# Dynamics of generalizations of the AGM continued fraction of Ramanujan. Part I: divergence.

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**Abstract**

We study several generalizations of the AGM continued fraction of Ramanujan inspired by a series of recent articles in which the validity of the AGM relation and the domain of convergence of the continued fraction were determined for certain complex parameters [4, 3, 2]. A study of the AGM continued fraction is equivalent to an analysis of the convergence of certain difference equations and the stability of dynamical systems. Using the matrix analytical tools developed in [2], we determine the convergence properties of deterministic difference equations and so divergence of their corresponding continued fractions.

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## 1 Introduction

For the sequence $a := (a_n)_{n=1}^{\infty}$, denote the continued fraction $\mathcal{S}_1(a)$ by

$$\mathcal{S}_1(a) = \cfrac{1^2 a_1^2}{1 + \cfrac{2^2 a_2^2}{1 + \cfrac{3^2 a_3^2}{1 + \ddots}}} \tag{1.1}$$

We study the convergence properties of this continued fraction for deterministic sequences $(a_n)$. The case of sequences of random variables is treated in a companion paper [5]. We derive our most

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general results from an examination of periodic sequences, that is, sequences satisfying $a_j = a_{j+c}$ for all $j$ and some finite $c$. In this case we will sometimes represent the sequence only by its base cycle $a = (a_1, a_2, \ldots, a_c)$. Our definition of $\mathcal{S}_1$ leads to a slight idiosyncrasy with respect to other definitions for the case $c = 2$ since these begin the continued fraction with the *second* element of the sequence. Many special cases of the above continued fraction for particular choices of $a$ have been determined in [3, 2]. In particular the cases (i) $a_n = const \in \mathbb{C}$, (ii) $a_n = -a_{n+1} \in \mathbb{C}$, (iii) $|a_{2n}| = 1,\ a_{2n+1} = i$, and (iv) $a_{2n} = a_{2m},\ a_{2n+1} = a_{2m+1}$ with $|a_n| = |a_m|\ \forall\ m, n \in \mathbb{N}$. In the present work we are interested in the convergence of $\mathcal{S}_1$ for arbitrary sequences of parameters.

To evaluate $\mathcal{S}_1$, we study the recurrence for the classical convergents $p_n/q_n$ to the fraction $\mathcal{S}_1$, For a general continued fraction of the form

$$\mathcal{S}_\eta(\gamma) = \eta_0 + \cfrac{\gamma_1}{\eta_1 + \cfrac{\gamma_2}{\eta_2 + \cfrac{\gamma_3}{\eta_3 + \ddots}}}$$

these are defined by the truncated continued fraction: $p_{-1} = 1,\ p_0 = \eta_0,\ q_{-1} = 0,\ q_0 = 1$ and

$$\begin{aligned}
\mathcal{S}_\eta(\gamma) &\approx \frac{p_1}{q_1} = \frac{\eta_1 p_0 + \gamma_1 p_1}{\eta_1 q_0} = \eta_0 + \frac{\gamma_1}{\eta_1} \qquad \text{first order,} \\
&\approx \frac{p_2}{q_2} = \frac{\eta_2 p_1 + \gamma_2 p_0}{\eta_2 q_1 + \gamma_2 q_0} = \eta_0 + \cfrac{\gamma_1}{\eta_1 + \cfrac{\gamma_1}{\eta_2/\gamma_2}} \qquad \text{second order,} \\
&\approx \ \ldots \\
&\approx \frac{p_n}{q_n} = \frac{\eta_n p_{n-1} + \gamma_n p_{n-2}}{\eta_n q_{n-1} + \gamma_n q_{n-2}} = \eta_0 + \cfrac{\gamma_1}{\eta_1 + \cfrac{\gamma_1}{\eta_2 \cfrac{}{\ddots + \cfrac{\ddots}{\eta_n/\gamma_n}}}} \qquad \text{n'th order.}
\end{aligned}$$

A simple induction argument establishes the general recurrence for the numerator and denominator $p_n$ and $q_n$ shown above, namely

$$p_n = \eta_n p_{n-1} + \gamma_n p_{n-2} \quad \text{and} \quad q_n = \eta_n q_{n-1} + \gamma_n q_{n-2}.$$

For the continued fraction $\mathcal{S}_1(a)$ we have

$$q_n = q_{n-1} + n^2 \alpha_n q_{n-2} \quad \text{where} \quad \alpha_n := a_n^2. \tag{1.2}$$

We will use $\alpha_n$ and $a_n^2$ interchangeably throughout. The $p_n$ terms of the classical convergents also satisfy Eq.(1.2).

Since the recurrence is a 2-step backward difference equation it is convenient to reformulate

Eq.(1.2) in terms of $2 \times 2$ matrices

$$q^{(n)} = Q_n q^{(n-1)} \quad \text{where} \quad Q_n := \begin{bmatrix} 1 & n^2\alpha_n \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad q^{(n)} := \begin{pmatrix} q_n \\ q_{n-1} \end{pmatrix}. \tag{1.3}$$

To analyze the case of cyclic parameters $a_n$ with periods of length $c$, we regroup the above recursion into blocks of length $c$

$$q^{(cn+m)} = \widehat{Q}_n^{(m)} q^{(c(n-1)+m)} \quad \text{where} \quad \widehat{Q}_n^{(m)} := \prod_{j=c(n-1)+m+1}^{cn+m} Q_j. \tag{1.4}$$

Throughout, we interpret the matrix product ascending from right to left. To avoid notational clutter, we will, without loss of generality, consider only the 0th term of the cycle, i.e. $m = 0$ in Eq.(1.4). In this case $\widehat{Q}_n := \widehat{Q}_n^{(0)}$.

Following [2], it is helpful to consider the renormalized sequences $(t_n)$ and $(v_n)$ where

$$t_n := \frac{q_{n-1}}{n!} \quad \text{and} \quad v_n := \frac{q_n}{\Gamma(n+3/2)a_n^{(n+1)}}. \tag{1.5}$$

As with $q^{(n)}$ define

$$t^{(n)} := \begin{pmatrix} t_n \\ t_{n-1} \end{pmatrix} \quad \text{and} \quad v^{(n)} := \begin{pmatrix} v_n \\ v_{n-1} \end{pmatrix}. \tag{1.6}$$

Then

$$t^{(cn)} = T_n t^{(c(n-1))} \quad \text{where} \quad T_n := [N_n]^{-1} \widehat{Q}_n N_{n-1} \tag{1.7}$$

for

$$N_n := \text{Diag}\left((cn)!, (cn-1)!\right). \tag{1.8}$$

Here we have also used the simplification of considering only the 0th term of the cycle in order to avoid cumbersome notation.

As we will see in Section 5 the sequence $(v_n)$ lends itself to a more general analysis that is independent of the cycle length. Accordingly, we do not group the corresponding recurrence into cycles as above but study, instead, the base sequence

$$v^{(n)} = Y_n v^{(n-1)} \quad \text{where} \quad Y_n := G_n^{-1} Q_n G_{n-1} \tag{1.9}$$

for

$$G_n := \text{Diag}\left(\Gamma\left(n+\frac{3}{2}\right)a_n^{(n+1)}, \Gamma\left(n+\frac{1}{2}\right)a_{n-1}^n\right). \tag{1.10}$$

By a standard identity [7, Eq.(1.2.10)], the separation of the convergents to $\mathcal{S}_1$ can be written as

$$\begin{aligned}\frac{p_{cn}}{q_{cn}} - \frac{p_{cn-1}}{q_{cn-1}} &= \frac{(-1)^{cn-1}(cn)!^2}{q_{cn}q_{cn-1}} \prod_{j=1}^{cn} a_j^2 \\ &= \frac{(-1)^{cn-1}(cn)!^2}{q_{cn}q_{cn-1}} \left(\prod_{j=1}^{c} \alpha_j\right)^n . \end{aligned} \tag{1.11}$$

In terms of the renormalized sequences $(t_n)$, and $(v_n)$, this is

$$\frac{p_{cn}}{q_{cn}} - \frac{p_{cn-1}}{q_{cn-1}} = \frac{(-1)^{cn-1}}{t_{cn+1}t_{cn}(cn+1)} \left(\prod_{j=1}^{c} \alpha_j\right)^n \tag{1.12}$$

$$= \frac{(-1)^{cn-1}}{v_{cn}v_{cn-1}a_{cn}^{(cn+1)}a_{cn-1}^{cn}} \left(\prod_{j=1}^{cn} \alpha_j\right) \left\{1 + O\left(\frac{1}{n}\right)\right\}. \tag{1.13}$$

Thus, for $|a_n| = |a_m| = b \neq 0$ for all $n, m \in \mathbb{N}$, the continued fraction $\mathcal{S}_1$ diverges – that is, the convergents separate – if

$$|t_n| \leq O\left(\frac{b^n}{\sqrt{n}}\right) \quad \text{or} \quad (v_n) \text{ is bounded}, \tag{1.14}$$

each of these being equivalent.

To begin, we focus our attention on cyclic parameters, that is $a_{n+c} = a_n$ for $c \geq 1$ and all $n$. We then broaden the analysis to infinite sequences. In Section 4 we investigate the first of the convergence criterion Eq.(1.14) using operator norms. The advantage of this criterion is that it yields an asymptotic estimate of the rate of divergence of $\mathcal{S}_1$ for the cases we consider. The second criterion allows for more general, albeit less detailed, analytical techniques, which we apply in Section 5. The case of random sequences $(a_n)$ builds upon the ideas developed in Section 5 but requires a slightly different perspective which we develop in [5]. A summary of our most attractive results is given in Theorem 6.1. Before proceeding with the analysis, however, we motivate this study in Section 2 with some numerical experiments of specific examples.

