HYDROLOGIC OPTICS

Volume IV. Imbeddings

R.W. PREISENDORFER

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16. ABSTRACT The 4th volume of a set of six volumes of the text on hydrologic optics provides detailed treatment on invariant imbedding techniques for light fields. Subjects presented are: differential equations governing the steady state R and T operators; differential equations governing the time dependent R and T operators; algebraic and analytic properties of the R and T operators; algebraic properties of the invariant imbedding operators; analytic properties of the invariant imbedding operators; special solution procedures for ~~R(a,b) and T(a,b) in plane-parallel media; method~~ of modules for deep homogeneous media; method of semigroups for deep homogeneous media; method of groups for deep homogeneous media; method of groups for general optical media; homogeneity, isotropy and related properties of optical media; functional relations for media with internal sources; and invariant imbedding and integral transform techniques. (OEIS)

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i-a

We owe to Ambarzumian the first introduction of a principle of invariance in the treatment of transfer problems.

S. Chandrasekhar
Ref. [43]

These ideas were further developed and extensively generalized by Chandrasekhar.

R.E. Bellman, R.E. Kalaba
M.C. Prestrud Ref. [15]

In three successive generalizations the theory ascended from Ambarzumian's concept of the invariance of a visual impression of brightness up to the concept of the invariance of a *set of radiance functions* under the application of each member of a *set of transformations* associated with a given *set of optical media*.

Ref. [251], p. 167

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PREFACE

It is a relatively rare occurrence in applied mathematics that one encounters a method of solution of a given type of equation that is both effective numerically and rich in physical imagery. With the advent of the principles of invariance into radiative transfer theory, the equation of transfer underlying the theory received its natural solution companion. Together the principles of invariance and the equation of transfer form a combination which illustrates that rare occurrence alluded to above.

In the present work we use this combination to explore the transfer of radiant energy through general optical media (exemplified by the atmosphere and the sea) and develop numerically effective procedures of strong intuitive content. The Method of Groups is a case in point. It is summarized in Equations (6)-(9) of Sec. 7.11 and shows in outline how the complex problem of radiant energy scattered in a general three-dimensional medium (such as a cloud) may be reduced to an ostensible one-dimensional sweep method--the hallmark of the invariant imbedding idea.

Over the years this useful combination of an equation of transfer and the principles of invariance has been extended to other fields of physics. In linear hydrodynamics, e.g., the counterpart to the equation of transfer is the set of dynamic and continuity equations. Instead of radiance (upward and downward into the sea) we have water wave elevation and fluid volume flux over the surface of a fluid basin, such as the sea. The principles of invariance go over essentially unchanged into the hydrodynamic setting. Consequently, all of the visualizable physical notions of invariant imbedding are transferable intact to hydrodynamics, such as the transmittances and reflectances of bodies of water--canals, bays, oceans. Moreover, the numerical efficiency of the imbedding technique is once again realized in this new setting. Work in this direction has proceeded far enough to show the thoroughgoing analogy between radiative transfer of light in optical media and the linear transport of water waves in natural bodies of water.[1-5]

The development of the invariant imbedding idea continues in still other fields and may be pursued in a recently compiled bibliography.[6]

The work in this volume was essentially done in the period 1964-1965 while I was with the Visibility Laboratory at Scripps Institution of Oceanography, and has been essentially unchanged in its conversion to manuscript form. My recent application of invariant imbedding to linear hydrodynamics has served to check the correctness of the theory below, and to reinforce my confidence in its universal applicability to all linear transport phenomena including light, ocean wave, electromagnetic or acoustic fields.

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The final manuscript was typed by Ms. Judith Marshall.

R.W.P.

Honolulu, Hawaii
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1. Preisendorfer, R.W., *Surface-Wave Transport in Nonuniform Canals* NOAA-JTRE-80, HIG-72-18 Hawaii Institute of Geophysics July 1972.

2. Preisendorfer, R.W. and F.I. Gonzalez, Jr., *Classic Canal Theory* NOAA-JTRE-83, HIG-73-14 Hawaii Institute of Geophysics August 1973 (vols. 1 and 2).

3. Preisendorfer, R.W., *Multimode Long Surface Waves in Two-Port Basins* NOAA-JTRE-125, HIG-75-4 Hawaii Institute of Geophysics January 1975.

4. Preisendorfer, R.W., *Directly Multimoded Two-Flow Long Surface Waves* NOAA-JTRE-131, HIG-75-12 May 1975.

5. Preisendorfer, R.W., *Marching Long Surface Waves Through Two-Port Basins I. Continuous Circuit Case* NOAA-JTRE-132 HIG-76-1 January 1976.

6. Scott, M.R., *A Bibliography on Invariant Imbedding and Related Topics* SLA-74-0284 Sandia Laboratories Albuquerque, N.M. June 1974.

CHAPTER 7

Invariant Imbedding Techniques for Light Fields

7.0 Introduction

We return in this chapter to the general circle of ideas introduced in Chapter 3, our purpose being to give detailed derivations of functional relations holding among the various interaction operators introduced there and discussions of how those operators can be evaluated in practice. For expository reasons we shall at first limit the discussions for the most part to the case of light fields in isotropic plane-parallel media. However, the techniques displayed are all extendable in principle to light fields in arbitrarily shaped anisotropic media. By carrying out the present program we not only add to the store of solution techniques for light fields discussed in Chapters 4-6, but illustrate within the domain of radiative transfer an important procedure of modern theoretical physics, the invariant imbedding solution technique. This technique gives rise to functional equations governing various physical processes by means of certain general group-theoretic and limit-theoretic arguments. These functional equation representatives of the physical processes give insight into the processes and occasionally result in useful numerical methods of determination of the processes. Some of these methods will be illustrated in this chapter.

