The First Exit Time of a Brownian Motion from an Unbounded Convex Domain

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The First Exit Time of a Brownian Motion from an Unbounded Convex Domain

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Abstract: Consider the first exit time, τ_D of a $(d+1)$-dimensional Brownian motion from an unbounded open domain $D = \{(x,y) \in \mathbb{R}^{d+1} : y > f(x), x \in \mathbb{R}^d\}$ starting at $(x_0, f(x_0)+1) \in \mathbb{R}^{d+1}$ for some $x_0 \in \mathbb{R}^d$, where the function $f(x)$ on $\mathbb{R}^d$ is convex and $f(x) \to \infty$ as the Euclidean norm $|x| \to \infty$. Very general estimates for the asymptotics of $\log \mathbb{P}(\tau_D > t)$ are given by using Gaussian techniques. In particular, for $f(x) = \exp\{|x|^p\}$, $p > 0$,

$$\lim_{t\to\infty} t^{-1}(\log t)^{2/p} \log \mathbb{P}(\tau_D > t) = -j_\nu^2/2$$

where $\nu = (d-2)/2$ and j_ν is the smallest positive zero of the Bessel function J_ν.

Abbreviated title: Exit Time from Unbounded Domain
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1 Introduction

Let $B(t) = (B_1(t), \cdots, B_d(t)) \in \mathbb{R}^d$, $t \geq 0$ be a standard d–dimensional Brownian motion, where $B_i(t)$, $1 \leq i \leq d$, are independent Brownian motions starting at 0. Consider the first exit time τ_D of a $(d+1)$-dimensional Brownian motion from the unbounded open domain

$$D = \left\{(x, y) \in \mathbb{R}^{d+1} : y > f(x), x \in \mathbb{R}^d\right\} \tag{1.1}$$

starting at the point $(x_0, f(x_0)+1) \in \mathbb{R}^{d+1}$ for some $x_0 \in \mathbb{R}^d$, where the function $f(x)$ on $\mathbb{R}^d$ is convex and $f(x) \to \infty$ as the Euclidean norm $|x| \to \infty$. That is, the exit time or the stopping time

$$\tau_D = \inf\{t : (x_0 + B(t), f(x_0) + 1 + W(t)) \notin D\} \tag{1.2}$$

which plays a key role in the probabilistic solution to the Dirichlet problem. Here and throughout the paper, $W(t)$ is a standard one-dimensional Brownian motion starting at 0, independent of $B(t)$.

It is well-known that for a bounded smooth open (connected) domain $\widetilde{D}$,

$$\lim_{t\to\infty} t^{-1} \log \mathbb{P}\left(\tau_{\widetilde{D}} > t\right) = -\lambda_1(\widetilde{D}) \tag{1.3}$$

where $\lambda_1(\widetilde{D}) > 0$ is the principal eigenvalue of $-\Delta/2$ in $\widetilde{D}$ with Dirichlet boundary condition. It is natural to consider what happens for an unbounded domain such as D in (1.1)?

When $f(x) = A \cdot |x|$, $A > 0$, the unbounded domain in (1.1), called $D(\theta)$, is the right circular cone of angle $\theta = \arccos(A/\sqrt{1+A^2})$. Burkholder (1977) proved that $\mathbb{E}\,(\tau^p_{D(\theta)}) < \infty$ if and only if $p < p(\theta, d)$, where $p(\theta, d)$ is a number defined in terms of the smallest zero of a certain hypergeometric function. There is also a close connection between moments of the exit time and the least harmonic majorant of $|x|^p$ in the domain. In DeBlassie (1987, 1988), Burkholder's result together with techniques from partial differential equations are used to find an exact formula for $\mathbb{P}\{\tau_{D(\theta)} > t\}$ as an infinite series involving confluent hypergeometric functions. Using this formula the exact asymptotics in t for $\mathbb{P}\{\tau_{D(\theta)} > t\}$ follow and the result also holds for more general cones. We should mention here that in $\mathbb{R}^2$ formulas for $\mathbb{P}\{\tau_{D(\theta)} > t\}$ have existed for many years. Indeed, Spitzer (1958) derives an expression for $\mathbb{P}\{\tau_{D(\theta)} > t\}$ in his study of the winding of two dimensional Brownian motion. Recently, a uniform treatment that covers all the above results was presented in Bañuelos and Smits (1997) for very general cones. In particular, for exit time τ_C from a cone C,

$$\mathbb{P}\{\tau_C > t\} \sim ct^{-\gamma_C}, \quad \text{as} \quad t \to \infty, \tag{1.4}$$

where $c >$ is a known constant and γ_C is determined by the first eigenvalue of the Dirichlet problem for the Laplace-Beltrami operator on a subset of the unit sphere determined by the cone C. The special geometric structure of the cone (scale invariance) is essential for these results.

When $f(x) = A \cdot |x|^2$, $A > 0$, the unbounded domain in (1.1), called D_2, is parabolic in shape. Denote the exit time from D_2 by τ_2. Bañuelos, DeBlassie and Smits (2001) proved that for $d = 1$, there are two positive constants A_1 and A_2 such that

$$-A_1 \leq \liminf_{t\to\infty} t^{-1/3} \log \mathbb{P}\left(\tau_2 > t\right) \leq \limsup_{t\to\infty} t^{-1/3} \log \mathbb{P}\left(\tau_2 > t\right) \leq -A_2. \tag{1.5}$$

Their techniques, based on conformal transformation and elementary principles of large deviations, are completely different from those used in the study of cones. Indeed, they change the problem to the study of the exit time of a degenerate diffusion from an infinite strip on $\mathbb{R}^2$, with singular generator

$$L = \frac{1}{8} \frac{1}{u^2 + v^2} \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right). \tag{1.6}$$

Their bounds are derived in terms of infinite series involving certain Feynman-Kac functionals. It is their work that motivates the present study and we thank them for providing a copy of the paper and useful discussions.

In this paper, we provide general upper and lower estimates for the asymptotic tail distribution of the exist time τ_D on $\mathbb{R}^{d+1}$. Our techniques, separate for upper and lower bounds, are completely different from those used before. The following simple fact is the key step for our upper bound estimates. It is based on a powerful Gaussian technique, Slepian's inequality, after putting pieces into context.

Proposition 1.1 *Let f be a function on $\mathbb{R}^d$. Then*

$$\begin{aligned} \mathbb{P}(\tau_D > t) &= \mathbb{P}(f(x_0 + B(s)) \le W(s) + f(x_0) + 1, 0 \le s \le t) \\ &\le \mathbb{P}\left(f(x_0 + B(s)) \le \sqrt{s}\xi + f(x_0) + 1, 0 \le s \le t\right) \end{aligned}$$

where ξ is a standard normal random variable independent of $\{B(t) \in \mathbb{R}^d, t \ge 0\}$.

