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Two-order superconvergence for a weak Galerkin method on rectangular and cuboid grids

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Abstract

This article introduces a particular weak Galerkin (WG)element on rectangular/cuboid partitions that uses kth order polynomial for weak finite element functions and (k+1)th order polynomials for weak derivatives. This WG elementis highly accurate with convergence two orders higher thanthe optimal order in an energy norm and the L^2 norm. The superconvergence is verified analytically and numerically. Furthermore, the usual stabilizer in the standard weak Galerkin formulation is no longer needed for this element.

Keywords
finite element, weak Galerkin method, second-order elliptic problems, stabilizer-free, rectangular mesh

1 | Introduction

The weak Galerkin finite element method is an effective and flexible numerical technique for solving partial differential equations. It is a natural extension of the standard Galerkin finite element method where classical derivatives were substituted by weakly defined derivatives on functions with discontinuity. The WG method was first introduced in [16, 17] and then has been applied to solve various PDEs such as second order elliptic equations, biharmonic equations, Stokes equations, Navier–Stokes equations, Brinkman equations, parabolic equations, Helmholtz equation, convection dominant problems, hyperbolic equations, and Maxwell's equations [3, 5–15, 18].

The main idea of weak Galerkin finite element methods is the use of weak functions and their corresponding weak derivatives in algorithm design. For the second order elliptic equation, weak functions have the form of $v = \{v_0, v_b\}$ with $v = v_0$ inside of each element and $v = v_b$ on the boundary of the element. Both v_0 and v_b can be approximated by polynomials in $P_\ell(T)$ and $P_s(e)$ respectively, where T stands for an element and e the edge or face of T, ℓ and s are non-negative integers with possibly different values. Weak derivatives are defined for weak functions in the sense of distributions. For example, one may approximate a weak gradient in the polynomial space $[P_m(T)]^d$. Various combination of $\left(P_\ell(T), P_s(e), [P_m(T)]^d\right)$ leads to different weak Galerkin methods tailored for specific partial differential equations.

In this article, we introduce a new WG method with the combination of $(P_k(T), P_k(e), [P_{k+1}(T)]^d)$ on rectangular/cuboid mesh because it outperforms others for solving the second order elliptic problem:

$$-\nabla \cdot (a\nabla u) = f \quad \text{in} \;\; \Omega, \tag{1}$$

$$u = g \quad \text{on} \;\; \partial\Omega, \tag{2}$$

where Ω is a convex polytopal domain in $\mathbb{R}^d$, $f \in L^2(\Omega)$, $g \in H^{1/2}(\partial\Omega)$, and a is a symmetric positive definite matrix of piecewise constants.

The standard weak Galerkin finite element formulation for the problem (1)–(2) has the form: find $u_h \in V_h$ such that $u_h = Q_b g$ on $\partial\Omega$ and satisfies

$$(a\nabla_w u_h, \nabla_w v) + s\,(u_h, v) = (f, v) \quad \forall v \in V_h^0, \tag{3}$$

where $s(\cdot,\cdot)$ is a parameter independent stabilizer. Most of the finite element methods with discontinuous approximation have one or more stabilizing terms to guarantee the well posedness and the convergence of the methods.

First we will prove that our new WG method has the following ultra-simple formulation, without a stabilizer $s(\cdot,\cdot)$:

$$(a\nabla_w u_h, \nabla_w v) = (f, v) \quad \forall v \in V_h^0. \tag{4}$$

Removing stabilizers from weak Galerkin finite element methods will simplify the formulation and reduce programming complexity. Stabilizer-free WG finite element methods have been studied in [1, 20]. The idea is to increase the connectivity of a weak function across element boundary by raising the degree of polynomials for computing weak derivatives. In [20], it has been proved that for a WG element $\left(P_k(T), P_k(e), \left[P_j(T)\right]^d\right)$, the condition of $j \geq k+n-1$ guarantees a stabilizer-free WG method, where n is the number of edges/faces of an element. Such condition has been improved in [1]. The results in [1, 20] have been extended in this article. The optimal order error estimates are established for the corresponding WG approximations in both a discrete H^1 norm and the L^2 norm, in [20].

Secondly, we will prove that this WG finite element $\left(P_k(T), P_k(e), [P_{k+1}(T)]^d\right)$ on rectangle/cuboid is highly accurate with two order higher convergence rates than the optimal order in an energy norm and in the L^2 norm, theoretically and numerically. This is one of surprising numerical results reported in [19].

The WG element $\left(P_k(T), P_k(e), [P_{k+1}(T)]^d\right)$ on rectangular/cuboid mesh is closely related to the *BDFM* (Brezzi–Douglas–Fortin–Marini) element [2] in the standard mixed finite element method. Owing to Lemma 7, the two methods are in fact identical for diffusive coefficient a that assumes constant value on each element. Our superconvergence result of order $k+2$ is comparable, but different to the one developed in [4] between the mixed finite element approximation and the Fortin projection of the flux variable. Both requires a postprocessing procedure to gain a numerical approximation with the accuracy of $\mathcal{O}\left(h^{k+2}\right)$.

In summary, the contribution of this article is to introduce a highly effective and highly accurate WG element $\left(P_k(T), P_k(e), [P_{k+1}(T)]^d\right)$ on rectangular/cuboid mesh. First this WG element leads to a stabilizer-free WG finite element method which is highly desirable in many applications without

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raising the degree of polynomial too high for the approximation of gradient. Secondly, it is highly accurate with order two superconvergence demonstrated by many numerical tests and proved in theory.

2 | WEAK GALERKIN FINITE ELEMENT SCHEMES

Let $\mathcal{T}_h$ be a partition of the domain Ω consisting of rectangles and cuboids. Denote by $\mathcal{E}_h$ the set of all edges and faces in $\mathcal{T}_h$, and let $\mathcal{E}_h^0 = \mathcal{E}_h \backslash \partial\Omega$ be the set of all interior edges and faces. For every element $T \in \mathcal{T}_h$, we denote by h_T its diameter and mesh size $h = \max_{T\in\mathcal{T}_h} h_T$ for $\mathcal{T}_h$.

