define our candidate bijection to send the box selection b to the *ancestral map* i.

An alternative description of the ancestral map associated to b is as the monotone map with domain $[\ell(\sigma)]$ and range

$$A = \{u \in [\ell(\tau)] : \tau_u \neq b_u\}.$$

To see this, note that the range of the ancestral map identifies the columns in τ that have descendants in σ, which are exactly the columns that survive the deconstruction. Combining this description with our previous one, it can be shown that the map sending $i \in \mathcal{I}_{\sigma,\tau}$ to the box selection

$$b_u = \begin{cases} \tau_u, & u \in [\ell(\tau)] \setminus \text{range } i, \\ \tau_u - \sigma_{i^{-1}(u)}, & u \in \text{range } i \end{cases}$$

is the inverse of the map $b \mapsto i$, and as a result, that these maps are bijections.

Having established a correspondence between $B_{\sigma,\tau}$ and $\mathcal{I}_{\sigma,\tau}$, we can rewrite (2.2) as

$$\begin{aligned} g(\sigma,\tau) &= \sum_{i \in \mathcal{I}_{\sigma,\tau}} \frac{(|\tau| - |\sigma|)!}{\prod\limits_{u \notin \text{range } i} \tau_u! \ \prod_{r=1}^{\ell(\sigma)} (\tau_{i_r} - \sigma_r)!} \\ &= (|\tau| - |\sigma|)! \sum_{i \in \mathcal{I}_{\sigma,\tau}} \frac{\prod_{r=1}^{\ell(\sigma)} \tau_{i_r}!}{\prod_{u=1}^{\ell(\tau)} \tau_u! \prod_{r=1}^{\ell(\sigma)} (\tau_{i_r} - \sigma_r)!} \\ &= \frac{(|\tau| - |\sigma|)!}{\prod_{u=1}^{\ell(\tau)} \tau_u!} \sum_{i \in \mathcal{I}_{\sigma,\tau}} \prod_{r=1}^{\ell(\sigma)} \frac{\tau_{i_r}!}{(\tau_{i_r} - \sigma_r)!} \\ &= \frac{|\tau|!}{|\tau|^{\downarrow|\sigma|} \prod_{u=1}^{\ell(\tau)} \tau_u!} \sum_{i \in \mathcal{I}_{\sigma,\tau}} \prod_{r=1}^{\ell(\sigma)} \tau_{i_r}^{\downarrow \sigma_r} \\ &= \frac{g(\tau)}{|\tau|^{\downarrow|\sigma|}} \sum_{i \in \mathcal{I}_{\ell(\sigma),\ell(\tau)}} \prod_{r=1}^{\ell(\sigma)} \tau_{i_r}^{\downarrow \sigma_r} \\ &= \frac{g(\tau)}{|\tau|^{\downarrow|\sigma|}} (m_\sigma^*)_{|\tau|}(\tau), \end{aligned}$$

establishing the identity when $g(\sigma,\tau) > 0$ and $\sigma, \tau \neq \varnothing$. The cases $\sigma = \varnothing$ and $\tau = \varnothing$ are trivial, and for the remaining case, simply observe that

$$g(\sigma,\tau) = 0 \iff B_{\sigma,\tau} = \emptyset \iff \mathcal{I}_{\sigma,\tau} = \emptyset \iff (m_\sigma^*)_{|\tau|}(\tau) = 0.$$

□

An immediate consequence of this identity is that a quasisymmetric function can be recovered from its actions on compositions.

Proposition 2.4.2. The map $q \mapsto \{q_n\}$ from Λ to $\prod_{n=0}^{\infty} C(\mathcal{C}_n)$ is injective (each $\mathcal{C}_n$ is equipped with the discrete topology).

Proof. Viewing $\prod_{n=0}^{\infty} C(\mathcal{C}_n)$ as a real vector space with standard sequence operations, the map $q \mapsto \{q_n\}$ is linear. As such, it will suffice to show that it has a trivial kernel. Let q be in this kernel and $\sum_{\sigma \in \mathcal{C}} a_\sigma m_\sigma^*$ be its expansion in the monomial-variant basis. By assumption, we have that

$$0 = \sum_{\sigma \in \mathcal{C}} a_\sigma (m_\sigma^*)_{|\tau|}(\tau)$$

for every composition τ. However, Proposition 2.4.1 gives us the equivalence

$$(m_\sigma^*)_{|\tau|}(\tau) \neq 0, \quad |\tau| \leq |\sigma| \qquad \Longleftrightarrow \qquad \sigma = \tau,$$

so the above sum simplifies to

$$0 = a_\tau (m_\tau^*)_{|\tau|}(\tau) + \sum_{\substack{\sigma \in \mathcal{C} \\ |\sigma| < |\tau|}} a_\sigma (m_\sigma^*)_{|\tau|}(\tau).$$

Now we proceed inductively. In the base case, we set $\tau = \varnothing$ above to obtain $a_\varnothing = 0$. For the inductive step, we fix $n \in \mathbb{N}$ and assume that $a_\sigma = 0$ whenever $|\sigma| < n$. Substituting any $\tau \in \mathcal{C}_n$ above leads to the conclusion that $a_\tau = 0$, so the assumption can be extended to the case $|\sigma| < n+1$. This gives us that $a_\sigma = 0$ for all σ, and hence, $q = 0$.

□

2.5 The Up-Down Factorization

In this section, we obtain explicit formulas for the transition operators of the up-down chains that make taking the limit feasible. We follow the general approach in [8, 38], factorizing a transition operator into an up- and down-operator and then handling these factors separately. The down-operator case is straightforward and is done in Proposition 2.5.1. The up-operator case is more challenging and is addressed in Proposition 2.5.2.

To begin, we equip each $\mathcal{C}_n$ with the discrete topology and each $C(\mathcal{C}_n)$ with the supremum norm. The transition operator of the process $\mathbf{X}_n^{(\alpha,\theta)}$ is the operator $\mathcal{T}_n^{(\alpha,\theta)}\colon C(\mathcal{C}_n) \to C(\mathcal{C}_n)$ given by

$$(\mathcal{T}_n^{(\alpha,\theta)} f)(\sigma) = \sum_{\sigma' \in \mathcal{C}_n} T_n^{(\alpha,\theta)}(\sigma,\sigma') f(\sigma').$$

Each transition operator can be factorized as $\mathcal{T}_n^{(\alpha,\theta)} = U_{n,n+1}^{(\alpha,\theta)} D_{n+1,n}$, where $U_{n,n+1}^{(\alpha,\theta)} : C(\mathcal{C}_{n+1}) \to C(\mathcal{C}_n)$ and $D_{n+1,n} : C(\mathcal{C}_n) \to C(\mathcal{C}_{n+1})$ are defined by

$$(U_{n,n+1}^{(\alpha,\theta)} f)(\sigma) = \sum_{\tau \in \mathcal{C}_{n+1}} p^{\uparrow}_{(\alpha,\theta)}(\sigma,\tau) f(\tau),$$
$$(D_{n+1,n} g)(\tau) = \sum_{\sigma \in \mathcal{C}_n} p^{\downarrow}(\tau,\sigma) g(\sigma).$$

We call these operators the *up-operator* and *down-operator*, respectively, and they can be thought of as transition operators associated with a single up-step or down-step. Explicit formulas for these operators and the transition operators are given below. For simplicity, we delay the proof of the up-operator formula until the end of the section.

Proposition 2.5.1. The actions of the down-operators are completely described by the formula

$$D_{n+1,n}(m^*_\rho)_n = \frac{n - |\rho| + 1}{n+1}(m^*_\rho)_{n+1}, \qquad n \geq 0, \rho \in \mathcal{C}.$$

Proof. The formula is trivial when $|\rho| > n$. When $|\rho| \leq n$, we use Proposition 2.4.1, the identity $p^{\downarrow}(\tau,\sigma) = \frac{g(\sigma)}{g(\tau)}\kappa(\sigma,\tau)$, and a standard path-counting identity to obtain the formula:

$$
\begin{aligned}
(D_{n+1,n}(m^*_\rho)_n)(\tau) &= \sum_{\sigma\in\mathcal{C}_n} p^\downarrow(\tau,\sigma)(m^*_\rho)_n(\sigma) \\
&= \sum_{\sigma\in\mathcal{C}_n} \frac{g(\sigma)}{g(\tau)}\kappa(\sigma,\tau)\frac{g(\rho,\sigma)|\sigma|^{\downarrow|\rho|}}{g(\sigma)} \\
&= \frac{n^{\downarrow|\rho|}}{g(\tau)} \sum_{\sigma\in\mathcal{C}_n} g(\rho,\sigma)\kappa(\sigma,\tau) \\
&= \frac{n^{\downarrow|\rho|}}{g(\tau)} g(\rho,\tau) \\
&= \frac{n-|\rho|+1}{n+1}(m^*_\rho)_{n+1}(\tau).
\end{aligned}
$$

To see that this is a complete description, note from Proposition 2.4.1 that

$$
(m^*_\sigma)_{|\sigma|}(\tau) = \frac{|\sigma|!}{g(\sigma)}\,\mathbb{1}(\sigma=\tau),
$$

so the collection $\{(m^*_\sigma)_n\}_{|\sigma|=n}$ is a basis for $C(\mathcal{C}_n)$.

□

Proposition 2.5.2. The actions of the up-operators are completely described by the formula

$$
\begin{aligned}
U^{(\alpha,\theta)}_{n,n+1}(m^*_\rho)_{n+1} = \frac{1}{n+\theta}\Bigg((n+|\rho|+\theta)(m^*_\rho)_n + \sum_{\substack{s=1\\ \rho_s=1}}^{\ell(\rho)} \eta_s (m^*_{\rho\ominus\Box_s})_n \\
+ \sum_{\substack{s=1\\ \rho_s\geq 2}}^{\ell(\rho)} \rho_s(\rho_s-1-\alpha)(m^*_{\rho-\Box_s})_n\Bigg), \qquad n\geq 0, \rho\in\mathcal{C},
\end{aligned}
$$

where $\eta_1=\theta$ and $\eta_s=\alpha$ otherwise. Alternatively, we have the factorization

$$
\begin{aligned}
U^{(\alpha,\theta)}_{n,n+1}(m^*_\rho)_{n+1} = \frac{n+|\rho|+\theta}{n+\theta}(m^*_\rho)_n \\
+ \frac{|\rho|(|\rho|-1+\theta)}{n+\theta} \sum_{\mu:\mu\nearrow\rho} p^\uparrow_{(\alpha,\theta)}(\mu,\rho)\frac{g(\mu)}{g(\rho)}(m^*_\mu)_n.
\end{aligned}
$$

Proposition 2.5.3. The actions of the transition operators are completely described by the formula

$$(\mathcal{T}_n^{(\alpha,\theta)} - \mathbf{1})(m^*_\rho)_n = -\frac{|\rho|(|\rho|-1+\theta)}{(n+\theta)(n+1)}(m^*_\rho)_n + \frac{(n-|\rho|+1)}{(n+\theta)(n+1)}\Bigg(\sum_{\substack{c=1\\ \rho_c=1}}^{\ell(\rho)} \eta_c (m^*_{\rho\ominus\square_c})_n + \sum_{\rho_c\geq 2}\rho_c(\rho_c-1-\alpha)(m^*_{\rho-\square_c})_n\Bigg),$$

where $\eta_1 = \theta$ and $\eta_s = \alpha$ otherwise, or the factorization

$$(\mathcal{T}_n^{(\alpha,\theta)} - \mathbf{1})(m^*_\rho)_n = \frac{|\rho|(|\rho|-1+\theta)}{(n+\theta)(n+1)}\Big(-(m^*_\rho)_n + (n-|\rho|+1)\sum_{\mu:\mu\nearrow\rho} p^{\uparrow}_{(\alpha,\theta)}(\mu,\rho)\frac{g(\mu)}{g(\rho)}(m^*_\mu)_n\Big).$$

Proof. Both formulas follow immediately from Propositions 2.5.1 and 2.5.2.

□

The remainder of this section is dedicated to proving Proposition 2.5.2. Letting

$$h(i,\rho,\tau) = \prod_{r=1}^{\ell(\rho)} \tau_{i_r}^{\downarrow\rho_r}$$

for $i \in \mathcal{I}_{\ell(\rho),\ell(\tau)}$, this amounts to evaluating the sum

$$\sum_{\tau:\sigma\nearrow\tau} p^{\uparrow}_{(\alpha,\theta)}(\sigma,\tau)(m^*_\rho)_{|\tau|}(\tau) = \sum_{\tau:\sigma\nearrow\tau}\sum_{i\in\mathcal{I}_{\ell(\rho),\ell(\tau)}} p^{\uparrow}_{(\alpha,\theta)}(\sigma,\tau)h(i,\rho,\tau) \tag{2.3}$$

for all compositions ρ and σ. To handle this sum, we rely on some bijections defined on classes of monotone functions and identities involving h.

To begin, we introduce some operations on monotone functions. Let k, l, and u be positive integers satisfying $k,u \leq l$ and define

$$\mathcal{I}_{k,l,u} = \{i \in \mathcal{I}_{k,l} : u \in \text{range } i\},$$

and

$$\mathcal{I}^c_{k,l,u} = \mathcal{I}_{k,l} \setminus \mathcal{I}_{k,l,u}.$$

For $i \in \mathcal{I}_{k,l,u}$, let $i \setminus u$ be the monotone function with domain and range given by $[k-1]$ and (range i) $\setminus \{u\}$, respectively. For $i \in \mathcal{I}^c_{k-1,l,u}$, let $i \cup u$ be the monotone function with domain and range given by $[k]$ and (range i) $\cup \{u\}$, respectively, and let i^u be the monotone function with domain $[k-1]$ and whose range is obtained from the range of i by decrementing by 1 the elements that are larger than u. Explicitly,

$$(i \setminus u)_r = i_{r+\mathbb{1}(i_r \geq u)},$$

and

$$i^u_r = i_r - \mathbb{1}(i_r > u).$$

Setting

$$\phi_u(i) = 1 + |\{z \in \text{range } i : z < u\}|$$

for all monotone functions i and positive integers u, it can be verified (see Proposition 2.5.4 below) that the position of u in $i \cup u$ is given by $\phi_u(i)$. Thus, we also have

$$(i \cup u)_r = \begin{cases} u, & r = \phi_u(i), \\ i_{r-\mathbb{1}(r>\phi_u(i))}, & \text{else.} \end{cases}$$

The following result summarizes the basic properties of the above operations.

Proposition 2.5.4. Let k, l, and u be positive integers satisfying $k, u \leq l$. The following statements hold:

(i) the map $i \mapsto i \setminus u$ is a bijection from $\mathcal{I}_{k,l,u}$ to $\mathcal{I}^c_{k-1,l,u}$,

(ii) the map $i \mapsto i \cup u$ is a bijection from $\mathcal{I}^c_{k-1,l,u}$ to $\mathcal{I}_{k,l,u}$,

(iii) the map $i \mapsto i^u$ is a bijection from $\mathcal{I}^c_{k-1,l,u}$ to $\mathcal{I}_{k-1,l-1}$, and

(iv) for $i \in \mathcal{I}_{k,l,u}$, we have the equalities

$$i^{-1}(u) = \phi_u(i) = \phi_{u+1}(i) - 1 = \phi_u(i \setminus u) = \phi_u((i \setminus u)^u) = \phi_{u+1}(i \setminus u).$$

Proof. Statements (i) and (ii) follow from the fact that the corresponding maps are inverses of eachother. The map in (iii) also has an inverse: the map sending $j \in \mathcal{I}_{k-1,l-1}$ to the function $i \in \mathcal{I}^c_{k-1,l,u}$ whose range is obtained from the range of j by incrementing the elements larger than $u-1$.

To obtain (iv), we set $s = i^{-1}(u)$ and observe the chain of equalities

$$\begin{aligned} i([s] \setminus \{s\}) &= \{z \in \text{range } i : z \le i_s\} \setminus \{i_s\} \\ &= \{z \in \text{range } i : z < u+1\} \setminus \{u\} \\ &= \{z \in \text{range } i : z < u\} \\ &= \{z \in (\text{range } i) \setminus \{u\} : z < u\} \\ &= \{z \in \text{range } (i \setminus u) : z < u\} \\ &= \{z \in \text{range } (i \setminus u)^u : z < u\} \\ &= \{z \in \text{range } (i \setminus u) : z \le u\} \\ &= \{z \in \text{range } (i \setminus u) : z < u+1\}. \end{aligned}$$

□

To handle the sum in (2.3), we need only one other ingredient: the following identities involving h.

Proposition 2.5.5. Let ρ and $\sigma \neq \varnothing$ be compositions satisfying $\ell(\rho) \le \ell(\sigma)$. For $i \in \mathcal{I}_{\ell(\rho),\ell(\sigma)}$ and $u \in [\ell(\sigma)]$, the following statements hold:

(i) if $\sigma_u > 1$, then

$$h(i,\rho,\sigma) = \begin{cases} h(i,\rho,\sigma - \square_u), & u \notin \text{range } i, \\ h(i,\rho,\sigma - \square_u) + \rho_s(\sigma_u - 1)^{\downarrow(\rho_s - 1)} \prod_{\substack{r=1 \\ r \neq s}}^{\ell(\rho)} \sigma_{i_r}^{\downarrow \rho_r}, & i_s = u, \end{cases}$$

(ii) if $\sigma_u = 1$, then

$$h(i,\rho,\sigma) = \begin{cases} h(i^u,\rho,\sigma \ominus \square_u), & u \notin \text{range } i, \\ h(i \cup u, \rho \oplus \square_{\phi_u(i)}, \sigma), & u \notin \text{range } i, \\ h(i \setminus u, \rho \ominus \square_s, \sigma)\mathbb{1}(\rho_s = 1), & i_s = u. \end{cases}$$

Proof. The case $\rho = \varnothing$ is trivial, so we assume $\ell(\rho) \geq 1$. Suppose that $\sigma_u > 1$ and $i_s = u$. Using the first property in (2.1), we obtain

$$\begin{aligned} h(i,\rho,\sigma) - h(i,\rho,\sigma - \Box_u) &= \prod_{r=1}^{\ell(\rho)} \sigma_{i_r}^{\downarrow \rho_r} - \prod_{r=1}^{\ell(\rho)} (\sigma - \Box_u)_{i_r}^{\downarrow \rho_r} \\ &= (\sigma_{i_s}^{\downarrow \rho_s} - (\sigma_{i_s} - 1)^{\downarrow \rho_s}) \prod_{\substack{r=1 \\ r \neq s}}^{\ell(\rho)} \sigma_{i_r}^{\downarrow \rho_r} \\ &= \rho_s (\sigma_{i_s} - 1)^{\downarrow (\rho_s - 1)} \prod_{\substack{r=1 \\ r \neq s}}^{\ell(\rho)} \sigma_{i_r}^{\downarrow \rho_r}, \end{aligned}$$

establishing the second statement in (i). For the first statement, notice that $u \notin \operatorname{range} i$ implies that $(\sigma - \Box_u)_{i_r} = \sigma_{i_r}$ for all r, so the above difference is zero.

Suppose now that $\sigma_u = 1$. Using the identity

$$\begin{aligned} (\sigma \ominus \Box_u)_{i_r^u} &= \sigma_{i_r^u + \mathbb{1}(i_r^u \geq u)} \\ &= \sigma_{i_r} \end{aligned}$$

for all $r \in [\ell(\rho)]$, we obtain the first statement in (ii). The third statement follows directly from the computation

$$\begin{aligned} \prod_{r=1}^{\ell(\rho)} \sigma_{i_r}^{\downarrow \rho_r} &= \sigma_{i_s}^{\downarrow \rho_s} \prod_{r=1}^{s-1} \sigma_{i_r}^{\downarrow \rho_r} \prod_{r=s+1}^{\ell(\rho)} \sigma_{i_r}^{\downarrow \rho_r} \\ &= \sigma_u^{\downarrow \rho_s} \prod_{r=1}^{s-1} \sigma_{i_r}^{\downarrow \rho_r} \prod_{r=s}^{\ell(\rho)-1} \sigma_{i_{r+1}}^{\downarrow \rho_{r+1}} \\ &= 1^{\downarrow \rho_s} \prod_{r=1}^{s-1} \sigma_{i_r}^{\downarrow (\rho \ominus \Box_s)_r} \prod_{r=s}^{\ell(\rho)-1} \sigma_{i_{r+1}}^{\downarrow (\rho \ominus \Box_s)_r} \\ &= \mathbb{1}(\rho_s = 1) \prod_{r=1}^{\ell(\rho)-1} \sigma_{i_{r+\mathbb{1}(i_r \geq u)}}^{\downarrow (\rho \ominus \Box_s)_r}. \end{aligned}$$

For the second statement, note that $u \notin \operatorname{range} i$ implies that $\ell(\rho) < \ell(\sigma)$, so $j = i \cup u$, $\rho' = \rho \oplus \Box_{\phi_u(i)}$, and σ fall into the third case of (ii). Combining that result with

Proposition 2.5.4 (iv) concludes the proof:

$$\begin{aligned} h(j,\rho',\sigma) &= h(j\setminus u, \rho' \ominus \square_{\phi_u(j)}, \sigma)\mathbb{1}(\rho'_{\phi_u(j)} = 1) \\ &= h(i,\rho,\sigma). \end{aligned}$$

□

Proposition 2.5.6. Let $\rho, \tau \in \mathcal{C}$ and set $k = \ell(\rho)$, $l = \ell(\tau)$, and $n = |\tau|$. The following identities hold:

$$\begin{aligned} \sum_{\sigma:\tau\nearrow\sigma^+} p^{\uparrow}_{(\alpha,\theta)}(\tau,\sigma)(m^*_\rho)_{n+1}(\sigma) &= \frac{1}{n+\theta}\Big((n+|\rho|-\alpha l)(m^*_\rho)_n(\tau) \\ &+ \sum_{\substack{s=1\\ \rho_s\geq 2}}^{k} \rho_s(\rho_s - 1 - \alpha)(m^*_{\rho-\square_s})_n(\tau) - \alpha \sum_{\substack{s=1\\ \rho_s=1}}^{k} \sum_{i\in\mathcal{I}_{k,l}} \prod_{r\neq s} \tau_{i_r}^{\downarrow\rho_r}\Big), \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} \sum_{\sigma:\tau\nearrow\sigma^{\oplus}} p^{\uparrow}_{(\alpha,\theta)}(\tau,\sigma)(m^*_\rho)_{n+1}(\sigma) &= \frac{1}{n+\theta}\Big((\alpha l+\theta)(m^*_\rho)_n(\tau) \\ &+ \sum_{\substack{s=1\\ \rho_s=1}}^{k} \eta_s (m^*_{\rho\ominus\square_s})_n(\tau) + \alpha \sum_{\substack{s=1\\ \rho_s=1}}^{k} \sum_{i\in\mathcal{I}_{k,l}} \prod_{r\neq s} \tau_{i_r}^{\downarrow\rho_r}\Big), \end{aligned} \tag{2.5}$$

where $\eta_1 = \theta$ and $\eta_s = \alpha$ otherwise.

