SPECTRAL TURÁN THEOREMS AND RELATED PROBLEMS

by

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A dissertation submitted to the Faculty of the University of Delaware in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics

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AND RELATED PROBLEMS

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DEDICATION

I am indebted to the sacrifice of several people who may or may not know me. I dedicate this to some of these who have willingly sacrificed of themselves for me.

Mom, Jesus Christ and Taylor

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ABSTRACT

This thesis investigates some graph theoretic problems on the spectral radii and Perron vectors of graphs with certain structural constraints.

For $k \geq 2$, the odd wheel, W_{2k+1}, is the graph formed by joining a vertex to an even cycle, C_{2k}. In Chapter 2, we study the maximum value of the spectral radius of adjacency matrices of graphs on n vertices that do not contain an odd wheel W_{2k+1}. For n sufficiently large, we determine the structure of the spectral extremal graphs when $k \geq 2$ and $k \notin \{4, 5\}$. We show that when $k = 2$, the spectral extremal graphs are among the Turán extremal graphs on n vertices that do not contain W_{2k+1} and have the maximum number of edges. However, when $k \geq 9$, we show that the family of spectral extremal graphs and the family of Turán-extremal graphs are disjoint. Chapter 2 comes from the paper, "The spectral radius of graphs with no odd wheels", with coauthors Sebastian Cioabă and Michael Tait [16].

For $k \geq 2$ and $r \geq 3$, a (k, r)-fan $F_{k,r}$ is a graph on $(r-1)k+1$ vertices consisting of k copies of the complete graph K_r, that intersect at exactly one common vertex. When $r = 3$, this is the friendship graph with $2k+1$ vertices and k triangles. In Chapter 3, we study the maximum value of the spectral radius of adjacency matrices of graphs on n vertices that do not contain $F_{k,r}$. For given $r > 3$ and n sufficiently large, we show that the family of spectral extremal graphs are among the Turán extremal graphs on n vertices that do not contain $F_{k,r}$ and have maximum number of edges. This extends a result of Cioabă, Feng, Tait, and Zhang [17] for $r = 3$. Chapter 3 comes from the paper, "Spectral extremal graphs for intersecting cliques", with coauthors Liying Kang, Yongtao Li, Zhenyu Ni, Michael Tait and Jing Wang [21].

For $k \geq 2$, let $S_{n,k}$ denote the graph on n vertices formed by joining a clique on k vertices K_k, to an independent set $\overline{K}_{n-k}$ on $n-k$ vertices. Let $S^+_{n,k}$ denote the graph obtained from $S_{n,k}$ by adding exactly one more edge in the independent set on $n-k$ vertices.

In Chapter 4, for $k \geq 2$, we study the maximum value of the spectral radius of adjacency matrices of graphs on n vertices that do not contain any C_{2k+2}. For n sufficiently large we prove that the spectral extremal graphs are isomorphic to $S^+_{n,k}$. This extends a result of Zhai and Lin [77] for $k = 2$. Along with results of Nikiforov [50] and Zhai and Wang [78], this covers all even cycles. When $k \geq 2$, we also investigate the maximum value of the spectral radius of adjacency matrices of graphs on n vertices that contain neither any C_{2k+1} nor C_{2k+2}. For n sufficiently large we prove that the spectral extremal graphs are isomorphic to $S_{n,k}$. These results settle a conjecture of Nikiforov (Conjecture 15 in [56]).

For any graph G, we call the graph obtained by joining a new vertex with G as the cone of G, and denote it by G^*. In Chapter 5 we investigate the Perron vector of graphs and determine the minimum value of $\lambda(G^*) - \lambda(G)$ among all simple graphs on n-vertices. We show that only the complete graph K_n attains this minimal increase, thus proving a conjecture of Akbari [2]. We generalize the result to obtain a similar result for loopless multigraphs with a bounded number of edges between any pair of vertices in G. We end with a generalization of cones to simple regular graphs, where we only join pn vertices of a regular graph, G, to a new vertex, for some $0 \leq p \leq 1$. We call this the p-cone of G, and denote it by G^{*_p}. We obtain a lower bound for $\lambda(G^{*_p}) - \lambda(G)$ which is tight for $p = 0, 1$ and reduces to our first result in this chapter.

Chapter 1

INTRODUCTION

Given a graph F, the *Turán number of* F, $\mathrm{ex}(n, F)$ is defined as the maximum number of edges among all graphs on n vertices that do not contain F as a subgraph. The family of F-free graphs with n vertices and $\mathrm{ex}(n, F)$ edges is denoted by $\mathrm{EX}(n, F)$. Many questions in extremal combinatorics may be rephrased as asking for a certain Turán number, and hence the study of the function $\mathrm{ex}(n, F)$ for various F (or, more generally, for various families of forbidden graphs) is one of the most important topics in graph theory and combinatorics. The area has been studied extensively since its introduction to the present (see surveys [37, 40, 62]).

In 1941, Turán [70] determined $\mathrm{ex}(n, K_{r+1})$ where K_{r+1} is the complete graph on $r+1$ vertices. Let $T_r(n)$ denote the complete r-partite graph on n vertices where its part sizes are as equal as possible, i.e., each part has size $\lfloor n/r \rfloor$ or $\lceil n/r \rceil$. Also, let $t_r(n)$ denote the number of edges in $T_r(n)$. Turán [70] extended a result of Mantel [45] for $r = 2$ from 1907 and obtained that $\mathrm{ex}(n, K_{r+1}) = t_r(n)$ and $\mathrm{EX}(n, K_{r+1}) = \{T_r(n)\}$ (see Figure 1.1).

There are many extensions and generalizations of Turán's result; see, e.g., [8, p. 294]. The problem of determining $\mathrm{ex}(n, F)$ is called a Turán-type problem. A celebrated generalization of Turán's theorem is the following result of Erdős, Stone and Simonovits [27, 28], which states that

$$\mathrm{ex}(n, F) = \left(1 - \frac{1}{\chi(F) - 1}\right) \frac{n^2}{2} + o(n^2), \tag{1.1}$$

where $\chi(F)$ is the vertex-chromatic number of F. This provides precise asymptotic estimates for the extremal numbers of non-bipartite graphs. However, for bipartite graphs, where $\chi(F) = 2$, it only gives the bound $\mathrm{ex}(n, F) = o(n^2)$. Kővári, Sós, and Turán [41] proved that

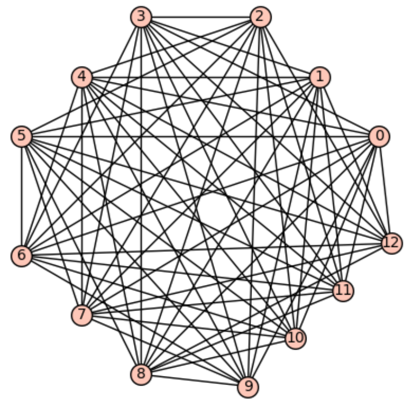

Figure 1.1: The Turán graph $T_4(13)$ on 13 vertices with four color classes.

if $K_{s,t}$ is the complete bipartite graph with vertex classes of size $s \geq t$, then $\mathrm{ex}(n, K_{s,t}) = O(n^{2-1/t})$ (see [32, 34] for more details). Determining the asymptotics of $\mathrm{ex}(n, C_{2k})$ is perhaps one of the most famous Turán-type open problem and is notoriously difficult. Bondy and Simonovits [10] proved that $\mathrm{ex}(n, C_{2k}) = O(n^{1+1/k})$ and several improvements to the upper bound with the same order of magnitude have been obtained since then [72, 39, 12], however constructions with matching asymptotic lower bounds have only been obtained in a handful of cases. The exact order of magnitude is only known for $k = 2, 3$ and 5 [33, 36, 42], and determining it for other k is called the even cycle problem. Although there have been numerous attempts to find better bounds of $\mathrm{ex}(n, F)$ for various bipartite graphs F, we know very little in this case. We refer the interested reader to the comprehensive survey by Füredi and Simonovits [37].

For a graph on n vertices G, the *adjacency matrix* $A = A(G)$ is the n-dimensional integer valued matrix, $A(G) = (a_{ij})_{n\times n}$, with $a_{i,j}$ equal to the number of edges between v_i and v_j. Therefore, $A(G)$ is a symmetric matrix and has n real eigenvalues, that may be denoted in descending order as follows: $\lambda_1(G) \geq \lambda_2(G) \geq \ldots \geq \lambda_n(G)$. The n-dimensional space $\mathbb{R}^n$ has an orthonormal basis formed by eigenvectors of A. We will also call $\lambda_1(G)$ the *spectral radius of* G, and we may interchangeably denote it by $\lambda(G)$, λ_1 or λ depending on how unambiguous the associated graph and eigenvalue order are from the context. As a consequence of having a basis of real eigenvectors and a complete set of eigenvalues, we get the Rayleigh characterization of the spectral radius:

$$\lambda_1(G) = \max_{\mathbf{x}\neq 0} \frac{\mathbf{x}^T A\mathbf{x}}{\mathbf{x}^T\mathbf{x}},$$

where $\mathbf{x}$ ranges over all non-zero n-dimensional vectors. Replacing $\mathbf{x}$ by $\mathbb{1}_n$, the n-dimensional all 1's vector, shows us that $\lambda_1(G)$ is greater than or equal to the average degree of vertices in G.

A square matrix B with rows and columns indexed by some finite set $\{1, 2, \dots, n\} =: [n]$, is said to be reducible if there exists a proper subset $X \subsetneq [n]$, such that $B_{x,y} = 0$ for every $x \in X$ and $y \in [n] \setminus X$. In contrast, B is said to be irreducible if it is not reducible. The adjacency matrix of a graph is irreducible if and only if the graph is connected. An important theorem related to the spectral radius is the celebrated Perron-Frobenius theorem (see Theorem 31.11 in [71] for more details).

Theorem 1.0.1 (Perron-Frobenius)**.** *Let A be an irreducible $n \times n$ non-negative matrix. There is, up to scalar multiples, a unique eigenvector $\mathbf{a} = (a_1, \dots, a_n)$ all of whose coordinates a_i are positive. The corresponding eigenvalue λ (which is called the dominant eigenvalue of A) has algebraic multiplicity one and has the property that $\lambda \geq |\mu|$ for any eigenvalue μ of A.*

Thus, the eigenspace of the spectral radius of a connected graph is one-dimensional. In honor of Oskar Perron who proved the above theorem for positive matrices in 1907, we call the unique (up to scalar multiples) positive eigenvector of connected graphs as the Perron vector. The spectral radius and Perron vector contain a lot of information about the structure of the graph which we shall exploit and uncover in this thesis.

First we study a spectral version of the Turán problem: given a graph F, let $\mathrm{spex}(n, F)$ denote the maximum value of $\lambda_1(G)$ over all n-vertex graphs G which do not contain F as a subgraph. We denote the family of F-free graphs on n vertices with spectral radius equal to $\mathrm{spex}(n, F)$ by $\mathrm{SPEX}(n, F)$. The study of $\mathrm{spex}(n, F)$ for various graphs F (or various families of forbidden subgraphs) was first proposed in generality by Nikiforov [56], though several sporadic results appeared earlier. In particular, Wilf [74] and Nikiforov [47] strengthened the spectral Turán theorem when the forbidden substructure was the complete graph using an inequality due to Motzkin and Straus [46]. Soon after, Nikiforov [49] showed that if G is a K_{r+1}-free graph on n vertices, then $\lambda(G) \leq \lambda(T_r(n))$, with equality if and only if $G = T_r(n)$. This in turn implied Turán's theorem by giving $e(G) \leq t_r(n)$, with equality if and only if $G = T_r(n)$, for any K_{r+1}-free graph, G.

Nikiforov [52] obtained a spectral strengthening of the Erdős–Stone–Simonovits theorem which greatly generalizes the spectral Turán theorem to other forbidden graphs, F, with $\chi(F) = \chi(K_{r+1})$, for some $r \geq 1$.

$$\text{spex}(n, F) = \left(1 - \frac{1}{\chi(F) - 1}\right) n + o(n). \tag{1.2}$$

However, for bipartite graphs F, this only gives that $\text{spex}(n, F) = o(n)$. Nikiforov [49] and Zhai and Wang [78] determined the maximum spectral radius of $K_{2,2}$-free graphs. Furthermore, Nikiforov [55], Babai and Guiduli [5] independently obtained the spectral generalization of the Kővari-Sós-Turán theorem when the forbidden graph is the complete bipartite graph $K_{s,t}$.

Nikiforov [56] determined the maximum spectral radius of graphs without paths and gave bounds for the maximum spectral radius of graphs without cycles of specified length. In addition, Fiedler and Nikiforov [30] obtained tight sufficient spectral conditions for graphs to be Hamiltonian or traceable. For many other spectral analogues of results in extremal graph theory see [57].

One motivation for studying such problems is that $\lambda_1(G)$ is an upper bound for the average degree of G, and hence any upper bound on $\text{spex}(n, F)$ also gives an upper bound on $\text{ex}(n, F)$. Indeed the results in [47, 55] imply Turán's theorem as well as Füredi's improvement to the Kővári-Sós-Turán theorem [34]. The function $\text{spex}(n, F)$ has been studied for many families of graphs (see, for example, [17, 44, 50, 61, 74, 76, 78, 79]). The study of $\text{spex}(n, F)$ fits into a broader framework of *Brualdi-Solheid problems* [11] investigating the maximum spectral radius over all graphs belonging to a specified family. Many results are known in this area (for example [7, 9, 23, 30, 58, 64, 65]).

1.1 Spectral Turán Theorem for Odd Wheels

Let W_t be the *wheel graph* on t vertices: the graph formed by joining a vertex to all of the vertices in a cycle on $t-1$ vertices. In [81], the authors determine the maximum spectral radius over n-vertex graphs which do not contain any wheels. They state that "it seems difficult to determine the maximum spectral radius of a $\{W_l\}$-free graph of order n". In this thesis, we study $\mathrm{spex}(n, W_{2k+1})$ and $\mathrm{SPEX}(n, W_{2k+1})$ and answer their question except when $k \in \{4, 5\}$ as described below. The Turán problem for odd wheels was recently resolved in [22] and [75] and we record these results here to compare them with ours.

Theorem 1.1.1 Dzido and Jastrzębski [22]**.** *Let W_5 be the wheel on 5 vertices. Then*

$$\mathrm{ex}(n, W_5) = \begin{cases} \frac{n^2}{4} + \frac{n}{2} - 1, & \text{if } n \equiv 2 \pmod 4 \\ \left\lfloor \frac{n^2}{4} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor, & \text{otherwise.} \end{cases}$$

The extremal graphs in $\mathrm{EX}(n, W_5)$ consist of a complete bipartite graph with an additional maximum matching in each part. When $n \not\equiv 2 \pmod 4$, the bipartite graph is as balanced as possible. When $n \equiv 2 \pmod 4$ there are two extremal graphs: choosing the bipartite graph to have parts of size $n/2 - 1$ and $n/2 + 1$ gives the same number of edges as choosing them to each be of size $n/2$. Our first theorem shows that the spectral extremal graphs are a subset of the Turán-extremal graphs.

Theorem 1.1.2. *For sufficiently large n, $\mathrm{SPEX}(n, W_5) \subset \mathrm{EX}(n, W_5)$.*

When $k \geq 3$, the structure of the extremal graphs changes.

Theorem 1.1.3 Yuan [75]**.** *Let $k \geq 3$ be an integer. For n sufficiently large,*

$$\mathrm{ex}(n, W_{2k+1}) = \max\left\{ n_0 n_1 + \left\lfloor \frac{(k-1)n_0}{2} \right\rfloor + 1 : n_0 + n_1 = n \right\}.$$

A graph is called *nearly $(k-1)$-regular* if every vertex but one has degree $k-1$ and the final vertex has degree $k-2$. Let $\mathcal{U}_{k,n}$ be the family of $(k-1)$-regular or nearly $(k-1)$-regular graphs on n vertices which do not contain a path on $2k-1$ vertices. This family is non-empty when $n \geq 2k$ (see Proposition 2.3 in [75]). In [75] it is shown that for $k \geq 3$, the family $\mathrm{EX}(n, W_{2k+1})$ consists of complete bipartite graphs with parts of size $\frac{n}{2} - r$ and $\frac{n}{2} + r$ along with a graph from $\mathcal{U}_{k,n/2+r}$ embedded in the larger part and a single edge embedded in the smaller part, where $\frac{n}{2} + r \in \left\{ \left\lfloor \frac{2n+k-1}{4} \right\rfloor, \left\lceil \frac{2n+k-1}{4} \right\rceil \right\}$. Our second theorem determines the structure of the graphs in $\mathrm{SPEX}(n, W_{2k+1})$ for $k \geq 3, k \notin \{4, 5\}$ and n sufficiently large.

Theorem 1.1.4. *Let $k \geq 3$, $k \notin \{4, 5\}$. For sufficiently large n, if $G \in \mathrm{SPEX}(n, W_{2k+1})$, then G is the union of a complete bipartite graph with parts L and R of size $\frac{n}{2} + s$ and $\frac{n}{2} - s$, respectively, and a graph from $\mathcal{U}_{k,n/2+s}$ embedded in $G[L]$ and exactly one edge in $G[R]$. Furthermore, $|s| \leq 1$.*

As in Theorems 1.1.1 and 1.1.2, the exact spectral extremal graph depends on the parity of n mod 4. Furthermore, in the case where $|L|(k-1)$ is odd, it may depend on which particular graph in $\mathcal{U}_{k,n/2+s}$ is embedded in L, making it complicated to determine $\mathrm{SPEX}(n, W_{2k+1})$ precisely. Theorem 1.1.3 and Theorem 1.1.4 imply that $\mathrm{SPEX}(n, W_{2k+1}) \cap \mathrm{EX}(n, W_{2k+1}) = \emptyset$ when $k = 7$ or $k \geq 9$ and n is sufficiently large. In Section 2.4 we give more information about the value of s for the spectral extremal graph, and in most cases this allows one to determine $\mathrm{spex}(n, W_{2k+1})$ precisely. In particular, we can determine the exact value of s if $n \not\equiv 2 \pmod 4$ or if k is odd, and in these cases we have that $\mathrm{spex}(n, W_{2k+1})$ is the root of a cubic polynomial. When $n \equiv 2 \pmod 4$ and k is even, we determine $\mathrm{spex}(n, W_{2k+1})$ up to an additive factor of $o(1/n)$.

Both problems for even wheels are solved in a more general setting. Let F be any graph of chromatic number $r + 1 \geq 3$ which contains an edge such that $\chi(F \setminus e) = r$. Simonovits [63] proved that for n large enough, the only graph in $\mathrm{EX}(n, F)$ is the Turán graph $T_{n,r}$. Nikiforov's work [53] implies a spectral version of this theorem; $\mathrm{SPEX}(n, F)$ is also the Turán graph $T_{n,r}$, for n sufficiently large. Since $\chi(W_{2k}) = 4$ and $\chi(W_{2k} \setminus e) = 3$ for any edge e of W_{2k}, these theorems apply.

1.2 Spectral Turán Theorem for Intersecting Cliques

A graph on $2k+1$ vertices consisting of k triangles which intersect in exactly one common vertex is called a *k-fan* (also known as the friendship graph) and denoted by F_k. Since $\chi(F_k) = 3$, the Erdős–Stone–Simonovits theorem in (1.1) implies that $\mathrm{ex}(n, F_k) = n^2/4 + o(n^2)$. In 1995, Erdős, Füredi, Gould and Gunderson [24] proved the following exact result.

Theorem 1.2.1 Erdős, Füredi, Gould and Gunderson [24]. *For every $k \geq 1$, and for every $n \geq 50k^2$, we have*

$$\mathrm{ex}(n, F_k) = \left\lfloor \frac{n^2}{4} \right\rfloor + \begin{cases} k^2 - k, & \text{if } k \text{ is odd,} \\ k^2 - \frac{3}{2}k, & \text{if } k \text{ is even.} \end{cases}$$

From the spectral analogues of the Erdős–Stone–Simonovits theorem (1.2), we know that $\mathrm{spex}(n, F_k) = n/2 + o(n)$ where F_k is the k-fan graph. Recently, Cioabă, Feng, Tait and Zhang [17] generalized this bound by improving the error term $o(n)$ to $O(1)$, and obtained a spectral counterpart of Theorem 1.2.1. More precisely, they showed that the extremal graphs that attain the maximum spectral radius in a graph on n vertices containing no copy of k-fan must be in $\mathrm{EX}(n, F_k)$ for n sufficiently large.

Theorem 1.2.2 Cioabă, Feng, Tait and Zhang [17]. *Let G be a graph of order n that does not contain a copy of a k-fan, $k \geq 2$. For sufficiently large n, if G has the maximal spectral radius, then*

$$G \in \mathrm{EX}(n, F_k).$$

A graph on $(r-1)k+1$ vertices consisting of k cliques each with r vertices, which intersect in exactly one common vertex, is called a (k,r)-fan and denoted by $F_{k,r}$. Clearly, when $r = 3$, $F_{k,3}$ is the friendship graph F_k. Note that $\chi(F_{k,r}) = r$. Similarly, the Erdős–Stone–Simonovits theorem also implies that $\mathrm{ex}(n, F_{k,r}) = (1 - \frac{1}{r-1})\frac{n^2}{2} + o(n^2) = t_{r-1}(n) + o(n^2)$. In 2003, Chen, Gould, Pfender and Wei [13] proved an exact answer and generalized Theorem 1.2.1 as follows.

Theorem 1.2.3 Chen, Gould, Pfender and Wei [13]. *For every $k \geq 1$ and $r \geq 2$, if $n \geq 16k^3r^8$, then*

$$\mathrm{ex}(n, F_{k,r}) = t_{r-1}(n) + \begin{cases} k^2 - k, & \text{if } k \text{ is odd,} \\ k^2 - \frac{3}{2}k, & \text{if } k \text{ is even.} \end{cases}$$

The extremal graphs of Theorem 1.2.3, denoted by $G_{n,k,r}$, are constructed by taking the $(r-1)$-partite Turán graph $T_{r-1}(n)$ and embedding a graph G_0 in one vertex part. If k is odd, G_0 is isomorphic to two vertex disjoint copies of K_k. If k is even, G_0 may be isomorphic to any nearly $(k-1)$-regular graph on $2k-1$ vertices.

The main result of Chapter 3 is the following theorem giving $\mathrm{SPEX}(n, F_{k,r}) \subset \mathrm{EX}(n, F_{k,r})$, giving a sharp bound on $\mathrm{spex}(n, F_{k,r})$.

Theorem 1.2.4. *Let k and $r \geq 2$ be two natural numbers. For n sufficiently large, if G is a graph on n vertices which does not contain a copy of $F_{k,r}$, and G has the maximal spectral radius, then*

$$G \in \mathrm{EX}(n, F_{k,r}).$$

Our theorem is a spectral analogue of the Turán extremal problem for $F_{k,r}$, Theorem 1.2.3. It may also be viewed as an extension of Theorem 1.2.2. To some extent, this result could be regarded as a continuation and development of [17]. However, we highlight that there are some differences in the approach compared from [17]. In [17], the extremal graph is constant edit distance from a bipartite graph. One of the key steps is to show that the extremal graph has a large bipartite subgraph. To do so, the authors prove a result, namely Lemma 7 in [17], that relates the number of edges to the spectral radius and the number of triangles in the graph and then use the triangle removal lemma and a stability theorem of Füredi [35]. Unfortunately, for this problem the extremal graph is constant edit distance from an $(r-1)$-partite graph and for $r > 3$ the same approach fails. Instead we use a spectral stability theorem of Nikiforov, Lemma 1.5.7 that helps us overcome the limitations of having many triangles and guarantees the presence of a large multipartite graph in G with $o(n^2)$ edit distance from $T_r(n)$, Corollary 1.5.8.

1.3 Spectral Turán Theorem for Even Cycles

In extremal combinatorics, the Turán problem for even cycles is a famously difficult open problem. In Chapter 4 we investigate the spectral version of this problem. For some fixed $k \geq 2$, let $S_{n,k} := K_k \vee \overline{K}_{n-k}$ and $S_{n,k}^{+} := K_k \vee (\overline{K}_{n-k-2} \cup K_2)$. The graph $S_{n,k}^{+}$ has n-vertices and does not contain any C_{2k+2}. Moreover, the graph $S_{n,k}$ has n-vertices and contains neither C_{2k+1} nor C_{2k+2}. In [56], Nikiforov studied the maximum value of the spectral radius for paths and cycles, and made the following conjecture.

Conjecture 1.3.1. *Let $k \geq 2$ and G be a graph of sufficiently large n.*
(a) if $\lambda(G) \geq \lambda(S_{n,k})$ then G contains C_{2k+1} or C_{2k+2} unless $G = S_{n,k}$;
(b) if $\lambda(G) \geq \lambda(S_{n,k}^{+})$ then G contains C_{2k+2} unless $G = S_{n,k}^{+}$.

In Chapter 4 we prove both parts of Conjecture 1.3.1, thus characterizing the extremal graphs for both problems. These are recorded as Theorems 4.0.1 and 4.0.2 in Chapter 4.

1.4 Spectral Extremal Problem for Cones of Graphs

We have mentioned how we wish to study the spectral radius and Perron vector of a graph to get structural information. We have seen how the spectral radius gives us information about the average degree and hence the edge density.

It is also well-known for connected graphs, that all the entries of the Perron vector are the same if and only if the graph is regular. Thus, for irregular graphs, the irregularities of the Perron vector may be thought of as a measure of the graph's irregularity. For example, the *principal ratio* is one such measure of irregularity. The principal ratio of a connected graph is the ratio of the largest Perron vector entry to the smallest Perron vector entry. The two entries are equal for a regular graph, and so their ratio is 1. Tait and Tobin [69] characterized the extremal graphs maximizing the principal ratio and obtained that the family of extremal graphs are *lollipop graphs*. These are graphs constructed by identifying one of the vertices of a clique with an end point of a path. Since we may measure departures of the Perron vector from a constant vector in many other ways, we can generate several extremal type problems with respect to this vector and then look for the most irregular graph with respect to each measure. Similarly, for any graph G, we may define a measure of irregularity in terms of the degrees of its vertices. As an example, we know that $\lambda_1(G) - \bar{d}(G) \geq 0$ with equality if and only if G is regular. Thus, this difference may be considered as a measure of irregularity. Bell [6] studied this difference for all graphs on n vertices and m edges and determined the extremal graphs. Tait and Tobin [68] studied this measure over the set of all connected graphs on n vertices instead, and found that the extremal graphs in this instance are *pineapple graphs*, graphs obtained by identifying one of the vertices of a clique to the center of a star graph. See [3, 4, 6, 18, 48, 80] for an extensive study. We note that these measures of irregularity are in general incomparable.