Slepian's inequality (lemma) and its variations provide a very useful tool in the theory of Gaussian processes and probability in Banach spaces. Very nice discussions with various applications can be found in Ledoux and Talagrand (1991) and Lifshits (1995). The simplest form of Slepian's lemma for centered Gaussian processes X_t and Y_t with index $t \in T$, states that if $\mathbb{E}\, X_t^2 = \mathbb{E}\, Y_t^2$ and $\mathbb{E}\, X_t X_s \le \mathbb{E}\, Y_t Y_s$ for any $t, s \in T$, then for any x

$$\mathbb{P}\left(\sup_{t \in T} X_t \le x\right) \le \mathbb{P}\left(\sup_{t \in T} Y_t \le x\right) \tag{1.7}$$

An interesting and useful extension of Slepian's inequality, including reversed direction under certain conditions, can be found in Li and Shao (2002b) with applications. In particular, a conjecture of Kesten (1992) on the random pursuit problem for Brownian particles is confirmed there, which can be viewed as the exit probability from a cone.

Returning to our exit time problem, we need further restrictions on f and notation to state our main results. Assume the convex $f(x)$ is symmetric with respect to the set of all orthogonal transformations of $\mathbb{R}^d$. Then as given in Rockafellar (1970), page 110, $f(x) = h(|x|)$ and $h(x)$ is a non-decreasing lower semi continuous convex function on $[0, \infty)$ with $h(0)$ finite. It is easy to see that asymptotic rate of $\mathbb{P}(\tau_D > t)$ depends neither on the fixed initial starting point $(x_0, f(x_0) + 1) = (x_0, h(|x_0|) + 1) \in \mathbb{R}^{d+1}$ nor any bounded part of the domain, so we will assume for simplicity $x_0 = 0$ and the conditions on $h(x)$ are only for $x \ge K > 0$. Here and throughout this paper, we use letter K and their modifications K_1, K_δ, etc for various positive constants which may be different from a line to another, and use $\log x$ for the natural logarithm. Also, let j_ν be the smallest positive zero of the Bessel function J_ν, $\nu = (d-2)/2$ and $j_{-1/2} = \pi/2$.

Define

$$H(x) = \int_K^x \left(h^{-1}(\sqrt{s} + h(0) + 1)\right)^{-2} ds. \tag{1.8}$$

Assume that for any $0 < \delta < 1$, and $0 < q = q(\delta) < 1$, there exists $u_0 = u_0(\delta, q)$ such that for all $x \ge u_0$,

$$H(x) \ge \delta^{-1} H(x^q). \tag{1.9}$$

Theorem 1.1 *If condition (1.9) holds for any $0 < \delta < 1$ and for some $\delta_0 > 0$ small,*

$$\inf_{x \ge u_0} \left(\frac{j_v^2}{2} \cdot \frac{tH(x)}{x} + \frac{x}{2t}\right) \ge t^{\delta_0} \tag{1.10}$$

for t large, then

$$\log \mathbb{P}(\tau_D > t) \le -(1-\delta) \inf_{x \ge u_0} \left(\frac{j_v^2}{2} \cdot \frac{tH(x)}{x} + \frac{x}{2t} \right). \tag{1.11}$$

In particular, for $h^{-1}(x) = Ax^\alpha (\log x)^\beta$, $x > 1$,

$$\log \mathbb{P}(\tau_D > t) \le \begin{cases} -(1-\delta)^2(1+\alpha)(2\alpha)^{-\alpha/(1+\alpha)}(2^{-1}(1+\alpha))^{2\beta/(1+\alpha)} C^{1/(1+\alpha)} & \\ \quad \cdot t^{(1-\alpha)/(1+\alpha)}(\log t)^{-2\beta/(1+\alpha)} & \text{if } 0 < \alpha < 1,\ \beta \in \mathbb{R} \\ -(1-\delta)^2 2^{-1} A^{-2} j_v^2 \cdot t(\log t)^{-2\beta} & \text{if } \alpha = 0,\ \beta > 0 \\ -K^{-1} \cdot (\log t)^{-\beta} & \text{if } \alpha = 1,\ \beta \le 0 \end{cases}$$

for t sufficiently large, where $C = (1-\alpha)^{-1} 2^{2\beta-1} A^{-2} j_\nu^2$.

To see that our upper estimates are reasonably sharp, we have the following general lower bound in the case $f = h(|x|)$. In fact, we conjecture that our lower bounds below in Theorem 1.2 are sharp in general. To state them, let us first define a class of functions $\mathcal{G}$ with

$$\mathcal{G} = \left\{ g \in C^2[0,\infty) : \ g(0) < h(0) + 1, g' \ge 0, g'' \le 0 \,\text{for } t \text{ large and } \lim_{t\to\infty} \sqrt{t} g'(t) = \infty \right\}. \tag{1.12}$$

Theorem 1.2 *Assume that $h' > 0$ and $h'' \ge 0$ for $x \ge K$. For any $\delta > 0$ fixed and $t > 0$ sufficiently large,*

$$\log \mathbb{P}(\tau_D > t) \ge -(1+\delta) \inf_{g \in \mathcal{G}} \left(\frac{j_v^2}{2} \int_K^t \frac{1}{(h^{-1}(g(s)))^2} ds + \frac{1}{2} \int_K^t (g'(s))^2 ds \right) \tag{1.13}$$

where $K > 1$ is a constant. In particular, for $h^{-1}(x) = Ax^\alpha (\log x)^\beta$, $x > 1$,

$$\log \mathbb{P}(\tau_D > t) \ge \begin{cases} -(1+\delta)^2 2^{-1}(1-\alpha)^{-1} \left(\alpha^{-\alpha}(1+\alpha)^{2\beta+2} A^{-2} j_v^2\right)^{1/(1+\alpha)} & \\ \quad \cdot t^{(1-\alpha)/(1+\alpha)}(\log t)^{-2\beta/(1+\alpha)} & \text{if } 0 < \alpha < 1,\ \beta \in \mathbb{R} \\ -(1+\delta)^2 2^{-1} A^{-2} j_v^2 \cdot t(\log t)^{-2\beta} & \text{if } \alpha = 0,\ \beta > 0 \\ -(1+\delta)^2 2^{\beta-1}(1-\beta)^{-1} A^{-1} j_v \cdot (\log t)^{1-\beta} & \text{if } \alpha = 1,\ \beta \le 0 \end{cases}$$

for t sufficiently large.