Proof. The first identity is trivial when $k = 0$ or $k > l$, so we assume $n, l \geq 1$ and $k \in [\,l\,]$. Recall from Proposition 2.2.1 that a composition obtained from τ via stacking has a unique representation as $\tau + \square_u$. Combining this with Proposition 2.5.5 (i), we obtain

$$\begin{aligned} &\sum_{\sigma:\tau\nearrow\sigma^+} p^{\uparrow}_{(\alpha,\theta)}(\tau,\sigma)(m^*_\rho)_{n+1}(\sigma) \\ &\quad= \sum_{u=1}^{l}\sum_{i\in\mathcal{I}_{k,l}} p^{\uparrow}_{(\alpha,\theta)}(\tau,\tau+\square_u) h(i,\rho,\tau+\square_u) \\ &\quad= \sum_{u=1}^{l}\sum_{i\in\mathcal{I}_{k,l}} \frac{\tau_u-\alpha}{n+\theta}\Bigg(h(i,\rho,\tau) + \sum_{s=1}^{k}\mathbb{1}(i_s = u)\rho_s\tau_u^{\downarrow(\rho_s-1)}\prod_{\substack{r=1\\ r\neq s}}^{k}\tau_{i_r}^{\downarrow\rho_r}\Bigg) \\ &\quad= \sum_{u=1}^{l}\frac{\tau_u-\alpha}{n+\theta}\Bigg((m^*_\rho)_n(\tau) + \sum_{i\in\mathcal{I}_{k,l}}\sum_{s=1}^{k}\mathbb{1}(i_s = u)\rho_s\tau_u^{\downarrow(\rho_s-1)}\prod_{\substack{r=1\\ r\neq s}}^{k}\tau_{i_r}^{\downarrow\rho_r}\Bigg). \end{aligned}$$

The first term above simplifies to $\frac{n-\alpha l}{n+\theta}(m^*_\rho)_n(\tau)$. Using the second property in (2.1), we rewrite the second term as

$$\frac{1}{n+\theta}\sum_{s=1}^k \sum_{i\in\mathcal{I}_{k,l}} \rho_s(\tau_{i_s}-\alpha)\tau_{i_s}^{\downarrow(\rho_s-1)}\prod_{\substack{r=1\\r\neq s}}^k \tau_{i_r}^{\downarrow\rho_r}\sum_{u=1}^l \mathbb{1}(i_s=u)$$
$$=\frac{1}{n+\theta}\sum_{s=1}^k \sum_{i\in\mathcal{I}_{k,l}} \rho_s(\tau_{i_s}-\alpha)\tau_{i_s}^{\downarrow(\rho_s-1)}\prod_{\substack{r=1\\r\neq s}}^k \tau_{i_r}^{\downarrow\rho_r}$$
$$=\frac{1}{n+\theta}\sum_{s=1}^k \sum_{i\in\mathcal{I}_{k,l}} \rho_s\big(\tau_{i_s}^{\downarrow\rho_s}+(\rho_s-1-\alpha)\tau_{i_s}^{\downarrow(\rho_s-1)}\big)\prod_{\substack{r=1\\r\neq s}}^k \tau_{i_r}^{\downarrow\rho_r}$$
$$=\frac{1}{n+\theta}\sum_{s=1}^k\left(\rho_s(m^*_\rho)_n(\tau)+\rho_s(\rho_s-1-\alpha)\sum_{i\in\mathcal{I}_{k,l}}\tau_{i_s}^{\downarrow(\rho_s-1)}\prod_{\substack{r=1\\r\neq s}}^k \tau_{i_r}^{\downarrow\rho_r}\right)$$
$$=\frac{1}{n+\theta}\Bigg(|\rho|(m^*_\rho)_n(\tau)+\sum_{\substack{s=1\\\rho_s\geq 2}}^k \rho_s(\rho_s-1-\alpha)(m^*_{\rho-\square_s})_n(\tau)$$
$$-\alpha\sum_{\substack{s=1\\\rho_s=1}}^k\sum_{i\in\mathcal{I}_{k,l}}\prod_{\substack{r=1\\r\neq s}}^k \tau_{i_r}^{\downarrow\rho_r}\Bigg),$$

establishing (2.4).

For the identity in (2.5), we first address the case $2\leq k\leq l$, when all of the results of Proposition 2.5.4 and Proposition 2.5.5 will be applicable. Recall from Proposition 2.2.1 that each composition obtained from τ via insertion can be written uniquely as $\tau\oplus\square_c$ for some c satisfying $\tau_{c-1}\neq 1$ or $c=1$. For such c, the expression

$$p^{\uparrow}_{(\alpha,\theta)}(\tau,\tau\oplus\square_c)=\frac{1}{n+\theta}\big(\alpha\kappa(\tau,\tau\oplus\square_c)+(\theta-\alpha)\mathbb{1}(\tau\oplus\square_1=\tau\oplus\square_c)\big)$$
$$=\frac{1}{n+\theta}\left(\alpha\sum_{u=1}^{l+1}\mathbb{1}(\tau\oplus\square_u=\tau\oplus\square_c)+(\theta-\alpha)\mathbb{1}(c=1)\right)$$

gives us that

$$\sum_{\sigma:\tau\nearrow\sigma^{\oplus}} p^{\uparrow}_{(\alpha,\theta)}(\tau,\sigma)(m^*_\rho)_{n+1}(\sigma)$$
$$=\frac{1}{n+\theta}\left(\alpha\sum_{u=2}^{l+1}(m^*_\rho)_{n+1}(\tau\oplus\square_u)+\theta(m^*_\rho)_{n+1}(\tau\oplus\square_1)\right).$$

As before, we proceed by employing the recursive identities involving h. Let $u \in [l+1]$. Applying Proposition 2.5.4 (iii) and Proposition 2.5.5 (ii), we obtain

$$\begin{aligned}\sum_{i\in\mathcal{I}^c_{k,l+1,u}} h(i,\rho,\tau\oplus\square_u) &= \sum_{i\in\mathcal{I}^c_{k,l+1,u}} h(i^u,\rho,\tau)\\ &= \sum_{j\in\mathcal{I}_{k,l}} h(j,\rho,\tau)\\ &= (m^*_\rho)_n(\tau),\end{aligned} \tag{2.6}$$

and by applying Proposition 2.5.4 (i), (iii), (iv), and Proposition 2.5.5 (ii), we obtain

$$\begin{aligned}\sum_{i\in\mathcal{I}_{k,l+1,u}} h(i,\rho,\tau\oplus\square_u) &= \sum_{s=1}^{k}\sum_{i\in\mathcal{I}_{k,l+1,u}} h(i,\rho,\tau\oplus\square_u)\mathbb{1}(i_s=u)\\ &= \sum_{\substack{s=1\\ \rho_s=1}}^{k}\sum_{i\in\mathcal{I}_{k,l+1,u}} h((i\setminus u)^u,\rho\ominus\square_s,\tau)\mathbb{1}(s=\phi_u((i\setminus u)^u))\\ &= \sum_{\substack{s=1\\ \rho_s=1}}^{k}\sum_{j\in\mathcal{I}_{k-1,l}} h(j,\rho\ominus\square_s,\tau)\mathbb{1}(s=\phi_u(j)).\end{aligned} \tag{2.7}$$

When $u=1$, noting that $\phi_1\equiv 1$ reduces the latter sum to

$$\begin{aligned}\sum_{i\in\mathcal{I}_{k,l+1,1}} h(i,\rho,\tau\oplus\square_1) &= \sum_{\substack{s=1\\ \rho_s=1}}^{k}\sum_{j\in\mathcal{I}_{k-1,l}} h(j,\rho\ominus\square_s,\tau)\mathbb{1}(s=1)\\ &= \mathbb{1}(\rho_1=1)\sum_{j\in\mathcal{I}_{k-1,l}} h(j,\rho\ominus\square_1,\tau)\\ &= \mathbb{1}(\rho_1=1)(m^*_{\rho\ominus\square_1})_n(\tau).\end{aligned} \tag{2.8}$$

Combining (2.6) and (2.8) gives us that

$$\begin{aligned}(m^*_\rho)_{n+1}(\tau\oplus\square_1) &= \sum_{i\in\mathcal{I}^c_{k,l+1,1}} h(i,\rho,\tau\oplus\square_1) + \sum_{i\in\mathcal{I}_{k,l+1,1}} h(i,\rho,\tau\oplus\square_1)\\ &= (m^*_\rho)_n(\tau) + \mathbb{1}(\rho_1=1)(m^*_{\rho\ominus\square_1})_n(\tau).\end{aligned}$$

When $u > 1$, this sum in (2.7) is handled by decomposing each j sum into its $\mathcal{I}_{k-1,l,u-1}$ and $\mathcal{I}^c_{k-1,l,u-1}$ parts. Let $s \in [k]$ such that $\rho_s = 1$. Applying Proposition 2.5.4 (iv), we write the first part of the corresponding j sum as

$$\begin{aligned}
&\sum_{j \in \mathcal{I}_{k-1,l,u-1}} h(j, \rho \ominus \square_s, \tau) \mathbb{1}(s = \phi_u(j)) \\
&\qquad = \sum_{j \in \mathcal{I}_{k-1,l,u-1}} h(j, \rho \ominus \square_s, \tau) \mathbb{1}(s = j^{-1}(u-1) + 1) \\
&\qquad = \sum_{j \in \mathcal{I}_{k-1,l}} h(j, \rho \ominus \square_s, \tau) \mathbb{1}(j_{s-1} = u-1)
\end{aligned} \tag{2.9}$$

for $s > 1$ (this sum is zero when $s = 1$). In the second part of the j sum, we can alter the $(u-1)^{st}$ column of τ since $u-1$ is not in the range of j. Applying then Proposition 2.5.4 (i), (iv), and Proposition 2.5.5 (ii), we have that

$$\begin{aligned}
&\sum_{j \in \mathcal{I}^c_{k-1,l,u-1}} h(j, \rho \ominus \square_s, \tau) \mathbb{1}(s = \phi_u(j)) \\
&\quad = \sum_{j \in \mathcal{I}^c_{k-1,l,u-1}} h(j, \rho \ominus \square_s, \tau / \square_{u-1}) \mathbb{1}(s = \phi_{u-1}(j)) \\
&\quad = \sum_{j \in \mathcal{I}^c_{k-1,l,u-1}} h(j \cup (u-1), \rho, \tau / \square_{u-1}) \mathbb{1}(s = \phi_{u-1}(j \cup (u-1))) \\
&\quad = \sum_{i \in \mathcal{I}_{k,l,u-1}} h(i, \rho, \tau / \square_{i_s}) \mathbb{1}(i_s = u-1) \\
&\quad = \sum_{i \in \mathcal{I}_{k,l}} h(i, \rho, \tau / \square_{i_s}) \mathbb{1}(i_s = u-1).
\end{aligned} \tag{2.10}$$

Combining (2.6), (2.9), and (2.10) gives us that

$$
\begin{aligned}
&\sum_{u=2}^{l+1}(m^*_\rho)_{n+1}(\tau\oplus\square_u)\\
&=\sum_{u=2}^{l+1}\Big(\sum_{i\in\mathcal{I}^c_{k,l+1,u}}h(i,\rho,\tau\oplus\square_u)+\sum_{i\in\mathcal{I}_{k,l+1,u}}h(i,\rho,\tau\oplus\square_u)\Big)\\
&=\sum_{u=2}^{l+1}\Big((m^*_\rho)_n(\tau)+\sum_{\substack{s=2\\ \rho_s=1}}^{k}\sum_{\substack{j\in\mathcal{I}_{k-1,l}\\ j_{s-1}=u-1}}h(j,\rho\ominus\square_s,\tau)+\sum_{\substack{s=1\\ \rho_s=1}}^{k}\sum_{\substack{i\in\mathcal{I}_{k,l}\\ i_s=u-1}}h(i,\rho,\tau/\square_{i_s})\Big)\\
&=l\,(m^*_\rho)_n(\tau)+\sum_{\substack{s=2\\ \rho_s=1}}^{k}\sum_{j\in\mathcal{I}_{k-1,l}}h(j,\rho\ominus\square_s,\tau)+\sum_{\substack{s=1\\ \rho_s=1}}^{k}\sum_{i\in\mathcal{I}_{k,l}}h(i,\rho,\tau/\square_{i_s})\\
&=l\,(m^*_\rho)_n(\tau)+\sum_{\substack{s=2\\ \rho_s=1}}^{k}(m^*_{\rho\ominus\square_s})_n(\tau)+\sum_{\substack{s=1\\ \rho_s=1}}^{k}\sum_{i\in\mathcal{I}_{k,l}}h(i,\rho,\tau/\square_{i_s}).
\end{aligned}
$$

Noting that $h(i,\rho,\tau/\square_{i_s})=\prod_{r\neq s}\tau_{i_r}^{\downarrow\rho_r}$ whenever $\rho_s=1$ establishes (2.5) for $2\le k\le l$. The cases $k=0$ and $k>l+1$ are trivial. When $k=1<l+1$, we can verify that (2.6) still holds and replace (2.7) by

$$
\begin{aligned}
\sum_{i\in\mathcal{I}_{1,l+1,u}}h(i,\rho,\tau\oplus\square_u)&=(\tau\oplus\square_u)_u^{\downarrow\rho_1}\\
&=1^{\downarrow\rho_1}\\
&=\mathbb{1}(\rho_1=1)
\end{aligned}
$$

to obtain

$$
\sum_{\sigma:\tau\nearrow\sigma^\oplus}p^{\uparrow}_{(\alpha,\theta)}(\tau,\sigma)(m^*_\rho)_{n+1}(\sigma)=\frac{\alpha l+\theta}{n+\theta}\Big((m^*_\rho)_n(\tau)+\mathbb{1}(\rho_1=1)\Big).
$$

When $k=l+1$, the conclusion of (2.6) still holds (the first and last quantities are both zero) and (2.7) still holds. Since $\mathcal{I}_{l,l}$ is the singleton containing the identity map, the latter simplifies to

$$\sum_{i\in\mathcal{I}_{k,l+1,u}} h(i,\rho,\tau\oplus\Box_u) = \sum_{\substack{s=1\\ \rho_s=1}}^{l+1}\sum_{j\in\mathcal{I}_{l,l}} h(j,\rho\ominus\Box_s,\tau)\mathbb{1}(s=\phi_u(j))$$
$$= \sum_{\substack{s=1\\ \rho_s=1}}^{l+1}\sum_{j\in\mathcal{I}_{l,l}} h(j,\rho\ominus\Box_s,\tau)\mathbb{1}(s=u)$$
$$= \mathbb{1}(\rho_u=1)(m^*_{\rho\ominus\Box_u})_n(\tau),$$

from which we obtain

$$\sum_{\sigma:\tau\nearrow\sigma^\oplus} p^\uparrow_{(\alpha,\theta)}(\tau,\sigma)(m^*_\rho)_{n+1}(\sigma) = \frac{1}{n+\theta}\sum_{\substack{u=1\\ \rho_u=1}}^{l+1}\eta_u(m^*_{\rho\ominus\Box_u})_n(\tau).$$

□

Let us remark that the final term in (2.4) and (2.5) does not correspond to a quasisymmetric function, except in the trivial case when ρ contains no ones.

Proof of Proposition 2.5.2. Summing together (2.4) and (2.5) gives the first formula in Proposition 2.5.2. The latter sum in that formula can be factorized using

$$p^\uparrow_{(\alpha,\theta)}(\rho-\Box_s,\rho)\frac{g(\rho-\Box_s)}{g(\rho)} = \frac{\rho_s-1-\alpha}{|\rho|-1+\theta}\frac{\rho_s}{|\rho|}.$$

For the other sum, we recall from Proposition 2.2.1 that the compositions obtained from ρ via uninsertion can be written uniquely as $\rho\ominus\Box_c$ for some c satisfying either $c=1$ or $\rho_{c-1}\neq 1$. Therefore, we can write

$$\sum_{\substack{s=1\\ \rho_s=1}}^{k}\eta_s(m^*_{\rho\ominus\Box_s})_n = \sum_{\sigma:\sigma\nearrow\rho^\oplus}\sum_{\substack{s=1\\ \rho_s=1}}^{k}\eta_s(m^*_{\rho\ominus\Box_s})_n\mathbb{1}(\sigma=\rho\ominus\Box_s)$$
$$= \sum_{\sigma:\sigma\nearrow\rho^\oplus}\sum_{\substack{s=1\\ \rho_s=1}}^{k}(m^*_\sigma)_n(\alpha+(\theta-\alpha)\mathbb{1}(s=1))\mathbb{1}(\sigma\oplus\Box_s=\rho)$$
$$= \sum_{\sigma:\sigma\nearrow\rho^\oplus}(m^*_\sigma)_n(\alpha\,\kappa(\sigma,\rho)+(\theta-\alpha)\mathbb{1}(\sigma\oplus\Box_1=\rho))$$
$$= (|\rho|-1+\theta)\sum_{\sigma:\sigma\nearrow\rho^\oplus}(m^*_\sigma)_n\,p^\uparrow_{(\alpha,\theta)}(\sigma,\rho).$$

Noting that $|\rho|g(\sigma)/g(\rho) = 1$ whenever $\sigma \nearrow \rho^{\oplus}$ concludes the proof.

□

2.6 The Spaces $\mathcal{U}$ and $C(\mathcal{U})$

In this section, we explore the properties of the metric spaces $\mathcal{U}$ and $C(\mathcal{U})$ introduced in Section 2.1. The results presented here are crucial to performing the limit computation. Of particular importance is Proposition 2.6.3, where we introduce a useful homomorphism from Λ to $C(\mathcal{U})$.

Recall from Section 2.1 that $\mathcal{U}$ denotes the collection of open subsets of $(0,1)$ equipped with the metric obtained from applying the Hausdorff metric on the complements of sets (complements are taken in $[0,1]$). That is, the distance between open sets $U, V \in \mathcal{U}$ is given by

$$d(U,V) = \inf\{\varepsilon \geq 0 : U^c \subset (V^c)_\varepsilon, V^c \subset (U^c)_\varepsilon\},$$

where X_ε denotes the ε-enlargement of a set X,

$$X_\varepsilon = \bigcup_{x\in X}\{y \in [0,1] : |y-x| \leq \varepsilon\}.$$

As shown in [23], $\mathcal{U}$ is compact under this topology.

We regard $\mathcal{C}$ as a subset of $\mathcal{U}$ by identifying $\varnothing$ with $\iota(\varnothing) = \emptyset$ and a non-empty composition σ with the open set

$$\iota(\sigma) = \left(0, \frac{\sigma_1}{|\sigma|}\right) \cup \left(\frac{\sigma_1}{|\sigma|}, \frac{\sigma_1+\sigma_2}{|\sigma|}\right) \cup \ldots \cup \left(\frac{|\sigma| - \sigma_{\ell(\sigma)}}{|\sigma|}, 1\right).$$

The set $\iota(\mathcal{C})$ is not only dense in $\mathcal{U}$, but has the following approximation property.

Proposition 2.6.1. Every open subset of $\mathcal{U}$ intersects all but finitely many of the sets $\iota(\mathcal{C}_n)$. In particular, for every $U \in \mathcal{U}$, there is a sequence $\{U_n\}_{n\geq 1}$ satisfying $U_n \in \iota(\mathcal{C}_n)$ and

$$d(U, U_n) \leq \frac{1}{n}.$$

Proof. Fix U and n as above and set $\varepsilon = n^{-1}$. Letting $E_n = \{\frac{1}{n}, \frac{2}{n}, ..., \frac{n-1}{n}\}$, note that every point in $[0,1]$ is at most a distance of n^{-1} from a point in E_n. In particular, for every $x \in U^c$ there is some $z \in E_n$ satisfying $|x - z| \leq \varepsilon$. Since this implies that $z \in (U^c)_\varepsilon$, we have the cover

$$U^c \subset \bigcup_{z \in (U^c)_\varepsilon \cap E_n} \{y \in [0,1] : |y - z| \leq \varepsilon\}.$$

Writing the above index set as $z_1 < ... < z_N$, a suitable choice for U_n is

$$U_n = (0, z_1) \cup (z_1, z_2) \cup ... \cup (z_N, 1).$$

Indeed, U_n lies in $\iota(\mathcal{C}_n)$ because each z_i lies in E_n. The containment $U_n^c \subset (U^c)_\varepsilon$ holds because each z_i lies in $(U^c)_\varepsilon$. Finally, $U^c \subset (U_n^c)_\varepsilon$ because the points $\{z_i\}_{i=1}^N$ index the above cover. As a result, $d(U, U_n) \leq \varepsilon$, concluding the proof.

□

The identification of $\mathcal{C}$ with $\iota(\mathcal{C})$ induces projections $\pi_n : C(\mathcal{U}) \to C(\mathcal{C}_n)$ given by

$$\pi_n f = f \circ \iota\big|_{\mathcal{C}_n}.$$

Since $\iota(\mathcal{C})$ is dense in $\mathcal{U}$, a continuous function f can be recovered from its projections $\{\pi_n f\}$. In the finite-dimensional setting, a stronger version of this property holds.

Proposition 2.6.2. Let F be a finite dimensional subspace of $C(\mathcal{U})$. Then the restricted projections $\pi_n\big|_F$ are injective for large n.

Proof. Given a sequence $\{f_n\}_{n\geq 1}$ with $f_n \in \ker \pi_n\big|_F$, define

$$\tilde{f}_n = \begin{cases} 0, & f_n = 0, \\ f_n / \|f_n\|, & \text{else}, \end{cases}$$

and consider an arbitrary subsequence $\{\tilde{f}_{n_k}\}_{k\geq 1}$. This subsequence lies in the unit ball of a finite-dimensional subspace, so it contains a convergent subsequence, say

$\tilde{f}_{n_{k_l}} \to \tilde{f}$. Given now any $U \in \mathcal{U}$, let $\{U_n\}_{n\geq 1}$ be a composition approximation of U, as in Proposition 2.6.1, and observe that $\tilde{f}_n(U_n) = 0$ for all n. Consequently,

$$\tilde{f}(U) = \lim_{l\to\infty} \tilde{f}_{n_{k_l}}(U_{n_{k_l}}) = 0, \qquad U \in \mathcal{U},$$

or $\tilde{f} = 0$. This establishes the convergence $\tilde{f}_n \to 0$, from which we find that $f_n = 0$ for large n.

□

We have seen with the maps $q \mapsto \{q_n\}$ and $f \mapsto \{\pi_n f\}$ that the elements of both Λ and $C(\mathcal{U})$ are uniquely identified by elements in $\prod_{n=0}^{\infty} C(\mathcal{C}_n)$. A natural way, then, to move from Λ to $C(\mathcal{U})$ would be to identify each $\{q_n\}$ as some $\{\pi_n f\}$. Unfortunately, this approach fails. A projection family $\{\pi_n f\}$ must be uniformly bounded while the actions of a quasisymmetric function $\{q_n\}$ easily are not (the norms of the family $\{(m_\sigma)_n\}$ grow at the rate $n^{|\sigma|}$). To remedy this, we introduce normalizing automorphisms $\{G_n\}_{n\geq 1}$ defined on monomials by

$$G_n m_\sigma = n^{-|\sigma|} m_\sigma.$$

Replacing $\{q_n\}_{n\geq 1}$ with the normalized family $\{(G_n q)_n\}_{n\geq 1}$, the above approach does work.

Proposition 2.6.3. There exists a homomorphism of algebras $\Psi : \Lambda \to C(\mathcal{U})$ so that $q^o := \Psi q$ satisfies

$$\pi_n(q^o) = (G_n q)_n, \quad n \geq 1$$

for all $q \in \Lambda$.

Proof. When q is a monomial, the existence of some q^o satisfying the above system follows from Proposition 10 in [23] ($p_n^u(\eta)$ there would be $g(\eta) m_\eta^o(u)$ here). We use this to define Ψ on monomials and then extend to all of Λ by linearity. Since the maps π_n, G_n, and $q \mapsto q_n$ are all linear, this extension continues to satisfy the given system. The fact that Ψ is a homomorphism of algebras follows from observing that each π_n, G_n, and $q \mapsto q_n$ is one. Indeed, for all $q, \bar{q} \in \Lambda$, we have

$$\begin{aligned}
\pi_n(q^o\bar{q}^o) &= \pi_n(q^o)\pi_n(\bar{q}^o) \\
&= (G_n q)_n (G_n \bar{q})_n \\
&= (G_n q\, G_n \bar{q})_n \\
&= (G_n(q\,\bar{q}))_n \\
&= \pi_n((q\,\bar{q})^o), \qquad n \geq 1,
\end{aligned}$$

from which we obtain $q^o\bar{q}^o = (q\,\bar{q})^o$.

□

We remark that the above construction is not just technically convenient but in fact natural. This is seen from the following formula (see [23]): for an open set of the form

$$U = (0, x_1) \cup (x_1, x_1 + x_2) \cup (x_1 + x_2, x_1 + x_2 + x_3) \cup \ldots,$$

where $\{x_i\}$ is a sequence in $[0, 1]$ summing to 1, we have

$$m_\sigma^o(U) = \sum_{i \in \mathcal{I}_{\ell(\sigma)}} \prod_{r=1}^{\ell(\sigma)} x_{i_r}^{\sigma_r}.$$

Let $\mathcal{F}$ denote the image of Λ under Ψ and $\mathcal{F}_k$ denote the image of Λ_k.