Towards a different direction, Papendieck and Recht [59] showed that the largest eigenvector entry of the unit Perron vector is at most $\frac{1}{\sqrt{2}}$, for any connected graph on G, with equality if and only if G is a star. As an extension, Cioabă [15] proved the following inequality for any connected graph, G, on n vertices with an independent set $\mathcal{I}$ and normalized Perron vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$:

$$\sum_{i \in \mathcal{I}} x_i^2 \leq 1/2, \tag{1.3}$$

with equality if and only if G is a bipartite graph and $\mathcal{I}$ is one of its color classes. Keeping the importance of the Perron vector along with spectral radius and degree sequence, towards capturing various aspects of graph structure in mind, we consider the effect of taking the *cone of a graph* on the spectral radius. For any graph $G = (V, E)$, we define the cone of G, denoted by $G^* = (V^*, E^*)$, to be the graph obtained from G by adding a new vertex to V which is adjacent to all the vertices of G. Thus, if we denote the new vertex by z, then $V^* = V \cup \{z\}$ and $E^* = E \cup \{vz | v \in V\}$. We know that $\lambda(G^*) - \lambda(G) > 0$ for all graphs, G, but it was not known what the minimum value of the difference could be. A conjecture of Professor Saieed Akbari [2] asks whether $\lambda(G^*) - \lambda(G) \geq 1$, and the extremal graphs achieving equality are all complete graphs. We prove this to be true in the following theorem.

Theorem 1.4.1. *Let G be a graph on n vertices, then $\lambda + 1 \leq \lambda^*$ with equality holding if and only if $G \cong K_n$.*

In our subsequent communication to Professor Akbari, he asked a similar conjecture for multigraphs. We generalized the proof of Theorem 1.4.1 to prove the following Theorem for multigraphs.

Theorem 1.4.2. *Let G be a loopless multigraph on n vertices where any pair of vertices have at most m edges between them, then $\lambda + \frac{1}{m} \leq \lambda^*$ with equality holding if and only if $G \cong K_n$, $m = 1$.*

We also study a generalized version of cones for regular graphs, see Section 5.3 for more details.

Organization and notation

In Section 1.5 we present several lemmas that we will use throughout Chapters 2, 3 and 4. Chapter 2 is organized into sections as follows. In Section 2.1, we give structural results that graphs in $\mathrm{SPEX}(n, W_{2k+1})$ must satisfy for all k. We then specialize to the $k = 2$ and $k > 2$ cases. In Section 2.2 we prove Theorem 1.1.2 and in Section 2.3 we prove Theorem 1.1.4. In Section 2.4, we discuss the exact sizes of the partitions in the spectral extremal W_{2k+1}-free graphs. We end with some concluding remarks and open problems. On similar lines, Chapter 3 consists of Section 3.1 where we prove Theorem 1.2.4. In Section 3.2 there are several structural results for graphs in $\mathrm{SPEX}(n, F_{k,r})$. Section 3.3 contains some concluding remarks and a problem we wish to consider in the future. In Chapter 4 we answer a two-part conjecture of Nikiforov related to spectral Turán problems on cycles. Section 4.2 contains lemmas compiled from various papers in spectral and extremal graph theory, along with some estimation on the spectral radius and Perron vectors of spectral extremal graphs in $\mathrm{SPEX}(n, C_{2k+2})$ and $\mathrm{SPEX}(n, \{C_{2k+1}, C_{2k+2}\})$, for $k \geq 2$. In Section 4.3, we prove some structural lemmas that allow us to show that $K_{k,n-k} \subset G$, where G is any graph in $\mathrm{SPEX}(n, C_{2k+2})$ or $\mathrm{SPEX}(n, \{C_{2k+1}, C_{2k+2}\})$. In Section 4.4 we prove Theorems 4.0.1 and 4.0.2, that resolve a conjecture of Nikiforov (Conjecture 1.3.1). Finally, we end this chapter with Section 4.5 stating another conjecture of Nikiforov (Conjecture 4.5.1) where we believe our techniques will be useful. We plan to pursue this problem in the near future. Chapter 5 consists of Section 5.1 where we prove a conjecture of Professor Saieed Akbari in Theorem 1.4.1. In Section 5.2 we prove Theorem 1.4.2 which is a generalization of Theorem 1.4.1 for connected multigraphs having at most m-edges between any pair of distinct vertices. In Section 5.3, we study a generalization of cones for regular graphs and prove Theorem 5.3.1. We end this chapter with Section 5.4 giving a related problem and some suggestions on how to tackle it.

In this thesis, we consider only simple and undirected graphs unless otherwise specified. For any undefined graph theoretic terms in the thesis, see [38, 71]. Let G be a simple connected graph with vertex set $V(G) = \{v_1, \ldots, v_n\}$ and edge set $E(G) = \{e_1, \ldots, e_m\}$. For a vertex $v \in V(G)$, we write $N(v)$ for the set of neighbors of v. Let $d(v)$ be the degree of a vertex v in G. That is, $d(v) = |N(v)|$. Also, let $\bar{d}(G)$ (or just $\bar{d}$ if it is clear from context) denote the average degree of vertices in $V(G)$. Let S be a set of vertices. We write $N_S(v)$ for the set of neighbors of v in the set S, and $d_S(v)$ for the number of neighbors of v in the set S, that is, $d_S(v) = |N_S(v)| = |N(v) \cap S|$. We denote by $e(S)$ the number of edges contained in S. We use $G[S]$ to denote the subgraph of G induced by the vertices and edges of S. For two graphs G and H, we use $G \cup H$ to denote their disjoint union, and $G \vee H$ to denote their join. We use G^* to denote the cone of G, that is, $G^* = G \vee K_1$.

We will use P_t, C_t, K_t and W_t to denote the path, the cycle, complete graph, and the wheel on t vertices, respectively.

1.5 Background Lemmas

In this section, we record several lemmas that will be useful for Chapters 2 and 3. We start with the Triangle Removal Lemma and a stability theorem of Füredi.

Lemma 1.5.1 Triangle Removal Lemma [24, 31, 60]**.** *For each $\epsilon > 0$, there exists a $\delta > 0$ and $N = N(\epsilon)$, such that every graph G on n vertices with $n \geq N$, and at most δn^3 triangles, can be made triangle-free by removing at most ϵn^2 edges.*

Lemma 1.5.2 Füredi Stability Theorem [35]**.** *Suppose G is a triangle-free graph on n vertices and s is a positive integer such that $e(G) = e(T_{n,2}) - s$. Then there exists a bipartite subgraph H, such that $e(H) \geq e(G) - s$.*

In fact, Füredi's stability theorem applies more generally to any K_r-free graphs, but we only apply it to triangle free graphs and so state it as above. Next, we will need the even-circuit theorem. We note that the best current bounds for $\mathrm{ex}(n, C_{2k})$ are given by He [39] (see also Bukh and Jiang [12]), but for our purposes the dependence of the multiplicative constant on k is not important. We use the following version because it makes the calculations slightly easier.

Lemma 1.5.3 Even Circuit Theorem [72]. *For $k \geq 2$ and n a natural number,*

$$\mathrm{ex}(n, C_{2k}) \leq 8(k-1)n^{1+1/k}.$$

The following useful lemma may be proved by the inclusion-exclusion principle.

Lemma 1.5.4. *If $A_1, \ldots, A_q$ are finite sets, then*

$$|A_1 \cap \ldots \cap A_q| \geq \sum_{i=1}^{q} |A_i| - (q-1)\left|\bigcup_{i=1}^{q} A_i\right|.$$

Lemma 1.5.5 [67] Theorem 4.4. *Let H_1 be a graph on n_0 vertices with maximum degree d and H_2 be a graph on $n - n_0$ vertices with maximum degree d'. H_1 and H_2 may have loops or multiple edges, where loops add 1 to the degree. Let H be the join of H_1 and H_2. Define*

$$B = \begin{bmatrix} d & n - n_0 \\ n_0 & d' \end{bmatrix}. \tag{1.4}$$

Then $\lambda_1(H) \leq \lambda_1(B)$.

Lemma 1.5.6 [17] Lemma 7. *If G has n vertices, t triangles and spectral radius $\lambda_1 > \frac{n}{2}$, then $e(G) > \lambda_1^2 - \frac{6t}{n}$.*

We now turn to a useful spectral stability lemma that helps us overcome the limitations of having many triangles.

Lemma 1.5.7 Nikiforov [54]. *Let $r \geq 2, 1/\ln n < c < r^{-8(r+21)(r+1)}, 0 < \varepsilon < 2^{-36}r^{-24}$ and G be a graph on n vertices. If $\lambda(G) > (1 - \frac{1}{r} - \varepsilon)n$, then one of the following statements holds:*

(a) G contains a $K_{r+1}(\lfloor c \ln n \rfloor, \ldots, \lfloor c \ln n \rfloor, \lceil n^{1-\sqrt{c}} \rceil)$;

(b) G differs from $T_r(n)$ in fewer than $(\varepsilon^{1/4} + c^{1/(8r+8)})n^2$ edges.

From the above theorem, one can easily get the following spectral analogue of the classical Erdős-Simonovits stability theorem [63, 35].

Corollary 1.5.8. *Let F be a graph with chromatic number $\chi(F) = r + 1$. For every $\varepsilon > 0$, there exist $\delta > 0$ and n_0 such that if G is an F-free graph on $n \geq n_0$ vertices with $\lambda(G) \geq (1 - \frac{1}{r} - \delta)n$, then G can be obtained from $T_r(n)$ by adding and deleting at most εn^2 edges.*

Let G be a simple graph with matching number $\beta(G)$ and maximum degree $\Delta(G)$. For given two integers β and Δ, define $f(\beta, \Delta) = \max\{e(G) : \beta(G) \leq \beta, \Delta(G) \leq \Delta\}$.

In 1976, Chvátal and Hanson [14] obtained the following result.

Lemma 1.5.9 Chvátal-Hanson [14]**.** *For every two integers $\beta \geq 1$ and $\Delta \geq 1$, we have*

$$f(\beta, \Delta) = \Delta\beta + \left\lfloor \frac{\Delta}{2} \right\rfloor \left\lfloor \frac{\beta}{\lceil \Delta/2 \rceil} \right\rfloor \leq \Delta\beta + \beta.$$

We will frequently use a special case proved by Abbott, Hanson and Sauer [1]:

$$f(k-1, k-1) = \begin{cases} k^2 - k, & \text{if } k \text{ is odd,} \\ k^2 - \frac{3}{2}k, & \text{if } k \text{ is even.} \end{cases}$$

Furthermore, the extremal graphs with matching number β and maximum degree Δ which have $f(k-1, k-1)$ edges are exactly those that we embedded into the Turán graph $T_{r-1}(n)$ to obtain the extremal $F_{k,r}$-free graphs.

Denote by $K_{n_1,n_2,\ldots,n_{r-1}}$ the complete $(r-1)$-partite graph on $n = \sum_{i=1}^{r-1} n_i$ vertices. For convenience, we assume that $n_1 \geq n_2 \geq \ldots \geq n_{r-1} > 0$. It is well-known [19, p. 74] or [20] that the characteristic polynomial of $K_{n_1,n_2,\ldots,n_{r-1}}$ is given as

$$\phi(K_{n_1,n_2,\ldots,n_{r-1}}, x) = x^{n-r+1} \left(1 - \sum_{i=1}^{r-1} \frac{n_i}{x + n_i}\right) \prod_{j=1}^{r-1} (x + n_j).$$

So the spectral radius $\lambda(K_{n_1,n_2,\ldots,n_{r-1}})$ satisfies the following equation:

$$\sum_{i=1}^{r-1} \frac{n_i}{\lambda(K_{n_1,n_2,\ldots,n_{r-1}}) + n_i} = 1 \tag{1.5}$$

Feng, Li and Zhang [29, Theorem 2.1] implicitly proved the following lemma, which was also proved by Stevanović, Gutman and Rehman [66].

Lemma 1.5.10. *If $n_i - n_j \geq 2$, then*

$$\lambda(K_{n_1,\ldots,n_i-1,\ldots,n_j+1,\ldots,n_{r-1}}) > \lambda(K_{n_1,\ldots,n_i,\ldots,n_j,\ldots,n_{r-1}}).$$

For a connected graph G on n vertices, let $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_n)^{\mathrm{T}}$ be an eigenvector of $A(G)$ corresponding to $\lambda(G)$. By the Perron–Frobenius Theorem 1.0.1, we can choose $\mathbf{x}$ as a positive real vector.

$$\lambda(G)\mathbf{x}_i = \sum_{j=1}^{n} a_{ij}\mathbf{x}_j = \sum_{j \in N_G(i)} \mathbf{x}_j, \text{ for any } i \in [n]. \tag{1.6}$$

Another useful result concerns the Rayleigh quotient:

$$\lambda(G) = \max_{\mathbf{x} \in \mathbb{R}^n} \frac{\mathbf{x}^{\mathrm{T}} A(G) \mathbf{x}}{\mathbf{x}^{\mathrm{T}}\mathbf{x}} = \max_{\mathbf{x} \in \mathbb{R}^n} \frac{2\sum_{\{i,j\} \in E(G)} \mathbf{x}_i \mathbf{x}_j}{\mathbf{x}^{\mathrm{T}}\mathbf{x}}. \tag{1.7}$$

Let G be a graph with a partition of the vertices into $r-1$ non-empty parts $V(G) = V_1 \cup V_2 \cup \ldots \cup V_{r-1}$. Let $E_{cr}(G) = \cup_{1 \leq i < j \leq r-1} E(V_i, V_j)$ be the crossing edges of G. The following lemma was proved in Chen, Gould, Pfender and Wei [13].

Lemma 1.5.11 Chen, Gould, Pfender and Wei [13]**.** *Suppose G is partitioned as above so that the following conditions are satisfied*

$$\sum_{j \neq i} \beta(G[V_j]) \leq k-1 \quad \textit{and} \quad \Delta(G[V_i]) \leq k-1, \tag{1.8}$$

$$d_{G[V_i]}(v) + \sum_{j \neq i} \beta(G[N(v) \cap V_j]) \leq k-1, \tag{1.9}$$

for any $i \in [r-1]$ and $v \in V_i$. If G is $F_{k,r}$-free, then

$$\sum_{i=1}^{r-1} |E(G[V_i])| - \left(\sum_{1 \leq i < j \leq r-1} |V_i||V_j| - |E_{cr}(G)| \right) \leq f(k-1, k-1).$$

Chapter 2

THE SPECTRAL RADIUS OF GRAPHS WITH NO ODD WHEELS

In this chapter, we will study the spectral Turán problem for odd wheels and prove tight bounds on $\mathrm{spex}(n, W_{2k+1})$. In most cases we determine $\mathrm{spex}(n, W_{2k+1})$ exactly by characterizing what graphs in $\mathrm{SPEX}(n, W_{2k+1})$ for all $k \geq 2$, $k \notin \{4, 5\}$ may look like. The main results are Theorems 1.1.2 and 1.1.4.

We begin with two straightforward but useful remarks.

Remark 1. By Theorem 1.1.3 and some straightforward computation, for fixed $k \geq 3$ and n sufficiently large we have

$$\mathrm{ex}(n, W_{2k+1}) \leq \frac{n^2}{4} + \frac{n(k-1)}{4} + \frac{(k-1)^2}{16} + 1 < \frac{n^2}{4} + \frac{nk}{4}. \tag{2.1}$$

Remark 2. A graph is W_{2k+1}-free if and only if the subgraph induced by the neighborhood of each vertex is C_{2k}-free.

2.1 Structural results for extremal graphs

In this section, we will assume that $k \geq 2$ is fixed and that $G \in$ SPEX(n, W_{2k+1}). We will also use auxiliary constants ϵ, p and θ. We will frequently assume that n is larger than some constant depending only on k, ϵ, p, and θ. Every lemma in this section holds for n large enough.

First, we will need a lower bound on $\mathrm{spex}(n, W_{2k+1})$.

Lemma 2.1.1. *Let $k \geq 2$ be an integer. Then,* $\lambda_1(G) > \frac{n+k-1}{2}$.

Proof. For $k = 2$, if $H \in \mathrm{EX}(n, W_5)$ then since G is a graph maximizing spectral radius over all W_5-free graphs, we have

$$\lambda_1(G) \geq \lambda_1(H) \geq \frac{\mathbf{1}^T A(H)\mathbf{1}}{\mathbf{1^T 1}} \geq 2\frac{\left(\frac{n^2}{4} + \frac{n}{2} - 1\right)}{n} = \frac{n}{2} + 1 - \frac{2}{n} > \frac{n+1}{2}. \tag{2.2}$$

For $k \geq 3$, let

$$Q = \begin{bmatrix} k-1 & \lfloor \frac{n}{2} \rfloor \\ \lceil \frac{n}{2} \rceil & 0 \end{bmatrix},$$

and let μ be the spectral radius of Q with eigenvector $\begin{bmatrix} 1 & \eta \end{bmatrix}^T$. By direct computation, $\mu = \frac{k-1+\sqrt{(k-1)^2+4\lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor}}{2} \geq \frac{k-1+\sqrt{(k-1)^2+n^2-1}}{2}$, and $\eta = \frac{\lceil \frac{n}{2} \rceil}{\mu}$. Thus, for any $\epsilon > 0$, we have that $|1 - \eta| < \epsilon$ for n large enough. Let $\mathbf{z}$ be the n-dimensional vector where the first $\lceil n/2 \rceil$ entries are 1 and the last $\lfloor n/2 \rfloor$ entries are η.

Now, let G_1 be a graph on $\lceil \frac{n}{2} \rceil$ vertices in $\mathrm{EX}(\lceil \frac{n}{2} \rceil, \{K_{1,k}, P_{2k-1}\})$. Therefore, G_1 is a P_{2k-1}-free graph and is $(k-1)$-regular if $(k-1)\lceil \frac{n}{2} \rceil$ is even and is $(k-1)$-nearly regular otherwise. Define $\gamma = 0$ if $(k-1)\lceil \frac{n}{2} \rceil$ is even and 1 if it is odd, and so $e(G_1) = (k-1)\lceil \frac{n}{2} \rceil/2 - \gamma/2$. Now let

$$\tilde{G} = G_1 \vee (K_2 \cup (\lfloor \frac{n}{2} \rfloor - 2)K_1).$$

Let $A(\tilde{G})$ be indexed so that the vertices corresponding to G_1 are first, and let $A(E_1)$ be a diagonal matrix with exactly γ entries equal to 1 (and the rest 0s) so that the principal submatrix of $A(\tilde{G}) + A(E_1)$ corresponding to the first $\lceil \frac{n}{2} \rceil$ vertices has constant row sum $(k-1)$. Since $\tilde{G}$ is W_{2k+1}-free, we have that

$$\begin{aligned} \lambda_1(G) \geq \lambda_1(\tilde{G}) \geq \frac{\mathbf{z}^T A(\tilde{G})\mathbf{z}}{\mathbf{z}^T\mathbf{z}} &= \mu - \frac{\mathbf{z}^T A(E_1)\mathbf{z}}{\mathbf{z}^T\mathbf{z}} + \frac{2\eta^2}{\mathbf{z}^T\mathbf{z}} > \mu - \frac{\gamma}{\mathbf{z}^T\mathbf{z}} + \frac{2(1-\epsilon)^2}{\mathbf{z}^T\mathbf{z}} \\ &> \frac{k-1+\sqrt{(k-1)^2+n^2-1}}{2} + \frac{2(1-\epsilon)^2-\gamma}{n} > \frac{n+k-1}{2}. \end{aligned} \tag{2.3}$$

□

To get the equality above, we use that

$$\frac{\mathbf{z}^T A(\tilde{G})\mathbf{z}}{\mathbf{z}^T\mathbf{z}} = \frac{\mathbf{z}^T(A(\tilde{G}) + A(E_1))\mathbf{z}}{\mathbf{z}^T\mathbf{z}} - \frac{\mathbf{z}^T A(E_1)\mathbf{z}}{\mathbf{z}^T\mathbf{z}}.$$

Next we show that G contains a large maximum cut.

Lemma 2.1.2. *For any $\epsilon > 0$, there is a partition $V(G) = S \sqcup T$ which forms a maximum cut satisfying*

$$e(S,T) \geq \left(\frac{1}{4} - \epsilon\right) n^2.$$

Furthermore,

$$\left(\frac{1}{2} - \sqrt{\epsilon}\right) n \leq |S|, |T| \leq \left(\frac{1}{2} + \sqrt{\epsilon}\right) n.$$

Proof. Fix $\epsilon > 0$, and let δ, N_1 be the constants that come from the Triangle Removal Lemma (Lemma 1.5.1) with respect to the constant $\epsilon/4$. That is, δ and N_1 are chosen so that any graph on $n \geq N_1$ vertices and at most δn^3 triangles can be made triangle-free by removing at most $\frac{\epsilon}{4}n^2$ edges. Now, since G is W_{2k+1}-free, the neighborhood of any vertex does not contain a C_{2k}. If t denotes the number of triangles in G, then we have

$$3t = \sum_{v \in V(G)} e(G[N(v)]) \leq \sum_{v \in V(G)} \text{ex}(d(v), C_{2k}) \leq \sum_{v \in V(G)} \text{ex}(n, C_{2k}) \leq 8(k-1)n^{2+1/k}, \tag{2.4}$$

where we use Lemma 1.5.3 for the last inequality. Thus, for n a large enough constant depending only on k and δ (and hence only on k and ϵ), we have that

$$t \leq \frac{8}{3}(k-1)n^{2+1/k} < \delta n^3.$$

By Lemma 1.5.1, for $n \geq N_1$, there is a triangle-free subgraph G_1 obtained by deleting at most $\frac{\epsilon}{4}n^2$ edges from G. Since G_1 is triangle-free, we may define $s = e(T_{n,2}) - e(G_1) \geq 0$. By Füredi's stability theorem (Lemma 1.5.2), G_1 contains a bipartite subgraph G_2 with at least $e(G_1) - s$ edges. We now have a bipartite subgraph G_2 of G such that $e(G_2) \geq e(G) - \frac{\epsilon}{4}n^2 - s$.

To lower bound the number of edges in G, we use Lemma 1.5.6, Lemma 2.1.1, and (2.4) to get that

$$e(G) > \lambda_1^2 - \frac{6t}{n} > \frac{n^2}{4} - 16(k-1)n^{1+1/k} > \frac{n^2}{4} - \frac{\epsilon}{4}n^2, \tag{2.5}$$

for n large enough. This implies that $e(G_1) \geq e(G) - \frac{\epsilon}{4}n^2 \geq \frac{n^2}{4} - \frac{\epsilon}{2}n^2$ and hence $s \leq \frac{\epsilon}{2}n^2$. Therefore we have a bipartite subgraph G_2 and any partition $V(G) = S \sqcup T$ which forms a maximum cut satisfies

$$e(S,T) \geq e(G_2) \geq e(G_1) - s \geq \frac{n^2}{4} - \epsilon n^2.$$

The bounds on the sizes of $|S|$ and $|T|$ follow from $|S||T| \geq e(G_2)$ and the inequality above. $\square$

Next we will show that most vertices have degree close to $\frac{n}{2}$. Define

$$P := \left\{ v : d(v) \leq \left(\frac{1}{2} - \frac{1}{p} \right) n \right\}.$$

Lemma 2.1.3. *Let p be a fixed natural number. Then the set P satisfies*

$$|P| \leq 16pkn^{1/k}.$$

Proof. We will ignore floors and ceilings throughout this proof. Assume to the contrary that the set of 'atypical' vertices, P, has cardinality greater than $16pkn^{1/k}$. Then consider any fixed subset $P' \subseteq P$ with $|P'| = 16pkn^{1/k}$. Using (2.5), it follows that

$$\begin{aligned}
&\text{ex}(n - 16pkn^{1/k}, W_{2k+1}) \geq e[G \setminus P'] \geq e(G) - \sum_{v \in P'} d(v) \\
&\geq \frac{n^2}{4} - 16(k-1)n^{1+1/k} - 16pkn^{1/k}\left(\frac{1}{2} - \frac{1}{p}\right)n \\
&= \left(\frac{(n - 16pkn^{1/k})^2}{4} + \frac{(n - 16pkn^{1/k})k}{4} \right) - \frac{(n - 16pkn^{1/k})k}{4} - 64p^2k^2n^{2/k} \\
&+ 8pkn^{1+1/k} - 16(k-1)n^{1+1/k} - 16pkn^{1/k}\left(\frac{1}{2} - \frac{1}{p}\right)n \\
&\geq \left(\frac{(n - 16pkn^{1/k})^2}{4} + \frac{(n - 16pkn^{1/k})k}{4} \right) - \frac{nk}{4} - 64p^2k^2n^{2/k} + 16n^{1+1/k} \\
&> \left(\frac{(n - 16pkn^{1/k})^2}{4} + \frac{(n - 16pkn^{1/k})k}{4} \right),
\end{aligned}$$

which contradicts (2.1). $\square$

For any vertex v, and any subset $A \subset V$, let $d_A(v) = |N(v) \cap A|$. Also, let $\theta > 0$ be arbitrary and define

$$M := \{v \in S : d_S(v) \geq \theta n\} \cup \{v \in T : d_T(v) \geq \theta n\}.$$

We will now see that M and P are empty sets. The following lemmas will prove this.

Lemma 2.1.4. *Let $\epsilon > 0$ be arbitrary. Then*

$$|M| \leq \frac{3\epsilon n}{\theta}$$

and $M \setminus P$ is empty.

Proof. We know from Lemma 2.1.2 that (for n large enough) G has a maximum cut with $e(S,T) \geq \left(\frac{1}{4} - \epsilon\right)n^2$. Hence, for $k = 2$,

$$e(S) + e(T) = e(G) - e(S,T) \leq \frac{n^2}{4} + \frac{n}{2} - \frac{n^2}{4} + \epsilon n^2 \leq \frac{n}{2} + \epsilon n^2,$$

and for $k \geq 3$ and n large enough,

$$e(S) + e(T) = e(G) - e(S,T) \leq \frac{n^2}{4} + \frac{n(k-1)}{4} + \frac{(k-1)^2}{16} + 1 - \frac{n^2}{4} + \epsilon n^2 \leq \frac{3}{2}\epsilon n^2.$$

On the other hand, if we let $M_1 = M \cap S$ and $M_2 = M \cap T$, then

$$2e(S) = \sum_{u \in S} d_S(u) \geq \sum_{M_1} d_S(u) \geq |M_1|\theta n$$

$$2e(T) = \sum_{u \in T} d_T(u) \geq \sum_{M_2} d_T(u) \geq |M_2|\theta n$$

So, $e(S) + e(T) \geq \frac{|M|\theta n}{2}$, and hence $\frac{|M|\theta n}{2} \leq \frac{3\epsilon n^2}{2}$. Therefore proving, $|M| \leq \frac{3\epsilon n}{\theta}$.