For a given integer $k \geq 1$, let V_h be the weak Galerkin finite element space associated with $\mathcal{T}_h$ defined as follows

$$V_h = \{v = \{v_0, v_b\} : \quad v_0|_T \in P_k(T), \quad v_b|_e \in P_k(e), \quad e \subset \partial T, T \in \mathcal{T}_h\} \tag{5}$$

and its subspace V_h^0 is defined as

$$V_h^0 = \{v : \quad v \in V_h, \quad v_b = 0 \ \text{ on } \ \partial\Omega\}. \tag{6}$$

We would like to emphasize that any function $v \in V_h$ has a single value v_b on each edge $e \in \mathcal{E}_h$.

For $v = \{v_0, v_b\} \in V_h$, a weak gradient $\nabla_w v$ is a piecewise vector valued polynomial such that on each $T \in \mathcal{T}_h$, $\nabla_w v \in [P_{k+1}(T)]^d$ satisfies

$$(\nabla_w v, \mathbf{q})_T = -(v_0, \nabla \cdot \mathbf{q})_T + \langle v_b, \mathbf{q}\cdot\mathbf{n}\rangle_{\partial T} \qquad \forall \mathbf{q} \in [P_{k+1}(T)]^d. \tag{7}$$

Let Q_0 and Q_b be the two element-wise defined L^2 projections onto $P_k(T)$ and $P_k(e)$ on each $T \in \mathcal{T}_h$ respectively. Define $Q_h u = \{Q_0 u, Q_b u\} \in V_h$. Let $\mathbb{Q}_h$ be the element-wise defined L^2 projection onto $[P_{k+1}(T)]^d$ on each element $T \in \mathcal{T}_h$.

For simplicity, we adopt the following notations,

$$(v, w)_{\mathcal{T}_h} = \sum_{T\in\mathcal{T}_h} (v, w)_T = \sum_{T\in\mathcal{T}_h} \int_T vw d\mathbf{x},$$
$$\langle v, w\rangle_{\partial\mathcal{T}_h} = \sum_{T\in\mathcal{T}_h} \langle v, w\rangle_{\partial T} = \sum_{T\in\mathcal{T}_h} \int_{\partial T} vw ds.$$

Algorithm 1. Weak Galerkin algorithm

A numerical approximation for (1)–(2) can be obtained by seeking $u_h = \{u_0, u_b\} \in V_h$ satisfying $u_b = Q_b g$ on $\partial\Omega$ and the following equation:

$$(a\nabla_w u_h, \nabla_w v) = (f, \ v_0) \quad \forall v = \{v_0, v_b\} \in V_h^0. \tag{8}$$

The following lemma will be used later in error analysis.

Lemma 1 *Let $\phi \in H^1(\Omega)$, then on any $T \in \mathcal{T}_h$,*

$$(\nabla_w (Q_h\phi), \mathbf{q})_T = (\mathbb{Q}_h \nabla\phi, \mathbf{q})_T - \langle \phi - Q_b\phi, \mathbf{q}\cdot\mathbf{n}\rangle_{\partial T} \quad \forall \mathbf{q} \in [P_{k+1}(T)]^2. \tag{9}$$

Proof. Using (7) and integration by parts, we have that for any $\mathbf{q} \in [P_{k+1}(T)]^d$

$$\begin{aligned}
(\nabla_w Q_h\phi, \mathbf{q})_T &= -(Q_0\phi, \nabla\cdot\mathbf{q})_T + \langle Q_b\phi, \mathbf{q}\cdot\mathbf{n}\rangle_{\partial T} \\
&= -(\phi, \nabla\cdot\mathbf{q})_T + \langle Q_b\phi, \mathbf{q}\cdot\mathbf{n}\rangle_{\partial T} \\
&= (\nabla\phi, \mathbf{q})_T - \langle \phi - Q_b\phi, \mathbf{q}\cdot\mathbf{n}\rangle_{\partial T} \\
&= (\mathbb{Q}_h\nabla\phi, \mathbf{q})_T - \langle \phi - Q_b\phi, \mathbf{q}\cdot\mathbf{n}\rangle_{\partial T},
\end{aligned}$$

which implies the Equation (9). ∎

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3 | WELL POSEDNESS

For any $v \in V_h$, two semi-norms are defined as follows

$$|||v|||^2 = (\nabla_w v, \nabla_w v), \tag{10}$$

$$|||v|||_1^2 = (a\nabla_w v, \nabla_w v). \tag{11}$$

As a is uniform positive definite, there exist two positive constants α and β such that

$$\alpha|||v||| \leq |||v|||_1 \leq \beta|||v|||. \tag{12}$$

We introduce a discrete H^1 norm as follows:

$$\|v\|_{1,h} = \left(\sum_{T\in\mathcal{T}_h} \left(\|\nabla v_0\|_T^2 + h_T^{-1}\|v_0 - v_b\|_{\partial T}^2\right)\right)^{\frac{1}{2}}. \tag{13}$$

For any function $\varphi \in H^1(T)$, the trace inequality holds true (see [17] for details):

$$\|\varphi\|_e^2 \leq C\left(h_T^{-1}\|\varphi\|_T^2 + h_T\|\nabla\varphi\|_T^2\right). \tag{14}$$

Next we will show that $|||\cdot|||$ also defines a norm for V_h^0 by proving the equivalence of $|||\cdot|||$ and $\|\cdot\|_{1,h}$ in V_h. First we need the following lemma.