From the great number of results on functional relations for various radiative transfer operators obtained in recent years by means of invariant imbedding techniques, we select the following for exposition in the present chapter: first, the derivations of the differential equations governing the reflectance and transmittance operators R and T for plane-parallel media. The steady state version of the derivation is given in Section 7.1, the time-dependent version is given in Section 7.2. A particularly interesting feature of these derivations is the statement of the local forms of the principles of invariance and their conceptual relation to the usual (global) forms of the principles of invariance. In Sections 7.3-7.5 it is shown how new and possibly useful functional relations can be discovered for the various interaction operators by treating the operators as algebraic entities and the equations in which they appear as algebraic statements which are occasionally subject to simple limit arguments. As a result of these heuristic manipulations three novel means of determining light fields in natural optical media, which occur in Sections 7.4-7.5, are selected for further study in Sections

7.6-7.8. An example of an actual numerical computation of the R and T operators based on the functional relations of Section 7.1 is given in Section 7.9 for the case of homogeneous source-free plane-parallel media with isotropic scattering. This numerical method is generalized in Section 7.10. In Section 7.11 the preceding results are consolidated and generalized. Section 7.12 is concerned with the conditions of homogeneity and isotropy and related ideas, which will help simplify theoretical and numerical work and help classify optical media in general. Section 7.13 develops some deep connections among the various standard and invariant imbedding operators within media with internal sources. Finally, in Section 7.14, it is observed how the Laplace and Fourier transform techniques, which have proved so useful in the classical formulation of the transport phenomena, can be combined with the invariant imbedding approach to simplify the functional relations of the latter approach and to encourage their applications to time-dependent problems, point source problems, and other transport problems which ordinarily involve higher numbers of variables.

7.1 Differential Equations Governing the Steady State, R and T Operators

In Sections 3.6 and 3.7 we saw how the R and T operators of plane-parallel (and other) media were used in both theory and practice to determine light fields in natural optical media. In this section we show how the four R and T operators generally associated with stratified plane-parallel media may be determined from knowledge of the volume scattering and volume attenuation functions within the medium. This will be done by deriving the differential equations governing the operators as a function of the thickness of the medium. Thus, if we know the R and T operators for a given layer of material the differential equation will show how the operators change by addition of a very thin layer of the material to the given layer. By letting the given layer grow continuously from some given thickness, we will therefore know how its R and T operators evolve from their given values, and how they may be computed in both theory and practice. We turn now to the details of the derivations.

Local Forms of the Principles of Invariance

We begin the derivations by casting the equation of transfer for a stratified plane-parallel medium into a pair of equations which are strongly reminiscent of the two main principles of invariance for such media (Ex. 3, Section 3.7); the main difference being the presence of derivatives of N in the new equations. Thus under the assumption that all functions (radiance distributions and optical properties) depend only on depth y in the medium (cf. Fig. 7.1) Equation (3) of Sec. 3.15 becomes:

$$(-\xi\cdot\mathbf{k})\ \frac{dN(y,\xi)}{dy} = -\alpha(y)N(y,\xi) + N_*(y,\xi) \tag{1}$$

where

$$N_*(y,\xi) = \int_{\Xi} N(y,\xi')\sigma(y;\xi';\xi)\ d\Omega(\xi') \quad . \tag{2}$$

Here $\mathbf{k}$ is the unit outward normal to the medium, ξ is an arbitrary direction in Ξ, and $a \le y \le b$.

To obtain the requisite form of the equation of transfer we restrict the radiance distribution $N(y,\cdot)$ to the two halves Ξ_+ and Ξ_- of Ξ, (cf. Fig. 7.1, and Sec. 2.4). We denote the restriction of $N(y,\cdot)$ to Ξ_+ as usual by "$N_+(y)$", and the restriction of $N(y,\cdot)$ to Ξ_- by "$N_-(y)$"; $a \le y \le b$. Next we write:

$$\text{"}\rho(y)\text{"} \quad \text{for} \quad \frac{1}{|\xi\cdot\mathbf{k}|}\int_{\Xi_+} [\ \ ]\sigma(y;\xi';\xi)\ d\Omega(\xi') \tag{3}$$

in which ξ is in Ξ_-; and :

$$\text{"}\tau(y)\text{"} \quad \text{for} \quad \frac{1}{|\xi\cdot\mathbf{k}|}\int_{\Xi_+} [\ \ ]\sigma(y;\xi';\xi)\ d\Omega(\xi') - \frac{1}{|\xi\cdot\mathbf{k}|}\ \alpha(y) \tag{4}$$

in which ξ is in Ξ_+, and in both of which $a \le y \le b$. Furthermore, we assume the medium to be isotropic, so that $\sigma(y;\xi';\xi)$ depends only on the value $\xi'\cdot\xi$ for each choice of ξ' and ξ. Hence for each ξ, y, the values of the integrals in each definition in (3) and (4) are unchanged if Ξ_+ is replaced by Ξ_-, and Ξ_- by Ξ_+ throughout. The operator $\rho(y)$ is the *local reflectance* operator and $\tau(y)$ is the *local transmittance* operator. In discussions where it is necessary to consider the possibility of anisotropic media, the operators $\rho(y)$ and $\tau(y)$ must be defined with specific reference to the domains of integration in (3) and (4). Thus "$\rho(y)$" in (3) becomes "$\rho_+(y)$" and "$\rho_-(y)$" denotes the same kind of integral but over Ξ_- with ξ in Ξ_+. Similarly (4) will define what we will call "$\tau_+(y)$" and a similar integral over Ξ_- with ξ in Ξ_- will be denoted by "$\tau_-(y)$". (See, Ref. [251], and Sec. 7.13 below.)

The local operators $\rho(y)$ and $\tau(y)$ are used as follows: In (1) let ξ be in Ξ_+, so that $N(y,\xi) = (N_+(y))(\xi)$. Furthermore, writing $N_*(y,\xi)$ in (1) as:

$$N_*(y,\xi) =$$

$$= \int_{\Xi_+} N(y,\xi')\sigma(y;\xi';\xi)\ d\Omega(\xi') + \int_{\Xi_-} N(y,\xi')\sigma(y;\xi';\xi)\ d\Omega(\xi') \ ,$$

we divide through each side of (1) by $|\xi\cdot\mathbf{k}|$, apply (3) and (4), and end up with:

$$\boxed{-\frac{dN_+(y)}{dy} = N_+(y)\tau(y) + N_-(y)\rho(y)} \quad . \tag{5}$$

Similarly, now with ξ in Ξ_-, so that $N(y,\xi) = \big(N_-(y)\big)(\xi)$, we obtain:

$$\boxed{\frac{dN_-(y)}{dy} = N_-(y)\tau(y) + N_+(y)\rho(y)} \quad . \tag{6}$$

Equations (5) and (6) are the desired *local (or integrodifferential) forms* of the principles of invariance for plane-parallel media. The striking similarity between the pair (5), (6) and the pair I, II of Ex. 3, Sec. 3.7, is evident once "T" is paired with "τ" and "R" with "ρ". These equations can be put into a more compact form by first writing:

$$\text{"}\mathcal{K}(y)\text{"} \quad \text{for} \quad \begin{pmatrix} -\tau(y) & \rho(y) \\ -\rho(y) & \tau(y) \end{pmatrix} \tag{7}$$

and

$$\text{"}N(y)\text{"} \quad \text{for} \quad \big(N_+(y), N_-(y)\big) \quad . \tag{8}$$

Then (5), (6) become:

$$\boxed{\frac{dN(y)}{dy} = N(y)\mathcal{K}(y)} \tag{9}$$

which is an alternate and equivalent rendition of the equation of transfer (1) via the local forms (5), (6) of the principles of invariance. We shall return to this form of the equation of transfer in subsequent sections, wherein it will play an important role in determining the radiance functions. For the present, we continue the derivation of the desired functional relations for the R and T operators.