Proposition 2.6.4. The subalgebra $\mathcal{F}$ is dense in $C(\mathcal{U})$.

Proof. Since $\mathcal{F}$ contains the constant $m_\varnothing^o = 1$, we need only to check that $\mathcal{F}$ separates points. This follows from Proposition 10 in [23], where it is shown that the map $U \mapsto \{m_\sigma^o(U)\}_{\sigma \in \mathcal{C}}$ is injective.

□

Proposition 2.6.5. For every composition $\mu \in \mathcal{C}$, we have the convergence

$$(n^{-|\mu|} G_n^{-1} m_\mu^*)^o \longrightarrow m_\mu^o$$

as $n \to \infty$. Consequently, for any sequence of compositions $\{\sigma_k\}_{k\geq 1}$ with $|\sigma_k| \to \infty$ and $\iota(\sigma_k) \to U$ as $k \to \infty$, we have that

$$\frac{g(\mu, \sigma_k)}{g(\sigma_k)} \longrightarrow m^o_\mu(U)$$

as $k \to \infty$.

Proof. The homogeneous component of largest degree in m^*_μ is m_μ, so its expansion in the monomial basis has the form

$$m^*_\mu = m_\mu + \sum_{|\lambda|<|\mu|} a_\lambda m_\lambda.$$

This provides the expansion

$$n^{-|\mu|} G_n^{-1} m^*_\mu = m_\mu + \sum_{|\lambda|<|\mu|} a_\lambda n^{|\lambda|-|\mu|} m_\lambda,$$

from which we can compute

$$\begin{aligned}
\left\|(n^{-|\mu|} G_n^{-1} m^*_\mu)^o - m^o_\mu\right\| &= \left\| \sum_{|\lambda|<|\mu|} a_\lambda n^{|\lambda|-|\mu|} m^o_\lambda \right\| \\
&\leq \sum_{|\lambda|<|\mu|} |a_\lambda|\, n^{|\lambda|-|\mu|} \left\|m^o_\lambda\right\| \\
&= O(n^{-1}),
\end{aligned}$$

and the first claim follows.

For the second claim, we set $n_k = |\sigma_k|$ and use Proposition 2.4.1 to rewrite the given ratio as

$$\begin{aligned}
\frac{g(\mu, \sigma_k)}{g(\sigma_k)} &= \frac{(m^*_\mu)_{n_k}(\sigma_k)}{n_k^{\downarrow|\mu|}} \\
&= \frac{\pi_{n_k}(G_{n_k}^{-1} m^*_\mu)^o(\sigma_k)}{n_k^{\downarrow|\mu|}} \\
&= \frac{n_k^{|\mu|}}{n_k^{\downarrow|\mu|}} (n_k^{-|\mu|} G_{n_k}^{-1} m^*_\mu)^o(\iota(\sigma_k)).
\end{aligned}$$

Applying the first claim concludes the proof.

□

2.7 The Limiting Process

In this section, we perform the limit computation, identify our diffusions, and establish our main result.

Recall from Proposition 2.5.3 that we have a formula for the transition operators:

$$(\mathcal{T}_n^{(\alpha,\theta)} - \mathbf{1})(m_\rho^*)_n = \frac{|\rho|(|\rho|-1+\theta)}{(n+\theta)(n+1)}\Big(-(m_\rho^*)_n + (n-|\rho|+1)\sum_{\mu\nearrow\rho} p^{\uparrow}_{(\alpha,\theta)}(\mu,\rho)\frac{g(\mu)}{g(\rho)}(m_\mu^*)_n\Big)$$

for all $n \geq 1$ and $\rho \in \mathcal{C}$. Since $(m_\rho^*)_n$ lies in $\pi_n(\mathcal{F}_k)$ whenever $|\rho| \leq k$, this formula shows that $\pi_n(\mathcal{F}_k)$ is invariant under $\mathcal{T}_n^{(\alpha,\theta)}$ for all n and k. When n is large enough, we identify $\pi_n(\mathcal{F}_k)$ with $\mathcal{F}_k$ (see Proposition 2.6.2), and regard $\mathcal{T}_n^{(\alpha,\theta)}$ as an operator on $\mathcal{F}_k$ by defining

$$\mathcal{T}_n^{(\alpha,\theta)}\Big|_{\mathcal{F}_k} = (\pi_n\big|_{\mathcal{F}_k})^{-1} \circ \mathcal{T}_n^{(\alpha,\theta)} \circ \pi_n\Big|_{\mathcal{F}_k}.$$

Using the identity

$$\pi_n\Big|_{\mathcal{F}_k}(G_n^{-1}m_\rho^*)^o = (m_\rho^*)_n, \quad |\rho| \leq k,$$

we have the explicit form

$$\begin{aligned}(\mathcal{T}_n^{(\alpha,\theta)}\Big|_{\mathcal{F}_k} - \mathbf{1})(G_n^{-1}m_\rho^*)^o = \frac{|\rho|(|\rho|-1+\theta)}{(n+\theta)(n+1)}\Big(&-(G_n^{-1}m_\rho^*)^o \\ &+ (n-|\rho|+1)\sum_{\mu\nearrow\rho} p^{\uparrow}_{(\alpha,\theta)}(\mu,\rho)\frac{g(\mu)}{g(\rho)}(G_n^{-1}m_\mu^*)^o\Big),\end{aligned} \tag{2.11}$$

which holds whenever $|\rho| \leq k$.

Proposition 2.7.1. For each k, we have the convergence

$$n^2(\mathcal{T}_n^{(\alpha,\theta)}\Big|_{\mathcal{F}_k} - \mathbf{1}) \longrightarrow \mathcal{A}\Big|_{\mathcal{F}_k}$$

as $n \to \infty$ in the strong operator topology, where $\mathcal{A} : \mathcal{F} \to \mathcal{F}$ is the linear operator satisfying

$$\mathcal{A}m_\rho^o = |\rho|(|\rho|-1+\theta)\Big(-m_\rho^o + \sum_{\mu\nearrow\rho} p^{\uparrow}_{(\alpha,\theta)}(\mu,\rho)\frac{g(\mu)}{g(\rho)}m_\mu^o\Big), \qquad \rho \in \mathcal{C}. \tag{2.12}$$

Proof. Fix k and take n large so that $\mathcal{T}_n^{(\alpha,\theta)}\big|_{\mathcal{F}_k}$ is well-defined. We claim that

$$n^2(\mathcal{T}_n^{(\alpha,\theta)}\big|_{\mathcal{F}_k} - \mathbf{1})(n^{-|\rho|}G_n^{-1}m_\rho^*)^o \\ \longrightarrow |\rho|(|\rho| - 1 + \theta)\Big(-m_\rho^o + \sum_{\mu\nearrow\rho} p^{\uparrow}_{(\alpha,\theta)}(\mu,\rho)\frac{g(\mu)}{g(\rho)}m_\mu^o\Big), \qquad |\rho| \le k,$$

and

$$n^2(\mathcal{T}_n^{(\alpha,\theta)}\big|_{\mathcal{F}_k} - \mathbf{1})(m_\rho^o - (n^{-|\rho|}G_n^{-1}m_\rho^*)^o) \longrightarrow 0, \qquad |\rho| \le k,$$

as $n \to \infty$. The first claim follows from the formula in (2.11) and Proposition 2.6.5. For the second claim, we use the expansion of m_ρ in the monomial-variant basis,

$$m_\rho = m_\rho^* + \sum_{|\lambda|<|\rho|} a_\lambda m_\lambda^*,$$

to obtain the expansions

$$m_\rho^o - (n^{-|\rho|}G_n^{-1}m_\rho^*)^o = (n^{-|\rho|}G_n^{-1}(m_\rho - m_\rho^*))^o \\ = \sum_{|\lambda|<|\rho|} a_\lambda n^{|\lambda|-|\rho|}(n^{-|\lambda|}G_n^{-1}m_\lambda^*)^o,$$

and

$$n^2(\mathcal{T}_n^{(\alpha,\theta)}\big|_{\mathcal{F}_k} - \mathbf{1})(m_\rho^o - (n^{-|\rho|}G_n^{-1}m_\rho^*)^o) \\ = \sum_{|\lambda|<|\rho|} a_\lambda n^{|\lambda|-|\rho|} n^2(\mathcal{T}_n^{(\alpha,\theta)}\big|_{\mathcal{F}_k} - \mathbf{1})(n^{-|\lambda|}G_n^{-1}m_\lambda^*)^o.$$

The latter expansion reveals that the second claim follows from the first. Combining the claims, we have, for $|\rho| \le k$, the convergence

$$n^2(\mathcal{T}_n^{(\alpha,\theta)}\big|_{\mathcal{F}_k} - \mathbf{1})\, m_\rho^o \longrightarrow |\rho|(|\rho| - 1 + \theta)\Big(-m_\rho^o + \sum_{\mu\nearrow\rho} p^{\uparrow}_{(\alpha,\theta)}(\mu,\rho)\frac{g(\mu)}{g(\rho)}m_\mu^o\Big),$$

as $n \to \infty$. Since this convergence extends to all of $\mathcal{F}_k$, the span of $\{m_\rho^o : |\rho| \le k\}$, for each k there exists a limit in the strong operator topology

$$\mathcal{A}_k = \lim_{n\to\infty} n^2(\mathcal{T}_n^{(\alpha,\theta)}\big|_{\mathcal{F}_k} - \mathbf{1}), \qquad k \ge 1.$$

Noting that $\mathcal{T}_n^{(\alpha,\theta)}\big|_{\mathcal{F}_{k+1}}$ is an extension of $\mathcal{T}_n^{(\alpha,\theta)}\big|_{\mathcal{F}_k}$ whenever both are well-defined, it follows that $\mathcal{A}_{k+1}$ is an extension of $\mathcal{A}_k$. Consequently, the family $\{\mathcal{A}_k\}$ has a common extension to $\mathcal{F}$. Taking $\mathcal{A}$ to be this extension concludes the proof.

□

We note here that an alternative formula for $\mathcal{A}$ can be obtained by using the expansion for the transition operators given in Proposition 2.5.3. The resulting formula is

$$\mathcal{A}m_\rho^o = -|\rho|(|\rho|-1+\theta)m_\rho^o + \sum_{\rho_c\geq 2}\rho_c(\rho_c-1-\alpha)m^o_{\rho-\square_c} + \sum_{\substack{\rho_c=1\\ c\geq 1}}\eta_c m^o_{\rho\ominus\square_c}, \quad \rho\in\mathcal{C}.$$

This formula can be used to show that, in some sense, $\mathcal{A}$ agrees with the operator in [38] on the image under Ψ of the subalgebra of symmetric functions.

The following result proves Theorem 2.1.1.

Proposition 2.7.2. The following statements hold:

(i) the operator $\mathcal{A}$ is closable in $C(\mathcal{U})$ and its closure $\overline{\mathcal{A}}$ generates a conservative Feller semigroup $\{\mathcal{T}^{(\alpha,\theta)}(t)\}_{t\geq 0}$ on $C(\mathcal{U})$,

(ii) the discrete semigroups $\{1,\mathcal{T}_n,\mathcal{T}_n^2,...\}_{n\geq 1}$ converge to $\{\mathcal{T}^{(\alpha,\theta)}(t)\}_{t\geq 0}$ in the following sense: for all $f\in C(\mathcal{U})$ and $t\geq 0$,

$$\left\|\mathcal{T}_n^{\lfloor n^2t\rfloor}\pi_n f - \pi_n\mathcal{T}^{(\alpha,\theta)}(t)f\right\|_{C(\mathcal{C}_n)} \longrightarrow 0,$$

as $n\to\infty$,

(iii) the convergence in (ii) is uniform in t on bounded intervals, and

(iv) if $(\iota(\mathbf{X}_n^{(\alpha,\theta)}(0)))_{n\geq 1}$ has a limiting distribution ν, then we have the convergence

$$\left(\iota\left(\mathbf{X}_n^{(\alpha,\theta)}\left(\lfloor n^2t\rfloor\right)\right)\right)_{t\geq 0} \longrightarrow_d \left(\mathbf{U}^{(\alpha,\theta)}(t)\right)_{t\geq 0},$$

in the Skorokhod space $D([0,\infty),\mathcal{U})$, where $\left(\mathbf{U}^{(\alpha,\theta)}(t)\right)_{t\geq 0}$ is the Feller diffusion with paths in $\mathcal{U}$, initial distribution ν, and semigroup $\{\mathcal{T}^{(\alpha,\theta)}(t)\}_{t\geq 0}$.

Proof. The compactness of $\mathcal{U}$, invariance of the transition operators on $\mathcal{F}_k$, and the results of Propositions 2.6.1, 2.6.4, and 2.7.1 verify the hypotheses of Proposition 1.4 in [8]. This establishes (i)-(iii). To obtain (iv), we then apply Chapter 4, Theorem 2.12 from [13] to obtain the convergence in distribution on the Skorokhod space. The fact that $\left(\mathbf{U}^{(\alpha,\theta)}(t)\right)_{t\geq 0}$ has continuous sample paths, and therefore is a diffusion, then follows from the observation that the size of the largest jump of $(\iota(\mathbf{X}_n^{(\alpha,\theta)}(\lfloor n^2 t\rfloor)))_{t\geq 0}$ tends to 0 as $n \to \infty$.

□

Chapter 3

THE LEFT-MOST COLUMN IN THE ORDERED CHINESE RESTAURANT PROCESS

3.1 Introduction

We continue our study of the oCRP by considering the evolution of the left-most column. More precisely, if $\phi : \mathcal{C} \setminus \{\varnothing\} \to \mathbb{N}$ maps a non-empty composition σ to its first component σ_1, our interest lies in the processes $Y_n^{(\alpha,\theta)} = \phi(\mathbf{X}_n^{(\alpha,\theta)})$ for $n \geq 1$. As mentioned in section 1.1, understanding these processes could give insight into the behavior of the diffusions of Chapter 2. Of particular interest is the examination of the Markov property. Here, we restrict to the case when $n > 2$ and $\theta = 0$, since it is clear that the Markov property holds when $n \leq 2$ and fails when $n > 2$ and $\theta \neq 0$.

Results. Our main result is Theorem 3.1.1 below, in which we identify a condition on $\mathbf{X}_n^{(\alpha,0)}$ that ensures that the left-most column process is Markovian. This condition also leads to a commutation relation between the transition kernels of the left-most column process and the corresponding up-down chain known as *intertwining* in the literature.

Theorem 3.1.1. *Let $n > 2$. There exists a transition kernel $\Lambda_n : [n] \to \mathcal{C}_n$ such that if the initial distribution of $\mathbf{X}_n^{(\alpha,0)}$ is of the form $\mu\Lambda_n$ for some distribution μ on $[n]$, then the process $Y_n^{(\alpha,0)}$ is Markovian. In this case, the relation*

$$\Lambda_n T_n^{(\alpha,0)} = Q_n^{(\alpha,0)} \Lambda_n$$

holds, where $Q_n^{(\alpha,0)}$ is the transition kernel of $Y_n^{(\alpha,0)}$. In other words, the processes $Y_n^{(\alpha,0)}$ and $\mathbf{X}_n^{(\alpha,0)}$ are intertwined.

We also obtain explicit descriptions for all the transition kernels appearing in Theorem 3.1.1. Surprisingly, the lifting kernel Λ_n is described by the stationary distribution of

$Y_n^{(\alpha,\alpha)}$. For this reason, we restrict our study of the left-most column to the (α,α) and $(\alpha,0)$ cases.

Outline. In Section 3.2, we look at the (α,α) case, paying special attention to asymptotic properties. In Section 3.3, we examine the $(\alpha,0)$ case, proving our main result along with some other curious properties. In the final section, we study the excursions of the process $Y_n^{(\alpha,0)}$.

Notation. In addition to the notation from Chapter 2, the following will be used throughout this chapter. For $x > -1$ and non-negative integer k, we denote the rising factorial by

$$(x)_k := \prod_{i=1}^{k}(x+i-1) = \frac{\Gamma(x+k)}{\Gamma(x)},$$

where $\Gamma(x)$ is the gamma function. Multinomial coefficients will be denoted using the shorthand

$$\binom{|\sigma|}{\sigma} := \begin{cases} \binom{|\sigma|}{\sigma_1, \ldots, \sigma_{\ell(\sigma)}}, & \sigma \neq \varnothing, \\ 1, & \sigma = \varnothing. \end{cases}$$

Except in the context of Theorem 3.1.1, where the initial distribution of $\mathbf{X}_n^{(\alpha,0)}$ is unspecified, we will consider the up-down chains to be running in stationarity.

3.2 The Left-Most Column in the (α,α) Chain

Proposition 3.2.1. Let $n \geq 0$. The composition $\mathbf{X}_n^{(\alpha,\alpha)}(0)$ has distribution

$$\mathbb{P}\{\mathbf{X}_n^{(\alpha,\alpha)}(0) = \sigma\} = M_n^{(\alpha,\alpha)}(\sigma) := \binom{n}{\sigma}\frac{1}{(\alpha)_n}\prod_{j=1}^{\ell(\sigma)} \alpha\,(1-\alpha)_{\sigma_j-1}, \quad \sigma \in \mathcal{C}_n.$$

Proof. It can be verified that the consistency conditions

$$\begin{aligned} M_n^{(\alpha,\alpha)}(\tau) &= \sum_{\sigma\in\mathcal{C}_{n-1}} M_{n-1}^{(\alpha,\alpha)}(\sigma) p^{\uparrow}_{(\alpha,\alpha)}(\sigma,\tau) \\ &= \sum_{\sigma\in\mathcal{C}_{n+1}} M_{n+1}^{(\alpha,\alpha)}(\sigma) p^{\downarrow}(\sigma,\tau) \end{aligned} \tag{3.1}$$

hold for $n \geq 1$. From this, we can establish, inductively, that $\{M_n^{(\alpha,\alpha)}\}_{n\geq 0}$ is a family of probability measures (note that the $n=0$ case is trivial), and that each $M_n^{(\alpha,\alpha)}$ is a stationary measure for $T_n^{(\alpha,\alpha)}$. Noting that $T_n^{(\alpha,\alpha)}$ has a unique stationary distribution concludes the proof.

□

Proposition 3.2.2. Let $n \geq 1$. The size of the left-most column in the composition $\mathbf{X}_n^{(\alpha,\alpha)}(0)$ has distribution

$$\mathbb{P}\{Y_n^{(\alpha,\alpha)}(0) = i\} = \nu_n^{(\alpha,\alpha)}(i) := \binom{n}{i} \frac{\alpha\,(1-\alpha)_{i-1}}{(n-i+\alpha)_i} \mathbb{1}(1 \leq i \leq n), \quad i \geq 0.$$

Proof. Let $1 \leq i \leq n$ and $\sigma \in \mathcal{C}_{n-i}$. It can easily be verified that

$$M_n^{(\alpha,\alpha)}(i,\sigma) = \nu_n^{(\alpha,\alpha)}(i) M_{n-i}^{(\alpha,\alpha)}(\sigma). \tag{3.2}$$

Summing over σ concludes the proof.

□

Fix $1 \leq i \leq n$ and $\sigma \in \mathcal{C}_{n-i}$. Consider taking an $(\alpha, 0)$ up-step from (i,σ) followed by a down-step. Let U be the event in which the up-step stacks a box into the first column of (i,σ), and let D be the event in which the down-step removes a box from the first column of a composition. Then, $r_{i,i+1} = \mathbb{P}(U \cap D^c)$, $r_{i,i-1} = \mathbb{P}(U^c \cap D)$, $r_{i,i}^{(1)} = \mathbb{P}(U^c \cap D^c)$, $r_{i,i}^{(2)} = \mathbb{P}(U \cap D)$, and $r_{i,i} = r_{i,i}^{(1)} + r_{i,i}^{(2)}$ do not depend on σ. Indeed, we have the formulas

$$\begin{aligned} r_{i,i-1} &= \tfrac{i(n-i+\alpha)}{n(n+1)}, & r_{i,i}^{(1)} &= \tfrac{(n-i+1)(n-i+\alpha)}{n(n+1)}, \\ r_{i,i+1} &= \tfrac{(i-\alpha)(n-i)}{n(n+1)}, & r_{i,i}^{(2)} &= \tfrac{(i-\alpha)(i+1)}{n(n+1)}. \end{aligned} \tag{3.3}$$

Using the above formulas to define the case $i > n$, we have the identity:

$$\frac{\alpha}{n+1} = \nu_n^{(\alpha,\alpha)}(1) r_{1,0} = r_{i,i-1} - r_{i,i+1}, \qquad n, i \geq 1. \tag{3.4}$$

Proposition 3.2.3. Let $n, i \geq 1$. The following identities hold:

$$\begin{aligned} \nu_n^{(\alpha,\alpha)}(i)\, r_{i,i+1} &= \nu_n^{(\alpha,\alpha)}(i+1)\, r_{i+1,i} \\ &= \nu_n^{(\alpha,\alpha)}(1) r_{1,0}\, \mathbb{P}\{Y_n^{(\alpha,\alpha)}(0) > i\}. \end{aligned}$$

Proof. The case $i \geq n$ is trivial. For $1 \leq i < n$, a direct computation yields the first identity:

$$\begin{aligned}\nu_n^{(\alpha,\alpha)}(i)\, r_{i,i+1} &= \binom{n}{i} \frac{\alpha(1-\alpha)_{i-1}(i-\alpha)}{(n-i+\alpha)_i} \frac{n-i}{n(n+1)} \\ &= \binom{n}{i+1} \frac{\alpha(1-\alpha)_i}{(n-i-1+\alpha)_{i+1}} \frac{(i+1)(n-i-1+\alpha)}{n(n+1)} \\ &= \nu_n^{(\alpha,\alpha)}(i+1)\, r_{i+1,i}.\end{aligned}$$

For the second identity, we multiply (3.4) by $\nu_n^{(\alpha,\alpha)}(j)$, take a sum, and use the first identity to obtain

$$\begin{aligned}\sum_{j=i+1}^{n} \nu_n^{(\alpha,\alpha)}(j)\nu_n^{(\alpha,\alpha)}(1) r_{1,0} &= \sum_{j=i+1}^{n} \nu_n^{(\alpha,\alpha)}(j) r_{j,j-1} - \sum_{j=i+1}^{n} \nu_n^{(\alpha,\alpha)}(j) r_{j,j+1} \\ &= \sum_{j=i}^{n-1} \nu_n^{(\alpha,\alpha)}(j+1) r_{j+1,j} - \sum_{j=i+1}^{n} \nu_n^{(\alpha,\alpha)}(j+1) r_{j+1,j} \\ &= \nu_n^{(\alpha,\alpha)}(i+1) r_{i+1,i} - \nu_n^{(\alpha,\alpha)}(n+1) r_{n+1,n}.\end{aligned}$$

Recalling that $\nu_n^{(\alpha,\alpha)}(n+1) = 0$ concludes the proof.