Note that from Theorem 1.1 and 1.2, we have for the constants in (1.5),

$$A_1 = 3^{1/3}(3/2)(j_\nu)^{4/3}, \quad A_2 = (3/2)(j_\nu)^{4/3}$$

which are not too far apart. Furthermore, for $f(x) = |x|^\gamma$, $\gamma > 1$,

$$\begin{aligned} \liminf_{t\to\infty} t^{-(\gamma-1)/(\gamma+1)} \log \mathbb{P}(\tau_D > t) &\ge -2^{-1}(1+\gamma)^{2\gamma/(\gamma+1)} \gamma^{(2-\gamma)/(\gamma+1)} (\gamma-1)^{-1} j_\nu^{2\gamma/(\gamma+1)} \\ \limsup_{t\to\infty} t^{-(\gamma-1)/(\gamma+1)} \log \mathbb{P}(\tau_D > t) &\le -2^{-1}(1+\gamma)(\gamma-1)^{-\gamma/(\gamma+1)} j_\nu^{2\gamma/(\gamma+1)} \end{aligned} \tag{1.14}$$

and for $f(x) = \exp\{|x|^p\}$, $p > 0$,

$$\lim_{t\to\infty} \frac{(\log t)^{2/p}}{t} \log \mathbb{P}(\tau_D > t) = -j_\nu^2/2. \tag{1.15}$$

Thus, as seen in (1.15), both our upper and lower estimates are sharp for fast exponential growing f. In the case of f growing slightly faster than linear, say $f = |x|(\log |x|)^p$, $p > 0$, our upper bound does not provide the correct rate and more work needs to be done in this direction. It seems a challenging problem to find the limiting constant in (1.14) assuming it exists. Also the case $f(x) = \langle x, Qx\rangle$ is of special interests where Q is positive definite matrix.

It is also worthy to point out a general framework for this type of problems which leads us to this work from the point of view of the theory of Gaussian processes. There are two types of probability estimates that have attracted a lot of attention recently for general Gaussian process X_t, $t \in T$. One is lower tail probabilities which study $\mathbb{P}\left(\sup_{t\in S}(X_t - X_{t_0}) \le x\right)$ as $x \to 0$, with $t_0 \in S$ fixed, see Li and Shao (2002a) for details. The other is small ball (small deviation) probabilities, see (2.3) with full discussions there. Both types of probability estimate can also be viewed as exit time problems if the Gaussian process has scaling property. In fact, our approach in this paper is based on this point of view together with related techniques.

Although our upper bound estimate is based on a pure Gaussian technique, namely, Slepian's inequality, it is possible to extend the approach to some non-Gaussian settings. For example, one could use the extension of Slepian's inequality for commuting semigroups given in Dudley and Stroock (1987). The stable processes are also of great interest.

Finally we want to point out the following connections with the heat equation. Let

$$v(x,t) = \mathbb{P}_x\{\tau_D \ge t\}, \qquad x \in \mathbb{R}^{d+1}.$$

Then v solves

$$\begin{cases} \frac{\partial v}{\partial t} = \frac{1}{2}\Delta v \quad \text{in } D \\ \\ v(x,0) = 1 \qquad x \in D. \end{cases} \tag{1.16}$$

So our results can be viewed as long time behavior of $\log v(x,t)$. Furthermore, a closly related and useful technique in studying certain asymptotic problems is the logarithmic transformation $V = -\log v(x,t)$ which changes (1.16) into a nonlinear evolution equation for V. This can then be viewed as a stochastic control problem, see, for example, Fleming and Soner (1993). For other connections and interplay between exit probability and principle eigenvalues, etc, see Chen and Li (2002) for discussions on a concrete and a relative simple example of a degenerate (non-uniform elliptic) generators.

The rest of the paper is organized as follows. In Section 2, we present several estimates on exit probabilities with moving boundary. They are necessary for the proofs of Theorem 1.1 and 1.2 and important in their own. Detailed discussions on the connection with small ball estimates are given with examples. In Section 3, we give the proof of Proposition 1.1 and Theorem 1.1 which requires detailed asymptotic analysis. The proof of Theorem 1.2 is presented in Section 4.

2 Exit probabilities with moving boundary

First we mention the following useful results due to Novikov (1979) for one dimensional Brownian motion. Consider the following stopping times

$$\sigma_1 = \inf\{t \ge 0 : \, W(t) \le g_1(t)\}$$

and

$$\sigma_2 = \inf\{t \ge 0 : \, |W(t)| \ge g_2(t)\}.$$

Lemma 2.1 *Let $g_1(t)$ be a continuous function, $g_1(0) < 0$, and let $K > 0$ be such that when $t \ge K$ the function $g_1''(t)$ is continuous and $g_1'(t) \ge 0$, $g_1''(t) \le 0$. Then, if $\sqrt{t}g_1'(t) \to \infty$ as $t \to \infty$,*

$$\log \mathbb{P}(\sigma_1 > t) = -\frac{1}{2}\int_K^t (g_1'(s))^2 \, ds(1 + o(1)) \quad \textit{as} \quad t \to \infty, \tag{2.1}$$

where $\mathbb{P}(\sigma_1 > t) = \mathbb{P}(W(s) \geq g_1(s), 0 \leq s \leq t)$.

Let $g_2(t)$ *be a continuous positive function and let* $K > 0$ *be such that for* $t \geq K$ *the function* $g_2''(t)$ *is continuous, and either* $g_2'(t) \geq 0$, $g_2''(t) \leq 0$, $g_2(t) \to \infty$, $\sqrt{t}g_2'(t) \to 0$ *as* $t \to \infty$, *or* $g_2'(t) \leq 0$, $g_2''(t) \geq 0$, $g_2(t) \to 0$ *as* $t \to \infty$. *Then*

$$\log \mathbb{P}(\sigma_2 > t) = -\frac{\pi^2}{8} \int_K^t \frac{1}{g_2^2(s)}\, ds + \frac{1}{2} \log g_2(t)\ (1 + o(1)) \tag{2.2}$$

as $t \to \infty$, *where* $\mathbb{P}(\sigma_1 > t) = \mathbb{P}(|W(s)| \leq g_2(s), 0 \leq s \leq t)$.

Indeed, sharper two-sided inequalities are also given in Novikov (1979) for $\mathbb{P}(\sigma_1 > t)$ and $\mathbb{P}(\sigma_2 > t)$. The arguments used combine the Girsanov transformation with Laplace transform techniques. Early work with various applications can be found in Mogulskii (1974), Lai (1977), Portnoy (1978), Lai and Wijsman (1979). Some approximations related to these boundary crossing probabilities for fixed t can be found in Novikov, Frishling and Kordzakhia (1999) along with references to other approaches and applications. Related results for processes with independent increments are given in Greenwood and Novikov (1986).

