DETERMINING THE TWIST OF AN OPTICAL FIBER

by

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A dissertation submitted to the Faculty of the University of Delaware in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematical Sciences

Fall 2014

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ACKNOWLEDGEMENTS

The author wishes to thank Professor Rakesh for his much-valued guidance and support.

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ABSTRACT

This research focuses on recovering the coefficient of a two speed hyperbolic system of partial differential equations from the reflection boundary data, where the source and the receiver are at the same location. We study the associated initial value problem (the forward problem) and then the coefficient determination inverse problem using a fixed point argument. We then implement the inversion scheme numerically.

We also study the inverse problem of recovering the coefficient of this system from the transmission boundary data, where the source and receiver are at different locations. We obtain an upper bound of the coefficient in terms of the transmission data, and we also obtain a relation between transmission and reflection data.

For multi-dimensional problems, we study the regularity at the origin of spherical harmonic expansions because solutions of some PDEs are constructed using spherical harmonic expansions.

Chapter 1

INTRODUCTION

The thesis is mainly devoted to the study of an inverse problem for a first order system of hyperbolic PDEs, in one space dimension, with two different speeds of propagation. This problem originates in the determination of the twist in an optical fiber. In this chapter we state the problem, survey earlier work on related problems, state the main results and then summarize the contents of each of the chapters.

1.1 Background

Consider a fiber stretching along the z axis, with two channels twisting around each other. Each channel has a different travel speed associated with it, and the twist is represented by a function $\beta(z)$, with $\beta(z) = 0$ for $z \leq 0$ (see Figure 1.1). The fiber is probed by a signal from the left end, and the response is measured at the same end. The goal is to determine the twist $\beta(z)$ for $z > 0$ from the medium response.

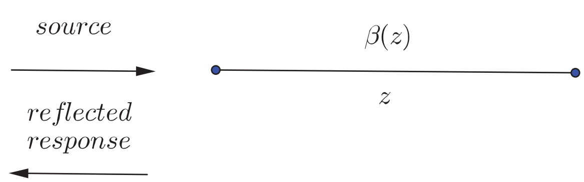

Figure 1.1: Fiber model

By scaling we may assume that the channels have two speeds c and 1, where $0 < c < 1$. Let $\mathbf{M}(z,t) = [M_1(z,t), M_2(z,t), M_3(z,t), M_4(z,t)]^T$ be a vector function, with M_1, M_3 denoting the left moving waves of speeds 1 and c respectively, and M_2, M_4 the right moving waves of speeds 1 and c (see Figure 1.2).

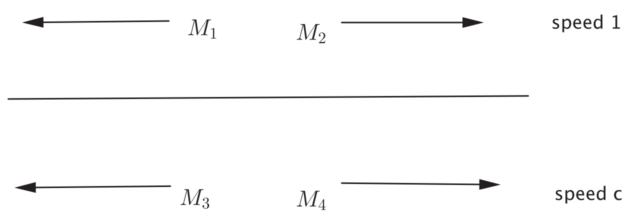

Figure 1.2: Left and right moving waves

We show in Chapter 2 that the intertwining of the waves due to the twist is governed by the following system of hyperbolic PDEs:

$$\mathbf{M}_t - A\mathbf{M}_z - \beta(z)B\mathbf{M} = 0, \quad (z,t) \in \mathbb{R}^2 \tag{1.1a}$$

where

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & -c \end{bmatrix} \quad B = \frac{1}{2}\begin{bmatrix} 0 & 0 & -1-c & -1+c \\ 0 & 0 & 1-c & 1+c \\ 1+c & -1+c & 0 & 0 \\ 1-c & -1-c & 0 & 0 \end{bmatrix} \tag{1.1b}$$

with initial condition

$$\mathbf{M}(z,t) = [0, \delta(t-z), 0, 0]^T, \quad t < 0. \tag{1.1c}$$

The initial condition represents a plane wave sent from the left end of the fiber along the faster channel (see Figure 1.3). We show in Chapter 2 that $(1.1a) - (1.1c)$ can be converted to the following IBVP

$$\mathbf{M}_t = A\mathbf{M}_z + \beta B\mathbf{M}, \quad z \geq 0, \quad t \in \mathbb{R}, \tag{1.2a}$$

$$M_2(0,t) = \delta(t), \quad M_4(0,t) = 0, \quad t \geq 0, \tag{1.2b}$$

$$\mathbf{M}(z,t) = \mathbf{0}, \quad t < 0, \quad z \geq 0. \tag{1.2c}$$

Our goal is to recover the twist $\beta(z)$ from the reflection boundary data $M_1(0,t)$ and $M_3(0,t)$.

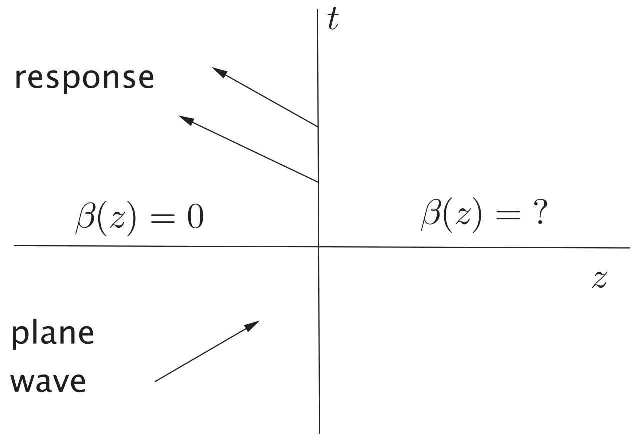

Figure 1.3: Signal and response

For the rest of the thesis, we use CBVP to stand for characteristic boundary value problem, and IBVP for initial boundary value problem. Also, $l \preceq r$ will mean that $l \leq Cr$ where C is a constant and

$$\mathcal{L} := I\partial_t - A\partial_z + \beta B.$$

1.2 Literature

Inverse problems for a hyperbolic PDE with a single speed of propagation were first studied by Gelfand and Levitan (1955) and others. Browning's thesis [7] and [10] contain a thorough survey of these results. The more recent book [21] describes results for many one dimensional inverse problems for hyperbolic PDEs with real valued coefficients, including those for systems of equations with multiple coefficients to be determined. Numerical methods for solving inverse problems for one dimensional hyperbolic PDEs with a single speed and real valued coefficients may be found in [8], [9] and [27].

For one speed problems in one space dimension, the inverse problem associated with the following initial boundary value problem was studied in [17]:

$$u_z - u_t = q(z)v, \quad (z,t) \in \mathbb{R}^2,$$
$$v_z + v_t = \bar{q}(z)u, \quad (z,t) \in \mathbb{R}^2$$

with the initial conditions

$$u(z,t) = 0, \quad v(z,t) = \delta(t-z), \quad t < 0.$$

From a knowledge of the reflection data $u(0,t)$, the coefficient $q(z)$ was recovered by applying the downward continuation method discussed in [25]. Also, good results for the real valued q case are presented in [7].

The study of inverse problems for hyperbolic systems with multiple speeds of propagation was initiated in [5] and [2] by Belishev. Let $E(z), G(z)$ be 2×2 coefficient matrices, and let λ, μ be any positive real numbers with $\lambda < \mu$. Consider the two speed IBVP:

$$\Lambda U := HU_{tt} - U_{zz} - EU_z - GU = 0, \quad (z,t) \in [0,\infty) \times \mathbb{R}, \tag{1.3a}$$
$$U = 0, \quad t < 0, \tag{1.3b}$$
$$U(0,t) = \delta(t) I_2, \quad t \in \mathbb{R} \tag{1.3c}$$

where $U(z,t) = [\mathbf{u}, \bar{\mathbf{u}}]$ is a 2×2 matrix of functions, $H = \begin{bmatrix} \lambda^2 & 0 \\ 0 & \mu^2 \end{bmatrix}$, and I_2 is the 2×2 identity matrix. Belishev studied the recovery of $E(z)$ and $G(z)$ from a knowledge of $U_z(0,t)$ and a special function $l(t)$ which we describe next.

For arbitrary $f_1(t), f_2(t)$, let $\mathbf{v}(z,t) \in \mathbb{R}^2$ be the solution of the IBVP

$$H\mathbf{v}_{tt} - \mathbf{v}_{zz} - E\mathbf{v}_z - G\mathbf{v} = 0, \quad (z,t) \in [0,\infty) \times \mathbb{R}, \tag{1.4a}$$
$$\mathbf{v} = 0, \quad t < 0, \tag{1.4b}$$
$$\mathbf{v}(0,t) = [f_1(t),\ f_2(t)]^T, \quad t \in \mathbb{R}. \tag{1.4c}$$

Knowing $U_z(0,t)$ is equivalent to knowing $\mathbf{v}_z(0,t)$ corresponding to all possible f_1, f_2. If f_1, f_2 are supported in the region $t \geq 0$ then, because of the finite speed of propagation, $\mathbf{v}(z,t)$ is supported in the fast region $t \geq \lambda z$ (see Figure 1.4). In [19], Belishev showed that there exists a unique $l(\cdot)$ such that $\mathbf{v}$ is supported in the slow region $t \geq \mu z$ if $f_2 = l * f_1$, where $*$ represents convolution with respect to t.

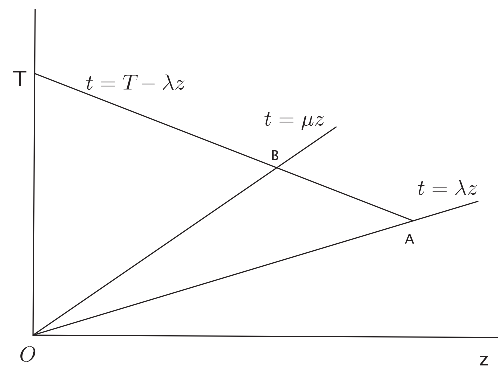

Figure 1.4: Fast and slow regions

Given the reflection data, for single speed inverse problems inversion is accomplished by creating a map, by solving a sideways forward problem, whose fixed point is the sought after coefficient. For two speed problems, such a map cannot be constructed because of the mismatch between the z interval over which the coefficient is to be recovered and the z interval over which the PDE must be solved to setup the map. This mismatch occurs because of the two speeds of propagation. If one knows $l(t)$ in addition to the reflection data, this mismatch can be avoided. It was shown in [3] that if Λ is self adjoint (the diagonal entries of E are zero and $G - G^T = E'$), then G can be recovered from $U_z(0,t)$ and $l(t)$.

Reference [19] gives a proof of the stability for the inverse problem of (1.3a) $-$ (1.3c), which suggests that given the reflection data $U_z(0,t)$ over the interval $[0,T]$, one should be able to recover some of the coefficients of E and G over the interval $[0, \frac{T}{2\mu}]$, determined by the slower speed of transmission.

Belishev in [4] showed that given E and the reflection data $\mathbf{v}_z(0,\cdot)$ (but $l(\cdot)$ not given), one can recover $l(\cdot)$ over a small interval, and therefore recover G over a small interval. The question of recovering $l(t)$, and hence $G(z)$, over the full interval, from the reflection data, remains open.

Our work focuses on recovering a single coefficient of a hyperbolic system of PDEs without the knowledge of $l(t)$. Since we are given less data, we focus on recovering only one coefficient $\beta(z)$. Applications of two speed problems on wave propagation in elastic solids are discussed in [1] and [14], and [15] is a good starting point for inverse problems for two speed systems.

1.3 Main Results

We study the question of recovering $\beta(z)$ from a knowledge of $M_1(0,t)$ and $M_3(0,t)$ or just one of these two functions. Before going into the details of the main results, we need to introduce some notation.

Let $c, K, T > 0$, $Y = \frac{2cT}{1+c}$, and define

$$\Theta := \{\beta \in C^1[0,T] \mid ||\beta(\cdot)||^2_{L^2[0,T]} \leq K\}.$$

Theorem 1.1. (**inverse stability**) *Let* $0 < c < 1$*, and* $\mathbf{M}, \tilde{\mathbf{M}}$ *be the solutions of* $(1.1a) - (1.1c)$ *corresponding to* $\beta, \tilde{\beta} \in \Theta$*, then*

$$||(\beta - \tilde{\beta})(\cdot)||^2_{L^2[0,Y]} \leq C(||(M_1 - \tilde{M}_1)(0,\cdot)||^2_{L^2[0,2T]} + ||(M_3 - \tilde{M}_3)(0,\cdot)||^2_{L^2[0,2T]})$$

where C *is a constant dependent only on* c, T, K*.*

If we define the forward map

$$F : \Theta \to C^1[0, 2T] \times C^1[0, 2T],$$
$$\beta(z) \mapsto [M_1(0, t), M_3(0, t)]$$

which maps the coefficient to the full reflection data, then Theorem 1.1 implies that F is injective, and F^{-1} is continuous.

The above result corresponds to an incoming source wave moving at the fast speed of 1 unit. One could attempt to obtain similar results if the incoming source wave was travelling at the slower speed c (i.e. the initial condition (1.1c) is replaced by $\mathbf{M}(z, t) = [0, 0, 0, \delta(t-z/c)]^T$), but we do not have any results for this case since downward continuation techniques do not apply to this situation because of the existence of precursor waves as noticed by Belishev.

Now consider the following CBVP

$$\mathcal{L}\mathbf{h} = 0, \quad 0 \le z \le T, \quad z \le t \le 2T - z, \tag{1.5a}$$
$$h_1(z, z) = h_3(z, z) = h_4(z, z) = 0, \quad 0 \le z \le T, \tag{1.5b}$$
$$h_2(0, t) = \phi(t), \quad h_4(0, t) = 0, \quad 0 \le t \le 2T. \tag{1.5c}$$

In Chapter 3 we show that (1.5a) – (1.5c) is well-posed. Define the reflection operator

$$R : C^1[0, 2T] \to C^1[0, 2T] \times C^1[0, 2T],$$
$$R(\phi) = [h_1(0, t), h_3(0, t)]$$

where $\mathbf{h}(z, t)$ is the solution of (1.5a) – (1.5c). R is completely determined by $M_1(0, \cdot)$ and $M_3(0, \cdot)$ because

$$R\phi(t) = \left[\int_0^t M_1(0, s)\phi(t - s)ds, \int_0^t M_3(0, s)\phi(t - s)ds \right].$$

As seen in Chapter 4,

Theorem 1.2. *If R is the reflection operator corresponding to $\beta \in C^1[0, T]$, then $||R|| \leq 1$.*

Theorem 1.2 gives a necessary condition on $M_1(0, t), M_3(0, t)$.

Theorem 1.3. (**Reconstruction**) *If the reflection data $(M_1(0, t), M_3(0, t))$ corresponds to a β which is in $C^1[0, T]$, then one can recover $\beta(z)$ from (1.1a) − (1.1c).*

Note that one can recover $\beta(z)$ for z in $[0, 2cT/(1 + c)]$ and not on the whole interval $[0, T]$.

The recovery of $\beta(z)$ is done by a fixed point argument, step by step in the z direction, first within the interval $[0, \delta]$, then $[\delta, 2\delta]$ and so on. The reconstruction scheme will terminate in a finite number of steps because δ has a positive lower bound.

1.4 Overview

Our research focuses on recovering the coefficient of a two speed hyperbolic system of partial differential equations given the reflection data. This involves first studying a general form of a characteristic boundary value problem (CBVP) for a two speed system of first order hyperbolic PDEs in one space dimension. Then we recover the coefficient using a fixed point argument.

Chapter 2 contains the derivation of (1.2a) − (1.2c) from Maxwell's equations and also shows how the solution of (1.2a) − (1.2c) may be reduced to solving a CBVP using a progressing wave expansion. This reduces a problem with singular solutions to one with no singularities.

Chapter 3 discusses the well posedness of the CBVP derived in Chapter 2. We show the existence and uniqueness of the classical and the weak solution for this CBVP.

To recover the coefficient, we need to study a CBVP with full Cauchy data on $z = 0$

and fewer conditions on the characteristics, called the sideways problem. This CBVP is different from the CBVP studied in Chapter 3. Chapter 4 shows the well posedness of this new CBVP. We use the sideways CBVP to construct a map whose fixed point is the coefficient that we wish to recover.

Chapter 5 discusses recovering the coefficient numerically from the reflection boundary data. We use the sideways CBVP discussed in Chapter 4 to construct a map whose fixed point is the coefficient that we wish to recover, and find the fixed point by recursively solving the sideways CBVP.

Chapter 6 discusses recovering the coefficient from the transmission boundary data, where the source and receiver are at different locations. We obtain an upper bound of the coefficient in terms of the transmission data, and we also obtain a relation between the transmission and the reflection data.

Chapter 7 studies the regularity at the origin of spherical harmonic expansions.

Chapter 2

MODEL DERIVATION

In this chapter we model the twist in the optical fiber as the solution of the IVP

$$\mathbf{M}_t = A\mathbf{M}_z + \beta(z)B\mathbf{M}, \quad z \geq 0, \quad t \in \mathbb{R}, \tag{2.1a}$$

$$M_2(0,t) = \delta(t), \quad M_4(0,t) = 0, \quad t \geq 0, \tag{2.1b}$$

$$\mathbf{M}(z,t) = \mathbf{0}, \quad t < 0, \quad z \geq 0 \tag{2.1c}$$

where

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & -c \end{bmatrix} \qquad B = \frac{1}{2}\begin{bmatrix} 0 & 0 & -1-c & -1+c \\ 0 & 0 & 1-c & 1+c \\ 1+c & -1+c & 0 & 0 \\ 1-c & -1-c & 0 & 0 \end{bmatrix}.$$

We also show how the solution of (2.1a) – (2.1c) may be reduced to solving a CBVP using a progressing wave expansion. This reduces a problem with singular solutions to one with no singularities.

2.1 Model

This model was proposed by Andrew Lacey during the MPI workshop at University of Delaware in 2000 (see [6] and [13]). Since this derivation is not readily available we have included the derivation in the thesis.

Consider a fiber stretching along the z axis. Let $\mathbf{E}(z,t), \mathbf{P}(z,t)$ be the electric field and the polarization at the point z units away from the left end of the fiber, and we assume that $\mathbf{E}$ and $\mathbf{P}$ have no component along the fiber. $\mathbf{E}$ and $\mathbf{P}$ obey Maxwell's equations

$$\nabla^2\mathbf{E} - \nabla(\nabla \cdot \mathbf{E}) = \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} + \frac{1}{\epsilon_0 c^2}\frac{\partial^2 \mathbf{P}}{\partial t^2} \tag{2.2}$$

where c is the speed of light in vacuum and ϵ_0 is the permittivity of free space. Noting that $\mathbf{E}$ and $\mathbf{P}$ depend only on z and t and have no components in the $\mathbf{k}$ direction, we have:

$$\nabla^2\mathbf{E} - \nabla(\nabla \cdot \mathbf{E}) = \mathbf{E}_{zz} - \nabla(0) = \mathbf{E}_{zz}.$$

So (2.2) reduces to

$$\mathbf{E}_{zz} = \frac{1}{c^2}\mathbf{E}_{tt} + \frac{1}{\epsilon_0 c^2}\mathbf{P}_{tt}. \tag{2.3}$$

At every point in the fiber, we can find two unit orthogonal vectors $\mathbf{v}_1(z)$ and $\mathbf{v}_2(z)$ perpendicular to the fiber, which represent the polarization directions of the two channels. As the fiber twists along its length, the polarization directions change. Since $\mathbf{v}_1$ and $\mathbf{v}_2$ are unit vectors in a plane and othogonal, $d\mathbf{v}_1/dz$ is orthogonal to $\mathbf{v}_1(z)$ and hence parallel to $\mathbf{v}_2(z)$, which implies that $d\mathbf{v}_1/dz = \beta(z)\mathbf{v}_2$ for some real valued function $\beta(z)$ that captures the twist in the fiber. Also, $d\mathbf{v}_2/dz$ is perpendicular to $\mathbf{v}_2$ so parallel to $\mathbf{v}_1$. But differentiating $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$, we have:

$$0 = \frac{d\mathbf{v}_1}{dz} \cdot \mathbf{v}_2 + \mathbf{v}_1 \cdot \frac{d\mathbf{v}_2}{dz} = \beta + \mathbf{v}_1 \cdot \frac{d\mathbf{v}_2}{dz}, \quad z \in \mathbb{R} \tag{2.4}$$

which implies that $d\mathbf{v}_2/dz = -\beta\mathbf{v}_1$. Summarizing, the principal directions propagate as

$$\frac{d\mathbf{v}_1}{dz} = \beta\mathbf{v}_2, \quad \frac{d\mathbf{v}_2}{dz} = -\beta\mathbf{v}_1. \tag{2.5}$$

Let E_1, E_2 be the components of $\mathbf{E}$ along the principal directions, so that $\mathbf{E} = E_1\mathbf{v}_1 + E_2\mathbf{v}_2$. We further assume that the polarization vector is related to the electric field via

$$\mathbf{P} = \epsilon_0(\alpha_1 E_1\mathbf{v}_1 + \alpha_2 E_2\mathbf{v}_2)$$

where α_1, α_2 are real constants. Now

$$\begin{aligned} \frac{1}{c^2}\mathbf{E}_{tt} + \frac{1}{\epsilon_0 c^2}\mathbf{P}_{tt} &= \frac{1}{c^2}(E_{1tt}\mathbf{v}_1 + E_{2tt}\mathbf{v}_2) + \frac{1}{c^2}(\alpha_1 E_{1tt}\mathbf{v}_1 + \alpha_2 E_{2tt}\mathbf{v}_2) \\ &= \frac{1+\alpha_1}{c^2}E_{1tt}\mathbf{v}_1 + \frac{1+\alpha_2}{c^2}E_{2tt}\mathbf{v}_2 \\ &= \frac{1}{c_1^2}E_{1tt}\mathbf{v}_1 + \frac{1}{c_2^2}E_{2tt}\mathbf{v}_2 \end{aligned} \tag{2.6}$$

where we have assumed that $1 + \alpha_i > 0$ and defined $c_i = c/\sqrt{1+\alpha_i}$. Now $\mathbf{E} = E_1\mathbf{v}_1 + E_2\mathbf{v}_2$ combined with (2.5) gives:

$$\mathbf{E}_z = E_{1z}\mathbf{v}_1 + E_{2z}\mathbf{v}_2 + \beta E_1\mathbf{v}_2 - \beta E_2\mathbf{v}_1 = (E_{1z} - \beta E_2)\mathbf{v}_1 + (E_{2z} + \beta E_1)\mathbf{v}_2, \tag{2.7a}$$

$$\begin{aligned}\mathbf{E}_{zz} &= (E_{1z} - \beta E_2)_z\mathbf{v}_1 + (E_{2z} + \beta E_1)_z\mathbf{v}_2 + \beta(E_{1z} - \beta E_2)\mathbf{v}_2 - \beta(E_{2z} + \beta E_1)\mathbf{v}_1 \\ &= \{(E_{1z} - \beta E_2)_z - \beta(E_{2z} + \beta E_1)\}\mathbf{v}_1 + \{(E_{2z} + \beta E_1)_z + \beta(E_{1z} - \beta E_2)\}\mathbf{v}_2.\end{aligned} \tag{2.7b}$$

Using (2.7*a*) – (2.7*b*) and (2.6) in (2.3) and matching the components of $\mathbf{v}_1, \mathbf{v}_2$, we have:

$$(E_{1z} - \beta E_2)_z - \beta(E_{2z} + \beta E_1) = \frac{1}{c_1^2}E_{1tt}, \tag{2.8a}$$

$$(E_{2z} + \beta E_1)_z + \beta(E_{1z} - \beta E_2) = \frac{1}{c_2^2}E_{2tt}. \tag{2.8b}$$

This is the end of the derivation due to Andrew Lacey.

We assume that the fiber is extended to the left of $z = 0$ and that $\beta(z) = 0$ for $z \leq 0$. The incoming wave is modeled as

$$E_1(t,z) = -H(c_1 t - z), \quad E_2(t,z) = 0, \quad t < 0. \tag{2.9}$$

We now convert (2.8*a*) – (2.8*b*) to a first order system of PDEs in the left and right moving waves. We start by defining the variables

$$N_1 = E_{1z} - \beta E_2, \quad N_2 = E_{2z} + \beta E_1, \quad N_3 = \frac{1}{c_1}E_{1t}, \quad N_4 = \frac{1}{c_2}E_{2t}.$$

Then (2.8*a*) – (2.8*b*) can be rewritten as

$$c_1 N_{1z} - c_1\beta N_2 = N_{3t},$$

$$c_2 N_{2z} + c_2\beta N_1 = N_{4t}$$

combined with the relations

$$N_{1t} = E_{1zt} - \beta E_{2t} = c_1 N_{3z} - \beta c_2 N_4,$$

$$N_{2t} = E_{2zt} + \beta E_{1t} = c_2 N_{4z} + \beta c_1 N_3.$$

So if $\mathbf{N} = [N_1, N_2, N_3, N_4]^T$, then (2.8$a$) – (2.8$b$) can be replaced by the first order system

$$\mathbf{N}_t = G\mathbf{N}_z + \beta H\mathbf{N} \tag{2.10}$$

where

$$G = \begin{bmatrix} 0 & 0 & c_1 & 0 \\ 0 & 0 & 0 & c_2 \\ c_1 & 0 & 0 & 0 \\ 0 & c_2 & 0 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 0 & 0 & 0 & -c_2 \\ 0 & 0 & c_1 & 0 \\ 0 & -c_1 & 0 & 0 \\ c_2 & 0 & 0 & 0 \end{bmatrix}.$$

Since G is symmetric, it may be diagonalized by an orthogonal matrix. In fact, $P^{-1}GP = A$ where

$$P = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \quad A = \begin{bmatrix} c_1 & 0 & 0 & 0 \\ 0 & -c_1 & 0 & 0 \\ 0 & 0 & c_2 & 0 \\ 0 & 0 & 0 & -c_2 \end{bmatrix}.$$

Note that P is almost orthogonal in that $P^T P = 2I$. So if we introduce a new dependent variable $\mathbf{M} = P^{-1}\mathbf{N}$, then (2.10) may be rewritten as

$$\mathbf{M}_t = A\mathbf{M}_z + \beta B\mathbf{M} \tag{2.11}$$

where

$$B = P^{-1}HP = \frac{1}{2}\begin{bmatrix} 0 & 0 & -c_1 - c_2 & -c_1 + c_2 \\ 0 & 0 & c_1 - c_2 & c_1 + c_2 \\ c_1 + c_2 & -c_1 + c_2 & 0 & 0 \\ c_1 - c_2 & -c_1 - c_2 & 0 & 0 \end{bmatrix}.$$

Note that $B^T = -B$.

Let $\mathbf{M} = [M_1, M_2, M_3, M_4]^T$. If we backtrack we obtain the following relation between M_i and E_i

$$2M_1 = E_{1z} - \beta E_2 + \frac{1}{c_1} E_{1t}, \quad 2M_2 = E_{1z} - \beta E_2 - \frac{1}{c_1} E_{1t}, \tag{2.12a}$$

$$2M_3 = E_{2z} + \beta E_1 + \frac{1}{c_2} E_{2t}, \quad 2M_4 = E_{2z} + \beta E_1 - \frac{1}{c_2} E_{2t} \tag{2.12b}$$

and because of the form of A, M_1, M_2 are, loosely speaking, the left and right moving waves propagating with speed c_1, and M_3, M_4 the left and right moving waves propagating with speed c_2. We may assume for convenience that $c_1 = 1, c_2 = c$ with $\mathbf{0} < \mathbf{c} < \mathbf{1}$ (see Figure 2.1), then we still have (2.11) but with A, B defined as in (1.1b).

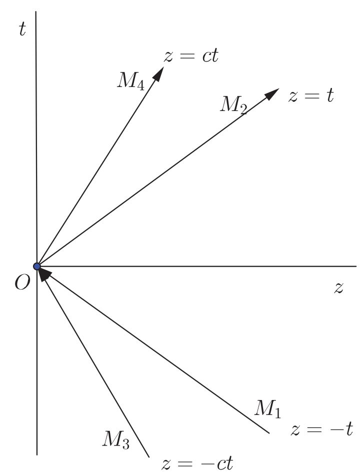

Figure 2.1: Left and right moving waves

2.2 Goal

The fiber is probed by a right moving wave coming from the left end and one of the components of the response is measured at the same end for a certain period of time, and the goal is the recovery of $\beta(z)$. Mathematically speaking, the measurement consists of either $E_1(0, t)$ or $E_2(0, t)$ over a certain interval $t \in [0, T]$. The goal is to recover $\beta(z)$.

The initial conditions for $\mathbf{E}$ in (2.9) and the relationship between E_i and M_i in (2.12a) – (2.12b) imply that

$$2\mathbf{M}(t,z) = [0, 2\delta(t-z), -\beta H(t-z), -\beta H(ct-z)]^T, \quad t<0,\ z\in\mathbb{R}.$$

Since $H(t-z)$ is supported in $t \geq z$, and $\beta(z)=0$ for $z\leq 0$, we may conclude that

$$\mathbf{M}(t,z) = [0, \delta(t-z), 0, 0]^T, \quad t<0,\ z\in\mathbb{R}.$$

Since $\beta(z)=0$ for $z\leq 0$, (2.11) implies that

$$M_t = AM_z, \quad z\leq 0,\ t\in\mathbb{R}.$$

Since A is diagonal, for $z\leq 0$, M has the form

$$\mathbf{M}(z,t) = [f_1(t+z), f_2(t-z), f_3(ct+z), f_4(ct-z)]^T, \quad z\leq 0,\ t\in\mathbb{R}$$

for some functions f_i. From the initial condition (2.1c), we know that $\mathbf{M}(0,t)=\mathbf{0}$ for $t<0$, which implies that f_i are supported in $t\geq 0$. Also, for any $\epsilon>0$, we have

$$\begin{aligned}\mathbf{M}(z,-\epsilon) &= [f_1(-\epsilon+z), f_2(-\epsilon-z), f_3(-c\epsilon+z), f_4(-c\epsilon-z)]^T\\ &= [0, f_2(-\epsilon-z), 0, f_4(-c\epsilon-z)]^T, \quad z\leq 0.\end{aligned}$$

So the initial condition of $\mathbf{M}$ and the support of f_2, f_4 imply that $f_2(t)=\delta(t)$, $f_4(t)=0$. Hence, we have

$$\mathbf{M}(z,t) = [f_1(t+z), \delta(t-z), f_3(ct+z), 0]^T, \quad z\leq 0,\ t\in\mathbb{R}$$

which implies that

$$\mathbf{M}(0,t) = [f_1(t), \delta(t), f_3(t), 0]^T, \quad t\in\mathbb{R} \tag{2.13}$$

with $f_1(t), f_3(t)$ supported in $t\geq 0$. Summarizing, $\mathbf{M}(z,t)$ is the solution of the IBVP

$$\mathbf{M}_t = A\mathbf{M}_z + \beta B\mathbf{M}, \quad z\geq 0, \quad t\in\mathbb{R}, \tag{2.14a}$$

$$M_2(0,t) = \delta(t), \quad M_4(0,t)=0, \quad t\geq 0, \tag{2.14b}$$

$$\mathbf{M}(z,t) = \mathbf{0}, \quad t<0, \quad z\geq 0. \tag{2.14c}$$

Our **goal** is that, given $M_1(0,t)$ or $M_3(0,t)$ or both within some interval $t\in[0,T]$, determine $\beta(z)$ within some interval $z\in[0,Z]$.