□

Proposition 3.2.4. The size of the left-most column in $\mathbf{X}_n^{(\alpha,\alpha)}(0)$ has expected value

$$\mathbb{E}(Y_n^{(\alpha,\alpha)}(0)) = \frac{\Gamma(1+\alpha)\Gamma(n+1)}{\Gamma(n+\alpha)}, \quad n \geq 1.$$

Proof. Let $2 \leq j \leq n$ and $\tau \in \mathcal{C}_{n+1-j}$. Using the consistency conditions (3.1) and the relation (3.2), we see that

$$\begin{aligned}M_{n+1}^{(\alpha,\alpha)}(j,\tau) &= \sum_{\sigma \in \mathcal{C}_n} M_n^{(\alpha,\alpha)}(\sigma) p^{\uparrow}_{(\alpha,\alpha)}(\sigma,(j,\tau)) \\ &= M_n^{(\alpha,\alpha)}(j-1,\tau) p^{\uparrow}_{(\alpha,\alpha)}((j-1,\tau),(j,\tau)) \\ &\quad + \sum_{\sigma' \in \mathcal{C}_{n-j}} M_n^{(\alpha,\alpha)}(j,\sigma') p^{\uparrow}_{(\alpha,\alpha)}((j,\sigma'),(j,\tau)) \\ &= \nu_n^{(\alpha,\alpha)}(j-1) M_{n+1-j}^{(\alpha,\alpha)}(\tau)\, \tfrac{j-1-\alpha}{n+\alpha} \\ &\quad + \sum_{\sigma' \in \mathcal{C}_{n-j}} \nu_n^{(\alpha,\alpha)}(j) M_{n-j}^{(\alpha,\alpha)}(\sigma')\, \tfrac{n-j+\alpha}{n+\alpha} p^{\uparrow}_{(\alpha,\alpha)}(\sigma',\tau) \\ &= \nu_n^{(\alpha,\alpha)}(j-1) M_{n+1-j}^{(\alpha,\alpha)}(\tau)\, \tfrac{j-1-\alpha}{n+\alpha} + \nu_n^{(\alpha,\alpha)}(j)\, \tfrac{n-j+\alpha}{n+\alpha} M_{n+1-j}^{(\alpha,\alpha)}(\tau).\end{aligned}$$

Dividing by $M^{(\alpha,\alpha)}_{n+1-j}(\tau)$ and applying (3.2) once again, we obtain the recursion

$$\nu^{(\alpha,\alpha)}_{n+1}(j) = \nu^{(\alpha,\alpha)}_{n}(j-1)\,\frac{j-1-\alpha}{n+\alpha} + \nu^{(\alpha,\alpha)}_{n}(j)\,\frac{n-j+\alpha}{n+\alpha},$$

which in fact holds for $2 \leq j$ and $1 \leq n$. The $j = 1$ case can be included with the addition of another term:

$$\begin{aligned}\nu^{(\alpha,\alpha)}_{n+1}(j) = \nu^{(\alpha,\alpha)}_{n}(j-1)\,\frac{j-1-\alpha}{n+\alpha} + \nu^{(\alpha,\alpha)}_{n}(j)\,\frac{n-j+\alpha}{n+\alpha}& \\ + \frac{\alpha}{n+\alpha}\,\mathbb{1}(j=1), \qquad\qquad & 1 \leq j, n.\end{aligned} \tag{3.5}$$

Adding a factor of $(n+\alpha)j$ and taking a sum gives us that

$$\begin{aligned}(n+\alpha)\sum_{j=1}^{n+1} \nu^{(\alpha,\alpha)}_{n+1}(j)\,j &= \alpha + \sum_{j=1}^{n+1} \nu^{(\alpha,\alpha)}_{n}(j-1)\,(j-1-\alpha)j \\ &\quad + \sum_{j=1}^{n+1} \nu^{(\alpha,\alpha)}_{n}(j)\,(n-j+\alpha)j \\ &= \alpha + \sum_{j=1}^{n} \nu^{(\alpha,\alpha)}_{n}(j)((j-\alpha)(j+1) + (n-j+\alpha)j) \\ &= \alpha + \sum_{j=1}^{n} \nu^{(\alpha,\alpha)}_{n}(j)\,((n+1)j-\alpha) \\ &= (n+1)\sum_{j=1}^{n} \nu^{(\alpha,\alpha)}_{n}(j)\,j,\end{aligned}$$

or simply

$$(n+\alpha)\mathbb{E}(Y^{(\alpha,\alpha)}_{n+1}(0)) = (n+1)\mathbb{E}(Y^{(\alpha,\alpha)}_{n}(0)), \quad n \geq 1. \tag{3.6}$$

To conclude the proof, note that the given formula satisfies this recursion and the initial condition $\mathbb{E}(Y^{(\alpha,\alpha)}_{1}(0)) = 1$.

□

Proposition 3.2.5. There exists a unique family of polynomials $\{P_k\}_{k\geq 1}$ such that

$$\mathbb{E}(Y^{(\alpha,\alpha)}_{n}(0)^k) = \mathbb{E}(Y^{(\alpha,\alpha)}_{n}(0))P_k(n), \quad n \geq 1.$$

Moreover, P_k has degree $k-1$ and real coefficients.

Proof. Using (3.5) and the expansion

$$(j+1)^k(j-\alpha) = (j+1)^{k+1} - (1+\alpha)(j+1)^k$$
$$= j^{k+1} + \sum_{m=0}^{k} j^m \big(\tbinom{k+1}{m} - (1+\alpha)\tbinom{k}{m}\big),$$

we obtain the moment recursion

$$\begin{aligned}
(n+\alpha)\mathbb{E}(Y_{n+1}^{(\alpha,\alpha)}(0))^k &= \alpha + \sum_{j=1}^{n+1} \nu_n^{(\alpha,\alpha)}(j) j^k (n-j+\alpha) \\
&\quad + \sum_{j=1}^{n+1} \nu_n^{(\alpha,\alpha)}(j-1) j^k (j-1-\alpha) \\
&= \alpha + \sum_{j=1}^{n} \nu_n^{(\alpha,\alpha)}(j)((n+\alpha)j^k - j^{k+1} + (j+1)^k(j-\alpha)) \\
&= \alpha + (n+\alpha)\mathbb{E}(Y_n^{(\alpha,\alpha)}(0))^k \\
&\quad + \sum_{m=0}^{k} \mathbb{E}(Y_n^{(\alpha,\alpha)}(0))^m \big(\tbinom{k+1}{m} - (1+\alpha)\tbinom{k}{m}\big) \\
&= (n+k)\mathbb{E}(Y_n^{(\alpha,\alpha)}(0))^k \\
&\quad + \sum_{m=1}^{k-1} \mathbb{E}(Y_n^{(\alpha,\alpha)}(0))^m \big(\tbinom{k+1}{m} - (1+\alpha)\tbinom{k}{m}\big),
\end{aligned}$$

which holds for $n, k \geq 1$. Defining $H_k(n) = \mathbb{E}(Y_n^{(\alpha,\alpha)}(0)^k)/\mathbb{E}(Y_n^{(\alpha,\alpha)}(0))$ and using (3.6), this becomes

$$(n+1)H_k(n+1) = (n+k)H_k(n) + \sum_{m=1}^{k-1} H_m(n)\big(\tbinom{k+1}{m} - (1+\alpha)\tbinom{k}{m}\big), \quad n, k \geq 1.$$

We will show that each H_k agrees with some polynomial. Since the family $\{H_k\}$ is characterized by the above recursions and the initial conditions $H_k(1) = 1$ for all k, it suffices to show that some family of polynomials satisfies the same conditions. We identify these polynomials inductively.

For the base case, we can take the polynomial $P_1 \equiv 1$. In the inductive step, we fix $k \geq 2$ and assume that P_m has been identified for $1 \leq m \leq k-1$. Consider the operator Δ defined by

$$\Delta f(x) = (x+1)f(x+1) - (x+k)f(x), \qquad f \in \mathbf{P}_{k-1},$$

where $\mathbf{P}_{k-1}$ denotes the space of polynomials of x with real coefficients and degree at most $k-1$. A direct computation reveals that the range of Δ lies in $\mathbf{P}_{k-2}$ and that its kernel is the one-dimensional subspace spanned by $g(x) = (x+1)_{k-1}$. Applying the Rank-Nullity Theorem, we conclude that the range of Δ is exactly $\mathbf{P}_{k-2}$. In particular, there exists some polynomial f_k of degree $k-1$ such that

$$\Delta f_k = \sum_{m=1}^{k-1} \left(\binom{k+1}{m} - (1+\alpha)\binom{k}{m}\right) P_m.$$

Taking $P_k = f_k + \frac{1-f_k(1)}{g(1)} g$ concludes the inductive step and establishes the fact that each H_k agrees with a polynomial having degree $k-1$ and real coefficients. The uniqueness of the polynomials is immediate from the identity.

□

Proposition 3.2.6. Let $s > \alpha$. Then, as $n \to \infty$,

$$\frac{\mathbb{E}(Y_n^{(\alpha,\alpha)}(0)^s)}{n^{s-\alpha}} \longrightarrow \frac{\Gamma(1+\alpha)\Gamma(s-\alpha)}{\Gamma(1-\alpha)\Gamma(s)}.$$

Proof. To obtain the result, we write the sequence as a sequence of integrals of step functions and apply the dominated convergence theorem. The relevant step functions are

$$g_n(t) = \mathbb{1}(\tfrac{1}{n} < t \leq \tfrac{n-1}{n}) n^2 \frac{\Gamma(\lceil nt \rceil - \alpha)\Gamma(n - \lceil nt \rceil + \alpha)}{\Gamma(\lceil nt \rceil + 1)\Gamma(n - \lceil nt \rceil + 1)},$$

$$f_n(t) = n^{-s} \lceil nt \rceil^s,$$

defined on $[0,1]$. Recalling that $\Gamma(z+r)/\Gamma(z) \sim z^r$ as $z \to \infty$, we can compute

$$\begin{aligned}
\frac{\mathbb{E}(Y_n^{(\alpha,\alpha)}(0))^s}{n^{s-\alpha}} &= n^\alpha \sum_{j=1}^{n} \nu_n^{(\alpha,\alpha)}(j)\Big(\frac{j}{n}\Big)^s \\
&= \frac{\alpha n^{\alpha-1}}{\Gamma(1-\alpha)} \frac{\Gamma(n+1)}{\Gamma(n+\alpha)} \sum_{j=1}^{n} \Big(\frac{j}{n}\Big)^s n^2 \frac{\Gamma(j-\alpha)\Gamma(n-j+\alpha)}{\Gamma(j+1)\Gamma(n-j+1)} \frac{1}{n} \\
&= O(n^{\alpha-s}) + O(n^{-\alpha}) + \frac{\alpha n^{\alpha-1}}{\Gamma(1-\alpha)} \frac{\Gamma(n+1)}{\Gamma(n+\alpha)} \sum_{j=2}^{n-1} f_n\Big(\frac{j}{n}\Big) g_n\Big(\frac{j}{n}\Big) \frac{1}{n} \\
&= O(n^{\alpha-s}) + O(n^{-\alpha}) + \frac{\alpha n^{\alpha-1}}{\Gamma(1-\alpha)} \frac{\Gamma(n+1)}{\Gamma(n+\alpha)} \int_0^1 f_n(t) g_n(t)\, dt \\
&\longrightarrow \frac{\alpha}{\Gamma(1-\alpha)} \int_0^1 t^{s-\alpha-1}(1-t)^{\alpha-1}\, dt \\
&= \frac{\alpha\, \Gamma(\alpha)\Gamma(s-\alpha)}{\Gamma(1-\alpha)\Gamma(s)}.
\end{aligned}$$

Notice that the final integral above can be evaluated by comparing its integrand to the density of a beta distribution.

To conclude the proof, we show that a multiple of the limiting function serves as a suitable bound:

$$|f_n(t) g_n(t)| \le \mathbb{1}(0 < t < 1)\, 2^s t^{s-\alpha-1}(1-t)^{\alpha-1}.$$

It will suffice to consider the case when $1 < nt \le n-1$. Here, we have that

$$2 - \lceil nt \rceil \le\ 0\ \le \lceil nt \rceil - nt \le\ 1\ \le n - \lceil nt \rceil,$$

which rearrange into

$$\frac{nt}{2} \le \lceil nt \rceil - 1, \quad \frac{n(1-t)}{2} \le n - \lceil nt \rceil.$$

Combining this with the lower bound of Gautschi's Inequality,

$$(x-1)^\delta < \frac{\Gamma(x)}{\Gamma(x-\delta)} < x^\delta, \qquad 0 < \delta < 1 < x,$$

we obtain the bound:

$$\begin{aligned}
f_n(t) g_n(t) &= n^{2-s} \lceil nt \rceil^{s-1} \frac{\Gamma(\lceil nt \rceil - \alpha)\Gamma(n - \lceil nt \rceil + \alpha)}{\Gamma(\lceil nt \rceil)\Gamma(n - \lceil nt \rceil + 1)} \\
&\le n^{2-s}(nt+1)^{s-1}(\lceil nt \rceil - 1)^{-\alpha}(n - \lceil nt \rceil)^{\alpha-1} \\
&\le n^{2-s}(2nt)^{s-1}(nt/2)^{-\alpha}(n(1-t)/2)^{\alpha-1}.
\end{aligned}$$

□

Proposition 3.2.7. Let $x \in (0,1]$. As $n \to \infty$,

$$n^{\alpha}\mathbb{P}\{Y_n^{(\alpha,\alpha)}(0) > nx\} \longrightarrow \frac{x^{-\alpha}(1-x)^{\alpha}}{\Gamma(1-\alpha)}.$$

Proof. For $x = 1$, the result is trivial. When $x < 1$, we use Proposition 3.2.3 to compute

$$\begin{aligned}
\mathbb{P}\{Y_n^{(\alpha,\alpha)}(0) > nx\} &= \mathbb{P}\{Y_n^{(\alpha,\alpha)}(0) > \lfloor nx\rfloor\} \\
&= \frac{n+1}{\alpha}\nu_n^{(\alpha,\alpha)}(\lfloor nx\rfloor)\, r_{\lfloor nx\rfloor,\lfloor nx\rfloor+1} \\
&= \frac{n+1}{\alpha}\binom{n}{\lfloor nx\rfloor}\frac{\alpha\,(1-\alpha)_{\lfloor nx\rfloor-1}}{(n-\lfloor nx\rfloor+\alpha)_{\lfloor nx\rfloor}}\,\frac{(\lfloor nx\rfloor-\alpha)(n-\lfloor nx\rfloor)}{n(n+1)} \\
&= \frac{\Gamma(n)}{\Gamma(\lfloor nx\rfloor+1)\Gamma(n-\lfloor nx\rfloor)}\,\frac{(1-\alpha)_{\lfloor nx\rfloor}}{(n-\lfloor nx\rfloor+\alpha)_{\lfloor nx\rfloor}} \\
&= \frac{\Gamma(n)\,\Gamma(\lfloor nx\rfloor+1-\alpha)\Gamma(n-\lfloor nx\rfloor+\alpha)}{\Gamma(\lfloor nx\rfloor+1)\Gamma(n-\lfloor nx\rfloor)\,\Gamma(1-\alpha)\Gamma(n+\alpha)} \\
&= \frac{1}{\Gamma(1-\alpha)}\,\frac{\Gamma(n)}{\Gamma(n+\alpha)}\,\frac{\Gamma(\lfloor nx\rfloor+1-\alpha)}{\Gamma(\lfloor nx\rfloor+1)}\,\frac{\Gamma(n-\lfloor nx\rfloor+\alpha)}{\Gamma(n-\lfloor nx\rfloor)} \\
&\sim \frac{1}{\Gamma(1-\alpha)}\,n^{-\alpha}\lfloor nx\rfloor^{-\alpha}(n-\lfloor nx\rfloor)^{\alpha},
\end{aligned}$$

from which the result follows. □

3.3 The Left-Most Column in the $(\alpha, 0)$ Chain

For $n \geq 1$, define transition kernels $\Lambda_n : [n] \to \mathcal{C}_n$ and $\Phi_n : \mathcal{C}_n \to [n]$ by

$$\Lambda_n(i,(i,\sigma)) = M_{n-i}^{(\alpha,\alpha)}(\sigma),$$

$$\Phi(\sigma,i) = \mathbb{1}(\sigma_1 = i).$$

We will show that $Y_n^{(\alpha,0)}$ is a time-homogeneous Markov chain with transition kernel $Q_n^{(\alpha,0)} = \Lambda_n T_n^{(\alpha,0)}\Phi_n$ whenever the initial distribution of $\mathbf{X}_n^{(\alpha,0)}$ is of the form $\mu\Lambda_n$. Since $\Lambda_n\Phi_n$ is the identity kernel on $[n]$, it will suffice (see Theorem 2 in [47]) to show that the intertwining condition

$$\Lambda_n T_n^{(\alpha,0)} = Q_n^{(\alpha,0)}\Lambda_n \tag{3.7}$$

holds. This is proved in Proposition 3.3.2. Before this, we present a useful identity relating the transition kernels of the $(\alpha, 0)$ and (α, α) chains.

Proposition 3.3.1. For $1 \le i, j \le n$, $\sigma \in \mathcal{C}_{n-i}$, and $\sigma' \in \mathcal{C}_{n-j}$, the following identity holds:

$$\begin{aligned} T_n^{(\alpha,0)}((i,\sigma),(j,\sigma')) = r_{i,i-1} p^{\uparrow}_{(\alpha,\alpha)}(\sigma,\sigma')\mathbb{1}(j=i-1) + r_{i,i+1} p^{\downarrow}(\sigma,\sigma')\mathbb{1}(j=i+1) \\ + (r^{(1)}_{i,i} T^{(\alpha,\alpha)}_{n-i}(\sigma,\sigma') + r^{(2)}_{i,i}\mathbb{1}(\sigma=\sigma'))\mathbb{1}(j=i) \\ + r_{1,0} p^{\uparrow}_{(\alpha,\alpha)}(\sigma,(j,\sigma'))\mathbb{1}(i=1). \end{aligned}$$

Proof. Fix (i,σ) and (j,σ') in $\mathcal{C}_n$. Let $\mathbf{C}^{\uparrow}$ be the composition obtained by performing an $(\alpha, 0)$ up-step from (i,σ) and $\mathbf{C}^{\downarrow}$ be the composition obtained by performing a down-step from $\mathbf{C}^{\uparrow}$. As before, let U be the event in which the up-step adds to the first column of a composition and D be the event in which the down-step removes from the first column of a composition. Then, we have that

$$U = \left\{\mathbf{C}^{\uparrow} = (i+1,\sigma)\right\}, \quad U^c = \{\mathbf{C}^{\uparrow}_1 = i\}, \quad D^c \subset \{\mathbf{C}^{\downarrow}_1 = \mathbf{C}^{\uparrow}_1\},$$

and

$$\begin{aligned} D \subset \left\{\mathbf{C}^{\uparrow}_1 > 1,\, \mathbf{C}^{\downarrow} = (\mathbf{C}^{\uparrow}_1 - 1, (\mathbf{C}^{\uparrow})_2^{\ell(\mathbf{C}^{\uparrow})})\right\} \\ \cup \left\{\mathbf{C}^{\uparrow}_1 = 1,\, \mathbf{C}^{\downarrow} = (\mathbf{C}^{\uparrow})_2^{\ell(\mathbf{C}^{\uparrow})}\right\}. \end{aligned}$$

To obtain the identity, we note that

$$T_n^{(\alpha,0)}((i,\sigma),(j,\sigma')) = \mathbb{P}\{\mathbf{C}^{\downarrow} = (j,\sigma')\},$$

and rewrite this probability by conditioning on the above sets. Of particular importance will be the following observations: the conditional distribution of $(\mathbf{C}^{\downarrow})_2^{\ell(\mathbf{C}^{\downarrow})}$ given $(\mathbf{C}^{\uparrow})_2^{\ell(\mathbf{C}^{\uparrow})}$ and D^c is $p^{\downarrow}((\mathbf{C}^{\uparrow})_2^{\ell(\mathbf{C}^{\uparrow})}, \cdot)$, and, conditionally given U^c, $(\mathbf{C}^{\uparrow})_2^{\ell(\mathbf{C}^{\uparrow})}$ is independent of D and has distribution $p^{\uparrow}_{(\alpha,\alpha)}(\sigma, \cdot)$. We also make use of the fact that the events $\{\mathbf{C}^{\uparrow} = (n+1-|\rho|, \rho)\}$ and $\{(\mathbf{C}^{\uparrow})_2^{\ell(\mathbf{C}^{\uparrow})} = \rho\}$ are identical, since the size of $\mathbf{C}^{\uparrow}$ is known to be $n+1$.

Our first conditional probability is given by

$$\begin{aligned}\mathbb{P}(\mathbf{C}^{\downarrow} = (j,\sigma')|U,D) &= \mathbb{P}(\mathbf{C}^{\downarrow} = (j,\sigma')|\mathbf{C}^{\uparrow} = (i+1,\sigma),D) \\ &= \mathbb{P}((i,\sigma) = (j,\sigma')|\mathbf{C}^{\uparrow} = (i+1,\sigma),D) \\ &= \mathbb{1}((j,\sigma') = (i,\sigma)).\end{aligned}$$

Next, we will condition on $U \cap D^c$. Notice that this is a null set when $i = n$. When $i < n$, we have

$$\begin{aligned}\mathbb{P}(\mathbf{C}^{\downarrow} = (j,\sigma')|U,D^c) &= \mathbb{P}(\mathbf{C}^{\uparrow}_1 = j, (\mathbf{C}^{\downarrow})_2^{\ell(\mathbf{C}^{\downarrow})} = \sigma'|\mathbf{C}^{\uparrow} = (i+1,\sigma),D^c) \\ &= \mathbb{1}(j = i+1)\mathbb{P}((\mathbf{C}^{\downarrow})_2^{\ell(\mathbf{C}^{\downarrow})} = \sigma'|(\mathbf{C}^{\uparrow})_2^{\ell(\mathbf{C}^{\uparrow})} = \sigma, D^c) \\ &= \mathbb{1}(j = i+1)p^{\downarrow}(\sigma,\sigma').\end{aligned}$$

Conditioning on $U^c \cap D$ will require two cases. For $1 < i$, we have

$$\begin{aligned}\mathbb{P}(\mathbf{C}^{\downarrow} = (j,\sigma')|U^c,D) &= \mathbb{P}(\mathbf{C}^{\downarrow} = (j,\sigma')|\mathbf{C}^{\uparrow}_1 = i,D) \\ &= \mathbb{P}((i-1,(\mathbf{C}^{\uparrow})_2^{\ell(\mathbf{C}^{\uparrow})}) = (j,\sigma')|\mathbf{C}^{\uparrow}_1 = i,D) \\ &= \mathbb{1}(j = i-1)\mathbb{P}((\mathbf{C}^{\uparrow})_2^{\ell(\mathbf{C}^{\uparrow})} = \sigma'|U^c,D) \\ &= \mathbb{1}(j = i-1)\mathbb{P}((\mathbf{C}^{\uparrow})_2^{\ell(\mathbf{C}^{\uparrow})} = \sigma'|U^c) \\ &= \mathbb{1}(j = i-1)p^{\uparrow}_{(\alpha,\alpha)}(\sigma,\sigma'),\end{aligned}$$

and for $i = 1$, we have

$$\begin{aligned}\mathbb{P}(\mathbf{C}^{\downarrow} = (j,\sigma')|U^c,D) &= \mathbb{P}(\mathbf{C}^{\downarrow} = (j,\sigma')|\mathbf{C}^{\uparrow}_1 = 1,D) \\ &= \mathbb{P}((\mathbf{C}^{\uparrow})_2^{\ell(\mathbf{C}^{\uparrow})} = (j,\sigma')|U^c,D) \\ &= \mathbb{P}((\mathbf{C}^{\uparrow})_2^{\ell(\mathbf{C}^{\uparrow})} = (j,\sigma')|U^c) \\ &= p^{\uparrow}_{(\alpha,\alpha)}(\sigma,(j,\sigma')).\end{aligned}$$

Finally, we condition on $U^c \cap D^c$. We have that

$$
\begin{aligned}
&\mathbb{P}(\mathbf{C}^{\downarrow} = (j, \sigma')|U^c, D^c) \\
&= \mathbb{P}(\mathbf{C}_1^{\uparrow} = j, (\mathbf{C}^{\downarrow})_2^{\ell(\mathbf{C}^{\downarrow})} = \sigma'|\mathbf{C}_1^{\uparrow} = i, D^c) \\
&= \mathbb{1}(j = i)\, \mathbb{P}((\mathbf{C}^{\downarrow})_2^{\ell(\mathbf{C}^{\downarrow})} = \sigma'|U^c, D^c) \\
&= \mathbb{1}(j = i) \sum_{\tau \in \mathcal{C}_{n+1-i}} \mathbb{P}((\mathbf{C}^{\uparrow})_2^{\ell(\mathbf{C}^{\uparrow})} = \tau|U^c, D^c)\, \mathbb{P}((\mathbf{C}^{\downarrow})_2^{\ell(\mathbf{C}^{\downarrow})} = \sigma'|\mathbf{C}^{\uparrow} = (i, \tau), D^c) \\
&= \mathbb{1}(j = i) \sum_{\tau \in \mathcal{C}_{n+1-i}} \mathbb{P}((\mathbf{C}^{\uparrow})_2^{\ell(\mathbf{C}^{\uparrow})} = \tau|U^c)\, \mathbb{P}((\mathbf{C}^{\downarrow})_2^{\ell(\mathbf{C}^{\downarrow})} = \sigma'|(\mathbf{C}^{\uparrow})_2^{\ell(\mathbf{C}^{\uparrow})} = \tau, D^c) \\
&= \mathbb{1}(j = i) \sum_{\tau \in \mathcal{C}_{n+1-i}} p^{\uparrow}_{(\alpha,\alpha)}(\sigma, \tau) p^{\downarrow}(\tau, \sigma') \\
&= \mathbb{1}(j = i)\, T_{n-i}^{(\alpha,\alpha)}(\sigma, \sigma').
\end{aligned}
$$

Collecting the terms above with the appropriate terms in (3.3) establishes the result.