However, conditions for (2.2) can be significantly weakened for the upper bounds as given in Theorem 2.1 below for the radial part of the d-dimensional Brownian motion by using piecewise approximation arguments. Similar arguments have been used in Shi (1996), Li (1999, 2001), Berthet and Shi (2000), Chen, Kuelbs and Li (2000), and all deal with small ball probabilities for various processes under weighted norms. In general, the small ball probability (or small deviation) studies the behavior of

$$\log \mathbb{P}\left(\|X\| \leq \varepsilon\right) \tag{2.3}$$

for a random process under various norm $\|\cdot\|$ as $\varepsilon \to 0$. In the last few years there has been considerable progress on the small ball estimate for Gaussian processes. As was established in Kuelbs and Li (1993) (see also Li and Linde (1999) for further developments), the behavior of (2.3) for Gaussian random element X is determined up to a constant by the metric entropy of the unit ball of the reproducing kernel Hilbert space associated with X, and vice versa. For other connections and applications of small ball probabilities, we refer readers to a recent survey paper by Li and Shao (2001).

To see a connection between small ball probability under weighted sup-norm and exit probability with moving boundary, the following simple example is instructive. Recall that $B(t)$ denotes a d-dimensional Brownian motion starting from 0, and $|B(t)| = \left(\sum_{i=1}^d B_i^2(t)\right)^{1/2}$ its radial part. It is well known now, see Li (1999) for example, that for $T > 0$ fixed,

$$\begin{aligned} & \log \mathbb{P}\left(\sup_{0 \leq s \leq T} |B(s)|/g(s) \leq \varepsilon\right) \\ = \ & \log \mathbb{P}\left(|B(s)| \leq \varepsilon \cdot g(s),\ 0 \leq s \leq T\right) \\ \sim \ & -\frac{j_v^2}{2} \int_0^T \frac{1}{g^2(s)} ds \cdot \frac{1}{\varepsilon^2} \quad \text{as} \quad \varepsilon \to 0 \end{aligned} \tag{2.4}$$

under regularity condition on the weight function g. On the other hand, consider the stopping time $\tau = \inf\{t > 0 : |B(t)| > t^\alpha\}$ with $\alpha < 1/2$. By using the Brownian scaling property,

$$\begin{aligned} \mathbb{P}\left(\tau > t\right) &= \mathbb{P}\left(|B(s)| \leq s^\alpha, 0 \leq s \leq t\right) \\ &= \mathbb{P}\left(|B(st)| \leq s^\alpha t^\alpha, 0 \leq s \leq 1\right) \end{aligned}$$

$$
\begin{aligned}
&= \mathbb{P}\left(|B(s)| \le s^{\alpha} t^{\alpha-1/2}, 0 \le s \le 1\right) \\
&= \mathbb{P}\left(\sup_{0\le s\le 1} \frac{|B(s)|}{s^{\alpha}} < t^{\alpha-1/2}\right)
\end{aligned}
$$

and thus the small ball estimate (2.4) implies as $t \to \infty$

$$
\log \mathbb{P}(\tau > t) \sim -2^{-1} j_{\nu}^{2} (1-2\alpha)^{-1} t^{1-2\alpha}. \tag{2.5}
$$

As can be seen, the simple argument above does not work for general moving boundary g. However, the basic ideas used in the small ball estimates for weighted norms can be used and indeed that is what we do in Theorem 2.1 below. Note that the time interval is not fixed in contrast with (2.4). We present detailed upper bounds since we need a uniform estimates for the proof of Theorem 1.1. These results are not available in the literature for Bessel process as far as we know, although Novikov's argument could be applied under much stronger smoothness conditions as can be seen in the proof of Theorem 2.2.

To get started, we need the following well-known exact distribution function given in Ciesielski and Taylor (1962)

$$
\mathbb{P}\left(\sup_{0\le s\le 1} |B(s)| \le x\right) = \frac{2^{1-\nu}}{\Gamma(\nu+1)} \sum_{n=1}^{\infty} \frac{1}{j_{\nu,n}^{1-\nu} J_{\nu+1}(j_{\nu,n})} \exp\left\{-\frac{j_{\nu,n}^{2}}{2x^{2}}\right\},
$$

where $\nu = (d-2)/2$, and $0 < j_{\nu,1} < j_{\nu,2} < \cdots$ are the positive zeros of the Bessel function J_ν (and of course $J_{\nu+1}$ denotes the Bessel function of index $\nu+1$). It implies the small ball estimate

$$
\mathbb{P}\left(\sup_{0\le s\le 1} |B(s)| \le x\right) \sim \frac{2^{1-\nu}}{\Gamma(\nu+1) j_{\nu}^{1-\nu} J_{\nu+1}(j_{\nu})} \exp\left\{-\frac{j_{\nu}^{2}}{2x^{2}}\right\}, \quad \text{as } x \to 0,
$$

where $j_\nu \equiv j_{\nu,1}$ is the smallest positive zero of the Bessel function J_ν, $\nu = (d-2)/2$. And further we have for any $x > 0$,

$$
K^{-1} \exp\left\{-\frac{j_{\nu}^{2}}{2x^{2}}\right\} \le \mathbb{P}\left(\sup_{0\le s\le 1} |B(s)| \le x\right) \le K \exp\left\{-\frac{j_{\nu}^{2}}{2x^{2}}\right\} \tag{2.6}
$$

for some universal constant $K > 0$.

Now we can state the following general upper bound estimates.

Theorem 2.1 *For any positive integer n and finite partition $P = \{t_i, 0 \le i \le n\}$ of $[0,t]$ such that*

$$
0 = t_0 < t_1 < \cdots < t_n = t,
$$

it follows that

$$
\mathbb{P}\left(|B(s)| \le g(s),\ 0 \le s \le t\right) \le K^{n} \exp\left\{-\frac{j_{\nu}^{2}}{2} \sum_{i=1}^{n} (t_i - t_{i-1}) \left(\sup_{t_{i-1}\le s\le t_i} g(s)\right)^{-2}\right\} \tag{2.7}
$$

where the absolute constant $K > 0$ is the one given in (2.6). And in particular, for non-decreasing function g and the uniform partition $t_i = in^{-1}t$, $0 \le i \le n$, we have

$$
\mathbb{P}\left(|B(s)| \le g(s),\ 0 \le s \le t\right) \le K^{n} \exp\left\{-\frac{j_{\nu}^{2}}{2} \int_{t/n}^{t} g^{-2}(s)\, ds\right\}. \tag{2.8}
$$