The Differential Equations for R and T

The main step in the derivation of the differential equations for the R and T operators will now be taken. We begin with the operator $R(y,b)$ for an arbitrary subslab $X(y,b)$ of the plane-parallel medium $X(a,b)$, $a \le y \le b$. (We now are using the notation of Section 3.7). We let $N_-(a)$ be an arbitrary incident radiance function over the plane upper boundary of $X(a,b)$ at level a. We set $N_+(b) = 0$, and assume that no sources of radiant flux are within $X(a,b)$. The two main principles of invariance for an arbitrary subslab $X(x,z)$, $a \le x \le y \le z \le b$, of $X(a,b)$ are as given in Ex. 3, Sec. 3.7:

$$\text{I.} \quad N_+(y) = N_+(z)T(z,y) + N_-(y)R(y,z)$$

$$\text{II.} \quad N_-(y) + N_-(x)T(x,y) + N_+(y)R(y,x) \quad .$$

We next set $z = b$ in principle I, which with the present boundary lighting conditions becomes:

$$N_+(y) = N_-(y)R(y,b) \quad . \tag{10}$$

Equation (10) states that the upward radiance distributions at level y in $X(a,b)$ consist of the reflected flux from $X(y,b)$ induced by the downward radiance distributions entering $X(y,b)$ at level y. We next take the derivative of each side of (10) with respect to y, thus:

$$\frac{dN_+(y)}{dy} = \frac{d}{dy}\left(N_-(y)R(y,b)\right)$$

$$= \frac{dN_-(y)}{dy} R(y,b) + N_-(y)\frac{dR(y,b)}{dy} \tag{11}$$

where we have written:

$$\text{"}\frac{dR(y,b)}{dy}\text{"} \quad \text{for} \quad \frac{1}{|\xi\cdot\mathbf{k}|}\int_{\Xi_-}[\ \ ]\frac{d}{dy}R(y,b;\xi';\xi)\ d\Omega(\xi') \tag{12}$$

and where $R(y,b;\xi';\xi)$ is defined in Example 3 of Sec. 3.7 (cf. also (8)-(11) of Sec. 3.6). Therefore $dR(y,b)/dy$ in (11) is an integral operator acting on $N_-(y)$. Further, $R(y,b)$ in (11) acts on the function $dN_-(y)/dy$. Thus all terms of (11) are well defined. Now, we are interested in $R(a,b)$, which we may envision as the limit of $R(y,b)$ as $y \to a$. Hence in (11) we let y approach a. Thus we are led to consider

$$\lim_{y\to a}\frac{dN_+(y)}{dy}$$

which by (5) is given as:

$$\lim_{y\to a}\frac{dN_+(y)}{dy} = -\left[N_+(a)\tau(a) + N_-(a)\rho(a)\right] \quad . \tag{13}$$

By principle III of Example 3, Sec. 3.7, which we repeat here for convenience:

$$\text{III.} \quad N_+(a) = N_+(b)T(b,a) + N_-(a)R(a,b) \quad ,$$

equation (13) becomes:

$$\lim_{y\to a}\frac{dN_+(y)}{dy} = -N_-(a)\left[R(a,b)\tau(a) + \rho(a)\right] \tag{14}$$

where we have used the boundary condition that $N_+(b) = 0$. In a similar manner, we find that the limit of the remaining derivative of $N_-(y)$ in (11) can be represented via (6) and principle III as:

$$\lim_{y\to a} \frac{dN_-(y)}{dy} = N_-(a)[\tau(a) + R(a,b)\rho(a)] \quad . \tag{15}$$

Let us agree to write:

$$\text{"}\frac{\partial R(a,b)}{\partial a}\text{"} \quad \text{for} \quad \lim_{y\to a} \frac{dR(y,b)}{dy} \quad . \tag{16}$$

Then applying the limit operation, $\lim_{y\to a}$, to each side of (11), we have:

$$-N_-(a)[R(a,b)\tau(a)+\rho(a)] =$$

$$= N_-(a)[\tau(a)+R(a,b)\rho(a)]R(a,b) + N_-(a)\frac{\partial R(a,b)}{\partial a} \quad . \tag{17}$$

This equation holds for every incident radiance function $N_-(a)$. Hence we can formally cancel "$N_-(a)$" from each side. After rearranging the resultant operator equation, we have:

$$\boxed{\begin{array}{l}\text{I}' \\ -\dfrac{\partial R(a,b)}{\partial a} = \rho(a)+\tau(a)R(a,b)+R(a,b)\tau(a)+R(a,b)\rho(a)R(a,b)\end{array}} \tag{18}$$

Equation I' is the requisite differential equation for R(a,b) as a function of the depth parameter a. Observe that I' has the form of a Riccati equation for the operator R(a,b) with known operators $\rho(a)$ and $\tau(a)$. Thus, (18) is in principle solvable for R(a,b) with the initial condition: $R(a,b) = 0$ whenever $a = b$, (cf. (30) of Sec. 3.7). Hence from I' we have:

$$-\frac{\partial R(a,b)}{\partial a} = \rho(a) \tag{19}$$

for $a = b$, showing that the initial rate of growth of R(a,b) is given directly by the local reflectance operator, $\rho(a)$, i.e., the integral operator with the volume scattering function as kernel.