## 2 Numerical Motivation

For different cases of the parameters $a_n$ in the continued fraction $\mathcal{S}_1$ we plot in the complex plane odd and even iterates of the recurrence

$$t_n = \frac{1}{n}t_{n-1} + \frac{n-1}{n}\alpha_{n-1}t_{n-2}, \tag{2.1}$$

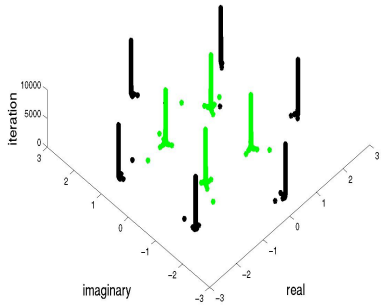



Figure 1: *Dynamics for cycles of length $c = 2$. Shown are the iterates $\widetilde{t}_n := \sqrt{n}t_n$ for $t_n$ given by Eq.(2.1) with $(a_1, a_2) = (\exp(i\pi/4), \exp(i\pi/6))$. Odd iterates are light, even iterates are dark.*

which follows directly from the rescaling Eq.(1.5). Our examples focus on the case $|a_n| = b$ for all $n$, and, in particular (without loss of generality) $|a_n| = 1$. In order to confirm the order of convergence of the iterates required by Eq.(1.14) (indicating the *divergence* of $\mathcal{S}_1$), we plot $\sqrt{n}t_n$. As a point of reference we reproduce in Fig. 1 the dynamics for periodic $(a_n)$ with cycle length 2, and each $a_1$ and $a_2$ being roots of unity. This was also demonstrated in [2]. For cycles of length $c = 2n$ $(n \geq 2)$, the rates of convergence appear also to be $O(1/\sqrt{n})$ with odd and even iterates easily distinguishable as shown in Fig. 2. These dynamics are explained principally in Section 4. Even if the sequences $(a_n)$ are random, the odd and even iterates demonstrate a surprising amount of structure as demonstrated by Fig. 3. We explain this remarkable regularity in [5].

When cycle lengths of $(a_n)$ are *odd*, however, the iterates of Eq.(2.1) display a much richer diversity of behaviors. We show examples of cycles of length 3 with parameter values on the unit circle. The iterates still appear to obey a regular odd/even behavior, however in the first case Fig. 4 it appears that the iterates scaled by $\sqrt{n}$ are diverging. This indicates that the order of convergence of the unscaled iterates, if they converge at all, is something greater than $O(1/\sqrt{n})$, which suggests, from Eq.(1.14), that it is possible that $\mathcal{S}_1$ converges for these parameters. On the other hand, for different parameter choices with cycles of length 3 shown in Fig. 5(a), it appears that the odd iterates scaled by $\sqrt{n}$ are converging, while the even iterates are diverging. In light of Eq.(1.14) it is unclear what this indicates about the continued fraction $\mathcal{S}_1$. We get a different picture if we look instead at the iterates $v_n$ of the corresponding recurrence

$$v_n = \frac{2}{a_n(2n+1)} \left(\frac{a_{n-1}}{a_n}\right)^n v_{n-1} + \frac{4n^2}{(2n-1)(2n+1)} \left(\frac{a_{n-2}}{a_n}\right)^{(n-1)} v_{n-2}. \tag{2.2}$$

As with Eq.(2.1) this recurrence follows directly from the rescaling Eq.(1.5). Fig. 5(b) shows the *unscaled* iterates $v_n$. It appears from this simulation that the sequence $(v_n)$ is indeed bounded, though the iterates process around a circle of radius slightly larger than 1 in the complex plane.

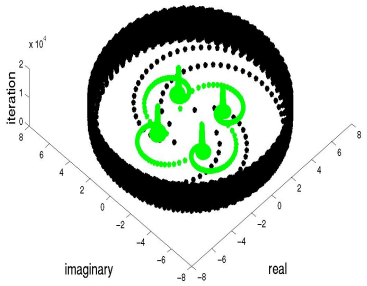
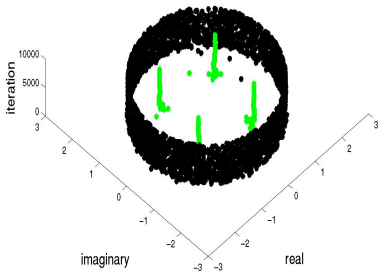
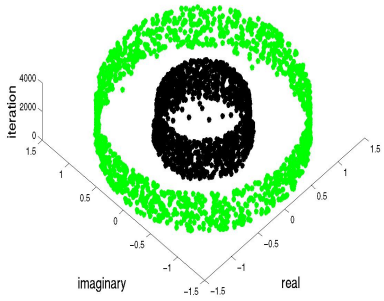



Figure 2: *Dynamics for cycles of length* $c = 4$. *Shown are the iterates* $\widetilde{t}_n := \sqrt{n}t_n$ *for* $t_n$ *given by Eq.(2.1) with cycle length* 4, $a_1 = a_3 = \exp(i\pi/4)$, $a_2 = \exp(i\pi/6)$, $a_4 = \exp(i(\pi/6 + 1/2))$. *Odd iterates are light, even iterates are dark.*




(a)




(b)

Figure 3: *Dynamics for random cycles. Shown are the iterates* $\widetilde{t}_n := \sqrt{n}t_n$ *for* $t_n$ *given by Eq.(2.1) with (a) cycle length* $\infty$ *with only one random strand* $\bmod 2$, $a_{2n+1} = \exp(i\pi/4)$, $a_{2n} = \exp(i\theta_n)$, $\theta_n \sim U[0, 2\pi]$, *and (b) cycle length* $\infty$ *(i.e.* $a_n = \exp(i\theta_n)$, $\theta_n \sim U[0, 2\pi]$ *for all* $n$*). Odd iterates are light, even iterates are dark.*

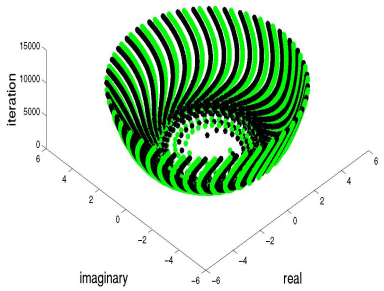



Figure 4: *Dynamics for cycles of length* 3. *Shown are the iterates* $\widetilde{t}_n := \sqrt{n}t_n$ *for* $t_n$ *given by Eq.(2.1) with* $(a_1, a_2, a_3) = (\exp(i\pi/4), \exp(i\pi/4), \exp(i\pi/4 + 1/\sqrt{2}))$. *Odd iterates are light, even iterates are dark.*

This indicates that the continued fraction $\mathcal{S}_1$ diverges for these parameter values. While it is not apparent from this example, the iterates $v_n$ display the same odd/even behavior as the rescaled iterates $t_n$. In our final example Fig. 6 we show the remarkable behavior of a length 3 cycle with well balanced parameters $a_n$. We explain exactly what we mean by "well balanced" in Section 5. In the first example, the scaled iterates of Eq.(2.1) appear to line up at specific locations in the complex plane. In the second example we see concentric orbits familiar from the even cycle examples.

Our object in the following analysis is to shed some light on some of these dynamics.

## 3 Convergence: the real case

It was conjectured in [3] and proved in [2] that:

**Theorem 3.1 (length 2 cycles)** *The original Ramanujan continued fraction*

$$\mathcal{R}_1(a_1, a_2) := \frac{a_1}{1 + \mathcal{S}_1(a_2, a_1)}$$

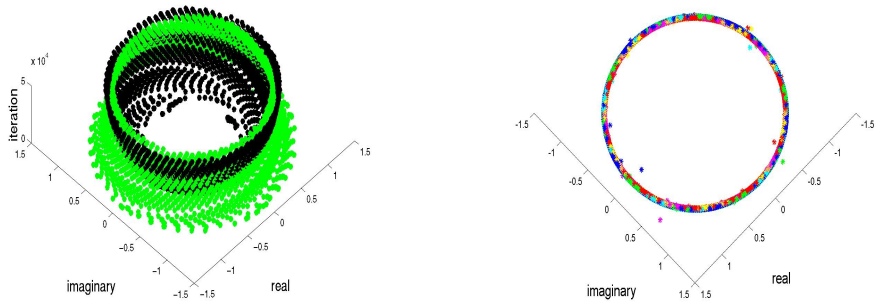
*diverges if and only if*

$$0 \neq a_2 = a_1 e^{i\phi} \text{ with } \cos^2\phi \neq 1$$

*or if*

$$a_1^2 = a_2^2 \in (-\infty, 0).$$

*In particular,* $\mathcal{R}_1(a_1, a_2)$ *converges for all real* $a_1$ *and* $a_2$.




(a) (b)

Figure 5: *Dynamics for cycles of length* 3. *Shown are the iterates (a)* $\widetilde{t}_n := \sqrt{n}t_n$ *for* $t_n$ *given by Eq.(2.1) and (b)* $v_n$ *given by Eq.(2.2). In both of these examples the parameter values are* $(a_1, a_2, a_3) = (\exp(i\pi/4), -\exp(i\pi/4), \exp(i\pi/4 + 1/\sqrt{2}))$. *Odd iterates are light, even iterates are dark.*

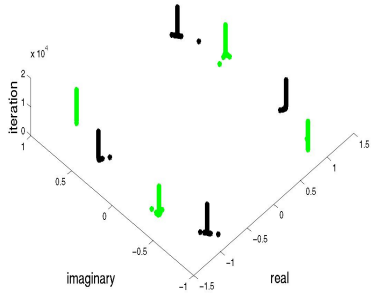



Figure 6: *Dynamics for cycle of length* $c = 3$. *Shown are the iterates* $\widetilde{t}_n := \sqrt{n}t_n$ *for* $t_n$ *given by Eq.(2.1) with* $(a_1, a_2, a_3) = (\exp(i\pi/2), \exp(i\pi/6), \exp(-i\pi/6))$. *Even iterates are light, odd iterates are dark.*

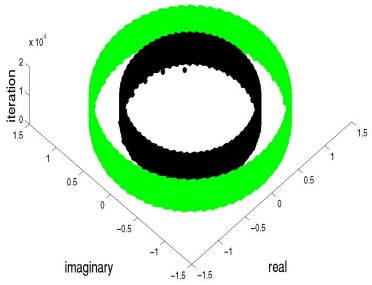



Figure 7: *Dynamics for cycle of length $c = 3$. Shown are the iterates $\widetilde{t}_n := \sqrt{n}t_n$ for $t_n$ given by Eq.(2.1) with $(a_1, a_2, a_3) = (\exp(i(\pi/3 + 0.05)), \exp(-i(\pi/3 + 0.05)), \exp(0.05i))$. Even iterates are light, odd iterates are dark.*

We note that only the case $a_1^2 = a_2^2 < 0$ leads to something other than convergence of the odd and even iterates to distinct values [2, Section 2].

Our main task in the rest of this paper is to establish a general divergence theorem capturing much of Theorem 3.1. Before we do, we establish a simple but fairly general real convergence result.