□

For $n \geq 1$, we define a transition kernel on $[n]$ by

$$
R_n(i, j) = \begin{cases} r_{i,j}, & |i - j| \leq 1, \\ 0, & \text{else.} \end{cases}
$$

Note that only for the special case $i = 1$, we obtain a sub-probability measure. Indeed, it can be checked that $r_{1,0} > 0$, $r_{n,n+1} = 0$, and

$$
\sum_{k=i-1}^{i+1} r_{i,k} = 1, \qquad 1 \leq i \leq n. \tag{3.8}
$$

Proposition 3.3.2 (Intertwining)**.** For $n \geq 1$, the intertwining condition (3.7) holds and the action of $Q_n^{(\alpha,0)}$ on a probability measure μ on $[n]$ is given by

$$
\mu Q_n^{(\alpha,0)} = \mu R_n + \mu(1) r_{1,0}\, \nu_n^{(\alpha,\alpha)}.
$$

Proof. Letting K_n be the kernel on $[n]$ defined by the right side of the above equation, we have that

$$
K_n(i, j) = r_{i,j} \mathbb{1}(|j - i| \leq 1) + r_{1,0} \nu_n^{(\alpha,\alpha)}(j) \mathbb{1}(i = 1),
$$

where, by convention, $r_{i,j} = 0$ whenever $|j-i| > 1$.

Fix $i, j \in [n]$ and $\sigma' \in \mathcal{C}_{n-j}$. Using Proposition 3.3.1 and the consistency conditions in (3.1), we obtain that

$$
\begin{aligned}
(\Lambda_n T_n^{(\alpha,0)})(i,(j,\sigma')) &= \sum_{\sigma \in \mathcal{C}_{n-i}} \Lambda_n(i,(i,\sigma)) T_n^{(\alpha,0)}((i,\sigma),(j,\sigma')) \\
&= r_{i,i-1} \mathbb{1}(j=i-1) \sum_{\sigma \in \mathcal{C}_{n-i}} M_{n-i}^{(\alpha,\alpha)}(\sigma) p^{\uparrow}_{(\alpha,\alpha)}(\sigma,\sigma') \\
&\quad + r_{i,i+1} \mathbb{1}(j=i+1) \sum_{\sigma \in \mathcal{C}_{n-i}} M_{n-i}^{(\alpha,\alpha)}(\sigma) p^{\downarrow}(\sigma,\sigma') \\
&\quad + r_{i,i}^{(1)} \mathbb{1}(j=i) \sum_{\sigma \in \mathcal{C}_{n-i}} M_{n-i}^{(\alpha,\alpha)}(\sigma) T_{n-i}^{(\alpha,\alpha)}(\sigma,\sigma') \\
&\quad + r_{i,i}^{(2)} \mathbb{1}(j=i) \sum_{\sigma \in \mathcal{C}_{n-i}} M_{n-i}^{(\alpha,\alpha)}(\sigma) \mathbb{1}(\sigma=\sigma') \\
&\quad + r_{1,0} \mathbb{1}(i=1) \sum_{\sigma \in \mathcal{C}_{n-i}} M_{n-i}^{(\alpha,\alpha)}(\sigma) p^{\uparrow}_{(\alpha,\alpha)}(\sigma,(j,\sigma')) \\
&= r_{i,i-1} \mathbb{1}(j=i-1) M_{n-j}^{(\alpha,\alpha)}(\sigma') \\
&\quad + r_{i,i+1} \mathbb{1}(j=i+1) M_{n-j}^{(\alpha,\alpha)}(\sigma') \\
&\quad + r_{i,i}^{(1)} \mathbb{1}(j=i) M_{n-j}^{(\alpha,\alpha)}(\sigma') \\
&\quad + r_{i,i}^{(2)} \mathbb{1}(j=i) M_{n-j}^{(\alpha,\alpha)}(\sigma') \\
&\quad + r_{1,0} \mathbb{1}(i=1) M_{n}^{(\alpha,\alpha)}(j,\sigma') \\
&= (r_{i,j} \mathbb{1}(j=i-1) + r_{i,j} \mathbb{1}(j=i+1) + r_{i,j} \mathbb{1}(j=i)) M_{n-j}^{(\alpha,\alpha)}(\sigma') \\
&\quad + r_{1,0} \mathbb{1}(i=1) \nu_n^{(\alpha,\alpha)}(j) M_{n-j}^{(\alpha,\alpha)}(\sigma') \\
&= K_n(i,j) \Lambda_n(j,(j,\sigma')) \\
&= (K_n \Lambda_n)(i,(j,\sigma')).
\end{aligned}
$$

Note that the final equality follows from the fact that $\Lambda_n(j,\cdot)$ is supported on $\{\sigma \in \mathcal{C}_n : \sigma_1 = j\}$. This establishes the identity $\Lambda_n T_n^{(\alpha,0)} = K_n \Lambda_n$, but since $\Lambda_n \Phi_n$ is the identity kernel, we have that

$$Q_n^{(\alpha,0)} = \Lambda_n T_n^{(\alpha,0)} \Phi_n = K_n \Lambda_n \Phi_n = K_n.$$

□

Proposition 3.3.3. The size of the left-most column in $\mathbf{X}_n^{(\alpha,0)}(0)$ has distribution

$$\mathbb{P}\{Y_n^{(\alpha,0)}(0) = i\} = \nu_n^{(\alpha,0)}(i) := a_n\, i\nu_n^{(\alpha,\alpha)}(i), \qquad i \geq 0,$$

where $a_n^{-1} = \mathbb{E}(Y_n^{(\alpha,\alpha)}(0))$.

Proof. It suffices to show that $\nu_n^{(\alpha,0)}$ is a stationary measure for $Q_n^{(\alpha,0)}$. Observe first that (3.4) can be written as

$$\nu_n^{(\alpha,\alpha)}(1) r_{1,0} = \sum_{i=j-1}^{j+1} (j-i) r_{j,i}, \qquad 1 \leq j \leq n.$$

Combining this with Propositions 3.3.2 and 3.2.3 and the identity (3.8), we have the result:

$$\begin{aligned}
(\nu_n^{(\alpha,0)} Q_n^{(\alpha,0)})(j) &= (\nu_n^{(\alpha,0)} R_n)(j) + \nu_n^{(\alpha,0)}(1) r_{1,0} \nu_n^{(\alpha,\alpha)}(j) \\
&= \sum_{i=1}^{n} \nu_n^{(\alpha,0)}(i) r_{i,j} \mathbb{1}(|j-i| \leq 1) + a_n \nu_n^{(\alpha,\alpha)}(1) r_{1,0} \nu_n^{(\alpha,\alpha)}(j) \\
&= a_n \sum_{i=1\vee(j-1)}^{n\wedge(j+1)} i\nu_n^{(\alpha,\alpha)}(i) r_{i,j} + a_n \nu_n^{(\alpha,\alpha)}(j) \sum_{i=j-1}^{j+1} (j-i) r_{j,i} \\
&= a_n \sum_{i=j-1}^{j+1} i\nu_n^{(\alpha,\alpha)}(j) r_{j,i} + a_n \nu_n^{(\alpha,\alpha)}(j) \sum_{i=j-1}^{j+1} (j-i) r_{j,i} \\
&= a_n\, j\nu_n^{(\alpha,\alpha)}(j) \sum_{i=j-1}^{j+1} r_{j,i} \\
&= a_n\, j\nu_n^{(\alpha,\alpha)}(j) \\
&= \nu_n^{(\alpha,0)}(j).
\end{aligned}$$

□

Remark. Proposition 3.3.3 establishes a connection between the moments of $Y_n^{(\alpha,\alpha)}(0)$ and $Y_n^{(\alpha,0)}(0)$. As a result, we obtain analogues of Propositions 3.2.5 and 3.2.6 for the $(\alpha, 0)$ chain.

For $n \geq 2$ and $1 \leq k \leq n$, we introduce hitting times and return times

$$\omega_k = \inf\{m \geq 0 : Y_n^{(\alpha,0)}(m) = k\},$$
$$\omega_k^+ = \inf\{m \geq 1 : Y_n^{(\alpha,0)}(m) = k\}.$$

In addition, let $\hat{\omega}_k$ and $\hat{\omega}_k^+$ be the hitting and return times of the process whose transition kernel is given by $Q_n^{(\alpha,0)}$ but with the re-entry term $\nu_n^{(\alpha,\alpha)}$ replaced by δ_1, the point mass at 1. Notice that this modified chain evolves exactly like $Y_n^{(\alpha,0)}$ until it reaches the state 1. As a result, we have the identities

$$\mathbb{E}_j(\omega_i) = \mathbb{E}_j(\hat{\omega}_i), \qquad \mathbb{P}_j\{\omega_i < \omega_j^+\} = \mathbb{P}_j\{\hat{\omega}_i < \hat{\omega}_j^+\}, \qquad i < j,$$

where $\mathbb{E}_y$ and $\mathbb{P}_y$ are the conditional expectations and conditional probabilities resulting from starting the relevant process at the state y.

Proposition 3.3.4. For $1 \leq j \leq n-1$, the following formulas hold:

$$\mathbb{E}_j(\hat{\omega}_{j+1}) = \frac{1}{\nu_n^{(\alpha,\alpha)}(j) r_{j,j+1}} - \frac{n+1}{\alpha},$$
$$\mathbb{E}_{j+1}(\hat{\omega}_j) = \frac{n+1}{\alpha},$$
$$\mathbb{E}_i(\omega_j) + \mathbb{E}_j(\omega_i) = \mathbb{E}\Big(\frac{Y_n^{(\alpha,\alpha)}(0)}{i \vee j}\Big)(\mathbb{E}_i(\hat{\omega}_j) + \mathbb{E}_j(\hat{\omega}_i)).$$

Proof. The first identity in Proposition 3.2.3 are exactly the detailed-balance equations for $\nu_n^{(\alpha,\alpha)}$ and the modified chain. Therefore, we obtain the first two formulas from Theorem 2.3 in [36] and the second identity in Proposition 3.2.3. For the third, we apply Corollary 2.8 in [1]. When $i < j$, we obtain

$$\begin{aligned}\mathbb{E}_i(\omega_j) + \mathbb{E}_j(\omega_i) &= \frac{1}{\nu_n^{(\alpha,0)}(j)\mathbb{P}_j\{\omega_i < \omega_j^+\}} \\ &= \frac{1}{a_n j \nu_n^{(\alpha,\alpha)}(j)\mathbb{P}_j\{\hat{\omega}_i < \hat{\omega}_j^+\}} \\ &= (a_n j)^{-1}(\mathbb{E}_i(\hat{\omega}_j) + \mathbb{E}_j(\hat{\omega}_i)).\end{aligned}$$

□

Proposition 3.3.5. Let $n \geq 1$. Viewing the measure $\nu_n^{(\alpha,\alpha)}$ as a vector in $\mathbb{R}^n$ and $Q_n^{(\alpha,0)}$ as a linear operator on $\mathbb{R}^n$, $\nu_n^{(\alpha,\alpha)}$ is a cyclic vector for $Q_n^{(\alpha,0)}$.

Proof. The $n = 1$ case is trivial, so we consider $n \geq 2$. Let $\{e_i\}_{i=1}^n$ denote the standard basis of $\mathbb{R}^n$. Since $\nu_n^{(\alpha,\alpha)}(n)$ is non-zero, we have that $e_n \in \text{span}\{e_1, ..., e_{n-1}, \nu_n^{(\alpha,\alpha)}\}$. As a result, it suffices to show that

$$e_i \in V_i = \text{span}\{\nu_n^{(\alpha,\alpha)}, \nu_n^{(\alpha,\alpha)} Q_n^{(\alpha,0)}, ..., \nu_n^{(\alpha,\alpha)} (Q_n^{(\alpha,0)})^i\}, \qquad i = 1, \ldots, n-1.$$

The $i = 1$ case is handled by the relation

$$\nu_n^{(\alpha,\alpha)} Q_n^{(\alpha,0)} = (1 + \tfrac{\alpha}{n+1})\nu_n^{(\alpha,\alpha)} - \tfrac{\alpha}{n+1} e_1,$$

and, if $n \geq 3$, the relation

$$e_1 Q_n^{(\alpha,0)} = \tfrac{n-1+\alpha}{n(n+1)} \nu_n^{(\alpha,\alpha)} + \left(\tfrac{n-1+\alpha}{n+1} + \tfrac{2(1-\alpha)}{n(n+1)}\right) e_1 + \tfrac{(1-\alpha)(n-1)}{n(n+1)} e_2$$

establishes the $i = 2$ case. For $n \geq 4$, we proceed inductively.

Suppose that $e_i \in V_i$ for $i = 1, \ldots, m$ and $2 \leq m \leq n-2$. Since $Q_n^{(\alpha,0)}$ is tridiagonal after its first row, the vector $e_m Q_n^{(\alpha,0)}$ can be written as a linear combination of e_{m-1}, e_m, and e_{m+1} using nonzero coefficients. As a result, e_{m+1} lies in the span of $\{e_{m-1}, e_m, e_m Q_n^{(\alpha,0)}\}$. Observe now that the $\{V_i\}$ are nested, so the vectors e_{m-1} and e_m lie in V_{m+1}. Moreover, $e_m Q_n^{(\alpha,0)} \in V_m Q_n^{(\alpha,0)} \subset V_{m+1}$. Therefore, e_{m+1} also lies in V_{m+1}. This concludes the inductive step and the proof.

□

3.4 Excursions of the $(\alpha, 0)$ chain

For each $n \geq 1$, let N^n be a Poisson Process with rate n^2 that is independent of $Y_n^{(\alpha,0)}$ and define $Z_n = n^{-1}(Y_n^{(\alpha,0)}(N^n) - 1)$. These processes are continuous time versions of the chains $\{Y_n^{(\alpha,0)}\}_{n\geq 1}$ that have been suitably transformed to lie in $[0, 1]$ and have $x = 0$ as a common boundary. Our interest lies in studying how these processes behave after visiting this boundary.

Let $f : [0, \infty) \to [0, 1]$ be a cádlág path. The hitting, exit, and return times of f from zero are defined by

$$\tau^h(f) = \inf\{t \geq 0 : f(t) = 0\},$$
$$\tau^e(f) = \inf\{t > 0 : f(t) > 0\},$$
$$\tau^r(f) = \inf\{t > \tau^e(f) : f(t) = 0\}.$$

Define also

$$E(f) = f((\,\cdot\, + \tau^e(f)) \wedge \tau^r(f)),$$

which provides the first excursion of f from zero whenever it exists, and is the constant path at zero otherwise. For an excursion from zero f, we let $\varphi(f) = f(0)$.

The excursion measure of Z_n is given by

$$\mathcal{N}_n(A) = \mathbb{P}\{E(Z_n) \in A | Z_n(0) = 0\}$$

for suitable sets A (Borel subsets of $\mathcal{E}$, the collection of non-constant, cádlág excursions endowed with the J_1 topology, but this will not be important). The following result takes a step toward determining the limit of the chains $\{Y_n^{(\alpha,0)}\}_{n\geq 1}$.

Theorem 3.4.1. *Let Z be a Feller process on $[0, 1]$ satisfying*

$$\tau^h(Z_n) \longrightarrow_d \tau^h(Z)$$

whenever $Z_n(0) = z_n$ is a deterministic sequence converging to $Z(0) = z$. Define

$$\mathcal{N}(A) = \frac{\alpha}{2\Gamma(2-\alpha)} \int_0^1 \mathbb{P}_s\{Z(\cdot \wedge \tau^h(Z)) \in A\} s^{-\alpha-1}(1-s)^{\alpha-1}\, ds,$$

for suitable sets A and let $\varepsilon, \lambda > 0$. Then, as $n \to \infty$, we have the convergence

(i) $n^\alpha \mathcal{N}_n(\varphi > \varepsilon) \longrightarrow \mathcal{N}(\varphi > \varepsilon)$, *and*

(ii) $n^\alpha \mathcal{N}_n(1 - e^{-\lambda\tau^h}; \varphi \leq \varepsilon) \longrightarrow \mathcal{N}(1 - e^{-\lambda\tau^h}; \varphi \leq \varepsilon)$.

Proof. Let $\sigma^e(f) = \inf\{m > 0 : f(m) \neq 1\}$ be the exit time of 1 for a sample path f of any of the Markov chains $Y_n^{(\alpha,0)}$ and δ_x be the point mass at x. Then,

$$\begin{aligned}
\mathcal{N}_n(\varphi \in ds) &= \mathbb{P}\{\varphi(E(Z_n)) \in ds | Z_n(0) = 0\} \\
&= \mathbb{P}\{Z_n(\tau^e(Z_n)) \in ds | Z_n(0) = 0\} \\
&= \mathbb{P}\{Y_n^{(\alpha,0)}(N^n_{\tau^e(Z_n)}) \in nds + 1 | Y_n^{(\alpha,0)}(0) = 1\} \\
&= \mathbb{P}\{Y_n^{(\alpha,0)}(\sigma^e(Y_n^{(\alpha,0)})) \in nds + 1 | Y_n^{(\alpha,0)}(0) = 1\} \\
&= \mathbb{P}\{Y_n^{(\alpha,0)}(1) \in nds + 1 | Y_n^{(\alpha,0)}(0) = 1, Y_n^{(\alpha,0)}(1) \neq 1\} \\
&= (1 - Q_n^{(\alpha,0)}(1,1))^{-1} \sum_{k \in [n]} Q_n^{(\alpha,0)}(1,k) \delta_{\frac{k-1}{n}}(ds) \\
&= (1 - Q_n^{(\alpha,0)}(1,1))^{-1} \Big(r_{1,2}\, \delta_{\frac{1}{n}}(ds) + r_{1,0} \sum_{k \in [n]} \nu_n^{(\alpha,\alpha)}(k) \delta_{\frac{k-1}{n}}(ds) \Big) \\
&= (1 - Q_n^{(\alpha,0)}(1,1))^{-1} (r_{1,2}\, \delta_{\frac{1}{n}}(ds) + r_{1,0} \mathbb{P}\{Y_n^{(\alpha,\alpha)}(0) \in nds + 1\}).
\end{aligned}$$

Therefore, for $\varepsilon > 0$ and large n, we have

$$\begin{aligned}
n^\alpha \mathcal{N}_n(\varphi > \varepsilon) &= n^\alpha (1 - Q_n^{(\alpha,0)}(1,1))^{-1} r_{1,0}\, \mathbb{P}\{Y_n^{(\alpha,\alpha)}(0) > n\varepsilon + 1\} \\
&\sim (2(1-\alpha))^{-1} n^\alpha (\mathbb{P}\{Y_n^{(\alpha,\alpha)}(0) > n\varepsilon\} + o(n^{-\alpha}))
\end{aligned}$$

Applying now Proposition 3.2.7, we have the limit

$$n^\alpha \mathcal{N}_n(\varphi > \varepsilon) \longrightarrow \frac{\varepsilon^{-\alpha}(1-\varepsilon)^\alpha}{2\Gamma(2-\alpha)},$$

which establishes (i).

To obtain (ii), we will condition on the value of φ. Using the Strong Markov property, we obtain the conditional distribution

$$\begin{aligned}
&\mathbb{P}\{\tau^h(E(Z_n)) \in dt | \varphi(E(Z_n)) = s, Z_n(0) = 0\} \\
&\quad = \mathbb{P}\{\tau^r(Z_n) - \tau^e(Z_n) \in dt | Z_n(\tau^e(Z_n)) = s, Z_n(0) = 0\} \\
&\quad = \mathbb{P}\{\tau^r(Z_n) \in dt | Z_n(0) = s\} \\
&\quad = \mathbb{P}\{\tau^h(Z_n) \in dt | Z_n(0) = s\}
\end{aligned}$$

for $s \in [n-1]/n$. For $\varepsilon \in (0,1]$ and large n, we thus have

$$\begin{aligned} n^\alpha \mathcal{N}_n(1-e^{-\lambda T}; \varphi \le \varepsilon) &= n^\alpha \int_{(0,\varepsilon]} \int_0^\infty (1-e^{-\lambda t}) \mathbb{P}_s\{\tau^h(Z_n) \in dt\} \mathcal{N}_n(\varphi \in ds) \\ &= n^\alpha \int_{(0,\varepsilon]} \mathbb{E}_s(1-e^{-\lambda \tau^h(Z_n)}) \mathcal{N}_n(\varphi \in ds) \\ &= (1-Q_n^{(\alpha,0)}(1,1))^{-1} r_{1,2} n^\alpha \mathbb{E}_{1/n}(1-e^{-\lambda\tau^h(Z_n)}) \\ &\quad + (1-Q_n^{(\alpha,0)}(1,1))^{-1} r_{1,0} n^\alpha \mathbb{E} f_n(n^{-1} Y_n^{(\alpha,\alpha)}(0)), \end{aligned}$$

where the functions $f_n : [n]/n \to \mathbb{R}$ are given by

$$f_n(x) = \mathbb{1}_{\left(\frac{1}{n}, \varepsilon + \frac{1}{n}\right]}(x) \mathbb{E}_{x-\frac{1}{n}}(1-e^{-\lambda\tau^h(Z_n)}).$$

To proceed, we assume that c is a constant independent of x and n satisfying

$$|f_n(x)| \le c\left(x - \tfrac{1}{n}\right). \tag{3.9}$$

We prove this bound later. Using this bound, we find that

$$n^\alpha \mathbb{E}_{\frac{1}{n}}(1-e^{-\lambda\tau^h(Z_n)}) \le c n^{\alpha-1},$$

and since $(1-Q_n^{(\alpha,0)}(1,1))^{-1} r_{1,2}$ is bounded, the first term above goes to zero. To analyze the second term, note first that $(1-Q_n^{(\alpha,0)}(1,1))^{-1} r_{1,0} \to (2(1-\alpha))^{-1}$. We proceed, as in Proposition 3.2.6, to write

$$\begin{aligned} n^\alpha \mathbb{E} f_n(n^{-1} Y_n^{(\alpha,\alpha)}(0)) &= f_n(1/n) O(n^\alpha) + f_n(1) O(n^{-\alpha}) \\ &\quad + \left(\frac{\alpha}{\Gamma(1-\alpha)} + o(1)\right) \int_0^1 f_n(\lceil nt \rceil/n) g_n(t)\, dt, \end{aligned}$$

and observe that the first two terms go to zero. The integral is handled by the dominated convergence theorem: we have the bounding sequence

$$|f_n(\lceil nt \rceil/n) g_n(t)| \le c n^{-1} \lceil nt \rceil\, g_n(t),$$

for which the integration and limit operations can be interchanged (see Proposition 3.2.6), and the pointwise limit

$$f_n(\lceil nt \rceil/n) \to \mathbb{1}_{(0,\varepsilon]}(t) \mathbb{E}_t(1-e^{-\lambda\tau^h(Z)})$$

follows from the hypothesis. Altogether, we have the limit

$$n^\alpha \mathcal{N}_n(1 - e^{-\lambda T}; \varphi \leq \varepsilon) \longrightarrow \frac{\alpha}{2\Gamma(2-\alpha)} \int_0^\varepsilon \mathbb{E}_t(1 - e^{-\lambda \tau^h(Z)}) t^{-\alpha-1}(1-t)^{\alpha-1}\, dt,$$

which establishes (ii).