2.3 Linearization at $\beta = 0$

In this section we analyze the linearized inverse problem corresponding to (2.14a) − (2.14c), around $\beta = 0$, to get an indication of what reflection data is needed for the nonlinear inverse problem. Define the forward map

$$F : C^1[0,T] \to C^1[0,2T] \times C^1[0,2T],$$
$$\beta(z) \mapsto [M_1(0,t), M_3(0,t)]$$

which maps the coefficient to the reflection data. We want to analyze the formal derivative of F at $\beta = 0$. Formally,

$$F' : C^1[0,T] \to C^1[0,2T] \times C^1[0,2T],$$
$$d\beta(z) \mapsto [dM_1(0,t), dM_3(0,t)]$$

where $d\beta$ is a small perturbation of β, and $d\mathbf{M}$ is the small perturbation of the solution of (2.14a)−(2.14c) with respect to β, that is, $\mathbf{M}+d\mathbf{M}$ is the solution of (2.14a)−(2.14c) with respect to $\beta + d\beta$. Then from (2.14a), we have

$$(\mathbf{M} + d\mathbf{M})_t - A(\mathbf{M} + d\mathbf{M})_z - (\beta + d\beta)B(\mathbf{M} + d\mathbf{M}) = 0. \tag{2.15}$$

If we subtract (2.14a) from (2.15) for $\beta = 0$, and only keep the linear terms, we get

$$(d\mathbf{M})_t - A(d\mathbf{M})_z = (d\beta)B\mathbf{M} \quad \text{in } \mathbb{R}^2 \tag{2.16}$$

with the initial condition

$$d\mathbf{M}(z,t) = 0, \quad t < 0. \tag{2.17}$$

When $\beta = 0$, the solution of (2.14a) − (2.14c) is $\mathbf{M}(z,t) = [0, \delta(t-z), 0, 0]^T$. Since

$$B[0, \delta(t-z), 0, 0]^T = \frac{\delta(t-z)}{2}[0, 0, c-1, -1-c]^T$$

so (2.16) becomes

$$(d\mathbf{M})_t - A(d\mathbf{M})_z = \frac{(d\beta)(z)}{2}\delta(t-z)[0, 0, c-1, -1-c]^T \quad \text{in } \mathbb{R}^2.$$

Since the equations decouple, the initial condition (2.17) implies that

$$(dM)_1 = (dM)_2 = 0 \quad \text{in } \mathbb{R}^2.$$

And $(dM)_3$ is the solution of

$$(dM)_{3t} - c(dM)_{3z} = \frac{c-1}{2}(d\beta)(z)\delta(t-z), \quad (z,t) \in \mathbb{R}^2,$$
$$(dM)_3(z,t) = 0, \quad t < 0, \quad z \in \mathbb{R}$$

which implies that

$$\frac{d}{ds}(dM)_3(z+ct-cs,s) = \frac{c-1}{2}(d\beta)(z+ct-cs)\delta(s-z-ct+cs), \tag{2.18a}$$
$$(dM)_3(z,t) = 0, \quad t < 0. \tag{2.18b}$$

Integrating (2.18a) over $(-\infty, t]$, for $t \geq 0$, we obtain

$$\begin{aligned}
(dM)_3(z,t) &= \frac{c-1}{2}\int_{-\infty}^{t}(d\beta)(z+ct-cs)\delta(s-z-ct+cs)\ ds \\
&= \frac{c-1}{2(c+1)}\int_{-\infty}^{(c+1)t}(d\beta)\left(z+ct-\frac{cr}{c+1}\right)\delta(r-z-ct)\ dr \\
&= \frac{c-1}{2(c+1)}(d\beta)\left(\frac{z+ct}{c+1}\right)H((c+1)t-(z+ct)) \\
&= \frac{c-1}{2(c+1)}(d\beta)\left(\frac{z+ct}{c+1}\right)H(t-z)
\end{aligned}$$

which implies that

$$(dM)_3(0,t) = \frac{c-1}{2(c+1)}(d\beta)\left(\frac{ct}{1+c}\right)H(t).$$

Similarly, $(dM)_4$ is the solution of

$$(dM)_{4t} + c(dM)_{4z} = \frac{-c-1}{2}(d\beta)(z)\delta(t-z), \quad (z,t) \in \mathbb{R}^2, \tag{2.19a}$$
$$(dM)_4(z,t) = 0, \quad t < 0, \quad z \in \mathbb{R}. \tag{2.19b}$$

Integrating (2.19a) over $(-\infty, t]$, for $t \geq 0$, we obtain

$$
\begin{aligned}
(dM)_4(z,t) &= \frac{-c-1}{2} \int_{-\infty}^{t} (d\beta)(z - ct + cs)\delta(s - z + ct - cs)\ ds \\
&= \frac{-c-1}{2(1-c)} \int_{-\infty}^{(1-c)t} (d\beta)\left(z - ct + \frac{cr}{1-c}\right)\delta(r - z + ct)\ dr \\
&= \frac{-c-1}{2(1-c)}(d\beta)\left(\frac{z-ct}{1-c}\right) H((1-c)t - (z - ct)) \\
&= \frac{-c-1}{2(1-c)}(d\beta)\left(\frac{z-ct}{1-c}\right) H(t-z)
\end{aligned}
$$

which implies that

$$
(dM)_4(0,t) = \frac{-c-1}{2(1-c)}(d\beta)\left(\frac{-ct}{1-c}\right) H(t) = 0, \quad t > 0.
$$

So to solve the linearized inverse problem about $\beta = 0$, the values of $(dM)_3(0,t)$ over the interval $[0, T]$ will determine the value of $(d\beta)(z)$ for $z \in [0, \frac{cT}{1+c}]$. Further, values of $(dM)_1(0,t), (dM)_2(0,t), (dM)_4(0,t)$ do not contain any information about $d\beta$. This suggests, for the original non-linear problem, that having $M_3(0,t)$ is crucial for the recovery of $\beta(z)$.

2.4 Progressing Wave Expansion

In this section, we show, using a progressing wave expansion, how the solution of (2.14a) – (2.14c) may be reduced to solving a CBVP . This reduces a problem with singular solutions to one with no singularities.

Let $\mathbf{M}(z,t)$ be the solution of (2.14a) – (2.14c). We look for a solution of the form

$$
\mathbf{M}(z,t) = \mathbf{p}(z)\delta(t-z) + \mathbf{q}(z,t)H(t-z) + \mathbf{r}(z)\delta\left(t - \frac{z}{c}\right) + \mathbf{s}(z,t)H\left(t - \frac{z}{c}\right) \qquad (2.20)
$$

where $\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s}$ are vector functions. Then we have

$$
\begin{aligned}
\mathbf{M}_t &= \mathbf{p}\delta'(t-z) + \mathbf{q}\delta(t-z) + \mathbf{q}_t H(t-z) + \mathbf{r}\delta'\left(t - \frac{z}{c}\right) + \mathbf{s}\delta\left(t - \frac{z}{c}\right) + \mathbf{s}_t H\left(t - \frac{z}{c}\right), \\
\mathbf{M}_z &= -\mathbf{p}\delta'(t-z) + (\mathbf{p}_z - \mathbf{q})\delta(t-z) + \mathbf{q}_z H(t-z) - \frac{\mathbf{r}}{c}\delta'\left(t - \frac{z}{c}\right) \\
&\quad + \left(\mathbf{r}_z - \frac{\mathbf{s}}{c}\right)\delta\left(t - \frac{z}{c}\right) + \mathbf{s}_z H\left(t - \frac{z}{c}\right).
\end{aligned}
$$

Recall that $\mathcal{L} = I\partial_t - A\partial_z - \beta B$, then we have

$$\begin{aligned}\mathcal{L}\mathbf{M} &= (I+A)\mathbf{p}\delta'(t-z) + ((I+A)\mathbf{q} + \mathcal{L}\mathbf{p})\delta(t-z) + \mathcal{L}\mathbf{q}H(t-z) \\ &\quad + \left(I + \frac{A}{c}\right)\mathbf{r}\delta'\left(t - \frac{z}{c}\right) + ((I + A/c)\mathbf{s} + \mathcal{L}\mathbf{r})\delta\left(t - \frac{z}{c}\right) + \mathcal{L}\mathbf{s}H\left(t - \frac{z}{c}\right).\end{aligned}$$

Since $\mathcal{L}\mathbf{M} = 0$, we have

$$(I+A)\mathbf{p} = 0 \text{ on } z = t, \tag{2.21a}$$

$$(I+A)\mathbf{q} + \mathcal{L}\mathbf{p} = 0 \text{ on } z = t, \tag{2.21b}$$

$$\left(I + \frac{A}{c}\right)\mathbf{s} + \mathcal{L}\mathbf{r} = 0 \text{ on } z = ct, \tag{2.21c}$$

$$\left(I + \frac{A}{c}\right)\mathbf{r} = 0 \text{ on } z = ct, \tag{2.21d}$$

$$\mathcal{L}\mathbf{q}H(t-z) + \mathcal{L}\mathbf{s}H\left(t - \frac{z}{c}\right) = 0 \text{ on } \mathbb{R}^2. \tag{2.21e}$$

Let p_i, q_i, r_i, s_i be the ith components of $\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s}$, then $(2.21a), (2.21d)$ imply that p_1, p_3, p_4 are zero on $t - z = 0$, and r_1, r_2, r_3 are zero on $t = \frac{z}{c}$, which imply that $p_1, p_3, p_4, r_1, r_2, r_3$ are zero for all $z \geq 0$. From (2.20), (2.13), we have

$$p_2(0) = 1, \quad r_4(0) = 0, \quad (q_2 + s_2)(0,t) = (q_4 + s_4)(0,t) = 0 \tag{2.22}$$

and $(2.21b), (2.21d)$ imply that

$$(I+A)\mathbf{q} = -p_{2z}[0,1,0,0]^T + \frac{\beta}{2}p_2[0,0,-1+c,-1-c]^T \text{ on } z = t, \tag{2.23a}$$

$$\left(I + \frac{A}{c}\right)\mathbf{s} = -cr_{4z}[0,0,0,1]^T + \frac{\beta}{2}r_4[-1+c,1+c,0,0]^T \text{ on } z = ct. \tag{2.23b}$$

From the second component of $(2.23a)$ and the fourth component of $(2.23b)$, we have

$$p_{2z} = 0 \text{ on } z = t,$$

$$cr_{4z} = 0 \text{ on } z = ct.$$

Using the initial conditions $p_2(0) = 1, r_4(0) = 0$, we conclude that $p_2(z) = 1, r_4(z) = 0$. Using this in the other three components of $(2.23a) - (2.23b)$, we have

$$q_1(z,t) = 0, \quad q_3(z,t) = \frac{c-1}{2(1+c)}\beta(z), \quad q_4(z,t) = \frac{1+c}{2(c-1)}\beta(z) \quad \text{on } z = t,$$

$$s_1(z,t) = 0, \quad s_2(z,t) = 0, \quad s_3(z,t) = 0 \quad \text{on } z = ct.$$

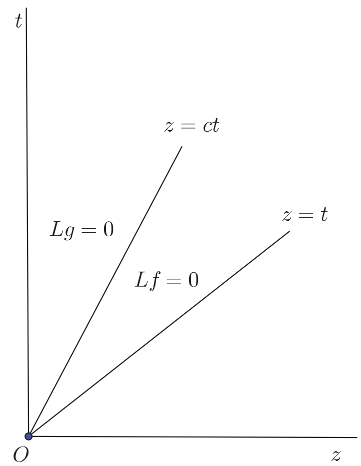

Figure 2.2: Ray geometry

Noting that $c < 1$, from the above calculations we conclude that (see Figure 2.2),

$$\mathbf{M}(z,t) = \delta(t-z)[0,1,0,0]^T + \mathbf{g}(z,t)H(t-z/c) + \mathbf{f}(z,t)(H(t-z) - H(t-z/c)) \tag{2.24}$$

where $\mathbf{g} = \mathbf{q} + \mathbf{s}, \mathbf{f} = \mathbf{q}$, are the solution of the following characteristic BVP

$$\mathcal{L}\mathbf{f} = \mathbf{0} \quad \text{on } 0 \leq ct \leq z \leq t, \tag{2.25a}$$

$$\mathcal{L}\mathbf{g} = \mathbf{0} \quad \text{on } 0 \leq z \leq ct, \tag{2.25b}$$

$$f_1(z,z) = 0, \quad f_3(z,z) = \frac{c-1}{2(1+c)}\beta(z), \quad f_4(z,z) = \frac{1+c}{2(c-1)}\beta(z), \quad z \geq 0, \tag{2.25c}$$

$$(g_1 - f_1)(z,t) = (g_2 - f_2)(z,t) = (g_3 - f_3)(z,t) = 0 \quad \text{on } z = ct, \ z \geq 0, \tag{2.25d}$$

$$g_2(0,t) = g_4(0,t) = 0, \quad t \geq 0. \tag{2.25e}$$

The IBVP $(2.1a) - (2.1c)$ corresponds to a source initially travelling right along the faster channel. For future use we also compute the progressing wave expansion of the solution corresponding to the situation when the source is initially travelling right but at the slower speed c.

Let $\overline{\mathbf{M}}(z,t)$ be the solution of the IBVP

$$\overline{\mathbf{M}}_t = A\overline{\mathbf{M}}_z + \beta B\overline{\mathbf{M}}, \quad z \geq 0, \quad t \in \mathbb{R}, \tag{2.26a}$$

$$\overline{M}_2(0,t) = 0, \quad \overline{M}_4(0,t) = \delta(t), \quad t \geq 0, \tag{2.26b}$$

$$\overline{\mathbf{M}}(z,t) = \mathbf{0}, \quad t < 0, \quad z \geq 0. \tag{2.26c}$$

Proceeding as above, we can show that

$$\overline{\mathbf{M}}(z,t) = \delta(t-z/c)[0,0,0,1]^T + \overline{\mathbf{g}}(z,t)H(t-z/c) + \overline{\mathbf{f}}(z,t)(H(t-z) - H(t-z/c)) \tag{2.27}$$

where $\overline{\mathbf{f}}, \overline{\mathbf{g}}$ is the solution of the following CBVP

$$\mathcal{L}\overline{\mathbf{g}} = 0 \text{ on } 0 \leq z \leq ct, \tag{2.28a}$$

$$\mathcal{L}\overline{\mathbf{f}} = 0 \text{ on } 0 \leq ct \leq z \leq t, \tag{2.28b}$$

$$(\overline{g}_1 - \overline{f}_1)(z,t) = \frac{c(c-1)}{2(c+1)}\beta(z), \quad (\overline{g}_2 - \overline{f}_2)(z,t) = \frac{c(c+1)}{2(c-1)}\beta(z)$$

$$(\overline{g}_3 - \overline{f}_3)(z,t) = 0 \text{ on } z = ct,\ z \geq 0, \tag{2.28c}$$

$$\overline{f}_1(z,t) = \overline{f}_3(z,t) = \overline{f}_4(z,t) = 0 \text{ on } z = t,\ z \geq 0, \tag{2.28d}$$

$$\overline{g}_2(0,t) = \overline{g}_4(0,t) = 0, \quad t \geq 0. \tag{2.28e}$$

Chapter 3

WELL-POSEDNESS OF THE CHARACTERISTIC BOUNDARY VALUE PROBLEM

In this chapter, we study the well-posedness of the IBVPs

$$\mathbf{M}_t = A\mathbf{M}_z + \beta B\mathbf{M}, \quad z \geq 0, \quad t \in \mathbb{R}, \tag{3.1a}$$

$$M_2(0,t) = \delta(t), \quad M_4(0,t) = 0, \quad t \geq 0, \tag{3.1b}$$

$$\mathbf{M}(z,t) = \mathbf{0}, \quad t < 0, \quad z \geq 0, \tag{3.1c}$$

and

$$\overline{\mathbf{M}}_t = A\overline{\mathbf{M}}_z + \beta B\overline{\mathbf{M}}, \quad z \geq 0, \quad t \in \mathbb{R}, \tag{3.2a}$$

$$\overline{M}_2(0,t) = 0, \quad \overline{M}_4(0,t) = \delta(t), \quad t \geq 0, \tag{3.2b}$$

$$\overline{\mathbf{M}}(z,t) = \mathbf{0}, \quad t < 0, \quad z \geq 0, \tag{3.2c}$$

where

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & -c \end{bmatrix} \quad B = \frac{1}{2}\begin{bmatrix} 0 & 0 & -1-c & -1+c \\ 0 & 0 & 1-c & 1+c \\ 1+c & -1+c & 0 & 0 \\ 1-c & -1-c & 0 & 0 \end{bmatrix}.$$

These are the two forward problems associated to our inverse problem, and their well-posedness is important for the solution of the inverse problem. The first problem corresponds to a plane wave initiated in the fast channel while the second one corresponds to a plane wave initiated in the slow channel.

From section 2.4 we know that (see Figure 3.1),

$$\mathbf{M}(z,t) = \delta(t-z)[0,1,0,0]^T + \mathbf{g}(z,t)H(t-z/c) + \mathbf{f}(z,t)(H(t-z) - H(t-z/c)) \tag{3.3}$$

where $\mathbf{g}$, $\mathbf{f}$ is the solution of the following CBVP

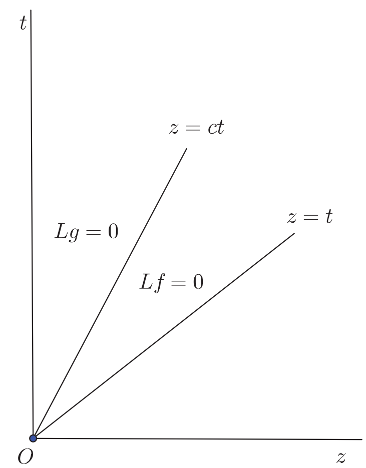

Figure 3.1: Ray geometry

$$\mathcal{L}\mathbf{f} = \mathbf{0} \quad \text{on } 0 \le ct \le z \le t, \tag{3.4a}$$

$$\mathcal{L}\mathbf{g} = \mathbf{0} \quad \text{on } 0 \le z \le ct, \tag{3.4b}$$

$$f_1(z,z) = 0, \quad f_3(z,z) = \frac{c-1}{2(1+c)}\beta(z), \quad f_4(z,z) = \frac{1+c}{2(c-1)}\beta(z), \quad z \ge 0, \tag{3.4c}$$

$$(g_1 - f_1)(z,t) = (g_2 - f_2)(z,t) = (g_3 - f_3)(z,t) = 0 \quad \text{on } z = ct,\ z \ge 0, \tag{3.4d}$$

$$g_2(0,t) = g_4(0,t) = 0, \quad t \ge 0. \tag{3.4e}$$

Similarly, from section 2.4, we have

$$\overline{\mathbf{M}}(z,t) = \delta(t-z/c)[0,0,0,1]^T + \overline{\mathbf{g}}(z,t)H(t-z/c) + \overline{\mathbf{f}}(z,t)(H(t-z) - H(t-z/c)) \tag{3.5}$$

where $\overline{\mathbf{f}}, \overline{\mathbf{g}}$ is the solution of the following CBVP

$$\mathcal{L}\overline{\mathbf{g}} = 0 \text{ on } 0 \leq z \leq ct, \tag{3.6a}$$

$$\mathcal{L}\overline{\mathbf{f}} = 0 \text{ on } 0 \leq ct \leq z \leq t, \tag{3.6b}$$

$$(\overline{g}_1 - \overline{f}_1)(z,t) = \frac{c(c-1)}{2(c+1)}\beta(z), \quad (\overline{g}_2 - \overline{f}_2)(z,t) = \frac{c(c+1)}{2(c-1)}\beta(z),$$

$$(\overline{g}_3 - \overline{f}_3)(z,t) = 0 \text{ on } z = ct,\ z \geq 0, \tag{3.6c}$$

$$\overline{f}_1(z,t) = \overline{f}_3(z,t) = \overline{f}_4(z,t) = 0 \text{ on } z = t,\ z \geq 0, \tag{3.6d}$$

$$\overline{g}_2(0,t) = \overline{g}_4(0,t) = 0, \quad t \geq 0. \tag{3.6e}$$

We study the well-posedness of $(3.4a) - (3.4e)$ and $(3.6a) - (3.6a)$. For $T > 0$, we first show the existence and uniqueness of the classical solutions for $\beta(z) \in C^1[0,T]$, then we show that the solution depends stably on $\beta(z) \in C^1[0,T]$. Lastly, we show the existence and uniqueness of the weak solution for $\beta(z) \in L^2[0,T]$.

3.1 Existence and Uniqueness of the Classical Solution

In this section we study the existence and uniqueness of the classical solutions of $(3.4a) - (3.4e)$ and $(3.6a) - (3.6a)$. We work on a more generalized CBVP that has jumps in the middle.

For $T > 0$, define (see Figure 3.2)

$$\begin{aligned} D_1 &:= \{(z,t) \mid 0 \leq z \leq ct, z + t \leq 2T\}, \\ D_2 &:= \{(z,t) \mid ct \leq z \leq t, z + t \leq 2T\}, \\ D &:= D_1 \cup D_2. \end{aligned}$$

The existence and uniqueness of classical solutions for $(3.4a) - (3.4e)$ and $(3.6a) - (3.6e)$

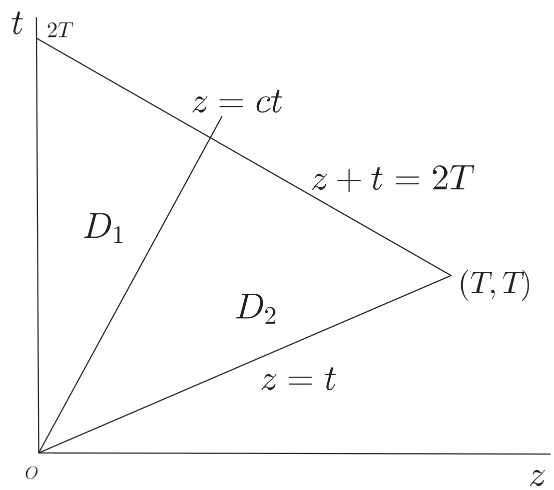

Figure 3.2: D_1 and D_2

will follow from the existence and uniqueness for the more general CBVP

$$\mathcal{L}\mathbf{f} = \mathbf{0} \text{ in } D_2, \tag{3.7a}$$

$$\mathcal{L}\mathbf{g} = \mathbf{0} \text{ in } D_1, \tag{3.7b}$$

$$(g_i - f_i)(ct, t) = h_i(t), \quad i = 1, 2, 3, \quad t \in \left[0, \frac{2T}{1+c}\right], \tag{3.7c}$$

$$f_1(t, t) = a(t), \quad f_3(t, t) = b(t), \quad f_4(t, t) = j(t), \quad t \in [0, T], \tag{3.7d}$$

$$g_2(0, t) = d(t), \quad g_4(0, t) = e(t), \quad t \in [0, 2T]. \tag{3.7e}$$

Proposition 3.1. *If $h_i(t) \in C^1[0, \frac{2T}{1+c}]$ for $i = 1, 2, 3$, $a(t), b(t), j(t) \in C^1[0, T]$, $d(t), e(t) \in C^1[0, 2T]$, and $\beta(z) \in C^1[0, T]$, then there exists a unique solution $\mathbf{f} \in C^1(D_2), \mathbf{g} \in C^1(D_1)$ of (3.7a)−(3.7e), and $|\mathbf{f}(\cdot)|, |\mathbf{g}(\cdot)|$ are bounded above by a function of c, T, N where*

$$N = \max\Big(\sup_{t\in[0,2T/(1+c)]} |h_1(t)|, \sup_{t\in[0,2T/(1+c)]} |h_2(t)|, \sup_{t\in[0,2T/(1+c)]} |h_3(t)|,$$
$$\sup_{t\in[0,T]} |a(t)|, \sup_{t\in[0,T]} |b(t)|, \sup_{t\in[0,T]} |j(t)|, \sup_{t\in[0,2T]} |d(t)|, \sup_{t\in[0,2T]} |e(t)|, \sup_{z\in[0,T]} |\beta(z)|\Big).$$

In addition, the solution of (3.7*a*) − (3.7*e*) *is in* $C^1(D)$ *if*

$$h_i(t) = 0, \quad t \in [0, 2T/(1+c)], \quad i = 1, 2, 3, \tag{3.8a}$$

$$e(0) = j(0), \quad (1-c)e'(0) = \frac{\beta(0)}{2}((1-c)a(0) - (1+c)d(0)) - cj'(0). \tag{3.8b}$$

Remark: We make an observation which we will use in the rest of the thesis. If $\beta \in C^1[0, T]$ such that $\beta(0) = \beta'(0) = 0$ then the compatibility conditions (3.8*a*)−(3.8*b*) for (3.4*a*) − (3.4*e*) are satisfied, and the solution is in $C^1(D)$, so (3.4*a*) − (3.4*e*) can be written as the following CBVP

$$\mathcal{L}\mathbf{m} = \mathbf{0} \text{ in } D, \tag{3.9a}$$

$$m_2(0, t) = m_4(0, t) = 0, \quad t \geq 0, \tag{3.9b}$$

$$m_1(z, z) = 0, \quad m_3(z, z) = \frac{c-1}{2(c+1)}\beta(z), \quad m_4(z, z) = \frac{c+1}{2(c-1)}\beta(z), \quad z \geq 0. \tag{3.9c}$$

The proof of Proposition 3.1 requires the existence and uniqueness for a system of Volterra-like integral equations described next.

Suppose $\zeta_l, \zeta_h, \tau_l, \tau_h \in \mathbb{R}$ such that $\zeta_l \leq \zeta_h$ and $\tau_l \leq \tau_h$; here l, h are mnemonics for low and high. Let $\alpha(z), \gamma(z) \in C^0[\zeta_l, \zeta_h]$ and $\xi(t), \eta(t) \in C^0[\tau_l, \tau_h]$ such that $\alpha(z) \leq \gamma(z)$ for $z \in [\zeta_l, \zeta_h]$ and $\xi(t) \leq \eta(t)$ for $t \leq [\tau_l, \tau_h]$. Suppose $\tilde{D} \in \mathbb{R}^2$ is a convex domain of the form I

$$\tilde{D} := \{(z, t) \mid \zeta_l \leq z \leq \zeta_h, \ \alpha(z) \leq t \leq \gamma(z)\}$$

or the form II

$$\tilde{D} := \{(z, t) \mid \tau_l \leq t \leq \tau_h, \ \xi(t) \leq z \leq \eta(t)\}$$

and l is a line cutting $\tilde{D}$ into two parts $\tilde{D}_1$ and $\tilde{D}_2$ with non-empty interiors. Note $\tilde{D}_1$ and $\tilde{D}_2$ will be convex (see Figure 3.3). For an arbitrary function $g(z, t) \in C^1(\tilde{D}_1) \cup C^2(\tilde{D}_2)$, define the C^1 norm

$$||g||_1 := \sup_{(z,t)\in\tilde{D}} |g(z, t)| + \sup_{(z,t)\in\tilde{D}} |g_t(z, t)| + \sup_{(z,t)\in\tilde{D}} |g_z(z, t)|,$$

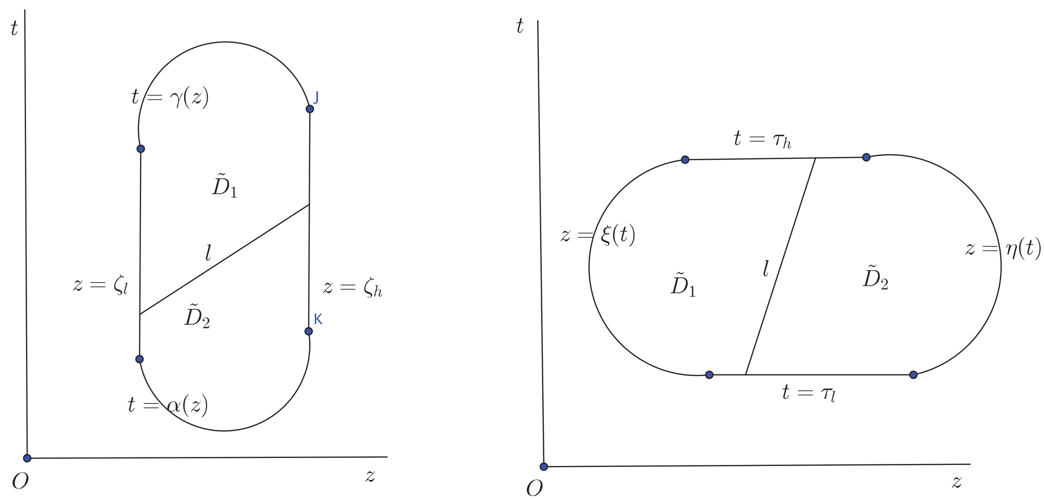

Figure 3.3: Two kinds of regions for $\tilde{D}$

and for any vector function $\mathbf{v}(z,t) \in C^1(\tilde{D}_2) \cup C^1(\tilde{D}_2)$ of dimension n, define

$$||\mathbf{v}||_1 := \sum_{i=1}^{n} ||v_i||_1.$$

Theorem 3.1. *Suppose $\tilde{D}$ is of form I and n an integer. If $E = (e_{ij}(z,t;s))$ is an $n \times n$ matrix with each $e_{ij}(z,t;s) \in C^1(\tilde{D} \times \mathbb{R})$, and $\mathbf{f}(z,t), \mathbf{a}(z,t), \mathbf{b}(z,t) \in C^1(\tilde{D}_1) \cup C^1(\tilde{D}_2)$, $\mathbf{d}(z,t;s) \in C^1(\tilde{D} \times \mathbb{R})$ are vector functions of dimension n, then the following system of Volterra-like integral equations*

$$v_i(z,t) = \int_{a_i(z,t)}^{b_i(z,t)} \sum_{j=1}^{n} e_{ij}(z,t;s) v_j(d_i(z,t;s),s)\ ds + f_i(z,t), \quad (z,t) \in \tilde{D}, \quad i = 1,...,n \tag{3.10}$$

has a unique solution $\mathbf{v}(z,t) \in C^1(\tilde{D}_1) \cup C^1(\tilde{D}_2)$, and $||\mathbf{v}||_1$ is bounded above by a function of $n, N, \tau_h - \tau_l$ where

$$N = \max\left(\sup_{1\le i,j\le n} ||e_{ij}||_1, \sup_{1\le i\le n} ||f_i||_1, \sup_{1\le i\le n} ||a_i||_1, \sup_{1\le i\le n} ||b_i||_1, \sup_{1\le i\le n} ||d_i||_1, 1\right)$$

if

$$(d_i(z,t;s),s) \in \tilde{D} \quad \textit{for all} \quad a_i(z,t) \le s \le b_i(z,t), \quad i=1,...,n, \quad (z,t) \in \tilde{D}$$

and one of the following two conditions hold

- $\tau_l \le a_i(z,t) \le b_i(z,t) \le t, \quad i=1,...,n, \quad (z,t) \in \tilde{D}$;
- $t \le a_i(z,t) \le b_i(z,t) \le \tau_h, \quad i=1,...,n, \quad (z,t) \in \tilde{D}$.

Also if $\tilde{D}$ is of the form II, the following system of Volterra-like integral equations

$$v_i(z,t) = \int_{a_i(z,t)}^{b_i(z,t)} \sum_{j=1}^{n} e_{ij}(z,t;s) v_j(s, d_i(z,t;s)) \, ds + f_i(z,t), \quad (z,t) \in \tilde{D}, \quad i=1,...,n \tag{3.11}$$

has a unique solution $\mathbf{v}(z,t) \in C^1(\tilde{D}_1) \cup C^1(\tilde{D}_2)$, and $||\mathbf{v}||_1$ is bounded above by a function of $n, N, \zeta_h - \zeta_l$ if

$$(s, d_i(z,t;s)) \in \tilde{D} \quad \textit{for all} \quad a_i(z,t) \le s \le b_i(z,t), \quad i=1,...,n, \quad (z,t) \in \tilde{D}$$

and one of the following two conditions hold

- $\zeta_l \le a_i(z,t) \le b_i(z,t) \le z, \quad i=1,...,n, \quad (z,t) \in \tilde{D}$;
- $z \le a_i(z,t) \le b_i(z,t) \le \zeta_h, \quad i=1,...,n, \quad (z,t) \in \tilde{D}$.

Proof. We prove Theorem 3.1 when $\tilde{D}$ has form I and $\tau_l \le a_i(z,t) \le b_i(z,t) \le t$, since the proofs for the other three cases are similar. Define a linear operator $\mathbb{A}$

$$\mathbb{A} : C^1(\tilde{D}_1) \cup C^1(\tilde{D}_2) \to C^1(\tilde{D}_1) \cup C^1(\tilde{D}_2),$$
$$(\mathbb{A}\mathbf{v})_i(z,t) = \int_{a_i(z,t)}^{b_i(z,t)} \sum_{j=1}^{n} e_{ij}(z,t;s) v_j(d_i(z,t;s), s) \, ds, \quad i=1,...,n$$

for any n-dimensional vector function $\mathbf{v}(z,t) \in C^1(\tilde{D}_1) \cup C^1(\tilde{D}_2)$, then solving the integral equation (3.10) corresponds to finding a fixed point of the map $\mathbf{v} \mapsto \mathbb{A}\mathbf{v} + \mathbf{f}$ in $C^1(\tilde{D}_1) \cup C^1(\tilde{D}_2)$.

Clearly, $(\mathbb{A}\mathbf{v}) \in C^1(\tilde{D}_1) \cup C^1(\tilde{D}_2)$, and for all $(z,t) \in \tilde{D}$ and $i = 1, ..., n$

$$
\begin{aligned}
|(\mathbb{A}\mathbf{v})_i(z,t)| &\leq \sum_{j=1}^{n} \int_{\tau_l}^{\tau_h} ||e_{ij}||_1 \cdot ||v_j||_1 \; ds \leq N(\tau_h - \tau_l)||\mathbf{v}||_1, \\
|(\mathbb{A}\mathbf{v})_{iz}(z,t)| &\leq \sum_{j=1}^{n} \left(\int_{\tau_l}^{\tau_h} ||e_{ij}||_1 \cdot ||v_j||_1 \cdot (1 + ||d_i||_1) \; ds + ||e_{ij}||_1 \cdot ||v_j||_1 \cdot (||a_i||_1 + ||b_i||_1) \right) \\
&\leq 2N^2(\tau_h - \tau_l + 1)||\mathbf{v}||_1, \\
|(\mathbb{A}\mathbf{v})_{it}(z,t)| &\leq \sum_{j=1}^{n} \left(\int_{\tau_l}^{\tau_h} ||e_{ij}||_1 \cdot ||v_j||_1 \cdot (1 + ||d_i||_1) \; ds + ||e_{ij}||_1 \cdot ||v_j||_1 \cdot (||a_i||_1 + ||b_i||_1) \right) \\
&\leq 2N^2(\tau_h - \tau_l + 1)||\mathbf{v}||_1,
\end{aligned}
$$

so

$$||(\mathbb{A}\mathbf{v})_i||_1 \leq 5N^2(\tau_h - \tau_l + 1)||\mathbf{v}||_1, \quad i = 1, ..., n,$$

and hence $\mathbb{A}$ is bounded. We construct a sequence $\mathbf{v}^j(z,t)$ with

$$\mathbf{v}^0 = \mathbf{f}, \quad \mathbf{v}^{j+1} = \mathbb{A}\mathbf{v}^j + \mathbf{f} \tag{3.12}$$

and the limit of $\mathbf{v}^j$ will be the fixed point we seek because of the continuity of $\mathbb{A}$.