We now prove that $M \setminus P$ is empty. Let us call $P_1 = P \cap S$, and $P_2 = P \cap T$. Suppose $M \setminus P \neq \emptyset$. We assume without loss of generality that there exists a vertex $u \in M_1 \setminus P_1$. As S and T form a maximum cut, $d_T(u) \geq \frac{d(u)}{2}$. Also, since $u \notin P$, it follows that $d(u) \geq \left(\frac{1}{2} - \frac{1}{p}\right)n$. Therefore, $d_T(u) \geq \left(\frac{1}{4} - \frac{1}{2p}\right)n$. On the other hand $|P| \leq 16pkn^{1/k}$. Hence, for fixed ϵ, θ, p and for n large enough, we have

$$|S \setminus (M \cup P)| \geq \left(\frac{1}{2} - \sqrt{\epsilon}\right)n - \frac{3\epsilon n}{\theta} - 16pkn^{1/k} > k. \tag{2.6}$$

Now suppose that u is adjacent to k distinct vertices $u_1, \ldots, u_k \in S \setminus (M \cup P)$. Since $u_i \notin P$, we have

$$d(u_i) \geq \left(\frac{1}{2} - \frac{1}{p}\right)n$$

On the other hand $d_S(u_i) \leq \theta n$. So,

$$d_T(u_i) = d(u_i) - d_S(u_i) \geq \left(\frac{1}{2} - \frac{1}{p}\right)n - \theta n$$

By Lemma 1.5.4 we have,

$$\begin{aligned} |N_T(u) \cap N_T(u_1) \cap \cdots \cap N_T(u_k)| &\geq |N_T(u)| + |N_T(u_1)| + \cdots + N_T(u_k)| \\ &\quad - k|N_T(u) \cup N_T(u_1) \cup \ldots \cup N_T(u_k)| \\ &\geq \left(\frac{1}{4} - \frac{1}{2p}\right) n + k \left(\frac{1}{2} - \frac{1}{p} - \theta\right) n - k|T| \qquad (2.7) \\ &\geq \left(\frac{1}{4} - \frac{2k+1}{2p} - k\theta - k\sqrt{\epsilon}\right) n \\ &> k, \end{aligned}$$

where the last inequality holds if we choose $p > 20k$, $\theta < \frac{1}{20k}$, $\epsilon < \frac{1}{100k^2}$ and n large enough. This implies that there are at least k distinct vertices $v_1, \ldots, v_k \in T$ such that $\{v_1, \ldots, v_k\} \subseteq |N_T(u) \cap N_T(u_1) \cap \ldots \cap N_T(u_k)|$. This is a contradiction as G should not contain a W_{2k+1}. Therefore, u can be adjacent to at most $k-1$ vertices in $S \setminus (M \cup P)$. Therefore,

$$\begin{aligned} d_S(u) &\leq |M| + |P| + k - 1 \\ &\leq \frac{3\epsilon n}{\theta} + 16pkn^{1/k} + k - 1 \qquad (2.8) \\ &< \theta n, \end{aligned}$$

where the last inequality holds by choosing $\epsilon < \frac{\theta^2}{6}$ and n large enough. This contradicts $u \in M$ and therefore $M_1 \setminus P_1$ must be empty, and hence $M \setminus P = \emptyset$. □

Lemma 2.1.5. *The set P is empty. $G[S]$ and $G[T]$ are $K_{1,k}$-free.*

Proof. In the proof for Lemma 2.1.4 we showed that there are no vertices in $M_1 \setminus P$ (or $M_2 \setminus P$) adjacent to at least k vertices in $S \setminus P$ (or $T \setminus P$, respectively). We will similarly show that $G[S \setminus P]$ and $G[T \setminus P]$ are $K_{1,k}$-free. Without loss of generality assume to the contrary that there exists a vertex $u \in S \setminus P$ that is adjacent to k distinct vertices $u_1, \ldots, u_k$ in $S \setminus P$. Then

$$\begin{aligned} |N_T(u) \cap N_T(u_1) \cap \ldots \cap N_T(u_k)| &\geq |N_T(u)| + |N_T(u_1)| + \ldots + |N_T(u_k)| \\ &\quad - k|N_T(u) \cup N_T(u_1) \cup \ldots \cup N_T(u_k)| \\ &\geq (k+1)\left(\frac{1}{2} - \frac{1}{p} - \theta\right)n - k\left(\frac{1}{2} + \sqrt{\epsilon}\right)n \\ &= \left(\frac{1}{2} - \frac{k+1}{p} - (k+1)\theta - k\sqrt{\epsilon}\right)n \\ &> k \end{aligned} \tag{2.9}$$

for sufficiently large n and p and sufficiently small θ and ϵ. This implies that there are at least k distinct vertices $v_1, \ldots, v_k \in T$ such that $\{v_1, \ldots, v_k\} \subseteq |N_T(u) \cap N_T(u_1) \cap \ldots \cap N_T(u_k)|$. This is a contradiction as G should not contain a W_{2k+1}. Therefore, u can be adjacent to at most $k-1$ vertices in $S \setminus P$. This implies that $G[S \setminus P]$ is $K_{1,k}$-free and similarly $G[T \setminus P]$ is $K_{1,k}$-free.

Next, let z be a vertex of G with largest Perron vector entry. By possible rescaling we may assume that $x_z = 1$. Therefore,

$$d(z) \geq \sum_{v \sim z} x_v = \lambda_1 x_z = \lambda_1 > \frac{n+k-1}{2}, \tag{2.10}$$

and hence $z \notin P$. Assume without loss of generality then, that $z \in S$. Also, since $G(S \setminus P)$ is $K_{1,k}$ free,

$$d_S(z) = d_{S \setminus P}(z) + d_{S \cap P}(z) \leq k - 1 + |S \cap P|$$

So,

$$\begin{aligned}
\lambda_1 = \lambda_1 \mathbf{x}_z &= \sum_{v \sim z} \mathbf{x}_v \\
&= \sum_{\substack{v \sim z \\ v \in S}} \mathbf{x}_v + \sum_{\substack{v \sim z \\ v \in T}} \mathbf{x}_v \\
&= \sum_{\substack{v \sim z \\ v \in S}} \mathbf{x}_v + \sum_{\substack{v \sim z \\ v \in P_2}} \mathbf{x}_v + \sum_{\substack{v \sim z \\ v \in T \setminus P_2}} \mathbf{x}_v \\
&\leq d_S(z) + |P_2| + \sum_{v \in T \setminus P} \mathbf{x}_v \\
&\leq k - 1 + |S \cap P| + |T \cap P| + \sum_{v \in T \setminus P} \mathbf{x}_v \\
&\leq k - 1 + 16pkn^{1/k} + \sum_{v \in T \setminus P} \mathbf{x}_v .
\end{aligned} \tag{2.11}$$

Therefore,

$$\sum_{v \in T \setminus P} x_v \geq \lambda_1 - 16pkn^{1/k} - k + 1 \tag{2.12}$$

Now to show $P = \emptyset$, first assume to the contrary that there exists some vertex $v \in P$ with $d(v) \leq \left(\frac{1}{2} - \frac{1}{p}\right) n$. Then consider the modified graph, G^+ with vertex set $V(G)$ and edge set $E(G^+) = E(G \setminus \{v\}) \cup \{vw : w \in T \setminus P\}$. That is, effectively we are deleting the vertex v and replacing it with another vertex that is adjacent to all the vertices in the set $T \setminus P$. This modification of G to G^+ preserves the property of being W_{2k+1}-free. If a wheel, W_{2k+1}, would be created after the modification, then either (i) v is the center of the wheel, or (ii) v is in the cycle part of the wheel. In the first case if v were the center of a wheel, then it would have $2k$ neighbors in $T \setminus P$ that induce a cycle. We can show, by choosing p sufficiently large, that the $2k$ vertices would already have had a common neighbor in S in this case, and therefore such a case would not be possible to begin with. On the other hand, in the second case, if $v \in N(c)$ where c denotes the center of the wheel created. Then v must be adjacent to at least two other vertices c_1 and c_2 in $T \setminus P$. Again choosing p large enough shows that $|N_S(c_1) \cap N_S(c_2) \cap N_S(c)| \geq 2k - 2$, and therefore, if the modification contained a W_{2k+1} then G itself would have contained a W_{2k+1}, which is a contradiction. Thus, G^+ is W_{2k+1}-free.

Now, using equation (2.12) we can say that

$$
\begin{aligned}
\lambda_1(G^+) - \lambda_1(G) &\geq \frac{\mathbf{x}^T(A(G^+) - A(G))\mathbf{x}}{\mathbf{x}^T\mathbf{x}} = \frac{2\mathbf{x}_v}{\mathbf{x}^T\mathbf{x}}\left(\sum_{w \in T \setminus P} x_w - \sum_{vw \in E(G)} x_w\right) \\
&\geq \frac{2\mathbf{x}_v}{\mathbf{x}^T\mathbf{x}}\left(\lambda_1 - 16pkn^{1/k} - k + 1 - d_G(v)\right) \\
&> \frac{2\mathbf{x}_v}{\mathbf{x}^T\mathbf{x}}\left(\frac{n+k-1}{2} - 16pkn^{1/k} - k + 1 - \left(\frac{1}{2} - \frac{1}{p}\right)n\right) \\
&= \frac{2\mathbf{x}_v}{\mathbf{x}^T\mathbf{x}}\left(\frac{n}{p} - 16pkn^{1/k} - \frac{k}{2} + \frac{1}{2}\right) \\
&> 0
\end{aligned} \tag{2.13}
$$

for n large enough. This contradicts the fact that G has the maximum spectral radius over all W_{2k+1}-free graphs. Hence, the set of atypical vertices, P, must be empty. Moreover, it follows from here that $G[S \setminus P] = G[S]$ and $G[T \setminus P] = G[T]$ are $K_{1,k}$-free.

□

Lemma 2.1.6. *We have the following bounds on the sizes of* $|S|$ *and* $|T|$.

$$\frac{n}{2} - \sqrt{\frac{3nk}{2}} \leq |S|, |T| \leq \frac{n}{2} + \sqrt{\frac{3nk}{2}}.$$

Proof. We know from Lemma 1.5.6 that $e(G) \geq \lambda_1^2 - \frac{6t}{n}$. Using Lemma 2.1.5 we may obtain an improved upper bound on the number of triangles in G, to obtain a lower bound on $e(G)$.

$$
\begin{aligned}
t &\leq \frac{n_0(k-1)}{2}\left(\frac{k-2}{3} + (n - n_0)\right) + \frac{(n-n_0)(k-1)}{2}\left(\frac{k-2}{3} + n_0\right) \\
&= \frac{n(k-1)(k-2)}{6} + \left(\frac{n^2}{4} - q^2\right)(k-1)
\end{aligned} \tag{2.14}
$$

where $n_0 = \frac{n}{2} + q$ is the size of the larger part. The above upper bound may be obtained by observing that any triangle in G contains an edge in either S or T and two more edges, which either lie in the same part or in $E(S,T)$. Note that by Lemma 2.1.5 the vertices of any edge in S or T have at most $k-2$ common neighbors in the same part, and any triangle lying entirely in one of the parts would be counted thrice depending on which of the three edges we chose to begin with initially.

We also have from (2.3) that $\lambda_1 > \frac{n+k-1}{2}$ for n large enough.

Therefore,

$$\begin{aligned} e(G) &> \left(\frac{n}{2}+\frac{k-1}{2}\right)^2 - \frac{6}{n}\left(\frac{n(k-1)(k-2)}{6} + \left(\frac{n^2}{4}-q^2\right)(k-1)\right) \\ &= \frac{n^2}{4} + \frac{n(k-1)}{2} + \frac{k^2-2k+1}{4} - \frac{3n(k-1)}{2} - (k-1)(k-2) + \frac{6q^2(k-1)}{n} \\ &\geq \frac{n^2}{4} - n(k-1) - \frac{3k^2-10k+7}{4} \end{aligned} \tag{2.15}$$

On the other hand, $e(G) = e(S) + e(T) + e(S,T) \leq \frac{n(k-1)}{2} + \frac{n^2}{4} - q^2$. So,

$$\frac{n(k-1)}{2} + \frac{n^2}{4} - q^2 \geq \frac{n^2}{4} - n(k-1) - \frac{3k^2-10k+7}{4}$$

and therefore,

$$\frac{3n(k-1)}{2} + \frac{3k^2-10k+7}{4} \geq q^2,$$

$$\frac{3nk}{2} > q^2$$

for n large enough. Thus giving $q < \sqrt{\frac{3nk}{2}}$.

□

We now show that all of the eigenvector entries are close to the maximum.

Lemma 2.1.7. *For all $u \in V(G)$ and $0 < \epsilon' < 1$, we have $x_u > 1 - \epsilon'$.*

Proof. Recall that the vertex z is the vertex with largest eigenvector entry, $x_z = 1$. Without loss of generality, assume that $z \in S$. Since $d(z) \geq \lambda_1 > \frac{n+k-1}{2}$, and $d(z) = d_T(z) + d_S(z)$, then $d_T(z) > \frac{n+k-1}{2} - (k-1) = \frac{n-k+1}{2}$. This implies that $|T| > \frac{n-k+1}{2}$ and $|S| < \frac{n+k-1}{2}$. Since $d_S(z) \leq k-1$, the amount of eigenweight in the neighborhood of z lying in T is greater than or equal to $\lambda_1 - (k-1) > \frac{n-k+1}{2}$.

Let z' be an arbitrary vertex of S. Then by Lemma 2.1.5, $d_T(z') \geq \frac{n}{2} - \frac{n}{p} - (k-1)$. We lower bound the amount of eigenweight lying in the common neighborhood of z and z'

by noting that an upper bound for the eigenweight of vertices in $N_T(z)$ but not in $N_T(z')$ is given by $|T| - d_T(z')$. Therefore,

$$
\begin{aligned}
\text{Eigenweight in } N_T(z) \cap N_T(z') &= \sum_{u \in N_T(z) \cap N_T(z')} \mathbf{x}_u \\
&\geq \left(\sum_{u \in N_T(z)} \mathbf{x}_u \right) - (|T| - d_T(z')) \\
&> \left(\frac{n-k+1}{2} \right) - (|T| - d_T(z')) \\
&\geq \left(\frac{n-k+1}{2} \right) \\
&\quad - \left(\left(\frac{n}{2} + \sqrt{\frac{3nk}{2}} \right) - \left(\frac{n}{2} - \frac{n}{p} - (k-1) \right) \right) \\
&= \frac{n-3k+3}{2} - \frac{n}{p} - \sqrt{\frac{3nk}{2}} \\
&\geq \frac{n}{2} - \frac{n}{p} - \sqrt{2nk},
\end{aligned} \tag{2.16}
$$

for n large enough. Therefore,

$$\lambda_1 \mathbf{x}_{z'} = \sum_{u \sim z'} \mathbf{x}_u \geq \frac{n}{2} - \frac{n}{p} - \sqrt{2nk}. \tag{2.17}$$

Since G is a subgraph of the union of a complete bipartite graph and a graph of maximum degree $k-1$, we have the upper bound, $\lambda_1 \leq \frac{n}{2} + k - 1$. Therefore,

$$
\begin{aligned}
\mathbf{x}_{z'} &\geq \frac{\frac{n}{2} - \frac{n}{p} - \sqrt{2nk}}{\lambda_1} \\
&\geq \frac{\frac{n}{2} - \frac{n}{p} - \sqrt{2nk}}{n\left(\frac{1}{2} + \frac{k-1}{n}\right)}.
\end{aligned} \tag{2.18}
$$

For n and p large enough, this gives us $\mathbf{x}_{z'} > 1 - \frac{\epsilon'}{2}$.

Similarly, let w be an arbitrary vertex in T. Then $d_S(w) \geq \frac{n}{2} - \frac{n}{p} - (k-1)$. Now, since every vertex in S has eigenvector entry greater than $1 - \frac{\epsilon'}{2}$, it implies that

$$\begin{aligned} \mathbf{x}_w &\geq \frac{\left(\frac{n}{2} - \frac{n}{p} - (k-1)\right)(1 - \frac{\epsilon'}{2})}{\frac{n}{2} + k - 1} \\ &\geq \frac{n\left(\frac{1}{2} - \frac{1}{p} - \frac{k-1}{n}\right)(1 - \frac{\epsilon'}{2})}{n\left(\frac{1}{2} + \frac{k-1}{n}\right)} \\ &> 1 - \epsilon' \end{aligned} \tag{2.19}$$

for n and p large enough. □

2.2 The proof of Theorem 1.1.2

To prove Theorem 1.1.2, assume that G is a graph in $\mathrm{SPEX}(n, W_5)$, and that n is large enough. Let S and T be the two parts of a maximum cut of G as in Section 2.1. Define the *internal degree* of a vertex u to be $d_S(u)$ if $u \in S$ or $d_T(u)$ if $u \in T$. We will show that $e(G) = \mathrm{ex}(n, W_5)$. By Lemma 2.1.5 we know that the two induced graphs, $G[S]$ and $G[T]$ are matchings. If one could increase the size of the matching in $G[S]$ or $G[T]$, this would not create any W_5 and would strictly increase the spectral radius. Therefore, G must have that $G[S]$ and $G[T]$ are matchings of size $\left\lfloor \frac{|S|}{2} \right\rfloor$ and $\left\lfloor \frac{|T|}{2} \right\rfloor$ respectively. Similarly, we must have that

$$E(S,T) = \Big\{ \{u, v\} \text{ for all } u \in S, v \in T \Big\}.$$

This is again because adding more edges to $E(S, T)$, if possible, will not create a W_5, but strictly increases the spectral radius. Say $|S| \leq |T|$ and let $|S| = \frac{n}{2} - q$ and $|T| = \frac{n}{2} + q$. We will now argue that $q \leq 1$ and so $|T| \leq |S| + 2$. For this we will use lower bounds on $\lambda_1(G)$, obtained similarly to how they were found in equation (2.2), and upper bounds on $\lambda_1(G)$. Let H be any graph in $\mathrm{EX}(n, W_5)$.

We break this argument into two cases based on when $n \equiv 0 \pmod 4$ and when $n \not\equiv 0 \pmod 4$.

Case 1. When $n \equiv 0 \pmod 4$

$$\lambda_1(G) \geq \lambda_1(H) \geq \frac{\mathbf{1}^T A(H)\mathbf{1}}{\mathbf{1}^{\mathbf{T}}\mathbf{1}} = 2\frac{\left\lfloor \frac{n^2}{4} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor}{n} = 2\frac{\left(\frac{n^2}{4} + \frac{n}{2}\right)}{n} = \frac{n}{2} + 1. \tag{2.20}$$

On the other hand,

$$\lambda_1(G) \le \sqrt{|S||T|} + 1 \le \sqrt{\left(\frac{n}{2} - q\right)\left(\frac{n}{2} + q\right)} + 1 = \sqrt{\left(\frac{n^2}{4} - q^2\right)} + 1 \tag{2.21}$$

where $q \in \mathbb{N} \cup \{0\}$. So we have, $\frac{n}{2} + 1 \le \lambda_1(G) \le \sqrt{\left(\frac{n^2}{4} - q^2\right)} + 1$. Therefore $q = 0$, meaning $|S| = |T|$.

Case 2. When $n \not\equiv 0 \pmod 4$

$$\lambda_1(G) \ge \lambda_1(H) \ge \frac{\mathbf{1}^T A(H)\mathbf{1}}{\mathbf{1^T 1}} \ge 2\frac{\left(\frac{n^2}{4} + \frac{n}{2} - 1\right)}{n} = \frac{n}{2} + 1 - \frac{2}{n}. \tag{2.22}$$

On the other hand,

$$\lambda_1(G) \le \sqrt{|S||T|} + 1 \le \sqrt{\left(\frac{n}{2} - q\right)\left(\frac{n}{2} + q\right)} + 1 = \sqrt{\left(\frac{n^2}{4} - q^2\right)} + 1 \tag{2.23}$$

where $2q \in \mathbb{N} \cup \{0\}$. So we have, $\frac{n}{2} + 1 - \frac{2}{n} \le \lambda_1(G) \le \sqrt{\left(\frac{n^2}{4} - q^2\right)} + 1$. Which gives $\frac{n^2}{4} - 2 + \frac{4}{n^2} \le \frac{n^2}{4} - q^2$, meaning $q^2 \le 2 - \frac{4}{n^2}$, and so $q < \sqrt{2}$. This implies that when n is odd, then $q = 0.5$ and we obtain that $|T| = |S| + 1$; and when $n \equiv 2 \pmod 4$, then $q \le 1$ and $|T| \le |S| + 2$. In fact, for $n \equiv 2 \pmod 4$, we may explicitly calculate the largest eigenvalues of the cases when $|T| = |S|$ (balanced case), and $|T| = |S| + 2$ (unbalanced case) and observe that the spectral radius is maximized in the unbalanced case. One may observe this by calculating the spectral radius of their respective equitable matrices,

$$B = \begin{bmatrix} \frac{n}{2} & 1 \\ \frac{n}{2} - 1 & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} 1 & \frac{n}{2} - 1 \\ \frac{n}{2} + 1 & 1 \end{bmatrix},$$

where the two parts of the balanced case (B) are the set of vertices that have internal degree 1, and the set of vertices that have internal degree 0; and the two parts of the unbalanced case (U) are the sets S and T.

In all cases, for large enough n, $\mathrm{SPEX}(n, W_5) \subseteq \mathrm{EX}(n, W_5)$, thus proving Theorem 1.1.2.

2.3 The proof of Theorem 1.1.4

In this section, we assume that $k \geq 3$ and that $G \in \text{SPEX}(n, W_{2k+1})$. We recall the notation from previous sections where S and T denote the two parts in the maximum cut of G. By Lemma 2.1.5 every vertex has internal degree at most $k-1$ and degree at least $\left(\frac{1}{2} - \frac{1}{p}\right) n$ where we may choose p to be a constant large enough for our needs. We have the following lemma related to the set of edges, $E(S, T)$.

Lemma 2.3.1. *Every vertex, u in S (or T), with internal degree $0 \leq d \leq k-1$, is adjacent to all but at most d vertices in T (or S respectively.)*

Proof. The proof is same for $u \in S$ as $u \in T$, so we will prove it in the case u is an arbitrary vertex in S only. Let u have internal degree d and by Lemma 2.1.5 we have $0 \leq d \leq k-1$. Let $\{u_1, u_2, \ldots, u_d\}$ be its neighbors in $G[S]$. If we now modify the graph G to G' by deleting the edges $\{u, u_i\}$ for all $1 \leq i \leq d$, and adding edges until u is adjacent to all vertices in T, then we claim that G' still does not have any W_{2k+1}.

G' has no W_{2k+1} because if u were the 'center' of a new cycle, then that would imply that $G[T]$ has C_{2k} as a subgraph. This is not possible because each vertex of the cycle has degree at least $(1/2 - 1/p)n$ and so the $2k$ vertices of the cycle would have at least one vertex in common in S, implying that G already contained W_{2k+1}. On the other hand, if u were part of the C_{2k} of a W_{2k+1} in G', then u would be adjacent to the center c and two more vertices v_1 and v_2 of the W_{2k+1}, all lying in T, in G'. If this were the case, c, v_1, and v_2 would similarly already be adjacent to at least $2k-2$ vertices in S, in G. This is not possible again because then G would have already had a W_{2k+1}. Hence, G' is W_{2k+1}-free.

Now since, $\lambda_1(G) \geq \lambda_1(G')$, it implies that

$$
\begin{aligned}
0 \leq \lambda_1(G) - \lambda_1(G') \leq \frac{\mathbf{x}^T(A(G) - A(G'))\mathbf{x}}{\mathbf{x}^T\mathbf{x}} &= \frac{2\mathbf{x}_u}{\mathbf{x}^T\mathbf{x}} \left(\sum_{i=1}^{d} x_{u_i} - \sum_{\substack{uw \notin E(G) \\ w \in T}} x_w \right) \\
&\leq \frac{2\mathbf{x}_u}{\mathbf{x}^T\mathbf{x}} \Big(d - (1-\epsilon)|W| \Big)
\end{aligned}
\tag{2.24}
$$

where $W = \{w : uw \notin E(G), w \in T\}$.

Therefore, $d - (1-\epsilon)(|W|) \geq 0$. Choosing $\epsilon < \frac{1}{k}$ implies $|W| \leq d$.

□

We can use Lemma 2.3.1, to say the following. Without loss of generality choose one of the parts, say S, and a set of vertices, $F \subset S$. If $|F| = f$, and every vertex in F has internal degree at most d, then for any $R \subset T$ there is a subset of R of size $|R| - fd$, such that each vertex in this set is adjacent to all the vertices of F.

Our next goal is to show that there is a vertex of internal degree $k-1$ (Lemma 2.3.3). In order to prove this, we use the following lemma that allows us to control the density inside each part.

Lemma 2.3.2. *Assume that C is a constant and that at most C edges may be removed from G so that the graph induced by S has maximum degree a and the graph induced by T has maximum degree b. Then for n large enough we must have $a + b \geq k - 1$. If $C = 0$ we must have $a + b \geq k$.*

Proof. By (2.3) and since G is extremal, for n large enough we must have on one hand that $\lambda_1(G) > \frac{k-1+\sqrt{(k-1)^2+n^2-1}}{2} + \frac{1}{2n}$. On the other hand, if G is the subgraph of a graph of maximum degree a joined to a graph of maximum degree b plus at most C edges, then we have by Lemmas 1.5.5 and 2.1.7

$$\lambda_1(G) = \frac{\mathbf{x}^T A(G)\mathbf{x}}{\mathbf{x}^T\mathbf{x}} \leq \lambda_1\left(\begin{bmatrix} a & |T| \\ |S| & b \end{bmatrix}\right) + \frac{2C}{\mathbf{x}^T\mathbf{x}} \leq \frac{a+b+\sqrt{(a+b)^2+n^2}}{2} + \frac{2C}{n(1-\epsilon)^2}.$$

For n large enough, combining the two inequalities gives that for $C = 0$ we must have $a + b \geq k$ and otherwise $a + b \geq k - 1$. □

Lemma 2.3.3. *For $k \geq 3, k \notin \{4, 5\}$, there exists at least one vertex in $G[S]$ or $G[T]$ with degree equal to $k - 1$.*

Proof. To prove this lemma, it suffices to show that there exists a vertex in $G[S]$ or $G[T]$, with degree at least $k - 1$. We prove this lemma by recursively applying Lemma 2.3.2 to show the existence of a vertex with higher and higher degrees in either $G[S]$ or $G[T]$. We begin by proving the following claim.