Proof. Since $B(t) = (B_1(t), \cdots, B_d(t)) \in \mathbb{R}^d$ has independent increments, we have

$$\begin{aligned}
& \mathbb{P}\left(|B(s)| \le g(s),\ 0 \le s \le t\right) \\
= \; & \mathbb{P}\left(|B(s)| \le g(s),\ t_{i-1} \le s \le t_i,\, 1 \le i \le n\right) \\
= \; & \mathbb{P}\left(\cap_{i=1}^{n-1} A_i,\, |B(s)| \le g(s),\, t_{n-1} \le s \le t_n\right) \\
= \; & \mathbb{E}\left(1_{\{\cap_{i=1}^{n-1} A_i\}} \mathbb{P}\left(|B(s) - B(t_{n-1}) + x| \le g(s),\ t_{n-1} \le s \le t_n\right) \Big| B(t_{n-1}) = x\right) \qquad (2.9)
\end{aligned}$$

where

$$A_i = \left\{|B(s)| \le g(s),\ t_{i-1} \le s \le t_i\right\}, \quad 1 \le i \le n.$$

By Anderson's inequality for Gaussian measure and scaling property of $B(t)$,

$$\begin{aligned}
& \mathbb{P}\left(|B(s) - B(t_{n-1}) + x| \le g(s),\ t_{n-1} \le s \le t_n\right) \\
\le \; & \mathbb{P}\left(|B(s) - B(t_{n-1})| \le g(s),\ t_{n-1} \le s \le t_n\right) \\
\le \; & \mathbb{P}\left(|B(s)| \le \sup_{t_{n-1} \le s \le t_n} g(s),\ 0 \le s \le t_n - t_{n-1}\right) \\
= \; & \mathbb{P}\left(\sup_{0 \le s \le 1} |B(s)| \le (t_n - t_{n-1})^{-1/2} \sup_{t_{n-1} \le s \le t_n} g(s)\right).
\end{aligned}$$

Put the above estimates into (2.9) and iterate, we obtain

$$\mathbb{P}\left(|B(s)| \le g(s),\ 0 \le s \le t\right) \le \prod_{i=1}^{n} \mathbb{P}\left(\sup_{0 \le s \le 1} |B(s)| \le (t_i - t_{i-1})^{-1/2} \sup_{t_{i-1} \le s \le t_i} g(s)\right) \qquad (2.10)$$

Next note that by (2.6)

$$\begin{aligned}
& \mathbb{P}\left(\sup_{0 \le s \le 1} |B(s)| \le (t_i - t_{i-1})^{-1/2} \sup_{t_{i-1} \le s \le t_i} g(s)\right) \\
\le \; & K \exp\left\{-\frac{j_\nu^2}{2} \cdot (t_i - t_{i-1}) \left(\sup_{t_{i-1} \le s \le t_i} g(s)\right)^{-2}\right\} \qquad (2.11)
\end{aligned}$$

and we have the upper bound (2.7).

If $g(s)$ is non-decreasing and the partition $t_i = in^{-1}t$, $0 \le i \le n$, then

$$\sum_{i=1}^{n} \frac{t_i - t_{i-1}}{g^2(t_i)} \ge \sum_{i=1}^{n-1} \frac{t_{i+1} - t_i}{g^2(t_i)} \ge \sum_{i=1}^{n-1} \int_{t_i}^{t_{i+1}} g^{-2}(s) ds = \int_{t/n}^{t} g^{-2}(s) ds.$$

and (2.8) follows.

Next we turn to the lower bounds for Bessel process. We use Novikov's argument this time due to its simplicity although a different one based on approximations can be given.

Theorem 2.2 *Let $g(t)$ be a continuous strictly positive function such that $g''(t) \le 0$ is continuous and $g'(t) \ge 0$. Then*

$$\mathbb{P}\left(|B(s)| \le g(s),\ 0 \le s \le t\right) \ge K^{-1} \exp\left\{\frac{d}{2} \log \frac{g(t)}{g(0)} - \frac{1}{2} \int_0^t (g'(s))^2 ds - \frac{j_\nu^2}{2} \int_0^t g^{-2}(s) ds\right\} \qquad (2.12)$$

And in particular, under the additional condition $\sqrt{t}g'(t) \to 0$ *as* $t \to \infty$,

$$\log \mathbb{P}\left(|B(s)| \leq g(s),\ 0 \leq s \leq t\right) \geq -(1+\delta)\frac{j_v^2}{2}\int_0^t g^{-2}(s)ds \tag{2.13}$$

for and $\delta > 0$ *and* t *large.*

Proof. We introduce the transformed process

$$\Xi(t) = g(t)\int_0^t G(s)dB(s) \tag{2.14}$$

where $G(s)$ is a diagonal matrix with diagonal elements equal to $1/g(s)$. By Ito's formula, the stochastic differential of the process $\Xi(t)$

$$d\Xi(t) = \frac{f'(t)}{f(t)}\Xi(t)dt + dB(t). \tag{2.15}$$

Since the matrix of the Gaussian diffusion process $\Xi(t)$ is the identity matrix, and the drift is square integrable, we see from Liptser and Shiryaev (1977) that the measures of $\Xi(s)$ and $B(s)$ are equivalent on $[0,t]$. In particular, we have

$$\mathbb{P}\left(|\Xi(s)| \leq g(s), 0 \leq s \leq t\right) = \mathbb{E}\, Z(t) I\{|B(s)| \leq g(s), 0 \leq s \leq t\} \tag{2.16}$$

where $I\{\cdot\}$ is the indicator function and

$$Z(t) = \exp\left\{\int_0^t \frac{g'(s)}{g(s)}\langle B(s), dB(s)\rangle - \frac{1}{2}\int_0^t \left|\frac{g'(s)}{g(s)}B(s)\right|^2 ds\right\}. \tag{2.17}$$