Lemma 2 *Let T be a shape-regular rectangle/cuboid of size h_T. Let $v_h \in V_h$ and $v_h = \{v_0, v_b\}$ on T. Then there is a polynomial $\mathbf{q} \in [P_{k+1}(T)]^d$ such that*

$$-(\nabla v_0, \mathbf{q})_T = 0, \tag{15}$$

$$\langle v_0 - v_b, \mathbf{q}\cdot\mathbf{n}\rangle_e = \|v_0 - v_b\|_e^2 \quad \forall e \subset \partial T, \tag{16}$$

$$\|\mathbf{q}\|_T^2 \leq Ch_T\|v_0 - v_b\|_{\partial T}^2. \tag{17}$$

Proof. In 2D we need to construct 2 components of $\mathbf{q}$, and 3 components in 3D, respectively. But the construction of other components are the same as that for q, the first component of $\mathbf{q}$. Let T be a cube. Let e_1 and e_2 be the left and right face of T, respectively, i.e., $e_1 = \{\mathbf{x} = (x_1, x_2, x_3)\,|x_1 = x_l,\ \ \mathbf{x} \in T\}$ and $e_2 = \{\mathbf{x} = (x_1, x_2, x_3)\,|x_1 = x_r,\ \ \mathbf{x} \in T\}$. We determine q by the linear system of equations,

$$\begin{aligned} &\langle q, p_1\rangle_{e_1} = -\langle v_0 - v_b, p_1\rangle_{e_1} && \forall p_1 \in P_{k+1}(e_1), \\ &\langle q, p_2\rangle_{e_2} = \langle v_0 - v_b, p_2\rangle_{e_2} && \forall p_2 \in P_k(e_2), \\ &(q, p_3)_T = 0 && \forall p_3 \in P_{k-1}(T). \end{aligned} \tag{18}$$

■

In above system, the number of equations is the same as the number of unknowns. The uniqueness would guarantee the existence of a solution. We let the right hand side functions be zero in (18). As $q|_{e_1} \in P_{k+1}(e_1)$, the first equation in (18) implies $q|_{e_1} = 0$. So $q = (x_1 - x_l)\,q_1$ for some $q_1 \in P_k(T)$. By the second equation in (18), as q_1 is P_k polynomial on the face e_2, we have $q_1|_{e2} = 0$. Thus $q = (x_1 - x_l)\,(x_r - x_1)\,q_2$ for some $q_2 \in P_{k-1}$. The third equation in (18) is a weighted $L^2(T)$ projection for q_2 with a positive weight $(x_1 - x_l)\,(x_r - x_1)$ on T. Therefore this equation determines $q_2 = 0$ and consequently $q \equiv 0$ on T. Similarly we define the second and the third components of $\mathbf{q}$.

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By (18), the constructed $\mathbf{q}$ satisfy the first two equations, (15) and (16). As all norms are equivalent on finite dimensional vector space $P_{k+1}(T)$, we have

$$\|q\|_T^2 \le Ch_T \left(\|v_0 - v_b\|_{e_1}^2 + \|v_0 - v_b\|_{e_2}^2 \right),$$
$$\|\mathbf{q}\|_T^2 \le Ch_T \|v_0 - v_b\|_{\partial T}^2,$$

where h_T comes from the scaling argument. The lemma is proved.

Lemma 3 *There exist two positive constants C_1 and C_2 such that for any $v = \{v_0, v_b\} \in V_h$, we have*

$$C_1 \|v\|_{1,h} \le |||v||| \le C_2 \|v\|_{1,h}. \tag{19}$$

Proof. For any $v = \{v_0, v_b\} \in V_h$, it follows from the definition of weak gradient (7) and integration by parts that on each $T \in \mathcal{T}_h$

$$(\nabla_w v, \mathbf{q})_T = (\nabla v_0, \mathbf{q})_T + \langle v_b - v_0, \mathbf{q} \cdot \mathbf{n} \rangle_{\partial T} \quad \forall \mathbf{q} \in [P_{k+1}(T)]^d. \tag{20}$$

By letting $\mathbf{q} = \nabla_w v|_T$ in (20) we arrive at

$$(\nabla_w v, \nabla_w v)_T = (\nabla v_0, \nabla_w v)_T + \langle v_b - v_0, \nabla_w v \cdot \mathbf{n} \rangle_{\partial T}.$$

Letting $\mathbf{q} = \nabla v_0|_T$ in (20), it follows

$$(\nabla_w v, \nabla v_0)_T = (\nabla v_0, \nabla v_0)_T + \langle v_b - v_0, \nabla v_0 \cdot \mathbf{n} \rangle_{\partial T}. \tag{21}$$

From the trace inequality (14) and the inverse inequality, we have

$$\begin{aligned} \|\nabla_w v\|_T^2 &\le \|\nabla v_0\|_T \|\nabla_w v\|_T + \|v_0 - v_b\|_{\partial T} \|\nabla_w v\|_{\partial T} \\ &\le \|\nabla v_0\|_T \|\nabla_w v\|_T + Ch_T^{-1/2} \|v_0 - v_b\|_{\partial T} \|\nabla_w v\|_T. \end{aligned}$$

Thus

$$\|\nabla_w v\|_T \le C \left(\|\nabla v_0\|_T + h_T^{-1/2} \|v_0 - v_b\|_{\partial T} \right),$$

and consequently

$$|||v||| \le C_2 \|v\|_{1,h}.$$

Next we will prove $C_1 \|v\|_{1,h} \le |||v|||$. First we prove

$$h_e^{-1/2} \|v_0 - v_b\|_e \le C \|\nabla_w v\|_T. \tag{22}$$

For $e \in \mathcal{E}_h$ and $T \in \mathcal{T}_h$ with $e \subset \partial T$, it has been proved in Lemma 2 that there exists $\mathbf{q}_0 \in [P_{k+1}(T)]^d$ such that

$$\left(\nabla v_0, \mathbf{q}_0 \right)_T = 0, \quad \langle v_b - v_0, \mathbf{q}_0 \cdot \mathbf{n} \rangle_{\partial T} = \|v_0 - v_b\|_{\partial T}^2, \tag{23}$$

and

$$\|\mathbf{q}_0\|_T \le Ch_T^{1/2} \|v_b - v_0\|_{\partial T}. \tag{24}$$

Substituting $\mathbf{q}_0$ into (20), we get

$$\left(\nabla_w v, \mathbf{q}_0 \right)_T = \|v_b - v_0\|_{\partial T}^2. \tag{25}$$