To obtain the bound in (3.9), first observe that

$$|f_n(x)| \leq \lambda \mathbb{E}_{x-\frac{1}{n}}(\tau^h(Z_n)).$$

Letting ΔN_i^n be the i^{th} interarrival time of N^n and recalling that ω_1 is the hitting time of 1 of the Markov chain $Y_n^{(\alpha,0)}$, we have the relation $\tau^h(Z_n) = \Delta N_i^n + \cdots + \Delta N_{\omega_1}^n$. Applying now Proposition 3.3.4, we can compute

$$\begin{aligned}
\mathbb{E}_{x-1/n}(\tau^h(Z_n)) &= \mathbb{E}_{nx}\Big(\sum_{i=1}^{\omega_1} \Delta N_i^n\Big) \\
&= \mathbb{E}_{nx}(\omega_1)\, \mathbb{E}(\Delta N_1^n) \\
&= \alpha^{-1}(n+1)(nx-1)n^{-2} \\
&\leq 2\alpha^{-1}\left(x - \tfrac{1}{n}\right),
\end{aligned}$$

which concludes the proof.

□

Chapter 4

THE MAXIMUM VERTEX DEGREE IN RANDOM DISSECTIONS

4.1 Introduction

We now turn our attention to the context of dissections. As mentioned in section 1.2, our motivation is to make progress on the conjecture in (1.3) and to identify analogues of Gao and Wormald's results for triangulations. The main object of interest here is the random variable that provides the maximum vertex degree in a uniformly drawn dissection of a convex n-gon.

Results. Our main result, Theorem 4.1.1 below, is a concentration inequality that confirms Curien and Kortchemski's conjecture. In addition, we obtain estimates on the moments of the number of vertices of a fixed degree (Lemma 4.2.1). Both results are analogues of results of Gao and Wormald.

Theorem 4.1.1. *Let $b = 1 + \sqrt{2}$, $\lambda_n = (\log n + \log\log n)/\log b$, and Δ_n denote the maximum vertex degree in a random dissection of an n-gon. Then, as $n \to \infty$,*

$$\mathbb{E}(\Delta_n) = \lambda_n + O(1),$$
$$\mathrm{Var}(\Delta_n) = O(\log n),$$

and

$$\mathbb{P}(|\Delta_n - \lambda_n| \le \Omega_n) \ge 1 - O\Big((\log n)^{-1} + b^{-\Omega_n}\Big)$$

for any $\Omega_n \to \infty$.

Techniques. Our proof of Theorem 4.1.1 follows the spirit of [20], combining counting arguments and standard probabilistic techniques with tools from analytic combinatorics. Our main departure from their method is in our counting, where we make use of dual trees to avoid dealing with dissections directly. The use of dual trees was inspired by the work in [9].

Outline. This chapter is organized as follows. In Section 4.2, we state estimates on the moments of some auxiliary random variables from which Theorem 4.1.1 will follow. The remainder of the chapter is dedicated to obtaining these estimates. In Section 4.3, we relate the moments to the sizes of certain classes of trees. In Section 4.4, we derive functional equations involving generating functions that correspond to our tree classes. In Section 4.5, we employ the techniques of analytic combinatorics to obtain the estimates of Section 4.2.

Notation. We will use the big-oh notation in the standard way – for example, $f(x) = O(g(x))$ as $x \to a$ means that the function f is bounded by a multiple of g in a neighborhood of a. We will also use this notation without an associated limit but with an associated set, and this is to mean that a bound of the same form holds on the given set. We use $[x^n]G(x)$ to denote the coefficient of x^n in a generating function $G(x)$. A class of combinatorial objects will always be denoted by some calligraphic symbol, and its size will be denoted by a non-calligraphic version of that symbol (e.g. the size of the class $\mathcal{A}_m$ is A_m). Sums over empty index sets are to take the value zero.

4.2 Dissections

In this section, we reduce the study of the maximum vertex degree to one of the number of vertices of a fixed degree. We begin by introducing the objects of interest.

Definition 4.2.1. For $n \geq 3$, let $\mathcal{P}_n$ denote the convex n-gon in the complex plane whose vertices are the n^{th} roots of unity. A subset of the plane d is a *dissection* of $\mathcal{P}_n$ if it is the union of the sides of $\mathcal{P}_n$ and a collection of diagonals that may intersect only at their endpoints. In this case, we define for $j = 1, 2, ..., n$, the j^{th} vertex of d as the point $v_j(d) = e^{2\pi i(n+1-j)/n}$, and the degree of $v_j(d)$, denoted by $\deg v_j(d)$, is the total number of diagonals and sides of $\mathcal{P}_n$ that lie in d and contain $v_j(d)$. See Figure 4.1. For convenience, we will often omit the argument of $v_j(d)$ when it is clear from context.

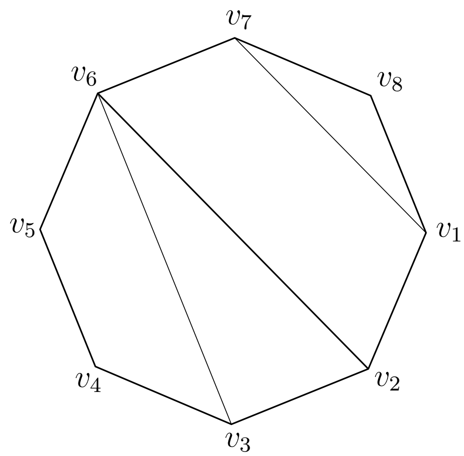

Figure 4.1: A dissection of an octagon.

We denote the set of all dissections of $\mathcal{P}_{n+1}$ by $\mathcal{D}_n$. The subset of these in which v_1 has degree k will be denoted by $\mathcal{D}_{n,k}$. The collection of pairs (d, v) in which d is a dissection from $\mathcal{D}_{n,k}$ and $v \neq v_1$ is a vertex in d with degree l will be denoted by $\mathcal{D}_{n,k,l}$. We will refer to such a pair as a dissection with a *distinguished vertex* v. The parameters n, k, and l are to take integer values no less than 2.

We construct, for each n, a probability space by equipping $\mathcal{D}_n$ with the uniform measure. On each space, we define a random variable Δ_n that maps a dissection d to the maximum vertex degree of d. In addition, we define a sequence of random variables $\{\zeta_{k,n}\}_{k\geq 2}$ by letting $\zeta_{k,n}$ (or ζ_k for short) map a dissection d to the number of vertices in d having degree k. Theorem 4.1.1 is a direct consequence of the following lemma.

Lemma 4.2.1. Let $b = 1+\sqrt{2}$. The following estimates hold (all constants are uniform in k and n):

(i) $\mathbb{E}(\zeta_k) = O(np^{-k})$ as $n \to \infty$ for $p \in (0, b)$ and all k,

(ii) $\mathbb{E}(\zeta_k) = 2knb^{-k}(1 + O(1/k))$, as $k, n \to \infty$, and $k = O(\log n)$,

(iii) $\mathbb{E}(\zeta_k(\zeta_k - 1)) = \mathbb{E}(\zeta_k)^2(1 + O(1/k))$, as $k, n \to \infty$, and $k = O(\log n)$.

Proof of Theorem 4.1.1. The argument is exactly like the proof of Theorem 1 in [20].

□

4.3 Dual Trees

In this section, we make the connection between the factorial moments of the ζ_k and the enumeration of trees. The first step towards this goal is expressing these moments in terms of the sizes of our dissection classes. This is given in the following result.

Lemma 4.3.1. For $n, k \geq 2$, the following identities hold:

(i) $\mathbb{E}(\zeta_k) = (n+1)\frac{D_{n,k}}{D_n}$,

(ii) $\mathbb{E}(\zeta_k(\zeta_k - 1)) = (n+1)\frac{D_{n,k,k}}{D_n}$.

Proof. Let $\mathcal{D}_n^{(i)}$ and $\mathcal{D}_{n,k,k}^{(i)}$ be the subsets of $\mathcal{D}_n$ and $\mathcal{D}_{n,k,k}$, respectively, containing those dissections that have i vertices of degree k. Let $\mathcal{S}$ consist of pairs $(d, (u, v))$ in which d is a dissection in $\mathcal{D}_n^{(i)}$ and (u, v) is an ordered pair of distinct vertices in d with degree k. It follows immediately that the size of $\mathcal{S}$ is given by $D_n^{(i)} i(i-1)$.

Now let $\eta : \mathcal{D}_{n,k,k}^{(i)} \to \mathcal{S}$ be the map $(d, v) \mapsto (d, (v_1, v))$, and for $m = 0, \ldots, n$, let $\varphi_m : \mathbb{C} \to \mathbb{C}$ be the rotation $z \mapsto ze^{-2\pi i m/(n+1)}$, which acts on elements of $\mathcal{S}$ as $(d, (u, v)) \mapsto (\varphi_m(d), (\varphi_m(u), \varphi_m(v)))$. Then, it can be verified that $\mathcal{S}$ is given by the disjoint union

$$\mathcal{S} = \bigsqcup_{m=0}^{n} \varphi_m(\eta(\mathcal{D}_{n,k,k}^{(i)})).$$

Noting that the maps $\varphi_0, \ldots, \varphi_n$ and η are injective yields the identity

$$D_n^{(i)} i(i-1) = D_{n,k,k}^{(i)}(n+1), \qquad i \geq 2.$$

Summing this relation over i and dividing by D_n yields the claim in (ii). The claim in (i) can be obtained in a similar manner. □

Definition 4.3.1. A *rooted ordered tree* is a graph-theoretic tree with a vertex designated as the *root* and an ordering among the children of any vertex. A non-root vertex in such a tree is a *leaf* if it has no children.

We will think of rooted ordered trees as subsets of the upper half of the complex plane by embedding them in such a way that the root is mapped to $z = 0$

and the clockwise orientation agrees with the ordering throughout the tree. Up to orientation-preserving homeomorphisms of the upper half plane, a rooted ordered tree then corresponds to a unique subset of the plane. We denote the root of a tree t with n leaves by $\ell_0(t) = \ell_{n+1}(t)$ and its leaves by $\ell_1(t), ..., \ell_n(t)$ (in clockwise order). As with the vertices of a dissection, we will often omit the argument t when it is clear from context.

For each dissection $d \in \mathcal{D}_n$, we construct a dual tree, t_d, in the following way: first, we place a vertex in each inner face of d and connect those vertices whose corresponding faces share an edge; then, we place n vertices in the outer face, assign each to a distinct edge of $\mathcal{P}_{n+1}$, and connect each to the vertex whose corresponding face shares the assigned edge; finally, we root the tree at the vertex assigned to the edge (v_1, v_{n+1}) by applying an orientation-preserving homeomorphism of the plane that maps this vertex to $z = 0$ and the remainder of the tree into the upper half plane (see Figure 4.2). In addition, we correspond a dissection with a distinguished vertex to a tree with a distinguished leaf by designating (t_d, ℓ_{j-1}) as the dual 'tree' of (d, v_j).

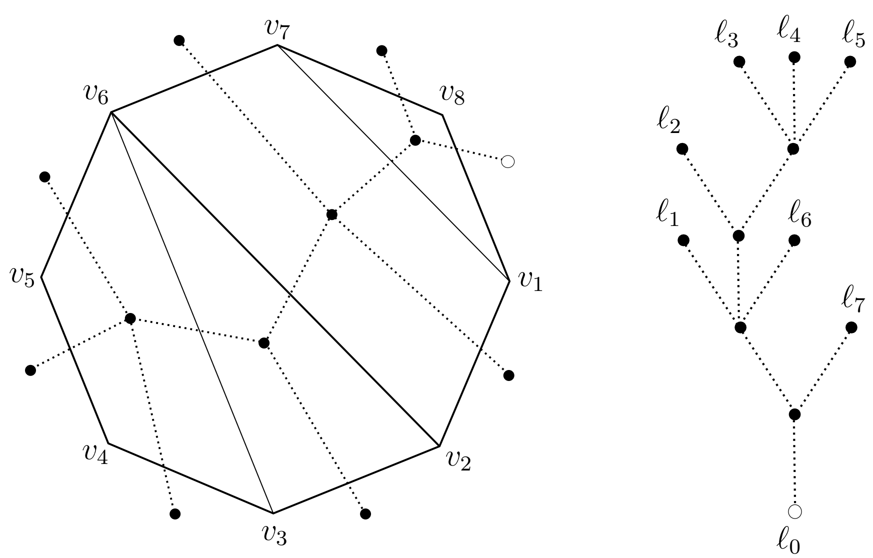

Figure 4.2: A dual tree. Its root is colored white.

As in [9], the map sending a dissection to its dual tree is bijective. The image of $\mathcal{D}_n$, which we will denote by $\mathcal{T}_n$, consists of rooted ordered trees that have n leaves, root degree one, and no non-root vertex with exactly one child. To determine the images of the other classes, we resort to the following result.

Proposition 4.3.1. Let $d \in \mathcal{D}_n$ and ρ be the graph metric on t_d. Then, for each $i = 1, ..., n+1$, we have that

$$\deg v_i = \rho(\ell_{i-1}, \ell_i).$$

Proof. Superimpose t_d onto d (as in Figure 4.2) and assign to each edge in d the unique edge in t_d that it intersects. This assignment maps the edges adjacent to v_i to the edges in t_d that surround v_i, or those that constitute the path from ℓ_{i-1} to ℓ_i. Thus, these two groups of edges are equal in number.

□

Proposition 4.3.1 reveals that the image of $\mathcal{D}_{n,k}$ under the duality map is the collection of $\mathcal{T}_n$ trees in which the path from the root to the leftmost leaf is of length k (contains k edges). We denote this set by $\mathcal{T}_{n,k}$. The image of $\mathcal{D}_{n,k,l}$, which we denote by $\mathcal{T}_{n,k,l}$, consists of pairs (t, ℓ_i) in which t lies in $\mathcal{T}_{n,k}$ and the path from the distinguished leaf ℓ_i to ℓ_{i+1} has length l. We will refer to this path as the *l-path* of such a tree.

4.4 Functional Equations

In this section, we derive functional equations involving the generating functions that correspond to our tree classes. Our arguments will be purely combinatorial and will require us to consider a number of additional tree classes. These classes will be introduced when we need them, but a list of their enumerative sequences and corresponding ordinary generating functions appear in the table below.

For positive integers n and k, we define $\mathcal{T}_n^*$ as the collection of rooted ordered trees having n leaves and no non-root vertex with exactly one child, and $\mathcal{T}_{n,k}^*$ as the subset of these in which the leftmost path has length k.

Proposition 4.4.1. The following relation holds true:

$$T(x,y) = \frac{xy^2\, T^*(x)}{1 - yT^*(x)}$$

Table 4.1: A list of all generating functions and their associated sequences.

Generating Function	Associated Sequence
$T^*(x)$	$T^*_n, \quad n \geq 1$
$T^*(x,y)$	$T^*_{n,k}, \quad n,k \geq 1$
$T(x)$	$T_n, \quad n \geq 2$
$T(x,y)$	$T_{n,k}, \quad n,k \geq 2$
$T^0(x,y,z)$	$T^0_{n,k,l}, \quad n,k,l \geq 2$
$T^1(x,y,z)$	$T^1_{n,k,l}, \quad n,k,l \geq 2$
$T^2(x,y,z)$	$T^2_{n,k,l}, \quad n,k,l \geq 2$
$T^R(x,y,z)$	$T^R_{n,k,l}, \quad n,k,l \geq 2$
$T^L(x,y,z)$	$T^L_{n,k,l}, \quad n,k,l \geq 2$
$T^N(x,y,z)$	$T^N_{n,k,l}, \quad n,k,l \geq 3$
$\overline{T}(x,y,z)$	$T_{n,k,l}, \quad n,k,l \geq 2$

Proof. Fix $n,k \geq 2$ and $t \in \mathcal{T}_{n,k}$. We obtain $k-1$ disjoint trees by deleting from t the root, the first leaf, and the edges of the leftmost path. We then root each of these trees at the vertex it contains from the leftmost path of t and have them inherit the ordering in t. In addition, we order these trees into the tuple $(t_1, ..., t_{k-1})$ so that the root of t_i is a vertex that was distance i from the root of t. These trees, as well as their copies within t, will be referred to as the *spinal subtrees* of t. See Figure 4.3.

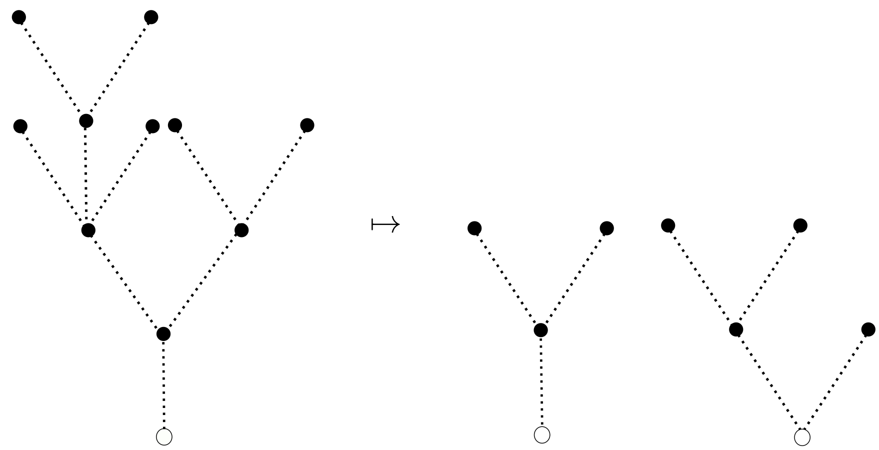

Figure 4.3: Decomposing a tree into its spinal subtrees.

Observe that each t_i lies in some $\mathcal{T}^*_{m_i}$ and these values must satisfy $\sum m_i = n-1$. On the other hand, all such $(k-1)$-tuples of trees are the spinal subtrees of a unique tree in $\mathcal{T}_{n,k}$. As a result, the decomposition of a tree into its spinal subtrees is a bijection. In particular, its domain and range have equal size:

$$\begin{aligned} T_{n,k} &= \sum_{m:\sum m_i = n-1} T^*_{m_1} \cdot \ldots \cdot T^*_{m_{k-1}}, \\ &= [x^{n-1}]\, T^*(x)^{k-1}. \end{aligned}$$

Notice that the above relation holds for $n, k \geq 2$. Multiplying by y^k and summing over k gives us that

$$\begin{aligned} \sum_{k\geq 2} T_{n,k} y^k = [x^n]T(x,y) &= [x^n]\, x \sum_{k\geq 2} T^*(x)^{k-1} y^k \\ &= [x^n]\, xy^2 T^*(x) \sum_{k\geq 2} T^*(x)^{k-2} y^{k-2} \\ &= [x^n]\, xy^2 T^*(x)(1 - yT^*(x))^{-1} \end{aligned}$$

for all $n \geq 2$. As a result, the corresponding generating functions are identical.

□

Proposition 4.4.2. The following relation holds true:

$$\begin{aligned} T^*(x,y) &= xy + T(x,y)(1+1/y) \\ &= xy\frac{1+T^*(x)}{1-yT^*(x)} \end{aligned}$$

Proof. Fix $n \geq 2$, $k \geq 1$, and $t \in \mathcal{T}^*_{n,k} \setminus \mathcal{T}_{n,k}$. Letting u denote the root of t, we construct a new tree $\hat{t}$ from t by inserting a new vertex v as the last child of u, and then designating v as the root of $\hat{t}$.

It follows from this construction that the tree $\hat{t}$ lies in $\mathcal{T}_{n,k+1}$ and the map $t \mapsto \hat{t}$ is a bijection onto this set. As a result, the identity

$$T^*_{n,k} = T_{n,k} + T_{n,k+1}$$

holds for $n \geq 2$ and $k \geq 1$. When $n = 1$ and $k \geq 1$, we have

$$T^*_{n,k} = \mathbb{1}(k=1)$$

since $\mathcal{T}^*_1$ contains only the tree whose leftmost path is exactly one edge. Setting $T_{1,k} = 0$ for all k, these relations can be written concisely as

$$T^*_{n,k} = \mathbb{1}(n=k=1) + T_{n,k} + T_{n,k+1}, \qquad n,k \geq 1.$$

The corresponding statement for generating functions,

$$T^*(x,y) = xy + T(x,y) + T(x,y)/y,$$

is the first of the given identities. We obtain the other by applying Proposition 4.4.1:

$$\begin{aligned} T^*(x,y) &= xy + \frac{xy^2\,T^*(x)}{1-yT^*(x)} + \frac{xy\,T^*(x)}{1-yT^*(x)}, \\ &= xy\frac{1+T^*(x)}{1-yT^*(x)}. \end{aligned}$$

□

For $i = 0, 1, 2$, let $\mathcal{T}^i_{n,k,l}$ be the subset of $\mathcal{T}_{n,k,l}$ in which the l-path and the leftmost path share exactly i vertices. Notice that these form a partition of $\mathcal{T}_{n,k,l}$. We further divide the trees in $\mathcal{T}^2_{n,k,l}$ into three classes: those where the root is a shared vertex, $\mathcal{T}^R_{n,k,l}$; those where the first leaf is a shared vertex, $\mathcal{T}^L_{n,k,l}$; and those where neither the root nor the first leaf is shared, $\mathcal{T}^N_{n,k,l}$.

Proposition 4.4.3. The following relation holds true:

$$\begin{aligned} T^L(x,y,z) &= yzT^*(x,z)(T(x,y)+xy) \\ &= \frac{1+T^*(x)}{1-zT^*(x)}\frac{x^2y^2z^2}{1-yT^*(x)} \end{aligned}$$

Proof. Fix a pair (t,ℓ_1) in $\mathcal{T}^L_{n,k,l}$ and let t_1 be the last spinal subtree of t. Construct a second tree, t_2, by deleting the copy of t_1 in t, and merging the last two edges of the leftmost path.

Observe that the leftmost path of t_1 is essentially the l-path of t with one edge removed. Therefore, t_1 lies in $\mathcal{T}^*_{m,l-1}$ for some m. Similarly, the leftmost path of t_2 is formed by merging two edges in the leftmost path of t, so t_2 lies in $\mathcal{T}_{n-m,k-1}$ when $k>2$ and in $\mathcal{T}^*_{1,1}$ when $k=2$. In either case, the map $(t,\ell_1)\mapsto(t_1,t_2)$ is a bijection between the relevant sets and we have the relations

$$T^L_{n,2,l} = T^*_{n-1,l-1}, \tag{4.1}$$

$$T^L_{n,k,l} = \sum_{m=1}^{n-2} T^*_{m,l-1}T_{n-m,k-1}, \tag{4.2}$$

which hold for $n,l\geq 2$ and $k\geq 3$. Introducing a factor of y^2z^l in (4.1) and summing over l, we have that

$$\begin{aligned} \sum_{l\geq 2} T^L_{n,2,l}y^2z^l &= y^2\sum_{l\geq 2} T^*_{n-1,l-1}z^l \\ &= [x^n]\,xy^2zT^*(x,z) \end{aligned}$$

for all n. Similarly, from (4.2) we obtain the identity

$$\begin{aligned}\sum_{\substack{l\geq 2\\k\geq 3}} T^L_{n,k,l} y^k z^l &= \sum_{m=1}^{n-2}\left(\sum_{l\geq 2} T^*_{m,l-1} z^l\right)\left(\sum_{k\geq 3} T_{n-m,k-1} y^k\right)\\ &= \sum_{m=1}^{n-2} [x^m]\, z\, T^*(x,z)[x^{n-m}]\, yT(x,y)\\ &= [x^n]\, yzT^*(x,z)T(x,y)\end{aligned}$$

for all n. We can combine these into the single relation

$$\sum_{\substack{l\geq 2\\k\geq 2}} T^L_{n,k,l} y^k z^l = [x^n] yzT^*(x,z)(xy + T(x,y))$$

which holds for all n. Writing this in terms of generating functions, we obtain the first result. Applying Propositions 4.4.1 and 4.4.2 yields the second result:

$$\begin{aligned}T^L(x,y,z) &= yz\,\frac{xz(1+T^*(x))}{1-zT^*(x)}\left(xy + \frac{xy^2\,T^*(x)}{1-yT^*(x)}\right),\\ &= \frac{xyz^2(1+T^*(x))}{1-zT^*(x)}\,\frac{xy}{1-yT^*(x)}.\end{aligned}$$

□

Proposition 4.4.4. The following relation holds true:

$$T^2(x,y,z) = 2T^L(x,y,z) + \frac{T^L(x,y,z)^2}{xyz}$$

Proof. Fix a pair (t,ℓ) in $\mathcal{T}^R_{n,k,l}$ and construct a new pair (t',ℓ') as follows. The tree t' is obtained from t by re-rooting at the first leaf and flipping the resulting tree from left to right. The distinguished leaf ℓ' is the first leaf of t'. See Figure 4.4.