From (3.12), we have

$$||v_i^0||_1 \leq N, \quad i = 1, ..., n.$$

We show that the sequence $\{\mathbf{v}^j\}_{j=0}^{\infty}$ converges in $C^1(\tilde{D}_1) \cup C^1(\tilde{D}_2)$ by proving that the series $\sum_{j=1}^{\infty} ||\mathbf{v}^j - \mathbf{v}^{j-1}||_1$ converges uniformly in $\tilde{D}$. First, for $i = 1, ..., n$, we have

$$
\begin{aligned}
|v_i^1(z,t) - v_i^0(z,t)| &= \sum_{i=1}^{n} \int_{a_i(z,t)}^{b_i(z,t)} |e_{ij}(z,t;s) v_j^0(d_i(z,t;s),s)| \; ds \\
&\leq \sum_{i=1}^{n} \int_{\tau_l}^{t} N^2 \; ds \leq nN^2(t - \tau_l), \quad (z,t) \in \tilde{D}.
\end{aligned}
$$

Given that for some $k \geq 1$,

$$|v_i^k(z,t) - v_i^{k-1}(z,t)| \leq \frac{N(nN(t-\tau_l))^k}{k!}, \quad (z,t) \in \tilde{D}, \quad i = 1, 2, ..., n, \tag{3.13}$$

we have

$$\begin{aligned}
|v_i^{k+1}(z,t) - v_i^k(z,t)| &= \sum_{j=1}^{n} \int_{a_i(z,t)}^{b_i(z,t)} |e_{ij}(z,t;s)||v_j^k(d_i(z,t;s),s) - v_j^{k-1}(d_i(z,t;s),s)|\ ds \\
&\leq \sum_{i=1}^{n} \int_{\tau_l}^{t} N \frac{n^k N^{k+1}(s-\tau_l)^k}{k!}\ ds \\
&= \frac{N(nN(t-\tau_l))^{k+1}}{(k+1)!}, \quad (z,t) \in \tilde{D}.
\end{aligned}$$

Hence by induction, (3.13) holds for all $k = 1, 2, ...$, implying

$$|\mathbf{v}^k(z,t) - \mathbf{v}^{k-1}(z,t)| = \sum_{i=1}^{n} |v_i^k(z,t) - v_i^{k-1}(z,t)| \leq nN \frac{(nN(t-\tau_l))^k}{k!}, \quad (z,t) \in \tilde{D}$$

and hence

$$\sum_{j=1}^{\infty} |\mathbf{v}^j(z,t) - \mathbf{v}^{j-1}(z,t)| \leq nN \sum_{j=1}^{\infty} \frac{(nN(t-\tau_l))^j}{j!} \leq nNe^{nN(\tau_h - \tau_l)}, \quad (z,t) \in \tilde{D}.$$

Now for $\mathbf{v}_z$, we have

$$\begin{aligned}
&|v_{iz}^1 - v_{iz}^0|(z,t) \\
\leq& |b_{iz}(z,t) \sum_{j=1}^{n} e_{ij}(z,t;b_i) f_j(d_i(z,t;b_i),b_i)| + |a_{iz}(z,t) \sum_{j=1}^{n} e_{ij}(z,t;a_i) f_j(d_i(z,t;a_i),a_i)| \\
&+ \int_{a_i(z,t)}^{b_i(z,t)} \sum_{j=1}^{n} |e_{ijz}(z,t;s) f_j(d_i(z,t;s),s) + e_{ij}(z,t;s) d_{iz}(z,t) f_{jz}(d_i(z,t;s),s)|\ ds \\
\leq& 2nN^3 + n(N^2 + N^3)(t-\tau_l) \\
\leq& 2nN^3(1 + (t-\tau_l)), \quad (z,t) \in \tilde{D}, \quad i = 1, 2, ..., n.
\end{aligned}$$

We show by induction that for all $(z,t) \in \tilde{D}$ and $i = 1, ..., n$

$$|v_{iz}^k - v_{iz}^{k-1}|(z,t) \leq 2kn^k N^{2k+1} \left(\frac{(t-\tau_l)^{k-1}}{(k-1)!} + \frac{(t-\tau_l)^k}{k!} \right). \tag{3.14}$$

(3.14) is true for $k = 1$. Now assume that (3.14) is true for some $k > 1$, then from (3.13)

$$
\begin{aligned}
&|v_{iz}^{k+1} - v_{iz}^{k}|(z,t) \\
&\quad \leq |b_{iz}(z,t) \sum_{j=1}^{n} e_{ij}(z,t;b_i)(v_j^k - v_j^{k-1})(d_i(z,t;b_i),b_i)| \\
&\qquad + |a_{iz}(z,t) \sum_{j=1}^{n} e_{ij}(z,t;a_i)(v_j^k - v_j^{k-1})(d_i(z,t;a_i),a_i)| \\
&\qquad + \int_{a_i(z,t)}^{b_i(z,t)} \sum_{j=1}^{n} |e_{ijz}(z,t;s)(v_j^k - v_j^{k-1})(d_i(z,t;s),s) \\
&\qquad\qquad\qquad + e_{ij}(z,t;s)d_{iz}(z,t)(v_{jz}^k - v_{jz}^{k-1})(d_i(z,t;s),s)|\, ds \\
&\quad \leq 2nN^2 \frac{N(nN(t-\tau_l))^k}{k!} + N\frac{(nN(t-\tau_l))^{k+1}}{(k+1)!} + 2kn^{k+1}N^{2k+3}\left(\frac{(t-\tau_l)^{k-1}}{(k-1)!} + \frac{(t-\tau_l)^k}{k!}\right) \\
&\quad \leq 2(k+1)n^k N^{2k+1}\left(\frac{(t-\tau_l)^{k-1}}{(k-1)!} + \frac{(t-\tau_l)^k}{k!}\right).
\end{aligned}
$$

Hence by induction, (3.14) holds for all $k = 1, 2, ...$, implying

$$
\begin{aligned}
|\mathbf{v}_z^k(z,t) - \mathbf{v}_z^{k-1}(z,t)| &= \sum_{i=1}^{n} |v_{iz}^k(z,t) - v_{iz}^{k-1}(z,t)| \\
&\leq 2kn^{k+1}N^{2k+1}\left(\frac{(t-\tau_l)^{k-1}}{(k-1)!} + \frac{(t-\tau_l)^k}{k!}\right), \quad (z,t) \in \tilde{D}
\end{aligned}
$$

and hence $\sum_{j=1}^{\infty} |\mathbf{v}_z^j(z,t) - \mathbf{v}_z^{j-1}(z,t)|$ is convergent and bounded above by a function of $n, N, \tau_h - \tau_l$. Similarly, one can show by induction that $\sum_{j=1}^{\infty} |\mathbf{v}_t^j(z,t) - \mathbf{v}_t^{j-1}(z,t)|$ is convergent and bounded above by a function of $n, N, \tau_h - \tau_l$, which implies that $\sum_{j=1}^{\infty} ||\mathbf{v}^j - \mathbf{v}^{j-1}||_1$ is convergent and bounded above by a function of $n, N, \tau_h - \tau_l$. So the sequence $\{\mathbf{v}^j\}_{j=0}^{\infty}$ has a limit $\mathbf{v} \in C^1(\tilde{D}_1) \cup C^1(\tilde{D}_2)$, and $||\mathbf{v}||_1$ is bounded above by a function of $n, N, \tau_h - \tau_l$.

Now we show the uniqueness of the solution of (3.10). If $\mathbf{v}$ and $\tilde{\mathbf{v}} \in C^1(\tilde{D}_1) \cup C^1(\tilde{D}_2)$ are the solutions of (3.10), and we define $\mathbf{u} := \mathbf{v} - \tilde{\mathbf{v}}$, then $\mathbf{u}$ satisfies

$$u_i(z,t) = \int_{a_i(z,t)}^{b_i(z,t)} \sum_{j=1}^{n} e_{ij}(z,t;s) u_j(d_i(z,t;s),s)\ ds, \quad i = 1, ..., n. \tag{3.15}$$

From (3.15), for $i = 1, ..., n$ and $(z,t) \in \tilde{D}$

$$|u_i(z,t)| \le \int_{a_i(z,t)}^{b_i(z,t)} \sum_{j=1}^{n} |e_{ij}(z,t;s) u_j(d_i(z,t;s),s)|\ ds \le N \int_{\tau_l}^{t} \sum_{j=1}^{n} |u_j(d_i(z,t;s),s)|\ ds,$$

which implies that for all $(z,t) \in \tilde{D}$

$$\sum_{i=1}^{n} |u_i(z,t)| \le N \sum_{i=1}^{n} \int_{\tau_l}^{t} \sum_{j=1}^{n} |u_j(d_i(z,t;s),s)|\ ds = Nn \int_{\tau_l}^{t} \sum_{j=1}^{n} |u_j(d_i(z,t;s),s)|\ ds. \tag{3.16}$$

Define $U(t) := \max\limits_{z \in [\xi(t), \eta(t)]} \sum_{i=1}^{n} |u_i(z,t)|$ for $t \in [\tau_l, \tau_h]$; noting that $(d_i(z,t,s),s) \in \tilde{D}$, from (3.16) we have

$$U(t) \le Nn \int_{\tau_l}^{t} U(s)\ ds, \quad t \in [\tau_l, \tau_h]$$

and hence from the Gronwall's inequality

$$U(t) \le 0, \quad t \in [\tau_l, \tau_h],$$

so that $\mathbf{u}(z,t) = 0$ for all $(z,t) \in \tilde{D}$, and the uniqueness is proved. $\square$

Proof of Proposition 3.1:

Proof. Define $\mathbf{r}(\mathbf{v}, z, t) := \beta(z) B \mathbf{v}(z,t)$ and pick an arbitrary point $P(z,t) \in D$. Integrating (3.7*a*) – (3.7*b*) along the characteristics and using the boundary conditions, we have (see Figure 3.4)

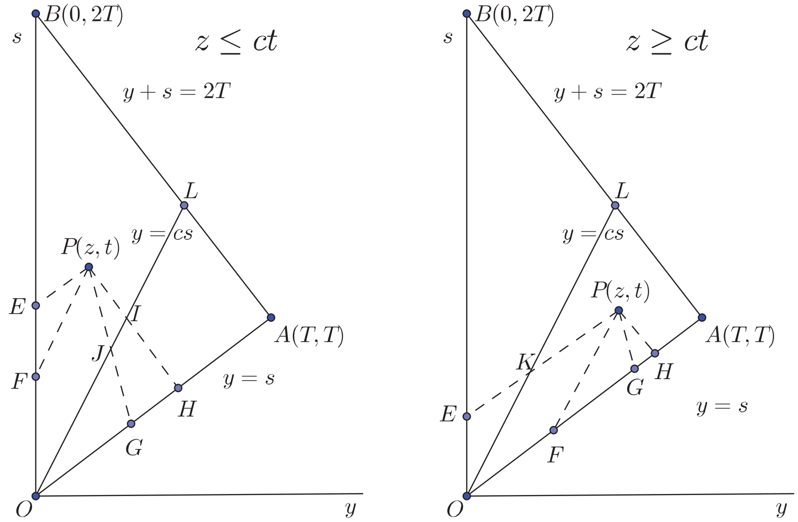

Figure 3.4: Downward moving lines through $P(z,t)$ with slopes ± 1 and $\pm 1/c$

$$v_1(z,t) = \begin{cases} \int_{s_H}^{t} r_1(\mathbf{v}, z+t-s, s)\ ds + a(s_H) + h_1(s_I) & \text{if } P \in D_1 \\ \int_{s_H}^{t} r_1(\mathbf{v}, z+t-s, s)\ ds + a(s_H) & \text{if } P \in D_2 \end{cases} \tag{3.17a}$$

$$v_2(z,t) = \begin{cases} \int_{s_E}^{t} r_2(\mathbf{v}, z+s-t, s)\ ds + d(s_E) & \text{if } P \in D_1 \\ \int_{s_E}^{t} r_2(\mathbf{v}, z+s-t, s)\ ds + d(s_E) - h_2(s_K) & \text{if } P \in D_2 \end{cases} \tag{3.17b}$$

$$v_3(z,t) = \begin{cases} \int_{s_G}^{t} r_3(\mathbf{v}, z+ct-cs, s)\ ds + b(s_G) + h_3(s_J) & \text{if } P \in D_1 \\ \int_{s_G}^{t} r_3(\mathbf{v}, z+ct-cs, s)\ ds + b(s_G) & \text{if } P \in D_2 \end{cases} \tag{3.17c}$$

$$v_4(z,t) = \begin{cases} \int_{s_F}^{t} r_4(\mathbf{v}, z+cs-ct, s)\ ds + e(s_F) & \text{if } P \in D_1 \\ \int_{s_F}^{t} r_4(\mathbf{v}, z+cs-ct, s)\ ds + j(s_F) & \text{if } P \in D_2. \end{cases} \tag{3.17d}$$

One may verify from Figure 3.4 that $0 \le s_E, s_F, s_G, s_H, s_I, s_J, s_K \le t$, so from Theorem 3.1, $(3.17a) - (3.17d)$ has a unique solution $\mathbf{v}(z,t) \in C^1(D_1) \cup C^1(D_2)$, which is the unique solution of $(3.7a) - (3.7e)$.

Clearly, v_1, v_2, v_3 are C^0 across $z = ct$ if $h_i = 0$ for $i = 1, 2, 3$. Since

$$s_F = \begin{cases} \frac{ct-z}{c} & (z,t) \in D_1 \\ \frac{z-ct}{1-c} & (z,t) \in D_2, \end{cases} \tag{3.18}$$

then from $(3.17d)$, v_4 is continuous across $z = ct$ if $e(0) = j(0)$, which means that $\mathbf{v} \in C^0(D)$. Now since $\mathbf{v} \in C^0(D)$, using $(3.7a) - (3.7b)$, the derivative of v_1, v_2, v_3 along the lines of slopes $-1, 1, -c$ respectively is continuous across $z = ct$. Since these lines are transversal to $z = ct$ and since the derivatives of v_1, v_2, v_3 along lines of slope c are also continuous across $z = ct, v_1, v_2, v_3$ are in $C^1(\tilde{D})$.

If the compatibility conditions $(3.8a) - (3.8b)$ hold then, on $z = ct$, v_4 is determined by $(3.7a) - (3.7b)$ and the boundary condition $(3.7c) - (3.7e)$. So v_4 is continuous across $z = ct$ and its derivative along $z = ct$ is also continuous across $z = ct$. From $(3.17d)$ and (3.18), and using the fact that $r_4(v, \cdot)$ involves only v_1, v_2, v_3 which are C^1 across $z = ct$, we can show that the derivative of v_4 in a direction transversal to $z = ct$ is continuous across $z = ct$ if the second equation of $(3.8b)$ is satisfied (see a similar derivation in the proof of Proposition 4.3). Hence the z and t derivatives of v_4 are C^1 across $z = ct$. $\square$

3.2 Stability

In this section we study the stability of the solution of $(3.9a) - (3.9c)$.

For arbitrary $T > 0, K > 0$, define

$$\dot{C}^1[0,T] := \{\beta(z) \in C^1[0,T] \mid \beta(0) = \beta'(0) = 0\},$$
$$\Theta := \{\beta(z) \in \dot{C}^1[0,T] \mid ||\beta(z)||^2_{L^2[0,T]} \leq K\}.$$

Proposition 3.2. (Stability for the forward problem) *If* $\mathbf{m}$ *and* $\tilde{\mathbf{m}}$ *are the solutions of* $(3.9a) - (3.9c)$ *corresponding to* β *and* $\tilde{\beta}$ *in* Θ*, then*

$$\sum_{i=1}^{4} \int_z^{2T-z} (m_i - \tilde{m}_i)^2(z,t)\, dt \preceq ||\beta - \tilde{\beta}||^2_{L^2[0,T]}, \quad 0 \leq z \leq T$$

with the constant dependent only on c, T and K.

Define

$$J(\mathbf{m}, z) := \int_z^{2T-z} (m_1^2 + m_2^2 + cm_3^2 + cm_4^2)(z,t)\ dt, \quad z \in [0, T].$$

The proof will require an upper bound on $J(\mathbf{m}, z)$ and that is what we obtain first.

Proposition 3.3. *If* $\mathbf{m}$ *is the solution of* (3.9a) − (3.9c) *corresponding to* $\beta \in \Theta$*, then*

$$J(\mathbf{m}, z) \leq C_0, \quad z \in [0, T], \tag{3.19}$$

where C_0 *is a constant dependent only on* c, T, K.

Proof. Since $\mathbf{m}$ is the solution of (3.9a) − (3.9c) and $B = -B^T$, we have

$$(m_1^2 + m_2^2 + m_3^2 + m_4^2)_t - (m_1^2 - m_2^2 + cm_3^2 - cm_4^2)_z = 0. \tag{3.20}$$

Integrating (3.20) over OAB (see Figure 3.5), we obtain

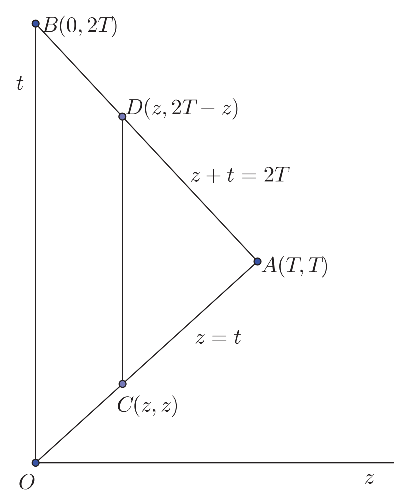

Figure 3.5: Energy function

$$C \int_{OA} \beta^2(z)\ dz = \int_{AB} 2m_2^2 + (1-c)m_3^2 + (1+c)m_4^2\ dt + \int_{OB} m_1^2 + cm_3^2\ dt, \tag{3.21}$$

where $C = \frac{(1+c)^2}{4(1-c)} + \frac{(1-c)^2}{4(1+c)}$, then from (3.21)

$$J(\mathbf{m}, 0) \preceq K, \quad \int_{AB} 2m_2^2 + (1-c)m_3^2 + (1+c)m_4^2 \; dt \preceq K, \tag{3.22}$$

with the constant dependent only on c.

Now if we multiply both sides of (3.9a) by $\mathbf{q}^T = [-m_1, \; m_2, \; -m_3, \; m_4]$, we have

$$(-m_1^2 + m_2^2 - m_3^2 + m_4^2)_t + (m_1^2 + m_2^2 + cm_3^2 + cm_4^2)_z = (1-c)\beta(m_1 m_4 + m_2 m_3) \tag{3.23}$$

Integrating the LHS (3.23) over $OCDB$, we have

$$\begin{aligned}
&\iint_{OCDB} \text{LHS of (3.23)} \; dA \\
&\qquad = k \int_{OC} \beta^2(z) \; dz - \int_{DB} 2m_2^2 - (1-c)m_3^2 + (1+c)m_4^2 \; dt - J(\mathbf{m}, z) + J(\mathbf{m}, 0),
\end{aligned}$$

where $k = \frac{(1+c)^2}{4(1-c)} - \frac{(1-c)^2}{4(1+c)}$, so from (3.22) – (3.23)

$$\begin{aligned}
J(\mathbf{m}, z) &\preceq K + \iint_{OCDB} |\beta(z)| \sum_{i=1}^{4} m_i^2 \; dA \\
&\preceq K + \int_0^z |\beta(y)| J(\mathbf{m}, y) \; dy, \quad z \in [0, T],
\end{aligned}$$

with the constant dependent only on c, and hence from Gronwall's inequality

$$J(\mathbf{m}, z) \preceq K e^{\int_0^z |\beta(y)| \; dy} \leq K e^{\sqrt{KT}},$$

and the proof is completed. □

Proof of Proposition 3.2:

Proof. Define $\mathbf{s} := \mathbf{m} - \tilde{\mathbf{m}}$ where $\mathbf{m}$ and $\tilde{\mathbf{m}}$ are the solutions of (3.9a) – (3.9c) corresponding to β and $\tilde{\beta}$ in Θ, then $s_2 = s_4 = 0$ on $z = 0$, and

$$\begin{aligned}
\mathbf{s}_t - A s_z - \beta B \mathbf{s} &= (\mathbf{m}_t - A\mathbf{m}_z - \beta B \mathbf{m}) - (\tilde{\mathbf{m}}_t - A\tilde{\mathbf{m}}_z - \tilde{\beta} B \tilde{\mathbf{m}}) + (\beta - \tilde{\beta}) B \tilde{\mathbf{m}} \\
&= (\beta - \tilde{\beta}) B \tilde{\mathbf{m}}.
\end{aligned} \tag{3.24}$$

If we multiply $2\mathbf{s}^T$ to both sides of (3.24), we have

$$(s_1^2 + s_2^2 + s_3^2 + s_4^2)_t - (s_1^2 - s_2^2 + cs_3^2 - cs_4^2)_z = 2\mathbf{s}^T(\beta - \tilde{\beta})B\tilde{\mathbf{m}} \tag{3.25}$$

Integrating (3.25) over OAB, from (3.19) we have

$$\begin{aligned}
&-C\int_{OA}(\beta-\tilde{\beta})^2(z)\ dz + \int_{AB} 2s_2^2 + (1-c)s_3^2 + (1+c)s_4^2\ dt + \int_{OB} s_1^2 + cs_3^2\ dt \\
\leq &2\iint_D |\mathbf{s}^T(\beta-\tilde{\beta})B\tilde{\mathbf{m}}|\ dA \leq \epsilon\int_0^T J(\mathbf{s},z)\ dz + \frac{4C_0}{\epsilon}\int_0^T(\beta-\tilde{\beta})^2(z)\ dz
\end{aligned}$$

for any $\epsilon > 0$, and so

$$\begin{aligned}
&\int_{AB} 2s_2^2 + (1-c)s_3^2 + (1+c)s_4^2\ dt + \int_{OB} s_1^2 + cs_3^2\ dt \\
&\qquad \leq a\int_0^T(\beta-\tilde{\beta})^2(z)\ dz + \epsilon\int_0^T J(\mathbf{s},z)\ dz,
\end{aligned} \tag{3.26}$$

where a is a constant dependent only on c, T, K, ϵ.

Now define $\mathbf{p}^T := [s_1, -s_2, s_3, -s_4]$ and multiplying $2\mathbf{p}^T$ to both sides of (3.24), we have

$$(s_1^2 - s_2^2 + s_3^2 - s_4^2)_t - (s_1^2 + s_2^2 + cs_3^2 + cs_4^2)_z = 2\mathbf{p}^T(\beta-\tilde{\beta})B\tilde{\mathbf{m}} - (1-c)\beta(s_1s_4 + s_2s_3). \tag{3.27}$$

Integrating (3.27) over $OCDB$, we have

$$\begin{aligned}
&-k\int_{OC}(\beta-\tilde{\beta})^2(y)\ dy + J(\mathbf{s},z) - J(\mathbf{s},0) + \int_{DB} 2s_2^2 - (1-c)s_3^2 + (1+c)s_4^2\ dt \\
\leq &\epsilon\int_0^T J(\mathbf{s},z)\ dz + \frac{4C_0}{\epsilon}\int_0^T(\beta-\tilde{\beta})^2(z)\ dz + \int_0^z |\beta(z)|J(\mathbf{s},y)\ dy, \quad z\in[0,T].
\end{aligned} \tag{3.28}$$

Define $J^* := \max_{z\in[0,T]} J(\mathbf{s},z)$, then from (3.26) and (3.28) we have

$$J^* \preceq 2a\int_0^T(\beta-\tilde{\beta})^2(z)\ dz + 2\epsilon\int_0^T J(\mathbf{s},z)\ dz + \int_0^z |\beta(z)|J(\mathbf{s},y)\ dy,$$

which implies that

$$(1-2\epsilon T)J^* \leq 2a\int_0^T(\beta-\tilde{\beta})^2(z)\ dz + \int_0^z |\beta(z)|J(\mathbf{s},y)\ dy. \tag{3.29}$$

Choose $\epsilon = \frac{1}{4T}$, then from (3.29) we have

$$J(\mathbf{s}, z) \preceq \int_0^T (\beta - \tilde{\beta})^2(z)\ dz + \int_0^z |\beta(z)| J(\mathbf{s}, y)\ dy, \quad z \in [0, T],$$

with the constant dependent only on c, T, K, hence from Gronwall's inequality

$$J(\mathbf{s}, z) \preceq \int_0^T (\beta - \tilde{\beta})^2(z)\ dz, \quad z \in [0, T],$$

with the constant dependent only on c, T, K. □

3.3 Existence and Uniqueness of the Weak Solution

In this section we study the existence and uniqueness of the weak solution of $(3.9a) - (3.9c)$.

Recall that

$$\begin{aligned} D_1 &:= \{(z,t) \mid 0 \leq z \leq ct, z + t \leq 2T\}, \\ D_2 &:= \{(z,t) \mid ct \leq z \leq t, z + t \leq 2T\}, \\ D &:= D_1 \cup D_2. \end{aligned}$$

Given $\beta(z) \in \dot{C}^1[0,T]$, let $\mathbf{m}(z,t) \in C^1(D)$ be the solution of $(3.9a) - (3.9c)$. We may define the forward map

$$\begin{aligned} F : \dot{C}^1[0,T] &\to C^1(D), \\ \beta(z) &\mapsto \mathbf{m}(z,t). \end{aligned}$$

Proposition 3.4. *F has a well-defined extension as a map from $L^2[0,T]$ to $L^2(D)$.*

Proof. Fix $R > 0$. For any $\beta \in L^2[0,T]$ with $||\beta||^2_{L^2} \leq R$, we can find a sequence $\{\beta_i\} \in \dot{C}^1[0,T]$ with $||\beta_i||^2_{L^2} \leq R + 1$ and $\beta_i \to \beta \in L^2[0,T]$. For each β_i, there is an $\mathbf{m}_i \in C^1(D)$ and from Proposition 3.2 with $K = R + 1$ we have

$$J(\mathbf{m}_i - \mathbf{m}_j, z) \leq C_{R+1} ||\beta_i - \beta_j||^2, \quad z \in [0, T].$$

So $\mathbf{m}_i$ is a Cauchy sequence in $L^2(D)$ and hence has a limit $\mathbf{m} \in L^2(D)$, and we define the extension

$$\begin{aligned} \tilde{F} : L^2[0,T] &\to L^2(D), \\ \beta(z) &\mapsto \mathbf{m}(z,t). \end{aligned}$$

Further if $\{\beta_i\}_{i=1}^{\infty}, \{\tilde{\beta}_i\}_{i=1}^{\infty}$ in $\dot{C}^1[0,T]$ are two sequences that converge to β in the L^2 norm, and $\mathbf{m}_i, \tilde{\mathbf{m}}_i \in L^2(D)$ are the solutions of (3.9a) − (3.9c) corresponding to $\beta_i, \tilde{\beta}_i$, then from Proposition 3.2

$$\begin{aligned} &\lim_{i\to\infty} ||\mathbf{m}_i - \tilde{\mathbf{m}}_i||_{L^2}^2 = \lim_{i\to\infty} \int_0^T J(\mathbf{m}_i - \tilde{\mathbf{m}}_i, z)\ dz \\ \leq &C_{R+1} T \lim_{i\to\infty} ||\beta_i - \tilde{\beta}_i||_{L^2}^2 \leq C_{R+1} T \lim_{n\to\infty} (||\beta_i - \beta||_{L^2}^2 + ||\tilde{\beta}_i - \beta||_{L^2}^2) = 0, \end{aligned}$$

so $\tilde{F}$ is well defined and is a continuous extension of F. □

We now motivate the definition of a weak solution of (3.9a) − (3.9c). Let (see Figure 3.6)

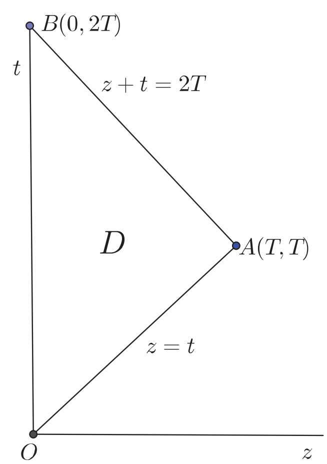

Figure 3.6: Region for the weak solution

$$\Lambda := \{\mathbf{N} \in C^1(D) \mid N_1 = N_3 = 0 \text{ on } OB,\ N_2 = N_3 = N_4 = 0 \text{ on } AB\}.$$

If $\beta(z) \in \dot{C}^1[0,T]$ and $\mathbf{m} \in C^1(D)$ is the solution of (3.9a) $-$ (3.9c) corresponding to β, then for any function $N \in \Lambda$ we have

$$\begin{aligned}0 &= \iint_D \mathbf{N}^T(\mathbf{m}_t - A\mathbf{m}_z - \beta B\mathbf{m})\, dA \\ &= \int_{\partial D} \mathbf{N}^T\mathbf{m}\, dz - \iint_D (\mathbf{N}_t^T - \mathbf{N}_z^T A + \mathbf{N}^T\beta B)\mathbf{m}\, dA - \int_{\partial D} \mathbf{N}^T A\mathbf{m}\, dt,\end{aligned}$$

which implies that

$$\begin{aligned}\iint_D (\mathcal{L}\mathbf{N})^T\mathbf{m}\, dA &= \int_{BA} \mathbf{N}^T(I-A)\mathbf{m}\, dz - \int_{OA} \mathbf{N}^T(I+A)\mathbf{m}\, dz + \int_{OB} \mathbf{N}^T A\mathbf{m}\, dt \\ &= \int_{OA} \mathbf{N}^T\left(\frac{1-c}{2}\beta + \frac{1+c}{2}\beta\right) dz = \int_{OA} \mathbf{N}^T\beta\, dz.\end{aligned}$$

This suggests the following definition.

Definition 3.1. *We say that* $\mathbf{w} \in L^2(D)$ *is a weak solution of* (3.9a) $-$ (3.9c) *corresponding to* $\beta \in L^2[0,T]$ *if*

$$\iint_D (\mathcal{L}\mathbf{N})^T\mathbf{w}\, dA = \int_{OA} \mathbf{N}^T\beta\, dz \tag{3.30}$$

for all $\mathbf{N} \in \Lambda$.

So far, we have shown the existence of the extension $\tilde{F}$; now we show the existence of a weak solution of (3.9a) $-$ (3.9c) by showing that $\tilde{F}(\beta)$ is a weak solution of (3.30).

Proposition 3.5. *If* $\beta(z) \in L^2[0,T]$, *there exists a weak solution of* (3.9a) $-$ (3.9c).

Proof. Since C^1 is dense in L^2, for any $\beta(z) \in L^2[0,T]$, there exists a sequence $\{\beta_i\}_{i=1}^{\infty}$ in $\dot{C}^1[0,T]$ that converges to $\beta(z)$ in the L^2 norm. For each β_i, there exists a unique solution $\mathbf{w}_i(z,t) \in C^1(D)$ of (3.9a)$-$(3.9c). From Proposition 3.2, $\{\mathbf{w}_i\}_{i=1}^{\infty}$ is a Cauchy sequence that converges to $\mathbf{w}$ in $L^2(D)$, which is $\tilde{F}(\beta)$. Due to the derivation of (3.30), (3.30) is satisfied for each β_i and $\mathbf{w}_i$ and fixed $N \in \Lambda$, then take the limit of both sides of (3.30), (3.30) is satisfied for β and $\mathbf{w}$, which implies that $\mathbf{w}$ is a weak solution of (3.9a) $-$ (3.9c) for $\beta \in L^2(D)$. $\square$

Now we show that the weak solution is unique. As a first step we study the existence of a solution of a CBVP.

Proposition 3.6. *Let* $\mathbf{F} \in C^1(D)$ *and* $\beta \in C^1[0,T]$*, then (see Figure 3.7)*

$$\mathcal{L}\mathbf{v} = \mathbf{F} \text{ in } D, \tag{3.31a}$$

$$v_1 = v_3 = 0 \text{ on } OB, \tag{3.31b}$$

$$v_2 = v_3 = v_4 = 0 \text{ on } AB, \tag{3.31c}$$

has a unique solution $\mathbf{v} \in C^1(D)$.