It follows from Cauchy-Schwarz inequality and (24) that

$$\|v_b - v_0\|_{\partial T}^2 \le C \|\nabla_w v\|_T \|\mathbf{q}_0\|_T \le Ch_T^{1/2} \|\nabla_w v\|_T \|v_0 - v_b\|_{\partial T},$$

and then

$$h_T^{-1/2} \|v_0 - v_b\|_{\partial T} \le C \|\nabla_w v\|_T. \tag{26}$$

It follows from (21), the trace inequality, the inverse inequality and (26),

$$\|\nabla v_0\|_T^2 \leq \|\nabla_w v\|_T \|\nabla v_0\|_T + Ch_T^{-1/2} \|v_0 - v_b\|_{\partial T} \|\nabla v_0\|_T \leq C \|\nabla_w v\|_T \|\nabla v_0\|_T,$$

that implies

$$\|\nabla v_0\|_T \leq C \|\nabla_w v\|_T.$$

Combining the above estimate and (26), we prove the lower bound of (19) and complete the proof of the lemma. ■

Lemma 4 *The weak Galerkin finite element scheme (8), defined in Algorithm 1, has a unique solution.*

Proof. Let $u_h^{(1)}$ and $u_h^{(2)}$ be the two solutions of (8), then $\varepsilon_h = u_h^{(1)} - u_h^{(2)} \in V_h^0$ would satisfy the following equation

$$(a\nabla_w \varepsilon_h, \nabla_w v) = 0, \qquad \forall v \in V_h^0.$$

Then by letting $v = \varepsilon_h$ in the above equation and (12), we arrive at

$$\alpha \, |||\varepsilon_h|||^2 \leq (a\nabla_w \varepsilon_h, \nabla_w \varepsilon_h) = 0.$$

It follows from (19) that $\|\varepsilon_h\|_{1,h} = 0$. Since $\|\cdot\|_{1,h}$ is a norm in V_h^0, one has $\varepsilon_h = 0$. This completes the proof of the lemma. ■

4 | ERROR ESTIMATES IN ENERGY NORM

The goal of this section is to establish some error estimates for the weak Galerkin finite element solution u_h arising from (8). For simplicity of analysis, we assume that the coefficient tensor a in (1) is a piecewise constant matrix with respect to the finite element partition $\mathcal{T}_h$.

Next we derive an equation that the error $e_h = Q_h u - u_h$ satisfies.

Lemma 5 *For any $v \in V_h^0$, the following error equation holds true*

$$(a\nabla_w e_h, \nabla_w v) = \ell_1(u, v) - \ell_2(u, v), \tag{27}$$

where

$$\ell_1(u, v) = \langle a\,(\nabla u - \mathbb{Q}_h \nabla u) \cdot \mathbf{n}, \;\; v_0 - v_b \rangle_{\partial \mathcal{T}_h}$$
$$\ell_2(u, v) = \langle u - Q_b u, \;\; a\nabla_w v \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h}.$$

Proof. For $v = \{v_0, v_b\} \in V_h^0$, testing (1) by v_0 and using the fact that $\langle a\nabla u \cdot \mathbf{n}, v_b \rangle_{\partial \mathcal{T}_h} = 0$, we have

$$(a\nabla u, \nabla v_0)_{\mathcal{T}_h} - \langle a\nabla u \cdot \mathbf{n}, v_0 - v_b \rangle_{\partial \mathcal{T}_h} = (f, v_0). \tag{28}$$

It follows from integration by parts, (7) and (9) that

$$\begin{aligned}
(a\nabla u, \nabla v_0)_{\mathcal{T}_h} &= (a\mathbb{Q}_h \nabla u, \nabla v_0)_{\mathcal{T}_h} \\
&= -(v_0, \nabla \cdot (a\mathbb{Q}_h \nabla u))_{\mathcal{T}_h} + \langle v_0, a\mathbb{Q}_h \nabla u \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h} \\
&= (a\mathbb{Q}_h \nabla u, \nabla_w v)_{\mathcal{T}_h} + \langle v_0 - v_b, a\mathbb{Q}_h \nabla u \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h} \\
&= (a\nabla_w Q_h u, \nabla_w v) + \langle u - Q_b u, a\nabla_w v \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h} \\
&\quad + \langle v_0 - v_b, a\mathbb{Q}_h \nabla u \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h}.
\end{aligned} \tag{29}$$

Combining (28) and (29) yields

$$(a\nabla_w Q_h u, \nabla_w v) = (f, v_0) + \ell_1(u, v) - \ell_2(u, v). \tag{30}$$

The error equation follows from subtracting (8) from (30),

$$(a\nabla_w e_h, \nabla_w v) = \ell_1(u, v) - \ell_2(u, v) \quad \forall v \in V_h^0.$$

This completes the proof of the lemma. ■

Next we will bound the two terms $\ell_1(u, v)$ and $\ell_2(u, v)$.

Lemma 6 *For any $w \in H^{k+3}(\Omega)$ and $v = \{v_0, v_b\} \in V_h^0$, we have*

$$|\ell_1(w, v)| \le Ch^{k+2}|w|_{k+3}|||v|||. \tag{31}$$

Proof. Using the Cauchy-Schwarz inequality, the trace inequality (14), and (19), we have

$$\begin{aligned}
|\ell_1(w, v)| &= \left| \sum_{T\in\mathcal{T}_h} \langle a\,(\nabla w - \mathbb{Q}_h \nabla w) \cdot \mathbf{n}, v_0 - v_b \rangle_{\partial T} \right| \\
&\le C \sum_{T\in\mathcal{T}_h} \|\nabla w - \mathbb{Q}_h \nabla w\|_{\partial T} \|v_0 - v_b\|_{\partial T} \\
&\le C \left(\sum_{T\in\mathcal{T}_h} h_T \|\,(\nabla w - \mathbb{Q}_h \nabla w)\,\|_{\partial T}^2 \right)^{\frac{1}{2}} \left(\sum_{T\in\mathcal{T}_h} h_T^{-1} \|v_0 - v_b\|_{\partial T}^2 \right)^{\frac{1}{2}} \\
&\le Ch^{k+2}|w|_{k+3}\,|||v|||.
\end{aligned}$$

We have proved the lemma. ■

To estimate the term $\ell_2(u, v)$, we need the following lemma.