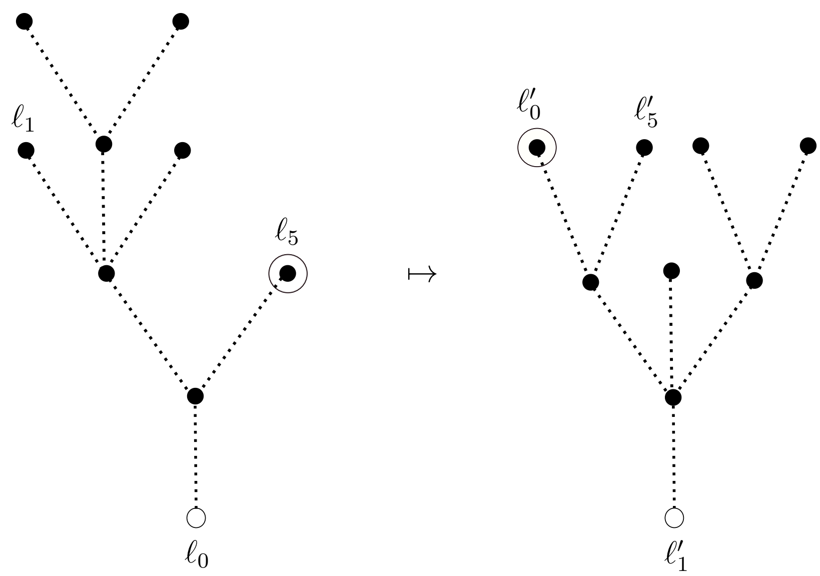

Figure 4.4: The map from (t, ℓ) to (t', ℓ'). The image of $\ell_i(t)$ is ℓ'_i.

Observe that the leftmost path and the l-path of (t', ℓ') correspond to the leftmost path and the l-path of (t, ℓ), respectively. As a result, we have that $(t', \ell') \in \mathcal{T}^L_{n,k,l}$. In fact, this set is the range of the bijection $(t, \ell) \mapsto (t', \ell')$, so we have the equality

$$T^R_{n,k,l} = T^L_{n,k,l}, \qquad n, k, l \geq 2. \tag{4.3}$$

Let $n, k, l \geq 3$ and fix a pair (t, ℓ) in $\mathcal{T}^N_{n,k,l}$. Let u and v be the vertices shared by the leftmost path and the l-path, with u being closest to the root. Next, we delete the edge (u, v) so that two new trees form. To the tree containing v, which we will denote by t_1, we add a vertex preceding v and root the tree there. To the other tree, which we will denote by t_2, we add a vertex descending from u so that it becomes the first leaf in this tree. We distinguish the last leaf in t_1 and the first leaf in t_2 to obtain pairs (t_1, ℓ_m) and (t_2, ℓ_1), where m is the number of leaves in t_1.

The leftmost paths of t_1 and t_2 are formed from the leftmost path of t via the removal of an edge and the addition of two edges. Similarly, the l-paths of (t_1, ℓ_m) and (t_2, ℓ_1) are formed from the l-path of (t, ℓ) via the removal of an edge and the addition of two edges. Consequently, (t_1, ℓ_m) lies in some $\mathcal{T}^R_{m,j,h}$, (t_2, ℓ_1) lies in some $\mathcal{T}^L_{m',j',h'}$, and these parameters must satisfy $m + m' = n + 1, j + j' = k + 1$, and $h + h' = l + 1$. In fact, the map $(t, \ell) \mapsto ((t_1, \ell_m), (t_2, \ell_1))$ is a bijection whose range consists of all such pairs. This yields the identity

$$T^N_{n,k,l} = \sum_{m=2}^{n-1} \sum_{j=2}^{k-1} \sum_{h=2}^{l-1} T^R_{m,j,h} T^L_{n+1-m,k+1-j,l+1-h}$$

for $n, k, l \geq 3$. However, it is easy to verify that equality holds when any of n, k, or l is equal to 2 (both sides are zero). Combining this with (4.3), the above becomes

$$\begin{aligned} T^N_{n,k,l} &= \sum_{m=2}^{n+1} \sum_{j=2}^{k+1} \sum_{h=2}^{l+1} T^L_{m,j,h} T^L_{n+1-m,k+1-j,l+1-h} \\ &= [x^n y^k z^l]\,(xyz)^{-1} T^L(x, y, z)^2. \end{aligned} \tag{4.4}$$

Adding $T^L_{n,k,l} + T^R_{n,k,l}$ and using (4.3) once again, we obtain the final relation

$$T^2_{n,k,l} = [x^n y^k z^l]\, 2T^L(x, y, z) + [x^n y^k z^l]\,(xyz)^{-1} T^L(x, y, z)^2,$$

which holds for all parameter values. This proves the result. □

Proposition 4.4.5. The following relation holds true:

$$T^1(x, y, z) = \frac{T^L(x, y, z)^2}{xy^2z^2}.$$

Proof. Fix a pair (t, ℓ_i) in $\mathcal{T}^1_{n,k,l}$ and let v denote the vertex shared between the leftmost and l-paths. We view the spinal subtree in t attached to v as being made up of two components: the path from v to ℓ_i and the structure to the left of it will be the left component, and the path from v to ℓ_{i+1} and the structure to the right of it will be the right component. Construct the tree t' from t by splitting v into two vertices, v_1, v_2, so that both lie on the leftmost path, are adjacent, and the spinal subtrees in t' attached to v_1, v_2 are the left and right components respectively. Finally, distinguish the i^{th} leaf in t' to obtain the pair (t', ℓ_i).

Since the pair (t', ℓ_i) has an $(l+1)$-path that shares two vertices with the leftmost path (v_1 and v_2), neither of which can be the root or the first leaf, (t', ℓ_i) lies in $\mathcal{T}^N_{n,k+1,l+1}$. In fact, this set is the range of the bijective map $(t, \ell_i) \mapsto (t', \ell_i)$, so the relation

$$T^1_{n,k,l} = T^N_{n,k+1,l+1}$$

holds for all values of n, k, and l. Using (4.4), we can rewrite this as

$$\begin{aligned} T^1_{n,k,l} &= [x^n y^{k+1} z^{l+1}]\,(xyz)^{-1}\,T^L(x,y,z)^2 \\ &= [x^n y^k z^l]\,x^{-1}(yz)^{-2}\,T^L(x,y,z)^2, \end{aligned}$$

from which we have the result. □

Proposition 4.4.6. The following relation holds true:

$$T^0(x,y,z) = T^1(x,y,z)\partial_x T^*(x).$$

Proof. Given a pair (t,ℓ) in $\mathcal{T}^0_{n,k,l}$, let $\bar{t}$ be the spinal subtree in t containing the l-path. Denote by u the vertex on the l-path that is closest to the 'root' of $\bar{t}$. Form the tree t_1 by taking a copy of t and replacing its copy of $\bar{t}$ with the subtree formed by u and its descendants. Form the tree t_2 by taking a copy of $\bar{t}$ and turning its copy of u, which we denote by u^*, into a leaf by deleting its descendancy. Distinguish the leaf in t_1 that is a copy of ℓ to obtain the pair (t_1,ℓ') and distinguish the newly created leaf u^* in t_2 to obtain the pair (t_2,u^*). See Figure 4.5.

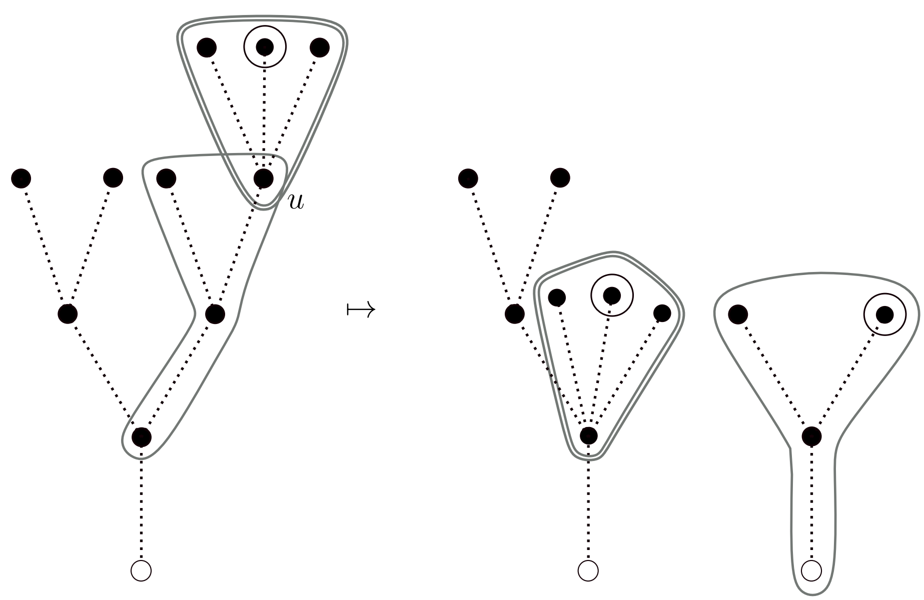

Figure 4.5: The decomposition of (t,ℓ) into $((t_1,\ell'),(t_2,u^*))$. Root vertices are white, distinguished leaves are circled, and circled subtrees on the left map to the corresponding circled subtree on the right.

The pair (t_1, ℓ') must lie in some $\mathcal{T}^1_{m_1,k,l}$ since its leftmost path and l-path are copies of the ones in (t, ℓ) except that the l-path of t_1 now shares one vertex with the leftmost path (some copy of u). The tree t_2 must lie in some $\mathcal{T}^*_{m_2}$ since it is derived from the spinal subtree t_2. This time, the leaf parameters satisfy $m_1 + m_2 = n + 1$ since all of the leaves in t_1 and t_2 are derived from leaves in t with the exception of the newly created leaf u. The bijection $(t, \ell) \mapsto ((t_1, \ell'), (t_2, u))$ yields the relation

$$T^0_{n,k,l} = \sum_{m=1}^{n} T^1_{n+1-m,k,l}\, m\, T^*_m, \qquad n, k, l \geq 2.$$

The entire parameter range is included here because both sides are zero when the bijection is not well-defined. Introducing a factor of $y^k z^l$ and summing over k and l, we obtain

$$\begin{aligned}
\sum_{k\geq 2}\sum_{l\geq 2} T^0_{n,k,l} y^k z^l &= \sum_{m=1}^{n} \left(\sum_{k,l\geq 2} T^1_{n+1-m,k,l} y^k z^l \right) m\, T^*_m \\
&= \sum_{m=1}^{n} [x^{n+1-m}] T^1(x, y, z)\, [x^m] x\partial_x T^*(x) \\
&= [x^{n+1}] T^1(x, y, z) x\partial_x T^*(x) \\
&= [x^n] T^1(x, y, z) \partial_x T^*(x)
\end{aligned}$$

for all n, which corresponds to the given generating function relation.

□

4.5 Asymptotic Expansions

In this section, we obtain the estimates on our generating functions' coefficients that lead to the estimates on the moments of the ζ_k in Lemma 4.2.1. In spirit, our approach is based on the classical tools in [15] but requires their multivariate versions developed in [20]. The general idea is to estimate a generating function's coefficients by identifying it as an analytic function on some subset of the complex plane, expanding it around its singularity of smallest modulus, and applying some analytic combinatorics machinery.

To begin, we recall some definitions from [20].

Definition 4.5.1. For every $\varepsilon > 0$ and $\phi \in (0, \frac{\pi}{2})$, we associate the δ-*domain*

$$\delta(\varepsilon, \phi) = \{x \in \mathbb{C} : |x| \leq 1 + \varepsilon, x \neq 1, |\mathrm{Arg}(x-1)| \geq \phi\}.$$

For every $\varepsilon > 0$, $\phi \in (0, \frac{\pi}{2})$, and positive integer m, we associate the $\mathcal{R}$-*domain*

$$\mathcal{R}^m(\varepsilon, \phi) = \{(x, y_1, ..., y_m) \in \mathbb{C}^{m+1} : x \in \delta(\varepsilon, \phi), |y_j| < 1 \text{ for all } j\}.$$

Since, the parameter m can be deduced from context, we will omit it in $\mathcal{R}^m(\varepsilon, \phi)$ and use $\mathbf{y}$ as a shorthand for a list of variables $y_1, ..., y_m$.

Definition 4.5.2. We write

$$f(x, \mathbf{y}) = \widetilde{O}\Big((1-x)^{-\alpha} \prod_{j=1}^{m} (1-y_j)^{-\beta_j}\Big)$$

if there exist some $\mathcal{R}(\varepsilon, \phi)$, a real number α', $\beta' \geq 0$, and $\gamma \geq 0$ such that

(i) f is analytic on $\mathcal{R}(\varepsilon, \phi)$,

(ii) $f(x, \mathbf{y}) = O\Big((1-x)^{-\alpha'} \prod_{j=1}^{m} (1-|y_j|)^{-\beta'}\Big)$ on $\mathcal{R}(\varepsilon, \phi)$, and

(iii) $f(x, \mathbf{y}) = O\Big((1-x)^{-\alpha} \prod_{j=1}^{m} (1-|y_j|)^{-\beta_j}\Big)$ as $(1-x)(1-y_i)^{-\gamma} \to 0$ for all i and $(x, \mathbf{y}) \in \mathcal{R}(\varepsilon, \phi)$.

Definition 4.5.3. We write

$$f(x, \mathbf{y}) \approx c(1-x)^{-\alpha} \prod_{j=1}^{m} (1-y_j)^{-\beta_j}$$

if $c \neq 0$ and there exist $\alpha' < \alpha$, non-negative numbers $\beta'_1, ..., \beta'_m$, and functions $C(\mathbf{y})$, $C_0(x, \mathbf{y})$, ..., $C_m(x, \mathbf{y})$, and $E(x, \mathbf{y})$ such that

(i) $f(x, \mathbf{y}) = C(\mathbf{y})(1-x)^{-\alpha} \prod_{j=1}^{m} (1-y_j)^{-\beta_j} + \sum_{j=0}^{m} C_j(x, \mathbf{y}) + E(x, \mathbf{y})$,

(ii) C is analytic and $C(\mathbf{y}) = c + O\left(\sum_{j=1}^{m} |1-y_j|\right)$ on some product of δ-domains,

(iii) each C_j is a polynomial in some variable, and

(iv) $E(x, \mathbf{y}) = \widetilde{O}\Big((1-x)^{-\alpha'} \prod_{j=1}^{m} (1-y_j)^{-\beta'_j}\Big)$.

Our analysis begins with the function $T^*(x)$. For convenience, we work with the rescaled version

$$\tau(x) = 2rs\,T^*(x/r), \tag{4.5}$$

where $r = 3 + 2\sqrt{2}$ and $s = (\sqrt{2} - 1)/2$. It will be useful to note the identities

$$s(r-1) = 1 = 2s\sqrt{r}. \tag{4.6}$$

The following result summarizes the properties of $\tau(x)$.

Proposition 4.5.1. The following statements hold:

(i) $\tau(x)$ is a generating function with non-negative coefficients, τ_n, and there exists a nonempty set of indices, J, so that $\tau_j > 0$ for $j \in J$ and $\gcd\{i - j : i, j \in J\} = 1$,

(ii) $\tau(x)$ can be regarded as a function on $\mathbb{C}$ that is analytic on every δ-domain,

(iii) $\tau(x)$ has the form

$$\tau(x) = 1 - qs(1-x)^{1/2} + s(1-x) - s(1-x)^{1/2}\sigma(x),$$

where $q = (r^2 - 1)^{1/2}$, $\sigma(x) = (r^2 - x)^{1/2} - q$, and $\sigma(x)(1-x)^{-1}$ is bounded on every δ-domain,

(iv) $|\tau(x)| \le 1$ on some $\delta(\varepsilon_0, \phi_0)$, and

(v) $\tau'(x)$ has the form

$$\tau'(x) = \frac{qs}{2}(1-x)^{-1/2} + u(x),$$

where $u(x)$ is analytic and bounded on every δ-domain.

Proof. To see that (i) holds, observe that $\tau_0 = 0$ and all other coefficients are positive. Thus, J can be taken to be $\mathbb{N}$.

Setting $y = 1$ in the identity in Proposition 4.4.2, we find that $T^*(x)$ must satisfy

$$T^*(x) = x\,\frac{1 + T^*(x)}{1 - T^*(x)}.$$

Together with the condition $T_0^* = 0$, this equation leads to the explicit form

$$\begin{aligned} T^*(x) &= \frac{1 - x - \sqrt{x^2 - 6x + 1}}{2} \\ &= \frac{1 - x - \sqrt{(1/r - x)(r - x)}}{2}. \end{aligned} \tag{4.7}$$

This representation allows us to regard $T^*(x)$ as a genuine function on $\mathbb{C}$ that is analytic on $\mathbb{C} \setminus [\frac{1}{r}, r]$. Consequently, $\tau(x)$ is analytic on $\mathbb{C} \setminus [1, r^2]$, and (ii) holds.

The expansion in (iii) follows from (4.7):

$$\begin{aligned} \tau(x) &= rs(1 - x/r - (1/r - x/r)^{1/2}(r - x/r)^{1/2}) \\ &= s(r - 1 + 1 - x - (1 - x)^{1/2}(q + \sigma(x)), \\ &= 1 + s(1 - x) - qs(1 - x)^{1/2} - s(1 - x)^{1/2}\sigma(x). \end{aligned}$$

To obtain the bound on $(1 - x)^{-1}\sigma(x)$, we first observe that $\sigma(1) = 0$ and $\sigma(x)$ is differentiable at $x = 1$. As a result, the quantity $(1 - x)^{-1}\sigma(x)$ is a difference quotient, whereby it must be bounded near $x = 1$. Away from $x = 1$, the quantity $(1 - x)^{-1}$ is clearly bounded, and within any δ-domain, $\sigma(x)$ is bounded.

The bound in (iv) follows directly from Lemma 4 in [20]. The boundedness claimed in (v) follows from the explicit form

$$\begin{aligned} u(x) &= \tau'(x) - \frac{qs}{2}(1 - x)^{-1/2} \\ &= -s - s(1 - x)^{1/2}\sigma'(x) + \frac{s}{2}(1 - x)^{-1/2}\sigma(x) \\ &= -s + \frac{s}{2}(1 - x)^{1/2}(r^2 - x)^{-1/2} + \frac{s}{2}(1 - x)^{-1/2}\sigma(x) \end{aligned}$$

and the boundedness established in (iii). The analyticity follows from the analyticity established in (ii), which $\tau'(x)$ inherits from $\tau(x)$.

□

The singular expansion in Proposition 4.5.1(iii) gives us our first coefficient estimate.

Proposition 4.5.2. The coefficients of $T(x)$ are given by

$$T_n = \frac{qs^2}{2\sqrt{\pi}} r^n n^{-3/2}(1 + O(1/n))$$

as $n \to \infty$.

Proof. Using Proposition 4.4.2 and (4.5), we find that $T(x)$ and $\tau(x)$ are related by the following equation:

$$T(x/r) = s\tau(x) - 2s^2 x.$$

Together with Proposition 4.5.1, this tells us that $T(x/r)$ can be regarded as an analytic function and has the expansion

$$T(x/r) = s - qs^2(1-x)^{1/2} + s^2(1-x) + O(1-x)^{3/2} - 2s^2 x$$

on some δ-domain. It is known (see Theorems VI.1-3 in [15]) that a function of this form has coefficients given by

$$\begin{aligned} r^{-n}T_n &= [x^n]T(x/r) \\ &= -qs^2[x^n](1-x)^{1/2} + O([x^n](1-x)^{3/2}) \\ &= \frac{qs^2}{2\sqrt{\pi}} n^{-3/2}(1 + O(1/n)) + O(n^{-5/2}) \\ &= \frac{qs^2}{2\sqrt{\pi}} n^{-3/2}(1 + O(1/n)) \end{aligned}$$

as $n \to \infty$. □

The second consequence of Proposition 4.5.1 is as follows.

Proposition 4.5.3. Under some rescaling, each generating function in Section 4.4 can be regarded as an analytic function on some $\mathcal{R}$-domain. In particular,

(i) $T(x/r, y/2s)$ and $T^*(x/r, y/2s)$ are analytic functions on $\mathcal{R}^1(\varepsilon_0, \phi_0)$, and

(ii) $T^L(x/r, y/2s, z/2s)$, $T^2(x/r, y/2s, z/2s)$, $T^1(x/r, y/2s, z/2s)$, and $T^0(x/r, y/2s, z/2s)$ are analytic functions on $\mathcal{R}^2(\varepsilon_0, \phi_0)$.

Proof. We first show that the generating function

$$f(x,y) = \left(1 - \frac{y}{2s}T^*(x/r)\right)^{-1} = \sum_{k\geq 0}\left(\frac{y}{2s}T^*(x/r)\right)^k$$

can be identified as an analytic function on $\mathcal{R}^1(\varepsilon_0, \phi_0)$. Since $T^*(x/r)$ has been identified as an analytic function on every δ-domain (see Proposition 4.5.1(ii)), it suffices to establish the bound

$$\left|\frac{y}{2s}T^*(x/r)\right| < 1$$

on $\mathcal{R}^1(\varepsilon_0, \phi_0)$. This bound follows from the second identity in (4.6) and the bound in Proposition 4.5.1(iv).

Given the analyticity of $T^*(x/r)$ and $f(x,y)$, Propositions 4.4.1, 4.4.2, and 4.4.3 establish (i) and the analyticity of $T^L(x/r, y/2s, z/2s)$. Applying Propositions 4.4.3, 4.4.4, 4.4.5, and 4.4.6 concludes the proof. □

To obtain singular expansions for $T(x/r, y/2s)$ and $\overline{T}(x/r, y/2s, z/2s)$, it will be worthwhile to first analyze the functions

$$A_k(x,y) = \frac{x^k}{1 - y\tau(x)} \qquad \text{and} \qquad B_k(x,y) = A_k(x,y)\,\tau(x),$$

defined for each non-negative integer k on $\mathcal{R}(\varepsilon_0, \phi_0)$. The basic properties of these functions are summarized in the following proposition.