Proof. Define $\mathbf{r}(\mathbf{v}, z, t) := \beta(z)B\mathbf{v}(z,t) + \mathbf{F}(z,t)$, and pick an arbitrary point $P(z,t) \in D$. Integrating (3.31*a*) along the characteristics and using the boundary conditions, we have (see Figure 3.7)

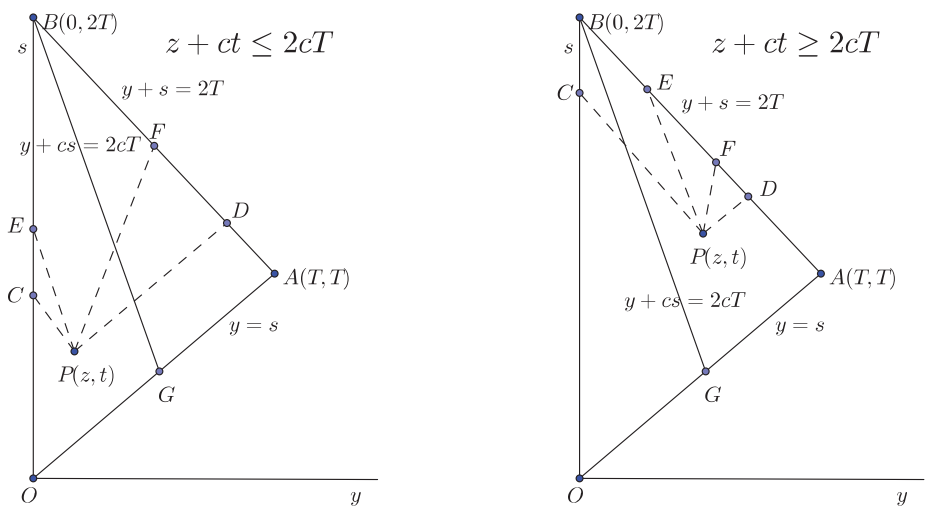

Figure 3.7: Upward moving lines through $P(z,t)$ with slopes ± 1 and $\pm 1/c$

$$v_1(z,t) = -\int_t^{s_C} r_1(\mathbf{v}, z + t - s, s)\, ds, \tag{3.32a}$$

$$v_2(z,t) = -\int_t^{s_D} r_2(\mathbf{v}, z + s - t, s)\, ds, \tag{3.32b}$$

$$v_3(z,t) = -\int_t^{s_E} r_3(\mathbf{v}, z + ct - cs, s)\, ds, \tag{3.32c}$$

$$v_4(z,t) = -\int_t^{s_F} r_4(\mathbf{v}, z + cs - ct, s)\, ds. \tag{3.32d}$$

One may verify from Figure 3.4 that $t \le s_E, s_F, s_G, s_H \le 2T$, so from Theorem 3.1, (3.32a) − (3.32d) has a unique solution $\mathbf{v}(z,t)$ which is C^1 on OBG and ABG. Noting that the compatibility condition is satisfied because of the boundary conditions (3.31b) − (3.31c), then $\mathbf{v} \in C^1(D)$, which is the unique solution of (3.31a) − (3.31c). □

Proposition 3.7. *There is exactly one weak solution of* (3.7a) − (3.7e) *that satisfies* (3.30).

Proof. Suppose $\mathbf{v}, \tilde{\mathbf{v}} \in L^2(D)$ are the weak solutions of (3.7a) − (3.7e). Define $\mathbf{p} := \mathbf{v} - \tilde{\mathbf{v}}$, then from (3.30), we have

$$\iint_D (\mathcal{L}\mathbf{N})^T \mathbf{p}\, dA = 0, \quad \forall\, N \in \Lambda.$$

Since C^1 is dense in L^2, there exists a sequence $\{\mathbf{p}_n\} \in C^1(D)$ such that $p_n \to p$ in $L^2(D)$. Also, from Proposition 3.6, there exists a sequence $\{\mathbf{N}_n\}_{n=1}^{\infty} \in \Lambda$ such that $\pounds N_n = p_n$. Hence we have

$$0 = \lim_{n\to\infty} \iint_D (\mathcal{L}\mathbf{N_n})^T \mathbf{p}\, dA = \lim_{n\to\infty} \iint_D \mathbf{p}_n^T \mathbf{p}\, dA = \iint_D \mathbf{p}^T \mathbf{p}\, dA,$$

which implies that $\mathbf{p} = 0$ in D. □

3.4 Summary of Main Results

In this section we summarize the results obtained. Based on Proposition 3.1-3.7, we have

Theorem 3.2. *If* $\beta \in L^2[0,T]$, *then* (3.4a) − (3.4e) *has a unique weak solution. Furthermore, if* $\beta \in \dot{C}^1[0,T]$, *then* (3.4a) − (3.4e) *has a unique solution in* $C^1(D)$.

Chapter 4

THE RECONSTRUCTION

Given a function $\beta(z) \in C^1[0,\infty)$, let $\mathbf{M}(z,t)$ be the solution of the following IBVP

$$\mathbf{M}_t = A\mathbf{M}_z + \beta B\mathbf{M}, \quad z \geq 0, t \in \mathbb{R}, \tag{4.1a}$$

$$M_2(0,t) = \delta(t), \quad M_4(0,t) = 0, \quad t \geq 0, \tag{4.1b}$$

$$\mathbf{M}(z,t) = \mathbf{0}, \quad t < 0, \ z \geq 0 \tag{4.1c}$$

where

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & -c \end{bmatrix} \qquad B = \frac{1}{2}\begin{bmatrix} 0 & 0 & -1-c & -1+c \\ 0 & 0 & 1-c & 1+c \\ 1+c & -1+c & 0 & 0 \\ 1-c & -1-c & 0 & 0 \end{bmatrix}.$$

In this chapter we study the inverse problem of recovering $\beta(z)$ from $M_1(0,t)$, $M_3(0,t)$. In the process we show that the map $\beta(\cdot) \to (M_1(0,\cdot), M_3(0,\cdot))$ is injective and has a Lipschitz continuous inverse. We obtain necessary conditions for a function to be in the range of this map. Finally, we give an algorithm for the reconstruction of β from $M_1(0,\cdot)$ and $M_3(0,\cdot)$ and we prove that this algorithm will recover β.

For arbitrary $T > 0, K > 0$, define $Y = \frac{2cT}{1+c}$ and (see Figure 4.1)

$$D := OAB = \{(z,t) \mid z \geq 0, \ z \leq t, \ z + t \leq 2T\},$$

$$\tilde{D} := OCB = \{(z,t) \mid 0 \leq z \leq Y, z \leq t \leq 2T - z/c\},$$

$$\dot{C}^1[0,T] := \{\beta(z) \in C^1[0,T] \mid \beta(0) = 0, \beta'(0) = 0\}.$$

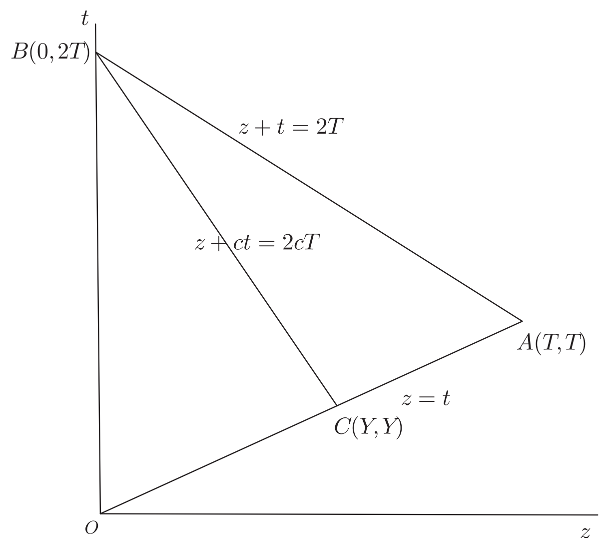

Figure 4.1: D and $\tilde{D}$

4.1 Inverse Stability from Single Reflection Data

We define the forward map $F : \beta \mapsto M_3(0,t)$ that maps $\beta(z)$ to the reflection data $M_3(0,t)$. In this section, we show that F is injective and its inverse is Lipschitz continuous in the *sup* norm. We note that the reflection data $M_3(0,t)$ corresponds to a left moving wave travelling with the slower speed c, where as the source is a right moving travelling with the faster speed 1.

Theorem 4.1. (*Injectivity and Stability*) *If* $\mathbf{M}$ *and* $\tilde{\mathbf{M}}$ *are the solutions of* (4.1*a*) − (4.1*c*) *corresponding to* $\beta, \tilde{\beta} \in \dot{C}^1[0,T]$, *then*

$$|(\beta - \tilde{\beta})(z)| \preceq \max_{[0,2T]} |(M_3 - \tilde{M}_3)(0,\cdot)|, \quad \forall z \in [0,Y]$$

with the constant dependent only on c, T *and* $||\beta||_{L^\infty[0,T]}, ||\tilde{\beta}||_{L^\infty[0,T]}$.

Proof. If $\mathbf{M}$ and $\tilde{\mathbf{M}}$ are the solutions of (4.1*a*) − (4.1*c*) corresponding to β and $\tilde{\beta}$ and $\mathbf{N} := \mathbf{M} - \tilde{\mathbf{M}}$, then $\mathbf{N}$ satisfies $\mathcal{L}\mathbf{N} = (\beta - \tilde{\beta})B\tilde{\mathbf{M}}$. Now $\forall \tau \in [0, 2T]$, define

$$\mathbf{M}^*(z,t) := [\overline{M}_2(z,\tau - t),\ \overline{M}_1(z,\tau - t),\ \overline{M}_4(z,\tau - t),\ \overline{M}_3(z,\tau - t)]^T$$

where $\overline{M}(z,t)$ is the solution of the IBVP

$$\overline{\mathbf{M}}_t = A\overline{\mathbf{M}}_z + \beta B\overline{\mathbf{M}}, \quad (z,t) \in [0,\infty) \times \mathbb{R}, \tag{4.2a}$$

$$\overline{M}_2(0,t) = 0, \quad \overline{M}_4(0,t) = \delta(t), \quad t \in \mathbb{R}, \tag{4.2b}$$

$$\overline{\mathbf{M}}(z,t) = \mathbf{0}, \quad t < 0,\ z \geq 0. \tag{4.2c}$$

One may verify that $\pounds\mathbf{M}^* = \mathbf{0}$ and as seen in (2.13),

$$\mathbf{M}(0,t) = [m_1(t),\ \delta(t),\ m_3(t),\ 0]^T, \tag{4.3}$$

$$\mathbf{M}^*(0,t) = [0,\ m_2^*(t),\ \delta(\tau - t),\ m_4^*(t)]^T \tag{4.4}$$

with m_1, m_3 supported in $t \geq 0$, and m_2^*, m_4^* supported in $(-\infty, \tau]$. For arbitrary vector functions $\mathbf{u}, \mathbf{v} \in C^1([0,\infty) \times \mathbb{R})$, since $B^T = -B$, we can verify that

$$\mathbf{u}^T(\mathcal{L}\mathbf{v}) + (\mathcal{L}\mathbf{u})^T\mathbf{v} = (\mathbf{u}^T\mathbf{v})_t - (\mathbf{u}^TA\mathbf{v})_z. \tag{4.5}$$

Since $\mathbf{N}(z,t) = \mathbf{0}$ for $t < 0$ and $\mathbf{M}^*(z,t) = \mathbf{0}$ for $t > \tau$, and $\mathbf{M}^*$ and $\mathbf{N}$ are compactly supported in z, t space when $0 \leq t \leq \tau$, we have

$$\begin{aligned}
\int_0^\infty \int_{-\infty}^\infty \mathbf{M}^{*T}(\beta - \tilde{\beta})B\tilde{\mathbf{M}}\ dt\ dz &= \int_0^\infty \int_{-\infty}^\infty \mathbf{M}^{*T}(\mathcal{L}\mathbf{N})\ dt\ dz \\
&= \int_0^\infty \int_{-\infty}^\infty \mathbf{M}^{*T}(\mathcal{L}\mathbf{N}) + (\mathcal{L}\mathbf{M}^*)^T\mathbf{N}\ dt\ dz \\
&= \int_0^\infty \int_{-\infty}^\infty (\mathbf{M}^{*T}\mathbf{N})_t - (\mathbf{M}^{*T}A\mathbf{N})_z\ dt\ dz \\
&= -\int_{-\infty}^\infty \mathbf{M}^{*T}(0,t)A\mathbf{N}(0,t)\ dt
\end{aligned} \tag{4.6}$$

where

$$\mathbf{N}^T(0,t) = [m_1(t) - \tilde{m}_1(t),\ 0,\ m_3(t) - \tilde{m}_3(t),\ 0]. \tag{4.7}$$

Estimating the LHS of (4.6) involves the following estimates for continuous vector functions $\mathbf{v}(z,t), \mathbf{w}(z,t)$, a continuous $\beta(z)$, and σ_1, σ_2 taking values 1 or $1/c$;

$$\int_0^\infty \int_{-\infty}^\infty \mathbf{v}^T \beta \mathbf{w} \delta(\tau - t - \sigma_1 z) H(t - \sigma_2 z)\ dt\ dz \preceq \int_0^\infty |\beta(z)| H(\tau - \sigma_1 z - \sigma_2 z)\ dz$$
$$\preceq \int_0^{\frac{\tau}{\sigma_1+\sigma_2}} |\beta(z)|\ dz, \tag{4.8}$$
$$\int_0^\infty \int_{-\infty}^\infty \mathbf{v}^T \beta \mathbf{w} H(\tau - t - \sigma_1 z) H(t - \sigma_2 z)\ dt\ dz \preceq \int_0^\infty |\beta(z)| H(\tau - \sigma_1 z - \sigma_2 z)\ dz$$
$$\preceq \int_0^{\frac{\tau}{\sigma_1+\sigma_2}} |\beta(z)|\ dz, \tag{4.9}$$

with the constant determined only by the upper bounds of $|\mathbf{v}|$ and $|\mathbf{w}|$ on $\{(z,t) \mid 0 \leq \sigma_2 z \leq t \leq \tau - \sigma_1 z\}$.

From the progressing wave expansions of $\mathbf{M}$ and $\mathbf{M}^*$ in (3.3) and (3.5),

$$\mathbf{M}(z,t) = \delta(t-z)\mathbf{e}_2 + \mathbf{g}(z,t)H(t - z/c) + \mathbf{f}(z,t)(H(t-z) - H(t - z/c)), \tag{4.10}$$
$$\mathbf{M}^*(z,t) = \delta(\tau - t - z/c)\mathbf{e}_3 + \mathbf{g}^*(z,t)H(\tau - t - z/c)$$
$$+ \mathbf{f}^*(z,t)(H(\tau - t - z) - H(\tau - t - z/c)) \tag{4.11}$$

where $\mathbf{e}_3 = [0,\ 0,\ 1,\ 0]^T, \mathbf{e}_2 = [0,\ 1,\ 0,\ 0]^T$, and $\mathbf{f}, \mathbf{g}, \mathbf{f}^*, \mathbf{g}^*$ satisfy CBVPs similar to $(3.4a) - (3.4e)$. Note that from Proposition 3.1, $|\mathbf{f}|, |\mathbf{g}|, |\mathbf{f}^*|, |\mathbf{g}^*|$ can be bounded above by c, T, $||\beta||_{L^\infty[0,T]}$ and $||\tilde{\beta}||_{L^\infty[0,T]}$. Using $(4.10) - (4.11)$, the LHS of (4.6) is the sum of integrals of products. The most significant term comes from the product of delta functions which is

$$\int_0^\infty \int_{-\infty}^\infty (\beta - \tilde{\beta}) \mathbf{e_3}^T B \mathbf{e_2} \delta\left(\tau - t - \frac{z}{c}\right) \delta(t - z)\ dt\ dz = \frac{c(-1+c)}{2(1+c)} (\beta - \tilde{\beta})(Z_\tau) \tag{4.12}$$

where $Z_\tau = \frac{c\tau}{1+c}$. All other product terms of the LHS of (4.6) may be estimated using $(4.8) - (4.9)$, which can be bounded by c, T, $||\beta||_{L^\infty[0,T]}$ and $||\tilde{\beta}||_{L^\infty[0,T]}$. Then from $(4.6) - (4.12)$,

$$|(\beta - \tilde{\beta})(Z_\tau)| \preceq |\text{RHS of } (4.6)| + \int_0^{Z_\tau} |\beta(z) - \tilde{\beta}(z)|\ dz \tag{4.13}$$

with the constant dependent only on c, T, $||\beta||_{L^\infty[0,T]}$ and $||\tilde{\beta}||_{L^\infty[0,T]}$. Also from (4.4) and (4.7),

$$\text{RHS of (4.6)} = -\int_{-\infty}^{\infty} c\delta(\tau - t)(m_3 - \tilde{m}_3)(t)\, dt = -c(m_3 - \tilde{m}_3)(\tau). \tag{4.14}$$

Then from (4.13) − (4.14), $\forall \tau \in [0, 2T]$,

$$\begin{aligned} |(\beta - \tilde{\beta})(Z_\tau)| &\preceq |(m_3 - \tilde{m}_3)(\tau)| + \int_0^{Z_\tau} |\beta(z) - \tilde{\beta}(z)|\, dz \\ &\preceq \max_{[0,2T]} |(m_3 - \tilde{m}_3)(\cdot)| + \int_0^{Z_\tau} |\beta(z) - \tilde{\beta}(z)|\, dz. \end{aligned}$$

Hence by Gronwall's Inequality,

$$|(\beta - \tilde{\beta})(Z_\tau)| \preceq \max_{[0,2T]} |(M_3 - \tilde{M}_3)(0, \cdot)|$$

with the constant dependent only on c, T, $||\beta||_{L^\infty[0,T]}$ and $||\tilde{\beta}||_{L^\infty[0,T]}$. As τ varies in $[0, 2T]$, Z_τ takes all values in $[0, 2cT/(1 + c)]$; hence the theorem is proved. □

4.2 Coefficient Recovery from the Full Reflection Data

In this section, we reconstruct $\beta(z)$ from the full reflection data $M_1(0, t)$ and $M_3(0, t)$, where M(z,t) is the solution of $(4.1a) - (4.1c)$.

Recall that $Y = 2cT/(1 + c)$ and

$$\begin{aligned} D &:= \{(z, t) \mid 0 \le z \le T, z \le t \le 2T - z\}, \\ \Theta &:= \{\beta(z) \in \dot{C}^1[0, T] \mid ||\beta||^2_{L^2[0,T]} \le K\}. \end{aligned}$$

For an arbitrary $\beta \in \Theta$, based on the progressing wave expansion (3.3), the solution $\mathbf{M}(z, t)$ of $(4.1a) - (4.1c)$ can be written as

$$\mathbf{M}(z, t) = \delta(t - z)\mathbf{e}_2 + \mathbf{m}(z, t)H(t - z),$$

where $\mathbf{m}(z,t) \in C^1(D)$ is the solution of the CBVP (see remark at the end of Proposition 3.1)

$$\mathcal{L}\mathbf{m} = \mathbf{0} \text{ in } D, \tag{4.15a}$$

$$m_2(0,t) = m_4(0,t) = 0, \quad t \in [0, 2T], \tag{4.15b}$$

$$m_1(z,z) = 0, \quad m_3(z,z) = \frac{c-1}{2(c+1)}\beta(z), \quad m_4(z,z) = \frac{c+1}{2(c-1)}\beta(z), \quad z \in [0,T]. \tag{4.15c}$$

Remark: Hence knowing $M_1(0,t), M_3(0,t)$ is equivalent to knowing $m_1(0,t), m_3(0,t)$.

We want to recover $\beta(z)$ from the full reflection data $m_1(0,t)$ and $m_3(0,t)$. We define the forward map

$$F : \beta(z) \mapsto (m_1(0,t), m_3(0,t))$$

that maps the coefficient to the full reflection data. F is already known to be injective and its inverse is Lipschitz continuous by Theorem 4.1 - just $m_3(\cdot)$ is enough - but we used the *max* norm there. For reconstruction we need the Lipschitz continuity of F^{-1} in the L^2 norm. We also obtain an upper bound of $||\beta||_{L^2[0,T]}$ in terms of $||m_1(0,t)||_{L^2[0,2T]}$ and $||m_3(0,t)||_{L^2[0,2T]}$ and the reflection operator R (defined by $m_1(0,t)$ and $m_3(0,t)$) which will be defined later. Further we give a necessary condition on the range of F.

To recover $\beta(z)$, we need to study a CBVP with full Cauchy data on $z = 0$ and fewer conditions on the characteristics than in the CBVP $(4.15a) - (4.15c)$ studied in Section 3.1-3.3. We show the well posedness of this new CBVP and use it to construct a map whose fixed point is the coefficient β that we wish to recover.

4.2.1 Inverse Stability

Here we show that that the inverse of the injective forward map $F : \beta(z) \mapsto (m_1(0,t), m_3(0,t))$ is locally Lipschitz continuous in the L^2 norm instead of the max norm in Theorem 4.1.

Theorem 4.2. (inverse stability) *If* $\mathbf{m}$ *and* $\tilde{\mathbf{m}}$ *are the solutions of* $(4.15a)-(4.15c)$ *corresponding to* β *and* $\tilde{\beta}$ *in* Θ*, then*

$$||(\beta-\tilde{\beta})(\cdot)||^2_{L^2[0,Y]} \preceq ||m_1(0,\cdot)-\tilde{m}_1(0,\cdot)||^2_{L^2[0,2T]} + ||m_3(0,\cdot)-\tilde{m}_3(0,\cdot)||^2_{L^2[0,2T]}$$

with the constant depending only on c, T, K*.*

Define

$$\tilde{D} := \{(z,t) \mid 0 \le z \le Y, z \le t \le 2T - z/c\}.$$

For any vector function $\mathbf{p}(z,t) \in C^1(\tilde{D})$ and $\epsilon \in (0,1]$, define (see Figure 4.2)

$$J(\mathbf{p},z) := \int_{CD} (p_1^2 + p_2^2 + cp_3^2 + c\epsilon p_4^2)(z,t)\, dt, \quad z \in [0,Y],$$

$$|\mathbf{p}(z,t)|^2 := \sum_{i=1}^{4} p_i^2(z,t), \quad (z,t) \in \tilde{D},$$

$$|(\mathcal{L}\mathbf{p})(z,t)|^2 := \sum_{i=1}^{4} |\mathcal{L}p_i|^2(z,t), \quad (z,t) \in \tilde{D}.$$

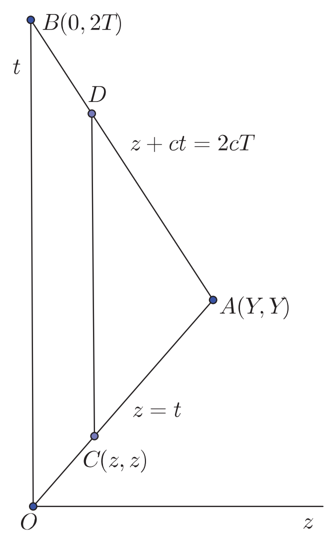

Figure 4.2: Energy function

The proof of Theorem 4.2 will require the following result.

Proposition 4.1. *If $\mathbf{p}(z,t) \in C^1(\tilde{D})$, then for all $\lambda > 0, \epsilon \in (0,1]$ and $z \in [0,Y]$ (see Figure 4.2)*

$$
\begin{aligned}
J(\mathbf{p},z) + \int_{OC} &(2p_1^2 + (1+c)p_3^2 - \epsilon(1-c)p_4^2)\ dy \\
&\leq J(\mathbf{p},0) + \lambda \iint_{OCDB} |(\mathcal{L}\mathbf{p})(y,t)|^2\ dA + \frac{1}{c\epsilon}\int_0^z \left(4|\beta(y)| + \frac{1}{\lambda}\right) J(\mathbf{p},y)\ dy.
\end{aligned}
\tag{4.16}
$$

Furthermore, if $\epsilon \leq \frac{(1-c)^3}{4(1+c)^3}$ and $\mathbf{p}$ satisfies

$$\mathcal{L}\mathbf{p} = 0, \quad \text{in } \tilde{D}, \tag{4.17a}$$

$$p_1(z,z) = 0, \quad p_3(z,z) = \frac{(c-1)^2}{(c+1)^2}p_4(z,z), \quad z \in [0,Y], \tag{4.17b}$$

then

$$J(\mathbf{p},z) \leq e^{4\sqrt{KY}/(c\epsilon)} J(\mathbf{p},0), \quad z \in [0,Y]. \tag{4.18}$$

Proof. Define $\mathbf{q} := [p_1, \ -p_2, \ p_3, \ -\epsilon p_4]^T$, and multiplying both sides of $\mathcal{L}\mathbf{p} = \mathbf{p}_t - A\mathbf{p}_z - \beta B\mathbf{p}$ by $-2\mathbf{q}^T$, we have

$$
\begin{aligned}
-2\mathbf{q}^T(\mathcal{L}\mathbf{p} + \beta B\mathbf{p}) =& 2\mathbf{q}^T(A\mathbf{p}_z - \mathbf{p}_t) \\
=& (p_1^2 + p_2^2 + cp_3^2 + \epsilon c p_4^2)_z - (p_1^2 - p_2^2 + p_3^2 - \epsilon p_4^2)_t.
\end{aligned}
\tag{4.19}
$$

Integrating the RHS of (4.19) over $OCDB$, we have

$$
\begin{aligned}
&\iint_{OCDB} (p_1^2 + p_2^2 + cp_3^2 + c\epsilon p_4^2)_z - (p_1^2 - p_2^2 + p_3^2 - \epsilon p_4^2)_t\ dA \\
=& \int_{\partial OCDB} (p_1^2 - p_2^2 + p_3^2 - \epsilon p_4^2)\ dz + \int_{\partial OCDB} (p_1^2 + p_2^2 + cp_3^2 + c\epsilon p_4^2)\ dt \\
=& \int_{OC} (p_1^2 + p_2^2 + cp_3^2 + c\epsilon p_4^2)\ dt + J(\mathbf{p},z) - J(\mathbf{p},0) + \int_{DB} (p_1^2 + p_2^2 + cp_3^2 + c\epsilon p_4^2)\ dt \\
&- \int_{DB} c(p_1^2 - p_2^2 + p_3^2 - \epsilon p_4^2)\ dt + \int_{OC} (p_1^2 - p_2^2 + p_3^2 - \epsilon p_4^2)\ dt \\
=& \int_{DB} ((1-c)p_1^2 + (1+c)p_2^2 + 2c\epsilon p_4^2)\ dt + \int_{OC} (2p_1^2 + (1+c)p_3^2 - \epsilon(1-c)p_4^2)\ dt \\
&+ J(\mathbf{p},z) - J(\mathbf{p},0).
\end{aligned}
\tag{4.20}
$$

Also

$$\iint_{OCDB} \text{LHS of } (4.19)\ dA$$
$$\leq \iint_{OCDB} \left(\frac{|\mathbf{p}(y,t)|^2}{\lambda} + \lambda |(\mathcal{L}\mathbf{p})(y,t)|^2 + 4|\beta(y)| \cdot |\mathbf{p}(y,t)|^2 \right)\ dA$$
$$\leq \lambda \iint_{OCDB} |(\mathcal{L}\mathbf{p})(y,t)|^2\ dA + \frac{1}{c\epsilon} \int_0^z \left(4|\beta(y)| + \frac{1}{\lambda} \right) J(\mathbf{p}, y)\ dy \qquad (4.21)$$

for all $\lambda > 0$ and $\epsilon \in (0, 1]$, so (4.16) follows directly from (4.20) − (4.21).

If $\mathbf{p}$ satisfies (4.17a) − (4.17b) and $\epsilon \leq \frac{(1-c)^3}{4(1+c)^3}$, then

$$\int_{OC} (2p_1^2 + (1+c)p_3^2 - \epsilon(1-c)p_4^2)\ dy \geq 0.$$

In (4.16) using (4.17a) and letting $\lambda \to \infty$, we have

$$J(\mathbf{p}, z) \leq J(\mathbf{p}, 0) + \frac{4}{c\epsilon} \int_0^z |\beta(y)| J(\mathbf{p}, y)\ dy,$$

so (4.18) follows directly from Gronwall's inequality. □

Proof of Theorem 4.2:

Proof. If $\mathbf{m}$ and $\tilde{\mathbf{m}}$ are the solutions of (4.15a) − (4.15c) corresponding to β and $\tilde{\beta}$ in Θ, and $\mathbf{p} = \mathbf{m} - \tilde{\mathbf{m}}$ then $\mathbf{p}$ satisfies $\mathcal{L}\mathbf{p} = (\beta - \tilde{\beta})B\tilde{\mathbf{m}}$. Choose $\epsilon = \frac{(1-c)^3}{8(1+c)^3}$, then

$$\int_{OC} ((1+c)p_3^2 - \epsilon(1-c)p_4^2)\ dy = \frac{(1-c)^3}{8(1+c)^3} \int_{OC} (\beta - \tilde{\beta})^2(y)\ dy,$$

so from (4.16) in Proposition 4.1 and (3.19) in Proposition 3.3 we have

$$J(\mathbf{p}, z) + \frac{(1-c)^3}{8(1+c)^3} \int_0^z (\beta - \tilde{\beta})^2(y)\ dy$$
$$\leq J(\mathbf{p}, 0) + 4C_0\lambda \int_0^z (\beta - \tilde{\beta})^2(y)\ dy + \frac{1}{c\epsilon} \int_0^z \left(4|\beta(z)| + \frac{1}{\lambda} \right) J(\mathbf{p}, y)\ dy, \qquad (4.22)$$

where C_0 is a constant dependent only on c, T, K. Choose $\lambda = \frac{(1-c)^3}{64C_0(1+c)^3}$, then from (4.22) we have

$$J(\mathbf{p}, z) + \frac{(1-c)^3}{16(1+c)^3} \int_0^z (\beta - \tilde{\beta})^2(y)\ dy \leq J(\mathbf{p}, 0) + \frac{1}{c\epsilon} \int_0^z \left(4|\beta(z)| + \frac{1}{\lambda} \right) J(\mathbf{p}, y)\ dy, \qquad (4.23)$$

which implies that

$$J(\mathbf{p},z) \le J(\mathbf{p},0) + \frac{1}{c\epsilon}\int_0^z \left(4|\beta(z)| + \frac{1}{\lambda}\right) J(\mathbf{p},y)\, dy,$$

and hence from Gronwall's inequality

$$J(\mathbf{p},z) \le CJ(\mathbf{p},0), \quad z \in [0,Y],$$

where C is a constant dependent only on c, T, K, so from (4.23) we have

$$\frac{(1-c)^3}{16(1+c)^3}\int_0^z (\beta-\tilde{\beta})^2(y)\, dy \le J(\mathbf{p},0) + \frac{C}{c\epsilon}\int_0^z \left(4|\beta(z)| + \frac{1}{\lambda}\right) J(\mathbf{p},0)\, dy, \quad z \in [0,Y],$$

which implies that

$$\int_0^Y (\beta-\tilde{\beta})^2(y)\, dy \preceq J(\mathbf{p},0) \le \int_0^{2T} ((m_1-\tilde{m}_1)^2 + (m_3-\tilde{m}_3)^2)(0,t)\, dt,$$

with the constant dependent only on c, T, K. □

4.2.2 Range Characterization

In this section we attempt to characterize the range of the forward map from the coefficient $\beta(z) \in \dot{C}^1[0,T]$ to the reflection data $m_1(0,t)$ and $m_3(0,t)$, where $\mathbf{m}(z,t)$ is the solution of $(4.15a) - (4.15c)$.

Define

$$\dot{C}^1[0,2T] := \{\phi \in C^1[0,2T] \mid \phi(0) = \phi'(0) = 0\},$$
$$D := \{(z,t) \mid 0 \le z \le T, z \le t \le 2T - z\}.$$

For an arbitrary $\phi(t) \in \dot{C}^1[0,2T]$ and $\beta \in \dot{C}^1[0,T]$, consider the CBVP

$$\mathcal{L}\mathbf{h} = 0 \text{ in } D, \tag{4.24a}$$

$$h_1(z,z) = h_3(z,z) = h_4(z,z) = 0, \quad 0 \le z \le T, \tag{4.24b}$$

$$h_2(0,t) = \phi(t), \quad h_4(0,t) = 0, \quad 0 \le t \le 2T. \tag{4.24c}$$

From Proposition 3.1, (4.24a) − (4.24c) has a unique solution in $C^1(D)$, so we define the reflection operator

$$\begin{aligned} R : \dot{C}^1[0,2T] &\mapsto C^1[0,2T] \times C^1[0,2T], \\ R(\phi) &= [h_1(0,t),\ \sqrt{c}\ h_3(0,t)]. \end{aligned} \tag{4.25}$$

Note that R is completely determined by $m_1(0,t)$ and $m_3(0,t)$ because if

$$\mathbf{H}(z,t) = \begin{cases} \mathbf{h}(z,t), & \text{if } t \geq z \\ 0, & \text{if } t < z, \end{cases}$$

then $\mathbf{H}(z,t)$ is the solution of (4.1a) − (4.1c) except with $\phi(t)$ replacing $\delta(t)$; here $\phi(t)$ has been extended by zero for $t < 0$. So $\mathbf{H}(z,t) = \mathbf{M}(z,t) * \phi(t)$ and in particular, for $t \geq 0$, $\mathbf{h}(0,t) = \mathbf{H}(0,t) = \mathbf{M}(0,t) * \phi(t)$, so

$$(R\phi)(t) = (m_1(0,t) * \phi(t), \sqrt{c}\ m_3(0,t) * \phi(t)).$$

Proposition 4.2. *If R is the reflection operator associated to $\beta(z) \in \dot{C}^1[0,T]$, then $||R|| \leq 1$, where $||R||$ represents the norm as an operator on L^2 functions.*

Remark: Proposition 4.2 gives a necessary condition on the range of the forward map $F : \beta(z) \to (m_1(0,t), m_3(0,t))$. For inverse problems for the single speed case, as in say [17], the range is characterized by the necessary and sufficient condition $||R|| < 1$. For our problem, necessary condition $||R|| \leq 1$ is probably not sharp and we don't believe $||R|| < 1$ is a sufficient condition.