**Theorem 3.2 (arbitrary real parameters)** *The generalized Ramanujan continued fraction $\mathcal{S}_1$ converges whenever all parameters $a_n$ are real and satisfy $0 < m \le |a_n| \le M < \infty$.*

*Proof.* The proof follows much as the two term case given in Theorem 2.1 of [4]. We write $\mathcal{S}_1$ as a reduced continued fraction $\widehat{\mathcal{S}}_1$ with coefficients $A_i > 0$, that is,

$$\widehat{\mathcal{S}}_1(a) = \cfrac{1}{A_1 + \cfrac{1}{A_2 + \cfrac{1}{A_3 + \ddots}}} \tag{3.1}$$

where

$$A_n = \begin{cases} \frac{n!^2}{(n/2)!^4} 4^{-n} \prod_{j=1}^{n/2} \frac{a_{2j-1}^2}{a_{2j}^2} \;=\; \left(\frac{2}{n\pi} + O\left(\frac{1}{n^2}\right)\right) \prod_{j=1}^{n/2} \frac{a_{2j-1}^2}{a_{2j}^2} & (n \text{ even}) \\ \frac{(((n-1)/2)!)^4}{n!^2} \frac{4^{n-1}}{a_n^2} \prod_{j=1}^{(n-1)/2} \frac{a_{2j}^2}{a_{2j-1}^2} \;=\; \left(\frac{\pi}{2n} + O\left(\frac{1}{n^2}\right)\right) \frac{1}{a_n^2} \prod_{j=1}^{(n-1)/2} \frac{a_{2j}^2}{a_{2j-1}^2} & (n \text{ odd}). \end{cases} \tag{3.2}$$

Note that for real $a_n$ satisfying $0 < m \le |a_n| \le M < \infty$ the sum of the coefficients $A_n$ is unbounded. Convergence then follows from the Seidel-Stern Theorem [7], which asserts that a reduced continued fraction converges if and only if $\sum A_i = \infty$. □

The case of complex convergence is much more vexing, largely because the Seidel-Stern result is not available.

## 4 Cyclic parameters: analysis of the renormalized sequence $(t_n)_{n=1}^{\infty}$

We look first at bounds on the sup-norm of the matrix $T_n$ defined by Eq.(1.7). For parameter cycles $a_n = a_{n+c}$ of length $c$, induction on $c$ shows that

$$t^{(cn)} = T_n t^{(c(n-1))} = \frac{2n-2}{2n-1}\left\{F_n + O(n^{-2})\right\} t^{(c(n-1))}. \tag{4.1}$$

Here, for $c$ even,

$$F_n := \begin{bmatrix} \widetilde{\alpha}_1 & \frac{1}{cn}\widetilde{\alpha}_2 \\ \frac{1}{cn}\widetilde{\alpha}_3 & \widetilde{\alpha}_4 \end{bmatrix} \tag{4.2}$$

with[1]

$$\widetilde{\alpha}_1 := \prod_{j=0}^{\frac{c}{2}-1} \alpha_{2j}, \qquad \widetilde{\alpha}_2 := \alpha_1 \sum_{\substack{c-2\geq j_{(\frac{c}{2}-1)}-2\geq \cdots \\ \cdots \geq j_1 \geq 3}} \prod_{i=1}^{\frac{c}{2}-1} \alpha_{j_i}, \tag{4.3}$$

$$\widetilde{\alpha}_3 := \sum_{\substack{c-3\geq j_{(\frac{c}{2}-1)}-2\geq \cdots \\ \cdots \geq j_1 \geq 2}} \prod_{i=1}^{\frac{c}{2}-1} \alpha_{j_i} \qquad \text{and} \qquad \widetilde{\alpha}_4 := \prod_{j=0}^{\frac{c}{2}-1} \alpha_{2j+1}. \tag{4.4}$$

For $c$ odd, however,

$$F_n := \begin{bmatrix} \frac{1}{cn}\widehat{\alpha}_1 & \left(1+\frac{1}{2cn}\right)\widehat{\alpha}_2 \\ \left(1-\frac{1}{2cn}\right)\widehat{\alpha}_3 & \frac{1}{cn}\widehat{\alpha}_4 \end{bmatrix} \tag{4.5}$$

[1] We define empty products to be equal to 1.

with

$$\widehat{\alpha}_1 := \sum_{\substack{c-2\geq j_{\frac{c-1}{2}}-2\geq\cdots \\ \cdots\geq j_1-2\geq 0}} \prod_{i=1}^{\frac{c-1}{2}} \alpha_{j_i}, \qquad \widehat{\alpha}_2 := \prod_{j=0}^{\frac{c-1}{2}} \alpha_{2j+1}, \tag{4.6}$$

$$\widehat{\alpha}_3 := \prod_{j=1}^{\frac{c-1}{2}} \alpha_{2j}, \qquad \text{and} \qquad \widehat{\alpha}_4 := \alpha_1 \sum_{\substack{c-3\geq j_{\frac{c-1}{2}}-2\geq\cdots \\ \cdots\geq j_1-2\geq 1}} \prod_{i=1}^{\frac{c-1}{2}} \alpha_{j_i}. \tag{4.7}$$

From Eq.(4.1) we obtain the bound

$$|t^{(cN)}| \leq \left( \prod_{n=2}^{N} \frac{2n-2}{2n-1} \left( |F_n| + O\left(n^{-2}\right) \right) \right) |t^{(c)}|. \tag{4.8}$$

By the Wallis/Stirling formula [1] we know that

$$\prod_{n=2}^{N} \frac{2n-2}{2n-1} = \sqrt{\frac{\pi}{4N}} + O\left(N^{-3/2}\right). \tag{4.9}$$

Moreover, the leading order behavior of $F_n$ is

$$|F_n| = \begin{cases} \frac{1}{\sqrt{2}} \sqrt{\widetilde{b}_1 + |\widetilde{b}_2|} + O\left(n^{-2}\right), & c \text{ even and } |\widetilde{\alpha}_1| \neq |\widetilde{\alpha}_4|, \\ \frac{1}{\sqrt{2}} \sqrt{\widehat{b}_1 + |\widehat{b}_2|} \left(1 - \frac{\widehat{b}_2}{2cn|\widehat{b}_2|}\right) + O\left(n^{-2}\right), & c \text{ odd and } |\widehat{\alpha}_3| \neq |\widehat{\alpha}_3| \end{cases} \tag{4.10}$$

where

$$\begin{aligned} \widetilde{b}_1 &:= |\widetilde{\alpha}_1|^2 + |\widetilde{\alpha}_4|^2, & \widetilde{b}_2 &:= |\widetilde{\alpha}_1|^2 - |\widetilde{\alpha}_4|^2, \\ \widehat{b}_1 &:= |\widehat{\alpha}_2|^2 + |\widehat{\alpha}_3|^2 \qquad \text{and} & \widehat{b}_2 &:= |\widehat{\alpha}_2|^2 - |\widehat{\alpha}_3|^2. \end{aligned}$$

For $c$ even, then, if $|\widetilde{\alpha}_1| \neq |\widetilde{\alpha}_4|$, the behavior of the product of matrix norms is

$$\prod_{n=2}^{N} \left( |F_n| + O\left(n^{-2}\right) \right) = O\left( \max\left( |\widetilde{\alpha}_1|,\ |\widetilde{\alpha}_4| \right)^N \right), \quad c \text{ even}. \tag{4.11}$$

For $c$ odd with $\widehat{b}_2 > 0$ we have

$$\prod_{n=2}^{N} \left( |F_n| + O\left(n^{-2}\right) \right) = O\left( \frac{|\widehat{\alpha}_2|^N}{N^{1/2c}} \right), \quad c \text{ odd}. \tag{4.12}$$

If, on the other hand, $\widehat{b}_2 < 0$, we see from Eq.(4.10) that the matrix product is unbounded as $N \to \infty$ and we lose any predictive power from this analysis. The behavior of the product of matrix norms apparently depends more intricately on the values of the sequence $(a_n)$ due to the $O\left(n^{-1}\right)$ term in Eq.(4.10). This also holds for the case $|\widetilde{\alpha}_1| = |\widetilde{\alpha}_4|$ or $|\widehat{\alpha}_3| = |\widehat{\alpha}_3|$ for $c$ even or odd respectively since, in this case,

$$|F_n| \quad = \quad \begin{cases} |\widetilde{\alpha}_1| + \frac{1}{2cn}\sqrt{|\widetilde{\alpha}_1|^2 + |\widetilde{\alpha}_2|^2 + |\widetilde{\alpha}_3|^2 + 2\mathrm{Re}\left(\frac{\overline{\widetilde{\alpha}_1}\widetilde{\alpha}_2\widetilde{\alpha}_3}{\widetilde{\alpha}_4}\right)} + O\left(n^{-2}\right), & |\widetilde{\alpha}_1| = |\widetilde{\alpha}_4| \\ |\widehat{\alpha}_2| + \frac{1}{2cn}\sqrt{|\widehat{\alpha}_1|^2 + |\widehat{\alpha}_2|^2 + |\widehat{\alpha}_4|^2 + 2\mathrm{Re}\left(\frac{\widehat{\alpha}_1\overline{\widehat{\alpha}_2}\widehat{\alpha}_4}{\widehat{\alpha}_3}\right)} + O\left(n^{-2}\right), & |\widehat{\alpha}_2| = |\widehat{\alpha}_3|. \end{cases} \tag{4.13}$$

A straight forward calculation shows that the $O(n^{-1})$ term in Eq.(4.13) disappears only when $F_n$ is the trivial zero matrix, hence there are no nontrivial sequences $(a_n)$ for which the product of matrices converges.

We summarize this discussion with the following theorem which generalizes [2, Theorem 5.1].