The determination of the differential equation for T(a,b) may be made next, starting with principle II in which $x = a$, the result being:

$$N_-(y) = N_-(a)T(a,y) + N_+(y)R(y,a) \quad . \tag{20}$$

Taking the derivative of each side with respect to y:

$$\frac{dN_-(y)}{dy} = N_-(a)\frac{dT(a,y)}{dy} + \frac{dN_+(y)}{dy}R(y,a) + N_+(y)\frac{dR(y,a)}{dy} \qquad (21)$$

Here we have written:

$$\text{"}\frac{dT(a,y)}{dy}\text{"} \quad \text{for} \quad \frac{1}{|\xi\cdot\mathbf{k}|}\int_{\Xi_-}[\ ]\frac{d}{dy}T(a,y;\xi';\xi)\,d\Omega(\xi') \qquad (22)$$

and where $T(a,y;\xi';\xi)$ is defined in Example 3 of Sec. 3.7 (cf. also (8)-(11) of Sec. 3.6). Thus $dT(a,y)/dy$ in (21) is an integral operator acting on $N_-(a)$. Now in (21) we consider:

$$\lim_{y\to b}\frac{dN_-(y)}{dy}$$

which by (6) is given as:

$$\lim_{y\to b}\frac{dN_-(y)}{dy} = [N_-(b)\tau(b) + N_+(b)\rho(b)] \quad . \qquad (23)$$

From principle IV of Example 3 Sec. 3.7:

$$\text{IV} \quad N_-(b) = N_-(a)T(a,b) + N_+(b)R(b,a) \ ,$$

which, applied to (23) yields:

$$\lim_{y\to b}\frac{dN_-(y)}{dy} = N_-(a)T(a,b)\tau(b) \quad . \qquad (24)$$

In a similar way we obtain for the derivative of $N_+(y)$ in (21):

$$\lim_{y\to b}\frac{dN_+(y)}{dy} = -N_-(a)T(a,b)\rho(b) \quad . \qquad (25)$$

Writing:

$$\text{"}\frac{\partial T(a,b)}{\partial b}\text{"} \quad \text{for} \quad \lim_{y\to b}\frac{dT(a,y)}{dy} \ , \qquad (26)$$

we here apply the limit operator $\lim_{y\to b}$ to each side of (21), the result being:

$$\boxed{\text{II}' \quad \frac{\partial T(a,b)}{\partial b} = T(a,b)[\tau(b) + \rho(b)R(b,a)]} \qquad (27)$$

in which we have used the fact that $N_-(a)$ is arbitrary. This shows that once $R(a,b)$ is known, the operator $T(a,b)$ is obtainable by a simple quadrature.

The pattern of derivation of the differential equations is now clear. By using next the second versions of principles III and IV in Example 3 of Sec. 3.7, together with (5) and (6) we arrive at:

$$\text{III}' \quad \frac{\partial R(a,b)}{\partial b} = T(a,b)\rho(b)T(b,a) \tag{28}$$

$$\text{IV}' \quad -\frac{\partial T(a,b)}{\partial a} = [\tau(a)+R(a,b)\rho(a)]T(a,b) \tag{29}$$

Discussion of the Differential Equations

Statements I'-IV' above are the desired differential equations for the R and T operators associated with the plane-parallel medium X(a,b). Observe how I' is autonomous with respect to R(a,b). Thus, as we already observed, I' in principle can yield R(a,b), starting with the initial condition R(a,b) = 0. By reversing "a" and "b" in I', (i.e., by literally turning X(a,b) upside down in the given coordinate system) we can also obtain R(b,a). Observe that if $\rho(y)$ and $\tau(y)$ vary with depth y in X(a,b), we generally will have $R(a,b) \neq R(b,a)$, i.e., the R operator will exhibit *polarity*. If X(a,b) is homogeneous, then R(a,b) = R(b,a) and clearly depends only on the difference b-a of the depth parameters. (Recall, we have assumed at the outset that X(a,b) is isotropic.) Once R(a,b) and R(b,a) have been found, T(a,b) and T(b,a) both follow from II' using first R(a,b) then R(b,a) by reversing "a" and "b" in II'. If polarity is the case for R-operators, then generally, the T operator will possess polarity also. Thus I' and II' are in principle sufficient to determine the four R and T operators. However, it is interesting to note that I'-IV' are sufficient, as they stand, to determine in principle all four operators R(a,b), T(a,b), T(b,a), R(b,a) in that order, by successively using I', IV', III', II', in corresponding order. Alternatively, the equations may be solved in the order I', IV', II', III'. For the general forms of these observations, in the context of general media the reader may consult section 25 and other relevant sections of Ref. [251].

Equations I'-IV' constitute a wealth of intuitive information about light fields in scattering media neatly summarized in symbolic form, and which the reader is invited to discover. Thus I' and III' considered together show the two distinct modes of growth of R(a,b) when the medium is altered by varying the parameter a, and then the parameter b. In other words R(a,b) grows differently when layers are added to X(a,b) from below, than when layers are added from above. The precise manner of growth in each case is clearly discernable from each differential equation and can be pictured in terms of the interaction of X(a,b) with an infinitely thin layer added to X(a,b), e.g., at level a, whose reflectance and transmittance are $\rho(a)$ and $\tau(a)$, respectively. The growth of $R(a,b)$ when a is varied is far more complex than when b is varied. Unfortunately this more complex growth is necessary to contend with in the task of determining $R(a,b)$. Remarks of a similar nature can be made about the general growth patterns of $T(a,b)$ using II' and IV'. In the case of $T(a,b)$ the difference of growth rates, depending on whether a or b is varied, are less subtle than that of $R(a,b)$, and rest mainly in the order of application of the operators in the square brackets with respect to $T(a,b)$.

Functional Relations for Decomposed Light Fields

Some radiative transfer investigations are simplified if one is able to treat separately the reduced and diffuse components of the radiance field. Thus in the classical researches of Chandrasekhar, the computations were limited to computing the diffuse radiance transmitted through plane-parallel media. In addition, our discussions of the point source problem were facilitated in Sec. 6.6 by adopting for study not N but the diffuse component N^* of N. Furthermore, as noted in the *Remarks on the Interaction Method* in Sec. 3.18, the AC property of general interaction operators is easily shown to hold for those operators whose response functions describe diffuse radiant flux, i.e., radiant flux which has been scattered at least once. With such observations in mind we are motivated to study some of the salient properties of the *decomposed* R and T operators for plane-parallel media, in particular the principles of invariance (both local and global) which govern them, and the differential equations they satisfy. The extensions of the results of the present discussion to more general geometries is straightforward and the present techniques are presented so as to readily serve as the prototype for such extensions.

We shall work with the setting already established and used in carrying out the discussion from (1) to (29) above. Thus Fig. 7.1 will represent the present geometrical setting. Now, the first step in the decomposition of the R and T operators is to decompose the light field at general level y into its reduced and diffuse components. The basis for this decomposition rests in (5), (6) of Sec. 3.13.