Proof. Let $\mathbf{h}(z,t) \in C^1(D)$ be the solution of (4.24a) − (4.24c). Multiplying both sides of (4.24a) by $\mathbf{h}^T$ and using $B = -B^T$, we have

$$(h_1^2 + h_2^2 + h_3^2 + h_4^2)_t - (h_1^2 - h_2^2 + ch_3^2 - ch_4^2)_z = 0. \tag{4.26}$$

Integrating (4.27) over D, we obtain

$$\begin{aligned} 0 = \int_{OA} (2h_1^2 + (1+c)h_3^2 + (1-c)h_4^2)\ dz - \int_{AB} (2h_2^2 + (1-c)h_3^2 + (1+c)h_4^2)\ dt \\ - \int_{OB} (h_1^2 - h_2^2 + ch_3^2 - ch_4^2)\ dt. \end{aligned} \tag{4.27}$$

Substituting the boundary values of $\mathbf{h}$, we obtain:

$$||\phi||^2_{L^2[0,2T]} = ||R(\phi)||^2_{L^2[0,2T]} + \int_{AB} (2h_2^2 + (1-c)h_3^2 + (1+c)h_4^2)\, dt \tag{4.28}$$

implying $||R|| \leq 1$. □

4.2.3 Well-posedness for the Sideways Problem

To recover the coefficient, we need to study a CBVP with full Cauchy data on z = 0 and fewer conditions on the characteristics, which we call the sideways problem. In this section, we discuss the well posedness of the sideways problem. For $T > 0$, we first show the existence and uniqueness of the classical solution for $\beta(z) \in C^1[0,T]$, then we show that the solution depends stably on $\beta(z) \in C^1[0,T]$. Lastly, we show the existence and uniqueness of the weak solution for $\beta(z) \in L^2[0,T]$.

Let $Y = \frac{2cT}{1+c}$ and recall that

$$\tilde{D} := \{(z,t) \mid 0 \leq z \leq Y, z \leq t \leq 2T - z/c\}.$$

For arbitrary functions $a(t), b(t), d(t), e(t) \in C^1[0,2T]$, consider the CBVP

$$\mathcal{L}\mathbf{h} = \mathbf{0} \text{ in } \tilde{D}, \tag{4.29a}$$

$$h_3(z,z) = \frac{(c-1)^2}{(c+1)^2} h_4(z,z), \quad 0 \leq z \leq Y, \tag{4.29b}$$

$$h_1(0,t) = a(t), \quad h_2(0,t) = b(t), \quad h_3(0,t) = d(t), \quad h_4(0,t) = e(t), \quad 0 \leq t \leq 2T. \tag{4.29c}$$

If $\mathbf{h}$ is a C^1 solution of $(4.29a) - (4.29c)$ then from $(4.29b)$ we obtain the matching conditions for a C^1 solution of $(4.29a) - (4.29c)$.

$$(c+1)^2 h_3(0,0) = (c-1)^2 h_4(0,0), \tag{4.30}$$

$$(c+1)^2 (h_{3z}(0,0) + h_{3t}(0,0)) = (c-1)^2 (h_{4z}(0,0) + h_{4t}(0,0)). \tag{4.31}$$

From the third and forth equation of $(4.29a)$ we have

$$h_{3t}(0,0) - ch_{3z}(0,0) = \frac{\beta(0)}{2}((1+c)a(0) - (1-c)b(0)), \tag{4.32}$$

$$h_{4t}(0,0) + ch_{4z}(0,0) = \frac{\beta(0)}{2}((1-c)a(0) - (1+c)b(0)). \tag{4.33}$$

Combining (4.30) − (4.33), and noting that $h_{3t}(0,0) = d'(0), h_{4t}(0,0) = e'(0)$, we obtain the matching conditions for a C^1 solution of (4.29a) − (4.29c);

$$(c+1)^2 d(0) = (1-c)^2 e(0), \tag{4.34a}$$

$$\begin{aligned}(c+1)^2 &\left((1+c)d'(0) - \frac{\beta(0)}{2}((1+c)a(0) - (1-c)b(0))\right) \\ &= (c-1)^2 \left((c-1)e'(0) + \frac{\beta(0)}{2}((1-c)a(0) - (1+c)b(0))\right).\end{aligned} \tag{4.34b}$$

Proposition 4.3. *If* $\beta \in C^1[0,Y]$, *and* $a(t), b(t), d(t), e(t) \in C^1[0,2T]$ *that satisfy* (4.34a) − (4.34b), *then there exists a unique solution* $\mathbf{h} \in C^1(\tilde{D})$ *of* (4.29a) − (4.29c).

Proof. Define $\mathbf{r}(\mathbf{h}, z, t) := \beta(z)B\mathbf{h}(z,t)$, and pick an arbitrary point $P(z,t) \in \tilde{D}$. Integrating (4.29a) along the characteristics and using the boundary conditions (4.29b) − (4.29c), we have (see Figure 4.3)

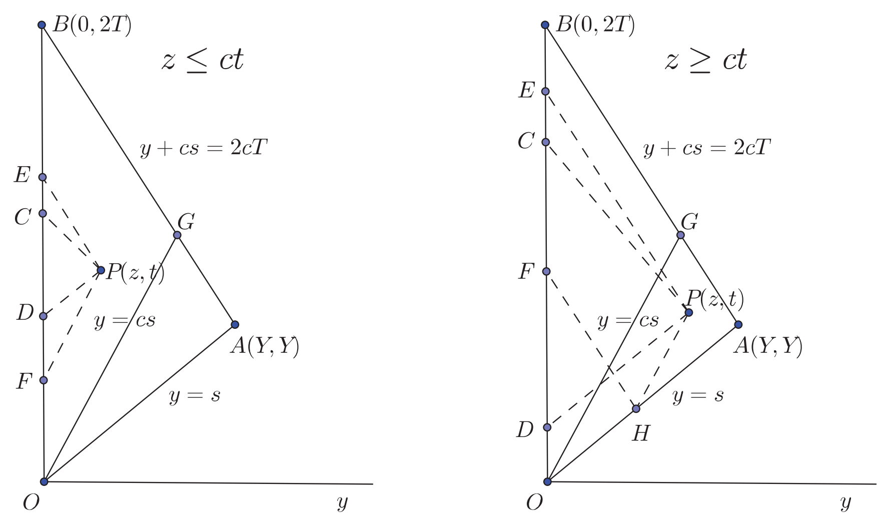

Figure 4.3: Leftward moving lines through $P(z,t)$ with slopes ± 1 and $\pm 1/c$

$$h_1(z,t) = \int_0^z r_1(\mathbf{h}, y, z+t-y)\ dy + a(s_C), \tag{4.35a}$$

$$h_2(z,t) = \int_0^z r_2(\mathbf{h}, y, y+t-z)\ dy + b(s_D), \tag{4.35b}$$

$$h_3(z,t) = \int_0^z r_3\left(\mathbf{h}, y, \frac{z+ct-y}{c}\right)\ dy + d(s_E), \tag{4.35c}$$

$$h_4(z,t) = \begin{cases} \int_0^z r_4\left(\mathbf{h}, y, \frac{y+ct-z}{c}\right)\ dy + e(s_F) & \text{if } z \le ct \\ \int_{y_H}^z r_4\left(\mathbf{h}, y, \frac{y+ct-z}{c}\right)\ dy + \frac{(1+c)^2}{(1-c)^2}(\int_0^{y_H} r_3(\mathbf{h}, y, \frac{y-(1-c)y_E}{c})\ dy + d(s_F)) & \text{if } z \ge ct. \end{cases} \tag{4.35d}$$

One may verify from Figure 4.3 that $0 \le y_C, y_D, y_E, y_F, y_H \le z$. So from Theorem 3.1, (4.35a) – (4.35d) has a unique solution $\mathbf{h}(z,t) \in C^1(OAG) \cup C^1(OBG)$. By repeating an argument similar to the one used in Prop 3.1, we can show that the compatibility conditions (4.34a) – (4.34b) imply that $\mathbf{h} \in C^1(\tilde{D})$. □

Now we study the stability of the solution of (4.29a) – (4.29c) with respect to the coefficient β.

Recall that $Y = \frac{2cT}{1+c}$ and

$$\Theta := \{\beta(z) \in \dot{C}^1[0,Y] \mid ||\beta||^2_{L^2[0,Y]} \le K\}.$$

Without loss of generality, we assume that

$$\max(||a(\cdot)||^2_{L^2[0,2T]}, ||b(\cdot)||^2_{L^2[0,2T]}, ||d(\cdot)||^2_{L^2[0,2T]}, ||e(\cdot)||^2_{L^2[0,2T]}) \le K.$$

Proposition 4.4. (stability) *If $\mathbf{h}, \tilde{\mathbf{h}} \in C^1(\tilde{D})$ are the solutions of* (4.29a) – (4.29c) *corresponding to $\beta, \tilde{\beta} \in \Theta$, then (see Figure 4.2)*

$$\sum_{i=1}^4 \int_{CD} (h_i - \tilde{h}_i)^2(z,t)\ dt \le C||\beta - \tilde{\beta}||^2_{L^2[0,Y]}, \quad \forall\ 0 \le z \le Y,$$

where C is a constant dependent only on c, T, K.

Proof. If $\mathbf{h} \in C^1(\tilde{D})$ is the solution of $(4.29a) - (4.29c)$ corresponding to β, then $\mathbf{h}$ satisfies $(4.17a) - (4.17b)$, so from (4.18) in Proposition 4.1 we have

$$J(\mathbf{h}, z) \leq J(\mathbf{h}, 0)e^{4\sqrt{KY}/(c\epsilon)} = C_0, \quad \forall z \in [0, Y], \tag{4.36}$$

where C_0 depends only on c, T, K.

If $\mathbf{h}$ and $\tilde{\mathbf{h}}$ are the solutions of $(4.29a)-(4.29c)$ corresponding to β and $\tilde{\beta}$ and $\mathbf{p} := \mathbf{h}-\tilde{\mathbf{h}}$ then $\mathbf{p}$ satisfies $\mathcal{L}\mathbf{p} = (\beta - \tilde{\beta})B\tilde{\mathbf{h}}$ with $\mathbf{p}(0, t) = 0$ for $t \in [0, 2T]$. Hence from (4.16) in Proposition 4.1 we have (taking $\lambda = 1$),

$$J(\mathbf{p}, z) \leq \iint_{OAB} |(\mathcal{L}\mathbf{p})(z, t)|^2 \, dA + \frac{1}{c\epsilon} \int_0^z (1 + 4|\beta(y)|) J(\mathbf{p}, y) \, dy;$$

so from (4.36) and Gronwall's inequality

$$J(\mathbf{p}, z) \leq e^{\frac{1}{c\epsilon}\int_0^Y (1+4|\beta(y)|) \, dy} \int_0^Y 4(\beta - \tilde{\beta})^2(y) J(\tilde{\mathbf{h}}, y) \, dy \preceq ||\beta - \tilde{\beta}||^2_{L^2[0,Y]}, \quad \forall z \in [0, Y],$$

with the constant dependent only on c, T, K. $\square$

Now we study the existence and uniqueness of the weak solution of $(4.29a) - (4.29c)$.

From Proposition 4.3, given $\beta(z) \in \dot{C}^1[0, Y]$, there is a unique $\mathbf{h}(z, t) \in C^1(\tilde{D})$ which solves $(4.29a) - (4.29c)$, so we may define the forward map

$$\begin{aligned} Q : \dot{C}^1[0, Y] &\to C^1(\tilde{D}), \\ \beta(z) &\mapsto \mathbf{h}(z, t). \end{aligned}$$

Since C^1 is dense in L^2 and Q is locally Lipschitz continuous in the L^2 norm because of Proposition 4.4, Q has a continuous extension $\tilde{Q}$

$$\begin{aligned} \tilde{Q} : L^2[0, Y] &\to L^2(\tilde{D}), \\ \beta(z) &\mapsto \mathbf{h}(z, t), \end{aligned}$$

where $\mathbf{h}$, as defined and shown below, is the weak solution of (4.29a) − (4.29c) corresponding to β. Now we motivate the definition of a weak solution of (4.29a) − (4.29c). Let (see Figure 4.2)

$$\Lambda := \{\mathbf{N} \in C^1(\tilde{D}) \mid N_1 = 0, (1-c)N_3 + (1+c)N_4 = 0 \text{ on } OA,\ N_1 = N_2 = N_4 = 0 \text{ on } AB\}.$$

If $\beta(z) \in \dot{C}^1[0, Y]$ and $\mathbf{h} \in C^1(\tilde{D})$ is the solution of (4.29a) − (4.29c) corresponding to β, then for all $N \in \Lambda$

$$\begin{aligned} 0 &= \iint_{\tilde{D}} \mathbf{N}^T(\mathbf{h}_t - A\mathbf{h}_z - \beta B\mathbf{h})\ dA \\ &= -\int_{\partial\tilde{D}} \mathbf{N}^T\mathbf{h}\ dz - \int_{\partial\tilde{D}} \mathbf{N}^T A\mathbf{h}\ dt + \iint_{\tilde{D}} (\mathbf{N}_z^T A - \mathbf{N}_t^T - \mathbf{N}^T\beta B)\mathbf{h}\ dA, \end{aligned}$$

which implies that

$$\iint_{\tilde{D}} (\mathcal{L}\mathbf{N})^T\mathbf{h}\ dA = \int_{AB} \mathbf{N}^T(cI - A)\mathbf{h}\ dt - \int_{OA} \mathbf{N}^T(I + A)\mathbf{h}\ dt + \int_{OB} \mathbf{N}^T A\mathbf{h}\ dt. \tag{4.37}$$

Since $N_1 = N_2 = N_4 = 0$ on AB, then $\mathbf{N}^T(cI - A) = \mathbf{0}$. Also, since $N_1 = 0$ and $(1-c)N_3 + (1+c)N_4 = 0$ on OA

$$\begin{aligned} \mathbf{N}^T(I + A)\mathbf{h} &= (1+c)N_3h_3 + (1-c)N_4h_4 = \frac{(1-c)^2}{1+c}N_3h_4 + (1-c)N_4h_4 \\ &= \frac{1-c}{1+c}h_4((1-c)N_3 + (1-c)N_4) = 0. \end{aligned}$$

Hence from (4.37)

$$\iint_{\tilde{D}} (\mathcal{L}\mathbf{N})^T\mathbf{h}\ dA = \int_{OB} \mathbf{N}^T A\mathbf{h}\ dt = \int_{OB} N_1a - N_2b + cN_3d - cN_4e\ dt.$$

This suggests the following definition.

Definition 4.1. *For $a, b, d, e \in C^1[0, 2T]$, we say that $\mathbf{h} \in L^2(\tilde{D})$ is a weak solution of (4.29a) − (4.29c) corresponding to $\beta \in L^2[0, Y]$ if (see Figure 4.3)*

$$\iint_{\tilde{D}} (\mathbf{N}_t - A\mathbf{N}_z - \beta B\mathbf{N})^T\mathbf{h}\ dA = \int_{OB} N_1a - N_2b + cN_3d - cN_4e\ dt \tag{4.38}$$

for all $\mathbf{N} \in \Lambda$.

The existence of a weak solution of ($4.29a$) – ($4.29c$) follows from an argument similar to the one used to prove Proposition 3.5; in fact the weak solution is $\tilde{Q}\beta$ where $\tilde{Q}$ is the extension of Q. Now we show that the weak solution is unique. As a first step we study the existence of a solution of a special adjoint CBVP.

Proposition 4.5. *If* $\mathbf{F} \in C^1(\tilde{D})$ *and* $\beta \in C^1[0, Y]$, *then* (*see Figure 4.4*)

$$\mathcal{L}\mathbf{v} = \mathbf{F} \text{ in } \tilde{D}, \tag{4.39a}$$

$$v_1 = 0, \quad (1-c)v_3 + (1+c)v_4 = 0 \text{ on } OA, \tag{4.39b}$$

$$v_1 = v_2 = v_4 = 0 \text{ on } AB. \tag{4.39c}$$

has a unique solution $\mathbf{v} \in C^1(\tilde{D})$.

Proof. Define $\mathbf{r}(\mathbf{v}, z, t) := \beta(z)B\mathbf{v}(z, t) + \mathbf{F}(z, t)$, and pick an arbitrary point $P(z, t) \in \tilde{D}$. Integrating ($4.39a$) along the characteristics and using the boundary conditions,

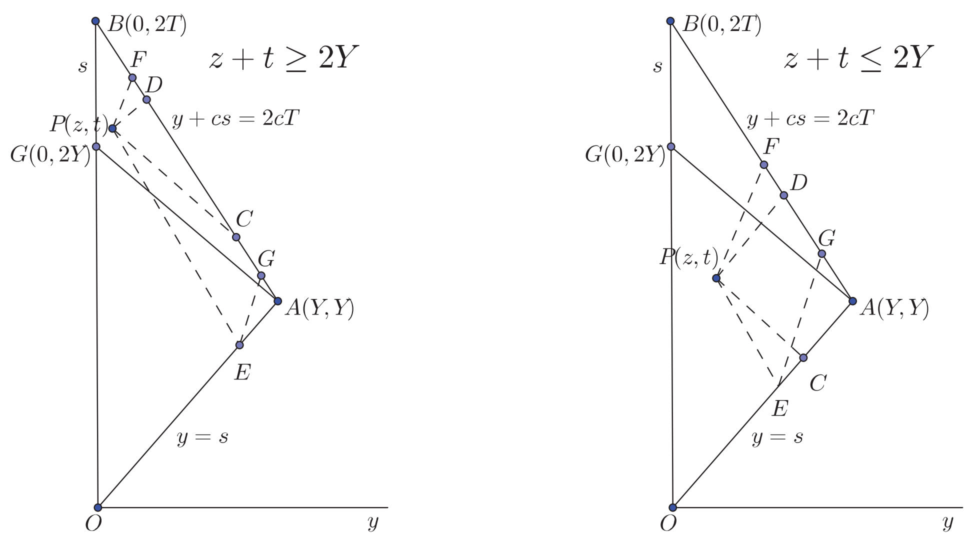

Figure 4.4: Rightward moving lines through $P(z, t)$ with slopes ± 1 and $\pm 1/c$

we have

$$v_1(z,t) = -\int_z^{y_C} r_1(\mathbf{v}, y, z+t-y)\ dy \tag{4.40a}$$

$$v_2(z,t) = -\int_z^{y_D} r_2(\mathbf{v}, y, y+t-z)\ dy \tag{4.40b}$$

$$v_3(z,t) = -\int_z^{y_E} r_3\left(\mathbf{v}, y, \frac{z+ct-y}{c}\right)\ dy + \frac{1+c}{1-c}\int_{y_E}^{y_G} r_4\left(\mathbf{v}, y, \frac{y+cy_E-y_E}{c}\right)\ dy \tag{4.40c}$$

$$v_4(z,t) = \int_z^{y_F} r_4\left(\mathbf{v}, y, \frac{y+ct-z}{c}\right)\ dy. \tag{4.40d}$$

One may verify from Figure 4.4 that $z \le y_E, y_F, y_C, y_D, y_G \le T$. So from Theorem 3.1, $(4.40a) - (4.40d)$ has a unique solution $\mathbf{v}(z,t) \in C^1(OGA) \cup C^1(BGA)$. From $(4.39b) - (4.39c)$, no compatibility conditions are needed for $\mathbf{v}$ to be in $C^1(\tilde{D})$. □

Applying techniques similar to those in the proof of Proposition 3.7, we obtain

Proposition 4.6. *There is exactly one weak solution of* $(4.29a) - (4.29c)$ *that satisfies* (4.38).

Based on Proposition 4.3-4.6, we have

Theorem 4.3. *If* $\beta \in \dot{C}^1[0,Y]$, *and* $a(t), b(t), c(t), d(t) \in C^1[0,2T]$ *that satisfy* $(4.34a)-(4.34b)$, *then there exists a unique solution* $\mathbf{h} \in C^1(\tilde{D})$ *of* $(4.29a) - (4.29c)$. *Furthermore, if* $\beta \in L^2[0,Y]$, *then there exists a unique weak solution* $\mathbf{h} \in L^2(\tilde{D})$ *of* $(4.29a) - (4.29c)$.

4.2.4 Reconstruction

We will reconstruct $\beta(z)$ over a small interval $[0,\delta]$, then over $[\delta, 2\delta]$, then $[2\delta, 3\delta]$, and so on. The crux of the reconstruction is a proposition about reconstructing $\beta(z)$ on $[Z, Z+\delta]$ if $\beta(z)$ is known on $[0,Z]$ - and crucially, δ is independent of Z.

Given $T > 0$ and $Z, \delta > 0$ such that $0 \le Z < Z+\delta \le Y$, define

$$\tilde{D}_{Z,\delta} := \{(z,t) \mid (z,t) \in \tilde{D}, Z \le z \le Z+\delta\}.$$

For any $K_Z > 0$, define the complete metric space (in the L^2 norm)

$$\Theta_Z := \{\beta \in L^2[Z, Z+\delta] \mid ||\beta||^2_{L^2[Z,Z+\delta]} \leq K_Z\},$$

which has a dense subset

$$\Lambda_Z := \{\beta \in C^1[Z, Z+\delta] \mid ||\beta||^2_{L^2[Z,Z+\delta]} \leq K_Z\}.$$

Proposition 4.7. *Let $0 \leq Z \leq Y$ and $d, e, a, b \in C^1[Z, \frac{2cT-Z}{c}]$ such that*

$$\begin{aligned}
e(Z) &= \frac{(1+c)^2}{(1-c)^2} d(Z), \\
(c+1)^2 &\left((1+c)d'(Z) - \frac{\beta(Z)}{2}((1+c)a(Z) - (1-c)b(Z))\right) \\
&= (c-1)^2 \left((c-1)e'(Z) + \frac{\beta(Z)}{2}((1-c)a(Z) - (1+c)b(Z))\right).
\end{aligned}$$

There exists a $\delta > 0$ and $K_Z > 0$ such that there is a unique $\beta(z) \in \Theta_Z$ with

$$h_3(z,z) = \frac{c-1}{2(c+1)}\beta(z), \quad Z \leq z \leq Z+\delta, \tag{4.41}$$

where $\mathbf{h}(z,t) \in L^2(\tilde{D}_{Z,\delta})$ is the unique weak solution of

$$\mathcal{L}\mathbf{h} = \mathbf{0} \text{ in } \tilde{D}_{Z,\delta}, \tag{4.42a}$$

$$h_3(z,z) = \frac{(c-1)^2}{(c+1)^2} h_4(z,z), \quad Z \leq z \leq Z+\delta, \tag{4.42b}$$

$$h_1(Z,t) = a(t), \quad h_2(Z,t) = b(t), \quad h_3(Z,t) = d(t), \quad h_4(Z,t) = e(t). \tag{4.42c}$$

Actually, it is sufficient that

$$K_Z \geq \frac{8(1+c)^2}{(1-c)^2} J_Z, \quad \delta = min\left(Y - Z, \frac{c^2\epsilon^2}{256K_Z}\right), \tag{4.43}$$

where $J_Z = ||a||^2_{L^2} + ||b||^2_{L^2} + c||d||^2_{L^2} + c\epsilon||e||^2_{L^2}$ and $\epsilon = \frac{c(1-c)^3}{(1+c)^4}$.

Proof. From Proposition 4.1 applied to $\mathbf{h}$ (see Figure 4.5) we obtain

$$\begin{aligned}
J(\mathbf{h}, z) &+ \int_{CF} (2h_1^2 + (1+c)h_3^2 - \epsilon(1-c)h_4^2)\, dy \\
&\leq J_Z + \lambda \iint_{DCFE} |(\mathcal{L}\mathbf{h})(y,t)|^2 \, dA + \frac{1}{c\epsilon}\int_Z^z \left(4|\beta(y)| + \frac{1}{\lambda}\right) J(\mathbf{h}, y)\, dy, \quad z \in [Z, Z+\delta].
\end{aligned} \tag{4.44}$$

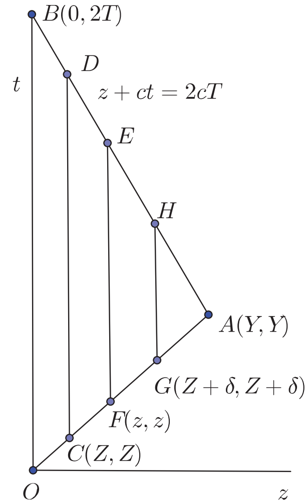

Figure 4.5: Local Reconstruction

If $\mathbf{h}$ is the solution of (4.42a) − (4.42c) and we choose $\epsilon = \frac{c(1-c)^3}{(1+c)^4}$ then

$$\int_{CF} ((1+c)h_3^2 - \epsilon(1-c)h_4^2)\ dt = \int_{CF} h_3^2\ dt. \tag{4.45}$$

So letting $\lambda \to \infty$ in (4.44) we have

$$J(\mathbf{h}, z) + \int_{CF} h_3^2 \le J_Z + \frac{4}{c\epsilon}\int_Z^z |\beta(y)| J(\mathbf{h}, y)\ dy, \quad z \in [Z, Z+\delta], \tag{4.46}$$

and hence from Gronwall's inequality

$$J(\mathbf{h}, z) \le e^{4\sqrt{K_Z\delta}/(c\epsilon)} J_Z, \quad z \in [Z, Z+\delta]. \tag{4.47}$$

Define $J^*(\mathbf{h}) := \max\limits_{y\in[Z,Z+\delta]} J(\mathbf{h}, y)$, then from (4.46) we have

$$J^*(\mathbf{h}) + \int_{CG} h_3^2\ dz \le 2J_Z + \frac{8}{c\epsilon}\int_Z^{Z+\delta} |\beta(y)| J^*(\mathbf{h})\ dy \le 2J_Z + \frac{8\sqrt{\delta K_Z}}{c\epsilon} J^*(\mathbf{h}),$$

which implies that

$$\left(1 - \frac{8\sqrt{\delta K_Z}}{c\epsilon}\right) J^*(\mathbf{h}) + \int_{CG} h_3^2\ dz \le 2J_Z.$$

So if $\delta \leq \frac{c^2\epsilon^2}{64K_Z}$, we have

$$\int_{CG} h_3^2 \, dz \leq 2J_Z. \tag{4.48}$$

Let $\mathbf{h}, \tilde{\mathbf{h}}$ be the unique C^1 solutions of (4.42a) – (4.42c) corresponding to β and $\tilde{\beta}$, and define $\mathbf{p} := \mathbf{h} - \tilde{\mathbf{h}}$; then $\mathcal{L}(\mathbf{p}) = (\beta - \tilde{\beta})B\tilde{\mathbf{h}}$ and $\mathbf{p}$ is zero on $z = Z$. Choose $\epsilon = \frac{c(1-c)^3}{(1+c)^4}$ and

$$\lambda = \frac{c\epsilon(1-c)^2}{64(1+c)^2 e^{4\sqrt{K_Z\delta}/(c\epsilon)} J_Z}, \tag{4.49}$$

then from (4.47) and Proposition 4.1 applied to $\mathbf{p}$ we obtain

$$\begin{aligned} J^*(\mathbf{p}) + \int_{CG} p_3^2 \, dz &\leq \frac{8\lambda}{c\epsilon} \int_Z^{Z+\delta} (\beta - \tilde{\beta})^2(y) J(\tilde{\mathbf{h}}, y) \, dy + \frac{2}{c\epsilon} \int_Z^{Z+\delta} \left(4|\beta(y)| + \frac{1}{\lambda}\right) J^*(\mathbf{p}) \, dy \\ &\leq \frac{(1-c)^2}{8(1+c)^2} \int_Z^{Z+\delta} (\beta - \tilde{\beta})^2(y) \, dy + \frac{2\delta/\lambda + 8\sqrt{\delta K_Z}}{c\epsilon} J^*(\mathbf{p}), \end{aligned}$$

which implies that

$$\left(1 - \frac{2\delta/\lambda + 8\sqrt{\delta K_Z}}{c\epsilon}\right) J^*(\mathbf{p}) + \int_{CG} p_3^2 \, dz \leq \frac{(1-c)^2}{8(1+c)^2} \int_Z^{Z+\delta} (\beta - \tilde{\beta})^2(y) \, dy.$$

So choosing $\delta \leq \min\left(\frac{c\lambda\epsilon}{4}, \frac{c^2\epsilon^2}{256K_Z}\right)$, we have

$$\int_{CG} p_3^2 \, dz \leq \frac{(1-c)^2}{8(1+c)^2} \int_Z^{Z+\delta} (\beta - \tilde{\beta})^2(y) \, dy. \tag{4.50}$$

Note that since $\delta \leq \frac{c^2\epsilon^2}{256K_Z}$, from (4.49) we have $\lambda \geq \frac{c\epsilon(1-c)^2}{64(1+c)^2 e^{1/4} J_Z}$, implying that (4.50) is satisfied if we choose

$$\delta = \min\left(\frac{c^2\epsilon^2(1-c)^2}{256(1+c)^2 e^{1/4} J_Z}, Y - Z, \frac{c^2\epsilon^2}{256K_Z}\right). \tag{4.51}$$

Define the map

$$\begin{aligned} &Q : \Lambda_Z \mapsto \Lambda_Z, \\ &(Q\beta)(z) = \frac{2(c+1)}{c-1} h_3(z, z). \end{aligned}$$

Note that if we choose $K_Z \geq \frac{8(1+c)^2}{(1-c)^2} J_Z$, from (4.48) we have

$$||Q\beta||^2_{L^2[Z,Z+\delta]} = \frac{4(1+c)^2}{(1-c)^2} \int_Z^{Z+\delta} h_3^2 \, dz \leq \frac{8(1+c)^2}{(1-c)^2} J_Z \leq K_Z,$$

which implies that the map Q is well defined, and from (4.50) we have

$$||Q\beta - Q\tilde{\beta}||^2_{L^2[Z,Z+\delta]} = \frac{4(1+c)^2}{(1-c)^2} ||(h_3 - \tilde{h}_3)(z,z)||^2_{L^2[Z,Z+\delta]} \leq \frac{1}{2} ||\beta - \tilde{\beta}||^2_{L^2[Z,Z+\delta]}, \tag{4.52}$$

and hence Q is a contraction. Since C^1 is dense in L^2, Q has an extension $\tilde{Q} : \Theta_Z \to \Theta_Z$ which is still a contraction. So $\tilde{Q}$ has a fixed point $\beta \in \Theta_Z$. Since $K_Z \geq \frac{8(1+c)^2}{(1-c)^2} J_Z$, we have

$$\frac{c^2\epsilon^2(1-c)^2}{256(1+c)^2 e^{1/4} J_Z} \geq \frac{c^2\epsilon^2}{256 K_Z}.$$

So from (4.51), one can choose $\delta = \min\left(Y - Z, \frac{c^2\epsilon^2}{256K_Z}\right)$.

Let $\{\beta_i\} \in \Lambda_Z$ be a sequence which converges to the fixed point β in the L^2 norm. Let $\mathbf{h}(z,t) \in L^2(\tilde{D}_{Z,\delta})$ be the weak solution of $(4.42a)-(4.42c)$ corresponding to $\beta(z) \in \Theta_Z$. If $\mathbf{h}_i \in C^1(\tilde{D}_{Z,\delta})$ is the the solution of $(4.42a) - (4.42c)$ corresponding to β_i, then from (4.50), $(h_i)_3(z,z)$ is a Cauchy sequence in $L^2[Z, Z+\delta]$ whose limit is $h_3(z,z)$, so

$$\tilde{Q}\beta = \lim_{i\to\infty} Q\beta_i = \lim_{i\to\infty} \frac{2(c+1)}{c-1}(h_i)_3(z,z) = \frac{2(c+1)}{c-1} h_3(z,z),$$

which implies that h_3 has a trace on $z = t$ and

$$\frac{2(c+1)}{c-1} h_3(z,z) = \beta(z).$$

□

Recall that we defined the forward map F

$$F : \dot{C}^1[0,T] \mapsto C^1[0,2T] \times C^1[0,2T],$$

$$(F\beta)(z) = [m_1(0,t), m_3(0,t)]$$

where $\mathbf{m}$ is the solution of $(4.15a) - (4.15c)$.

Theorem 4.4. *If $\phi(t), \psi(t)$ is in the range of F, then we can construct the unique $\beta(z) \in \dot{C}^1[0, Y]$ that corresponds to this ϕ, ψ and a function $\mathbf{h}(z, t) \in C^1(\tilde{D})$ so that*

$$\mathcal{L}\mathbf{h} = 0 \quad \text{in } \tilde{D}, \tag{4.53a}$$

$$h_2(0, t) = h_4(0, t) = 0, \quad h_1(0, t) = \phi(t), \quad h_3(0, t) = \psi(t), \qquad 0 \le t \le 2T, \tag{4.53b}$$

$$h_3(z, z) = \frac{(c-1)^2}{(c+1)^2} h_4(z, z), \qquad 0 \le z \le Y, \tag{4.53c}$$

for which

$$h_3(z, z) = \frac{c-1}{2(c+1)} \beta(z), \quad 0 \le z \le Y. \tag{4.54}$$

Note that if $\phi(t), \psi(t)$ is in the range of F, and (4.54) is satisfied, then from the uniqueness of the solution of the sideways problem $(4.29a) - (4.29c)$, the remaining boundary condition

$$h_1(z, z) = 0$$

is also satisfied.

Proof. The reconstruction of β is done by a repeated application of Proposition 4.7 with a predetermined constant $\delta > 0$.