**Theorem 4.1 (convergence/divergence rates for cyclic parameters)** *Let the coefficients of the continued fraction $\mathcal{S}_1(a)$ be given by $a = (a_1, a_2, \ldots, a_c) \in \mathbb{C}^c$. For $c$ even with $|\widetilde{\alpha}_1| \neq |\widetilde{\alpha}_4|$, and $\widetilde{\alpha}_j$ defined by Eq.(4.3)-Eq.(4.4), any solution of the recurrence Eq.(1.12) has the asymptotic behavior*

$$|t_{cn}| \leq O\left(\frac{\max(|\widetilde{\alpha}_1|, |\widetilde{\alpha}_4|)^n}{\sqrt{n}}\right),$$

*and the convergents to $\mathcal{S}_1(a)$ satisfy, for $\gamma > 0$ constant,*

$$\left|\frac{p_{2cn}}{q_{2cn}} - \frac{p_{2cn-1}}{q_{2cn-1}}\right| \geq \gamma \min\left(\left|\frac{\widetilde{\alpha}_4}{\widetilde{\alpha}_1}\right|, \left|\frac{\widetilde{\alpha}_1}{\widetilde{\alpha}_4}\right|\right)^n.$$

*For $c$ odd and $|\widehat{\alpha}_2| > |\widehat{\alpha}_3|$ where $\widehat{\alpha}_j$ are defined by Eq.(4.6)-Eq.(4.7), any solution of recurrence Eq.(1.12) has the asymptotic behavior*

$$|t_{cn}| \leq O\left(\frac{|\widehat{\alpha}_2|^n}{n^{(c+1)/(2c)}}\right),$$

*and the convergents to $\mathcal{S}_1(a)$ satisfy, for $\gamma > 0$ constant,*

$$\left|\frac{p_{2cn}}{q_{2cn}} - \frac{p_{2cn-1}}{q_{2cn-1}}\right| \geq \gamma \left(\frac{|\widehat{\alpha}_3|}{|\widehat{\alpha}_2|}\right)^n n^{1/c}.$$

# 5 General deterministic parameters: analysis of the renormalized sequence $(v_n)_{n=1}^{\infty}$

The analysis of the previous section provides upper bounds on the rate of convergence of the sequence $(t_n)$ for $|\widetilde{\alpha}_1| \neq |\widetilde{\alpha}_4|$ in the case of cycles of even length, and for odd-length cycles when

$|\widehat{\alpha}_2| > |\widehat{\alpha}_3|$. The case $|\widetilde{\alpha}_1| = |\widetilde{\alpha}_4|$ for periodic $(a_n)$, and, more generally, the case of infinite or *random* sequences $(a_n)$ requires different analytical tools which we study in this section. Since stochastic notions intersect only obliquely with deterministic phenomena, we consider exclusively deterministic sequences. The stochastic analysis detailed in [5] builds naturally upon the deterministic ideas.

## 5.1 Matrix Products

We pursue here a matrix analysis of $\mathcal{S}_1$ based on the renormalized sequence $(v^{(n)})$ defined by Eq.(1.9). Though the basic framework of our analysis makes no use of the notion of a cycle, the sequence $(v^{(n)})$ still exhibits an odd/even behavior. To exploit this, we define the matrix $\widehat{Y}_n$ by

$$\widehat{Y}_n := Y_{2n} Y_{2n-1}. \tag{5.1}$$

This has the explicit representation

$$\widehat{Y}_n = \begin{bmatrix} \left(\frac{\alpha_{2n-2}}{\alpha_{2n}}\right)^n \left(\frac{1}{\alpha_{2n-2}\alpha_{2n}}\right)^{1/2} \left(\frac{1+4n^2\alpha_{2n}}{4n^2-1/4}\right) & \left(\frac{\alpha_{2n-3}}{\alpha_{2n}}\right)^n \left(\frac{\alpha_{2n-1}}{\alpha_{2n-3}\alpha_{2n}^{1/2}}\right) \left(\frac{(2n-1)^2}{(2n-3/2)(4n^2-1/4)}\right) \\ \left(\frac{\alpha_{2n-2}}{\alpha_{2n-1}}\right)^n \frac{1}{\alpha_{2n-2}^{1/2}} \frac{1}{(2n-1/2)} & \left(\frac{\alpha_{2n-3}}{\alpha_{2n-1}}\right)^{n-1} \left(\frac{(2n-1)^2}{(2n-1)^2-1/4}\right) \end{bmatrix}. \tag{5.2}$$

The determinant of this general $\widehat{Y}_n$ is

$$\det(\widehat{Y}_n) = \left(\frac{\alpha_{2n-2}}{\alpha_{2n}}\right)^{n-1/2} \left(\frac{\alpha_{2n-3}}{\alpha_{2n-1}}\right)^{n-1} \frac{64n^2(2n-1)^2}{(4n-3)(4n-1)^2(4n+1)}.$$

Not only is the odd/even behavior manifest in the $\alpha_j$ terms above, but, as it turns out, the identity

$$\prod_{n=2}^{\infty} \frac{64n^2(2n-1)^2}{(4n-3)(4n-1)^2(4n+1)} = \frac{\pi}{2}$$

follows readily from the Wallis formula. Hence

$$\lim_{n\to\infty} \det(\mathcal{Y}_n) = \frac{\pi}{2}\beta \quad \text{where} \quad \mathcal{Y}_n = \prod_{j=1}^{n} \widehat{Y}_n. \tag{5.3}$$

and

$$\beta := \prod_{n=2}^{\infty} \left(\frac{\alpha_{2n-2}}{\alpha_{2n}}\right)^{n-1/2} \left(\frac{\alpha_{2n-3}}{\alpha_{2n-1}}\right)^{n-1} = \lim_{n\to\infty} \frac{\alpha_2^{1/2}}{\alpha_{2n}^{n-1/2}\alpha_{2n-1}^{n-1}} \prod_{j=1}^{2n-2} \alpha_j. \tag{5.4}$$

Existence of $\lim_{n\to\infty} \det(\mathcal{Y}_n)$ therefore depends on the product Eq.(5.4). Nevertheless, convergence of the determinant is no guarantee of the same for the matrices $\mathcal{Y}_n$. Proving that the matrices converge is the object of the analysis that follows. Assume, for the moment however, that

$\lim_{n\to\infty} \mathcal{Y}_n = \mathcal{Y}^\infty$ where $\mathcal{Y}^\infty$ is a finite complex matrix. We then have the following generalization of [2, Theorem 4.1] concerning the convergence of odd and even parts of $\mathcal{S}_1(a)$.

**Theorem 5.1 (odd and even convergents of continued fractions)** *Let the nonzero complex sequence $(a_n)_{n=1}^\infty$ satisfy*

$$\beta := \lim_{n\to\infty} \frac{a_2}{a_{2n}^{2n-1} a_{2n-1}^{2n-2}} \prod_{j=1}^{2n-2} a_j^2 \neq 0.$$

*For the corresponding continued fraction $\mathcal{S}_1(a)$ defined by Eq.(1.1), let $(u_n)$ be the analog to $(v_n)$ in Eq.(1.5) with $q_n$ replaced by $p_n$. Let the matrix $\mathcal{Y}_n$ defined by Eq.(5.3) converge to the matrix $\mathcal{Y}^\infty$. For the standard initial conditions*

$$(u_{-1}, u_0, v_{-1}, v_0) = \left( \frac{1}{\sqrt{\pi}}, 0, 0, \frac{2}{a_0\sqrt{\pi}} \right), \tag{5.5}$$

*the even and odd parts of $\mathcal{S}_1(a)$ are given by*

$$\mathcal{S}_1^{(even)}(a) = \frac{a_0 y_{1,2}^\infty}{2y_{1,1}^\infty}, \quad \text{and} \quad \mathcal{S}_1^{(odd)}(a) = \frac{a_0 y_{2,2}^\infty}{2y_{2,1}^\infty}. \tag{5.6}$$

*These limits are not equal, thus $\mathcal{S}_1$ diverges. Indeed, the separation of odd and even limits is given explicitly by*

$$\mathcal{S}_1^{(even)}(a) - \mathcal{S}_1^{(odd)}(a) = -\frac{a_0^2\pi}{4a_2 y_{1,1}^\infty y_{2,1}^\infty}\beta. \tag{5.7}$$

*Proof.* The first relation Eq.(5.6) is immediate from the definition of the classical convergents. The limits cannot be equal since otherwise we would have

$$\frac{a_0 y_{1,2}^\infty}{2y_{1,1}^\infty} = \frac{a_0 y_{2,2}^\infty}{2y_{2,1}^\infty} \quad \Longrightarrow \quad y_{1,1}^\infty y_{2,2}^\infty - y_{1,2}^\infty y_{2,1}^\infty = 0$$

whence, from Eq.(5.3), $\beta = 0$. But this contradicts the assumption that $\beta \neq 0$. To see Eq.(5.7) note that, by Eq.(1.13) and the initial condition $(v_{-1}, v_0) = (0, 2/(a_0\sqrt{\pi})$,

$$\begin{aligned} \mathcal{S}_1^{(even)}(a) - \mathcal{S}_1^{(odd)}(a) &= \lim_{n\to\infty} -\frac{\left(\prod_{j=1}^{2n} \alpha_j\right)}{v_{2n} v_{2n-1} \alpha_{2n}^{n+1/2} \alpha_{2n-1}^n}, \\ &= \lim_{n\to\infty} -\frac{a_0^2\pi}{4y_{1,1}^{(n)} y_{2,1}^{(n)}} \frac{\left(\prod_{j=1}^{2n} \alpha_j\right)}{\alpha_{2n}^{n+1/2} \alpha_{2n-1}^n}, \end{aligned}$$

where $y_{i,j}^{(n)}$ is the $ij$-th element of the matrix $\mathcal{Y}_n$ defined by Eq.(5.3). The limit above, together with Eq.(5.4), yields

$$\mathcal{S}_1^{(even)}(a) - \mathcal{S}_1^{(odd)}(a) = -\frac{a_0^2\pi}{4a_2 y_{1,1}^\infty y_{2,1}^\infty}\beta.$$

□

**Remark 5.2** This result makes no use of the specific nature of the parameters $a_n$ beyond the qualification that $\lim_{n\to\infty} \mathcal{Y}_n$ exists and is nonsingular. If $\beta = 0$, then the analysis is indeterminate. Formally from the definition of the classical convergents we have

$$\frac{2}{a_0\sqrt{\pi}} y_{1,1}^{\infty} \mathcal{S}_1^{(even)} = \frac{1}{\sqrt{\pi}} y_{1,2}^{\infty} \quad \text{and} \quad \frac{2}{a_0\sqrt{\pi}} y_{2,1}^{\infty} \mathcal{S}_1^{(odd)} = \frac{1}{\sqrt{\pi}} y_{2,2}^{\infty}.$$

Multiplying the equation on the left by $y_{2,1}^{\infty}$ and the right by $y_{1,1}^{\infty}$ and subtracting yields

$$\frac{1}{\sqrt{\pi}} \left( y_{1,1}^{\infty} y_{2,2}^{\infty} - y_{1,2}^{\infty} y_{2,1}^{\infty} \right) = \frac{2}{a_0\sqrt{\pi}} y_{2,1}^{\infty} y_{1,1}^{\infty} \left( \mathcal{S}_1^{(even)} - \mathcal{S}_1^{(odd)} \right).$$