Claim 2.3.3.1. There exists at least one vertex in $G[S]$ or $G[T]$ with degree at least $\frac{k}{2}$.

Proof. Assume to the contrary that there do not exist any such vertices in G. Then every vertex has 'internal degree' at most $\frac{k-1}{2}$. Then G must be a subgraph of some graph H of the form of Lemma 2.3.2, where H is the join of a graph H_1 with maximum degree a and another graph H_2 with maximum degree b, where $a = b \leq \frac{k-1}{2}$, and $n_0 = |S|$. Then $a + b \leq k - 1$, which contradicts Lemma 2.3.2, since $C = 0$. Hence, there must exist a vertex in $G[S]$ or $G[T]$ with degree greater than $\frac{k-1}{2}$. □

It follows from Claim 2.3.3.1 that if k is odd, then in fact there must exist a vertex with 'internal degree' at least $\frac{k+1}{2}$. This proves the lemma for $k = 3$. Now to be precise about which part contains a vertex of large internal degree, we will use the following notation. Let $L \in \{S, T\}$ be a part of G such that $G[L]$ has a vertex, v, of degree at least $\frac{k}{2}$. Let $R := L^c$.

Let $\mathcal{N} = \{v_1, v_2, \ldots, v_{\lceil \frac{k}{2} \rceil}\}$, be a set of $\lceil \frac{k}{2} \rceil$ distinct vertices in $N_L(v)$. Then all the vertices in $\mathcal{N} \cup \{v\}$ must be adjacent to a set $R' \subset R$, of minimum size $|R| - \lceil \frac{k+2}{2} \rceil (k-1)$.

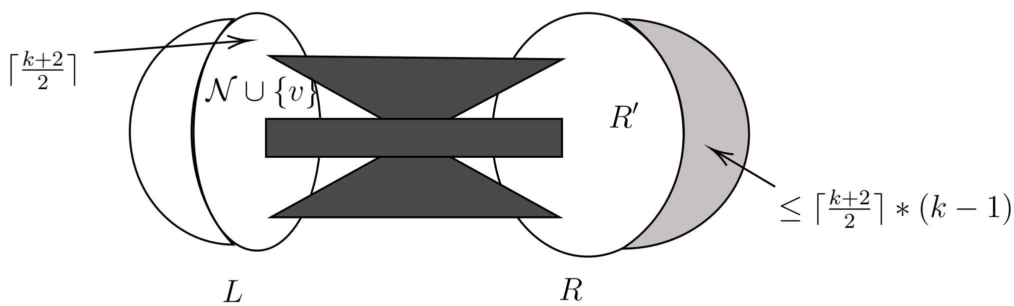

Figure 2.1: $\mathcal{N} \cup \{v\}$ is a subset of L which is adjacent to $R' \subset R$ of size at least $|R| - \lceil \frac{k+2}{2} \rceil (k-1)$

Next, observe that $G[R']$ has at most $\lfloor \frac{3k-2}{2} \rfloor$ vertices in any $\lceil \frac{k}{2} \rceil$ disjoint paths. This is because $C_{2k} \not\subset G[N(v)] \subset G$. In particular, there cannot be $\lceil \frac{k}{2} \rceil$ vertex disjoint paths on 3 vertices or more in $G[R']$. Now, observe that any two vertices u_1 and u_2 having $d_{R'}(u_i) \geq 2$ and lying at distance 3 or more from each other in $G[R']$ are contained in two disjoint P_3's. Since there are at most k^2 vertices in $G[R']$ at a distance less than or equal to 2 from any fixed vertex, it implies that there must be less than $\frac{k^3}{2}$ vertices of degree 2 or more in $G[R']$.

Thus, $e(G[R']) < \frac{|R'|}{2} + \frac{k^4}{2}$ (where, up to $\frac{|R'|}{2}$ edges may come from a maximal matching in R' and the remaining edges from those unaccounted edges adjacent to the set of vertices with degree at least 2 in $G[R']$).

Therefore, there are at most $\frac{|R|}{2} + \frac{k^4}{2} + \lceil\frac{k+2}{2}\rceil(k-1)^2$ many edges in $G[R]$, and at least $|R| - \frac{k^3}{2} - \lceil\frac{k+2}{2}\rceil(k-1)^2$ vertices in $G[R]$ have degree at most 1.

Next we prove the following claim while applying the same argument as in Claim 2.3.3.1.

Claim 2.3.3.2. There exists at least one vertex $v \in L$ with $d_L(v) \geq k-2$

Proof. Assume to the contrary that there do not exist any such vertices in L. Then every vertex in $G[L]$ has degree at most $k-3$. Observe that G must be a subgraph of some graph H of the form of Lemma 2.3.2, where H is the join of a graph H_1 with maximum degree a and another graph H_2 with maximum degree b; plus at most $\frac{k^4}{2} + \lceil\frac{k+2}{2}\rceil(k-1)^2$ more edges embedded in H_2, where $a = k-3$, $b = 1$, and $n_0 = |L|$. Then $a + b \leq k-2$ with $C = \frac{k^4}{2} + \lceil\frac{k+2}{2}\rceil(k-1)^2$, which contradicts Lemma 2.3.2 for n large enough. Thus, there must exist a vertex either in $G[S]$ or $G[T]$ with degree at least $k-2$. □

Now let $\mathcal{N}' = \{v_1, v_2, \ldots, v_{k-2}\}$, be a set of $k-2$ distinct vertices in $N_L(v)$. Then all the vertices in $\mathcal{N}' \cup \{v\}$ must be adjacent to a set $R'' \subset R$, of minimum size $|R| - (k-1)^2$. It follows from our arguments above that $G[R'']$ has at most $k+1$ vertices in any $k-2$ disjoint paths. Thus, for $k \geq 6$, $G[R'']$ cannot have 4 vertex disjoint edges. Lemma 2.1.5 implies that any edge in $G[R'']$ is adjacent to at most $2(k-2)$ other edges.

It follows from this that for $k \geq 6$, there must be at most $3(2k-3) + (k-1)(k-1)^2$ many edges in $G[R]$ and at least $|R| - 3(2k-2) - (k-1)(k-1)^2$ vertices in $G[R]$ have degree equal to 0.

Finally, to show the existence of a vertex in L with internal degree $k-1$, assume to the contrary that there do not exist any such vertices in L. Then every vertex in $G[L]$ has degree at most $k-2$. Then G must be a subgraph of some graph H of the form of Lemma 2.3.2, where H is the join of a graph H_1 with maximum degree a and another graph H_2 with maximum degree b; plus at most $3(2k-3) + (k-1)(k-1)^2$ more edges embedded in H_2, where $a = k-2$, $b = 0$, and $n_0 = |L|$. Then $a+b \leq k-2$ with $C = 3(2k-3)+(k-1)(k-1)^2$, which contradicts Lemma 2.3.2 for n large enough. Thus, there must exist a vertex either in $G[S]$ or $G[T]$ with degree at least $k-1$. □

Lemma 2.3.4. *There exist at least $4k^2 + 1$ vertices, $v \in L$, such that $d_L(v) = k - 1$ and at most one edge in $G[R]$.*

Proof. If there are at most $4k^2$ vertices, $v \in L$, with $d_L(v) = k-1$, then deleting at most $4k^2$ edges of $G[L]$ makes its maximum degree go down to $k-2$. Then, applying Lemma 2.3.2 gives us a contradiction as $a = k-2$, $b = 0$ and $C \leq 3(2k-3) + (k-1)(k-1)^2 + 4k^2$. Hence, L has at least $4k^2 + 1$ vertices, v, with $d_L(v) = k - 1$.

Now suppose $G[R]$ has 2 or more edges, $\{u_1, u_2\}$ and $\{v_1, v_2\}$. Let $\mathcal{I} := N_L(u_1) \cap N_L(u_2) \cap N_L(v_1) \cap N_L(v_2)$. Then, by Lemmas 2.3.1 and 2.3.3, there are at most $4(k-1)$ vertices in $L \setminus \mathcal{I}$. Now, since there are at least $4k^2 + 1$ vertices, $c \in L$, with $d_L(c) = k - 1$, there exists at least one vertex, $c \in I$, with $d_{\mathcal{I}}(c) = k - 1$. This is because there are at most $4(k-1)(k-1) < 4k^2$ vertices $w \in L$, such that $N_L(w) \cup \{w\} \not\subset \mathcal{I}$. Say $\{c_1, c_2, \ldots, c_{k-1}\} = N_L(c)$. Observe that $G[N_R(c) \cap N_R(c_1) \cap \ldots \cap N_R(c_{k-1})] \supset \{u_1, u_2\} \cup \{v_1, v_2\} \sqcup (k-2)K_1$, since $|N_R(c) \cap N_R(c_1) \cap \ldots \cap N_R(c_{k-1})| > |R| - k^2$. Hence, there exist $W_{2k+1} \subset G[\{c\} \cup N_G(c)] \subset G$, with c as the center. This is a contradiction. Hence, $G[R]$ has at most one edge. □

Lemma 2.3.5. *Let $\mathcal{C} \subset V$, such that $|\mathcal{C}| = c \geq 2$. If $\hat{G}$ is any graph obtained by modifying G by only changing the edges contained in $E(G[\mathcal{C}])$, such that $e(\hat{G}[\mathcal{C}]) - e(G[\mathcal{C}]) = m > 0$. Then $\lambda_1(\hat{G}) - \lambda_1(G) > 0$ for $\epsilon < \frac{m}{c(k-1)}$.*

Proof. Take n and p large enough so that $\epsilon < \frac{m}{c(k-1)}$. Then

$$\begin{aligned}
\lambda_1(\hat{G}) - \lambda_1(G) &\geq \frac{\mathbf{x}^T(A(\hat{G}) - A(G))\mathbf{x}}{\mathbf{x}^T\mathbf{x}} \geq \frac{2}{\mathbf{x}^T\mathbf{x}}\left(e(\hat{G}[\mathcal{C}])(1-\epsilon)^2 - e(G[\mathcal{C}])\right) \\
&> \frac{2}{\mathbf{x}^T\mathbf{x}}\left(e(\hat{G}[\mathcal{C}])(1-2\epsilon) - e(G[C])\right) = \frac{2}{\mathbf{x}^T\mathbf{x}}\left(m - 2e(\hat{G}[\mathcal{C}])\epsilon\right) \qquad (2.25)\\
&\geq \frac{2}{\mathbf{x}^T\mathbf{x}}\left(m - c(k-1)\epsilon\right) > 0
\end{aligned}$$

□

Lemma 2.3.6. *The number of edges in $E(L,R)$ is $|L||R|$, that is, $E(L,R) = \{(l,r)$ for all $l \in L, r \in R\}$*

Proof. If $G[R]$ has no edges, then the result follows from Lemma 2.3.1. So we assume that there exists exactly one edge, $\{r_1, r_2\}$ in $G[R]$. It also follows from Lemma 2.3.1 that $e(L,R) \geq |L||R| - 2$, and at most two edges $\{l_1, r_1\}$ and $\{l_2, r_2\}$ are missing from $E(L,R)$.

We will first show that at most one edge may be missing. Assume to the contrary that both are missing. Then modifying G to the graph $\overline{G}$ by deleting the edge $\{r_1, r_2\}$ and adding edges $\{l_1, r_1\}$ and $\{l_2, r_2\}$, strictly increases the spectral radius as may be seen by consequence of Lemma 2.3.5. However, observe that $\overline{G}$ is W_{2k+1}-free. This may be understood by first noting that no vertex in R may be the center of a W_{2k+1} as this requires the occurrence of a $C_{2k} \subset \overline{G}[L] = G[L]$; and second that no vertex in L may be the center of a W_{2k+1} as every vertex in $\overline{G}[L]$ is $K_{1,k}$ free and there are no edges in $\overline{G}[R]$. Thus, $\overline{G}[N(l)]$ is C_t free for all $t \geq 2k-1$ and $l \in L$. This contradicts the fact that $\lambda_1(G) \geq \lambda_1(\overline{G})$. So, $e(L,R) \geq |L||R| - 1$, and at most one edge $\{l_1, r_1\}$ may be missing from $E(L,R)$.

Now, the neighborhood of r_2 may not contain a C_{2k}. Observe that this implies that there are no C_{2k} contained in $G[L \cup \{r_1\}]$, which further implies that there are no P_{2k-1} in $G[L]$ that do not have l_1 as one of the end points. It follows that no P_{2k} can be contained in $G[L]$, as we could then select a path on $2k-1$ vertices without l_1 as one of its end points, whose vertices along with r_1, r_2, induce a graph with W_{2k+1} as a subgraph.

Let G^+ be the modification of G obtained by adding the edge $\{l_1, r_1\}$. Then $\lambda_1(G^+) > \lambda_1(G)$, which implies that $G^+ \supset W_{2k+1}$.

Note that either:

(i) for some $l \in L$, the edge $\{l_1, r_1\}$ is part of a $W_{2k+1} \subset G^+[N(l) \cup \{l\}]$, or

(ii) r_1 is the center of a W_{2k+1} in G^+ with $r_2, l_1 \in C_{2k} \subset G^+[N(r_1)]$, or

(iii) the edge $\{l_1, r_1\}$ is part of a $C_{2k} \subset G^+[N(r_2)]$.

In the first case, if $l \in L$ is such that $\{l_1, r_1\}$ is part of a $W_{2k+1} \subset G^+[N(l) \cup \{l\}]$. Then note that the $C_{2k} \subset G^+[N(l)]$ has at most $k-1$ vertices that lie in L. Therefore, there are at least $k+1$ vertices of R in the C_{2k}. Now observe that other than the edge $\{r_1, r_2\}$, no two vertices of R are adjacent to each other in the C_{2k}. Therefore, the number of vertices of R lying in the C_{2k} is maximized when the vertices of the C_{2k} alternate between the left, L, and right, R, parts as we go along a path on $2k$ vertices in the C_{2k}, starting at r_2 and ending at r_1. Therefore, there are at most k vertices from R in the C_{2k}. This is a contradiction. Hence the first case is not possible.

In the second case, if $d_L(l_1) \geq 2$, then either there exists a P_{2k} in $G^+[L] = G[L]$, or the vertices of the C_{2k} in G^+ induce a graph with another $C_{2k} \ni r_1$ in G. The vertices of this new cycle are all adjacent to r_2, and hence we can say that G would contain a W_{2k+1} if $d_L(l_1) \geq 2$ under the first case.

Similarly, if the third case were to be true and $d_L(l_1) \geq 2$, then either there exists a P_{2k} in $G^+[L] = G[L]$, or the vertices of the C_{2k} induce a graph with another C_{2k} as a subgraph. All of these vertices are adjacent to r_2, and hence again, we would already have had a W_{2k+1} in G.

Next, let us consider the situation when $d_L(l_1) = 1$. Let $\mathcal{C}$ be the connected component containing l_1 in $G^+[L]$. Now $|\mathcal{C}| \geq 2k-1$ and has no P_{2k}. If $|\mathcal{C}| \geq 2k$, then we can further modify the graph G^+ to $\hat{G}$, such that $\hat{G}[\mathcal{C}]$ has no P_{2k-1} and every vertex in $\mathcal{C}$ has internal degree $k-1$ with at most one vertex having internal degree $k-2$. This implies that $\hat{G}$ has at least $\frac{|\mathcal{C}|(k-1)-1}{2} - \frac{|\mathcal{C}|(k-1)-(k-2)}{2} = \frac{k-3}{2}$ more edges than G^+, and at least $\frac{k-1}{2}$ edges more than G. Also, note that the following loose upper bound $|\mathcal{C}| < k^k$ holds since $\mathcal{C}$ has no P_{2k-1} as a subgraph.

Therefore, using Lemma 2.3.5 with $m = \frac{k-1}{2}$, $c = k^k$ and $\epsilon < \frac{1}{2k^k}$, $\lambda_1(\hat{G}) - \lambda_1(G) > 0$ for n large enough. Hence, $\lambda_1(\hat{G}) > \lambda_1(G)$. However $G[L]$ has no P_{2k-1} or $K_{1,k}$ as subgraphs, and hence G has no W_{2k+1} as a subgraph, which is a contradiction. Therefore, G must contain the edge $\{l_1, r_1\}$.

Finally, if $|\mathcal{C}| = 2k-1$, consider the following. Let $\mathcal{D}$ be a different connected component in $G^+[L]$. Then $|\mathcal{C} \cup \mathcal{D}| \geq 2k$. Let $\mathcal{E} := \mathcal{C} \cup \mathcal{D} \ni l_1$. Now similarly modify G^+ to $\tilde{G}$ such that $\tilde{G}[\mathcal{E}]$ has no P_{2k-1} and every vertex in $\mathcal{E}$ has internal degree $k-1$ with at most one vertex having internal degree $k-2$. This again implies that $\tilde{G}$ has at least $\frac{|\mathcal{E}|(k-1)-1}{2} - \frac{|\mathcal{E}|(k-1)-(k-2)}{2} = \frac{k-3}{2}$ more edges than G^+, and at least $\frac{k-1}{2}$ edges more than G.

Here too $|\mathcal{E}| < k^k$ holds since D has no P_{2k-1} as a subgraph.

The rest of the proof follows the same arguments as above. Therefore, $E(L, R)$ must contain the edge $\{l_1, r_1\}$, and $e(L, R) = |L||R|$.

□

Lemma 2.3.7. *If $(k-1)|L|$ is even then $G[L]$ is a $(k-1)$-regular graph and otherwise $G[L]$ is a $(k-1)$-nearly regular graph. Furthermore, $e(G[R]) = 1$.*

Proof. By way of contradiction, assume that the statement is not true. Thus, either $G[L]$ has at least two vertices of degree not more than $k-2$ or at least one vertex of degree at most $k-3$, or $G[R]$ has no edge. Let E_2 be a set of edges such that $G \cup E_2$ is a (potentially not simple) graph that induces one edge on R and where the graph induced by L is $(k-1)$-regular. By the assumption, the sum of the entries in E_2 is at least 2. Now, by Lemma 2.1.7,

$$\begin{aligned}\lambda_1(G) = \frac{\mathbf{x}^T A(G)\mathbf{x}}{\mathbf{x}^T\mathbf{x}} &\le \mu - \frac{\mathbf{x}^T A(E_2)\mathbf{x}}{\mathbf{x}^T\mathbf{x}} + \frac{\mathbf{x}^T A(G[R])\mathbf{x}}{\mathbf{x}^T\mathbf{x}} \\ &< \mu - \frac{2(1-\epsilon)^2}{\mathbf{x}^T\mathbf{x}} + \frac{2(1+\epsilon)^2}{\mathbf{x}^T\mathbf{x}} \\ &< \mu + \frac{8\epsilon}{(1-\epsilon)^2 n}.\end{aligned}$$

This is a contradiction to (2.3) for ϵ small enough. □

Finally, we show that $|L|$ and $|R|$ differ by at most 2. This completes the proof of Theorem 1.1.4.

Lemma 2.3.8. *For $G \in \mathrm{SPEX}(n, W_{2k+1})$, with maximum cut L, R, we have*

$$\frac{n}{2} - 1 \le |L|, |R| \le \frac{n}{2} + 1.$$

Proof. Let $|L| = \frac{n}{2} + s$ and $|R| = \frac{n}{2} - s$. We will show $|s| \le 1$. Let $B = \begin{bmatrix} k-1 & \frac{n}{2} - s \\ \frac{n}{2} + s & 0 \end{bmatrix}$. We know that G is a complete bipartite graph with maximum cut L, R and $e(G[R]) \le 1$ and $G[L]$ a $k-1$ regular or nearly graph. Hence, $\lambda_1(G) \le \lambda_1(B) + \frac{2}{\mathbf{x}^T\mathbf{x}}$ by Lemma 1.5.5. Combining this with (2.3) gives

$$\frac{k-1+\sqrt{(k-1)^2+n^2-1}}{2} < \frac{k-1+\sqrt{(k-1)^2+(n^2-4s^2)}}{2} + \frac{2}{\mathbf{x}^T\mathbf{x}}$$
$$\leq \frac{k-1+\sqrt{(k-1)^2+(n^2-4s^2)}}{2} + \frac{2}{(1-\epsilon)^2 n}.$$

Simplifying shows that $|s| \leq 1$ for n large enough.

□

2.4 The sizes of $|L|$ and $|R|$ in Theorem 1.1.4

In this section we will show that when k is odd, then $|L| = \lceil \frac{n}{2} \rceil$ and when k is even, then $|L|$ is constrained as follows, depending on the value of $n \pmod 4$.

(i) For $n \equiv 0 \pmod 4$, $|L| = \frac{n}{2}$;

(ii) For $n \equiv 1 \pmod 4$, $|L| = \lfloor \frac{n}{2} \rfloor$;

(iii) For $n \equiv 2 \pmod 4$, $|L| \in \{\frac{n}{2}, \frac{n}{2}+1\}$;

(iv) For $n \equiv 3 \pmod 4$, $|L| = \lceil \frac{n}{2} \rceil$.

We will now try to fine-tune our argument a bit to show (i), (iii), and (iv), by using quotient graphs with three parts instead of just two parts, as done so far.

Let

$$\Pi_s = \begin{bmatrix} k-1 & \frac{n}{2}-s-2 & 2 \\ \frac{n}{2}+s & 0 & 0 \\ \frac{n}{2}+s & 0 & 1 \end{bmatrix}$$

and $P_s(\lambda) = \lambda^3 - k\lambda^2 - \left(\frac{n^2}{4} - s^2 - k + 1\right)\lambda + \frac{n^2}{4} - s^2 - n - 2s$, be its characteristic polynomial. Then for $\alpha \in \{0.5, 1\}$, we have $P_\alpha(\lambda) - P_{-\alpha}(\lambda) = -4\alpha$; and $P_0(\lambda) - P_1(\lambda) = 3 - \lambda < 0$, for λ near $\lambda_1(\Pi_0)$. Since the coefficient of λ^3 in P_s is positive, all three roots of P_s are real and the largest root is simple, it implies that $\lambda_1(\Pi_\alpha) > \lambda_1(\Pi_{-\alpha})$ and $\lambda_1(\Pi_0) > \lambda_1(\Pi_1)$.

Let $\mathcal{G}_s$ be the family of graphs that consist of a complete bipartite graph with parts of size $\frac{n}{2}+s$ and $\frac{n}{2}-s$ along with a graph from $\mathcal{U}_{k,n/2+s}$ embedded in the first part with $\frac{n}{2}+s$ vertices and a single edge embedded in the other part. For any arbitrary graph $G_s \in \mathcal{G}_s$, we can say that $\lambda_1(G_s) \leq \lambda_1(\Pi_s)$ (as Π_s is a quotient matrix for the adjacency matrix of G_s

with at most one loop added to make every vertex in the first part have same degree) with equality if and only if $(k-1)(\frac{n}{2}+s)$ is even. We know by Theorem 1.1.4 that $G \in \mathcal{G}_s$ for some s satisfying $-1 \leq s \leq 1$. Then, the previous paragraph implies that $s \in \{0, 0.5\}$ (i.e. $|L| = \lceil \frac{n}{2} \rceil$), whenever k is odd or $\lceil \frac{n}{2} \rceil$ is even (implying (i) and (iv)). Further, $s \in \{0, 1\}$ (i.e. $|L| = \frac{n}{2}$ or $\frac{n}{2}+1$), if $\frac{n}{2}$ is odd (implying (iii)).

When k is even and $n \equiv 1, 2 \pmod 4$ we argue similarly to Lemma 2.1.1. Let

$$Q_s = \begin{bmatrix} k-1 & \frac{n}{2}-s \\ \frac{n}{2}+s & 0 \end{bmatrix}.$$

Let uv be the edge that is embedded in R and let u_ℓ be the vertex that has internal degree $k-2$ in the case that $(\frac{n}{2}+s)(k-1)$ is odd. Let μ_s be the spectral radius of Q_s with eigenvector $\begin{bmatrix} 1 & \eta \end{bmatrix}^T$ and let $\mathbf{z}$ be the n-dimensional vector where the first $\frac{n}{2}+s$ entries are 1 and the last $\frac{n}{2}-s$ entries are η. Note that $\eta = 1-o(1)$ as $n \to \infty$. Let E be the adjacency matrix of the edge uv and (if u_ℓ exists) the loop u_ℓ with weight -1. That is, E is a matrix with exactly two entries equal to 1 and if $(n/2+s)(k-1)$ is odd a single diagonal entry equal to -1. Then

$$\begin{aligned} \frac{\mathbf{z}^T(A(G)-E)\mathbf{z}}{\mathbf{z}^T\mathbf{z}} + \frac{\mathbf{z}^T E\mathbf{z}}{\mathbf{z}^T\mathbf{z}} = \frac{\mathbf{z}^T(A(G))\mathbf{z}}{\mathbf{z}^T\mathbf{z}} \leq \lambda_1 &= \frac{\mathbf{x}^T(A(G))\mathbf{x}}{\mathbf{x}^T\mathbf{x}} \\ &= \frac{\mathbf{x}^T(A(G)-E)\mathbf{x}}{\mathbf{x}^T\mathbf{x}} + \frac{\mathbf{x}^T E\mathbf{x}}{\mathbf{x}^T\mathbf{x}}. \end{aligned} \tag{2.26}$$

Since Q_s is the quotient matrix of an equitable partition of the graph G minus edge uv and plus (if u_ℓ exists) loop u_ℓ, and since $\mathbf{z}$ is an eigenvector for $A(G)-E$, we have

$$\mu_s + \frac{\mathbf{z}^T E\mathbf{z}}{\mathbf{z}^T\mathbf{z}} \leq \lambda_1 \leq \mu_s + \frac{\mathbf{x}^T E\mathbf{x}}{\mathbf{x}^T\mathbf{x}} \tag{2.27}$$

Proposition 2.4.1. *When $n \equiv 1 \pmod 4$ and k is even, then $s = -1/2$.*

Proof. If $n \equiv 1 \pmod 4$ then the vertex u_ℓ exists if $s = 1/2$ and does not exist if $s = -1/2$. Let $G_{1/2}$ and $G_{-1/2}$ be arbitrary graphs in $\mathcal{G}_s$ in the cases that $s = 1/2$ and $s = -1/2$ respectively. Then by (2.27), we have

$$\lambda_1(G_{-1/2}) \geq \mu_{-1/2} + \frac{2\eta^2}{\mathbf{z}^T\mathbf{z}} = \mu_{-1/2} + \frac{2-o(1)}{n}.$$

On the other hand, if $G_{1/2}$ were extremal we would have by (2.27) and Lemma 2.1.7 that

$$\lambda_1(G_{1/2}) \leq \mu_{1/2} + \frac{2\mathbf{x}_u\mathbf{x}_v - {\mathbf{x}_{\mathbf{u}_\ell}}^2}{\mathbf{x}^T\mathbf{x}} = \mu_{1/2} + \frac{1+o(1)}{n}.$$

Noting that $\mu_{-1/2} = \mu_{1/2}$ completes the proof.