Lemma 7 *For any $T \in \mathcal{T}_h$ and $e \subset \partial T$, we have for $v = \{v_0, v_b\} \in V_h^0$,*

$$\text{degree}\,(\nabla_w v \cdot \mathbf{n}|_e) \le k. \tag{32}$$

Proof. For simplicity, we let $d = 2$ and $a = I$. The proof can be easily extended to cuboid mesh. For any $v \in V_h$, we have $v = \{v_0, v_b\} = v_1 + v_2 + v_3 + v_4 + v_5$ where $v_5 = \{v_0, 0\}$ and $v_i = \left\{0, v_b^i\right\}$ with v_b^i is only nonzero on e_i, $i = 1, 2, 3, 4$, shown in Figure 1. Next we will show degree $(\nabla_w v_i \cdot \mathbf{n}|_e) \le k$, $i = 1, 2, 3, 4, 5$. First we prove degree $(\nabla_w v_5 \cdot \mathbf{n}|_e) \le k$ for $e \subset \partial T$. Without loss of generality, let $e = e_1$. First we investigate degree $\left(\nabla_w v_5 \cdot \mathbf{n}|_{e_1}\right)$. Let

$$\nabla_w v_5 = \mathbf{q} = (q_1, q_2)^T, \quad \mathbf{q} \in [P_{k+1}(T)]^2.$$

Let

$$q_2 = c_0 x^{k+1} + p_1(y)x^k + \cdots + p_{k+1}(y)x^0, \quad p_i(y) \in P_i([0, 1]), \ \ i \le k + 1. \tag{33}$$

Then

$$\nabla_w v_5 \cdot \mathbf{n}|_{e_1} = -q_2. \tag{34}$$

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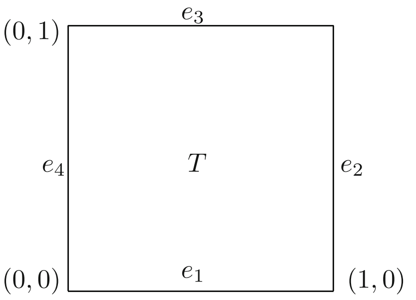

FIGURE 1 One rectangle T and its 4 edges e_i

It follows from (7) and the definition of v_5,

$$(\nabla_w v_5, \mathbf{r})_T = -(v_0, \nabla \cdot \mathbf{r})_T \quad \forall \mathbf{r} \in [P_{k+1}(T)]^2. \tag{35}$$

Letting $\mathbf{r} = (0, L_{k+1}(x))^T$, where L_{k+1} is the $k+1$st Legendre polynomial on $[0,1]$ satisfying $\int_0^1 x^i L_{k+1}(x)dx = \delta_{i,k+1}$, in (35) gives, by (33),

$$0 = (-v_0, \ 0)_T = \left(\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \begin{pmatrix} 0 \\ L_{k+1}(x) \end{pmatrix} \right)_T = \int_0^1 L_{k+1}(x) \left(\int_0^1 q_2(x,y)dy \right) dx = c_0.$$

Thus, by (33) and (34), we have proved degree $\left(\nabla_w v_5 \cdot \mathbf{n}|_{e_1} \right) \le k$.

Without loss of generality, we will prove degree$\left(\nabla_w v_3 \cdot \mathbf{n}|_e \right) \le k$ for $e \subset \partial T$. First we show degree $\left(\nabla_w v_3 \cdot \mathbf{n}|_{e_3} \right) \le k$. By the definitions of v_3 and weak gradient (7), we have

$$\nabla_w v_3|_e = (0, r)^T \quad e \subset \partial T. \tag{36}$$

Next we will show that degree$(r(x,1)) \le k$. Let $\phi_0(x), \phi_1(x), \cdots, \phi_{k+1}(x)$ form an orthonormal basis respect to the inner product $\langle \cdot, \cdot \rangle_{e_3}$, where degree$(\phi_i(x)) = i$, $i = 0, \cdots, k+1$.

Letting $\mathbf{q} = (0, \phi_{k+1})^T$ in (7) gives

$$(\nabla_w v_3, \mathbf{q})_T = \left\langle v_b^3, \phi_{k+1} \right\rangle_{e_3} = 0,$$

which implies

$$\begin{aligned} 0 = (\nabla_w v_3, \mathbf{q})_T &= \int_0^1 \phi_{k+1}(x) \left(\int_0^1 r dy \right) dx \\ &= \int_0^1 \phi_{k+1}(x) \left(a_0\phi_0 + a_1\phi_1 + \cdots + a_{k+1}\phi_{k+1} \right) dx \\ &= a_{k+1} \int_0^1 [\phi_{k+1}(x)]^2 dx. \end{aligned}$$

We conclude from the equation above $a_{k+1} = 0$, that is, degree$(r(x,1)) \le k$ and degree $\left(\nabla_w v_3|_{e_3} \right) \le k$. Similarly we can have degree $\left(\nabla_w v_3|_{e_1} \right) \le k$. For $e = e_2, e_4$, (36) implies $\nabla_w v_3 \cdot \mathbf{n}|_e = 0$. Thus we have proved (32).

Using Lemma 7, we can estimate $\ell_2(w,v)$ in the following Lemma.