But, since $\beta = 0$, by Eq.(5.3) we have $y_{1,1}^{\infty} y_{2,2}^{\infty} - y_{1,2}^{\infty} y_{2,1}^{\infty} = 0$. and so $y_{2,1}^{\infty} y_{1,1}^{\infty} \left( \mathcal{S}_1^{(even)} - \mathcal{S}_1^{(odd)} \right) = 0$. We cannot determine from this analysis whether the separation of the odd and even convergents is zero as would be the case if $\mathcal{S}_1$ were to converge. □

It remains to determine whether (or in what sense) $\mathcal{Y}_n$ converges as $n \to \infty$. We begin by extracting the leading-order behavior. Expanding $\widehat{Y}_n$ in powers of $n^{-1}$ yields

$$\widehat{Y}_n = K_n + \frac{1}{2n} W_n + O\left(n^{-2}\right)$$

where

$$K_n = \begin{bmatrix} \left(\frac{\alpha_{2n-2}}{\alpha_{2n}}\right)^{n-1/2} & 0 \\ 0 & \left(\frac{\alpha_{2n-3}}{\alpha_{2n-1}}\right)^{n-1} \end{bmatrix}, \tag{5.8}$$

and

$$W_n = \begin{bmatrix} 0 & \frac{1}{a_{2n}} \left(\frac{\alpha_{2n-3}}{\alpha_{2n}}\right)^{n} \left(\frac{\alpha_{2n-1}}{\alpha_{2n-3}}\right) \\ \frac{1}{a_{2n-2}} \left(\frac{\alpha_{2n-1}}{\alpha_{2n-2}}\right)^{-n} & 0 \end{bmatrix}. \tag{5.9}$$

As we shall see in Theorem 5.3 below, we need only focus our attention on the leading order behavior of the matrix product $\mathcal{Y}_n$. Define

$$\mathcal{U}_n := \prod_{j=2}^{n} K_j + \frac{1}{2j} W_j.$$

Then

$$\mathcal{Y}_n = \prod_{j=2}^{n} \left( K_j + \frac{1}{2j} W_j + O(j^{-2}) \right) = \mathcal{U}_n + O(n^{-2}) \tag{5.10}$$

Induction on $n$ shows that $\mathcal{U}_n$ factors as

$$\mathcal{U}_n = \left( \prod_{j=2}^{n} K_j \right) \prod_{j=2}^{n} \left( I + \frac{1}{2j} \widehat{W}_j \right), \tag{5.11}$$

where

$$\widehat{W}_n := \left( \prod_{j=2}^{n} K_j \right)^{-1} \left( \prod_{j=2}^{n-1} \widehat{K}_j \right) W_n$$

for

$$\widehat{K}_j = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} K_j \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Happily, $\widehat{W}_n$ has a simple explicit representation:

$$\widehat{W}_n = \frac{1}{a_{2n}} \begin{bmatrix} 0 & \omega_n \\ \omega_n^{-1} & 0 \end{bmatrix}, \quad \text{where} \quad \omega_n = \frac{a_{2n}}{a_2} \prod_{j=1}^{n} \frac{\alpha_{2j-1}}{\alpha_{2j}}. \tag{5.12}$$

At this point one can easily see how the parity of the cycle lengths (odd or even) might have a profound impact on the dynamics of the recursion for the classical convergents of $\mathcal{S}_1$. For cycle length $c$ even we have

$$\omega_n = \frac{a_{2n \bmod c}}{a_2} \left( \prod_{j=1}^{n \bmod c/2} \frac{\alpha_{2j-1}}{\alpha_{2j}} \right) \left( \prod_{j=1}^{c/2} \frac{\alpha_{2j-1}}{\alpha_{2j}} \right)^{\lfloor 2n/c \rfloor} \qquad (c \text{ even}). \tag{5.13}$$

For odd-length cycles, on the other hand, the power disappears through cancellation:

$$\omega_n = \frac{a_{2n \bmod c}}{a_2} \prod_{j=1}^{n \bmod c} \frac{\alpha_{2j-1}}{\alpha_{2j}} \qquad (c \text{ odd}). \tag{5.14}$$

We will return to this in the next sections.

To ease the computations, we focus our attention on the rotated product

$$\widehat{U}_n := \left( \prod_{j=2}^{n} K_j \right)^{-1} \mathcal{U}_n \; = \; \prod_{j=2}^{n} \left( I + \frac{1}{2j} \widehat{W}_j \right). \tag{5.15}$$

The justification for looking at these rotated products follows from the next theorem.

**Theorem 5.3 (invertible matrix products)** *Let $(A_n)$ and $(C_n)$ be sequences of $m \times m$ complex matrices. Suppose that (i) $\left( \prod_{j=1}^{n} C_j \prod_{j=1}^{n} A_j \right)$ converges to the invertible matrix $L_1$ as $n \to \infty$, and (ii) $\prod_{j=1}^{n} C_n$ converges to an invertible matrix $L_2$. Then, as $n \to \infty$, the matrix product $\prod_{j=1}^{n} A_n$*

*converges to* $L_2^{-1}L_1$. *Moreover, if* $(B_n)$ *is a sequence of* $m \times m$ *complex matrices satisfying (iii)* $\sum_{j=1}^{\infty} |B_j| < \infty$, *then* $\prod_{j=1}^{n}(A_j + B_j)$ *converges to a finite complex matrix.*

*Proof.* By hypotheses (i), given any $\epsilon$ there is an $N_1$ such that $n \geq N_1$ implies that

$$\left| \prod_{j=1}^{n} C_j \prod_{j=1}^{n} A_j - L_1 \right| < \frac{\epsilon}{2M}$$

where $M \geq \max_{k \geq N_1} \left| \left( \prod_{j=1}^{k} C_j \right)^{-1} \right|$ (which exists by (ii)). The Cauchy-Schwarz inequality then yields

$$\left| \prod_{j=1}^{n} A_j - \left( \prod_{j=1}^{n} C_j \right)^{-1} L_1 \right| < \frac{\epsilon}{2}. \tag{5.16}$$

On the other hand, by (ii) and [2, Lemma 6.3b], $(C_j)$ is tail Cauchy, that is, $\prod_{j=p+1}^{q} C_j \to I$ as $q > p \to \infty$, thus there is an $N_2$ such that $n \geq N_2$ implies

$$\left| L_2^{-1} \right| \left| \prod_{j=n+1}^{\infty} C_j - I \right| |L_1| < \frac{\epsilon}{2}.$$

Here, the Cauchy-Schwarz inequality gives

$$\left| \left( \prod_{j=1}^{n} C_j \right)^{-1} L_1 - L_2^{-1} L_1 \right| < \frac{\epsilon}{2}. \tag{5.17}$$

For $n \geq N = \max\{N_1, N_2\}$, adding Eq.(5.16) to Eq.(5.17) and applying the triangle inequality establishes the first statement of the theorem. This fact, along with hypothesis (iii) and [2, Theorem 6.1] yields the second statement and completes the proof. □

The application to continued fractions is an immediate corollary.

**Corollary 5.4** *If* $\widehat{U}_n \to \widehat{U}_\infty$ *and* $\prod_{j=2}^{n} K_j \to \mathcal{K}_\infty$ *where both* $\widehat{U}_\infty$ *and* $\mathcal{K}_\infty$ *are nonsingular, then* $\mathcal{U}_n \to \mathcal{K}_\infty^{-1}\widehat{U}_\infty$ *and* $\mathcal{Y}_n \to \mathcal{Y}^\infty$, *a finite matrix.*

**Remark 5.5 (parameter qualifications)** Before proceeding, a brief summary of our strategy and the attendant restrictions is in order. Most of the restrictions on the sequences $(a_n)$ come from the invertibility assumptions in Theorem 5.1 and Corollary 5.4. The first of these, that $\beta \neq 0$ where $\beta$ is defined by Eq.(5.4) was discussed in Remark 5.2. The assumption that $\prod_{j=1}^{n} K_j$ converges to

an invertible matrix is equivalent to the condition

$$0 \neq \lim_{n\to\infty} \frac{\alpha_2^{1/2}}{\alpha_{2n}^{n-1/2}\alpha_{2n-1}^{n-1}} \prod_{j=1}^{2n-2} \alpha_j < \infty \tag{5.18}$$

The remaining invertibility assumption of Corollary 5.4 is that $\widehat{U}_n$ converges to an invertible matrix. From Eq.(5.12) this is equivalent to the condition

$$0 \neq \det \prod_{j=2}^{\infty} \left( I + \frac{1}{2ja_{2j}} \begin{bmatrix} 0 & \omega_n \\ \omega_n^{-1} & 0 \end{bmatrix} \right) < \infty$$

or, more simply,

$$0 \neq \prod_{j=2}^{\infty} \left( 1 - \frac{1}{(2ja_{2j})^2} \right) < \infty. \tag{5.19}$$

Conditions Eq.(5.18) and Eq.(5.19) are central to our analysis.

## 5.2 Exponential-sums

The problem of determining the convergence or divergence of $\mathcal{S}_1$ has been reduced to determining the convergence or divergence of $\widehat{U}_n$ defined by Eq.(5.15). In [2] an exponential sum analysis was applied to such matrix products for the case of cycles of length $c = 2$ in order to obtain detailed results about the convergence of $\widehat{U}_n$. Though this analysis does not appear to be tractable in general, we set out the formal basis from which useful special cases may be gleaned.