Using representation (2.14) for $\Xi(t)$ in the left hand side of (2.16) and making the time change $u = \int_0^t 1/g^2(s)ds$ in the components of the stochastic integral $\int_0^t G(s)dB(s)$, we obtain

$$\begin{aligned}
\mathbb{P}\left(|\Xi(s)| \leq g(s), 0 \leq s \leq t\right) &= \mathbb{P}\left(|B(s)| \leq 1, 0 \leq s \leq \int_0^t 1/g^2(s)ds\right) \\
&= \mathbb{P}\left(\sup_{0\leq s\leq 1}|B(s)| \leq \left(\int_0^t 1/g^2(s)ds\right)^{-1/2}\right) \\
&\geq K^{-1}\exp\left\{-\frac{j_\nu^2}{2}\int_0^t \frac{1}{g^2(s)}ds\right\}
\end{aligned} \tag{2.18}$$

by Brownian scaling and (2.6). On the other hand, by Ito's formula and integration by parts,

$$\begin{aligned}
\int_0^t \frac{g'(s)}{g(s)}\langle B(s), dB(s)\rangle &= \frac{1}{2}\int_0^t \frac{g'(s)}{g(s)}d(|B(s)|^2 - (2\nu+2)s) \\
&= -(\nu+1)\log\frac{g(t)}{g(0)} + \frac{1}{2}\frac{g'(t)}{g(t)}|B(t)|^2 - \frac{1}{2}\int_0^t |B(s)|^2 d\left(\frac{g'(s)}{g(s)}\right).
\end{aligned}$$

where $\nu = (d-2)/2$ as used before in order to avoid confusion. Therefore, from representation for $Z(t)$ in (2.17), it follows on the set $\{|B(s)| \leq g(s), 0 \leq s \leq t\}$,

$$\begin{aligned}
Z(t) &= \exp\left\{-\frac{d}{2}\log\frac{g(t)}{g(0)} + \frac{1}{2}\frac{g'(t)}{g(t)}|B(t)|^2 - \frac{1}{2}\int_0^t |B(s)|^2\frac{g''(s)}{g(s)}ds\right\} \\
&\leq \exp\left\{-\frac{d}{2}\log\frac{g(t)}{g(0)} + \frac{1}{2}g'(t)g(t) - \frac{1}{2}\int_0^t g''(s)g(s)ds\right\} \\
&= \exp\left\{-\frac{d}{2}\log\frac{g(t)}{g(0)} + \frac{1}{2}g'(0)g(0) + \frac{1}{2}\int_0^t (g'(s))^2 ds\right\}
\end{aligned} \tag{2.19}$$

for $g' \geq 0$ and $g'' \leq 0$. Thus (2.12) follows from (2.16), (2.18) and (2.19). The estimate (2.13) is clear under the condition $\sqrt{t}g'(t) \to 0$ as $t \to \infty$ and we finished the proof.

Combining Theorem 2.1 and 2.2 with appropriate modifications to the proofs, the following results follow easily and we omit the details.

Corollary 2.1 *Let $g(t)$ be a continuous strictly positive function such that $g''(t) \leq 0$ is continuous, $g'(t) \geq 0$ and $\sqrt{t}g'(t) \to 0$ as $t \to \infty$. Then*

$$\log \mathbb{P}\left(|B(s)| \leq g(s),\ 0 \leq s \leq t\right) \sim -\frac{j_v^2}{2}\int_0^t g^{-2}(s)ds \quad as \quad t \to \infty.$$

In fact, if $g(t) = At^p(\log t)^q$ for $t \geq t_0 \geq 2$ and $g(t) = g(t_0) > 0$ for $0 \leq t \leq t_0$, then as $t \to \infty$

$$\log \mathbb{P}\left(|B(s)| \leq g(s),\ 0 \leq s \leq t\right) \sim \begin{cases} -2^{-1}A^{-2}j_\nu^2(1-2p)^{-1}t^{1-2p}(\log t)^{-2q} & \text{if } 0 < p < 1/2,\ q \in \mathbb{R} \\ -2^{-1}A^{-2}j_\nu^2 t(\log t)^{-2q} & \text{if } p = 0,\ q \geq 0 \\ -2^{-1}A^{-2}j_\nu^2(1-2q)^{-1}\cdot(\log t)^{1-2q} & \text{if } p = 1/2,\ q < 0. \end{cases}$$

We conjecture that the above Corollary also holds for the non-integer dimensional Bessel process. Finally, some related exit probabilities investigated in the literature can be found in Uchiyama (1980), Bass and Cranston (1983), Durbin (1985, 1992), Lerche (1986), Anderson and Pitt (1997). Our approach can also be used to obtain related estimates.

3 Upper bound estimates

First we need to prove Proposition 1.1. Let τ_D be defined as in (1.2). Then

$$\begin{aligned} \mathbb{P}\left(\tau_D > t\right) &= \mathbb{P}\left(f(x_0 + B(s)) < W(s) + f(x_0) + 1, 0 \leq s \leq t\right) \\ &= \mathbb{P}\left(\sup_{0 \leq s \leq t} f(x_0 + B(s)) - W(s) < f(x_0) + 1\right) \\ &\leq \mathbb{P}\left(\sup_{0 \leq s \leq t} f(x_0 + B(s)) - \sqrt{s}\xi < f(x_0) + 1\right) \end{aligned}$$

where the inequality follows from Slepian's Lemma by conditioning on $B(s), 0 \leq s \leq t$. To justify it, we simply note that $\mathrm{Var}(-W(s)) = \mathrm{Var}(-\sqrt{s}\xi)$ and

$$\mathbb{E}\, W(s)W(s') = \min(s, s') \leq \sqrt{ss'} = \mathbb{E}\,(\sqrt{s}\xi\sqrt{s'}\xi).$$

Thus Proposition 1.1 follows from Slepian's Lemma, see (1.7) or Slepian (1961). Note that in most papers and books, Slepian's Lemma is proved and used for mean zero Gaussian random vectors. Here we in fact use a form that the same mean depends on index parameter. Standard proof can be modify to cover this case. A direct proof, together with other applications of Slepian's Lemma, can be found in an elegant paper of Dudley and Stroock (1987).

Next we turn to the proof of Theroem 1.1. It follows from Proposition 1.1 that

$$\begin{aligned} \mathbb{P}\left(\tau_D > t\right) &= \mathbb{P}\left(h(|B(s)|) \leq W(s) + h(0) + 1, 0 \leq s \leq t\right) \\ &\leq \mathbb{P}\left(h(|B(s)|) \leq \sqrt{s}\xi + h(0) + 1, 0 \leq s \leq t\right) \\ &= \mathbb{P}\left(h(|B(s)|) \leq \sqrt{s}\xi + h(0) + 1, \xi \geq (u_0/t)^{1/2}, 0 \leq s \leq t\right) \\ &\quad + \mathbb{P}\left(h(|B(s)|) \leq \sqrt{s}\xi + h(0) + 1, \xi < (u_0/t)^{1/2}, 0 \leq s \leq t\right) \\ &= I + II \quad \text{(say)} \end{aligned}$$

Note that the second term II above can be easily bounded by

$$\begin{aligned} II &= \mathbb{P}\left(h(|B(s)|) \le \sqrt{s}\xi + h(0) + 1, \xi < (u_0/t)^{1/2}, 0 \le s \le t\right) \\ &\le \mathbb{P}\left(h(|B(s)|) \le u_0^{1/2} + h(0) + 1, \xi < (u_0/t)^{1/2}, 0 \le s \le t\right) \\ &= \mathbb{P}\left(\xi < (u_0/t)^{1/2}\right) \cdot \mathbb{P}\left(\sup_{0\le s\le t} |B(s)| \le h^{-1}(u_0^{1/2} + h(0) + 1)\right) \\ &\le \exp\{-K \cdot t\} \end{aligned}$$

for t large, where the last inequality follows from scaling property of Bessel process and (2.6).