Thus, $N_-(y)$, e.g., may be written:

$$N_-(y) = N_-^o(y) + N_-^*(y) \quad , \tag{30}$$

for every y, $a \leq y \leq b$, where $N_-^o(y)$ is the reduced (or residual) radiance distribution over the directions of Ξ_- at level y. $N_-^*(y)$ is the diffuse radiance distribution over the same direction set and at the same level. A similar decomposition holds for $N_+(y)$. The incident radiance distributions $N_-(a)$ and $N_+(b)$ on the slab $X(a,b)$ will, by convention, be of reduced form, i.e., we will assume $N_-^*(a) = 0$ and $N_+^*(b) = 0$. (See the Principle of Relative Scattering Order in Sec. 22 of Ref. [251]). Hence $N_-^o(a)$ and $N_+^o(b)$ serve as the incident radiance distributions on $X(a,b)$.

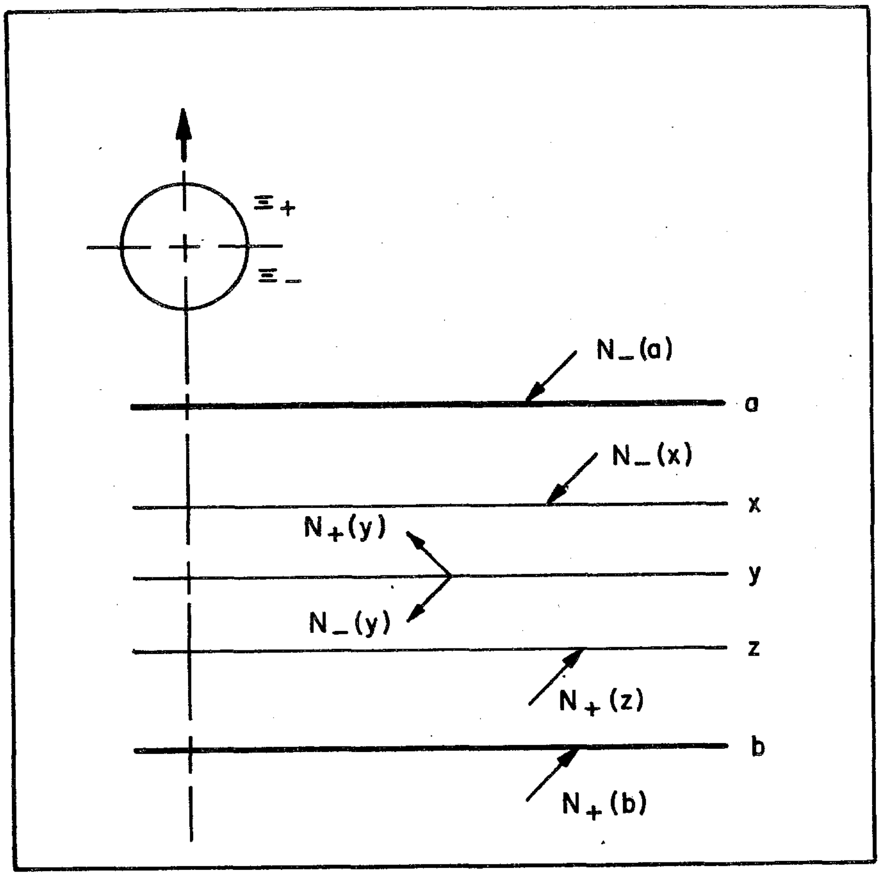


FIG. 7.1 Plane-parallel setting for the local and global forms of the Principles of Invariance.

The connection between the initial radiance function $N^o_-(a)$ over the upper boundary of $X(a,b)$ and the residual radiance function $N^o_-(y)$ over level y within $X(a,b)$ is readily established, using the results (4) of Sec. 3.10 and (3) of Sec. 3.11. Thus we have in general:

$$N^o_-(z) = N^o_-(x)T^o(x,z) \quad , \tag{31}$$

where we have written:

$$\text{"}T^o(x,z)\text{"} \quad \text{for} \quad \int_{\Xi_-} [\ \ ]T_r(p',\xi')\delta(\xi-\xi')\, d\Omega(\xi') \quad , \tag{32}$$

for $a \leq x \leq z \leq b$, and where p' is a point (i.e., an ordered triple of real numbers) in level x, and q is a point in level z such that:

$$q - p' = r\xi' \tag{33}$$

where r is determined by:

$$r = |z-x| / |\xi\cdot\mathbf{k}| \quad .$$

A companion equation to (31), written for $N_+^o(x)$, is readily stated. To see the way in which (32) is used, suppose $x = a$ and $z = y$, and that the value of $N_-^o(a)$ at p' and ξ' is specifically of the form $N_-^o(p',\xi')$. Further, let $N_-^o(q,\xi)$ be the residual radiance at point q induced by $N_-^o(a)$. Then $T^o(a,y)$ acting on $N_-^o(a)$ yields the radiance:

$$N_-^o(q,\xi) = \int_{\Xi_-} N_-^o(p',\xi')T_r(p',\xi')\delta(\xi-\xi')\, d\Omega(\xi')$$

$$= N_-^o(p,\xi)T_r(p,\xi) \quad , \tag{34}$$

where $p = q-r\xi$, in which the distance r is determined by $r = |y-a|/|\xi\cdot\mathbf{k}|$.

Recalling that y is the depth parameter for X(a,b), i.e., the distance to the upper boundary of X(a,b), it follows from (31) and (2) of Sec. 3.11 that:

$$\frac{dN_-^o(y)}{dy} = -\frac{1}{|\xi\cdot\mathbf{k}|}\cdot\alpha(y)N_-^o(y) \quad . \tag{35}$$

Suppose we write:

$$\text{"}\tau^o(y)\text{"} \quad \text{for} \quad -\frac{1}{|\xi\cdot\mathbf{k}|}\alpha(y) \quad . \tag{36}$$

Then (35) becomes:

$$\boxed{\frac{dN_-^o(y)}{dy} = N_-^o(y)\tau^o(y)} \quad . \tag{37}$$

A similar equation may be shown to hold for $N_+^o(y)$:

$$\boxed{-\frac{dN_+^o(y)}{dy} = N_+^o(y)\tau^o(y)} \quad . \tag{38}$$

The number $\tau^o(y)$ defined in (36) (and which acts as a multiplicative operator on radiance, as in (37), (38)) is called the *local residual (or reduced) transmittance operator.* Observe the analogous roles played by $\tau^o(y)$ and $\tau(y)$ in (37), (38) and (5), (6). This observation prompts us to write:

$$\text{"}\tau^*(y)\text{"} \quad \text{for} \quad \tau(y) - \tau^o(y) \tag{39}$$

which we call the *local diffuse transmittance operator.*

In view of (39), we have:

$$\boxed{\tau(y) = \tau^o(y) + \tau^*(y)} \tag{40}$$

which is the decomposition of the local transmittance operator into its residual and diffuse parts. This should be compared with (4), so that $\tau^*(y)$ is seen to be the integral operator part of (4).