□

Proposition 2.4.2. *When $n \equiv 2 \pmod 4$ and k is even, we have*

$$\mathrm{spex}(n, W_{2k+1}) = \frac{k-1+\sqrt{(k-1)^2+n^2}}{2} + \frac{1+o(1)}{n}.$$

Proof. Let G_1 be the graph when $s = 1$ and G_0 be the graph when $s = 0$. Note that the vertex u_ℓ exists in G_0 and does not exist in G_1. Hence if G_1 is extremal, by (2.27) we have

$$\lambda_1(G_1) = \mu_1 + \frac{2-o(1)}{n},$$

and if G_0 is extremal we have

$$\lambda_1(G_0) = \mu_0 + \frac{1+o(1)}{n}.$$

Comparing these two quantities shows that they differ by $o(1/n)$. □

Finally we notice that if $n \not\equiv 2 \pmod 4$ or if k is odd, then the extremal graph with the appropriately chosen s has an equitable partition with Π_s as the quotient matrix, and hence $\lambda_1(G)$ is equal to the largest root of $P_s(\lambda)$ in these cases.

2.5 Conclusion

In this chapter, we determined the structure of the graphs in $\mathrm{SPEX}(n, W_{2k+1})$ for all $k \notin \{4, 5\}$ and for n large enough. We believe that the extremal graphs when $k \in \{4, 5\}$ have the same structure, and it would be interesting to prove this. The main technical hurdle is to prove Lemma 2.3.3 when $k \in \{4, 5\}$. We note that it is a bit delicate. For example, for $k = 4$, the join of a disjoint union of triangles with a matching has spectral radius very close to a complete bipartite graph with disjoint copies of K_4 in one side and a single edge in the other side.

It would also be interesting to determine how large n needs to be as a function of k for our theorems to hold. Our use of the Triangle Removal Lemma means that our "sufficiently large n" is likely much larger than it needs to be.

Finally, we end with a general conjecture.

Conjecture 2.5.1. *Let F be any graph such that the graphs in $\mathrm{EX}(n, F)$ are Turán graphs plus $O(1)$ edges. Then $\mathrm{SPEX}(n, F) \subset \mathrm{EX}(n, F)$ for n large enough.*

We say that F is edge-color-critical if there exists an edge e of F such that $\chi(F - e) < \chi(F)$. Let F be an edge-color-critical graph with $\chi(F) = r + 1$. By a result of Simonovits [63] and a result of Nikiforov [53], we know that $\mathrm{EX}(n, F) = \mathrm{SPEX}(n, F) = \{T_r(n)\}$ for sufficiently large n. This shows that Conjecture 2.5.1 is true for all edge-color-critical graphs, when the $O(1)$ is replaced by 0. We believe that similar methods to what is presented here and in [17] would help to prove the conjecture for any fixed graph F satisfying the hypotheses. Recently, after this thesis was written, a solution to Conjecture 2.5.1 was announced in [73].

Chapter 3

THE SPECTRAL RADIUS OF GRAPHS WITH NO INTERSECTING CLIQUES

In this chapter, we will study the spectral Turán problem for intersecting cliques and obtain the $\mathrm{spex}(n, F_{k,r})$ and $\mathrm{SPEX}(n, F_{k,r})$ for all $k \geq 3$, $r \geq 3$. In doing so, we will prove Theorem 1.2.4.

3.1 Proof of Theorem 1.2.4

Throughout this chapter, we always assume that G is a graph on n vertices containing no $F_{k,r}$ as a subgraph and attaining the maximum spectral radius. The aim of this section is to prove that $e(G) = \mathrm{ex}(n, F_{k,r})$ for n large enough.

First of all, we note that G must be connected since adding an edge between different components will increase the spectral radius without creating any $F_{k,r}$. By the Perron–Frobenius Theorem, we know that there is a Perron vector with maximum entry equal to 1, we denote such an eigenvector by $\mathbf{x}$. Let z be a vertex such that $\mathbf{x}_z = 1$. If there are multiple such vertices, we choose and fix z arbitrarily among them.

In the sequel, we shall prove Theorem 1.2.4 iteratively, giving successively more precise estimates on both the structure of G and the eigenvector entries of the vertices, until finally we can show that $e(G) = \mathrm{ex}(n, F_{k,r})$.

The proof of Theorem 1.2.4 is outlined as follows.

- ♠ We apply Corollary 1.5.8 to give a lower bound $e(G) \geq t_{r-1}(n) - o(n^2)$. Moreover, G has a very large multipartite subgraph on parts $V_1, \ldots, V_{r-1}$ such that $\frac{n}{r-1} - o(n) \leq |V_i| \leq \frac{n}{r-1} + o(n)$; see Lemma 3.2.2.
- ♡ We show that the number of vertices that have $\Omega(n)$ neighbors in their own part is bounded by $o(n)$, and the number of vertices that have degree less than $(\frac{r-2}{r-1} - o(1))n$ is also bounded by $o(n)$; see Lemmas 3.2.3 and 3.2.4 respectively. Furthermore, we will prove that such vertices do not exist, and each $G[V_i]$ is $K_{1,k}$-free and M_k-free; see Lemmas 3.2.5 and 3.2.7.

♣ Based on the previous lemmas, we shall refine the structure of G, and show that almost all vertices in V_i are adjacent to every vertex in V_i^c, implying the presence of a large complete $(r-1)$-partite subgraph in G; see Lemma 3.2.8. Moreover, we shall prove that $\mathbf{x}_u = 1 - o(1)$ for every $u \in V(G)$; see Lemma 3.2.9.

♢ Once we know that all vertices have eigenvector entry close to 1, we can show that the $(r-1)$-partition is balanced; see Lemma 3.2.10. Invoking these facts, we finally show that $e(G) = \mathrm{ex}(n, F_{k,r})$.

3.2 Structural results for extremal graphs

Lemma 3.2.1. *Let G be an $F_{k,r}$-free graph on n vertices with maximum spectral radius. Then*

$$\lambda(G) \geq \left(1 - \frac{1}{r-1}\right) n - \frac{r-1}{4n}.$$

Proof. Let H be an $F_{k,r}$-free graph on n vertices with maximum number of edges. Since G is the graph maximizing the spectral radius over all $F_{k,r}$-free graphs, in view of Theorem 1.2.3, we can see by the Rayleigh quotient that

$$\lambda(G) \geq \lambda(H) \geq \frac{\mathbf{1}^T A(H) \mathbf{1}}{\mathbf{1}^T \mathbf{1}} = \frac{2(t_{r-1}(n) + f(k-1, k-1))}{n}.$$

Note that $t_{r-1}(n) \geq (1 - \frac{1}{r-1})\frac{n^2}{2} - \frac{r-1}{8}$, so we have $\lambda(G) \geq (1 - \frac{1}{r-1})n - \frac{r-1}{4n}$. □

Applying Lemma 1.5.7 and Corollary 1.5.8, we obtain the asymptotic structure of G. Roughly speaking, we can find a large $(r-1)$-partite subgraph in G.

Lemma 3.2.2 (Approximate structure)**.** *Let G be an $F_{k,r}$-free graph on n vertices with maximum spectral radius. For every $\epsilon > 0$, there is an integer n_0 such that if $n \geq n_0$, then*

$$e(G) \geq t_{r-1}(n) - \epsilon n^2.$$

Furthermore, there exists $\epsilon_1 = \sqrt{6\epsilon}$ such that G has a maximum $(r-1)$-cut $V = V_1 \sqcup \ldots \sqcup V_{r-1}$ with

$$\sum_{1 \leq i < j \leq r-1} e(V_i, V_j) \geq t_{r-1}(n) - \epsilon n^2,$$

and for each $i \in [r-1]$,

$$\left(\frac{1}{r-1} - \epsilon_1\right) n \leq |V_i| \leq \left(\frac{1}{r-1} + \epsilon_1\right) n.$$

Proof. As suggested above, it follows from Lemma 1.5.7 and Corollary 1.5.8 that for any given $\epsilon > 0$, we can take a large enough n such that $e(G) \geq t_{r-1}(n) - \epsilon n^2$. The same results also provide that there is a partition of $V(G) = U_1 \sqcup \ldots \sqcup U_{r-1}$ with $\sum_{i=1}^{r-1} e(U_i) \leq \epsilon n^2$, $\sum_{1\leq i<j\leq r-1} e(U_i, U_j) \geq t_{r-1}(n) - \epsilon n^2$ and $\lfloor \frac{n}{r-1} \rfloor \leq |U_i| \leq \lceil \frac{n}{r-1} \rceil$ for each $i \in [r-1]$. Thus, any maximum $(r-1)$-cut of $V = V_1 \sqcup \ldots \sqcup V_{r-1}$ must have $\sum_{i=1}^{r-1} e(V_i) \leq \sum_{i=1}^{r-1} e(U_i) \leq \epsilon n^2$ and $\sum_{1\leq i<j\leq r-1} e(V_i, V_j) \geq \sum_{1\leq i<j\leq r-1} e(U_i, U_j) \geq t_{r-1}(n) - \epsilon n^2$.

Furthermore, since G has edit distance at most ϵn^2 from some graph isomorphic to $T_{r-1}(n)$, we may let $a = \max\left\{\left||V_j| - \frac{n}{r-1}\right|, j \in [r-1]\right\}$. Without loss of generality, we assume that $\left||V_1| - \frac{n}{r-1}\right| = a$. Then

$$\begin{aligned}
e(G) &\leq \sum_{1\leq i<j\leq r-1} |V_i||V_j| + \sum_{i=1}^{r-1} e(V_i) \\
&\leq |V_1|(n - |V_1|) + \sum_{2\leq i<j\leq r-1} |V_i||V_j| + \epsilon n^2 \\
&= |V_1|(n - |V_1|) + \frac{1}{2}\left((\sum_{j=2}^{r-1} |V_j|)^2 - \sum_{j=2}^{r-1} |V_j|^2 \right) + \epsilon n^2 \\
&\leq |V_1|(n - |V_1|) + \frac{1}{2}(n - |V_1|)^2 - \frac{1}{2(r-2)}(n - |V_1|)^2 + \epsilon n^2 \\
&< -\frac{r-1}{2(r-2)}a^2 + \frac{r-2}{2(r-1)}n^2 + \epsilon n^2,
\end{aligned}$$

where the last second inequality holds by Hölder's inequality, and the last inequality holds since $\left||V_1| - \frac{n}{r-1}\right| = a$. On the other hand,

$$e(G) \geq t_{r-1}(n) - \epsilon n^2 \geq (1 - \frac{1}{r-1})\frac{n^2}{2} - \frac{r-1}{8} - \epsilon n^2 > \frac{r-2}{2(r-1)}n^2 - 2\epsilon n^2,$$

as n is large enough. Therefore, $\frac{r-1}{2(r-2)}a^2 < 3\epsilon n^2$, which implies that $a < \sqrt{\frac{6(r-2)\epsilon}{r-1}n^2} < \sqrt{6\epsilon}n = \epsilon_1 n$. The proof is completed. □

Lemma 3.2.3. *Let ϵ and θ be two sufficiently small constants with $\epsilon < \theta^2/3$. We denote*

$$W := \bigsqcup_{i=1}^{r-1} \{v \in V_i : |N_G(v) \cap V_i| \geq \theta n\}. \tag{3.1}$$

For sufficiently large n, we have

$$|W| \leq \frac{2\theta}{3}n + \frac{2k^2}{\theta n} < \theta n.$$

Proof. We obtain from Lemma 3.2.2 that $\sum_{1\leq i<j\leq r-1} e(V_i, V_j) \geq t_{r-1}(n) - \epsilon n^2$. Hence,

$$\sum_{i=1}^{r-1} e(V_i) = e(G) - \sum_{1\leq i<j\leq r-1} e(V_i, V_j) \leq t_{r-1}(n) + k^2 - t_{r-1}(n) + \epsilon n^2 \leq \epsilon n^2 + k^2.$$

On the other hand, if we let $W_i := W \cap V_i$ for all $i \in [r-1]$, then

$$2e(V_i) = \sum_{u\in V_i} d_{V_i}(u) \geq \sum_{u\in W_i} d_{V_i}(u) \geq |W_i|\theta n$$

Thus

$$\sum_{i=1}^{r-1} e(V_i) \geq \sum_{i=1}^{r-1} \frac{|W_i|}{2}\theta n = \frac{|W|}{2}\theta n.$$

Therefore, we have that $\frac{|W|}{2}\theta n \leq \epsilon n^2 + k^2$. This proves that $|W| \leq \frac{2\theta}{3}n + \frac{2k^2}{\theta n} < \theta n$. □

Lemma 3.2.4. *Let $k \geq 2$ and $\frac{2(r-2)}{r-1}\epsilon < \epsilon_2^2 \leq \theta$ with ϵ_2 small enough. We denote*

$$L := \left\{ v \in V(G) : d(v) \leq \left(1 - \frac{1}{r-1} - \epsilon_2\right) n \right\}. \tag{3.2}$$

Then $|L| \leq \epsilon_3 n$, where $\epsilon_3 \leq \epsilon_2$ is a sufficiently small constant satisfying $\frac{r-2}{2(r-1)}\epsilon_3^2 - \epsilon_2\epsilon_3 + \epsilon < 0$.

Proof. To prove this, assume to the contrary that the cardinality of L is greater than $\epsilon_3 n$. Then there exists a subset $L' \subseteq L$ with $|L'| = \lfloor \epsilon_3 n \rfloor$. Therefore,

$$\begin{aligned} e[G \setminus L'] &\geq e(G) - \sum_{v\in L'} d(v) \geq t_{r-1}(n) - \epsilon n^2 - \epsilon_3 n^2 \left(1 - \frac{1}{r-1} - \epsilon_2\right) \\ &> \frac{(n - \lfloor \epsilon_3 n \rfloor)^2}{2} \left(1 - \frac{1}{r-1}\right) + k^2 \\ &\geq t_{r-1}(n - \lfloor \epsilon_3 n \rfloor) + k^2. \end{aligned}$$

However, this is a contradiction as the above lower bound for $e[G \setminus L']$ exceeds the upper bound on the number of edges in any $F_{k,r}$-free graph on $n - |L'|$ vertices. □

Lemma 3.2.5. *Let W and L be the sets of vertices defined in (3.1) and (3.2). Then $W \subseteq L$.*

Proof. Suppose on the contrary that there exists a vertex $u_0 \in W$ and $u_0 \notin L$. Without loss of generality, we may assume that $u_0 \in V_1$. Since $V_1, \ldots, V_{r-1}$ form a maximum $(r-1)$-partite subgraph, we have $d_{V_1}(u_0) \leq d_{V_i}(u_0)$ for each $i \in [2, r-1]$. Indeed, otherwise, we can move the vertex u_0 into some part V_i and strictly increase the number of edges between V_1 and V_i. Thus, we can get $d(u_0) \geq (r-1)d_{V_1}(u_0)$, which implies

$$d_{V_2}(u_0) \geq d(u_0) - d_{V_1}(u_0) - (r-3)n\left(\frac{1}{r-1} + \epsilon_1\right).$$

On the other hand, invoking the fact that $u_0 \notin L$, we get $d(u_0) > (1 - \frac{1}{r-1} - \epsilon_2)n$. So

$$\begin{aligned} d_{V_2}(u_0) &\geq \left(1 - \frac{1}{r-1}\right) d(u_0) - (r-3)n\left(\frac{1}{r-1} + \epsilon_1\right) \\ &\geq \frac{n}{(r-1)^2} - \frac{r-2}{r-1}\epsilon_2 n - (r-3)\epsilon_1 n. \end{aligned}$$

Recall from Lemmas 3.2.3 and 3.2.4 that $|W| < \theta n$ and $|L| \leq \epsilon_3 n$. Hence, for fixed θ, ϵ_3 and sufficiently large n, we have

$$|V_i \setminus (W \cup L)| \geq \left(\frac{1}{r-1} - \epsilon_1\right)n - \theta n - \epsilon_3 n \geq k.$$

Claim 3.2.5.1. The vertex u_0 is adjacent to at most $k-1$ vertices in $V_1 \setminus (W \cup L)$.

Proof. Suppose that u_0 is adjacent to k vertices $u_1^{(1)}, u_2^{(1)}, \ldots, u_k^{(1)}$ in $V_1 \setminus (W \cup L)$. Since $u_j^{(1)} \notin L$, we have

$$d(u_j^{(1)}) > \left(1 - \frac{1}{r-1} - \epsilon_2\right)n.$$

On the other hand, we have $d_{V_1}(u_j^{(1)}) < \theta n$ because $u_j^{(1)} \notin W$. So for each $j \in [k]$,

$$\begin{aligned} d_{V_2}(u_j^{(1)}) &\geq d(u_j^{(1)}) - d_{V_1}(u_j^{(1)}) - (r-3)\left(\frac{1}{r-1} + \epsilon_1\right)n \\ &\geq \frac{n}{r-1} - \epsilon_2 n - \theta n - (r-3)\epsilon_1 n. \end{aligned}$$

By Lemma 1.5.4, we consider the common neighbors of $u_0, u_1^{(1)}, \ldots, u_k^{(1)}$ in V_2,

$$\begin{aligned}
& \left|N_{V_2}(u_0) \cap N_{V_2}(u_1^{(1)}) \cap \cdots \cap N_{V_2}(u_k^{(1)}) \setminus (W \cup L)\right| \\
\geq \quad & d_{V_2}(u_0) + \sum_{j=1}^{k} d_{V_2}(u_j^{(1)}) - k\,|V_2| - |W| - |L| \\
\geq \quad & \tfrac{n}{(r-1)^2} - \tfrac{r-2}{r-1}\epsilon_2 n - (r-3)\epsilon_1 n + k\left(\tfrac{n}{r-1} - \epsilon_2 n - \theta n - (r-3)\epsilon_1 n\right) \\
- \quad & k\left(\tfrac{1}{r-1} + \epsilon_1\right) n - \theta n - \epsilon_3 n \\
\geq \quad & \frac{n}{(r-1)^2} - o(n) > k
\end{aligned}$$

for sufficiently large n. So there exist k vertices $u_1^{(2)}, u_2^{(2)}, \ldots, u_k^{(2)}$ in $V_2 \setminus (W \cup L)$ such that the subgraph formed by two partitions $\left\{u_1^{(1)}, \ldots, u_k^{(1)}\right\}$ and $\left\{u_1^{(2)}, \ldots, u_k^{(2)}\right\}$ is a complete bipartite graph. It is easy to see that the subgraph of G formed by the vertex u_0 together with such a complete bipartite graph can contain a copy of $F_{k,3}$ centered at the vertex u_0. In the sequel, we shall extend this copy to the intersecting cliques $F_{k,r}$. Let $s \in [2, r-2]$ be a positive integer. Assume that we have found the vertices $u_1^{(i)}, u_2^{(i)}, \ldots, u_k^{(i)} \in V_i \setminus (W \cup L), (i = 1, 2, \ldots, s)$ such that these vertices form a complete s-partite subgraph in G. We next consider the common neighbors of these vertices in V_{s+1}. Similarly, we get that for each $i \in [s]$ and $j \in [k]$,

$$\begin{aligned}
d_{V_{s+1}}(u_j^{(i)}) &\geq d(u_j^{(i)}) - d_{V_i}(u_j^{(i)}) - (r-3)\left(\frac{1}{r-1} + \epsilon_1\right) n \\
&\geq \frac{n}{r-1} - \epsilon_2 n - \theta n - (r-3)\epsilon_1 n.
\end{aligned}$$

By Lemma 1.5.4 again, we can obtain

$$\begin{aligned}
& \left|N_{V_{s+1}}(u_0) \cap \left(\cap_{i\in[s], j\in[k]} N_{V_{s+1}}(u_j^{(i)})\right) \setminus (W \cup L)\right| \\
\geq \quad & d_{V_{s+1}}(u_0) + \sum_{i\in[s], j\in[k]} d_{V_{s+1}}(u_j^{(i)}) - ks\,|V_{s+1}| - |W| - |L| \\
\geq \quad & \tfrac{n}{(r-1)^2} - \tfrac{r-2}{r-1}\epsilon_2 n - (r-3)\epsilon_1 n + ks\left(\tfrac{n}{r-1} - \epsilon_2 n - \theta n - (r-3)\epsilon_1 n\right) \\
- \quad & ks\left(\tfrac{1}{r-1} + \epsilon_1\right) n - \theta n - \epsilon_3 n \\
\geq \quad & \frac{n}{(r-1)^2} - o(n) > k
\end{aligned}$$

Thus we can find k vertices $u_1^{(s+1)}, u_2^{(s+1)}, \ldots, u_k^{(s+1)} \in V_{s+1} \setminus (W \cup L)$, which together with the previous vertices $u_j^{(i)} \in V_i \setminus (W \cup L), (i \in [s], j \in [k])$ form a complete $(s+1)$-partite subgraph in G. Thus, for each $i \in [r-1]$, we can find k vertices from every vertex part $V_i \setminus (W \cup L)$ such that these vertices together with u_0 form a copy of $F_{k,r}$ centered at u_0, this is a contradiction. Therefore u_0 is adjacent to at most $k-1$ vertices in $V_1 \setminus (W \cup L)$. $\square$

Hence, applying Lemmas 3.2.3 and 3.2.4 again, we have

$$\begin{aligned} d_{V_1}(u_0) &\leq |W| + |L| + k - 1 \\ &< \frac{2\theta}{3} n + \frac{2k^2}{\theta n} + \epsilon_3 n + k - 1 \\ &< \theta n \end{aligned}$$

for sufficiently large n. This is a contradiction to the fact that $u_0 \in W$. Hence $W \subseteq L$. $\square$

Lemma 3.2.6. *For each i, there exists an independent set $I_i \subseteq V_i$ such that*

$$|I_i| \geq |V_i| - \epsilon_3 n - k^2.$$

Proof. Since $V_i \setminus L$ is large enough by Lemma 3.2.4, we first prove that there exists a large complete multipartite subgraph between $V_1, V_2, \ldots, V_{r-1}$. Let $u_1^{(1)}, u_2^{(1)}, \ldots, u_{2k}^{(1)}$ be $2k$ vertices chosen *arbitrarily* from $V_1 \setminus L$. Then $u_j^{(1)} \notin L$ which implies that $d(u_j^{(1)}) > \left(1 - \frac{1}{r-1} - \epsilon_2\right) n$. Note that $W \subseteq L$ by Lemma 3.2.5, so $u_j^{(1)} \notin W$, then $d_{V_1}(u_j^{(1)}) < \theta n$. Hence

$$d_{V_2}(u_j^{(1)}) \geq \frac{n}{r-1} - \epsilon_2 n - \theta n - (r-3)\epsilon_1 n.$$

Furthermore, by Lemma 1.5.4, we have

$$\begin{aligned} &\left| N_{V_2}(u_1^{(1)}) \cap N_{V_2}(u_2^{(1)}) \cap \cdots \cap N_{V_2}(u_{2k}^{(1)}) \setminus L \right| \\ \geq\ & \sum_{j=1}^{2k} d_{V_2}(u_j^{(1)}) - (2k-1)\,|V_2| - |L| \\ \geq\ & 2k\left(\tfrac{n}{r-1} - \epsilon_2 n - \theta n - (r-3)\epsilon_1 n\right) - (2k-1)\left(\tfrac{1}{r-1} + \epsilon_1\right) n - \epsilon_3 n \\ \geq\ & \frac{n}{r-1} - o(n) > 2k \end{aligned}$$

for sufficiently large n. Hence there exist $2k$ vertices $u_1^{(2)}, u_2^{(2)}, \dots, u_{2k}^{(2)} \in V_2$ such that the subgraph formed between the two parts $\{u_1^{(1)}, \dots, u_{2k}^{(1)}\}$ and $\{u_1^{(2)}, \dots, u_{2k}^{(2)}\}$ is a complete bipartite graph. Let $s \in [2, r-2]$ be a positive integer. Assume that we have found the vertices $u_1^{(i)}, u_2^{(i)}, \dots, u_{2k}^{(i)} \in V_i \setminus L, (i = 1, 2, \dots, s)$ such that these vertices form a complete s-partite subgraph in G. We next consider the common neighbors of these vertices in V_{s+1}. Similarly, we get that for each $i \in [s]$ and $j \in [2k]$,

$$d_{V_{s+1}}(u_j^{(i)}) \geq \frac{n}{r-1} - \epsilon_2 n - \theta n - (r-3)\epsilon_1 n.$$

By Lemma 1.5.4 again, we can obtain

$$\begin{aligned} & \left| \left(\cap_{i \in [s], j \in [2k]} N_{V_{s+1}}(u_j^{(i)}) \right) \setminus L \right| \\ \geq \quad & \sum_{i \in [s], j \in [2k]} d_{V_{s+1}}(u_j^{(i)}) - (2ks - 1) \left| V_{s+1} \right| - |L| \\ \geq \quad & 2ks \left(\tfrac{n}{r-1} - \epsilon_2 n - (r-3)\epsilon_1 n \right) - (2ks-1) \left(\tfrac{1}{r-1} + \epsilon_1 \right) n - \epsilon_3 n \\ \geq \quad & \frac{n}{r-1} - o(n) > 2k \end{aligned}$$

Thus we can find $2k$ vertices $u_1^{(s+1)}, u_2^{(s+1)}, \dots, u_{2k}^{(s+1)} \in V_{s+1} \setminus L$, which together with the vertices $u_j^{(i)} \in V_i \setminus L, (i \in [s], j \in [2k])$ form a complete $(s+1)$-partite subgraph in G. Thus, for any $2k$ vertices in $V_1 \setminus L$, we can find $2k$ vertices from $V_i \setminus L$ for each $i \in [2, r-1]$ such that all these vertices form a complete $(r-1)$-partite subgraph in G.

Claim 3.2.6.1. The induced graph $G[V_1 \setminus L]$ is both $K_{1,k}$-free and M_k-free.