It is harder to handle the first term I. we have

$$\begin{aligned} I &= \mathbb{P}\left(h(|B(s)|) \le \sqrt{s}\xi + h(0) + 1, \xi \ge (u_0/t)^{1/2}, 0 \le s \le t\right) \\ &= \frac{1}{2\pi}\int_{(u_0/t)^{1/2}}^{\infty} \mathbb{P}\left(|B(s)| \le h^{-1}(\sqrt{s}u + h(0) + 1), 0 \le s \le t\right) \cdot \exp\{-\frac{u^2}{2}\}du \end{aligned} \tag{3.1}$$

Now using scaling property of Bessel process and (2.6), we see that

$$\begin{aligned} & \mathbb{P}\left(|B(s)| \le h^{-1}(\sqrt{s}u + h(0) + 1), 0 \le s \le t\right) \\ =\ & \mathbb{P}\left(|B(s)| \le u \cdot h^{-1}(\sqrt{s} + h(0) + 1), 0 \le s \le u^2 t\right) \\ \le\ & \exp\{(u^2t)^{1-q}\log K\} \cdot \exp\left\{-2^{-1}j_v^2 u^{-2}\left(H(u^2t) - H((u^2t)^q)\right)\right\} \end{aligned} \tag{3.2}$$

by choosing $n = (u^2t)^{1-q}$ in the estimate (2.8).

For $u^2t \ge u_0$, it is clear from (1.9),

$$H(u^2t) - H((u^2t)^q) \ge (1-\delta)H(u^2t). \tag{3.3}$$

Hence by combining (3.1), (3.2) and (3.3), we obtain for t large

$$\begin{aligned} I &\le \int_{(u_0/t)^{1/2}}^{\infty} \exp\{(u^2t)^{1-q}\log K\} \cdot \exp\left\{-(1-\delta)2^{-1}j_v^2u^{-2}H(u^2t)\right\} \cdot \exp\{-\frac{u^2}{2}\}du \\ &\le \exp\left\{-(1-\delta)2^{-1}\inf_{u\ge(u_0/t)^{1/2}}\left(j_v^2u^{-2}H(u^2t) + u^2\right)\right\} \cdot \int_{(u_0/t)^{1/2}}^{\infty} \exp\{(u^2t)^{1-q}\log K - \delta\frac{u^2}{2}\}du \\ &\le \exp\{K_\delta t^{(1-q)/q}\} \cdot \exp\left\{-(1-\delta)2^{-1}\inf_{x\ge u_0}\left(j_v^2 tH(x)/x + x/t\right)\right\} \end{aligned}$$

where we used the estimate

$$\begin{aligned} & \int_{(u_0/t)^{1/2}}^{\infty} \exp\{(u^2t)^{1-q}\log K - \delta\frac{u^2}{2}\}du \\ \le\ & \int_{(u_0/t)^{1/2}}^{Q} \exp\{(u^2t)^{1-q}\log K\}du + \int_Q^{\infty} \exp\{-\delta\frac{u^2}{4}\}du \\ \le\ & \exp\{K_\delta t^{(1-q)/q}\} \end{aligned}$$

for some $K_\delta > 0$ and t large, where $Q = (4\delta^{-1}\log K)^{1/(2q)}t^{(1-q)/(2q)}$. We thus finished the general estimate (1.11).

Next we apply the estimate to the special case $h^{-1}(x) = Ax^{\alpha}(\log x)^{\beta}$ with $0 \le \alpha < 1$. It is easy to see from (1.8), as $x \to \infty$

$$\begin{aligned} H(x) &\sim \int_K^x A^{-2}s^{-\alpha}(\log\sqrt{s})^{-2\beta}ds \\ &\sim (1-\alpha)^{-1}2^{2\beta}A^{-2}x^{1-\alpha}(\log x)^{-2\beta} \end{aligned}$$

and thus condition (1.9) is satisfied. Hence in the case $0 < \alpha < 1$ and $\beta \in \mathbb{R}$, we have for t large,

$$\begin{aligned} & \inf_{x \ge u_0} \left(\frac{j_v^2}{2} \cdot \frac{tH(x)}{x} + \frac{x}{2t} \right) \\ \ge \;& (1-\delta)(1+\alpha)(2\alpha)^{-\alpha/(1+\alpha)}(2^{-1}(1+\alpha))^{2\beta/(1+\alpha)}C^{1/(1+\alpha)}t^{(1-\alpha)/(1+\alpha)}(\log t)^{-2\beta/(1+\alpha)} \end{aligned} \tag{3.4}$$

where $C = (1-\alpha)^{-1}2^{2\beta-1}A^{-2}j_{\nu}^2$. Note that the inf in (3.4) is attained around

$$x = (2\alpha C)^{1/(\alpha+1)}(2^{-1}(1+\alpha))^{2\beta/(1+\alpha)} \cdot t^{2/(1+\alpha)}(\log t)^{-2\beta/(1+\alpha)}$$

for t large.

In the case $\alpha = 0$ and $\beta > 0$, we have for t large,

$$\inf_{x \ge u_0} \left(\frac{j_v^2}{2} \cdot \frac{tH(x)}{x} + \frac{x}{2t} \right) \ge (1-\delta)2^{-1}A^{-2}j_{\nu}^2 \cdot t(\log t)^{-2\beta} \tag{3.5}$$

similarly to the estimate in (3.4).

Finally, in the case $\alpha = 1$ and $\beta \le 0$, our condition (1.9) is not satisfied. But the basic idea of our argument works and we can obtain for t large

$$\log \mathbb{P}(\tau_D > t) \le -K^{-1} \cdot (\log t)^{-\beta} \tag{3.6}$$

for some constant $K > 0$. Since the correct rate for $\beta = 0$ is $\log t$, the above result is not sharp and we omit the details of its proof. Note that the rate $(\log t)^{-\beta}$ is the best we can do via the method we used. In other words, we lost the correct rate in the key estimate, Proposition 1.1, as can be seen from the case $\alpha = 1$ and $\beta = 0$, where the upper bound goes to a constant as $t \to \infty$.