To begin, note that even products of $\widehat{W}_j$ are diagonal matrices, and odd products are skew matrices. To wit, we have

$$\prod_{j=1}^{2n+1} \widehat{W}_j = \left( \prod_{j=1}^{2n+1} \frac{1}{a_{2j}} \right) \begin{bmatrix} 0 & \prod_{j=1}^{2n+1} \omega_j^{(-1^{j+1})} \\ \prod_{j=1}^{2n+1} (\omega_j^{-1})^{(-1^{j+1})} & 0 \end{bmatrix} \tag{5.20}$$

while

$$\prod_{j=1}^{2n} \widehat{W}_j = \left( \prod_{j=1}^{2n} \frac{1}{a_{2j}} \right) \begin{bmatrix} \prod_{j=1}^{2n} \omega_j^{(-1^{j})} & 0 \\ 0 & \prod_{j=1}^{2n} (\omega_j^{-1})^{(-1^{j})} \end{bmatrix} \tag{5.21}$$

Let $\underline{\omega}_n = (\omega_1, \ldots, \omega_n)$ and

$$\mathcal{T}_j(n, \underline{\omega}_n) := \sum_{m_j > \cdots > m_1 \geq 1}^{n} \left( \prod_{k=1}^{j} \frac{\omega_{m_k}^{(-1^{(j \bmod 2)+k})}}{m_k a_{2m_k}} \right) \tag{5.22}$$

where the sum is empty (0) if $j > n$ and, by definition, $\mathcal{T}_0(n, \underline{\omega}_n) := 1$. In general we have

$$\widehat{U}_n = I + \sum_{j=1}^{n} \left( \sum_{m_j > \cdots > m_1 \geq 1}^{n} \left( \prod_{k=1}^{j} \frac{1}{2m_k} \widehat{W}_{m_k} \right) \right) = \begin{bmatrix} \mathcal{A}_n(\underline{\omega}_n) & \mathcal{B}_n(\underline{\omega}_n) \\ \mathcal{B}_n(\underline{\omega}_n^{-1}) & \mathcal{A}_n(\underline{\omega}_n^{-1}) \end{bmatrix},$$

where

$$\mathcal{A}_n(\underline{\omega}_n) := \sum_{j=0}^{\infty} 2^{-2j} \mathcal{T}_{2j}(n, \underline{\omega}_n) \quad \text{and} \quad \mathcal{B}_n(\underline{\omega}_n) := \sum_{j=0}^{\infty} 2^{-2j} \mathcal{T}_{2j+1}(n, \underline{\omega}_n).$$

This formulation is difficult to work with in general, though for some special cases it yields explicit bounds on matrix elements as the next example illustrates.

**Example 5.6** Let $a_n = a_{n+2} \quad (n = 1, 2, \dots)$. From Eq.(5.12) we have

$$\omega_n = \frac{a_{2n}}{a_2} \prod_{j=1}^{n} \frac{\alpha_{2j-1}}{\alpha_{2j}} = \left( \frac{\alpha_1}{\alpha_2} \right)^n$$

which we will write as $\omega^n$. Thus

$$\mathcal{T}_j(n, \underline{\omega}) = \frac{1}{a_2^j} \sum_{m_j > \cdots > m_1 \geq 1}^{n} \left( \prod_{k=1}^{j} \frac{\omega^{(-1^{(j \bmod 2)+k})m_k}}{m_k} \right).$$

This form of the exponential sum, it turns out, is tractable. Indeed, this can be rewritten as

(5.23)

$$\mathcal{T}_j(n, \underline{\omega}) = \left( \prod_{k=1}^{j} \omega^{(-1^{(j \bmod 2)+k})} \right) \int_0^1 \cdots \int_0^1 dx_1 \cdots dx_j S_j(n; \omega^{(-1^{(j \bmod 2)+1})} x_1, \dots, \omega^{(-1^{(j \bmod 2)+j})} x_j),$$

where

$$S_j(n; z_1, \dots, z_j) = \sum_{m_j > \cdots > m_1 \geq 1}^{n} z_n^{m_j} z_{j-1}^{m_{j-1}} \cdots z_1^{m_1}.$$

In particular,

$$S_j(\infty; z_1, \dots, z_j) = \frac{z_j^{j-1}}{1 - z_j} \frac{z_{j-1}^{j-2}}{1 - z_j z_{j-1}} \cdots \frac{1}{1 - z_j z_{j-1} \cdots z_1}.$$

Using these identities, it can been shown that if $|\omega| = 1$ with $\omega \neq 1$, then the matrix $\mathcal{U}_n = \widehat{U}_n$ converges as $N \to \infty$ with explicit bounds on the limit $\mathcal{U}_\infty$ (see [2, Theorem 7.5]). □

An interesting open problem is to find an integral representation similar to Eq.(5.23) for generalized Lerch sums of the form Eq.(5.22) with parameters $\underline{\omega} = (\omega_1, \dots, \omega_n)$ involving more complicated

behavior. Given the simplicity of $w_n$ given by Eq.(5.12), it seems quite likely that tractable reformulations can be extracted from Eq.(5.22).

### 5.3 General Matrix Analysis

Given the difficulty of working with generalized exponential sums of the form Eq.(5.22), we pursue a more general approach. In this section we shall prove following.

**Theorem 5.7 (matrix products)** *Let the sequences $(\zeta_j)$ and $(\zeta'_j)$ satisfy*

$$\sup_k \left| \sum_{j \geq n}^{k} \zeta_j \right| < \infty \quad \text{and} \quad \sup_k \left| \sum_{j \geq n}^{k} \zeta'_j \right| < \infty, \tag{5.24}$$

*and let $(\eta_j)$ be a real nonnegative square summable sequence decreasing monotonically to 0. Then the matrix product*

$$\widehat{T}_n := \prod_{j=1}^{n} \left( I + \eta_j \begin{bmatrix} 0 & \zeta_j \\ \zeta'_j & 0 \end{bmatrix} \right) \tag{5.25}$$

*converges to a finite matrix as $n \to \infty$. If, in addition,*

$$\left| 1 - \eta_j^2 \zeta'_j \zeta_j \right| \geq m > 0 \quad \forall\ j, \tag{5.26}$$

*then $\widehat{T}_n$ converges invertibly.*

**Remark 5.8** Compare this result to a similar result by Trench [8, Theorem 4] which states that for any sequence of $m \times m$ matrices $(A_n)$ the product $\prod_{n=1}^{\infty} (I + A_n)$ converges invertibly if $\sum^{\infty} |A_n| < \infty$. Condition Eq.(5.24) is less restrictive than the requirement that the corresponding matrix norms be summable, however our result is much less general. In light of other sufficient conditions developed by Trench [8, Theorems 5-6] it would be interesting to see if the techniques presented here can be applied to more general matrix products.

As in [2], denote the limit of $(a_n)$ by $a_\infty$, and denote by $(a_n) \prec (\epsilon_n)$ convergence of $(a_n)$ when this is provided by $|a_n - a_\infty| = O(\epsilon_n)$.

**Lemma 5.9** *Let $(a_n)$ and $(b_n)$ be complex sequences, let $(\epsilon_n)$ be a positive sequence and let $(z_n)$ with $|z_n| = z \in \mathbb{R}_+\ \forall\ n = 1, 2, \ldots$ be any complex number. Suppose that*

$$(a_n) \prec (\epsilon_n) \quad \text{and} \quad (b_n) \prec (\epsilon_n),$$

*then*

$$(a_n + b_n) \prec (\epsilon_n), \qquad (a_n b_n) \prec (\epsilon_n) \quad \text{and} \quad (z_n a_n) \prec (\epsilon_n)$$

*Proof.* The first two relations are clear. The last relation follows immediately from $|z_n a_n - z_n a_\infty| = z|a_n - a_\infty|$. □

The following lemma yields a key bound on partial sums which appear in our analysis and is a simple consequence of Abel's transformation [9, Eq.(I.2.1)],

$$\sum_{j=n}^{m} \eta_j \zeta_j = \sum_{j=n}^{m-1} \left( \sum_{k=n}^{j} \zeta_k \right) (\eta_j - \eta_{j+1}) + \eta_m \sum_{k=n}^{m} \zeta_k \qquad (n < m). \tag{5.27}$$

**Lemma 5.10** *Let $(\eta_n)$ be a decreasing sequence of non-negative real numbers that converges to 0 and let $(\zeta_n)$ be a sequence satisfying Eq.(5.24). Then*

$$\left( \sum_{j=1}^{n} \eta_j \zeta_j \right) \prec (\eta_n). \tag{5.28}$$

*Proof.* This is the content of [9, Theorem I.2.2]. Indeed, the absolute value of the right-hand side of Eq.(5.27) is bounded by

$$\eta_n \sup_j \left| \sum_{k=n}^{j} \zeta_k \right| = O(\eta_n).$$

□

**Lemma 5.11 (product convergence)** *Let $(\eta_j)$ be a nonnegative real decreasing sequence converging to 0 with $\sum_{j>0} \eta_j^2 < \infty$, and let $(\zeta_k)$ satisfy Eq.(5.24). Then the product*

$$\prod_{j=1}^{n} (1 + \eta_j \zeta_j) \tag{5.29}$$

*converges as $n \to \infty$.*

*Proof.* By Lemma 5.10, the sequences $(\eta_n)$ and $(\zeta_n)$ satisfy Eq.(5.28). Also, since $(\zeta_n)$ satisfying Eq.(5.24) is bounded and $(\eta_n)$ is square summable, then $(\eta_j \zeta_j)$ is square summable, thus by [6, pp.225] the product Eq.(5.29) converges. □

We are now ready to proceed with the proof of the main result of this section.

*Proof of Theorem 5.7* Our proof follows the same pattern as that of [2, Theorem 8.1]. We split the matrices in the infinite product into upper and lower triangular pieces and show that the resulting

submatrices and their products converge. Let

$$U := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad L := \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

We write

$$\left(I + \eta_n \begin{bmatrix} 0 & \zeta_n \\ \zeta_n' & 0 \end{bmatrix}\right) = (I + \eta_n \zeta_n U)\left(I + \eta_n \zeta_n' L\right) - \eta_n^2 \zeta_n \zeta_n' UL$$

and define the partial product

$$\Pi_{UL}^n = \prod_{j=1}^{n} \left((I + \eta_j \zeta_j U)\left(I + \eta_j \zeta_j' L\right)\right).$$

For $n \in \mathbb{N}$ let

$$\Pi_U^n := \prod_{j=1}^{n} (I + \eta_j \zeta_j U), \qquad \Pi_L^n := \prod_{j=1}^{n} \left(I + \eta_j \zeta_j' L\right)$$

$$\Sigma_U^n := \sum_{j=1}^{n} \eta_j \zeta_j, \qquad \text{and} \qquad \Sigma_L^n := \sum_{j=1}^{n} \eta_j \zeta_j'.$$

We interpret $\Sigma_U^0$ and $\Sigma_L^0$ to be zero. By [2, Lemma 8.6] (replace their "$zm_j\omega^j$" by "$\eta_j\zeta_j$" and "$zm_j\omega^{-j}$" by "$\eta_j\zeta_j'$") $\Pi_{UL}^n$ can be rewritten as

$$\Pi_{UL}^n = \Pi_U^n \Pi_L^n \prod_{j=1}^{n} (I + R_j),$$

where

$$R_n := \eta_n \zeta_n' \begin{bmatrix} -\Sigma_U^{n-1} - \left(\Sigma_U^{n-1}\right)^2 \Sigma_L^{n-1} & -\left(\Sigma_U^{n-1}\right)^2 \\ \Sigma_U^{n-1}\Sigma_L^n + \Sigma_U^{n-1}\Sigma_L^{n-1} + \Sigma_L^{n-1}\Sigma_L^n \left(\Sigma_U^{n-1}\right)^2 & \Sigma_U^{n-1} + \left(\Sigma_U^{n-1}\right)^2 \Sigma_L^n \end{bmatrix}.$$

By the definitions of $\Sigma_U^0$ and $\Sigma_L^0$, we have $R_1 := 0$. The partial sums $\Sigma_U^n$ and $\Sigma_L^n$ converge by Lemma 5.10. By induction it can be shown that

$$\Pi_U^n = \begin{bmatrix} 1 & \Sigma_U^n \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \Pi_L^n = \begin{bmatrix} 1 & \Sigma_L^n \\ 0 & 1 \end{bmatrix},$$

thus the sequences of matrices $(\Pi_U^n)$ and $(\Pi_L^n)$ converge. Hence, if $\prod_{j=1}^{n}(I + R_j)$ converges, then the sequence $(\Pi_{UL}^n)$ converges. Convergence of the product $\prod_{j=1}^{n}(I + R_j)$ follows from Lemmas 5.9 and 5.11, and [2, Lemma 8.7]. Finally, for $(\zeta_n)$ and $(\zeta_n')$ satisfying Eq.(5.24) and $\eta_n$ square summable, we have

$$\sum_{j=1}^{n} \left|\eta_j^2 \zeta_j \zeta_j' UL\right| < \infty,$$

thus, from Theorem 5.3, the product Eq.(5.25) must converge.