Lemma 8 *For any $w \in H^{k+3}(\Omega)$ and $v = \{v_0, v_b\} \in V_h^0$, we have*

$$\ell_2(w,v) = 0. \tag{37}$$

Proof. It follows from (32) that for any $v \in V_h^0$

$$\ell_2(w, v) = \langle w - Q_b w, \ \ a\nabla_w v \cdot \mathbf{n}\rangle_{\partial\mathcal{T}_h} = 0.$$

We complete the proof. ∎

Theorem 1 *Let $u_h \in V_h$ be the weak Galerkin finite element solution of (8). Assume the exact solution $u \in H^{k+3}(\Omega)$. Then, there exists a constant C such that*

$$|||Q_h u - u_h||| \leq Ch^{k+2}|u|_{k+3}. \tag{38}$$

Proof. By letting $v = e_h$ in (27) and using (12), we have

$$\alpha|||e_h|||^2 \leq (a\nabla_w e_h, \nabla_w e_h) = |\ell_1(u, e_h) - \ell_2(u, e_h)| \ . \tag{39}$$

It follows from (31) and (37) that

$$|||e_h|||^2 \leq Ch^{k+2}|u|_{k+3}|||e_h|||,$$

which implies (38). This completes the proof. ∎

5 | ERROR ESTIMATES IN L^2 NORM

We use the duality argument to obtain L^2 error estimate. Recall $e_h = \{e_0, e_b\} = Q_h u - u_h$. The corresponding dual problem seeks $\Phi \in H_0^1(\Omega)$ satisfying

$$-\nabla \cdot a\nabla\Phi = e_0 \quad \text{in} \ \ \Omega. \tag{40}$$

Assume that the following H^2-regularity holds

$$\|\Phi\|_2 \leq C\|e_0\|. \tag{41}$$

Theorem 2 *Let $u_h \in V_h$ be the weak Galerkin finite element solution of (8). Assume that the exact solution $u \in H^{k+3}(\Omega)$ and (41) holds true. Then, there exists a constant C such that*

$$\|Q_0 u - u_0\| \leq Ch^{k+3}|u|_{k+3}. \tag{42}$$

Proof. Testing (40) by e_0, we obtain

$$\begin{aligned}\|e_0\|^2 &= -(\nabla \cdot (a\nabla\Phi), e_0)\\ &= (a\nabla\Phi, \ \ \nabla e_0)_{\mathcal{T}_h} - \langle a\nabla\Phi \cdot \mathbf{n}, \ \ e_0 - e_b\rangle_{\partial\mathcal{T}_h}, \end{aligned}\tag{43}$$

where we have used the fact that $e_b = 0$ on $\partial\Omega$. Setting $u = \Phi$ and $v = e_h$ in (29) yields

$$\begin{aligned}(a\nabla\Phi, \ \ \nabla e_0)_{\mathcal{T}_h} &= (a\nabla_w Q_h\Phi, \ \ \nabla_w e_h) + \langle \Phi - Q_b\Phi, a\nabla_w e_h \cdot \mathbf{n}\rangle_{\partial\mathcal{T}_h}\\ &\quad + \ \langle (a\mathbb{Q}_h\nabla\Phi) \cdot \mathbf{n}, \ \ e_0 - e_b\rangle_{\partial\mathcal{T}_h}. \end{aligned}\tag{44}$$

Substituting (44) into (43) and using (37) give

$$\begin{aligned}\|e_0\|^2 &= (a\nabla_w e_h, \ \ \nabla_w Q_h\Phi) + \langle \Phi - Q_b\Phi, a\nabla_w e_h \cdot \mathbf{n}\rangle_{\partial\mathcal{T}_h}\\ &\quad + \ \langle a\,(\mathbb{Q}_h\nabla\Phi - \nabla\Phi) \cdot \mathbf{n}, \ \ e_0 - e_b\rangle_{\partial\mathcal{T}_h}\\ &= (a\nabla_w e_h, \ \ \nabla_w Q_h\Phi) - \ell_1(\Phi, e_h) + \ell_2(\Phi, e_h)\end{aligned}$$

$$
\begin{aligned}
&= \ell_1(u, Q_h\Phi) - \ell_2(u, Q_h\Phi) - \ell_1(\Phi, e_h) + \ell_2(\Phi, e_h) \\
&= \ell_1(u, Q_h\Phi) - \ell_1(\Phi, e_h). \qquad (45)
\end{aligned}
$$

Let us bound the two terms on the right hand side of (45). Using the triangle inequality, we obtain

$$
\begin{aligned}
|\ell_1(u, Q_h\Phi)| &= \left| \sum_{T\in\mathcal{T}_h} \langle a(\nabla u - \mathbb{Q}_h \nabla u)\cdot \mathbf{n},\ \ Q_0\Phi - Q_b\Phi\rangle_{\partial T} \right| \\
&\le C \sum_{T\in\mathcal{T}_h} \|\nabla u - \mathbb{Q}_h\nabla u\|_{\partial T} \|Q_0\Phi - Q_b\Phi\|_{\partial T} \\
&\le C\left(\sum_{T\in\mathcal{T}_h} \|\nabla u - \mathbb{Q}_h\nabla u\|_{\partial T}^2 \right)^{1/2} \left(\sum_{T\in\mathcal{T}_h} \|Q_0\Phi - \Phi\|_{\partial T}^2 \right)^{1/2} \qquad (46)
\end{aligned}
$$

From the trace inequality (14) we have

$$
\left(\sum_{T\in\mathcal{T}_h} \|Q_0\Phi - \Phi\|_{\partial T}^2 \right)^{1/2} \le Ch^{\frac{3}{2}} \|\Phi\|_2
$$

and

$$
\left(\sum_{T\in\mathcal{T}_h} \|a(\nabla u - \mathbb{Q}_h\nabla u)\|_{\partial T}^2 \right)^{1/2} \le Ch^{k+\frac{3}{2}} \|u\|_{k+3}.
$$

Combining the above two estimates with (46) gives

$$
|\ell_1(u, Q_h\Phi)| \le Ch^{k+3}|u|_{k+3}\|\Phi\|_2. \qquad (47)
$$

It follows from (31) and (38),

$$
|\ell_1(\Phi, e_h)| \le Ch|\Phi|_2 |||e_h||| \le Ch^{k+3}|u|_{k+3}\|\Phi\|_2. \qquad (48)
$$

Substituting (47) and (48) into (45) yields

$$
\|e_0\|^2 \le Ch^{k+3}|u|_{k+3}\|\Phi\|_2.
$$

Using the estimate above, the regularity assumption (41) and the triangle inequality, we obtain the optimal order error estimate (42). ■

6 | NUMERICAL EXPERIMENTS

6.1 | Example 1

Consider problem (1) with $\Omega = (0,1)^2$. Taking $a = A_i, i = 1,2$, respectively, where $A_1 = I$ and $A_2 = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$. The source term f and the boundary value g are chosen so that the exact solution is

$$
u(x,y) = \sin(x+y)\cos(x-y).
$$

We subdivided the domain Ω into $N \times N$ uniform squares as our meshes. The error and the order of convergence are listed in Tables 1 and 2, where we have order two superconvergence for $k \ge 1$ in both L^2 norm and H^1-like triple-bar norm. For P_0 finite element, we have order 1 and order 2 superconvergence in L^2 and H^1-like norms, respectively.