Proof. Recall that G contains a large complete $(r-1)$-partite subgraph with each part in $V_i \setminus L$. If $G[V_1 \setminus L]$ contains a copy of $K_{1,k}$ centered at a vertex $u_0 \in V_1$ with leaves $u_1^{(1)}, u_2^{(1)}, \dots, u_k^{(1)}$, then by the discussion above, we can embed the $F_{k,r}$ into G. Therefore, $G[V_1 \setminus L]$ is $K_{1,k}$-free. Now, we assume that $\{u_1^{(1)}u_2^{(1)}, u_3^{(1)}u_4^{(1)}, \dots, u_{2k-1}^{(1)}u_{2k}^{(1)}\}$ is a matching of size k. Then for each $j \in [k]$, the vertices $u_{2j-1}^{(1)}, u_{2j}^{(1)}, u_1^{(2)}, u_j^{(3)} \dots, u_j^{(r-1)}$ form a clique of order r, and these r cliques intersect at the vertex $u_1^{(2)}$. So $G[V_1 \setminus L]$ is M_k-free. □

Hence both the maximum degree and the maximum matching number of $G[V_1 \setminus L]$ are at most $k-1$, respectively. By Theorem 1.5.9,

$$e(G[V_1 \setminus L]) \leq f(k-1, k-1).$$

The same argument gives that for each $j \in [2, r-1]$,

$$e(G[V_j \setminus L]) \le f(k-1, k-1).$$

For each $i \in [r-1]$, since $G[V_i \setminus L]$ has at most $f(k-1, k-1)$ edges, then the subgraph obtained from $G[V_i \setminus L]$ by deleting one vertex of each edge in $G[V_i \setminus L]$ contains no edges, which is an independent set of $G[V_i \setminus L]$. By Lemma 3.2.4, there exists an independent set $I_i \subseteq V_i$ such that

$$|I_i| \ge |V_i \setminus L| - f(k-1, k-1) \ge |V_i| - \epsilon_3 n - k^2.$$

This completes the proof. □

Lemma 3.2.7. *L is empty, and each $G[V_i]$ is $K_{1,k}$-free and M_k-free.*

Proof. Recall that $A\mathbf{x} = \lambda(G)\mathbf{x}$ and z is defined as a vertex with maximum eigenvector entry and satisfies $\mathbf{x}_z = 1$. So we have

$$d(z) \ge \sum_{w \sim z} \mathbf{x}_w = \lambda(G)\mathbf{x}_z = \lambda(G) \ge \left(1 - \frac{1}{r-1} - \frac{r-1}{4n^2}\right) n > \left(1 - \frac{1}{r-1} - \epsilon_2\right) n,$$

as n is large enough. Hence $z \notin L$. Without loss of generality, we may assume that $z \in V_1$. Since the maximum degree in the induced subgraph $G[V_1 \setminus L]$ is at most $k-1$ (containing no $K_{1,k}$), from Lemma 3.2.4, we have $|L| \le \epsilon_3 n$ and

$$d_{V_1}(z) = d_{V_1 \cap L}(z) + d_{V_1 \setminus L}(z) \le \epsilon_3 n + k - 1.$$

Therefore, by Lemma 3.2.6, we have

$$\begin{aligned}
\lambda(G) &= \lambda(G)\mathbf{x}_z = \sum_{v \sim z} \mathbf{x}_v = \sum_{\substack{v \in V_1 \\ v \sim z}} \mathbf{x}_v + \sum_{\substack{v \in V_2 \sqcup \cdots \sqcup V_{r-1} \\ v \sim z}} \mathbf{x}_v \\
&= \sum_{\substack{v \in V_1 \\ v \sim z}} \mathbf{x}_v + \sum_{\substack{v \in I_2 \sqcup \cdots \sqcup I_{r-1} \\ v \sim z}} \mathbf{x}_v + \sum_{\substack{v \in \sqcup_{i=2}^{r-1} V_i \setminus I_i \\ v \sim z}} \mathbf{x}_v \\
&\le d_{V_1}(z) + \sum_{v \in I_2 \sqcup \cdots \sqcup I_{r-1}} \mathbf{x}_v + |\sqcup_{i=2}^{r-1} V_i \setminus I_i| \\
&\le \epsilon_3 n + k - 1 + \sum_{v \in I_2 \sqcup \cdots \sqcup I_{r-1}} \mathbf{x}_v + (r-2)(\epsilon_3 n + k^2).
\end{aligned}$$

By Lemma 3.2.1, we can get

$$\sum_{v\in I_2\sqcup\cdots\sqcup I_{r-1}} \mathbf{x}_v \geq \left(1-\frac{1}{r-1}-\frac{r-1}{4n^2}\right)n-(r-1)\epsilon_3 n-(r-2)k^2-k+1. \quad (3.3)$$

Next we are going to prove $L=\varnothing$.

By way of contradiction, assume that there is a vertex $v\in L$, so $d_G(v)\leq (1-\frac{1}{r-1}-\epsilon_2)n$. Consider the graph G^+ with vertex set $V(G)$ and edge set $E(G^+)=E(G\backslash\{v\})\cup\{vw: w\in \sqcup_{i=2}^{r-1} I_i\}$. Roughly speaking, in this process, the number of added edges is greater than the number of deleted edges. Note that adding a vertex incident with vertices in I_i does not create any cliques, and so G^+ is $F_{k,r}$-free. Note that $\mathbf{x}$ is a vector such that $\lambda(G)=\frac{\mathbf{x}^T A(G)\mathbf{x}}{\mathbf{x}^T\mathbf{x}}$, and the Rayleigh theorem implies $\lambda(G^+)\geq \frac{\mathbf{x}^T A(G^+)\mathbf{x}}{\mathbf{x}^T\mathbf{x}}$. Furthermore,

$$\begin{aligned}\lambda(G^+)-\lambda(G) &\geq \frac{\mathbf{x}^T\left(A(G^+)-A(G)\right)\mathbf{x}}{\mathbf{x}^T\mathbf{x}} = \frac{2\mathbf{x}_v}{\mathbf{x}^T\mathbf{x}}\left(\sum_{w\in I_2\sqcup\cdots\sqcup I_{r-1}}\mathbf{x}_w-\sum_{uv\in E(G)}\mathbf{x}_u\right)\\ &\overset{(3.3)}{\geq} \frac{2\mathbf{x}_v}{\mathbf{x}^T\mathbf{x}}\left(\epsilon_2 n-\frac{r-1}{4n}-(r-1)\epsilon_3 n-(r-2)k^2-k+1\right)>0,\end{aligned}$$

where the last inequality holds for n large enough and $\epsilon_3\leq\epsilon_2$, ϵ_3 small enough. This contradicts G having the largest spectral radius over all $F_{k,r}$-free graphs, so L must be empty. Furthermore, the claim in the proof of Lemma 3.2.6 implies that each $G[V_i]$ is $K_{1,k}$-free and M_k-free. □

Lemma 3.2.8. *For any $i\in[r-1]$, let $B_i=\{u\in V_i: d_{V_i}(u)\geq 1\}$ and $C_i=V_i\setminus B_i$. Then*

(1) $|B_i|\leq 2k^2+1$;

(2) For every vertex $u\in C_i$, u is adjacent to all vertices of $V\setminus V_i$.

Proof. We prove the assertions by contradiction.

(1) If there exists a $j\in[r-1]$ such that $|B_j|>2k^2+1$, then $\sum_{u\in B_j} d_{V_j}(u)>2k^2+1$. Since $G[V_j]$ is both $K_{1,k}$-free and M_k-free, $e(G[V_j])\leq f(k-1,k-1)<k^2$. Therefore,

$$2k^2+1<\sum_{u\in B_j} d_{V_j}(u)=\sum_{u\in V_j} d_{V_j}(u)=2e(G[V_j])<2k^2,$$

which is a contradiction.

(2) If there exists a vertex $v \in C_1$ such that there is a vertex $w_{1,1} \notin V_1$ and $vw_{1,1} \notin E(G)$. Let G' be the graph with $V(G') = V(G)$ and $E(G') = E(G) \cup \{vw_{1,1}\}$. We claim that G' is $F_{k,r}$-free. Otherwise, G' contains a copy of $F_{k,r}$, say F_0, as a subgraph, then $vw_{1,1} \in E(F_0)$. We may assume that v is the center of F_0 (The case that v is not the center of F_0 can be proved similarly). As v is the center of F_0, there exist vertices $w_{1,1}, w_{1,2}, \cdots, w_{1,r-1}, w_{2,1}, \cdots, w_{2,r-1}, \ldots, w_{k,1}, \cdots, w_{k,r-1} \notin V_1$ such that for any $i \in [k]$, the vertex set $\{w_{i,1}, w_{i,2}, \ldots, w_{i,r-1}\}$ induces a copy of K_{r-1} in G. Therefore, for any $i \in [k]$ and $j \in [r-1]$, we have

$$d_{V_1}(w_{i,j}) = d(w_{i,j}) - d_{V \setminus V_1}(w_{i,j}) \geq \delta(G) - (k-1) - (r-3)\left(\frac{n}{r-1} + \epsilon_1 n\right),$$

where the last inequality holds as $G[V_s]$ is $K_{1,k}$-free, $|V_s| \leq \frac{n}{r-1} + \epsilon_1 n$ for any $s \in [r-1]$. Since L is empty by Lemma 3.2.7, we have $\delta(G) > (\frac{r-2}{r-1} - \epsilon_2)n$. It follows that

$$d_{V_1}(w_{i,j}) > \frac{n}{r-1} - o(n).$$

Using Lemma 1.5.4, we get

$$\begin{aligned}
&\Big|\bigcap_{i=1}^{k}\bigcap_{j=1}^{r-1} N_{V_1}(w_{i,j}) \setminus B_1\Big| \\
\geq\ & \sum_{i=1}^{k}\sum_{j=1}^{r-1} |(N_{V_1}(w_{i,j})| - (k(r-1)-1)\Big|\bigcup_{i=1}^{k}\bigcup_{j=1}^{r-1} N_{V_1}(w_{i,j})\Big| - |B_1| \\
\geq\ & \sum_{i=1}^{k}\sum_{j=1}^{r-1} d_{V_1}(w_{i,j}) - (kr-k-1)|V_1| - |B_1| \\
>\ & k(r-1)\left(\frac{n}{r-1} - o(n)\right) - (kr-k-1)\left(\frac{n}{r-1} + o(n)\right) - (2k^2+1) \\
\geq\ & \frac{n}{r-1} - o(n) > 1.
\end{aligned}$$

Then there exists $v' \in C_1$ such that v' is adjacent to $w_{1,1}, \ldots, w_{1,r-1}, \ldots, w_{k,1}, \ldots, w_{k,r-1}$. Then $(F_0 \setminus \{v\}) \cup \{v'\}$ is a copy of $F_{k,r}$ in G, which is a contradiction. Thus G' is $F_{k,r}$-free. From the construction of G', we see that $\lambda(G') > \lambda(G)$, which contradicts the assumption that G has the maximum spectral radius among all $F_{k,r}$-free graphs on n vertices. $\square$

Lemma 3.2.9. *For any* $u \in V(G)$, $\mathbf{x}_u \geq 1 - \frac{20k^2r^2}{n}$.

Proof. Recall that $\mathbf{x}_z = \max\{\mathbf{x}_i : i \in V(G)\} = 1$. Without loss of generality, we may assume that $z \in V_1$. Then

$$\begin{aligned}\lambda(G)\mathbf{x}_z = \sum_{w \sim z} \mathbf{x}_w &= \sum_{w \sim z, w \in V_1} \mathbf{x}_w + \sum_{i=2}^{r-1}\left(\sum_{w \sim z, w \in V_i} \mathbf{x}_w\right) \\ &= \sum_{w \sim z, w \in V_1} \mathbf{x}_w + \sum_{i=2}^{r-1}\left(\sum_{w \sim z, w \in B_i} \mathbf{x}_w + \sum_{w \sim z, w \in C_i} \mathbf{x}_w\right),\end{aligned}$$

which implies that

$$\begin{aligned}\sum_{i=2}^{r-1}\left(\sum_{w \sim z, w \in C_i} \mathbf{x}_w\right) &= \lambda(G) - \sum_{w \sim z, w \in V_1} \mathbf{x}_w - \sum_{i=2}^{r-1}\left(\sum_{w \sim z, w \in B_i} \mathbf{x}_w\right) \\ &\geq \lambda(G) - d_{V_1}(z) - \sum_{i=2}^{r-1}\left(\sum_{w \in B_i} 1\right) \\ &\geq \lambda(G) - (k-1) - (r-2)(2k^2+1), \qquad (3.4)\end{aligned}$$

where (3.4) holds as $G[V_1]$ is $K_{1,k}$-free, and $|B_i| \leq 2k^2 + 1$ for any $i \in [r-1]$.

We will prove this lemma by contradiction. Suppose that there is a vertex $v \in V(G)$ with $\mathbf{x}_v < 1 - \frac{20k^2r^2}{n}$. Let G' be the graph with $V(G') = V(G)$ and $E(G') = E(G \setminus \{v\}) \cup \{vw : w \in N(z) \cap (\cup_{i=2}^{r-1} C_i)\}$. Since C_i is an independent set for any $i \in [r-1]$, one may observe that G' is $F_{k,r}$-free. By (3.4), we have

$$\begin{aligned}\lambda(G') - \lambda(G) &\geq \frac{\mathbf{x}^T(A(G') - A(G))\mathbf{x}}{\mathbf{x}^T\mathbf{x}} \\ &= \frac{2\mathbf{x}_v}{\mathbf{x}^T\mathbf{x}}\left(\sum_{i=2}^{r-1}\Big(\sum_{w \sim z, w \in C_i} \mathbf{x}_w\Big) - \sum_{uv \in E(G)} \mathbf{x}_u\right) \\ &= \frac{2\mathbf{x}_v}{\mathbf{x}^T\mathbf{x}}\left(\sum_{i=2}^{r-1}\Big(\sum_{w \sim z, w \in C_i} \mathbf{x}_w\Big) - \lambda(G)\mathbf{x}_v\right) \\ &> \frac{2\mathbf{x}_v}{\mathbf{x}^T\mathbf{x}}\left(\lambda(G) - (k-1) - (r-2)(2k^2+1) - \lambda(G)\Big(1 - \frac{20k^2r^2}{n}\Big)\right) \\ &\geq \frac{2\mathbf{x}_v}{\mathbf{x}^T\mathbf{x}}\left(\frac{r-2}{r-1}20k^2r^2 - \frac{r-1}{4n}\frac{20k^2r^2}{n} - k + 1 - (r-2)(2k^2+1)\right) > 0,\end{aligned}$$

where the last inequality follows by $\lambda(G) \geq (1 - \frac{1}{r-1})n - \frac{r-1}{4n}$ by Lemma 3.2.1. This contradicts the assumption that G has the maximum spectral radius among all $F_{k,r}$-free graphs on n vertices. Thus $\mathbf{x}_u \geq 1 - \frac{20k^2r^2}{n}$ for any $u \in V(G)$. $\square$

Let $G_{in} = \cup_{i=1}^{r-1} G[V_i]$. For any $i \in [r-1]$, let $|V_i| = n_i$ and $F = K_{n_1,n_2,\ldots,n_{r-1}}$ be the complete $(r-1)$-partite graph on $V_1, V_2, \ldots, V_{r-1}$. Let G_{out} be the graph with $V(G_{out}) = V(G)$ and $E(G_{out}) = E(F) \setminus E(G)$.

Lemma 3.2.10. *For any* $1 \leq i < j \leq r-1$, $||V_i| - |V_j|| \leq 1$.

Proof. Suppose $n_1 \geq n_2 \geq \ldots \geq n_{r-1}$. We prove the assertion by contradiction. Assume that there exist i_0, j_0 with $1 \leq i_0 < j_0 \leq r-1$ such that $n_{i_0} - n_{j_0} \geq 2$.

Claim 3.2.10.1. There exists a constant $c_1 > 0$ such that $\lambda(T_{r-1}(n)) - \lambda(F) \geq \frac{c_1}{n}$.

Proof. Let $F' = K_{n_1,\ldots,n_{i_0}-1,\ldots,n_{j_0}+1,\ldots,n_{r-1}}$. Assume $F' \cong K_{n'_1,n'_2,\ldots,n'_{r-1}}$, where $n'_1 \geq n'_2 \geq \ldots \geq n'_{r-1}$. By (1.5), we have

$$1 = \sum_{i=1}^{r-1} \frac{n_i}{\lambda(F) + n_i} = \frac{n_{i_0}}{\lambda(F) + n_{i_0}} + \frac{n_{j_0}}{\lambda(F) + n_{j_0}} + \sum_{i \in [r-1] \setminus \{i_0, j_0\}} \frac{n_i}{\lambda(F) + n_i}, \tag{3.5}$$

and

$$1 = \sum_{i=1}^{r-1} \frac{n'_i}{\lambda(F') + n'_i} = \frac{n_{i_0} - 1}{\lambda(F') + n_{i_0} - 1} + \frac{n_{j_0} + 1}{\lambda(F') + n_{j_0} + 1} + \sum_{i \in [r-1] \setminus \{i_0, j_0\}} \frac{n_i}{\lambda(F') + n_i}. \tag{3.6}$$

Subtracting (3.6) from (3.5), we get

$$\begin{aligned}
& \frac{2(n_{i_0} - n_{j_0} - 1)\lambda^2(F) + (n_{i_0} + n_{j_0})(n_{i_0} - n_{j_0} - 1)\lambda(F)}{(\lambda(F) + n_{i_0} - 1)(\lambda(F) + n_{i_0})(\lambda(F) + n_{j_0} + 1)(\lambda(F) + n_{j_0})} \\
= & \sum_{i \in [r-1] \setminus \{i_0, j_0\}} \frac{n_i(\lambda(F') - \lambda(F))}{(\lambda(F) + n_i)(\lambda(F') + n_i)} + \frac{(n_{i_0} - 1)(\lambda(F') - \lambda(F))}{(\lambda(F) + n_{i_0} - 1)(\lambda(F') + n_{i_0} - 1)} \\
& + \frac{(n_{j_0} + 1)(\lambda(F') - \lambda(F))}{(\lambda(F) + n_{j_0} + 1)(\lambda(F') + n_{j_0} + 1)} \\
\leq & \frac{\lambda(F') - \lambda(F)}{\lambda(F) + n'_{r-1}} \left(\sum_{i \in [r-1] \setminus \{i_0, j_0\}} \frac{n_i}{\lambda(F') + n_i} + \frac{n_{i_0} - 1}{\lambda(F') + n_{i_0} - 1} + \frac{n_{j_0} + 1}{\lambda(F') + n_{j_0} + 1} \right) \\
= & \frac{\lambda(F') - \lambda(F)}{\lambda(F) + n'_{r-1}},
\end{aligned}$$

where the inequality holds as $n'_{r-1} \le \min\{n_1, \dots, n_{i_0} - 1, \dots, n_{j_0} + 1, \dots, n_{r-1}\}$, and the last equality is by (3.6). Combining with the assumption $n_{i_0} - n_{j_0} \ge 2$, we obtain

$$\frac{2\lambda^2(F) + (n_{i_0} + n_{j_0})\lambda(F)}{(\lambda(F) + n_{i_0} - 1)(\lambda(F) + n_{i_0})(\lambda(F) + n_{j_0} + 1)(\lambda(F) + n_{j_0})} \le \frac{\lambda(F') - \lambda(F)}{\lambda(F) + n'_{r-1}}. \tag{3.7}$$

In view of the construction of F, we see that

$$n - \left(\frac{n}{r-1} + \epsilon_1 n\right) \le \delta(F) \le \lambda(F) \le \Delta(F) \le n - \left(\frac{n}{r-1} - \epsilon_1 n\right),$$

thus $\lambda(F) = \Theta(n)$. From (3.7), it follows that there exists a constant $c_1 > 0$ such that $\lambda(F') - \lambda(F) \ge \frac{c_1}{n}$. Therefore, by Lemma 1.5.10, $\lambda(T_{r-1}(n)) - \lambda(F) \ge \lambda(F') - \lambda(F) \ge \frac{c_1}{n}$. □

Claim 3.2.10.2.

$$\lambda(G) \ge \lambda(T_{r-1}(n)) + \frac{2f(k-1, k-1)}{n}\left(1 - \frac{2}{n}\right).$$

Proof. Let $\mathbf{y}$ be an eigenvector of $T_{r-1}(n)$ corresponding to $\lambda(T_{r-1}(n))$, $a = n - (r-1)\left\lfloor \frac{n}{r-1} \right\rfloor$. Since $T_{r-1}(n)$ is a complete $(r-1)$-partite graph on n vertices where each partite set has either $\left\lfloor \frac{n}{r-1} \right\rfloor$ or $\left\lceil \frac{n}{r-1} \right\rceil$ vertices, we may assume $\mathbf{y} = (\underbrace{\mathbf{y}_1, \dots, \mathbf{y}_1}_{a\lceil \frac{n}{r-1} \rceil}, \underbrace{\mathbf{y}_2, \dots, \mathbf{y}_2}_{n-a\lceil \frac{n}{r-1} \rceil})^{\mathrm{T}}$. Thus by (1.6), we have

$$\lambda(T_{r-1}(n))\mathbf{y}_1 = (r-a-1)\left\lfloor \frac{n}{r-1} \right\rfloor \mathbf{y}_2 + (a-1)\left\lceil \frac{n}{r-1} \right\rceil \mathbf{y}_1, \tag{3.8}$$

and

$$\lambda(T_{r-1}(n))\mathbf{y}_2 = (r-a-2)\left\lfloor \frac{n}{r-1} \right\rfloor \mathbf{y}_2 + a\left\lceil \frac{n}{r-1} \right\rceil \mathbf{y}_1. \tag{3.9}$$

Combining (3.8) and (3.9), we obtain

$$\left(\lambda(T_{r-1}(n)) + \left\lceil \frac{n}{r-1} \right\rceil\right)\mathbf{y}_1 = \left(\lambda(T_{r-1}(n)) + \left\lfloor \frac{n}{r-1} \right\rfloor\right)\mathbf{y}_2.$$

Without loss of generality, we assume that $\mathbf{y}_2 = 1$. Then

$$\mathbf{y}_2 \ge \mathbf{y}_1 = \frac{\lambda(T_{r-1}(n)) + \lfloor \frac{n}{r-1} \rfloor}{\lambda(T_{r-1}(n)) + \lceil \frac{n}{r-1} \rceil} \ge 1 - \frac{1}{\lambda(T_{r-1}(n)) + \lceil \frac{n}{r-1} \rceil}.$$

Since $\lambda(T_{r-1}(n)) \geq \delta(T_{r-1}(n)) \geq n - \lceil \frac{n}{r-1} \rceil$, $\mathbf{y}_1 \geq 1 - \frac{1}{n}$.

Let $H \in \mathrm{EX}(n, F_{k,r})$. By Theorem 1.2.3, H is constructed from $T_{r-1}(n)$ by embedding a graph G_0 in one of the parts. Then $e(H) = \mathrm{ex}(n, F_{k,r}) = \mathrm{ex}(n, K_r) + f(k-1, k-1)$. Therefore

$$\begin{aligned}
\lambda(G) \geq \lambda(H) &\geq \frac{\mathbf{y}^{\mathrm{T}} A(H) \mathbf{y}}{\mathbf{y}^{\mathrm{T}} \mathbf{y}} \\
&\geq \frac{\mathbf{y}^{\mathrm{T}} A(T_{r-1}(n)) \mathbf{y}}{\mathbf{y}^{\mathrm{T}} \mathbf{y}} + \frac{2 \sum_{ij \in E(G_0)} \mathbf{y}_i \mathbf{y}_j}{\mathbf{y}^{\mathrm{T}} \mathbf{y}} \\
&\geq \lambda(T_{r-1}(n)) + \frac{2f(k-1, k-1)}{\mathbf{y}^{\mathrm{T}} \mathbf{y}} \left(1 - \frac{1}{n}\right)^2 \\
&\geq \lambda(T_{r-1}(n)) + \frac{2f(k-1, k-1)}{n} \left(1 - \frac{2}{n}\right). \qquad (3.10)
\end{aligned}$$

□

Claim 3.2.10.3. $e(G_{in}) - e(G_{out}) \leq f(k-1, k-1)$.

Proof. It follows from the definitions of G_{in} and G_{out}, that we have $e(G_{in}) = \sum_{i=1}^{r-1} |E(G[V_i])|$ and $e(G_{out}) = \sum_{1 \leq i < j \leq r-1} |V_i||V_j| - |E_{cr}(G)|$. To get the claim, we need to prove (1.8) and (1.9) by Lemma 1.5.11. Obviously (1.9) implies (1.8), so it is sufficient to prove (1.9). We prove (1.9) by contradiction. Without loss of generality, suppose that there exists a vertex $u \in V_1$ such that

$$d_{G[V_1]}(u) + \sum_{j=2}^{r-1} \beta(G[N(u) \cap V_j]) \geq k.$$

Let $\{w_1 w_2, \ldots, w_{2\ell-1} w_{2\ell}\}$ be an ℓ-matching of $\cup_{j=2}^{r-1} G[N(u) \cap V_j]$ and $u_1, \ldots, u_{k-\ell} \in V_1$ be in the neighborhood of u. By Lemma 3.2.8, there exist $v_1, \ldots, v_{k-\ell} \in C_2$ such that $\{u, u_1, \ldots, u_{k-\ell}, v_1, \ldots, v_{k-\ell}, w_1, \ldots, w_{2\ell}\}$ induce an F_k of G. For each $u_i v_i$ $(1 \leq i \leq k-\ell)$, there exist $r-3$ vertices $t_3 \in C_3, t_4 \in C_4, \ldots, t_{r-1} \in C_{r-1}$ such that $u, u_i, v_i, t_3, t_4, \ldots, t_{r-1}$ induce a K_r of G. For any $w_{i-1} w_i \in \{w_1 w_2, \ldots, w_{2\ell-1} w_{2\ell}\}$, without loss of generality, suppose that $w_{i-1} w_i \subseteq E(G[V_2])$, then there exist $r-3$ vertices $z_3 \in C_3, z_4 \in C_4, \ldots, z_{r-1} \in C_{r-1}$ such that the vertices $u, w_{i-1}, w_i, z_3, z_4, \ldots, z_{r-1}$ induce a K_r of G. Thus we find a copy of $F_{k,r}$ from the above F_k, a contradiction.