We see from (3) and (4) that the local reflectance operator $\rho(y)$ is already in diffuse form, i.e., that it already consists of just an integral of σ over Ξ_+. This fact lies at the base of the fundamental distinction between reflectance and transmittance operators whenever decomposed light fields are considered. This distinction may be carried on up to the global level where R(a,b) is necessarily already in diffuse form and where T(a,b) may be rendered into reduced and diffuse parts by writing in general:

$$\text{"}T^*(x,z)\text{"} \quad \text{for} \quad T(x,z) - T^o(x,z) \tag{41}$$

for $a \le x \le z \le b$; so that:

$$\boxed{T(x,z) = T^o(x,z) + T^*(x,z)} \quad . \tag{42}$$

A similar definition holds for upward transmittances. It follows immediately from (42), (32), and from (29), (30) of Sec. 3.7 that:

$$T^*(y,y) = 0 \tag{43}$$

for every y, $a \le y \le b$. That is, the diffuse transmittance operator $T^*(x,z)$ reduces to the zero operator whenever x = z. This shows that we may generally picture $T^*(x,z)$ as a "soft" operator in the same sense that the reflectance operator R(x,z) for the same slab is "soft". The precise mathematical description of this "softness" of $T^*(x,z)$ and R(x,z) is that they possess the AC property with respect to depth measure. By contrast with $T^*(x,z)$, the operator T(x,z), owing to its component $T^o(x,z)$, is "hard" in the sense that:

$$T^o(y,y) = I \tag{44}$$

for every y, $a \le y \le b$, as may be seen by (32). That is, $T^o(x,z)$ (and hence T(x,z)) reduces to the identity operator, and certainly does not have the AC property with respect to depth measure.

The terms "soft" and "hard" as used above to describe the AC properties of operators pictorially go back to certain everyday observable phenomena of light fields in the air or the sea. For example consider a slightly hazy morning when the sky is otherwise clear. The radiance distribution of the haze appears softly variable with direction except for the bright sharp sun image discernable through the haze. If one were to describe the diffuse light field constituting the haze only (thus omitting the residual radiance of the sun) the description would use the operator $T^*(x,z)$ acting on the incident sunlight at the top of the haze layer, where x may now be the altitude of the haze layer and z the altitude of the ground. As the haze "burns off", and the haze layer becomes thinner (optically or geometrically, or both) the altitude x approaches z, and with the vanishing difference z-x so too vanishes $T^*(x,z)$. On the other hand the residual radiance from the sun transmitted through the haze layer is described by $T^o(x,z)$, which approaches the identity operator with decreasing difference z-x thereby depicting the hardening or sharpening of the sun's image as seen through the dissipating mists.

The preceding discussion has made plausible the demonstrable fact that the operator $T^*(a,b)$ for an arbitrary medium $X(a,b)$ has the AC property with respect to depth measure. This can be shown to imply, via Theorem B of Sec. 3.16, the existence of an interaction kernel function S^* for $T^*(a,b)$ such that:

$$T^*(a,b) = \int_{\Xi_-} \int_{X_a} [\ ] S^*(X;x',\xi';x,\xi)\, dA(x')\, d\Omega(\xi') . \quad (45)$$

This may be compared with (9) of Sec. 3.6. As in the earlier case "X" stands for the medium at hand--$X(a,b)$ in this case. Further, X_a is the plane boundary of $X(a,b)$ at level a. A similar integral representation for $T^*(b,a)$ can be obtained.

We now have covered all the prerequisites for the main part of the present discussion, namely for the derivation of the appropriate forms of the local and global principles of invariance for decomposed light fields, along with the differential equations for $T^*(a,b)$. We begin the main discussion with the derivation of the local forms of the principles of invariance for $N^*_\pm(y)$.

Starting with (5) in which we decompose $N_+(y)$, $N_-(y)$ and $\tau(y)$ into their reduced and diffuse parts, we perform the following calculations:

$$-\frac{d\left(N^o_+(y) + N^*_+(y)\right)}{dy} = \left(N^o_+(y) + N^*_+(y)\right)\left(\tau^o(y) + \tau^*(y)\right) + \left(N^o_-(y) + N^*_-(y)\right)\rho(y) \quad .$$

In view of (38), this may be simplified and rearranged into the form:

$$-\frac{dN^*_+(y)}{dy} = N^*_+(y)\tau(y) + N^*_-(y)\rho(y) + \left[N^o_+(y)\tau^*(y) + N^o_-(y)\rho(y)\right] \quad (46)$$

This shows how the rate of change of $N_+^*(y)$ with depth depends on the counter-flowing scattered radiances $N_\pm^*(y)$ and the counter-flowing residual radiances $N_\pm^0(y)$. In a similar way we obtain:

$$\frac{dN_-^*(y)}{dy} = N_-^*(y)\tau(y) + N_+^*(y)\rho(y) + \left[N_-^0(y)\tau^*(y) + N_+^0(y)\rho(y)\right] . \tag{47}$$

Equations (46) and (47) constitute the local forms of the principles of invariance for the diffuse light field $N^*(y)$, where we have written:

$$\text{"}N^*(y)\text{"} \quad \text{for} \quad (N_+^*(y), N_-^*(y)) \quad . \tag{48}$$

Equations (46) and (47) together are equivalent to the equation of transfer (1) for $N(y,\cdot)$. The equation of transfer for $N^*(y,\cdot)$ was studied earlier in (7) of Sec. 5.2. With only slight modifications, the preceding derivations of (46) and (47) can be made directly from (2) and (7) of Sec. 5.2.