Proposition 4.5.4. On $\mathcal{R}(\varepsilon_0, \phi_0)$, each A_k and B_k

(i) is analytic,

(ii) is bounded by a multiple of $D(y) = (1 - |y|)^{-1}$, and

(iii) has a singular expansion given by

$$A_k(x,y) = \frac{1}{1-y} - \frac{qsy(1-x)^{1/2}}{(1-y)^2} + \widetilde{O}((1-x)(1-y)^{-3}), \qquad (4.8)$$

or

$$B_k(x,y) = \frac{1}{1-y} - \frac{qs(1-x)^{1/2}}{(1-y)^2} + \widetilde{O}((1-x)(1-y)^{-3}). \qquad (4.9)$$

Proof. The analyticity in (i) can be established as in Proposition 4.5.3. The bound in (ii) follows from Proposition 4.5.1(iv). To verify the expansion for $A_0(x,y)$, we define

$$E(x,y) = A_0(x,y) - \frac{1}{1-y} + \frac{qsy(1-x)^{1/2}}{(1-y)^2}.$$

It follows from (i) that $E(x,y)$ is analytic on $\mathcal{R}(\varepsilon_0,\phi_0)$. To obtain a bound on $E(x,y)$, we write

$$\begin{aligned} E(x,y)(1-y)(1-y\tau(x)) &= (1-y)-(1-y\tau(x)) + qsy(1-x)^{1/2}\frac{1-y\tau(x)}{1-y} \\ &= y(\tau(x)-1) + qsy(1-x)^{1/2}\left(1+y\,\frac{1-\tau(x)}{1-y}\right) \\ &= sy(1-x) - sy(1-x)^{1/2}\sigma(x) \\ &\quad + qsy^2(1-x)^{1/2}\,\frac{1-\tau(x)}{1-y}, \end{aligned}$$

and observe that both $\sigma(x)$ and $1-\tau(x)$ can be bounded by a constant times $|1-x|^{1/2}$ on $\delta(\varepsilon_0,\phi_0)$. From this, it follows that

$$|E(x,y)| \le M\frac{|1-x|}{(1-|y|)^3}$$

on $\mathcal{R}(\varepsilon_0,\phi_0)$ for some $M>0$, establishing the expansion for $A_0(x,y)$.

To verify that $A_k(x,y)$ and $A_0(x,y)$ have the same expansion, we simply note that their difference is negligible:

$$\begin{aligned} A_k(x,y) - A_0(x,y) &= \frac{x^k-1}{1-y\tau(x)} \\ &= \widetilde{O}(1-x)(1-y)^{-1}. \end{aligned}$$

Similarly, the relationship given by

$$\begin{aligned} A_k(x,y) - B_k(x,y) &= A_k(x,y)(1-\tau(x)) \\ &= \left((1-y)^{-1} + \widetilde{O}(1-x)^{1/2}(1-y)^{-3}\right) \\ &\quad \times \left(qs(1-x)^{1/2} + O(1-x)\right) \\ &= qs(1-x)^{1/2}(1-y)^{-1} + \widetilde{O}(1-x)(1-y)^{-3} \end{aligned}$$

verifies that the expansions for the $B_k(x,y)$ hold.

□

Our next singular expansion follows immediately from the previous result.

Proposition 4.5.5. The following holds true:

(i) $T(x/r, y/2s) \approx -2qs^2(1-x)^{1/2}(1-y)^{-2}$,

(ii) $T_{n,k} = qs^2\pi^{-1/2}kn^{-3/2}r^{n-k/2}(1+O(1/k))$ as $k, n \to \infty$ and $k = O(\log n)$,

(iii) $T_{n,k} = O(n^{-3/2}p^{-k}r^n)$ as $n \to \infty$ for $p \in (0, r^{1/2})$ and all k.

Proof. Combining Propositions 4.4.1 and 4.5.4, we obtain the expansion

$$\begin{aligned} T(x/r, y/2s) &= \frac{2sxy^2\tau(x)}{1-y\tau(x)} \\ &= 2sy^2B_1(x,y) \\ &= \frac{2sy^2}{1-y} - \frac{2qs^2y^2(1-x)^{1/2}}{(1-y)^2} + \widetilde{O}(1-x)(1-y)^{-3}. \end{aligned}$$

Letting $C_0(x,y) = 2sy^2(1-y)^{-1}$ and $C(y) = -2qs^2y^2$, we have that $C_0(x,y)$ is a polynomial in x, $C(1) = -2qs^2 \neq 0$, C is analytic on every δ-domain, and $C(y) = C(1) + O(1-y)$ on every δ-domain. This establishes (i).

The estimates in (ii) and (iii) then follow from applying Lemma 3 in [20]:

$$\begin{aligned} T_{n,k} &= r^n(2s)^k\,[x^ny^k]T(x/r, y/2s) \\ &= r^n(2s)^k\left(-2qs^2\frac{n^{-3/2}}{\Gamma(-\frac{1}{2})}\frac{k}{\Gamma(2)}\Big(1+O(1/k)\Big)\right) \\ &= r^{n-k/2}\left(\frac{qs^2}{\sqrt{\pi}}n^{-3/2}k\Big(1+O(1/k)\Big)\right) \end{aligned}$$

as $k, n \to \infty$ and $k = O(\log n)$, and

$$\begin{aligned} T_{n,k} &= r^n(2s)^k\,[x^ny^k]T(x/r, y/2s) \\ &= r^nr^{-k/2}O(n^{-3/2}(1-\varepsilon')^{-k}) \end{aligned}$$

for all k, n and for $\varepsilon' \in (0,1)$. This concludes the proof. □

Our final expansion is obtained in the next two results.

Proposition 4.5.6. The following holds true:

(i) $T^2(x/r, y/2s, z/2s) = \widetilde{O}\left((1-y)^{-6}(1-z)^{-6}\right)$,

(ii) $T^1(x/r, y/2s, z/2s) = \widetilde{O}\left((1-y)^{-6}(1-z)^{-6}\right)$, and

(iii) $T^0(x/r, y/2s, z/2s) \approx 2qs^2(1-x)^{-1/2}(1-y)^{-2}(1-z)^{-2}$.

Proof. Truncating the expansions in Proposition 4.5.4, we obtain the expansion

$$\begin{aligned}
&A_0(x,y)(A_1(x,z) + 2sB_1(x,z)) \\
&\quad = \left(\frac{1}{1-y} + \widetilde{O}\left((1-x)^{1/2}(1-y)^{-3}\right)\right)\left(\frac{1+2s}{1-z} + \widetilde{O}\left((1-x)^{1/2}(1-z)^{-3}\right)\right) \\
&\quad = \frac{1+2s}{(1-y)(1-z)} + \widetilde{O}\left((1-x)^{1/2}(1-y)^{-3}(1-z)^{-3}\right).
\end{aligned}$$

Combining this with the identity $1+2s = q^2s^2$ and Proposition 4.4.3 results in the expansion

$$\begin{aligned}
&T^L(x/r, y/2s, z/2s) \\
&\quad = xy^2z^2(1+2s\tau(x))\frac{1}{1-y\tau(x)}\frac{x}{1-z\tau(x)} \\
&\quad = xy^2z^2(1+2s\tau(x))A_0(x,y)A_1(x,z) \\
&\quad = xy^2z^2A_0(x,y)(A_1(x,z)+2sB_1(x,z)) \\
&\quad = xy^2z^2\left(\frac{q^2s^2}{(1-y)(1-z)} + \widetilde{O}\left((1-x)^{1/2}(1-y)^{-3}(1-z)^{-3}\right)\right).
\end{aligned}$$

Substituting this into the identity of Proposition 4.4.5, we obtain (ii):

$$\begin{aligned}
&T^1(x/r, y/2s, z/2s) \\
&\quad = 4s^2x^{-1}y^{-2}z^{-2}T^L(x/r, y/2s, z/2s)^2 \\
&\quad = 4s^2xy^2z^2\left(\frac{q^2s^2}{(1-y)(1-z)} + \widetilde{O}\left((1-x)^{1/2}(1-y)^{-3}(1-z)^{-3}\right)\right)^2 \\
&\quad = 4s^2xy^2z^2\left(\frac{q^4s^4}{(1-y)^2(1-z)^2} + \widetilde{O}\left((1-x)^{1/2}(1-y)^{-6}(1-z)^{-6}\right)\right) \qquad (4.10) \\
&\quad = (1-(1-x))\left(\frac{4q^4s^6y^2z^2}{(1-y)^2(1-z)^2} + \widetilde{O}\left((1-x)^{1/2}(1-y)^{-6}(1-z)^{-6}\right)\right) \\
&\quad = \frac{4q^4s^6y^2z^2}{(1-y)^2(1-z)^2} + \widetilde{O}\left((1-x)^{1/2}(1-y)^{-6}(1-z)^{-6}\right).
\end{aligned}$$

Using the above expansions in the identity of Proposition 4.4.4 establishes (i):

$$\begin{aligned} T^2(x/r, y/2s, z/2s) &= 2T^L(x/r, y/2s, z/2s) + ryzT^1(x/r, y/2s, z/2s), \\ &= \widetilde{O}\left((1-y)^{-3}(1-z)^{-3}\right) + \widetilde{O}\left((1-y)^{-6}(1-z)^{-6}\right) \\ &= \widetilde{O}\left((1-y)^{-6}(1-z)^{-6}\right). \end{aligned}$$

Substituting (4.10) and a variation of the expansion in Proposition 4.5.1(v),

$$\begin{aligned} (\partial_x T^*)(x/r) &= (2s)^{-1}\,\tau'(x) \\ &= (2s)^{-1}\,\left(qs\,2^{-1}(1-x)^{-1/2} + u(x)\right), \\ &= q\,4^{-1}(1-x)^{-1/2} + (2s)^{-1}u(x), \end{aligned}$$

into Proposition 4.4.6, we obtain our final expansion

$$\begin{aligned} T^0(x/r, y/2s, z/2s) &= T^1(x/r, y/2s, z/2s)(\partial_x T^*)(x/r) \\ &= \left(\frac{4q^4s^6y^2z^2}{(1-y)^2(1-z)^2} + \widetilde{O}\left((1-x)^{1/2}(1-y)^{-6}(1-z)^{-6}\right)\right) \\ &\quad \times \left(q4^{-1}(1-x)^{-1/2} + (2s)^{-1}u(x)\right) \\ &= q^5s^6y^2z^2\frac{(1-x)^{-1/2}}{(1-y)^2(1-z)^2} + \widetilde{O}\left((1-y)^{-6}(1-z)^{-6}\right). \end{aligned}$$

To obtain (iii), we write

$$\begin{aligned} C(y,z) &\coloneqq q^5s^6y^2z^2 \\ &= q^5s^6 - q^5s^6(1-y^2) - q^5s^6y^2(1-z^2), \end{aligned}$$

and observe that $C(1,1) \neq 0$, $C(y,z) = C(1,1) + O(|1-y| + |1-z|)$, $C(y,z)$ is analytic in any product of δ-domains, and $q^4s^4 = 2$.

□

Proposition 4.5.7. The following holds true:

(i) $\overline{T}(x/r, y/2s, z/2s) \approx 2qs^2\,(1-x)^{-1/2}(1-y)^{-2}(1-z)^{-2}$, and

(ii) $T_{n,k,k} = 2qs^2\pi^{-1/2}k^2n^{-1/2}r^{n-k}(1 + O(1/k))$ as $k, n \to \infty$ and $k = O(\log n)$.

Proof. Combining Proposition 4.5.6 with the relationship $\overline{T}(x,y,z) = T^0(x,y,z) + T^1(x,y,z) + T^2(x,y,z)$ establishes (i). Applying Lemma 3 in [20] then establishes (ii):

$$\begin{aligned} T_{n,k,k} &= r^n (2s)^{2k} [x^n y^k z^k] T(x/r, y/2s, z/2s) \\ &= r^n (2s)^{2k}\, 2qs^2 \frac{n^{-1/2}}{\Gamma(\frac{1}{2})} \frac{k^2}{\Gamma(2)^2} \,(1 + O(1/k)) \\ &= r^{n-k}\, 2qs^2 \pi^{-1/2} n^{-1/2} k^2 \,(1 + O(1/k)) \end{aligned}$$

as $k, n \to \infty$ and $k = O(\log n)$.

□

Having these coefficient estimates at our disposal, we can now prove Lemma 4.2.1.

Proof of Lemma 4.2.1. Combining Lemma 4.3.1(i), the bijection between our dissection and tree classes, and Propositions 4.5.2 and 4.5.5(iii), we have that

$$\begin{aligned} \mathbb{E}(\zeta_k) &= (n+1) \frac{T_{n,k}}{T_n} \\ &= (n+1) \frac{2\sqrt{\pi}\, O(n^{-3/2} p^{-k} r^n)}{qs^2 n^{-3/2} r^n (1 + O(1/n))} \\ &= O(np^{-k}) \end{aligned}$$

as $n \to \infty$ for $p \in (0, r^{1/2})$ and all k. Observing that $b = r^{1/2}$ establishes (i).

Using the estimate from Proposition 4.5.5(ii) instead, the above computation becomes

$$\begin{aligned} \mathbb{E}(\zeta_k) &= (n+1) \frac{T_{n,k}}{T_n} \\ &= (n+1) \frac{qs^2 \pi^{-1/2}\, k n^{-3/2} r^{n-k/2} (1 + O(1/k))}{2^{-1} \pi^{-1/2} qs^2\, n^{-3/2} r^n (1 + O(1/n))} \\ &= (n+1) \frac{2kr^{-k/2}(1 + O(1/k))}{(1 + O(1/n))} \end{aligned}$$

as $n, k \to \infty$ and $k = O(\log n)$. Observing that

$$\begin{aligned}(1+O(1/k))\frac{n+1}{1+O(1/n)} &= n(1+O(1/k))\frac{n+1}{n+O(1)} \\ &= n(1+O(1/k))\left(1+\frac{1-O(1)}{n+O(1)}\right) \\ &= n(1+O(1/k))(1+O(1/n)) \\ &= n(1+O(1/k))\end{aligned}$$

as $n, k \to \infty$ and $k = O(\log n)$ then establishes (ii).

The computation for the higher moment is nearly identical. From Lemma 4.3.1(i), the dissection-tree bijection, and Propositions 4.5.2 and 4.5.7(ii), we obtain the estimate

$$\begin{aligned}\mathbb{E}(\zeta_k(\zeta_k-1)) &= (n+1)\frac{T_{n,k,k}}{T_n} \\ &= (n+1)\frac{2qs^2\pi^{-1/2}\,k^2n^{-1/2}r^{n-k}(1+O(1/k))}{2^{-1}\pi^{-1/2}qs^2\,n^{-3/2}r^n(1+O(1/n))} \\ &= (n+1)\frac{4k^2nr^{-k}(1+O(1/k))}{(1+O(1/n))} \\ &= 4k^2n^2r^{-k}(1+O(1/k))\end{aligned}$$

as $n, k \to \infty$ and $k = O(\log n)$. Notice that the final equality above follows from our earlier observation. The computation

$$\begin{aligned}\frac{\mathbb{E}(\zeta_k(\zeta_k-1))}{\mathbb{E}(\zeta_k)^2} &= \frac{4k^2n^2r^{-k}(1+O(1/k))}{4k^2n^2r^{-k}(1+O(1/k))^2} \\ &= \frac{1+O(1/k)}{1+O(1/k)} \\ &= 1+\frac{O(1/k)}{1+O(1/k)} \\ &= 1+O(1/k)\end{aligned}$$

as $n, k \to \infty$ and $k = O(\log n)$ establishes (iii) and concludes the proof.

□

BIBLIOGRAPHY

[1] David Aldous and James Allen Fill. Reversible markov chains and random walks on graphs, 2002. Unfinished monograph, recompiled 2014, available at http://www.stat.berkeley.edu/$\sim$aldous/RWG/book.html.

[2] Jan Ambjø rn. Quantum gravity represented as dynamical triangulations. *Classical Quantum Gravity*, 12(9):2079–2134, 1995.

[3] Omer Angel and Oded Schramm. Uniform infinite planar triangulations. *Comm. Math. Phys.*, 241(2-3):191–213, 2003.

[4] Nicla Bernasconi, Konstantinos Panagiotou, and Angelika Steger. On properties of random dissections and triangulations. *Combinatorica*, 30(6):627–654, 2010.

[5] P. Billingsley. On the distribution of large prime divisors. *Period. Math. Hungar.*, 2:283–289, 1972.

[6] David M. Blei, Michael I. Jordan, Thomas L. Griffiths, and Joshua B. Tenenbaum. Hierarchical topic models and the nested chinese restaurant process. In *Proceedings of the 16th International Conference on Neural Information Processing Systems*, NIPS'03, pages 17–24, Cambridge, MA, USA, 2003. MIT Press.

[7] David M. Blei, Andrew Y. Ng, and Michael I. Jordan. Latent dirichlet allocation. *J. Mach. Learn. Res.*, 3(1):993–1022, 2003.

[8] Alexei Borodin and Grigori Olshanski. Infinite-dimensional diffusions as limits of random walks on partitions. *Probab. Theory Related Fields*, 144(1-2):281–318, 2009.

[9] Nicolas Curien and Igor Kortchemski. Random non-crossing plane configurations: A conditioned galton-watson tree approach. *Random Structures Algorithms*, 45(2):236–260, 2014.

[10] L. Devroye, P. Flajolet, F. Hurtado, M. Noy, and W. Steiger. Properties of random triangulations and trees. *Discrete & Computational Geometry*, 22(1):105–117, 1999.

[11] Peter Donnelly and Paul Joyce. Consistent ordered sampling distributions: Characterization and convergence. *Advances in Applied Probability*, 23(2):229–258, 1991.

[12] S. N. Ethier and Thomas G. Kurtz. The infinitely-many-neutral-alleles diffusion model. *Adv. in Appl. Probab.*, 13(3):429–452, 1981.

[13] Stewart N. Ethier and Thomas G. Kurtz. *Markov processes: characterization and convergence*. Wiley series in probability and mathematical statistics. J. Wiley & Sons, New York, Chichester, 2005.

[14] W. J. Ewens. The sampling theory of selectively neutral alleles. *Theoret. Population Biol.*, 3, 1972.

[15] Philippe Flajolet and Robert Sedgewick. *Analytic Combinatorics*. Cambridge University Press, 2009.

[16] Noah Forman, Soumik Pal, Douglas Rizzolo, and Matthias Winkel. Diffusions on a space of interval partitions: Poisson–Dirichlet stationary distributions. *to appear in Ann. Probab.*, 2020+. preprint available as arXiv:1910.07626 [math.PR].

[17] Noah Forman, Douglas Rizzolo, Quan Shi, and Matthias Winkel. A two-parameter family of measure-valued diffusions with Poisson-Dirichlet stationary distributions. arXiv:2007.05250, 2020.

[18] Noah Forman, Douglas Rizzolo, Quan Shi, and Matthias Winkel. Diffusions on a space of interval partitions: The two-parameter model. arXiv:2008.02823, 2020.

[19] Zhicheng Gao and Nicholas C. Wormald. The size of the largest components in random planar maps. *SIAM J. Discrete Math.*, 12(2):217–228, 1999.

[20] Zhicheng Gao and Nicholas C. Wormald. The distribution of the maximum vertex degree in random planar maps. *J. Combin. Theory Ser. A*, 89(2):201–230, 2000.

[21] Zhicheng Gao and Nicholas C. Wormald. Sharp concentration of the number of submaps in random planar triangulations. *Combinatorica*, 23(3):467–486, 2003.

[22] Alexander Gnedin and Jim Pitman. Regenerative composition structures. *Ann. Probab.*, 33(2):445–479, 2005.

[23] Alexander V. Gnedin. The representation of composition structures. *Ann. Probab.*, 25(3):1437–1450, 07 1997.

[24] F. Hurtado and M. Noy. Graph of triangulations of a convex polygon and tree of triangulations. *Comput. Geom.*, 13(3):179–188, 1999.

[25] Hemant Ishwaran and Lancelot F. James. Gibbs sampling methods for stick-breaking priors. *J. Amer. Statist. Assoc.*, 96(453):161–173, 2001.

[26] Hemant Ishwaran and Lancelot F. James. Generalized weighted Chinese restaurant processes for species sampling mixture models. *Statist. Sinica*, 13(4):1211–1235, 2003.

[27] Lancelot F. James. Poisson calculus for spatial neutral to the right processes. *Ann. Statist.*, 34(1):416–440, 2006.

[28] J. F. C. Kingman. Random discrete distributions. *Journal of the Royal Statistical Society. Series B (Methodological)*, 37(1):1–22, 1975.

[29] J. F. C. Kingman. Random partitions in population genetics. *Proc. Roy. Soc. London Ser. A*, 361(1704):1–20, 1978.

[30] J. F. C. Kingman. The representation of partition structures. *J. London Math. Soc. (2)*, 18(2):374–380, 1978.

[31] J. F. C. Kingman. The coalescent. *Stochastic Process. Appl.*, 13(3):235–248, 1982.

[32] Jean-François Le Gall. Random geometry on the sphere. In *Proceedings of the International Congress of Mathematicians—Seoul 2014. Vol. 1*, pages 421–442. Kyung Moon Sa, Seoul, 2014.

[33] Xiaonan Liu and Xingxing Yu. Number of Hamiltonian cycles in planar triangulations. *SIAM J. Discrete Math.*, 35(2):1005–1021, 2021.

[34] Nicola Martinov. A combinatorial characterization of planar triangulations. *Annuaire Univ. Sofia Fac. Math. Méc.*, 78(1):82–86 (1988), 1984.

[35] Grigori Olshanski. Random permutations and related topics. In *The Oxford handbook of random matrix theory*, pages 510–533. Oxford Univ. Press, Oxford, 2011.

[36] José Luis Palacios and Prasad Tetali. A note on expected hitting times for birth and death chains. *Statistics & Probability Letters*, 30(2):119–125, 1996.

[37] Mihael Perman, Jim Pitman, and Marc Yor. Size-biased sampling of Poisson point processes and excursions. *Probab. Theory Related Fields*, 92(1):21–39, 1992.

[38] L. A. Petrov. A two-parameter family of infinite-dimensional diffusions on the Kingman simplex. *Funktsional. Anal. i Prilozhen.*, 43(4):45–66, 2009.

[39] J. Pitman. *Combinatorial stochastic processes*, volume 1875 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2006. Lectures from the 32nd Summer School on Probability Theory held in Saint-Flour, July 7–24, 2002, With a foreword by Jean Picard.

[40] Jim Pitman. Exchangeable and partially exchangeable random partitions. *Probab. Theory Related Fields*, 102(2):145–158, 1995.

[41] Jim Pitman. Random discrete distributions invariant under size-biased permutation. *Adv. in Appl. Probab.*, 28(2):525–539, 1996.

[42] Jim Pitman. Partition structures derived from Brownian motion and stable subordinators. *Bernoulli*, 3(1):79–96, 1997.

[43] Jim Pitman and Matthias Winkel. Regenerative tree growth: binary self-similar continuum random trees and Poisson–Dirichlet compositions. *Ann. Probab.*, 37(5):1999–2041, 2009.

[44] Jim Pitman and Marc Yor. The two-parameter Poisson-Dirichlet distribution derived from a stable subordinator. *Ann. Probab.*, 25(2):855–900, 1997.

[45] Kelvin Rivera-Lopez and Douglas Rizzolo. Diffusive limits of two-parameter ordered Chinese Restaurant Process up-down chains. *arXiv:2011.06577v2 [math.PR]*, 2021.

[46] Dane Rogers and Matthias Winkel. A Ray-Knight representation of up-down Chinese restaurants, 2020.

[47] L. C. G. Rogers and J. W. Pitman. Markov functions. *The Annals of Probability*, 9(4):573–582, 1981.

[48] Quan Shi and Matthias Winkel. Two-sided immigration, emigration and symmetry properties of self-similar interval partition evolutions. *arXiv preprint arXiv:2011.13378*, 2020.

[49] Quan Shi and Matthias Winkel. Up-down ordered Chinese restaurant processes with two-sided immigration, emigration and diffusion limits. *arXiv preprint arXiv:2012.15758*, 2020.

[50] Yee Whye Teh, Michael I. Jordan, Matthew J. Beal, and David M. Blei. Hierarchical Dirichlet processes. *J. Amer. Statist. Assoc.*, 101(476):1566–1581, 2006.

[51] W. T. Tutte. Counting rooted triangulations. In *Theory and practice of combinatorics*, volume 60 of *North-Holland Math. Stud.*, pages 243–253. North-Holland, Amsterdam, 1982.

Appendix

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The work appearing in Chapter 2 is largely based on the work in [45], which was written jointly by the present author and Douglas Rizzolo. Rizzolo has granted the present author permission to use the joint work in this dissertation in an email. That email reads as follows.

Douglas Rizzolo
to me

Jul 31, 2021, 4:14 PM (1 day ago)   Reply

Dear Kelvin,

You have my permission to use material from our joint article "Diffusive limits of two-parameter ordered Chinese Restaurant Process up-down chains" (arXiv:2011.06577) in your dissertation.

Best,
Douglas Rizzolo