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TABLE 1 Example 1: $P_k - P_k - \left[P_{k+1}\right]^2$ element, $a = A_1$, uniform square mesh

k	N	$\|Q_h u - u_h\|$	Rate	$\|\|\|Q_h u - u_h\|\|\|$	Rate
0	8	6.5259E−04	1.96	2.9936E−03	2.00
	16	1.6435E−04	1.99	7.4832E−04	2.00
	32	4.1164E−05	2.00	1.8707E−04	2.00
	64	1.0296E−05	2.00	4.6767E−05	2.00
1	8	2.0456E−06	4.01	8.1794E−05	2.99
	16	1.2758E−07	4.00	1.0241E−05	3.00
	32	7.9693E−09	4.00	1.2807E−06	3.00
	64	4.9812E−10	4.00	1.6011E−07	3.00
2	8	1.4731E−08	4.95	9.2424E−07	3.98
	16	4.6670E−10	4.98	5.8152E−08	3.99
	32	1.4674E−11	4.99	3.6461E−09	4.00
	64	5.2034E−13	4.82	2.2837E−10	4.00

TABLE 2 Example 1: $P_k - P_k - \left[P_{k+1}\right]^2$ element, $a = A_2$, uniform square mesh

k	N	$\|Q_h u - u_h\|$	Rate	$\|\|\|Q_h u - u_h\|\|\|$	Rate
0	8	6.6746E−04	1.95	4.7776E−03	2.00
	16	1.6820E−04	1.99	1.1945E−03	2.00
	32	4.2133E−05	2.00	2.9862E−04	2.00
	64	1.0538E−05	2.00	7.4653E−05	2.00
1	8	2.0355E−06	4.01	1.2928E−04	2.99
	16	1.2698E−07	4.00	1.6190E−05	3.00
	32	7.9318E−09	4.00	2.0249E−06	3.00
	64	4.9552E−10	4.00	2.5315E−07	3.00
2	8	1.4776E−08	4.95	1.4637E−06	3.98
	16	4.6744E−10	4.98	9.2032E−08	3.99
	32	1.4687E−11	4.99	5.7678E−09	4.00
	64	5.0840E−13	4.85	3.6117E−10	4.00

6.2 | Example 2

Consider problem (1) with $\Omega = (0,1)^2$ and $a = I$. The source term f and the boundary value g are chosen so that the exact solution is

$$u(x,y) = \sin(\pi x)\sin(\pi y).$$

We use the same meshes as Example 1. The result is listed in Table 3. The superconvergence phenomena are the same as those in Example 1.

6.3 | Example 3

We compare results of the standard weak Galerkin finite element and the new weak Galerkin finite element on square meshes, in this example. Consider problem (1) with $\Omega = (0,1)^2$ and $a = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$.

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TABLE 3 Example 2: $P_k - P_k - \left[P_{k+1}\right]^2$ element, $a = I$, uniform rectangular mesh

k	N	$\|Q_h u - u_h\|$	Rate	$\|\|\|Q_h u - u_h\|\|\|$	Rate
0	8	6.2942E−03	1.91	2.8253E−02	1.95
	16	1.5981E−03	1.98	7.1185E−03	1.99
	32	4.0108E−04	1.99	1.7831E−03	2.00
	64	1.0037E−04	2.00	4.4599E−04	2.00
1	8	9.7469E−05	3.93	8.1226E−04	3.51
	16	6.1678E−06	3.98	8.6592E−05	3.23
	32	3.8667E−07	4.00	1.0296E−05	3.07
	64	2.4186E−08	4.00	1.2699E−06	3.02
2	8	5.4375E−07	5.85	7.6001E−05	4.07
	16	1.0533E−08	5.69	4.6895E−06	4.02
	32	2.5298E−10	5.38	2.9215E−07	4.00
	64	7.1980E−12	5.13	1.8245E−08	4.00

TABLE 4 Example 3: $P_k - P_k - \left[P_{k+1}\right]^2$ element, $a = I$, $e_h = Q_h u - u_h$, uniform rectangular mesh

$\mathcal{T}_l$	$\|e_h\|$	R	$\|\|\|e_h\|\|\|$	R	$\|e_h\|$	R	$\|\|\|e_h\|\|\|$	R
	$P_0 - P_0 - [P_1]^2$ element				$Q_0 - Q_0 - \left[RT_0\right]$ element			
6	0.352E−03	1.98	0.270E−02	2.00	0.352E−03	1.98	0.270E−02	2.00
7	0.882E−04	2.00	0.676E−03	2.00	0.882E−04	2.00	0.676E−03	2.00
8	0.221E−04	2.00	0.169E−03	2.00	0.221E−04	2.00	0.169E−03	2.00
	$P_1 - P_1 - [P_2]^2$ element				$Q_1 - Q_1 - [RT_1]$ element			
6	0.439E−06	3.99	0.657E−04	2.99	0.145E−04	2.99	0.359E−02	1.99
7	0.275E−07	4.00	0.823E−05	3.00	0.181E−05	3.00	0.899E−03	2.00
8	0.172E−08	4.00	0.103E−05	3.00	0.227E−06	3.00	0.225E−03	2.00
	$P_2 - P_2 - \left[P_3\right]^2$ element				$Q_2 - Q_2 - [RT_2]$ element			
5	0.516E−07	4.97	0.642E−05	3.98	0.157E−05	3.99	0.349E−03	2.99
6	0.162E−08	4.99	0.403E−06	3.99	0.985E−07	3.99	0.437E−04	3.00
7	0.509E−10	5.00	0.252E−07	4.00	0.617E−08	4.00	0.547E−05	3.00
	$P_3 - P_3 - [P_4]^2$ element				$Q_3 - Q_3 - \left[RT_3\right]$ element			
5	0.161E−07	5.96	0.156E−05	4.98	0.575E−06	4.95	0.828E−04	3.97
6	0.256E−09	5.98	0.489E−07	4.99	0.182E−07	4.98	0.521E−05	3.99
7	0.406E−11	5.98	0.153E−08	5.00	0.570E−09	4.99	0.326E−06	4.00