□

According to the definitions of G_{in}, G_{out} and F, we have $e(G) = e(G_{in}) + e(F) - e(G_{out})$. By Lemma 3.2.8, for any $i \in [r-1]$, and every vertex $u \in C_i$, u is adjacent to all vertices of $V \setminus V_i$. Thus

$$e(G_{out}) \leq \sum_{1\leq i<j\leq r-1} |B_i||B_j| \leq \binom{r-1}{2}(2k^2+1)^2 \leq 9k^4r^2.$$

Then

$$\begin{aligned}
\lambda(G) &= \frac{\mathbf{x}^{\mathrm{T}}A(G)\mathbf{x}}{\mathbf{x}^{\mathrm{T}}\mathbf{x}} \\
&= \frac{2\sum_{ij\in E(F)}\mathbf{x}_i\mathbf{x}_j}{\mathbf{x}^{\mathrm{T}}\mathbf{x}} + \frac{2\sum_{ij\in E(G_{in})}\mathbf{x}_i\mathbf{x}_j}{\mathbf{x}^{\mathrm{T}}\mathbf{x}} - \frac{2\sum_{ij\in E(G_{out})}\mathbf{x}_i\mathbf{x}_j}{\mathbf{x}^{\mathrm{T}}\mathbf{x}} \\
&\leq \lambda(F) + \frac{2e(G_{in})}{\mathbf{x}^{\mathrm{T}}\mathbf{x}} - \frac{2e(G_{out})(1-\frac{20k^2r^2}{n})^2}{\mathbf{x}^{\mathrm{T}}\mathbf{x}} \\
&\leq \lambda(F) + \frac{2(e(G_{in})-e(G_{out}))}{\mathbf{x}^{\mathrm{T}}\mathbf{x}} + \frac{2e(G_{out})\frac{40k^2r^2}{n}}{\mathbf{x}^{\mathrm{T}}\mathbf{x}} \\
&\leq \lambda(F) + \frac{2f(k-1,k-1)}{\mathbf{x}^{\mathrm{T}}\mathbf{x}} + \frac{\frac{720k^6r^4}{n}}{\mathbf{x}^{\mathrm{T}}\mathbf{x}} \qquad (3.11)
\end{aligned}$$

Using (3.10), (3.11) and $\mathbf{x}^{\mathrm{T}}\mathbf{x} \geq n(1-\frac{20k^2r^2}{n})^2 \geq n - 40k^2r^2$, we get

$$\begin{aligned}
&\lambda(T_{r-1}(n)) - \lambda(F) \\
\leq\ & \frac{2f(k-1,k-1)}{\mathbf{x}^{\mathrm{T}}\mathbf{x}} - \frac{2f(k-1,k-1)}{n} + \frac{4f(k-1,k-1)}{n^2} + \frac{\frac{720k^6r^4}{n}}{\mathbf{x}^{\mathrm{T}}\mathbf{x}} \\
\leq\ & \frac{2f(k-1,k-1)}{n-40k^2r^2} - \frac{2f(k-1,k-1)}{n} + \frac{4f(k-1,k-1)}{n^2} + \frac{\frac{720k^6r^4}{n}}{n-40k^2r^2} \\
\leq\ & \frac{80k^2r^2f(k-1,k-1)}{n(n-40k^2r^2)} + \frac{4f(k-1,k-1)}{n^2} + \frac{720k^6r^4}{n(n-40k^2r^2)} \\
\leq\ & \frac{c_2}{n^2},
\end{aligned}$$

where c_2 is a positive constant.

Combining with Claim 1, we have

$$\frac{c_1}{n} \leq \lambda(T_{r-1}(n)) - \lambda(F) \leq \frac{c_2}{n^2},$$

which is a contradiction when n is sufficiently large. Thus $||V_i| - |V_j|| \leq 1$ for any $1 \leq i < j \leq r-1$.

$\square$

Proof of Theorem 1.2.4: Now we prove that $e(G) = \mathrm{ex}(n, F_{k,r})$. Otherwise, we assume that $e(G) \leq \mathrm{ex}(n, F_{k,r}) - 1$. Let H be an $F_{k,r}$-free graph with $e(H) = \mathrm{ex}(n, F_{k,r})$ and $V(H) = V(G)$. By Lemma 3.2.10, we may assume that $V_1, \ldots, V_{r-1}$ induce a complete $(r-1)$-partite graph in H. Let $E_1 = E(G) \setminus E(H)$, $E_2 = E(H) \setminus E(G)$, then $E(H) = (E(G) \cup E_2) \setminus E_1$, and

$$|E(G) \cap E(H)| + |E_1| = e(G) < e(H) = |E(G) \cap E(H)| + |E_2|,$$

which implies that $|E_2| \geq |E_1| + 1$. Furthermore, by Lemma 3.2.8, we have

$$|E_2| \leq f(k-1, k-1) + \sum_{1 \leq i < j \leq r-1} |B_i||B_j| \leq k^2 + \binom{r-1}{2}(2k^2+1)^2 \leq 10k^4r^2. \quad (3.12)$$

According to (1.7) and (3.12), we deduce, for sufficiently large n, that

$$\begin{aligned}
\lambda(H) &\geq \frac{\mathbf{x}^{\mathrm{T}} A(H) \mathbf{x}}{\mathbf{x}^{\mathrm{T}} \mathbf{x}} \\
&= \frac{\mathbf{x}^{\mathrm{T}} A(G) \mathbf{x}}{\mathbf{x}^{\mathrm{T}} \mathbf{x}} + \frac{2\sum_{ij \in E_2} \mathbf{x}_i \mathbf{x}_j}{\mathbf{x}^{\mathrm{T}} \mathbf{x}} - \frac{2\sum_{ij \in E_1} \mathbf{x}_i \mathbf{x}_j}{\mathbf{x}^{\mathrm{T}} \mathbf{x}} \\
&= \lambda(G) + \frac{2}{\mathbf{x}^{\mathrm{T}} \mathbf{x}} \left(\sum_{ij \in E_2} \mathbf{x}_i \mathbf{x}_j - \sum_{ij \in E_1} \mathbf{x}_i \mathbf{x}_j \right) \\
&\geq \lambda(G) + \frac{2}{\mathbf{x}^{\mathrm{T}} \mathbf{x}} \left(|E_2|(1 - \frac{20k^2r^2}{n})^2 - |E_1| \right) \\
&\geq \lambda(G) + \frac{2}{\mathbf{x}^{\mathrm{T}} \mathbf{x}} \left(|E_2| - \frac{40k^2r^2}{n}|E_2| - |E_1| \right) \\
&\geq \lambda(G) + \frac{2}{\mathbf{x}^{\mathrm{T}} \mathbf{x}} \left(1 - \frac{40k^2r^2}{n}|E_2| \right) \\
&\geq \lambda(G) + \frac{2}{\mathbf{x}^{\mathrm{T}} \mathbf{x}} \left(1 - \frac{40k^2r^2}{n}10k^4r^2 \right) \\
&> \lambda(G),
\end{aligned}$$

which contradicting the assumption that G has the maximum spectral radius among all $F_{k,r}$-free graphs on n vertices. Hence $e(G) = \mathrm{ex}(n, F_{k,r})$. □

3.3 Concluding remarks

To avoid unnecessary calculations, we did not attempt to get the best bound on the order of graphs in the proof. It would be interesting to determine how large n needs to be for our result.

As we mentioned before, Theorem 1.2.2 says that Conjecture 2.5.1 holds for the k-fan graph F_k. Moreover, the result in [43] implies that Conjecture 2.5.1 also holds for the flower graph $H_{s,k}$, the graph defined by intersecting s triangles and k odd cycles of length at least 5 in exactly one common vertex. In addition, our main result (Theorem 1.2.4) tells us that Conjecture 2.5.1 also holds for the intersecting cliques $F_{k,r}$. Note that F_k, $H_{s,k}$ and $F_{k,r}$ are not edge-color-critical.

Chapter 4

SPECTRAL EXTREMAL GRAPHS FOR EVEN CYCLES

In this chapter we study the spectral Turán problem for even cycles and determine $\mathrm{spex}(n, C_{2k+2})$ and $\mathrm{SPEX}(n, C_{2k+2})$ for all $k \geq 2$, and sufficiently large n. The same proof also allows us to determine $\mathrm{spex}(n, \{C_{2k+1}, C_{2k+2}\})$ and $\mathrm{SPEX}(n, \{C_{2k+1}, C_{2k+2}\})$ for all $k \geq 2$, and sufficiently large n. This fully resolves both parts of Conjecture 1.3.1 of Nikiforov [56] in the affirmative. We record our results as Theorem 4.0.1 and 4.0.2 given below which we prove in subsequent sections.

Theorem 4.0.1. *Let $k \geq 2$ and n be sufficiently large, then* $\mathrm{SPEX}(n, C_{2k+2}) = \{S^+_{n,k}\}$.

Theorem 4.0.1 was proved to be true for $k = 2$ by Zhai and Lin [77]. We will make estimations on the Perron vector entries and use them to prove the remaining cases with $k > 2$. Next we will prove the following theorem which resolves Part(a) of the conjecture.

Theorem 4.0.2. *Let $k \geq 2$ and n be sufficiently large, then* $\mathrm{SPEX}(n, \{C_{2k+1}, C_{2k+2}\}) = \{S_{n,k}\}$.

4.1 Organization and Notation

For some fixed $k \geq 2$, let $H_k \in \mathrm{SPEX}(n, C_{2k+2})$ be a spectral extremal graph when forbidding the even cycle of size $2k+2$ and let $H_k' \in \mathrm{SPEX}(n, \{C_{2k+1}, C_{2k+2})$ be a spectral extremal graph when forbidding both C_{2k+1} and C_{2k+2}. All of our arguments in Sections 4.2 and 4.3 apply with identical proofs to both H_k and H_k' as we will only be using the fact that the graph is extremal and C_{2k+2}-free, and so for brevity we will only state the results for H_k until the proofs of Theorems 4.0.1 and 4.0.2 (Section 4.4).

Let v be the Perron vector of the adjacency matrix with maximum entry $\mathrm{v}_x = 1$, where v_u denotes the coordinate of v with respect to some vertex u. For any vertex u, let

$d(u) = d_1(u)$ denote its degree, $N_i(u)$ denote the set of vertices at distance i from u, with $d_i(u) := |N_i(u)|$. We will fix a small constant α below and we define L to be the following set of vertices of large weight, $L := \{u \in V(H_k)|\mathrm{v}_u > \alpha\}$. Similarly, let S denote its complementary set of vertices with small weights, $S := V(H_k)\backslash L = \{u \in V(H_k)|\mathrm{v}_u \leq \alpha\}$. Additionally, we will also use the following set in the proof of Lemma 4.3.1. Let $M := \{u \in V(H_k)|\mathrm{v}_u \geq \alpha/3\}$. Finally we define a subset of L called L' by $L' := \{u|\mathrm{v}_u \geq \eta\}$ where $\eta > \alpha$ is a constant defined below.

For some vertex u, let $L_i(u) := L \cap N_i(u)$, $S_i(u) := S \cap N_i(u)$ and $M_i(u) := M \cap N_i(u)$. If the vertex is unambiguous from context, we will use L_i, S_i and M_i instead. For two subsets $A, B \subset V(H_k)$, let $E(A, B)$ denote the subset of edges of $E(H_k)$, with one endpoint in A and the other in B, and let $e(A, B) := |E(A, B)|$.

With foresight, we choose η, ϵ, and α to be any positive constants satisfying

$$\eta < \left\{\frac{1}{k+1}, 1 - \frac{1}{16k^3}, \frac{1}{4} - \frac{1}{16k^2}\right\} \tag{4.1}$$

$$\epsilon < \min\left\{\frac{1}{16k^3}, \frac{\eta}{2}, \frac{\eta}{32k^3+2}\right\} \tag{4.2}$$

$$\alpha < \frac{\epsilon^2}{10k}. \tag{4.3}$$

We note that many of the above inequalities are redundant but we leave them so that it is easier to see exactly what inequalities we are using throughout the proofs. In Sections 4.2 and 4.3 we prove lemmas showing the structural properties of H_k and H'_k. We reiterate that every lemma applies to both H_k and H'_k with identical proofs and so the proofs are only written for H_k. In Section 4.4 we complete the proofs of Theorems 4.0.1 and 4.0.2.

4.2 Lemmas from spectral and extremal graph theory

In this section, we record several lemmas that we will use. Some calculations may only apply for n large enough without being explicitly stated. We start with a standard result from linear algebra which serves as a tool to bound the spectral radius of non-negative matrices.

Lemma 4.2.1. *For a non-negative symmetric matrix M, a non-negative non-zero vector* y *and a positive constant* c*, if* $M\mathrm{y} \geq c\mathrm{y}$ *entrywise, then* $\lambda(M) \geq c$*.*

Proof. Assume that $M\mathrm{y} \geq c\mathrm{y}$ entrywise, with the same assumptions for M, y and c as in the statement of the theorem. Then $\mathrm{y}^T M\mathrm{y} \geq \mathrm{y}^T c\mathrm{y}$ and we have a lower bound on $\lambda(M)$ from the Rayleigh quotient,

$$\lambda(M) \geq \frac{\mathrm{y}^T M\mathrm{y}}{\mathrm{y}^T\mathrm{y}} \geq c.$$

□

Next, we will use the Erdős-Gallai Theorem on the Turán number of a path [26].

Lemma 4.2.2. *Any graph on* n *vertices with no subgraph isomorphic to a path on* ℓ *vertices has at most* $\dfrac{(\ell-2)n}{2}$ *edges.*

We will be using Part B of the following lemma which appears as Lemma 1 in [51].

Lemma 4.2.3. *Suppose that* $k \geq 1$ *and let the vertices of a graph* G *be partitioned into two sets* U *and* W*.*

(A) If

$$2e(U) + e(U,W) > (2k-2)|U| + k|W|, \tag{4.4}$$

then there exists a path of order $2k$ *or* $2k+1$ *with both ends in* U*.*

(B) If

$$2e(U) + e(U,W) > (2k-1)|U| + k|W|, \tag{4.5}$$

then there exists a path of order $2k+1$ *with both ends in* U*.*

Let z be any vertex in H_k. Since our graph is C_{2k+2}-free, $N_1(z)$ may not contain a P_{2k+1}. Hence by Lemma 4.2.2 we have

$$e(N_1(z)) \leq \frac{2k-1}{2} d(z) < kn. \tag{4.6}$$

Similarly the bipartite subgraph between $N_1(z)$ and $N_2(z)$ may not contain a P_{2k+3}, otherwise there is a P_{2k+1} with both endpoints in $N_1(z)$ and hence a C_{2k+2}. Therefore, by Lemma 4.2.2 (forbidding P_{2k+3} in the bipartite subgraph) and Lemma 4.2.3 (forbidding P_{2k+1} with both endpoints in $N_1(z)$), we have

$$e(N_1(z), N_2(z)) \leq \min\left\{\frac{2k+1}{2}n, (2k-1)d(z) + k\left(n - d(z) - 1\right)\right\} \tag{4.7}$$

The spectral radius of $S_{n,k}$ gives a lower bound for $\lambda(H_k)$. We will modify the proof of a theorem of Nikiforov (Proof of Theorem 3 in [56]) to obtain an upper bound for $\lambda(H_k)$.

Lemma 4.2.4. $\sqrt{kn} \leq \frac{k-1+\sqrt{(k-1)^2+4k(n-k)}}{2} \leq \lambda(H_k) \leq \sqrt{2k(n-1)}$.

Proof. Here the inner lower bound is precisely $\lambda(S_{n,k})$. To prove the upper bound, let $u \in V(H_k)$ and consider Lemma 4.2.3 over the graph $H_k[N_1(u) \cup N_2(u)]$ with $U = N_1(u)$ and $W = N_2(u)$. We know that $|N_1(u)| = d(u)$. Also, there cannot be any path on $2k+1$ vertices in $H_k[N_1(u) \cup N_2(u)]$ with both end points in $N_1(u)$. So (4.5) implies that

$$\begin{aligned} 2e(H_k[N_1(u)]) + e(H_k[N_1(u), N_2(u)]) &\leq (2k-1)d(u) + kd_2(u) \\ &\leq (2k-1)d(u) + k(n - d(u) - 1) \\ &= (k-1)d(u) + k(n-1). \end{aligned} \tag{4.8}$$

Now, using the fact that the spectral radius of a non-negative matrix is at most the maximum row-sum, we can bound λ^2 in terms of the maximum row-sum of A^2 and then use (4.8).

$$\begin{aligned} \lambda^2 &\leq \max_{u \in V(G)} \Big\{ \sum_v A^2_{u,v} \Big\} = \max_{u \in V(G)} \Big\{ \sum_{v \in N(u)} d(v) \Big\} \\ &= \max_{u \in V(G)} \Big\{ d(u) + 2e(H_k[N_1(u)]) + e(H_k[N_1(u), N_2(u)]) \Big\} \\ &\leq kd(u) + k(n-1) \leq 2k(n-1). \end{aligned} \tag{4.9}$$

Thus, $\lambda \leq \sqrt{2k(n-1)}$. □

Next we upper bound the number of vertices in L. We use the same technique as in the proof of Lemma 8 in [68]. For this we will need the even-circuit theorem [72].

Lemma 4.2.5. $|L| \leq \dfrac{16k^{1/2}n^{(k+3)/(2k+2)}}{\alpha}$ *and* $|M| \leq \dfrac{48k^{1/2}n^{(k+3)/(2k+2)}}{\alpha}$.

Proof. For any vertex $u \in V(H_k)$, we have the following equation relating the spectral radius and Perron vector entries,

$$\lambda \mathrm{v}_u = \sum_{v \sim u} \mathrm{v}_v. \tag{4.10}$$

So,

$$\sqrt{kn}\mathrm{v}_u \le \lambda \mathrm{v}_u \le d_u,$$

and

$$\frac{|L|\sqrt{kn}\alpha}{2} \le \frac{1}{2}\sum_{u\in L}\lambda \mathrm{v}_u \le \frac{1}{2}\sum_{u\in L} d_u \le \frac{1}{2}\sum_{u\in V(H_k)} d_u \le \mathrm{ex}(n, C_{2k+2}) \le 8kn^{(k+2)/(k+1)}.$$

This gives

$$|L| \le \frac{16k^{1/2}n^{(k+3)/(2k+2)}}{\alpha}. \tag{4.11}$$

The bound for $|M|$ is obtained similarly by replacing α by $\alpha/3$ everywhere above.

□

We will also use the following result of Nikiforov (Theorem 2 of [51]) to further bound the size of L in Lemma 4.3.1.

Lemma 4.2.6. *Let G be a graph with n vertices and m edges. If G does not contain a C_{2k+2} then*

$$\sum_{u\in V(G)} d_G^2(u) \le 2km + k(n-1)n.$$

Using Lemma 4.2.4 we obtain the following lower bound for entries in the Perron vector of the extremal graphs by modifying a proof of Tait and Tobin (proof of Lemma 10 in [68]).

Lemma 4.2.7. *For any vertex $u \in V(H_k)$, $\mathrm{v}_u \ge \frac{1}{\lambda(H_k)} \ge \frac{1}{\sqrt{2k(n-1)}}$.*

Proof. Towards a contradiction, assume that there exists a vertex $u \in V(H_k)$, such that $\mathrm{v}_u < \frac{1}{\lambda(H_k)}$. Then by (4.10), u cannot be adjacent to any vertex x such that $\mathrm{v}_x = 1$. Let $\hat{H}_k$ be the graph obtained by modifying H_k by removing all the edges adjacent to u and making u adjacent to x. Then using the Rayleigh quotient, we have $\lambda(\hat{H}_k) > \lambda(H_k)$. But adding a vertex of degree 1 to a graph cannot create a cycle, $\hat{H}_k$ does not contain any subgraph isomorphic to C_{2k+2}, contradicting that H_k is extremal. □

4.3 Structural results for extremal graphs

In this section, we will assume that $k \geq 2$ is fixed. We will be working with subgraphs in H_k and due to lack of ambiguity we will drop H_k from some notations now onward. We will continue to use auxiliary constants α, ϵ, and η and we will frequently assume that n is larger than some constant depending only on $\alpha, k, \epsilon, \eta$. Every lemma in this section holds only for n large enough.

Lemma 4.3.1. *For any vertex $z \in L$, we have $d(z) \geq \frac{\alpha}{20k}n$. Also, $|L| \leq \frac{k+1}{(\alpha/20k)^2}$.*

Proof. For some vertex $z \in L$ such that $\mathrm{v}_z = c$, consider the following second degree eigenvalue-eigenvector equations relating λ^2, v, and entries in the z'th row of A^2:

$$\begin{aligned} knc \leq \lambda^2 c = \lambda^2 \mathrm{v}_z = \sum_{u\sim z}\sum_{w\sim u} \mathrm{v}_w &\leq d(z)c + 2(N_1(z)) + \sum_{u\sim z}\sum_{\substack{w\sim u\\ w\in N_2(z)}} \mathrm{v}_w \\ &\leq 2kd(z) + \sum_{u\sim z}\sum_{\substack{w\sim u\\ w\in N_2(z)}} \mathrm{v}_w, \end{aligned} \tag{4.12}$$

where the last inequality is by (4.6). Now assume to the contrary that there is a vertex $z \in L$ with $d(z) < \frac{\alpha}{20k}n$. Substituting this into the above equation and using $\alpha < c$ since $z \in L$, we have

$$(k-0.1)nc < \sum_{u\sim z}\sum_{\substack{w\sim u\\ w\in N_2(z)}} \mathrm{v}_w.$$

Next we show that many of the terms in the double sum come from vertices in $M_2(z)$ via the following claim.

Claim 4.3.1.1. There are at least $0.9nc$ terms v_w with $w \in M_2$ in the sum

$$\sum_{u\sim z}\sum_{\substack{w\sim u\\ w\in N_2(z)}} \mathrm{v}_w.$$

Proof. Assume to the contrary that there are less than $0.9nc$ terms v_w where $w \in M_2$. Then

$$(k-0.1)nc < \sum_{u\sim z}\sum_{\substack{w\sim u\\ w\in N_2(z)}} \mathrm{v}_w = \sum_{u\sim z}\sum_{\substack{w\sim u\\ w\in M_2}} \mathrm{v}_w + \sum_{u\sim z}\sum_{\substack{w\sim u\\ w\in S_2\setminus M_2}} \mathrm{v}_w < 0.9nc + e(N_1, S_2 \setminus M_2)\frac{\alpha}{3},$$

and so

$$(k-1)nc < e(N_1, S_2 \setminus M_2)\frac{\alpha}{3}.$$

Since $\alpha < c$ and by (4.7), we have

$$(k-1)n < \frac{2k+1}{2}\frac{1}{3}n,$$

a contradiction for $k \geq 2$. $\square$

Therefore $e(N_1(z), M_2(z)) \geq 0.9nc > 0.9n\alpha$. Then since $H_k[N_1(z) \cup M_2(z)]$ has no P_{2k+3}, by Lemma 4.2.2, we must have $0.9n\alpha \leq \frac{2k+1}{2}(|N_1(z) \cup M_2(z)|) < \frac{2k+1}{2}\left(\frac{n\alpha}{20k} + |M_2(z)|\right)$. So,

$$|M_2(z)| > \left(.9\alpha - \frac{(2k+1)\alpha}{40k}\right)\left(\frac{2}{2k+1}\right)n.$$

This contradicts the bound in Lemma 4.2.5 for n sufficiently large. Thus for n sufficiently large we have that $d(z) \geq \frac{\alpha}{20k}n$ for all $z \in L$. Combined with Lemma 4.2.6 this gives us that $|L| \leq \frac{k+1}{(\alpha/20k)^2}$.

$\square$

We now refine the lower bound on the degrees of vertices in L'.

Lemma 4.3.2. *If z is a vertex of L' with $\mathrm{v}_z = c$, then $d(z) \geq cn - \epsilon n$.*

Proof. Given $z \in L$ with $\mathrm{v}_z = c$, observe that

$$\begin{aligned} knc \leq \lambda^2 c = \sum_{u\sim z}\sum_{w\sim u} \mathrm{v}_w &= d(z)c + \sum_{u\sim z}\sum_{\substack{w\sim u\\ w\neq z}} \mathrm{v}_w \\ &\leq d(z)c + e(S_1, L_1 \cup L_2)) + 2e(S_1)\alpha + 2e(L) + e(S_1, L_1)\alpha + e(N_1, S_2)\alpha. \end{aligned}$$

Since N_1 is P_{2k+1}-free and the bipartite graph between N_1 and N_2 is P_{2k+3}-free,we have by (4.6) and (4.7) that

$$2e(S_1) + 2e(L_1) \leq 2e(N_1) < 2kn,$$

$$e(L_1, S_1) \leq e(N_1) < kn,$$

$$e(N_1, S_2) < 2kn.$$

Hence we have

$$\begin{aligned} knc &< d(z)c + \left(\sum_{\substack{u \sim z \\ u \in S_1}} \sum_{\substack{w \sim u \\ w \in L_1 \cup L_2}} \mathrm{v}_w \right) + 5kn\alpha \leq d(z)c + e(S_1, L_1 \cup L_2) + 5kn\alpha \\ &< d(z)c + e(S_1, L_1 \cup L_2) + \frac{\epsilon^2 n}{2}, \end{aligned} \tag{4.13}$$

by the choice of α in (4.3).

So, if $d(z) \leq (c - \epsilon)n$ for some $\epsilon > 0$, then

$$(k - c + \epsilon)nc \leq (kn - d(z))c \leq e(S_1, L_1 \cup L_2) + \frac{\epsilon^2 n}{2}. \tag{4.14}$$

Since $z \in L'$ we have $c \geq \epsilon$. Rearranging and using $\epsilon \leq c \leq 1$, we have

$$e(S_1, L_1 \cup L_2) \geq (k-1)nc + \frac{\epsilon^2 n}{2}. \tag{4.15}$$

We will show that $H_k[S_1, L_1 \cup L_2]$ contains a P_{2k+1} with both endpoints in S_1, thus contradicting the fact that H_k is C_{2k+2}-free. To show this we prove the following claim.

Claim 4.3.2.1. Let $\delta := \frac{\epsilon(\alpha/20k)^2}{k+1}$. Then there are δn vertices inside S_1 with degree at least k in $H_k[S_1, L_1 \cup L_2]$.

Proof. Assume to the contrary that at most δn vertices in S_1 have degree at least k in $H_k[S_1, L_1 \cup L_2]$. Then, $e(S_1, L_1 \cup L_2) < (k-1)|S_1| + |L|\delta n \leq (k-1)(c-\epsilon)n + \epsilon n$, because $|S_1| \leq d(z)$ and by Lemma 4.3.1. This contradicts (4.15). □

Hence, there is some subset of vertices $B \subset S_1$ such that any vertex in B has degree at least k in $H_k[S_1, L_1 \cup L_2]$ and $|B| = \delta n$. Since there are only $\binom{|L|}{k}$ options for every vertex in B to choose a set of k neighbors from, we have that there is some set of k vertices in $L_1 \cup L_2$ with at least $\delta n / \binom{|L|}{k}$ common neighbours in B. Therefore, by Lemma 4.2.2 and Lemma 4.3.1, for n sufficiently large we have a path on $2k+1$ vertices with both end points in the common neighbourhood contained in B, a contradiction. □

Thus, for the vertex x such that $\mathrm{v}_x = 1$, we have $d(x) \geq n - \epsilon n$ and $N_1(x)$ contains all but at most ϵn many vertices. Since every vertex in L' has degree more than ϵn (by the definition of L' and Lemma 4.3.2), this also gives that $L' \setminus \{x\} \subset L_1(x) \cup L_2(x)$. The arguments in the proof of Lemma 4.3.2 also allow us to show that all vertices of L' have degrees close to n and thus obtain $|L'| = k$.