Since $\phi(\cdot), \psi(\cdot)$ is in the range of F, there is a $\beta \in \dot{C}^1[0, Y]$ and an $\mathbf{m}(z, t) \in C^1(\tilde{D})$ so that $(4.15a) - (4.15c)$ holds and $m_1(0, t) = \phi(t)$ and $m_3(0, t) = \psi(t)$. Let $\mathbf{h}(t, z) = \mathbf{m}(t, z)$; then $\mathbf{h}(t, z)$ is the solution of $(4.29a) - (4.29c)$ corresponding to β. Hence from (4.18) in Proposition 4.1

$$\begin{aligned} J(\mathbf{h}, z) &\le J(\mathbf{h}, 0) e^{4\sqrt{Y}||\beta||_{L^2[0,Y]}/(c\epsilon)} \\ &= (||\phi(\cdot)||^2_{L^2[0,2T]} + ||\psi(\cdot)||^2_{L^2[0,2T]}) e^{4\sqrt{Y}||\beta||_{L^2[0,Y]}/(c\epsilon)}, \quad \forall z \in [0, Y]. \end{aligned} \tag{4.55}$$

(4.55), combined with (4.43), suggests how to define K_0, δ_0 that will work for all $Z \in [0, Y]$. Define K_0, δ_0 by

$$\begin{aligned} K_0 &= \frac{8(1+c)^2}{(1-c)^2} (||\phi||^2_{L^2[0,2T]} + ||\psi||^2_{L^2[0,2T]}) e^{4\sqrt{Y}||\beta||_{L^2[0,Y]}/(c\epsilon)}, \\ \delta_0 &= \min\left(\frac{c^2 \epsilon^2}{256 K_0}, Y \right). \end{aligned}$$

Now we proceed to construct the unique $\mathbf{h}(z,t)$ and $\beta(z)$ satisfying (4.53a)−(4.54). We start by applying Proposition 4.7 with $Z = 0, \delta = \delta_0, d = \phi, f = \psi, e = g = 0$, then one can recover $\beta(z)$ within the interval $[0, \delta_0]$. Then similarly, for the next step, one can apply Proposition 4.7 with $Z = \delta_0, \delta = \delta_0, d = h_1(\delta_0, t), e = h_2(\delta_0, t), f = h_3(\delta_0, t), g = h_4(\delta_0, t)$, and recover $\beta(z)$ through $[\delta_0, 2\delta_0]$, and so on. Finally, the iteration will stop after $N = \frac{Y}{\delta_0}$ steps, and one can recover $\beta(z)$ in the whole interval $[0, Y]$. □

4.3 Summary

We summarize the main results obtained for the inverse problem. If the source comes from the fast channel, we can reconstruct β from the full reflection data $m_1(0,\cdot)$ and $m_3(0,\cdot)$ (knowing only one of them is not enough). Given only $m_3(0,\cdot)$, we can only show the injectivity and continuity of β corresponding to the reflection data in the max norm, and can not reconstruct β.

Chapter 5

NUMERICAL WORK

In this chapter we recover $\beta(z)$ numerically from the reflection boundary data. Let $0 < c < 1, T > 0$ and $Y = \frac{2cT}{1+c}$. Define (see Figure 5.1)

$$D_f := \{(z,t) \mid 0 \leq z \leq T, z \leq t \leq 2T - z\},$$

$$D_s := \{(z,t) \mid 0 \leq z \leq Y, z \leq t \leq 2T - z/c\},$$

the domains for the forward and the sideways problem.

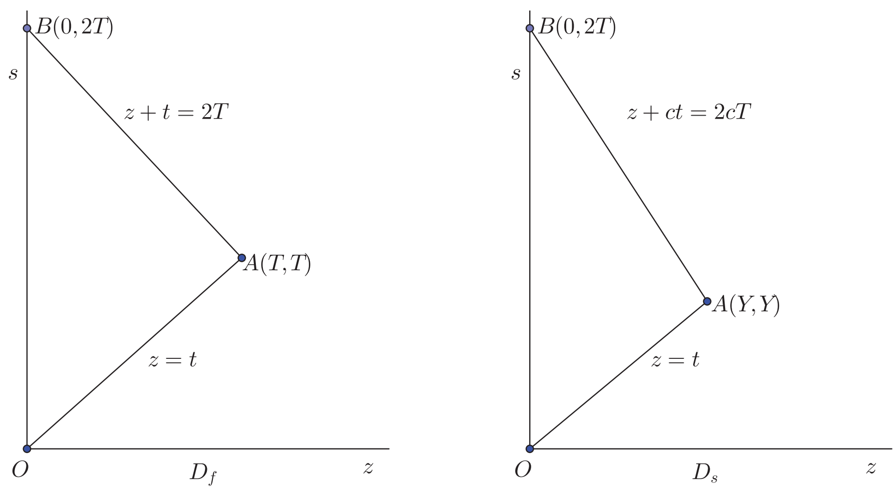

Figure 5.1: Domains for the forward and sideways problem

Let $\mathbf{m}(z,t)$ be the solution of the CBVP

$$\mathcal{L}\mathbf{m} = \mathbf{0} \text{ in } D_f, \tag{5.1a}$$

$$m_2(0,t) = m_4(0,t) = 0, \quad t \in [0,2T], \tag{5.1b}$$

$$m_1(z,z) = 0, \quad m_3(z,z) = \frac{c-1}{2(c+1)}\beta(z), \quad m_4(z,z) = \frac{c+1}{2(c-1)}\beta(z), \quad z \in [0,T]. \tag{5.1c}$$

We recover $\beta(z)$ on $z \in [0,Y]$ from the knowledge of $m_1(0,t)$ and $m_3(0,t)$ for $t \in [0,2T]$. Given the full reflection data $d_1(t) = m_1(0,t)$ and $d_3(t) = m_3(0,t)$, we define the map Q

$$\begin{aligned} &C^1[0,Y] \mapsto C^1[0,Y], \\ &\beta(z) \to \frac{2(c+1)}{c-1} p_3(z,z), \end{aligned}$$

where $\mathbf{p}(z,t)$ is the solution of the following sideways CBVP

$$\mathcal{L}\mathbf{p} = \mathbf{0} \text{ in } D_s, \tag{5.2a}$$

$$p_2(0,t) = p_4(0,t) = 0, \quad p_1(0,t) = d_1(t), \quad p_3(0,t) = d_3(t), \quad t \in [0,2T], \tag{5.2b}$$

$$p_4(z,z) = \frac{(c+1)^2}{(c-1)^2} p_3(z,z), \quad z \in [0,Y]. \tag{5.2c}$$

We have shown in Proposition 4.7 that Q has a unique fixed point, and we recover $\beta(z)$ by finding the unique fixed point of Q via the following recursive scheme

$i)$ guess a β;

$ii)$ solve the sideways problem $(5.2a) - (5.2c)$;

$iii)$ if $||\beta - \frac{2(c+1)}{c-1} p_3(z,z)||_{L^\infty[0,Y]}$ is within some tolerance (usually 0 numerically), then stop, and this is the correct β, otherwise set $\beta(z) = \frac{2(c+1)}{c-1} p_3(z,z)$, and go to step ii.

5.1 The Forward Problem

To generate the reflection data, we build a forward solver for the following inhomogeneous CBVP for arbitrary functions $\mathbf{F}(z,t) \in C^1(D_f)$ and $f(t) \in C^1[0,2T], g(t) \in \dot{C}^1[0,2T]$ and $h_1(z), h_3(z) \in C^1[0,T],, h_4(z) \in \dot{C}^1[0,T]$ and $\beta(z) \in C^1[0,T]$.

$$\mathcal{L}\mathbf{u} = \mathbf{F} \quad \text{in} \quad D_f, \tag{5.3a}$$

$$u_2(0,t) = f(t), \quad u_4(0,t) = g(t), \quad t \in [0,2T], \tag{5.3b}$$

$$u_1(z,z) = h_1(z), \quad u_3(z,z) = h_3(z), \quad u_4(z,z) = h_4(z), \quad z \in [0,T]. \tag{5.3c}$$

We rewrite ($5.3a$) as:

$$\frac{d\mathbf{u}}{d\mathbf{c}} = \beta C\mathbf{u} + \mathbf{F}, \tag{5.4}$$

where $\frac{d\mathbf{u}}{d\mathbf{c}}$ denote differentiation along the characteristics. We solve (5.4) using finite differences and the method of characteristics together with interpolation. Please refer to Figure 5.2 for the details of the scheme.

We implemented the Crank-Nicolson method to solve the ODE along the characteristics, which gives second order convergence rate in space and time, and unconditional stability. We tested the forward solver with various choices of elementary functions for $\beta, \mathbf{F}$ and boundary conditions, e.g. polynomials, trig, log, exponential, and their composition.

We applied our scheme to the following example. We chose $c = 0.5, T = 2\pi$, and

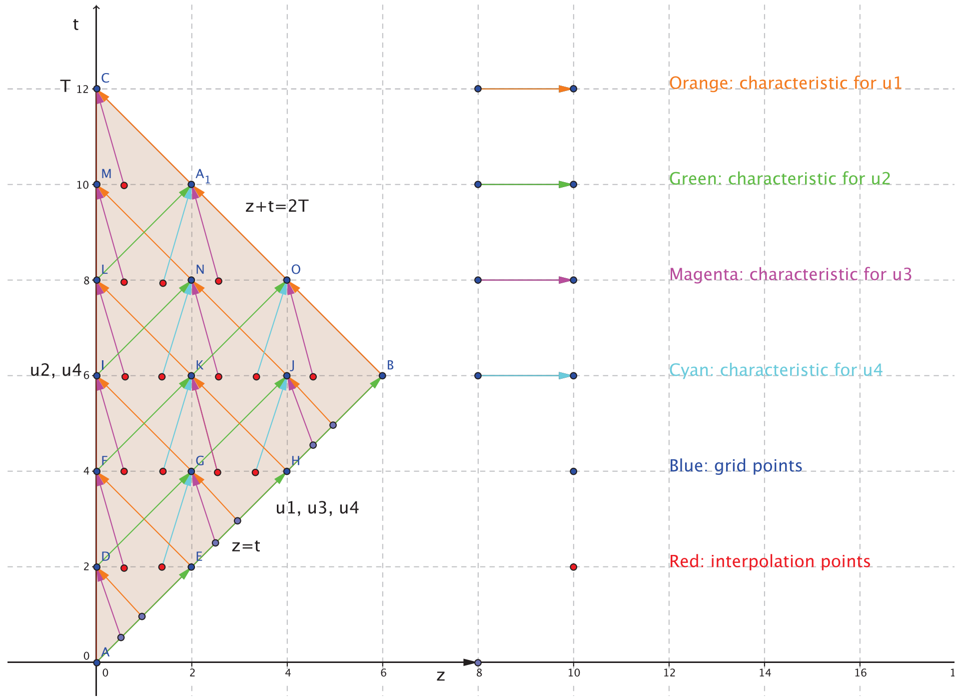

Figure 5.2: Solution of the forward problem

$$\beta(z) = z\sin(100z)\log(z+1), \tag{5.5a}$$

$$u_1(z,t) = 2z^2 + 3t^2 + 4zt + z + t + 1, \tag{5.5b}$$

$$u_2(z,t) = e^{z+t}, \tag{5.5c}$$

$$u_3(z,t) = 1 + \log(z + ct + 1), \tag{5.5d}$$

$$u_4(z,t) = \cos(100z - ct), \tag{5.5e}$$

$$\mathbf{F} = \mathcal{L}\mathbf{u}. \tag{5.5f}$$

Let N_t, N_z be the number of grid points in t and z directions, and let $\{z_i\}, i = 1, ..., N_z$ and $\{t_j\}, j = 1, ..., N_t$ be the grid points in t and z directions. Let $\mathbf{u}, \tilde{\mathbf{u}}$ be the exact and numerical solutions of $(5.3a) - (5.3c)$. Define the L^2 error vectors $\mathbf{E}, \tilde{\mathbf{E}}$ and the maximum relative error vectors $\mathbf{R}, \tilde{\mathbf{R}}$ in t and z directions as follows

$$
\begin{aligned}
E_k(t_j) &:= \sqrt{\frac{T}{N_z}\sum_{i=1}^{N_z}(u_k-\tilde{u}_k)^2(z_i,t_j)}, \quad k=1,...,4, \quad j=1,...,N_t,\\
\tilde{E}_k(z_i) &:= \sqrt{\frac{T}{N_t}\sum_{j=1}^{N_t}(u_k-\tilde{u}_k)^2(z_i,t_j)}, \quad k=1,...,4, \quad i=1,...,N_z,\\
R_k(t_j) &:= \max_{u_k(z_i,t_j)\neq 0}\left|\frac{(u_k-\tilde{u}_k)(z_i,t_j)}{u_k(z_i,t_j)}\right|, \quad k=1,...,4, \quad j=1,...,N_t,\\
\tilde{R}_k(z_i) &:= \max_{u_k(z_i,t_j)\neq 0}\left|\frac{(u_k-\tilde{u}_k)(z_i,t_j)}{u_k(z_i,t_j)}\right|, \quad k=1,...,4, \quad i=1,...,N_z.
\end{aligned}
$$

Figure 5.3 plots $\tilde{R}_1(0), \tilde{R}_3(0), \tilde{E}_1(0), \tilde{E}_3(0)$; Figure 5.4 plots $R_k(T), k=1,...,4$; Figure 5.5 plots $E_k(T), k=1,...,4$ for different number of grid points. All the plots are in *log* scale.

5.2 The Sideways Problem

Given the relfection data $m_1(0,t)$ and $m_3(0,t)$, the recursive scheme to obtain $\beta(z)$ requires solving the sideways problem (5.2a) – (5.2c). So we build a sideways problem solver by numerically implementing the following more generalized sideways problem

$$\mathcal{L}\mathbf{u} = \mathbf{F} \quad \text{in} \quad D_s, \tag{5.7a}$$

$$u_i(0,t) = f_i(t), \quad i=1,2,3,4, \quad t\in[0,2T], \tag{5.7b}$$

$$u_4(z,z) = \frac{(1+c)^2}{(1-c)^2}u_3(z,z), \quad z\in[0,Y] \tag{5.7c}$$

for arbitrary functions $\mathbf{F}(z,t)\in C^1(D_s)$ and $\mathbf{f}(t)\in C^1[0,2T]$ and $g(z), \beta(z)\in C^1[0,Y]$. Again, we use Crank-Nicolson method with interpolation to solve (5.4) (see Figure 5.6)

We test the algorithm for the sideways problem by applying it to the example in (5.5a) – (5.5f). Let N be the number of grid points, and $\{z_i\}, i=1,...,N$ be the grid

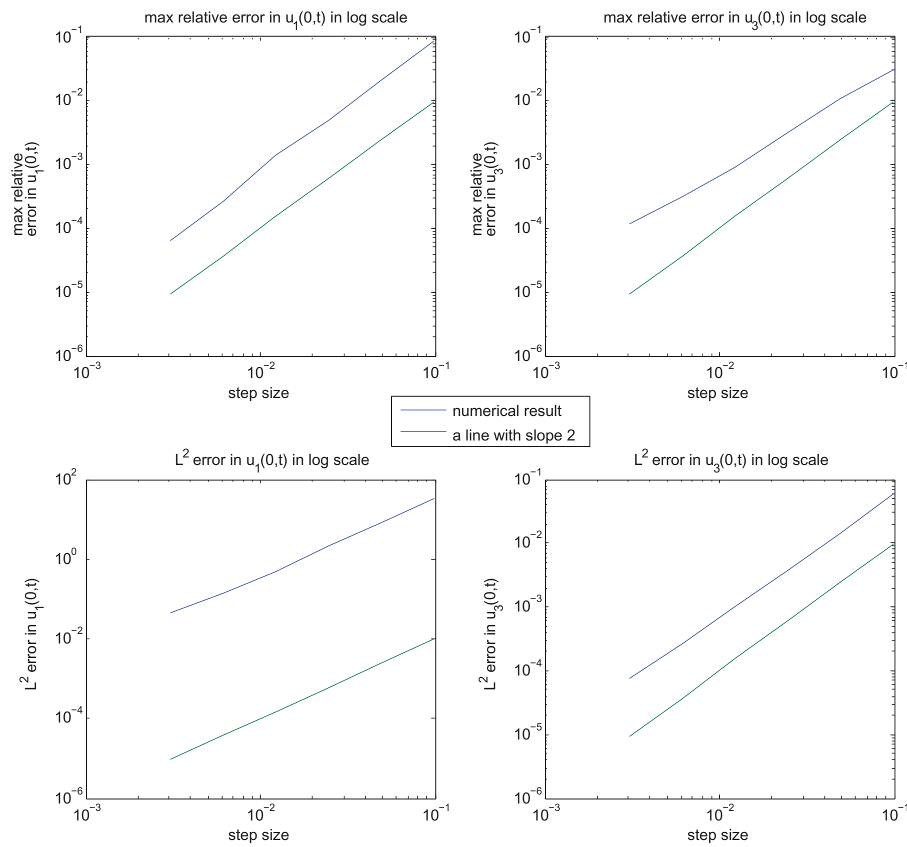

Figure 5.3: $\tilde{R}_1(0), \tilde{R}_3(0), \tilde{E}_1(0), \tilde{E}_3(0)$

points in z direction. Let $\mathbf{u}$ and $\tilde{\mathbf{u}}$ be the exact and numerical solution of (5.7a)$-$(5.7c), define the L^2 and maximum relative errors vectors $\mathbf{L}, \mathbf{R}$ as follows

$$L_k := \sqrt{\frac{Y}{N}\sum_{i=1}^{N}(u_k - \tilde{u}_k)^2(z_i, z_i)}, \quad k = 1, 2, 3, 4,$$

$$R_k := \max_{u_k(z_i,z_i)\neq 0}\left|\frac{(u_k - \tilde{u}_k)(z_i, z_i)}{u_k(z_i, z_i)}\right|, \quad k = 1, 2, 3, 4.$$

Figure 5.7 plots $P_k, k = 1, 2, 3, 4$; Figure 5.8 plots $L_k, k = 1, 2, 3, 4$. All plots are in *log* scale.

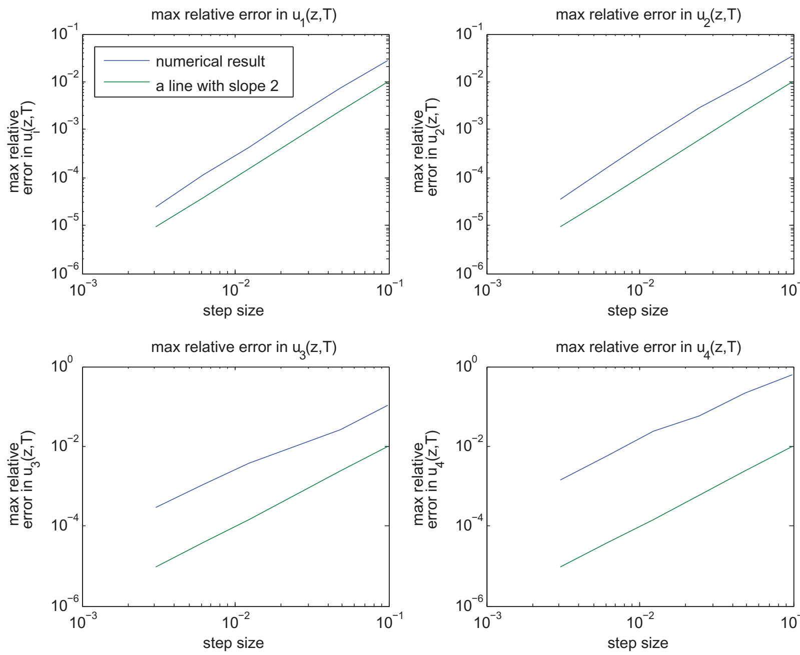

Figure 5.4: $R_k(T), k = 1, ..., 4$

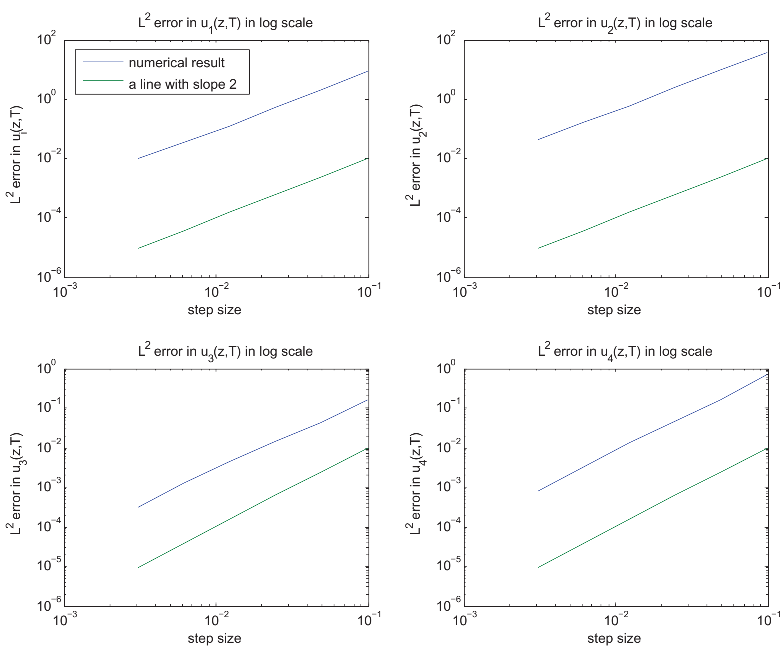

Figure 5.5: $E_k(T), k = 1, ..., 4$

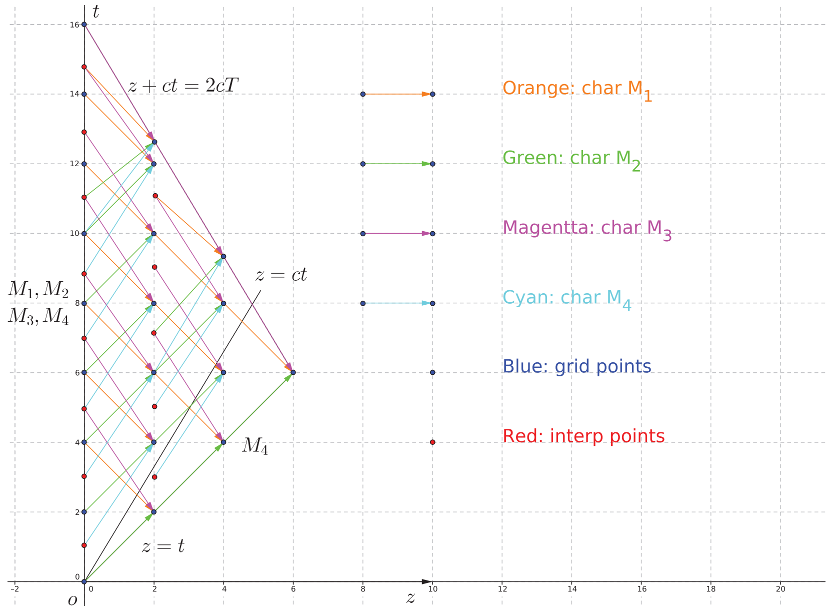

Figure 5.6: Solution of the sideways problem

5.3 Coefficient Recovery

We want to recover $\beta(z)$ numerically from the following CBVP

$$\mathcal{L}\mathbf{m} = \mathbf{0} \text{ in } D_f, \tag{5.8a}$$

$$m_2(0,t) = m_4(0,t) = 0, \quad t \in [0, 2T], \tag{5.8b}$$

$$m_1(z,z) = 0, \quad m_3(z,z) = \frac{c-1}{2(c+1)}\beta(z), \quad m_4(z,z) = \frac{c+1}{2(c-1)}\beta(z), \quad z \in [0,T]. \tag{5.8c}$$

We first solve the forward problem $(5.8a) - (5.8c)$ to get the full reflection data $d_1(t) = m_1(0,t)$ and $d_3(t) = m_3(0,t)$, then we use the reflection data $d_1(t)$ and $d_3(t)$ to solve the sideways CBVP

$$\mathcal{L}\mathbf{p} = \mathbf{0} \text{ in } D_s, \tag{5.9a}$$

$$p_2(0,t) = p_4(0,t) = 0, \quad p_1(0,t) = d_1(t), \quad p_3(0,t) = d_3(t), \quad t \in [0, 2T], \tag{5.9b}$$

$$p_4(z,z) = \frac{(c+1)^2}{(c-1)^2} p_3(z,z), \quad z \in [0, Y]. \tag{5.9c}$$

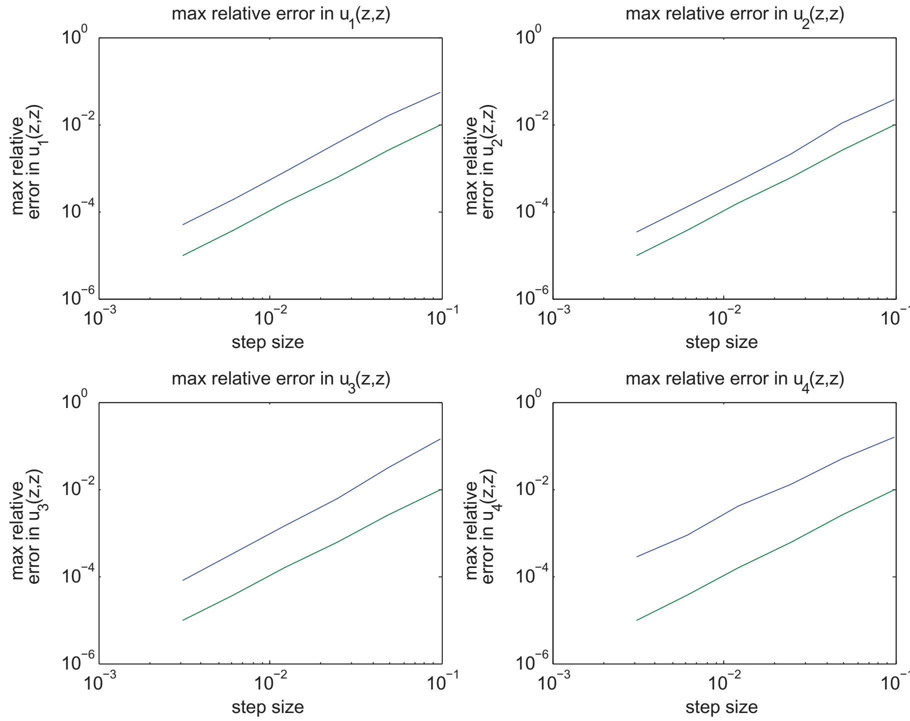

Figure 5.7: $R_k, k = 1, 2, 3, 4$

for different β and then use a fixed point method to determine the correct β. Consider the map Q

$$\begin{aligned} C^1[0, Y] &\to C^1[0, Y], \\ \beta(z) &\to \frac{2(c+1)}{c-1} p_3(z, z). \end{aligned}$$

We have shown in Proposition 4.7 that β is the unique fixed point of Q and we find β using the following recursive scheme:

$i)$ guess a β;

$ii)$ solve the sideways problem (5.2a) $-$ (5.2c);

$iii)$ if $||\beta - \dfrac{2(c+1)}{c-1} p_3(z, z)||_{L^\infty[0,Y]}$ is within some tolerance (usually 0 numerically), then stop, and this is the correct β, otherwise set $\beta(z) = \dfrac{2(c+1)}{c-1} p_3(z, z)$, and go to step ii.

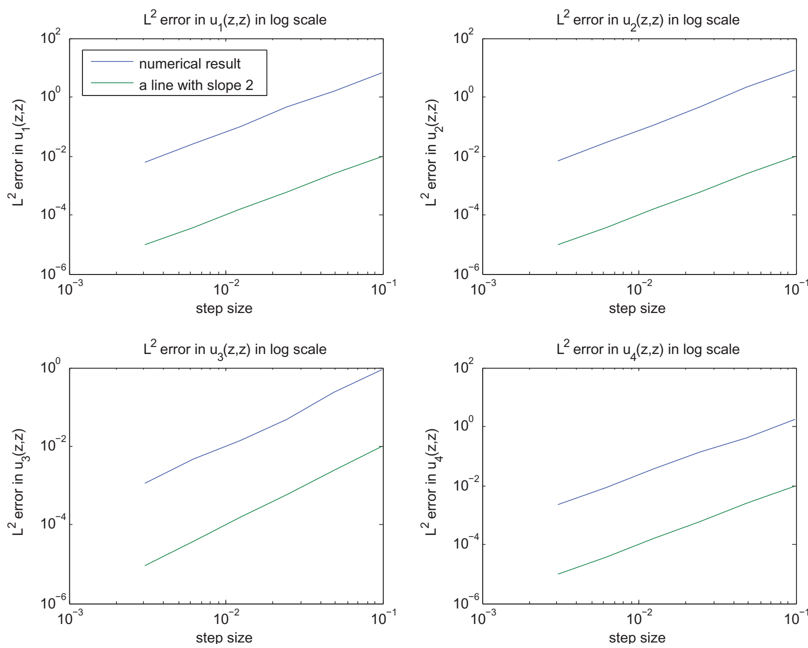

Figure 5.8: $L_k, k = 1, 2, 3, 4$

We use the Crank-Nicolson method with interpolation to solve (5.8a) − (5.8c) and (5.9a) − (5.9c) as in section 5.1 and 5.2. In the following examples, we take $c = 0.5, T = \frac{\pi}{2}$. Let N be the number of grid points, and $\{z_i\}, i = 1, ..., N$ be the grid points in z direction. Let β and $\tilde{\beta}$ be the exact and numerical value of the coefficient, define the L^2 error L and the maximum relative error R as follows

$$L := \sqrt{\frac{T}{N}\sum_{i=1}^{N}(\beta - \tilde{\beta})^2(z_i)},$$

$$R := \max_{\beta(z_i)\neq 0}\left|\frac{(\beta - \tilde{\beta})(z_i)}{\beta(z_i)}\right|.$$

5.3.1 First Example

Here we choose $\beta(z) = z\sin(100z)\log(z+1)$, and the initial guess of β is $\beta_{ig}(z) = z$. Figure 5.9 plots L and P in *log* scale for different step sizes; Table 5.1 shows the number of iterations for each step size; Figure 5.10 plots the exact and numerical result of $\beta(z)$.

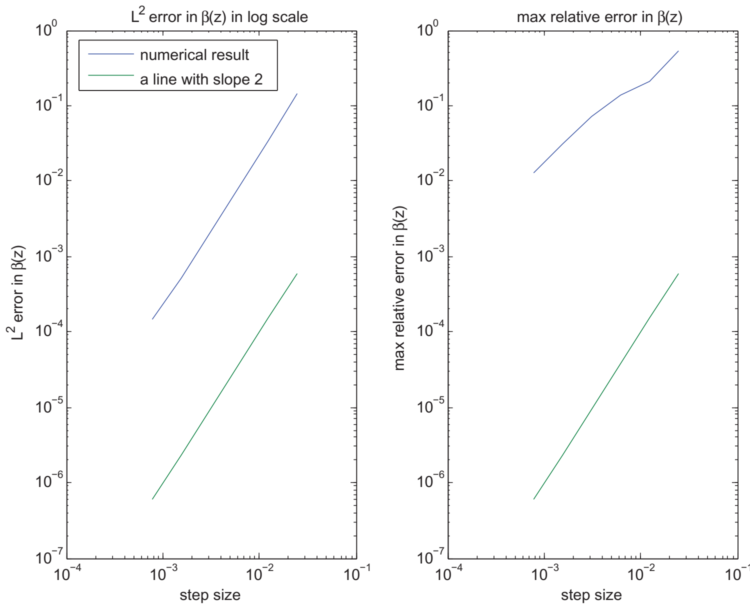

Figure 5.9: L and R errors

# of grid points in z direction	2^6	2^7	2^8	2^9	2^{10}	2^{11}
# of iterations	17	17	17	17	17	17

Table 5.1: Iterations needed for the inversion

5.3.2 Second Example

Here we choose $\beta(z) = 9z^2 \cos(100z) \log(z+1)$, and the initial guess of β is $\beta_{ig}(z) = z$. Figure 5.11 plots L and P in *log* scale for different step sizes; Table 5.2 shows the number of iterations for each step size; Figure 5.12 plots the exact and numerical result of $\beta(z)$.