We may cast (46), (47) into the decomposition counterpart to (9). Thus writing:

$$\text{"}N^0(y)\text{"} \quad \text{for} \quad (N_+^0(y), N_-^0(y)) \tag{49}$$

$$\text{"}\mathcal{K}^*(y)\text{"} \quad \text{for} \quad \begin{pmatrix} -\tau^*(y) & \rho(y) \\ -\rho(y) & \tau^*(y) \end{pmatrix} \tag{50}$$

we can write the system (46), (47) as:

$$\boxed{\frac{dN^*(y)}{dy} = N^*(y)\mathcal{K}(y) + N^0(y)\mathcal{K}^*(y)} \quad . \tag{51}$$

Next we look into the matter of the (global) principles of invariance for decomposed light fields. Going solely on the strength of the analogy between the pair (5),(6) and the pair I, II of Example 3 of Sec. 3.7, we should be able to immediately write down the present decomposed counterparts to I, II of Sec. 3.7, using (46), (47) as a basis. Thus we write:

I^*.

$$\boxed{N_+^*(y) = N_+^*(z)T(z,y) + N_-^*(y)R(y,z) + N_+^0(z)T^*(z,y) + N_-^0(y)R(y,z)}$$

and (52)

II^*.

$$\boxed{N_-^*(y) = N_-^*(x)T(x,y) + N_+^*(y)R(y,x) + N_-^0(x)T^*(x,y) + N_+^0(y)R(y,x)}$$

(53)

where $a \leq x \leq y \leq z$.

We can rigorously establish these two main principles of invariance for diffuse radiance directly from I, II of Example 3 of Sec. 3.7. Thus, to establish, say, I* we write I of Example 3 of Sec. 3.7 as:

$$\left(N_+^o(y) + N_+^*(y)\right) = N_+^*(z)T(z,y) + N_-^*(y)R(y,z) + N_+^o(z)T^o(z,y)$$
$$+ N_+^o(z)T^*(z,y) + N_-^o(y)R(y,z) \quad .$$

Using the + counterpart to (31), namely the representation:

$$N_+^o(y) = N_+^o(z)T^o(z,y) \quad , \tag{54}$$

in the preceding expanded form of principle I, I* follows immediately after a rearrangement of terms. The process of rewriting all the principles and functional relations of radiative transfer for decomposed light fields can now be readily carried out systematically by the interested reader. We shall leave this matter here and go on to the next observation of immediate concern.

From principles I* and II* above we can deduce the other two principles of invariance for the diffuse radiances $N_+^*(a)$, $N_-^*(b)$ on $X(a,b)$. Thus, in particular, we have on setting $y = a$, $z = b$, in I*:

$$\text{III}^*. \quad \boxed{N_+^*(a) = N_+^o(b)T^*(b,a) + N_-^o(a)R(a,b)} \quad , \tag{55}$$

and on setting $x = a$, $y = b$, in II*:

$$\text{IV}^*. \quad \boxed{N_-^*(b) = N_-^o(a)T^*(a,b) + N_+^o(b)R(b,a)} \quad . \tag{56}$$

The final matter to be taken up in this dicussion of decomposed light fields is the derivation of the differential equations for the diffuse transmittance operator $T^*(a,b)$. The present derivations can be patterned directly after the steps (20)-(27) of the derivation of the differential equation for $T(a,b)$. Thus, starting with principle II* in which $x = a$ and in which $N_-^*(a) = 0$ and $N_+(b) = 0$, we have:

$$N_-^*(y) = N_-^o(a)T^*(a,y) + N_+^*(y)R(y,a) \quad . \tag{57}$$

Taking the derivatives of each side with respect to y:

$$\frac{dN_-^*(y)}{dy} = N_-^o(a)\frac{dT^*(a,y)}{dy} + \frac{dN_+^*(y)}{dy}R(y,a) + N_+^*(y)\frac{dR(y,a)}{dy} \tag{58}$$

Here we have written:

$$\text{"}\frac{dT^*(a,y)}{dy}\text{"} \quad \text{for} \quad \frac{1}{|\xi\cdot k|}\int_{\Xi_-}[\ ]\frac{d}{dy}T^*(a,y;\xi';\xi)\ d\Omega(\xi') \qquad (59)$$

where, in turn, we have written (in analogy to (32) of Sec. 3.7)

$$\text{"}T^*(x,z;\xi';\xi)\text{"} \quad \text{for} \quad |\xi\cdot k|\int_{X_x}[\ ]S^*(X;y',\xi';y,\xi)\ dA(y') \qquad (60)$$

and where "X" now stands for $X(x,z)$, $a \le x \le z \le b$, and X_x is the plane at depth x. Point y is in X_z. Now in (58) consider:

$$\lim_{y\to b}\frac{dN_-^*(y)}{dy}$$

which by (47) is given as:

$$\lim_{y\to b}\frac{dN_-^*(y)}{dy} = \left[N_-^*(b)\tau(b) + N_-^o(b)\tau^*(b)\right] \quad . \qquad (61)$$

Using principle IV* above and (31) with y = b, we can reduce this limit to:

$$\lim_{y\to b}\frac{dN_-^*(y)}{dy} = N_-^o(a)\left[T^*(a,b)\tau(b) + T^o(a,b)\tau^*(b)\right]. \qquad (62)$$

In a similar way we obtain:

$$\lim_{y\to b}\frac{dN_+^*(y)}{dy} = -N_-^o(a)\left[T^*(a,b)\rho(y) + T^o(a,b)\rho(y)\right]. \qquad (63)$$

Writing

$$\text{"}\frac{\partial T^*(a,b)}{\partial b}\text{"} \quad \text{for} \quad \lim_{y\to b}\frac{dT^*(a,y)}{dy} \quad , \qquad (64)$$

we now apply the limit operator, $\lim_{y\to b}$, to each side of (58), the result being:

$$\boxed{\begin{aligned}\frac{\partial T^*(a,b)}{\partial b} &= T^*(a,b)\left[\tau(b) + \rho(b)R(b,a)\right]\\ &\quad + T^o(a,b)\left[\ \tau^*(b) + \rho(b)R(b,a)\right]\end{aligned}} \qquad (65)$$

which is the decomposed counterpart to (27). Observe that the structure of (65) is an inhomogeneous version of (27). That is, the gestalt of (65) is that of:

$$\frac{dT^*}{dy} = T^*A + B \tag{66}$$

while the gestalt of (27) is that of:

$$\frac{dT}{dy} = TA \tag{67}$$

where A and B are known operators.

This transition from homogeneity to nonhomogeneity of an operator equation is the general earmark of a transition from the representation of undecomposed to decomposed light fields, in which the latter are represented by N^*. This phenomenon was already seen in I^* and II^*, and earlier still in (46) and (47) (whose homogeneous counterparts are (5) and (6)); similarly with (51) and (9). A still earlier example of this transition is the transfer equation for N^* in Sec. 5.2.