Finally note that

$$\det \widehat{T}_n = \det \prod_{j=1}^{n} \left( I + \eta_j \begin{bmatrix} 0 & \zeta_j \\ \zeta'_j & 0 \end{bmatrix} \right) = \prod_{j=1}^{n} \det \left( I + \eta_j \begin{bmatrix} 0 & \zeta_j \\ \zeta'_j & 0 \end{bmatrix} \right) = \prod_{j=1}^{n} (1 - \eta_j^2 \zeta'_j \zeta_j).$$

This product is nonzero if $\left| 1 - \eta_j^2 \zeta'_j \zeta_j \right| \geq m > 0 \quad \forall\, j$, in which case $\widehat{T}_n$ converges invertibly. □

## 5.4 Application to continued fractions

In this section we establish the link between the generalized matrix analysis of the preceding section and the sequences of interest defined by Eq.(5.13)-(5.14).

**Example 5.12 (odd cycles/arbitrary sequences)** Consider the case of odd-length cycles or, more generally, arbitrary sequences $(a_n)$. Here, for $c \in \mathbb{N} \cup \{\infty\}$ not even,

$$\zeta_n = \zeta_{n \bmod c} := \frac{1}{a_{2n \bmod c}} \omega_n = \frac{1}{a_2} \left( \prod_{j=1}^{n \bmod c} \frac{\alpha_{2j-1}}{\alpha_{2j}} \right). \tag{5.30}$$

For the inverse, we have

$$\zeta'_n = \zeta'_{n \bmod c} := \frac{1}{a_{2n \bmod c}} \omega_n^{-1} = \frac{a_2}{\alpha_{2n \bmod c}} \left( \prod_{j=1}^{n \bmod c} \frac{\alpha_{2j-1}}{\alpha_{2j}} \right)^{-1}. \tag{5.31}$$

(see Eq.(5.14)). If $\zeta_n$ and $\zeta'_n$ satisfy

$$\sup_k \left| \sum_{n=1}^{k} \zeta_n \right| < \infty, \quad \text{and} \quad \sup_k \left| \sum_{n=1}^{k} \zeta'_n \right| < \infty. \tag{5.32}$$

then by Lemma 5.10,

$$\left( \sum_{j=1}^{n} \frac{1}{2j a_{2j}} \omega_j \right) \prec \left( \frac{1}{n} \right) \quad \text{and} \quad \left( \sum_{j=1}^{n} \frac{1}{2j a_{2j}} \omega_j^{-1} \right) \prec \left( \frac{1}{n} \right). \tag{5.33}$$

Theorems 5.1, 5.4, and 5.7 then yield the following general result concerning the continued fraction of Ramanujan.

**Theorem 5.13 (continued fractions with arbitrary parameters)** *Let the nonzero complex*

*sequence* $a := (a_n)_{n=1}^{\infty}$ *satisfy Eq.(5.18)-(5.19) in addition to*

$$\sup_k \left| \sum_{n=1}^{k} \frac{1}{a_2} \left( \prod_{j=1}^{n} \frac{\alpha_{2j-1}}{\alpha_{2j}} \right) \right| < \infty \quad \textit{and} \quad \sup_k \left| \sum_{n=1}^{k} \frac{a_2}{\alpha_{2n}} \left( \prod_{j=1}^{n} \frac{\alpha_{2j-1}}{\alpha_{2j}} \right)^{-1} \right| < \infty \tag{5.34}$$

*where* $\alpha_n := a_n^2$. *Then the iterates* $v_n$ *of the corresponding difference equation Eq.(2.2) are bounded and the Ramanujan continued fraction* $\mathcal{S}_1(a)$ *defined by Eq.(1.1) diverges with the even/odd parts of* $\mathcal{S}_1(a)$ *converging to separate limits.*

Condition Eq.(5.34), while nontrivial, is not difficult to satisfy. It is certainly satisfied by any sequence $(\zeta_n)$ which processes "evenly" around the unit circle in the complex plane, or, to anticipate the stochastic analysis, any bounded $(\zeta_n)$ with mean equal to zero[2]. We have already seen instances of such sequences in Section 2. For example, for the case of $(a_n)$ with odd-length cycles, say $c = 3$, then

$$\zeta_1 = \frac{a_1^2}{a_2^3}, \qquad \zeta_2 = \frac{a_3^2}{a_2^3} \quad \text{and} \quad \zeta_3 = \frac{1}{a_2}$$

while

$$\zeta_1' = \frac{a_2}{a_1^2}, \qquad \zeta_2' = \frac{a_2^3}{a_1^2 a_3^2} \quad \text{and} \quad \zeta_1' = \frac{a_2}{a_3^2}.$$

The parameters $a_i$ then must satisfy

$$a_1^2 + a_2^2 + a_3^2 = 0.$$

In Fig. 6 the parameters for iteration Eq.(2.1) with a cycle length 3 are $(a_1, a_2, a_3) = (\exp(\mathrm{i}\pi/2), \exp(\mathrm{i}\pi/6), \exp(-\mathrm{i}\pi/6))$ The dynamics of the corresponding sequence of partial sums is depicted in Fig. 8. In Fig. 7 the parameters are $a_1 = \pi/3 + 0.05$, $a_2 = -\pi/3 + 0.05$, and $a_3 = 0.05$. The dynamics of the corresponding sequence of partial sums are similar to those shown in Fig. 8.

**Example 5.14 (even cycles)** Let $a_j = a_{j+c}$ for all $j$ and $c$ finite and even. Define $\omega_n$ by Eq.(5.13), and define

$$\zeta_k := \frac{1}{a_{2k}} \omega_k = \gamma^{\lfloor 2k/c \rfloor} \xi_k \quad \text{where} \quad \xi_k = \xi_{k \bmod c/2} := \frac{1}{a_2} \left( \prod_{j=1}^{k \bmod c/2} \frac{\alpha_{2j-1}}{\alpha_{2j}} \right)$$

and

$$\gamma := \left( \prod_{j=1}^{c/2} \frac{\alpha_{2j-1}}{\alpha_{2j}} \right).$$

[2]The reason we cannot immediately extend these results to random sequences is because the partial sums Eq.(5.32) are not bounded above.

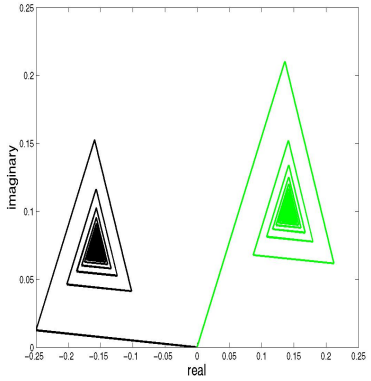



Figure 8: *Sequence of partial sums given by Eq.(5.33) for a cycle length 4 with parameters* $(a_1, a_2, a_3) = (\exp(\mathrm{i}\pi/2), \exp(\mathrm{i}\pi/6), \exp(-\mathrm{i}\pi/6))$ *corresponding to Fig. 6. The dark line corresponds to the partial sums of* $\frac{1}{a_{2j}}\omega_j$ *and the light line to the partial sums of* $\frac{1}{a_{2j}}\omega_j^{-1}$.

Similarly, define

$$\zeta'_k := \frac{1}{a_{2k}}\omega_k^{-1} = \gamma^{-\lfloor 2k/c \rfloor}\xi'_k \quad \text{where} \quad \xi'_k = \xi'_{k \bmod c/2} := \frac{a_2}{\alpha_{2k \bmod c}} \left( \prod_{j=1}^{k \bmod c/2} \frac{\alpha_{2j-1}}{\alpha_{2j}} \right)^{-1}.$$

The following Lemma yields a nice specialization of Theorem 5.13.

**Lemma 5.15** *Let* $\xi_j, \xi'_j \in \mathbb{C}$ *satisfy* $|\xi_j| \le z < \infty$ , $|\xi'_j| \le z' < \infty \ \forall\ j$ *and let* $|\gamma| = 1$ *with* $\gamma \ne 1$. *Then for any positive* $d \in \mathbb{N}$, *we have*

$$\sup_k \left| \sum_{j=0}^{k} \gamma^{\lfloor j/d \rfloor} \xi_j \right| < \infty \quad \textit{and} \quad \sup_k \left| \sum_{j=0}^{k} \gamma^{-\lfloor j/d \rfloor} \xi'_j \right| < \infty.$$

*Proof.* This follows immediately for $d$ finite and $\gamma \ne 1$ since $|\xi_j| \le z < \infty$ and $|\xi'_j| \le z' < \infty$ for all $j$ and $\gamma^{\pm j}$ is a (nonstationary) rotation around the unit disk in $\mathbb{C}$. □

If $|\gamma| = 1$ with $\gamma \ne 1$ then Lemma 5.10 and Lemma 5.15 guarantee that the parameters satisfy Eq.(5.33). If, in addition, $(a_n)$ satisfies Eq.(5.18)-(5.19) then Theorems 5.1, 5.4, and 5.7, together with Lemma 5.15 assure that $\mathcal{S}_1$ diverges for this class of parameters. This yields the following corollary.