The source term f and the boundary value g are chosen so that the exact solution is

$$u(x, y) = x^7 - y^7.$$

We use the same meshes as Example 1. That is, the first level mesh consists one square, the domain. Each square is refined into 4 to define next level mesh. The results of two types of weak Galerkin finite element methods are listed in Table 4. The superconvergence phenomena for the new element are the same as those in Example 1, two orders of superconvergence. The standard $Q_k - Q_k - [RT_k]$ element is also superconvergent, but of order 1. To see the superconvergence phenomena we plot the error of the new $P_2 - P_2 - [P_3]^2$ element and the error of the $Q_2 - Q_2 - [RT_2]$ element on the fifth level mesh,

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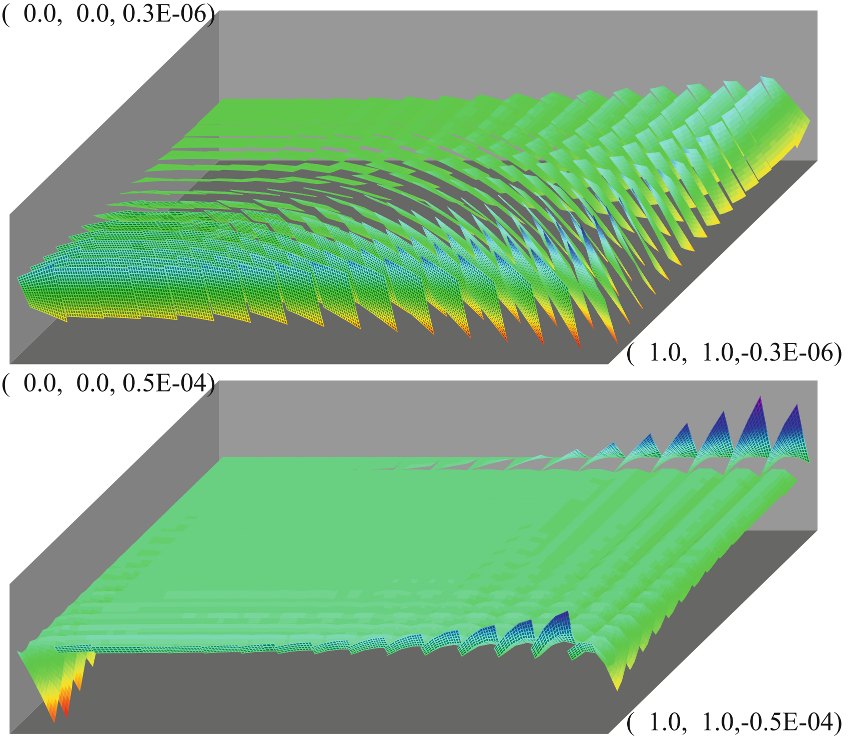

FIGURE 2 Example 3: The error plot of the $P_2 - P_2 - [P_3]^2$ element (top), and the error plot of the $Q_2 - Q_2 - [RT_2]$ element

TABLE 5 Example 4: $P_k - P_k - [P_{k+1}]^3$ element, $a = I$, uniform cubic mesh

k	$\mathcal{T}_l$	$\|Q_h u - u_h\|$	Rate	$\|\|\|Q_h u - u_h\|\|\|$	Rate
0	4	0.5713E−02	2.1	0.1343E+00	1.9
	5	0.1289E−02	2.1	0.3484E−01	1.9
	6	0.3027E−03	2.1	0.8850E−02	2.0
1	4	0.6404E−03	4.0	0.4240E−01	2.8
	5	0.4035E−04	4.0	0.5545E−02	2.9
	6	0.2542E−05	4.0	0.7070E−03	3.0
2	4	0.1072E−03	4.7	0.9585E−02	3.8
	5	0.3649E−05	4.9	0.6270E−03	3.9
	6	0.1178E−06	5.0	0.3999E−04	4.0
3	3	0.8274E−03	5.3	0.5007E−01	4.5
	4	0.1418E−04	5.9	0.1754E−02	4.8
	5	0.2324E−06	5.9	0.5749E−04	4.9
4	3	0.1764E−03	6.6	0.1519E−01	5.6
	4	0.1720E−05	6.7	0.2644E−03	5.8
	5	0.1460E−07	6.9	0.4325E−05	5.9

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in Figure 2. We can see that the error of the $P_2 - P_2 - [P_3]^2$ element crosses the plane $y = 0$ nearly symmetrically on each element.

6.4 | Example 4

Consider problem (1) with $\Omega = (0, 1)^3$ and $a = I_{3\times3}$. The source term f and the boundary value g are chosen so that the exact solution is

$$u(x, y, z) = \sin(\pi x)\sin(\pi y)\sin(\pi z).$$

The first level mesh consists one cube, the domain. Each cube is refined into 8 equal-size cubes to define next level mesh. The results of the $P_k - P_k - [P_{k+1}]^3$ weak Galerkin finite element methods are listed in Table 5. The superconvergence phenomena for the new element are the same as those in 2D, in first three examples, two orders of superconvergence.

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DATA AVAILABILITY STATEMENT

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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