4 Proof of Theorem 1.2

For any $g \in \mathcal{G}$, we have

$$\begin{aligned} \mathbb{P}(\tau_D > t) &= \mathbb{P}(h(|B(s)|) \le W(s) + h(0) + 1, 0 \le s \le t) \\ &\ge \mathbb{P}(h(|B(s)|) \le g(s) \le W(s) + h(0) + 1, 0 \le s \le t) \\ &= \mathbb{P}\left(|B(s)| \le h^{-1}(g(s)), 0 \le s \le t\right) \cdot \mathbb{P}(W(s) \ge g(s) - h(0) - 1, 0 \le s \le t). \end{aligned}$$

Thus for $t > 0$ sufficiently large, we have from Theorem 2.2,

$$\log \mathbb{P}\left(|B(s)| \le h^{-1}(g(s)), 0 \le s \le t\right) \ge -(1+\delta)\frac{j_v^2}{2}\int_K^t \frac{1}{(h^{-1}(g(s)))^2}ds \tag{4.1}$$

For the second term, we have from (2.1)

$$\log \mathbb{P}(W(s) \ge g(s) - h(0) - 1, 0 \le s \le t) \ge -\frac{1+\delta}{2}\int_K^t (g'(s))^2 ds. \tag{4.2}$$

Combining the above estimates, we have (1.13).

For the remaining part of the proof, we need to find a "good" function g corresponding to $h^{-1}(x) = Ax^\alpha(\log x)^\beta$. It is nature to require the integrals in (1.13) grow at about the same rate, i.e. $g' \cdot h^{-1}(g) \approx 1$. This relation forces

$$g(x) \approx x^{1/(1+\alpha)}(\log x)^{-\beta/(1+\alpha)}.$$

To determine the best constant $H > 0$ in front of the above function, let

$$g(x) = Hx^{1/(1+\alpha)}(\log x)^{-\beta/(1+\alpha)}$$

which is in $\mathcal{G}$. Note that the condition $\beta \leq 0$ fits naturally for the case $\alpha = 1$.

Next we need some detailed estimates for the integrals in (1.13). In the case $0 \leq \alpha < 1$, it is easy to see as $t \to \infty$,

$$\begin{aligned}
\frac{1}{2}\int_K^t (g'(s))^2 ds &\sim \frac{H^2}{2(1+\alpha)^2}\int_K^t s^{-2\alpha/(1+\alpha)}(\log s)^{-2\beta/(1+\alpha)} ds \\
&\sim \frac{H^2}{2(1+\alpha)(1-\alpha)} t^{(1-\alpha)/(1+\alpha)}(\log t)^{-2\beta/(1+\alpha)}
\end{aligned} \tag{4.3}$$

and

$$\begin{aligned}
\frac{j_\nu^2}{2}\int_K^t \frac{1}{(h^{-1}(g(s)))^2} ds &= \frac{j_\nu^2}{2A^2}\int_K^t g^{-2\alpha}(s)(\log g(s))^{-2\beta} ds \\
&\sim \frac{(1+\alpha)^{2\beta} j_\nu^2}{2H^{2\alpha}A^2}\int_K^t s^{-2\alpha/(1+\alpha)}(\log s)^{-2\beta/(1+\alpha)} ds \\
&\sim \frac{(1+\alpha)^{2\beta+1} j_\nu^2}{2(1-\alpha)H^{2\alpha}A^2} t^{(1-\alpha)/(1+\alpha)}(\log t)^{-2\beta/(1+\alpha)}.
\end{aligned} \tag{4.4}$$

In the case $\alpha = 1$ and $\beta < 0$, we have as $t \to \infty$,

$$\frac{1}{2}\int_K^t (g'(s))^2 ds \sim \frac{H^2}{8}\int_K^t s^{-1}(\log s)^{-\beta} ds \sim \frac{H^2}{8(1-\beta)}(\log t)^{1-\beta} \tag{4.5}$$

and

$$\begin{aligned}
\frac{j_\nu^2}{2}\int_K^t \frac{1}{(h^{-1}(g(s)))^2} ds &= \frac{j_\nu^2}{2A^2}\int_K^t g^{-2}(s)(\log g(s))^{-2\beta} ds \\
&\sim \frac{2^{2\beta-1} j_\nu^2}{H^2A^2}\int_K^t s^{-1}(\log s)^{-\beta} ds \\
&\sim \frac{2^{2\beta-1} j_\nu^2}{(1-\beta)H^2A^2}(\log t)^{1-\beta}.
\end{aligned} \tag{4.6}$$

Hence in the case $0 < \alpha < 1$ and $\beta \in \mathbb{R}$, we have by combining estimates 1.13, 4.3 and 4.4, for t large,

$$\begin{aligned}
&\log \mathbb{P}(\tau_D > t) \\
\geq\ & -(1+\delta)^2 \inf_{H>0}\left(\frac{H^2}{2(1+\alpha)(1-\alpha)} + \frac{(1+\alpha)^{2\beta+1} j_\nu^2}{2(1-\alpha)H^{2\alpha}A^2}\right) \cdot t^{(1-\alpha)/(1+\alpha)}(\log t)^{-2\beta/(1+\alpha)} \\
=\ & -(1+\delta)^2 2^{-1}(1-\alpha)^{-1}\left(\alpha^{-\alpha}(1+\alpha)^{2\beta+2}A^{-2}j_\nu^2\right)^{1/(1+\alpha)} \cdot t^{(1-\alpha)/(1+\alpha)}(\log t)^{-2\beta/(1+\alpha)}
\end{aligned} \tag{4.7}$$

where we used the fact that $\inf_{x>0}(x + Cx^{-q}) = (1+q)q^{-1}(Cq)^{1/(1+q)}$ for $C > 0$ and $q > 0$.

In the case $\alpha = 0$ and $\beta > 0$, we have by combining estimates (1.13), (4.3) and (4.4) as in (4.7), and by taking $H > 0$ sufficient small,

$$\log \mathbb{P}(\tau_D > t) \geq -(1+\delta)^2 \frac{j_\nu^2}{2A^2} \cdot t(\log t)^{-2\beta} \tag{4.8}$$

for t large.

Finally, in the case $\alpha = 1$ and $\beta \leq 0$, we see for t large

$$\begin{aligned} \log \mathbb{P}(\tau_D > t) &\geq -(1+\delta)^2 \inf_{H>0} \left(\frac{H^2}{8(1-\beta)} + \frac{2^{2\beta-1} j_\nu^2}{(1-\beta)H^2 A^2} \right) \cdot (\log t)^{1-\beta} \\ &= -(1+\delta)^2 \frac{2^{\beta-1} j_\nu}{(1-\beta)A} (\log t)^{1-\beta}. \end{aligned} \tag{4.9}$$

This finishes the proof of the lower bound.

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