**Corollary 5.16 (continued fractions with even parameter cycles)** *Let the nonzero complex sequence* $a := (a_n)_{n=1}^{\infty}$ *be periodic with even period, that is,* $a_n = a_{n+c}$ *for all* $n$ *and a fixed* $c$

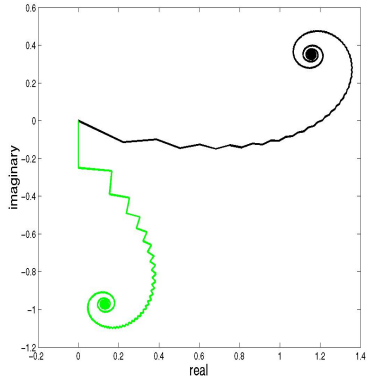



Figure 9: *Sequence of partial sums given by Eq.(5.33) for a cycle length 4 with parameters* $(a_1, a_2, a_3, a_4) = (\exp(\mathrm{i}\pi/4), \exp(\mathrm{i}\pi/6), \exp(\mathrm{i}\pi/4), \exp(\mathrm{i}(\pi/6 + 1/2)))$ *corresponding to Fig. 2. The light line corresponds to the partial sums of* $\frac{1}{a_{2j}}\omega_j$ *and the dark line to the partial sums of* $\frac{1}{a_{2j}}\omega_j^{-1}$.

*even. Define*

$$\gamma := \left( \prod_{n=1}^{c/2} \frac{a_{2n-1}^2}{a_{2n}^2} \right). \tag{5.35}$$

*If* $(a_n)$ *satisfies Eq.(5.18)-(5.19) and* $|\gamma| = 1$ *with* $\gamma \neq 1$*, then the iterates* $v_n$ *of the corresponding difference equation Eq.(2.2) are bounded and the Ramanujan continued fraction* $\mathcal{S}_1(a)$ *defined by Eq.(1.1) diverges with the even/odd parts of* $\mathcal{S}_1(a)$ *converging to separate limits. Conversely, if* $c = 2$ *and* $\mathcal{S}_1(a)$ *diverges, then either*

$$(i) \quad |\gamma| = 1 \quad with \quad \gamma \neq 1 \qquad or \qquad (ii) \quad a_1^2 = a_2^2 \in (-\infty, 0).$$

*Proof.* We have already proved the sufficient conditions for the boundedness of the iterates $v_n$ and corresponding divergence of $\mathcal{S}_1$ in the preceding discussion. In the case $c = 2$, $\beta = a_2/a_1$ where $\beta$ is defined by Eq.(5.4), so condition Eq.(5.18) is automatically satisfied for all nonzero $(a_1, a_2)$. The necessary conditions (i) and (ii) follow immediately from Theorem 3.1. □

The dynamics of the sequence of partial sums corresponding to iteration Eq.(2.1) for a cycle length 4 with parameters $(a_1, a_2, a_3, a_4) = (\exp(\mathrm{i}\pi/4), \exp(\mathrm{i}\pi/6), \exp(\mathrm{i}\pi/4), \exp(\mathrm{i}(\pi/6 + 1/2)))$ (see Fig. 2) is depicted in Fig. 9.

## 5.5 A generalization of $\mathcal{S}_1$

We note in passing an immediate generalization of the convergence theory for continued fractions of the form $\mathcal{S}_1$. Recall that we define $a = (a_1, a_2, \dots)$. In [2] the continued fraction

$$\mathcal{S}_1(a,b) = \cfrac{1^b a_1^2}{1 + \cfrac{2^b a_2^2}{1 + \cfrac{3^b a_3^2}{1 + \ddots}}} \tag{5.36}$$

was studied. It was shown that this leads to the rescaled sequence $(v_n^{(b)})$, analogous to Eq.(1.5),

$$v_n^{(b)} := \frac{q_n}{\Gamma^{b/2}(n+3/2) a_n^{(n+1)}}. \tag{5.37}$$

The difference equation Eq.(2.2) then becomes

$$v_n^{(b)} = \left(\frac{2}{2n+1}\right)^{b/2} \frac{1}{a_n} \left(\frac{a_{n-1}}{a_n}\right)^n v_{n-1}^{(b)} + \left(\frac{4}{(2n-1)(2n+1)}\right)^{b/2} n^2 \left(\frac{a_{n-2}}{a_n}\right)^{(n-1)} v_{n-2}^{(b)}, \tag{5.38}$$

and the matrix product

$$\mathcal{U}_n^{(b)} = \left(\prod_{j=2}^{n} K_j\right) \prod_{j=2}^{n} \left(I + \frac{1}{(2j)^{b/2}} \widehat{W}_j\right). \tag{5.39}$$

Thus we have the following generalization:

**Theorem 5.17 (generalized continued fractions of Ramanujan)** *Let the nonzero complex sequence $a := (a_n)$ satisfy Eq.(5.18), Eq.(5.34) and, for $b > 1$,*

$$0 \neq \prod_{j=2}^{\infty} \left(1 - \frac{1}{(2j)^b a_{2j}^2}\right) < \infty. \tag{5.40}$$

*Then the iterates $v_n^{(b)}$ of the difference equation Eq.(5.38) are bounded and the corresponding Ramanujan continued fraction $\mathcal{S}_1(a,b)$ defined by Eq.(5.36) diverges with the even/odd parts of $\mathcal{S}_1(a,b)$ converging to separate limits.*

*Proof.* Define $\eta_n := (2n)^{-b/2}$. The result then follows from Theorems 5.1, 5.4 and 5.7 with the exception that

$$\lim_{n\to\infty} \det(\mathcal{Y}_n^{(b)}) = \left(\frac{\pi}{2}\right)^{b/2} \beta \quad \text{where} \quad \mathcal{Y}_n^{(b)} = \prod_{j=1}^{n} K_j + \frac{1}{(2j)^{b/2}} W_j + O\left(j^{-b}\right) \tag{5.41}$$

for $\beta$, $K_j$ and $W_j$ given by Eq.(5.4), Eq.(5.8) and Eq.(5.9). $\square$

# 6 Concluding Remarks and Open Problems

We begin with a recapitulation of particularly clean versions of our main results.

**Theorem 6.1 (summary)** *Let the nonzero complex sequence of parameters $a := (a_n)$ satisfy*

$$0 \neq \prod_{n=1}^{\infty} \left(1 - \frac{1}{(2n)^b a_{2n}^2}\right) < \infty \quad \textit{and} \quad 0 \neq \lim_{n\to\infty} \frac{a_2}{a_{2n}^{2n-1} a_{2n-1}^{2n-2}} \prod_{j=1}^{2n-2} a_j^2 < \infty.$$

*For all $b > 1$, the iterates $v_n^{(b)}$ of the corresponding difference equation Eq.(5.38) are bounded and the Ramanujan continued fraction $\mathcal{S}_1(a, b)$ defined by Eq.(5.36) diverges with the even/odd parts of $\mathcal{S}_1(a, b)$ converging to separate limits in the following cases:*

(i) Even periodic parameters: *If $a_n = a_{n+c}$ for all $n$ and fixed $c$ even, and $|\gamma| = 1$ with $\gamma \neq 1$ where*

$$\gamma := \left(\prod_{n=1}^{c/2} \frac{a_{2n-1}^2}{a_{2n}^2}\right).$$

(ii) General deterministic parameters:

$$\sup_k \left|\sum_{j\geq n}^{k} \frac{1}{a_2} \prod_{i=1}^{j} \frac{a_{2i-1}^2}{a_{2i}^2}\right| < \infty \quad \textit{and} \quad \sup_k \left|\sum_{j\geq n}^{k} \frac{a_2}{a_{2j}^2} \prod_{i=1}^{j} \frac{a_{2i}^2}{a_{2i-1}^2}\right| < \infty.$$

While the principal application of interest in this work has been the determination of the divergence of continued fractions, our analysis touches on many different areas of mathematics, from difference equations, to dynamical systems, to matrix theory. We noted in Section 5.2 a direction for further research is to find an integral representation similar to Eq.(5.23) for generalized Lerch sums of the form Eq.(5.22) with parameters $\underline{\omega} = (\omega_1, \ldots, \omega_n)$ involving more complicated behavior. With regard to infinite products of matrices, it would be interesting to see if the techniques presented in Section 5 can be applied to general random matrix products. This would have important implications for Markov processes and random walks. While the continued fractions we considered here lead only to 2 term difference equations such as Eq.(1.2), one could conceive of more general difference equations in and of themselves, for example recursions of the form

$$q_n = (n + 1 - m)\alpha_n \left(\begin{array}{c} n \\ n-m \end{array}\right) q_{n-m-1} + \sum_{j=0}^{m-1} \left(\begin{array}{c} n \\ n-j \end{array}\right) q_{n-(j+1)}$$

and their corresponding renormalized difference equations

$$t_{n+1} = \frac{n+1-m}{n+1}\alpha_n t_{n-m} + \frac{1}{n+1}\sum_{j=0}^{m-1} t_{n-(j+1)}.$$

Such generalizations would lead to an $(m+1)\times(m+1)$ matrix analysis analogous to that pursued here. One possibly far reaching issue is whether or not the general recurrence relations above admit generating functions. If so, what can be said about these generating functions, the sequences they encode and the functions they characterize?

Finally, the attentive reader will note that we have left out any mention of the parameter values corresponding to the simulations shown in Fig. 4 and Fig. 5[3]. It is easy to verify that for the parameter values in these examples the partial sums corresponding to Eq.(5.32) are not bounded. However, condition Eq.(5.32) is only sufficient, thus we cannot determine from our analysis whether or not the continued fraction $\mathcal{S}_1$ converges for these parameter values. Our analysis, while quite general, still leaves undetermined the *necessary* conditions for the matrix products $\mathcal{Y}_n$ to converge.

Indeed, as noted in Section 3, the case of complex convergence is much more vexing, largely because the Seidel-Stern result is not available. Thus, we leave for a sequel the question of a more complete analogue to Theorem 3.1.

## Acknowledgments

We thank Ian Coope and Richard Crandall for many valuable discussions. Special thanks to Ray Mayer who laid the groundwork for the original and novel matrix theory in [2] that has proved so fruitful here.

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[3] We didn't mention Fig. 3 either, but this is treated in the companion paper [5].

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## Footnotes

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1. We define empty products to be equal to 1.

2. The reason we cannot immediately extend these results to random sequences is because the partial sums Eq.(5.32) are not bounded above.

3. We didn't mention Fig. 3 either, but this is treated in the companion paper [5].