Lemma 4.3.3. *For any vertex $z \in L'$ with $\mathrm{v}_z \geq 1 - \epsilon$, we have $(k - 2\epsilon)n \leq e(S_1, L) \leq (k + \epsilon)n$.*

Proof. To obtain the lower bound we refine (4.13). Using $1 - \epsilon \leq \mathrm{v}_z \leq 1$ and $d(z) \leq n$, we get

$$kn(1 - \epsilon) < d(z) + e(S_1, L_1 \cup L_2) + \frac{\epsilon^2 n}{2} = e(S_1, L) + \frac{\epsilon^2 n}{2}.$$

Thus $e(S_1, L) > (1 - 2\epsilon)kn$.

To obtain the upper bound, assume to the contrary that $e(S_1, L) > kn + \epsilon n$. We will show that $H_k[S_1, L \setminus \{z\}]$ contains a P_{2k+1} with both endpoints in S_1, thus contradicting the fact that H_k is C_{2k+2}-free. To show this we prove the following claim.

Claim 4.3.3.1. Let $\delta := \frac{\epsilon(\alpha/20k)^2}{k+1}$. Then there are δn vertices inside S_1 with degree at least k in $H_k[S_1, L \setminus \{z\}]$.

Proof. Assume to the contrary that at most δn vertices of S_1 have degree at least k in $H_k[S_1, L \setminus \{z\}]$. Then, $e(S_1, L \setminus \{z\}) < (k-1)|S_1| + |L|\delta n) \leq (k-1)n + |L|\delta n$, because $|S_1| \leq d(z)$. This contradicts our assumption that $e(S_1, L) \geq kn + \epsilon n$. □

Hence, there is some subset of vertices $B \subset S_1$ such that any vertex in B has degree at least k in $H_k[S_1, L \setminus \{z\}]$ and $|B| = \delta n$. Since there are only $\binom{|L|}{k}$ options for every vertex in B to choose a set of k neighbors from, we have that there is some set of k vertices in $L \setminus \{x\}$ with at least $\delta n / \binom{|L|}{k}$ common neighbours in B. Therefore, by Lemma 4.2.2 and Lemma 4.3.1, for n sufficiently large we have a path on $2k + 1$ vertices with both end points in the common neighbourhood contained in B, a contradiction.

□

Lemma 4.3.4. *For all vertices $z \in L'$, we have $d(z) \geq \left(1 - \frac{1}{8k^3}\right) n$ and $\mathrm{v}_z \geq 1 - \frac{1}{16k^3}$. Moreover, $|L'| = k$.*

Proof. If we show that every vertex $z \in L'$ has Perron entry $\mathrm{v}_z \geq 1 - \frac{1}{16k^3}$, then it follows from Lemma 4.3.2 and (4.2) that $d(z) \geq \left(1 - \frac{1}{8k^3}\right) n$. If all vertices in L' have degree at least $n - \frac{n}{8k^3}$, then $|L'| \leq k$, else there exists a $K_{k+1,k+1}$ in H_k, a contradiction. Also, if $|L'| \leq k-1$, then by (4.13) and Lemma 4.3.3, we have

$$\begin{aligned} kn = kn\mathrm{v}_x \leq \lambda^2 &\leq e(S_1(x), L'(x)) + e(S_1(x), L(x) \setminus L'(x))\eta + \frac{\epsilon^2 n}{2} \\ &\leq (k-1)n + (k+\epsilon)n \cdot \eta + \frac{\epsilon^2 n}{2}, \end{aligned} \tag{4.16}$$

a contradiction by (4.1) and (4.2). Hence, all we need to show is that every vertex in L' has Perron entry at least $1 - \frac{1}{16k^3}$.

By way of contradiction, assume that $z \in L'$ and $\mathrm{v}_z < 1 - \frac{1}{16k^3}$. Refining (4.13) applied to the vertex x we have

$$\begin{aligned} kn &< e(S_1(x), L_1(x) \setminus \{z\}) + |N_1(x) \cap N_1(z)|\mathrm{v}_z + \frac{\epsilon^2 n}{2} \\ &\leq (k+\epsilon)n - |N_1(x) \cap N_1(z)| + \left(1 - \frac{1}{16k^3}\right)|N_1(x) \cap N_1(z)| + \frac{\epsilon^2 n}{2} \\ &= kn + \epsilon n + \frac{\epsilon^2 n}{2} - \frac{|N_1(x) \cap N_1(z)|}{16k^3}, \end{aligned}$$

where the second inequality is by Lemma 4.3.3 and the bound on v_z. Therefore we have

$$\frac{|N_1(x) \cap N_1(z)|}{16k^3} < 2\epsilon n.$$

On the other hand, since $z \in L'$ we have $\mathrm{v}_z \geq \eta$ and so by Lemma 4.3.2 we have $|N_1(x) \cap N_1(z)| \geq (\eta - 2\epsilon)n$. Combining the two inequalities is a contradiction by (4.2). □

Now that we have $|L'| = k$ and every vertex in L' has degree at least $\left(1 - \frac{1}{8k^3}\right) n$, it follows that the common neighborhood of L' has size at least $\left(1 - \frac{1}{8k^2}\right) n$. That is, there are at most $\frac{n}{8k^2}$ vertices not adjacent to all of L'. Call this set of "exceptional vertices" E. That is,

$$E := \{v \in V(H_k) \setminus L' : |N_1(v) \cap L'| \leq k-1\}.$$

Let $R = V(H_k) \setminus (L' \cup E)$ be the remaining vertices. So we have that $V(H_k)$ is the disjoint union of L', R, and E with $|L'| = k$ and $|E| \leq \frac{n}{8k^2}$. In the next two lemmas we will show that $E = \emptyset$ and this will allow us to prove Theorems 4.0.1 and 4.0.2. Note that because R has size larger than $2k+2$, adding a new vertex adjacent only to the vertices in L' cannot create a C_{2k+2}, otherwise there would have already been one.

Lemma 4.3.5. *For any vertex $u \in V(H_k)$, the Perron weight in the neighborhood of u satisfies $\sum_{w\sim u} \mathrm{v}_w \geq k - \frac{1}{16k^2}$.*

Proof. Assume to the contrary that there exists a vertex u with $\sum_{w\sim u} \mathrm{v}_w < k - \frac{1}{16k^2}$. Note that because $\sum_{w\sim u} \mathrm{v}_w = \lambda \mathrm{v}_u \geq \sqrt{kn}\mathrm{v}_u$, we have that $u \notin L'$. Now modify the neighborhood of u by deleting all the edges adjacent to u and joining u to all the vertices of L'. Call the resultant graph, H_k^*. The neighborhood of u in H_k^* has Perron weight at least $k - \frac{1}{16k^2}$ by Lemma 4.3.4 thus, $\lambda(H_k^*) > \lambda(H_k)$ by the Rayleigh quotient. Moreover H_k^* does not contain any C_{2k+2}, a contradiction. □

With this we may show that E is empty.

Lemma 4.3.6. *The set E is empty and H_k contains the complete bipartite graph $K_{k,n-k}$.*

Proof. Assume to the contrary that $E \neq \emptyset$. Note that $e(R) \leq 1$ and every vertex in E has at most 1 neighbor in R, else we can embed a C_{2k+2} in H_k. Any vertex $r \in R$, satisfies $\mathrm{v}_r < \eta$. Therefore, for any vertex $u \in E$ we have that

$$\sum_{u\sim w} \mathrm{v}_w = \lambda \mathrm{v}_u = \sum_{\substack{w\sim u \\ w\in L'\cup R}} \mathrm{v}_w + \sum_{\substack{w\sim u \\ w\in E}} \mathrm{v}_w.$$

By Lemma 4.3.5 and using that vertices in E have at most $k-1$ neighbors in L', we have

$$\sum_{\substack{w\sim u \\ w\in E}} \mathrm{v}_w \geq 1 - \frac{1}{16k^2} - \eta, \tag{4.17}$$

Since the Perron weight in E is at least $1 - \frac{1}{16k^2} - \eta > \frac{3}{4}$, the Perron weight outside of E is at most $k - 1 + \eta$, and the total Perron weight is $\lambda \mathrm{v}_u$, we have that

$$\sum_{\substack{w\sim u \\ w\in E}} \mathrm{v}_w \geq \frac{3}{4k} \lambda \mathrm{v}_u$$

Now, applying Lemma 4.2.1, with $M = A(H_k[E])$, and $\mathrm{y} = \mathrm{v}_{|_E}$ (the restriction of v to the set E), we have that for any $u \in E$,

$$M\mathrm{y}_u = \sum_{\substack{w \sim u \\ w \in E}} \mathrm{v}_w \geq \frac{3}{4k}\lambda \mathrm{v}_u = \frac{3}{4k}\lambda \mathrm{y}_u. \tag{4.18}$$

Hence, by Lemma 4.2.1, $\lambda(M) \geq \frac{3}{4k}\lambda \geq \frac{3}{4}\sqrt{\frac{n}{k}}$. This contradicts Lemma 4.2.4 because $\lambda(M) \leq \sqrt{2k(|E|-1)} < \sqrt{\frac{n}{4k}}$ as E induces a C_{2k+2}-free graph. Thus, $E = \emptyset$. Therefore $R = V(H_k) \setminus L'$ and H_k must contain a $K_{k,n-k}$. □

4.4 Proofs of Theorems 4.0.1 and 4.0.2

With Lemma 4.3.6 in hand we may complete the proofs of Theorems 4.0.1 and 4.0.2. Lemma 4.3.6 gives us that $K_{k,n-k}$ is a subgraph of both H_k and H'_k and the results follow quickly from this.

Proof of Theorem 4.0.1. We have shown that $K_{k,n-k} \subset H_k$, where the part with k vertices is L' and the other part with $n-k$ vertices is R. Thus $E(L', R) = \{lr | l \in L', r \in R\}$. Now we know that $e(R) \leq 1$ and $H_k[L']$ is isomorphic to some subgraph of K_k. Thus, H_k is a subgraph of $S^+_{n,k}$ and by the monotonicity of the spectral radius over subgraphs, we have that $H_k \cong S^+_{n,k}$. □

Proof of Theorem 4.0.2. We have $E(L', R) = \{lr | l \in L', r \in R\}$. In addition, $e(R) = 0$, otherwise we can embed a C_{2k+1}. Also, $H_k[L']$ is isomorphic to some subgraph of K_k. Thus, H_k is a subgraph of $S_{n,k}$ and by the monotonicity of the spectral radius over subgraphs, we have that $H_k \cong S_{n,k}$. □

4.5 Conclusion

It would be interesting to see if the techniques used here can be used to determine $\mathrm{SPEX}(n, T)$, where T is any tree. In particular the techniques in this chapter may be used to investigate the following conjecture of Nikiforov [56]. This conjecture is a spectral analog of the famous Erdős-Sós conjecture made in the early 1960s (see [25]) and still open.

Conjecture 4.5.1 Conjecture 16 in [56]**.** *Let $k \geq 2$ and G be a graph of sufficiently large n.*

(a) if $\lambda(G) \geq \lambda(S_{n,k})$ then G contains all trees of order $2k+2$ unless $G = S_{n,k}$;

(b) if $\lambda(G) \geq \lambda(S^+_{n,k})$ then G contains all trees of order $2k+3$ unless $G = S^+_{n,k}$.

Chapter 5

LOWER BOUNDS FOR THE SPECTRAL RADIUS OF THE CONE OF A GRAPH

In this chapter, we study an important operation over graphs on n-vertices, called the cone of a graph. We obtain lower bounds on the increase in spectral radius under such an operation, and characterize the graphs achieving this extremal increase in spectral radius. This research was prompted by conjectures of Professor Saieed Akbari discussed with Professor Sebastian Cioabă [2].

5.1 A new conjecture on spectral radius answered

Fix an arbitrary connected graph $G = (V, E)$. Let $A = A(G)$ denote the adjacency matrix of G with respect to some arbitrarily fixed ordering on the vertices. Say A has spectral radius $\lambda = \lambda(G)$ and Perron vector $\mathbf{x} = [x_v]_{v\in V}$, corresponding to λ. Let $G^* := G \vee K_1 = G \vee \{z\}$ be the graph obtained by joining a new vertex z to G (also called the *cone* of G). Let us use $A^* = A^*(G)$ to denote the adjacency matrix of G^*, where we are ordering the matrix so that z corresponds to the last row and column. Also let λ^* denote the spectral radius of G^*. We will prove the following result for the spectral radius of G^*.

Theorem 1.4.1. Let G be a graph on n vertices, then $\lambda + 1 \leq \lambda^*$ with equality holding if and only if $G \cong K_n$.

It is not difficult to see that for $G \cong K_n$, we have $G^* \cong K_{n+1}$ and $\lambda^* = \lambda+1$. We will provide some more notation before beginning with the rest of the proof. Let $X := \sum_{v\in V} x_v$ and $X_{(2)} := \sum_{v\in V} x_v^2$. As is common practice, we use $\mathbb{1}_n$ to denote the n-coordinate column vector with all entries equal to 1.

Proof of Theorem 1.4.1. Let $M := \max\{x_v | v \in V\}$. Now consider the vector $\hat{\mathbf{x}} = [\mathbf{x}^T, M]^T$. Then,

$$\lambda^* \geq \frac{\hat{\mathbf{x}}^T A^* \hat{\mathbf{x}}}{\hat{\mathbf{x}}^T \hat{\mathbf{x}}}, \text{ where } A^* = \begin{bmatrix} A & \mathbb{1}_n \\ \mathbb{1}_n^T & 0 \end{bmatrix}. \tag{5.1}$$

It follows from (5.1) that

$$\begin{aligned} \lambda^* &\geq \frac{\mathbf{x}^T A \mathbf{x} + 2\mathbf{x}^T (M \mathbb{1}_n)}{\mathbf{x}^T \mathbf{x} + M^2} \\ &= \frac{\lambda \mathbf{x}^T \mathbf{x} + 2MX}{X_{(2)} + M^2} = \frac{\lambda X_{(2)} + 2MX}{X_{(2)} + M^2} \\ &= \frac{\lambda X_{(2)} + 2MX + \lambda M^2 - \lambda M^2}{X_{(2)} + M^2} = \lambda + \frac{M(2X - \lambda M)}{X_{(2)} + M^2}. \end{aligned} \tag{5.2}$$

Thus, to prove our theorem, it is sufficient to show that

$$\frac{M(2X - \lambda M)}{X_{(2)} + M^2} \geq 1.$$

Let u_0 be a vertex of G such that $x_{u_0} = M$. Then,

$$\begin{aligned} \lambda M = \sum_{v \sim_G u_0} x_v &= \sum_{v \sim_G u_0} x_v + x_{u_0} - x_{u_0} \\ &\leq X - M, \end{aligned}$$

and

$$\lambda \leq \frac{X - M}{M}. \tag{5.3}$$

So,

$$\frac{M(2X - \lambda M)}{X_{(2)} + M^2} \geq \frac{M(2X - (\frac{X-M}{M})M)}{X_{(2)} + M^2} = \frac{MX + M^2}{X_{(2)} + M^2} \geq 1. \tag{5.4}$$

The first inequality follows using (5.3). To observe the last inequality, note that for any $u \in V$, $Mx_u \geq x_u^2$ since $M = \max\{x_v | v \in V\}$ and thus $MX \geq X_{(2)}$.

Combining equations (5.2) and (5.4) gives $\lambda^* \geq \lambda + 1$. To show that for equality to occur, we need $G \cong K_n$, note that $\hat{\mathbf{x}}$ must be a non-negative eigenvector of A^* (consequently of G^*), as we need equality in (5.1). Also, for any $u \in V$,

$$\lambda x_u = \sum_{v \sim_G u} x_v, \tag{5.5}$$

$$(\lambda + 1)x_u = \lambda^* \hat{x}_u = \sum_{v \sim_{G^*} u} \hat{x}_v = \sum_{v \sim_G u} x_v + \hat{x}_z = \sum_{v \sim_G u} x_v + M \tag{5.6}$$

Subtracting (5.5) from (5.6) gives $x_u = \hat{x}_u = M$, where u is any arbitrary vertex of G. Thus $\hat{\mathbf{x}} = M\mathbb{1}_{n+1}$ which implies that G^* must be a regular graph. Since z has degree n in G^*, this proves the result. □

5.2 The spectral radius of the cone of a multigraph

In this section we will try to generalize our result to the case when $G = (V, E)$ is some loopless multigraph, with the constraint that the maximum number of edges between any pair of vertices is m. Thus, its adjacency matrix, $A = ((A_{u,v}))_{u,v \in V}$, has maximum entry equal to m. Like before, $G^* := G \vee K_1 = G \vee \{z\}$ denotes the cone of G and A^* its adjacency matrix, similar to how we used it previously. Recall that $\lambda = \lambda(G)$ denotes the spectral radius of G and $\lambda^* = \lambda(G^*)$ is the spectral radius of the cone of G. For calculation purposes, we will use **x** to denote the normalized Perron vector of G with respect to λ, such that $M := \max\{x_v | v \in V\} = m$, that is we are taking the maximum entry of the Perron vector, M, to be equal to the maximum number of edges between any pair of distinct vertices, m. We have the following result lower bounding $\lambda^* - \lambda$, in this setting.

Theorem 1.4.2. Let G be a loopless multigraph on n vertices where any pair of vertices have at most m edges between them, then $\lambda + \frac{1}{m} \leq \lambda^*$ with equality holding if and only if $G \cong K_n$, $m = 1$.

One may observe that as $n \to \infty$, $\lambda^* \to \lambda + \frac{1}{m}$ for G isomorphic to the n vertex graph with exactly m edges between any pair of distinct vertices. Thus the bound in Theorem 1.4.2 is asymptotically achieved. Before we begin with the proof, recall that X and $X_{(2)}$ denote $\sum_{v \in V} x_v$ and $\sum_{v \in V} x_v^2$, respectively.

Proof of Theorem 1.4.2. We begin by constructing the vector $\hat{\mathbf{x}} = [\mathbf{x}^T, 1]^T$. Then,

$$\lambda^* \geq \frac{\hat{\mathbf{x}}^T A^* \hat{\mathbf{x}}}{\hat{\mathbf{x}}^T \hat{\mathbf{x}}}, \text{ where } A^* = \begin{bmatrix} A & \mathbb{1}_n \\ \mathbb{1}_n^T & 0 \end{bmatrix}. \tag{5.7}$$

It follows from (5.7) that

$$\begin{aligned} \lambda^* &\geq \frac{\mathbf{x}^T A \mathbf{x} + 2\mathbf{x}^T(\mathbb{1}_n)}{\mathbf{x}^T \mathbf{x} + 1} \\ &= \frac{\lambda \mathbf{x}^T \mathbf{x} + 2X}{X_{(2)} + 1} = \frac{\lambda X_{(2)} + 2X}{X_{(2)} + 1} \\ &= \frac{\lambda X_{(2)} + 2X + \lambda - \lambda}{X_{(2)} + 1} = \lambda + \frac{2X - \lambda}{X_{(2)} + 1}. \end{aligned} \tag{5.8}$$

Thus, to prove our theorem, it is sufficient to show that

$$\frac{2X - \lambda}{X_{(2)} + 1} \geq \frac{1}{m}.$$

Let u_0 be a vertex of G such that $x_{u_0} = m$. Then,

$$\begin{aligned} \lambda m &= \sum_{v \sim_G u_0} A_{v,u_0} x_v \\ &= \sum_{v \sim_G u_0} A_{v,u_0} x_v + m x_{u_0} - m x_{u_0} \\ &\leq mX - m^2, \end{aligned}$$

and

$$\lambda \leq X - m. \tag{5.9}$$

So,

$$\frac{2X - \lambda}{X_{(2)} + 1} \geq \frac{2X - (X - m)}{X_{(2)} + 1} = \frac{X + m}{X_{(2)} + 1} \geq \frac{X + m}{mX + 1} \geq \frac{1}{m}. \tag{5.10}$$

The first inequality follows using (5.9). To observe the second inequality, note that for any $u \in V$, $mx_u \geq x_u^2$ since $m = \max\{x_v | v \in V\}$ and thus $mX \geq X_{(2)}$.

Combining equations (5.8) and (5.10) gives $\lambda^* \geq \lambda + \frac{1}{m}$. To show that for equality to occur, we need $G \cong K_n$, note that $\hat{\mathbf{x}}$ must be a non-negative eigenvector of A^* (consequently of G^*), as we need equality in (5.7). Also, for any $u \in V$,

$$\lambda x_u = \sum_{v \sim_G u} x_v, \tag{5.11}$$

$$(\lambda+1)x_u = \lambda^* \hat{x}_u = \sum_{v \sim_{G^*} u} \hat{x}_v = \sum_{v \sim_G u} x_v + \hat{x}_z = \sum_{v \sim_G u} x_v + 1 \tag{5.12}$$

Subtracting (5.11) from (5.12) gives $x_u = \hat{x}_u = 1$, where u is any arbitrary vertex of G. Thus $\hat{\mathbf{x}} = \mathbb{1}_{n+1}$ and so $m = 1$. This implies that G^* must be a regular graph, and there is at most one edge between any pair of vertices of G. Since z has degree n in G^*, this proves the result. □

5.3 A generalization for regular graphs

We have so far considered the joins of (multi)graphs with a K_1. We will now be looking at a generalization of cones for simple connected regular graphs, where we instead only attach a portion of the vertices to the K_1. For a graph $G = (V, E)$ on n vertices along with some ordering, $v_1, \ldots, v_n$, on its vertices, choose some $0 \leq p \leq 1$ so that $G^{*_p} = (V^{*_p}, E^{*_p})$ denotes the graph obtained from G by adding a new vertex to V that is adjacent to only the first pn vertices of G. Thus, $V^{*_p} = V \cup \{z\}$ and $E^{*_p} = E \cup \{v_i z | 1 \leq i \leq pn\}$.

Let $\lambda^{*_p} := \lambda(G^{*_p})$. Modifying the proof of Theorem 1.4.1 gives the following lower bound for $\lambda^{*_p} - \lambda$.

Theorem 5.3.1. *Let G be a connected regular graph on n vertices. For $0 \leq p \leq 1$, we have $\lambda + p^2 \leq \lambda^{*_p}$ with equality if and only if $p = 0$ or $G \cong K_n$ and $p = 1$.*

Before we begin with the proof, note that the lower bound is tight for $p = 0$ for any connected graph G, and for K_n with $p = 1$, by Theorem 1.4.1. Therefore, it remains only to show that $\lambda + p^2$ is indeed a lower bound for λ^{*_p} when $0 < p < 1$, with equality impossible.

Proof. We know that $\mathbb{1}_n$ is a Perron vector of G since G is a connected regular graph. Now construct the vector $\hat{\mathbf{x}} = [\mathbb{1}_n^T, \hat{x}_z = p]^T$. Let $\mathbb{1}_P$ denote the characteristic column vector for P. Then,

$$\lambda^* \geq \frac{\hat{\mathbf{x}}^T A^* \hat{\mathbf{x}}}{\hat{\mathbf{x}}^T \hat{\mathbf{x}}}, \text{ where } A^* = \begin{bmatrix} A & \mathbb{1}_P \\ \mathbb{1}_P^T & 0 \end{bmatrix} \text{ for some set of } pn \text{ vertices } P \subset V. \tag{5.13}$$

Thus,

$$\begin{aligned}
\lambda^* &\geq \frac{\mathbb{1}_n^T A \mathbb{1}_n + 2\mathbb{1}_P^T(p\mathbb{1}_n)}{\mathbb{1}_n^T \mathbb{1}_n + p^2} \\
&= \frac{\lambda \mathbb{1}_n^T \mathbb{1}_n + 2p^2 n}{\mathbb{1}_n^T \mathbb{1}_n + p^2} = \frac{\lambda n + 2p^2 n}{n + p^2} \\
&= \frac{\lambda n + 2p^2 n + \lambda p^2 - \lambda p^2}{n + p^2} = \lambda + \frac{2p^2 n - \lambda p^2}{n + p^2}. \\
&\geq \lambda + \frac{p^2(n+1)}{n+p^2} \geq \lambda + p^2.
\end{aligned} \tag{5.14}$$

□

We should note here that this is not the best possible lower bound one may find for bounding the spectral radius for λ^{*_p}. While the bound is tight for $p = 1$, and $p = p^2$ for $p = 1$, we have $p - p^2 > 0$ for $0 < p < 1$ and since we cannot extend our eigenvectors in the same way we have been doing in previous sections, for arbitrary regular graphs, this method may not yield the tightest bounds for $p \neq 0, 1$. Nonetheless, we have obtained the bound in Theorem 5.3.1.

5.4 Concluding remarks

In the previous sections, we have joined all the vertices of G with a K_1. It may be a good idea to ask what happens if we only join some 'nice' set of vertices of G to the K_1. In the proof of Theorem 1.4.1, we make use of the fact that the neighborhood of z is all of G and so the last column of A^* is $[\mathbb{1}_n^T, 0]^T =: C_{n+1}^*$. So we get that $\hat{\mathbf{x}}^T C_{n+1}^* = X$.

It would be interesting to obtain a generalization of our result for connected graphs when we join the K_1 to a portion of the vertices of G to obtain G', such that $\hat{\mathbf{x}}^T C'_{n+1} = cX$, for some $0 \leq c \leq 1$; that is, can we obtain some tight lower bounds for the increase in spectral radius, so that $\lambda + f(c) \leq \lambda'$ for some function f? We know now that $f(1) = 1$ and one may also observe that $f(0) = 0$ because $G' = G \cup K_1$ then. Perhaps modifying (5.2) with an appropriate substitute of M may help in this direction, for some $c \in (0, 1)$.

Another similar question to consider is what happens if G is a k-partite graph and we only join a few of the color classes to a K_1 to obtain the graph G''. Can we get some nice lower bounds on the increase in spectral radius in this case as well in terms of k? It may help to start with the case of bipartite graphs, G, with parts S and T where we only join all the vertices of S with a K_1. Thus G'' remains a bipartite graph. It may help us to observe from Cioabă [15] that $\sum_{v\in S} x_v^2 = \frac{X_{(2)}}{2}$ and $\sum_{v\in S}(x''_v)^2 = \frac{X''_{(2)}}{2}$ where $[x''_v]_{v\in V(G'')}$ is a Perron vector of G'' and $X'' := \sum_{v\in V(G'')}(x''_v)^2$. It may be helpful to somehow bound X in terms of $X_{(2)}$ and apply a modification of (5.2).

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