# of grid points in z direction	2^6	2^7	2^8	2^9	2^{10}	2^{11}
# of iterations	14	14	14	14	14	14

Table 5.2: Iterations needed for the inversion

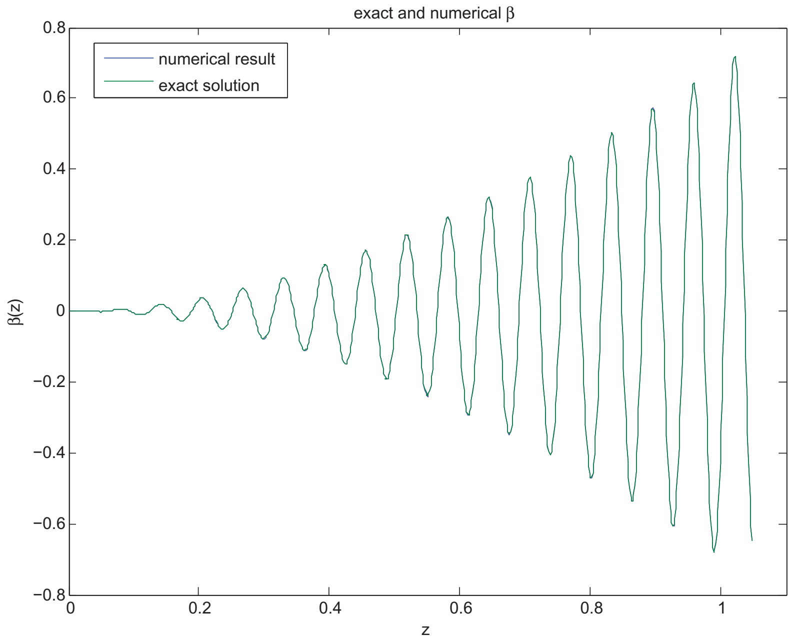

Figure 5.10: Comparing exact β with reconstructed β

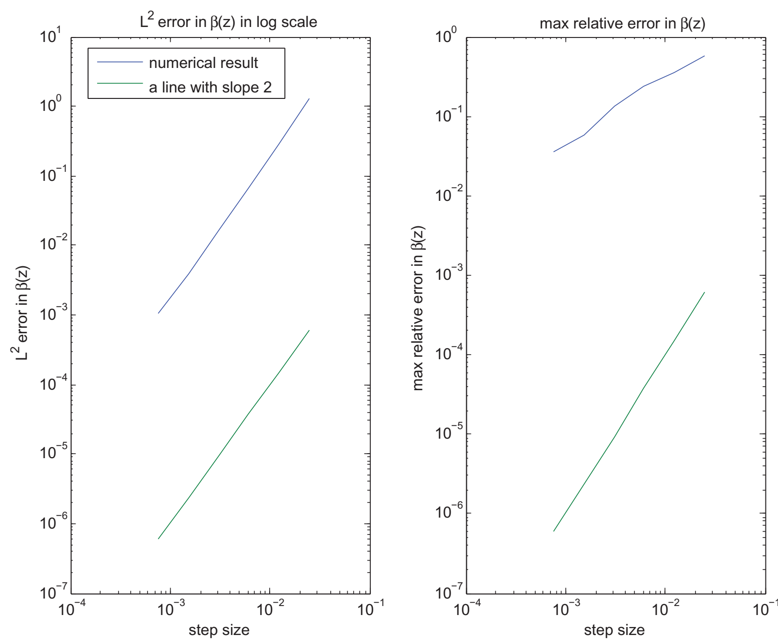

Figure 5.11: L and P errors

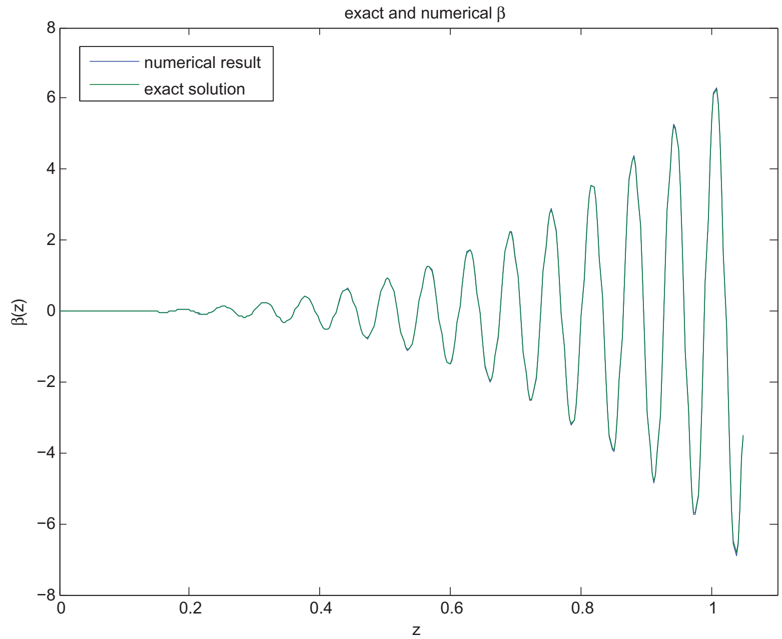

Figure 5.12: Comparing exact β with reconstructed β

5.3.3 Third Example

Here we choose $\beta(z) = z\sin(100z)e^{az}$ for $a \in \mathbb{N}$, and the initial guess of β is $\beta_{ig}(z) = z$. Table 5.3 shows the maximum value of a such that the algorithm will coverge. So when the exponential function e^{az} is involved, the algorithm needs more grid points to find the fixed point, while in the same time increasing the running time.

# of grid points in z direction	2^6	2^7	2^8	2^9	2^{10}	2^{11}
max value of a	3	4	4	5	5	6

Table 5.3: Max value of a for the algorithm to converge

Chapter 6

INVERSE PROBLEM WITH TRANSMISSION DATA

In previous chapters, we discussed the recovery of the coefficient of a hyperbolic system of PDEs from the reflection data, where the source and receiver are at the same end. In this chapter, we discuss the recovery of $\beta(z)$ from the transmission data, where the source and receiver are at different ends.

6.1 Introduction

Let $0 < c < 1, Z > 0$ and define

$$\dot{C}_Z^1[0,\infty) := \{\beta(z) \in \dot{C}^1[0,\infty) \mid \beta(z) = 0 \text{ for } z \geq Z\}.$$

Consider the IBVP

$$\mathbf{M}_t - A\mathbf{M}_z - \beta B\mathbf{M} = \mathbf{0}, \quad z \geq 0, \quad t \in \mathbb{R}, \tag{6.1a}$$

$$(M_1 - M_2)(0,t) = \delta(t), \quad (M_3 - M_4)(0,t) = 0, \quad t \in \mathbb{R}, \tag{6.1b}$$

$$\mathbf{M}(z,t) = \mathbf{0}, \quad t < 0, \tag{6.1c}$$

where

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & -c \end{bmatrix}, \quad B = \frac{1}{2}\begin{bmatrix} 0 & 0 & -1-c & -1+c \\ 0 & 0 & 1-c & 1+c \\ 1+c & -1+c & 0 & 0 \\ 1-c & -1-c & 0 & 0 \end{bmatrix},$$

with M_1, M_3 denoting the left moving waves with speed 1 and c, and M_2, M_4 denoting the right moving waves with speed 1 and c. (See Figure 6.1)

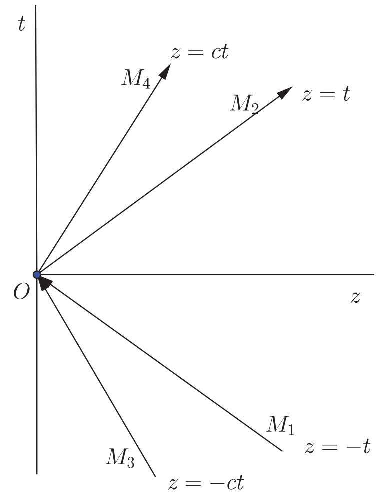

Figure 6.1: Left and right moving waves

Note that the boundary condition (6.1b) is different from the boundary condition (4.1b) used for the reflection data inverse problem. (4.1b) corresponds to an incoming delta function plane wave of speed 1 with the $z = 0$ boundary being transparent. The boundary condition (6.1b) also corresponds to an incoming plane of speed 1 but the boundary $z = 0$ is now fully reflecting. The boundary $z = 0$ being fully reflecting is important for the solution of the transmission data inverse problem.

Since the source is localized to $(z = 0, t = 0)$, from the speed of propagation we obtain that $M_1(Z,t) = 0, M_3(Z,t) = 0$ for $t < Z$. Since $\beta(z) = 0$ for $z \geq Z$, we have $M_1(Z,t) = M_3(Z,t) = 0, t \in \mathbb{R}$ because nothing comes from the right of the boundary $z = Z$. So the **goal** is to recover $\beta(z)$ from the transmission data $M_2(Z,t)$ and $M_4(Z,t)$ or more, which means that we may also need the transmission data $\tilde{M}_2(Z,t)$

and $\tilde{M}_4(Z,t)$ where $\tilde{\mathbf{M}}(z,t)$ is the solution of the IBVP

$$\tilde{\mathbf{M}}_t - A\tilde{\mathbf{M}}_z - \beta B\tilde{\mathbf{M}} = \mathbf{0}, \quad z \geq 0, \quad t \in \mathbb{R}, \tag{6.2a}$$

$$(\tilde{M}_1 - \tilde{M}_2)(0,t) = 0, \quad (\tilde{M}_3 - \tilde{M}_4)(0,t) = \delta(t), \quad t \in \mathbb{R}, \tag{6.2b}$$

$$\tilde{\mathbf{M}}(z,t) = \mathbf{0}, \quad t < 0. \tag{6.2c}$$

Much work has been done on one dimensional inverse problems for the wave equation, but most of it has been on inversion from reflection data, where the source and receiver are at the same location. Inversion from reflection data has been thoroughly analyzed, see [22] and [26] for reference. Inversion from transmission plus reflection data is analyzed in [12]. Inversion from reflection data is in some sense a local problem since $\mathbf{M}(0,t)$ for $t \in [0,Z]$ is influenced by the value of $\beta(z)$ only if $z \in [0, Z/2]$. This allows the use of so-called layer stripping techniques to resolve the inverse reflection problem. In contrast, for inversion from transmission data $\mathbf{M}(Z,t)$, even the earliest signal sensed at $z = Z$ has been influenced by the complete medium $z \in [0,Z]$, which makes inversion from transmission data more difficult.

Rakesh and Sacks in [18] showed how to recover coefficients from transmission data for the Webster's Horn equation, based on an idea of Claerbout in [11] for solving the discrete version of that problem. Rakesh in [16] proved an analogue of the result in [18] for the one dimensional plasma equation. Suppose $q(z) \in C[0,\infty)$ and is zero for $z \geq Z$ for some known positive number Z. Consider the IBVP

$$u_{tt} - u_{zz} + q(z)u = 0, \quad z \geq 0, \quad t \in \mathbb{R}, \tag{6.3a}$$

$$u = 0, \quad t < 0, \tag{6.3b}$$

$$u_z(0,t) = -\delta(t), \quad t \in \mathbb{R}. \tag{6.3c}$$

The author showed that q may be reconstructed from the transmission data $m(t) = u(Z,t) - H(t-Z)$ provided we are given an upper bound on $||q||_\infty$. The main step is to relate the transmission data to the reflection data for a related problem, for which one can apply downward continuation method. Consider the Goursat problem

$$F_{tt} - F_{zz} + qF = 0, \quad |t| \le z, \tag{6.4a}$$

$$F(z, \pm z) = \frac{1}{4}\int_0^z q(\sigma)\, d\sigma, \quad z \ge 0. \tag{6.4b}$$

Rakesh in [16] showed that

$$F(Z,t) = \frac{1}{2}[\alpha(Z+t) + \alpha(Z-t) - \alpha(2Z)], \tag{6.5a}$$

$$F_z(Z,t) = \frac{1}{2}[\alpha'(Z+t) + \alpha'(Z-t)], \tag{6.5b}$$

where

$$m'(t+Z) + \alpha(t) + \int_0^t m'(Z+t-s)\alpha(s)\, ds = 0, \quad 0 \le t \le 2Z. \tag{6.6}$$

So given the transmission data $m(t)$ for $t \in [Z, 3Z]$, one can solve the Volterra equation (6.6) to get $\alpha(t)$ for $t \in [0, 2Z]$, then from $(6.5a) - (6.5b)$, one can obtain the data $F(Z,t)$ and $F_z(Z,t)$ in $t \in [0, 2Z]$ for the reflection problem $(6.4a) - (6.4b)$. So one can apply downward continuation method to recover $q(z)$ for $z \in [0, Z]$.

So far, for the transmission data inverse problem, all work has been for a single speed PDE. Here we study the two speed inverse transmission data problem.

In this chapter we obtain an upper bound of the coefficient in terms of the transmission data in an infinite interval.

If $\mathbf{M}(z,t)$ satisfies $(6.1a) - (6.1c)$ and $\tilde{\mathbf{M}}(z,t)$ satisfies $(6.2a) - (6.2c)$ corresponding to $\beta \in \dot{C}^1_Z[0,\infty)$, then using the progressing wave expansion

$$\mathbf{M}(z,t) = \delta(t-z)[0,-1,0,0]^T + \mathbf{m}(z,t)H(t-z), \quad z \ge 0, \quad t \in \mathbb{R}, \tag{6.7a}$$

$$\tilde{\mathbf{M}}(z,t) = \delta(t-z/c)[0,0,0,-1]^T + \tilde{\mathbf{m}}(z,t)H(t-z), \quad z \ge 0, \quad t \in \mathbb{R}, \tag{6.7b}$$

where $\mathbf{m}(z,t)$ is supported in $t \geq z$ and satisfies the CBVP

$$\mathbf{m}_t - A\mathbf{m}_z - \beta B\mathbf{m} = \mathbf{0}, \quad 0 \leq z \leq t, \tag{6.8a}$$

$$(m_1 - m_2)(0,t) = 0, \quad (m_3 - m_4)(0,t) = 0, \quad t \geq 0, \tag{6.8b}$$

$$m_1(z,z) = 0, \quad m_3(z,z) = \frac{1-c}{2(c+1)}\beta(z), \quad m_4(z,z) = \frac{c+1}{2(1-c)}\beta(z), \quad z \geq 0. \tag{6.8c}$$

and $\tilde{\mathbf{m}}(z,t) = \begin{cases} \tilde{\mathbf{g}}(z,t), & 0 \leq z \leq ct \\ \tilde{\mathbf{f}}(z,t), & 0 \leq ct \leq z \leq t \end{cases}$ is supported in $t \geq z$ and satisfies the CBVP

$$\mathcal{L}\tilde{\mathbf{g}} = 0, \quad 0 \leq z \leq ct, \tag{6.9a}$$

$$\mathcal{L}\tilde{\mathbf{f}} = 0, \quad 0 \leq ct \leq z \leq t, \tag{6.9b}$$

$$\begin{aligned} &(\tilde{g}_1 - \tilde{f}_1)(z, z/c) = \frac{c(1-c)}{2(1+c)}\beta(z), \quad (\tilde{g}_2 - \tilde{f}_2)(z, z/c) = \frac{c(1+c)}{2(1-c)}\beta(z), \\ &(\tilde{g}_3 - \tilde{f}_3)(z, z/c) = 0, \quad z \geq 0, \end{aligned} \tag{6.9c}$$

$$\tilde{f}_1(z,z) = \tilde{f}_3(z,z) = \tilde{f}_4(z,z) = 0, \quad z \geq 0, \tag{6.9d}$$

$$\tilde{g}_1(0,t) = \tilde{g}_2(0,t), \quad \tilde{g}_3(0,t) = \tilde{g}_4(0,t), \quad t \geq 0. \tag{6.9e}$$

Also note that since $M_1(Z,\cdot) = M_3(Z,\cdot) = 0$ and $\tilde{M}_1(Z,\cdot) = \tilde{M}_3(Z,\cdot) = 0$ we will have

$$m_1(Z,t) = 0, \quad m_3(Z,t) = 0, \quad t \geq Z. \tag{6.10a}$$

$$\tilde{m}_1(Z,t) = 0, \quad \tilde{m}_3(Z,t) = 0, \quad t \geq Z.. \tag{6.10b}$$

Theorem 6.1. *If* $\mathbf{m}(z,t)$ *is the solution of* (6.8a) $-$ (6.8c) *for* $\beta(z) \in \dot{C}^1_Z[0,\infty)$*, then*

$$C||\beta||^2_{L^2[0,Z]} = \int_Z^\infty (m_2^2 + cm_4^2)(Z,t)\; dt, \tag{6.11}$$

where $C = \frac{1}{4}\left(\frac{(1-c)^2}{1+c} + \frac{(1+c)^2}{1-c}\right)$.

We also obtain a relation between the reflection data $m_1(0,\cdot), m_3(0,\cdot)$ and the transmission data $m_2(Z,\cdot), m_4(Z,\cdot)$ and $\tilde{m}_2(Z,\cdot), \tilde{m}_4(Z,\cdot)$.

Theorem 6.2. *If $\mathbf{m}(z,t)$ and $\tilde{\mathbf{m}}(z,t)$ are the solution of (6.8a) − (6.8c) and (6.9a) − (6.9e) corresponding to $\beta(z) \in \dot{C}^1_Z[0,\infty)$, then for $s > 0$ we have*

$$m_1(0,s) = m_2(Z, Z+s) - m_2(Z,-s) * m_2(Z,s) - cm_4(Z,-s) * m_4(Z,s), \tag{6.12a}$$

$$cm_3(0,s) = cm_4(Z, Z/c+s) - m_2(Z,s) * \tilde{m}_2(Z,-s) - cm_4(Z,s) * \tilde{m}_4(Z,-s), \tag{6.12b}$$

where $*$ stands for convolution with respect to s.

The boundary condition (6.1b) and (6.2b) play an important role in obtaining (6.12a)−(6.12b). One can not obtain a relation of the type (6.12a) − (6.12b) if (6.1b) and (6.2b) were replaced by the boundary condition used in Chapters 2-5 for the inversion from the reflection data problem, namely,

$$M_2(0,t) = \delta(t), \quad M_4(0,t) = 0, \quad t \in \mathbb{R}.$$

Given the transmission data $M_2(Z,t)$ and $M_4(Z,t)$ for $t \in \mathbb{R}$, one can compute the reflection data $M_1(0,s)$ and $M_3(0,s)$ for $s \in \mathbb{R}$ from Theorem 6.2. One can then apply a downward continuation method similar to the one used in Chapter 4 to recover $\beta(z)$ step by step through the z direction. However, this conversion of a transmission data inverse problem to a reflection data inverse problem requires knowledge of the transmission data over an infinite time interval. We would prefer to be able to do this with transmission data over a finite time interval as done in [18] for the single speed problem, but we are unable to do that. A lower bound on the step size is guaranteed because we obtain an upper bound of the coefficient $\beta(z)$ in terms of the transmission data $M_2(Z,t)$ and $M_4(Z,t)$ in Theorem 6.1.

In the following sections, we first show the well-posedness of (6.1a) − (6.1c), then we analyze the linearized problem of (6.1a) − (6.1c) for $\beta = 0$, which gives an indication of what transmission data we need. Next we show that the solution $\mathbf{m}(z,t)$ of (6.8a) − (6.8c) decays as $t \to \infty$, which is essential for obtaining the upper bound of β in terms of the transmission data. Lastly, we show a relation between reflection and transmission data.

6.2 Well-posedness of the Transmission Problem

In this section we study the existence and uniqueness of the solution of (6.8a) – (6.8c).

Let $0 < c < 1, Z > 0$, and define (see Figure 6.2)

$$D_Z := \{(z,t) \mid 0 \le z \le Z, z \le t \le 2Z - z\}.$$

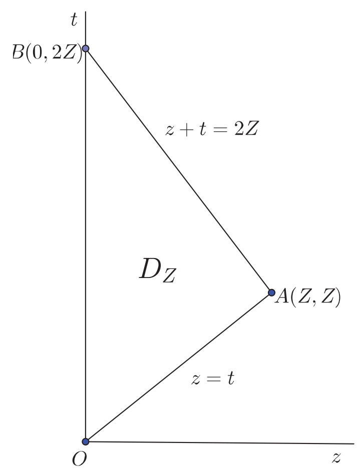

Figure 6.2: The region D_Z

Proposition 6.1. *If* $\beta(z) \in \dot{C}^1_Z[0,\infty)$*, then* (6.8$a$) – (6.8$c$) *has a unique solution* $\mathbf{m}(z,t) \in C^1(D_Z)$ *for* $Z > 0$.

Proof. Define $\mathbf{r}(\mathbf{m}, z, t) := \beta(z)B\mathbf{m}(z,t)$ and pick an arbitrary point $P(z,t) \in D_Z$ (see Figure 6.3). Integrating (6.8a) and using the boundary conditions, we have

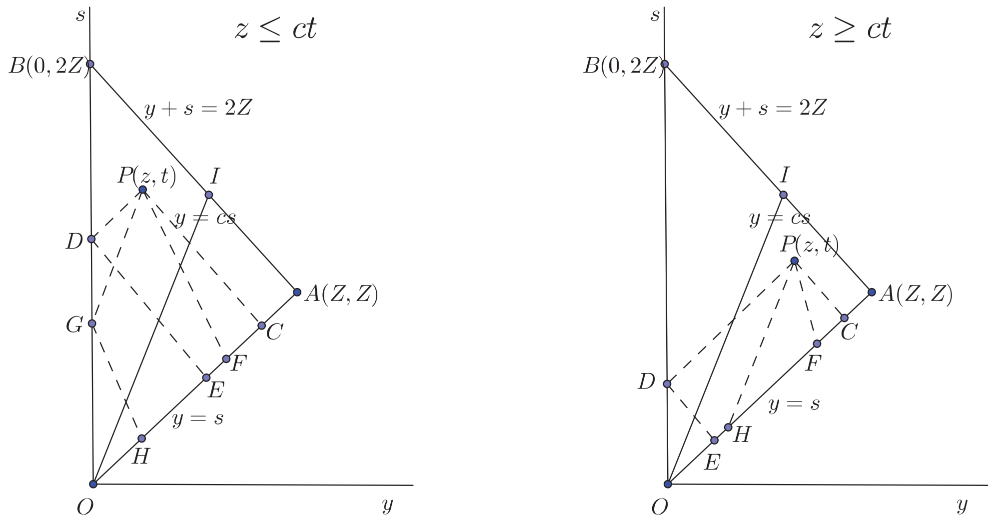

Figure 6.3: Downward moving lines through $P(z,t)$ with slopes ± 1 and $\pm 1/c$

$$m_1(z,t) = \int_{s_C}^{t} r_1(\mathbf{m}, z+t-s, s)\; ds, \tag{6.13a}$$

$$m_2(z,t) = \int_{s_D}^{t} r_2(\mathbf{m}, s+z-t, s)\; ds + \int_{s_E}^{s_D} r_1(\mathbf{v}, z+t-s, s)\; ds, \tag{6.13b}$$

$$m_3(z,t) = \int_{s_F}^{t} r_3(\mathbf{m}, z+ct-cs, s)\; ds + \frac{1-c}{2(c+1)}\beta(y_F), \tag{6.13c}$$

$$m_4(z,t) = \begin{cases} \int_{s_H}^{t} r_4(\mathbf{m}, cs+z-ct, s)\; ds + \frac{c+1}{2(1-c)}\beta(y_H), & \text{if } z \ge ct \\ \int_{s_G}^{t} r_4(\mathbf{m}, cs+z-ct, s)\; ds + \int_{s_H}^{s_G} r_3(\mathbf{m}, z+ct-cs, y)\; ds + \frac{1-c}{2(c+1)}\beta(y_H), & \text{if } z \le ct \end{cases} \tag{6.13d}$$

One may verify from Figure 6.3 that $0 \le s_C, s_D, s_E, s_F, s_G, s_H \le t$, so from Theorem 3.1, $(6.13a) - (6.13d)$ has a unique solution $\mathbf{m}(z,t) \in C^1(OBI) \cup C^1(OAI)$. Since $\beta \in \dot{C}^1_Z[0,\infty)$, one can use an argument similar to the proof of Proposition 3.1 to show that $\mathbf{m} \in C^1(D_Z)$. □

6.3 Linearization on $\beta = 0$

In this section we analyze the linearization of $(6.8a) - (6.8c)$ around $\beta = 0$ to get an indication of what transmission data we need for the inverse problem. Define the forward map

$$F : \dot{C}^1_Z[0,\infty) \to C^1[Z,\infty) \times C^1[Z,\infty),$$
$$\beta(z) \to (m_2(Z,t), m_4(Z,t)),$$

which maps the coefficient to the transmission data. We analyze the formal derivative of F at $\beta = 0$, namely

$$F' : \dot{C}^1_Z[0,\infty) \to C^1[Z,\infty) \times C^1[Z,\infty),$$
$$d\beta(z) \to (dm_2(Z,t), dm_4(Z,t)),$$

where $d\beta$ is a small perturbation of β and $d\mathbf{m}$ is the small perturbation of the solution of $(6.8a) - (6.8c)$ corresponding to β, that is, $\mathbf{m} + d\mathbf{m}$ is the solution of $(6.8a) - (6.8c)$ corresponding to $\beta + d\beta$. Hence from $(6.8a)$

$$(\mathbf{m} + d\mathbf{m})_t - A(\mathbf{m} + d\mathbf{m})_z - (\beta + d\beta)B(\mathbf{m} + d\mathbf{m}) = 0. \tag{6.14}$$

If we subtract $(6.8a)$ from (6.14) for $\beta = 0$, and only keep the linear terms, we have

$$(d\mathbf{m})_t - A(d\mathbf{m})_z = (d\beta)B\mathbf{m} \quad \text{in } z \geq 0, \quad t \geq z. \tag{6.15}$$

When $\beta = 0$, the solution of $(6.8a) - (6.8c)$ is $\mathbf{m}(z,t) = \mathbf{0}$. So (6.15) becomes

$$(d\mathbf{m})_t - A(d\mathbf{m})_z = 0 \quad \text{in } z \geq 0, \quad t \geq z, \tag{6.16}$$

with the boundary conditions

$$(dm)_1(0,t) = (dm)_2(0,t), \quad (dm)_3(0,t) = (dm)_4(0,t), \quad t \geq 0, \tag{6.17a}$$
$$(dm)_1(z,z) = 0, \quad (dm)_3(z,z) = \frac{1-c}{2(c+1)}(d\beta)(z), \quad (dm)_4(z,z) = \frac{c+1}{2(1-c)}(d\beta)(z), \quad z \geq 0. \tag{6.17b}$$

Integrating the $(dm)_2$ and $(dm)_4$ equations of (6.16) and using $(6.17a) - (6.17b)$, we have (see Figure 6.4)

$$(dm)_2(Z,t) = \int_{s_D}^{t} 0 \; ds + \int_{s_E}^{s_D} 0 \; ds + (dm)_1(y_E, y_E) = 0, \quad t \geq Z,$$
$$(dm)_4(Z,t) = \int_{s_F}^{t} 0 \; ds + (dm)_4(y_F, y_F) = \frac{c+1}{2(1-c)}(\delta\beta)\left(\frac{Z - ct}{1-c}\right), \quad t \geq Z,$$

which implies that one needs $m_4(Z,t)$ in $[Z, Z/c]$ to recover β in $[0, Z]$.

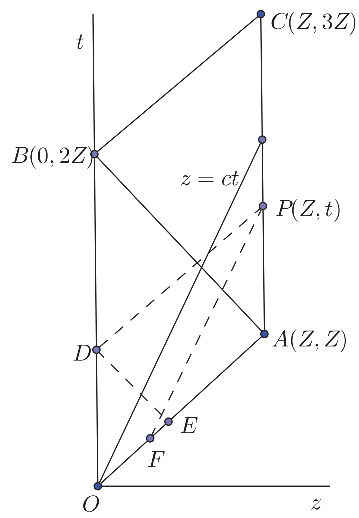

Figure 6.4: Rays for the linearized problem

6.4 Estimating β by the Transmission Data

We now prove Theorem 6.1 whose proof will also need an energy decay result which is also proved here.

For $Z > 0$, define (see Figure 6.5)

$$D := \{(z,t) \mid 0 \le z \le t\},$$

$$D_Z := \{(z,t) \mid 0 \le z \le Z, t \ge z\}.$$

For any four-dimensional vector function $\mathbf{m}(z,t) \in C([0,Z]\times\mathbb{R})$, define the time energy

$$E(\mathbf{m},t) := \sum_{i=1}^{4} \int_0^Z m_i(z,t)\ dz, \quad t \ge Z.$$

Proof of Theorem 6.1:

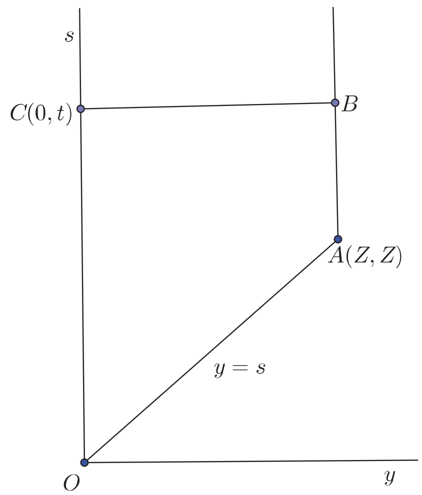

Figure 6.5: The region D_Z

Proof. Let $\mathbf{m}(z,t) \in C^1(D)$ be the solution of (6.8a) − (6.8c), then multiplying both sides of (6.8a) by $2\mathbf{m}^T$ we have

$$U_t - V_z = 2\beta \mathbf{m}^T B \mathbf{m} = 0, \tag{6.18}$$

where $U = m_1^2 + m_2^2 + m_3^2 + m_4^2, \quad V = m_1^2 - m_2^2 + cm_3^2 - cm_4^2$. Integrating (6.18) over the trapezoidal region $OABC$ (see Figure 6.5), we have

$$\begin{aligned}
0 =& \int_{\partial OABC} (m_1^2 + m_2^2 + m_3^2 + m_4^2)\ dz + \int_{\partial OABC} (m_1^2 - m_2^2 + cm_3^2 - cm_4^2)\ dt \\
=& \int_{OA} (2m_1^2 + (1+c)m_3^2 + (1-c)m_4^2)\ dt - \int_Z^t (m_2^2 + cm_4^2)(Z,s)\ ds - E(\mathbf{m},t) \\
=& C||\beta||^2_{L^2[0,Z]} - \int_Z^t (m_2^2 + cm_4^2)(Z,s)\ ds - E(\mathbf{m},t),
\end{aligned}$$

where $C = \frac{1}{4}\left(\frac{(1-c)^2}{1+c} + \frac{(1+c)^2}{1-c}\right)$, which implies that

$$C||\beta||^2_{L^2[0,Z]} = \int_Z^t (m_2^2 + cm_4^2)(Z,s)\ ds + E(\mathbf{m},t), \quad t \geq Z. \tag{6.19}$$

Since $\mathbf{m}(Z,t) = 0$ for $t < Z$, from (6.10a) and (6.19) we have

$$\int_{-\infty}^{\infty} (m_1^2 + m_2^2 + m_3^2 + cm_4^2)(Z,s)\ ds = \int_Z^{\infty} (m_2^2 + cm_4^2)(Z,s)\ ds \leq C||\beta||^2_{L^2[0,Z]},$$

implying that $\mathbf{m}(Z,t) \in L^2[Z,\infty)$, and since (as shown in Proposition 6.2 below) $\lim_{t\to\infty} E(\mathbf{m},t) = 0$, we obtain (6.11). $\square$

Proposition 6.2. *If $\beta \in \dot{C}^1_Z[0,\infty)$ and $\mathbf{m}(z,t) \in C^1(D)$ is the solution of (6.8a) – (6.8c), then $\lim_{t\to\infty} E(\mathbf{m},t) = 0$.*

Proof. Fix $T > \frac{1+c}{c}Z$ and choose $a > 0$ so that $T - a \geq \frac{1+c}{c}Z$. Define (see region $ABCD$ in Figure 6.6)

$$D(a,T) := \{(y,s) \mid 0 \leq y \leq Z, -y + c(T-a) \leq cs \leq y + c(T+a)\}.$$

In Figure 6.6, CB and HG are lines of slope $-1/c$, and DA is a line of slope $1/c$.

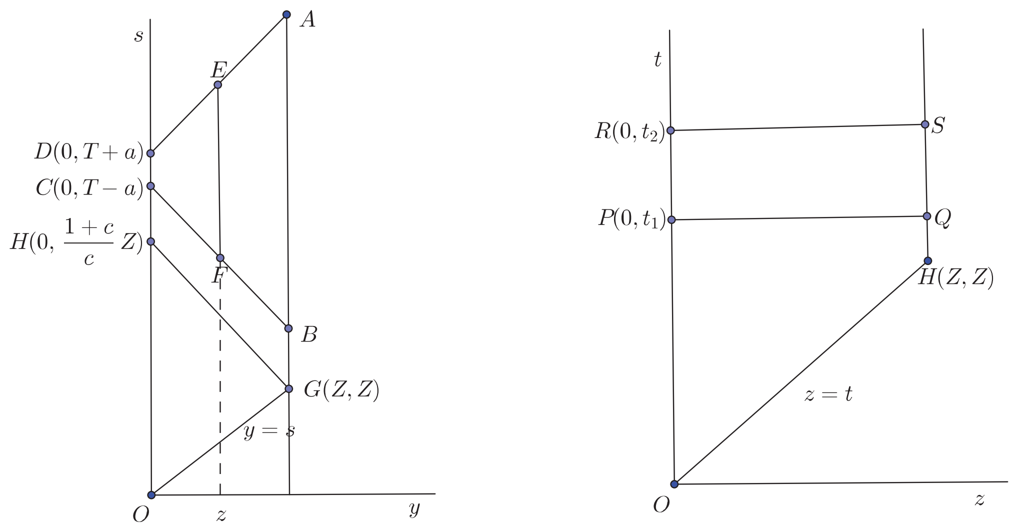

Figure 6.6: Energy Decay

Let $\mathbf{m}(z,t)$ be the solution of (6.8a) – (6.8c) and define $\mathbf{u} := [m_1, -m_2, m_3, -m_4]^T$. Multiplying both sides of (6.8a) by $2\mathbf{u}^T$, we have

$$H_t - G_z = 2\beta \mathbf{u}^T B \mathbf{m}, \tag{6.20}$$

where $H = m_1^2 - m_2^2 + m_3^2 - m_4^2, \quad G = m_1^2 + m_2^2 + cm_3^2 + cm_4^2$. Define (see Figure 6.6)

$$V(\mathbf{m}, z; T, a) := \int_{FE} (m_1^2 + m_2^2 + cm_3^2 + cm_4^2)(z, s)\ ds, \quad z \in [0, Z].$$

Integrating both sides of (6.20) over $ABFE$ we have

$$\begin{aligned}
\iint_{ABFE} &\text{LHS of (6.20) } dA \\
&= -\int_{BA} G\ dt + \int_{BF} G\ dt + \int_{FE} G\ dt + \int_{EA} G dt - c\int_{BF} H\ dt + c\int_{EA} H\ dt \\
&= -V(\mathbf{m}, Z; T, a) + V(\mathbf{m}, z; T, a) + \int_{EA} ((1+c)m_1^2 + (1-c)m_2^2 + 2cm_3^2)\ dt \\
&\quad + \int_{BF} ((1-c)m_1^2 + (1+c)m_2^2 + 2cm_4^2)\ dt
\end{aligned} \tag{6.21}$$

and

$$\begin{aligned}
\iint_{ABFE} |\text{RHS of (6.20)}|\ dA &\le 4\sum_{i=1}^{4} \iint_{ABFE} |\beta(y)| m_i^2(y, t)\ dA \\
&\le \frac{4}{c}\int_z^Z |\beta(y)| V(\mathbf{m}, y; T, a)\ dy.
\end{aligned} \tag{6.22}$$

Then from (6.21) – (6.22)

$$V(\mathbf{m}, z; T, a) \le V(\mathbf{m}, Z; T, a) + \frac{4}{c}\int_z^Z |\beta(y)| V(\mathbf{m}, y; T, a)\ dy,$$

and hence from Gronwall's inequality

$$V(\mathbf{m}, z; T, a) \le V(\mathbf{m}, Z; T, a) e^{\frac{4}{c}\int_0^Z |\beta(z)| dz} \le e^{\frac{4}{c}||\beta||_{L^2[0,Z]}\sqrt{Z}} V(\mathbf{m}, Z; T, a), \quad z \in [0, Z]. \tag{6.23}$$

Similarly, multiplying both sides of (6.8a) by $2\mathbf{m}^T$ we have

$$U_t - V_z = 2\beta \mathbf{m}^T B \mathbf{m} = 0, \tag{6.24}$$

where $U = m_1^2 + m_2^2 + m_3^2 + m_4^2, \quad V = m_1^2 - m_2^2 + cm_3^2 - cm_4^2$. Integrating (6.24) over $PQSR$ (see Figure 6.6), and using (6.8b) and (6.10a) we have

$$\begin{aligned}
0 &= \int_{\partial PQSR} (m_1^2 + m_2^2 + m_3^2 + m_4^2)\ dz + (m_1^2 - m_2^2 + cm_3^2 - cm_4^2)\ dt \\
&= -\int_{QS} (m_2^2 + cm_4^2)\ dt - E(\mathbf{m}, t_2) + E(\mathbf{m}, t_1),
\end{aligned}$$

which implies that

$$E(\mathbf{m}, t_2) \leq E(\mathbf{m}, t_1), \quad 0 \leq Z \leq t_1 \leq t_2. \tag{6.25}$$

From (6.25) and (6.23), we have

$$\begin{aligned} aE(\mathbf{m}, T) &\leq \int_{T-a}^{T} E(\mathbf{m}, s)\, ds \leq \sum_{i=1}^{4} \iint_{ABCD} m_i^2 \, dA \leq \frac{1}{c} \int_0^Z V(\mathbf{m}, z; T, a)\, dz \\ &\leq \frac{1}{c} e^{\frac{4}{c}||\beta||_{L^2[0,Z]}\sqrt{Z}} \int_0^Z V(\mathbf{m}, Z; T, a)\, dz = \frac{Z}{c} e^{\frac{4}{c}||\beta||_{L^2[0,Z]}\sqrt{Z}} V(\mathbf{m}, Z; T, a) \\ &= \frac{Z e^{\frac{4}{c}||\beta||_{L^2[0,Z]}\sqrt{Z}}}{c} \int_{T-(\frac{Z}{c}+a)}^{T+(\frac{Z}{c}+a)} (m_1^2 + m_2^2 + cm_3^2 + cm_4^2)(Z, s)\, ds. \end{aligned}$$

Since $\mathbf{m}(Z, t) \in L^2[Z, \infty)$, the integral on the right approaches zero as $t \to \infty$, and hence $E(\mathbf{m}, t)$ approaches zero as t approaches infinity. $\square$

6.5 Relation between Reflection and Transmission Data

In this section we obtain the relation between reflection and transmission data for $(6.1a) - (6.1c)$ and $(6.2a) - (6.2c)$.

Proof of Theorem 6.2:

Proof. We give a formal proof using the distribution solutions $\mathbf{M}(z, t)$. A rigorous proof using the smoother part $\mathbf{m}(z, t)$ can be constructed along the same lines except instead of obtaining functional values obtained by integrating a dirac delta distribution, one would use the trace of $\mathbf{m}$ on $z = t$.

For arbitrary $\mathbf{u}(z, t), \mathbf{w}(z, t)$ on $\mathbb{R}^2$, we have

$$\begin{aligned} \mathbf{w}^T \mathcal{L}\mathbf{u} =& \mathbf{w}^T \mathbf{u}_t - \mathbf{w}^T A\mathbf{u}_z - \beta \mathbf{w}^T B\mathbf{u} \\ =& (\mathbf{w}^T \mathbf{u})_t - (\mathbf{w}^T A\mathbf{u})_z - \mathbf{w}_t^T \mathbf{u} + \mathbf{w}_z^T A\mathbf{u} - \beta \mathbf{w}^T B\mathbf{u} \\ =& (\mathbf{w}^T \mathbf{u})_t - (\mathbf{w}^T A\mathbf{u})_z - \mathbf{u}^T \mathbf{w}_t + \mathbf{u}^T A\mathbf{w}_z - \beta \mathbf{u}^T B^T \mathbf{w} \\ =& (\mathbf{w}^T \mathbf{u})_t - (\mathbf{w}^T A\mathbf{u})_z - \mathbf{u}^T \mathbf{w}_t + \mathbf{u}^T A\mathbf{w}_z + \beta \mathbf{u}^T B\mathbf{w} \\ =& (\mathbf{w}^T \mathbf{u})_t - (\mathbf{w}^T A\mathbf{u})_z - \mathbf{u}^T \mathcal{L}\mathbf{w}, \end{aligned}$$

which implies that

$$\mathbf{w}^T\mathcal{L}\mathbf{u} + \mathbf{u}^T\mathcal{L}\mathbf{w} = (\mathbf{w}^T\mathbf{u})_t - (\mathbf{w}^T A\mathbf{u})_z. \tag{6.26}$$

Next, if $\mathbf{M}(z,t)$ is the solution of (6.1a) $-$ (6.1c) and we take $\mathbf{u}(z,t) = \mathbf{M}(z,t)$ and $\mathbf{w}(z,t;s) = \mathbf{M}(z,t+s)$ then $\mathcal{L}\mathbf{u} = 0$ and $\mathcal{L}\mathbf{w} = 0$ on $[0,Z]\times\mathbb{R}$ and $\mathbf{u}(z,t) = 0$ for $t<0, \mathbf{w}(z,t;s) = 0$ for $t<-s$. So we have

$$\begin{aligned} 0 &= \int_{-\infty}^{T}\int_0^Z (\mathbf{w}^T\mathbf{u})_t - (\mathbf{w}^T A\mathbf{u})_z \; dz\; dt \\ &= \int_{-\infty}^{T} f(0,t;s)\; dt - \int_{-\infty}^{T} f(Z,t;s)\; dt + \sum_{i=1}^{4}\int_0^Z M_i(z,s+T)M_i(z,T)\; dz, \end{aligned} \tag{6.27}$$

where

$$\begin{aligned} f(z,t;s) =& M_1(z,t)M_1(z,t+s) - M_2(z,t)M_2(z,t+s) + cM_3(z,t)M_3(z,t+s) \\ & - cM_4(z,t)M_4(z,t+s). \end{aligned}$$

Fix a real number s and take an arbitrary T so that $T>Z$ and $T+s>Z$. We have $\mathbf{M}(z,T) = m(z,T)$ and $M(z,T+s) = m(z,T+s)$, so from Proposition 6.2 we have

$$\sum_{i=1}^{4}\int_0^Z M_i(z,s+T)M_i(z,T)\; dz \le E(\mathbf{m},s+T) + E(\mathbf{m},T) \to 0, \quad \text{as } T\to\infty,$$

implying that

$$0 = \int_{-\infty}^{\infty} f(0,t;s)\; dt - \int_{-\infty}^{\infty} f(Z,t;s)\; dt. \tag{6.28}$$

From (6.7a) we have

$$\begin{aligned} M_1(z,t) &= m_1(z,t), \quad M_2(z,t) = -\delta(t-z) + m_2(z,t), \\ M_3(z,t) &= m_3(z,t), \quad M_4(z,t) = m_4(z,t), \end{aligned}$$

and from (6.1b) we have

$$M_2(0,t) = M_1(0,t) - \delta(t), \quad M_4(0,t) = M_3(0,t).$$

Hence

$$\begin{aligned}\int_{-\infty}^{\infty} f(0,t;s)\ dt &= \int_{-\infty}^{\infty} M_1(0,t)M_1(0,t+s) - (M_1(0,t)-\delta(t))(M_1(0,t+s)-\delta(t+s))\ dt \\ &= \int_{-\infty}^{\infty} M_1(0,t)\delta(t+s) + M_1(0,t+s)\delta(t) - \delta(t)\delta(t+s)\ dt \\ &= M_1(0,-s) + M_1(0,s) - \delta(s) \\ &= m_1(0,-s) + m_1(0,s) - \delta(s), \quad s \in \mathbb{R}.\end{aligned}$$

Further, since $M_1(Z,t) = M_3(Z,t) = 0$ we have

$$\begin{aligned}\int_{-\infty}^{\infty} f(Z,t;s)\ dt &= -\int_{-\infty}^{\infty} M_2(Z,t)M_2(Z,t+s) + cM_4(Z,t)M_4(Z,t+s)\ dt \\ &= -\int_{-\infty}^{\infty} (-\delta(t-Z) + m_2(Z,t))(-\delta(t+s-Z) + m_2(Z,t+s)) \\ &\quad + cm_4(Z,t)m_4(Z,t+s)\ dt \\ &= m_2(Z,Z+s) + m_2(Z,Z-s) - \delta(s) \\ &\quad - \int_{-\infty}^{\infty} m_2(Z,t)m_2(Z,t+s) + cm_4(Z,t)m_4(Z,t+s)\ dt \\ &= m_2(Z,Z+s) + m_2(Z,Z-s) - \delta(s) \\ &\quad - \int_{-\infty}^{\infty} m_2(Z,-t)m_2(Z,-t+s) + cm_4(Z,-t)m_4(Z,-t+s)\ dt \\ &= m_2(Z,Z+s) + m_2(Z,Z-s) - \delta(s) \\ &\quad - m_2(Z,-s) * m_2(Z,s) - cm_4(Z,-s) * m_4(Z,s), \quad s \in \mathbb{R}.\end{aligned}$$

So from (6.28) we have

$$\begin{aligned}0 &= \int_{-\infty}^{\infty} f(0,t;s)\ dt - \int_{-\infty}^{\infty} f(Z,t;s)\ dt \\ &= m_1(0,-s) + m_1(0,s) - m_2(Z,Z+s) - m_2(Z,Z-s) \\ &\quad + m_2(Z,-s) * m_2(Z,s) + cm_4(Z,-s) * m_4(Z,s), \quad s \in \mathbb{R}.\end{aligned}$$

Now $m_1(0,-s) = 0$ and $m_2(Z,Z-s) = 0$ for $s > 0$ so we obtain

$$m_1(0,s) - m_2(Z,Z+s) + m_2(Z,-s) * m_2(Z,s) + m_4(Z,-s) * m_4(Z,s) = 0, \quad s > 0,$$

proving (6.12a).

Similarly, let $\mathbf{u}(z,t) = \mathbf{M}(z,t)$ and $\mathbf{w}(z,t;s) = \tilde{\mathbf{M}}(z,t+s)$, where $\tilde{\mathbf{M}}$ is the solution of (6.2a) – (6.2c), then $\pounds\mathbf{w} = 0$ on $[0,Z]\times\mathbb{R}$ and $\mathbf{w}(z,t;s) = 0$ for $t < -s$. So we have

$$0 = \int_{-\infty}^{\infty} \tilde{f}(0,t;s)\ dt - \int_{-\infty}^{\infty} \tilde{f}(Z,t;s)\ dt, \tag{6.29}$$

where

$$\begin{aligned}\tilde{f}(z,t;s) =& M_1(z,t)\tilde{M}_1(z,t+s) - M_2(z,t)\tilde{M}_2(z,t+s) + cM_3(z,t)\tilde{M}_3(z,t+s)\\ &- cM_4(z,t)\tilde{M}_4(z,t+s).\end{aligned}$$

From (6.7b) we have

$$\begin{aligned}&\tilde{M}_1(z,t) = \tilde{m}_1(z,t), \quad \tilde{M}_2(z,t) = \tilde{m}_2(z,t),\\ &\tilde{M}_3(z,t) = \tilde{m}_3(z,t), \quad \tilde{M}_4(z,t) = -\delta(t - z/c) + \tilde{m}_4(z,t),\end{aligned}$$

and from (6.2b) we have

$$\tilde{M}_2(0,t) = \tilde{M}_1(0,t), \quad \tilde{M}_4(0,t) = \tilde{M}_3(0,t) - \delta(t).$$

Hence

$$\begin{aligned}\int_{-\infty}^{\infty} \tilde{f}(0,t;s)\ dt &= \int_{-\infty}^{\infty} \tilde{M}_1(0,t+s)(M_1(0,t) - M_2(0,t))\\ &\qquad + cM_3(0,t)(\tilde{M}_3(0,t+s) - \tilde{M}_4(0,t+s))\ dt\\ &= \int_{-\infty}^{\infty} \tilde{M}_1(0,t+s)\delta(t) + cM_3(0,t)\delta(t+s)\ dt\\ &= \tilde{M}_1(0,s) + cM_3(0,-s)\\ &= \tilde{m}_1(0,s) + cm_3(0,-s).\end{aligned}$$

Further, since $M_1(Z,t) = M_3(Z,t) = 0$ we have

$$\begin{aligned}
\int_{-\infty}^{\infty} \tilde{f}(Z,t;s)\ dt &= -\int_{-\infty}^{\infty} M_2(Z,t)\tilde{M}_2(Z,t+s) + cM_4(Z,t)\tilde{M}_4(Z,t+s)\ dt \\
&= -\int_{-\infty}^{\infty} (-\delta(t-Z) + m_2(Z,t))\tilde{m}_2(Z,t+s) \\
&\quad + cm_4(Z,t)(-\delta(t+s-Z/c) + \tilde{m}_4(Z,t+s))\ dt \\
&= \tilde{m}_2(Z,Z+s) + cm_4(Z,Z/c-s) \\
&\quad - \int_{-\infty}^{\infty} m_2(Z,t)\tilde{m}_2(Z,t+s) + cm_4(Z,t)\tilde{m}_4(Z,t+s)\ dt \\
&= \tilde{m}_2(Z,Z+s) + cm_4(Z,Z/c-s) \\
&\quad - m_2(Z,-s) * \tilde{m}_2(Z,s) - cm_4(Z,-s) * \tilde{m}_4(Z,s).
\end{aligned}$$

So from (6.29) we have

$$\begin{aligned}
0 &= \tilde{m}_1(0,s) + cm_3(0,-s) - \tilde{m}_2(Z,Z+s) - cm_4(Z,Z/c-s) \\
&\quad + m_2(Z,-s) * \tilde{m}_2(Z,s) + cm_4(Z,-s) * \tilde{m}_4(Z,s), \quad s \in \mathbb{R},
\end{aligned}$$

or equivalently

$$\begin{aligned}
0 &= \tilde{m}_1(0,-s) + cm_3(0,s) - \tilde{m}_2(Z,Z-s) - cm_4(Z,Z/c+s) \\
&\quad + m_2(Z,s) * \tilde{m}_2(Z,-s) + cm_4(Z,s) * \tilde{m}_4(Z,-s), \quad s \in \mathbb{R}.
\end{aligned}$$

Since $\tilde{m}_1(0,t) = 0$ for $t < 0$ and $\tilde{m}_2(Z,t) = 0$ for $t < Z$ we obtain

$$\begin{aligned}
0 &= cm_3(0,s) - cm_4(Z,Z/c+s) \\
&\quad + m_2(Z,s) * \tilde{m}_2(Z,-s) + cm_4(Z,s) * \tilde{m}_4(Z,-s), \quad s > 0,
\end{aligned}$$

proving (6.12*b*). $\square$

Chapter 7

SPHERICAL HARMONIC EXPANSION

If the coefficients of certain linear PDEs have spherical harmonic expansions then one may attempt to construct solutions of these PDEs in the form of a spherical harmonic expansion. The regularity of the solution constructed in this manner, at all points except the origin, is governed by the decay rate of the coefficients of the spherical harmonic expansion. The regularity at the origin needs more careful attention and that is what we study in this chapter, for the finite expansion case.

Let n be a positive integer and S the unit sphere in $\mathbb{R}^n$. For every $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{x} \neq \mathbf{0}$ we define $r = |\mathbf{x}|$ and $\boldsymbol{\theta} = \mathbf{x}/|\mathbf{x}|$; conversely, to every $r \geq 0$ and $\boldsymbol{\theta} \in S$ we can associate a unique $\mathbf{x} = r\boldsymbol{\theta}$ in $\mathbb{R}^n$. For each non-negative integer k, let H_k be the (finite dimensional) subspace of $L^2(S)$ consisting of the restrictions to S of all the homogeneous polynomials of degree k which are harmonic and R_k the restriction to S of polynomials of degree k. It is known that

1. H_k is orthogonal to $H_{k'}$ if $k \neq k'$.
2. $R_k = \displaystyle\sum_{j=0}^{k} H_j$.
3. The direct sum of the H_k is dense in $L^2(S)$.

Hence, we can find an orthonormal basis $\{\phi_i(\boldsymbol{\theta})\}_{i=1}^{\infty}$ of $L^2(S)$ with each $\phi_i(\boldsymbol{\theta})$ the restriction to S of a homogeneous harmonic polynomial $\phi_i(\mathbf{x})$. Below d_i will denote the degree of homogeneity of $\phi_i(\mathbf{x})$ and we arrange the index i so that $d_i \leq d_{i'}$ if $i < i'$. Please see [23] for details.

Given a continuous function $f(\mathbf{x})$ on $\mathbb{R}^n$, its restriction to the sphere of radius $r > 0$, results in a continuous function on S via $\boldsymbol{\theta} \to f(r\boldsymbol{\theta})$. Hence this function has an expansion in terms of the $\{\phi_i\}_{i=1}^{\infty}$, namely

$$f(r\boldsymbol{\theta}) = \sum_{i=1}^{\infty} f_i(r)\, \phi_i(\boldsymbol{\theta}), \qquad r > 0, \quad \boldsymbol{\theta} \in S \tag{7.1}$$

where

$$f_i(r) = \int_S f(r\boldsymbol{\theta})\, \phi_i(\boldsymbol{\theta})\, d\boldsymbol{\theta}, \qquad r \geq 0.$$

Note that $f_i(r)$ makes sense even when $r = 0$ and in fact $f_1(0) = f(0)$, $f_i(0) = 0$ for $i > 1$. The expansion (7.1) is called the spherical harmonic expansion of f.

We are interested in studying the relationship between the regularity of $f(\mathbf{x})$ and the regularity of $f_i(r)$. For finite expansions, the relationship is fairly straightforward at points $\mathbf{x} \neq \mathbf{0}$ but not so obvious at $\mathbf{x} = \mathbf{0}$, and that is what we study in this short chapter.

Spherical harmonic expansions are frequently used to analyze the forward and inverse problems of multi-dimensional functions. Given $R > 0$ and $\sigma > 0$, let $A_\sigma(R)$ denote the Banach space of C^∞ functions in the open ball of radius R in $\mathbb{R}^n$ for which

$$||f||_{\sigma,R} := \sup_{\mathbf{x}\in\mathbb{R}^n, |\mathbf{x}|\leq R} \sum_{|\alpha|=0}^{\infty} \frac{\sigma^{|\alpha|}}{\alpha!} |(D^\alpha f)(\mathbf{x})|$$

is finite. One can refer to [20] for some properties of $A_\sigma(R)$, which are listed below

- $A_\sigma(R)$ is a Banach space under the $||\cdot||_{\sigma,R}$ norm

- If $0 \leq \sigma' \leq \sigma$, then $A_\sigma(R) \subset A_{\sigma'}(R)$ and $||f||_{\sigma',R} \leq ||f||_{\sigma,R}$
- If $f \in A_\sigma(R)$ for some $\sigma > 0$, then $\triangle f \in A_{\sigma'}(R)$ for all $0 < \sigma' < \sigma$ and

$$|| \triangle f||_{\sigma',R} \leq C \frac{||f||_{\sigma,R}}{|\sigma - \sigma'|^2}$$

  with C independent of f and σ

- If $f, g \in A_\sigma(R)$, then $fg \in A_\sigma(R)$ and
$$||fg||_{\sigma,R} \le ||f||_{\sigma,R}||g||_{\sigma,R}.$$

Inverse problems for multi-dimensional hyperbolic equations were studied by Romanov in [20]. Consider the following IBVP

$$u_{tt} - u_{zz} - \Delta u - q(\mathbf{x}, z)u = 0, \quad x \in \mathbb{R}^n, \quad z \ge 0, \quad t \in \mathbb{R}, \tag{7.2a}$$

$$u|_{t<0} = 0, \quad u_z|_{z=0} = -g(\mathbf{x})\delta'(t). \tag{7.2b}$$

Romanov worked on the inverse problem of (7.2a) – (7.2b), that is, given the reflection data

$$u|_{z=0} = F(\mathbf{x}, t), \quad (\mathbf{x}, t) \in \mathbb{R}^{n+1},$$

the goal is to recover q. Based on the progressive wave expansion, the reflection data $F(\mathbf{x}, t)$ can be represented in the form

$$F(\mathbf{x}, t) = g(\mathbf{x})\delta(t) + f(\mathbf{x}, t)H(t).$$

Romanov claimed that if $f(\mathbf{x}, +0) = 0$, and $g(\mathbf{x}), 1/g(\mathbf{x}) \in A_{s_0}(r)$, and $f(\mathbf{x}, t), f_t(\mathbf{x}, t) \in C(A_{s_0}(r), [0, T])$ for some $T > 0$, then there exists a number a such that for any $s \in (0, s_0)$, there exists a unique solution to the inverse problem of (7.2a) – (7.2b), such that $q, u \in C(A_{s_0}(r), G_s)$ where $G_s = \{(z, t) \mid z \in [0, a(s_0 - s)], 0 \le z \le t \le T - z\}$. That is, one can recover $q(\mathbf{x}, z)$ for $z \in [0, a(s_0 - s)]$.

We now state the main result of this chapter.

Theorem 7.1. *If m, n, N are nonnegative integers such that $n \ge 2, N \ge 1$ and $f(\mathbf{x}) = f(r\boldsymbol{\theta}) = \sum_{i=1}^{N} f_i(r)\phi_i(\boldsymbol{\theta})$, then $f(\mathbf{x}) \in C^m(\mathbb{R}^n)$ if and only if for $i = 1, ..., N$*

- *$f_i(r) \in C^m[0, \infty)$,*
- *for all $0 \le k \le m$, $f_i^{(k)}(0) = 0$ for all $k \ge d_i$ such that $k - d_i$ is odd, and for all $k < d_i$.*

The proof of Theorem 7.1 requires the following result.

Lemma 7.1. *If $f(\mathbf{x}) \in C^0(\mathbb{R}^n) \cap C^m(\mathbb{R}^n \backslash \mathbf{0})$ for some $m, n \in \mathbb{N}$, and $\lim_{\mathbf{x} \to \mathbf{0}} D^\alpha f(\mathbf{x})$ exists for all $|\alpha| \leq m$, then $f(\mathbf{x}) \in C^m(\mathbb{R}^n)$.*

Proof. We show it by induction on m. First for $m = 1$, let

$$\lim_{\mathbf{x} \to \mathbf{0}} D_j f(\mathbf{x}) = L, \quad j = 1, ..., n.$$

Since $f(\mathbf{x}) \in C^0(\mathbb{R}^n) \cap C^1(\mathbb{R}^n \backslash \mathbf{0})$, for any $h \in \mathbb{R}$, by the mean value theorem, there exists a constant θ_h between 0 and h such that

$$D_j f(\theta_h \mathbf{e}_j) = \frac{f(h\mathbf{e}_j) - f(\mathbf{0})}{h}, \quad j = 1, ..., n.$$

Take the limits on both sides, we have $D_j f(\mathbf{0})$ exists and

$$L = \lim_{h \to 0} D_j f(\theta_h \mathbf{e}_j) = \lim_{h \to 0} \frac{f(h\mathbf{e}_j) - f(\mathbf{0})}{h} = D_j f(\mathbf{0}),$$

so by the definition of continuity, $D_j f(\mathbf{x})$ is continuous at $\mathbf{x} = \mathbf{0}$, implying that $f(\mathbf{x}) \in C^1(\mathbb{R}^n)$.

Now assume the result is true for some $m \geq 1$. Let $f(\mathbf{x}) \in C^0(\mathbb{R}^n) \cap C^{m+1}(\mathbb{R}^n \backslash \mathbf{0})$ and $\lim_{\mathbf{x} \to \mathbf{0}} D^\alpha f(\mathbf{x})$ exists for all $|\alpha| \leq m+1$, then by assumption we know that $f(\mathbf{x}) \in C^m(\mathbb{R}^n)$. Fix α such that $|\alpha| \leq m$ and define

$$g(\mathbf{x}) := D^\alpha f(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^n; \tag{7.3}$$

then $g(\mathbf{x}) \in C^0(\mathbb{R}^n) \cap C^1(\mathbb{R}^n \backslash \mathbf{0})$. Also, $\lim_{\mathbf{x} \to \mathbf{0}} D_j g(\mathbf{x}) = \lim_{\mathbf{x} \to \mathbf{0}} D_j D^\alpha f(\mathbf{x})$ exists because $|\alpha| + 1 \leq m + 1$. Hence by the $m = 1$ case, $g(\mathbf{x}) \in C^1(\mathbb{R}^n)$, implying that $f(\mathbf{x}) \in C^{m+1}(\mathbb{R}^n)$. □

Proof of Theorem 7.1:

Proof. We first show the necessity of the conditions. Since

$$f_i(r) = \int_{|\boldsymbol{\theta}|=1} f(r\boldsymbol{\theta}) \phi_i(\boldsymbol{\theta}) \, d\boldsymbol{\theta}, \quad i = 1, ..., N, \tag{7.4}$$

by the theorem about differentiation of parameter dependent integrals we have $f_i(r) \in C^m[0,\infty)$ for $i = 1, ..., N$, and

$$f_i^{(k)}(r) = \int_{|\boldsymbol{\theta}|=1} \sum_{|\alpha|=k} \binom{k}{\alpha} \boldsymbol{\theta}^\alpha (D^\alpha f)(r\boldsymbol{\theta})\phi_i(\boldsymbol{\theta})\ d\boldsymbol{\theta}, \quad 0 \le k \le m,$$

implying that

$$f_i^{(k)}(0) = \sum_{|\alpha|=k} \binom{k}{\alpha} (D^\alpha f)(\mathbf{0}) \int_{|\boldsymbol{\theta}|=1} \boldsymbol{\theta}^\alpha \phi_i(\boldsymbol{\theta})\ d\boldsymbol{\theta}. \tag{7.5}$$

For any $0 \le k \le m$ and $|\alpha| = k$, since $\boldsymbol{\theta}^\alpha$ is the restriction to S of a polynomial of degree k, we have $\boldsymbol{\theta}^\alpha \in \sum_{j=0}^{k} H_j$, and so by the orthogonality we have

$$\int_{|\boldsymbol{\theta}|=1} \boldsymbol{\theta}^\alpha \phi_i(\boldsymbol{\theta})\ d\boldsymbol{\theta} = 0, \quad |\alpha| = k < d_i,$$

implying that

$$f_i^{(k)}(0) = 0, \quad 0 \le k < d_i.$$

For $k \ge d_i$ and $|\alpha| = k$,

$$\int_{|\boldsymbol{\theta}|=1} \boldsymbol{\theta}^\alpha \phi_i(\boldsymbol{\theta})\ d\boldsymbol{\theta} = \sum_{|\beta|=d_i} c_\beta \int_{|\boldsymbol{\theta}|=1} \boldsymbol{\theta}^\alpha \boldsymbol{\theta}^\beta\ d\boldsymbol{\theta} = 0$$

if $k + d_i$ is odd. So from (7.5), $f_i^{(k)}(0) = 0$ if $k - d_i$ is odd.

We now show that the necessary conditions are sufficient. We are given that $f_i(r) \in C^m[0,\infty)$ and for all $0 \le k \le m$, $f_i^{(k)}(0) = 0$ for all $k < d_i$ and for all $k \ge d_i$ such that $k - d_i$ is odd. The Taylor expansion (see [24]) of $f_i(r)$ is

$$f_i(r) = \sum_{j=0}^{m} \frac{f_i^{(j)}(r)}{j!} r^j + R_m(r), \quad i = 1, ..., N, \tag{7.6}$$

where $R_m(r) \in C^m[0,\infty)$ and

$$R_m(r) = o(r^m), \quad r \to 0^+.$$

We also observe that $R_m^{(k)}(r) \in C^{m-k}[0,\infty)$ and

$$R_m^{(k)}(r) = o(r^{m-k}), \quad r \to 0^+ \tag{7.7}$$

as shown next. For all $0 \le k \le m$ we have

$$\begin{aligned} f_i^{(k)}(r) &= \sum_{j=k}^{m} \frac{f_i^{(j)}(r)}{j!} \frac{j!}{(j-k)!} r^{j-k} + R_m^{(k)}(r) \\ &= \sum_{j=k}^{m} \frac{f_i^{(j)}(r)}{(j-k)!} r^{j-k} + R_m^{(k)}(r) = \sum_{j=0}^{m-k} \frac{f_i^{(k+j)}(r)}{j!} r^j + R_m^{(k)}(r), \quad i = 1, ..., N. \end{aligned}$$

So $R_m^{(k)}(r)$ is the remainder of the Taylor expansion of $f_i^{(k)}(r)$ after $m-k$ terms, implying (7.7).

From (7.6) we have

$$\begin{aligned} f_i(r)\phi_i(\boldsymbol{\theta}) &= \sum_{j=0}^{m} \frac{f_i^{(j)}(r)}{j!} r^j \phi_i(\boldsymbol{\theta}) + R_m(r)\phi_i(\boldsymbol{\theta}) \\ &= \sum_{j=0}^{m} \frac{f_i^{(j)}(r)}{j!} r^{j-d_i} \phi_i(\mathbf{x}) + R_m(r)\phi_i(\boldsymbol{\theta}), \quad i = 1, ..., N \end{aligned} \tag{7.8}$$

with the first term a sum of polynomials since r^{j-d_i} is a polynomial if $j - d_i$ is even, so we need only analyze the regularity of the second term of (7.8):

$$e(\mathbf{x}) := R_m(r)\phi_i(\boldsymbol{\theta}) = R_m(r) r^{-d_i} \phi_i(\mathbf{x}).$$

Now $e(\mathbf{x}) \in C^0(\mathbb{R}^n) \cap C^m(\mathbb{R}^n \backslash \mathbf{0})$. Further, for $\mathbf{x} \neq \mathbf{0}$, noting that $\phi_i(\mathbf{x})$ is a homogeneous polynomial of degree d_i, we have

$$D_j e(\mathbf{x}) = (R_m'(r) r^{-d_i} - d_i R_m(r) r^{-d_i-1}) \frac{x_j}{r} \phi_i(\mathbf{x}) + R_m(r) r^{-d_i} \frac{\partial \phi_i}{\partial x_j}(\mathbf{x}), \quad j = 1, ..., n,$$

and hence

$$|D_j e(\mathbf{x})| \le C(|R_m'(r)| + |R_m(r)|/r) = o(r^{m-1}), \quad r \to 0^+, \quad j = 1, ..., n.$$

Similarly, for $0 \le k \le m$ and $|\alpha| = k$

$$|D^\alpha e(\mathbf{x})| \le C \sum_{j=0}^{k} |R_m^{(j)}(r)| r^{j-k} = o(r^{m-k}), \quad r \to 0^+,$$

implying that

$$\lim_{\mathbf{x}\to\mathbf{0}} D^{\alpha} e(\mathbf{x}) = 0, \quad |\alpha| \leq m.$$

Hence from Lemma 7.1, $e(\mathbf{x}) \in C^m(\mathbb{R}^n)$, and from (7.8), $f_i(r)\phi_i(\boldsymbol{\theta}) \in C^m(\mathbb{R}^n)$ for $i = 1, ..., N$, implying that $f(\mathbf{x}) \in C^m(\mathbb{R}^n)$. $\square$

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