The decomposed counterpart to (29) may now be written down by inspection of (65), using (29) as a guide:

$$\boxed{\begin{aligned} -\frac{\partial T^*(a,b)}{\partial a} &= [\tau(a) + R(a,y)\rho(a)]T^*(a,b) \\ &\quad + [\tau^*(a) + R(a,b)\rho(a)]T^{o}(a,b) \end{aligned}} \tag{68}$$

This differential equation may also be derived from first principles after the manner of (65); the details are left as an exercise for the reader. In analogy to (19), equation (68) shows that $-\partial T^*(a,b)/\partial a$ for $a = b$ is given by

$$-\frac{\partial T^*(a,b)}{\partial a} = \tau^*(a) \quad . \tag{69}$$

Thus when $a = b$, $T^*(a,b) = 0$ and its 'rate of growth' is precisely the magnitude of the local diffuse transmittance operator $\tau^*(b)$. Thus the analogy between (19) and (69) is perfect.

7.2 Differential Equations Governing the Time Dependent R and T Operators

We now extend the formulations of the preceding section to the time dependent case. The geometric setting and optical properties of the medium are unchanged except that now all functions in addition vary with time. The first step in such an extension is the derivation of the time dependent version of the local forms of the principles of invariance for a plane-parallel medium.

Time Dependent Local Forms of the Principles of Invariance

We begin with the time dependent equation of transfer (4) of Sec. 3.15. For every y, such that $a \le y \le b$ and ξ in Ξ, and time t in an interval $E = (t_0, t_1)$:

$$\frac{1}{v}\,\frac{DN(y,\xi,t)}{Dt} = -\alpha(y,t)N(y,\xi,t) + \int_{\Xi} N(y,\xi',t)\sigma(y;\xi';\xi;t)\, d\Omega(\xi') \tag{1}$$

where we have written:

$$\text{"}\frac{D}{Dt}\text{"} \quad \text{for} \quad v(-\xi\cdot\mathbf{k})\frac{\partial}{\partial y} + \frac{\partial}{\partial t} \quad . \tag{2}$$

Hence (1) above differs from (1) of Sec. 7.1 in only one essential respect: the presence of the time derivative term. Therefore the transition to the time dependent versions of (5), (6), and (9) of Sec. 7.1 should be a straightforward matter. Thus, let $N_+(y,t)$, and $N_-(y,t)$ be the upward and downward radiance distributions restricted to Ξ_+, Ξ_-, respectively, at level y in $X(a,b)$ and at time t in the time interval E. E may be finite or infinite and is generally of the form (t_0, t_1), where $t_0 \le t_1$. Furthermore we write:

$$\text{"}\rho(y,t)\text{"} \quad \text{for} \quad \frac{1}{|\xi\cdot\mathbf{k}|}\int_{\Xi_+} [\ ]\sigma(y;\xi';\xi;t)\, d\Omega(\xi') \tag{3}$$

in which ξ is in Ξ_-, and:

$$\text{"}\tau(y,t)\text{"} \quad \text{for} \quad \frac{1}{|\xi\cdot\mathbf{k}|}\int_{\Xi_+} [\ ]\sigma(y;\xi';\xi;t)\, d\Omega(\xi') - \frac{1}{|\xi\cdot\mathbf{k}|}\alpha(y,t) \tag{4}$$

in which ξ is in Ξ_+,

and in both of which $a \le y \le b$, and t is in E. The requisite pair of equations now follows directly from (1):

$$\boxed{\begin{aligned} \frac{D_-N_+(y,t)}{Dy} &= N_+(y,t)\tau(y,t) + N_-(y,t)\rho(y,t) \qquad (5) \\ \frac{D_+N_-(y,t)}{Dy} &= N_-(y,t)\tau(y,t) + N_+(y,t)\rho(y,t) \qquad (6) \end{aligned}}$$

where we have written:

$$\text{"}\frac{D_\pm}{Dy}\text{"} \quad \text{for} \quad \pm\frac{\partial}{\partial y} + \frac{1}{|\xi\cdot\mathbf{k}|v}\cdot\frac{\partial}{\partial t} \quad . \tag{7}$$

Equations (5), (6) are the required *time dependent local forms of the principles of invariance.* As in the steady state case, each is obtained from the other by an interchange of + and - subscript signs throughout; the only salient difference between the time dependent and steady state sets is the time derivative term, as can be seen from (7). If we write:

$$\text{"}\mathcal{X}(y,t)\text{"} \quad \text{for} \quad \begin{bmatrix} -\tau(y,t) & \rho(y,t) \\ -\rho(y,t) & \tau(y,t) \end{bmatrix} \tag{8}$$

and:

$$\text{"}N(y,t)\text{"} \quad \text{for} \quad \left(N_+(y,t), N_-(y,t)\right) \quad , \tag{9}$$

(5) and (6) become:

$$\boxed{\frac{D_o N(y,t)}{Dy} = N(y,t)\,\mathcal{X}(y,t)} \tag{10}$$

where we have written:

$$\frac{\text{"}D_o\text{"}}{Dy} \quad \text{for} \quad \frac{\partial}{\partial y} + \frac{1}{|\xi\cdot\mathbf{k}|v}\frac{\partial}{\partial t}\,C \tag{11}$$

and

$$\text{"}C\text{"} \quad \text{for} \quad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad . \tag{12}$$

In applying D_o/Dy to $N(y,t)$, all operations proceed as usual except that in the case of the time derivative term the derivative operator $\partial/\partial t$ acts (say) first and then C acts on the resultant derivative. Equation (10) is the time dependent version of (9). For brevity of notation we will subsequently write:

$$\text{"}\mu\text{"} \quad \text{for} \quad |\xi\cdot\mathbf{k}| \quad .$$

Time Dependent Invariant Imbedding Relation

The next step in the present discussion can be made on any one of several levels of generality. Since our present goal is a set of time dependent versions of (18), (27), (28), (29), of Sec. 7.1, the most immediate route is the development of the time dependent principles of invariance, along with the R and T operators they govern. We could develop the latter principles very much after the pattern set in Examples 2 and 3 of Sec. 3.7. However, with only slightly more effort we could outline the development of the time dependent version of the more general invariant imbedding relation, following the pattern of miscellaneous Example (iv) of Sec. 3.17. This we now do, as it affords some further illustrations of the interaction method of Chapter 3 as we make our way towards the present goal. It will also allow us to illustrate once again how the principles of invariance (now in time dependent form) are derivable from the more basic invariant imbedding relation.

Following the three main stages of Sec. 3.18, for Stage I, let $X(x,z)$ be the isolated subset of the optical medium $X(a,b)$. Let the current set of radiometric functions be radiance distributions defined over the plane surfaces X_y of $X(x,z)$, $x \le y \le z$ and over a time interval E. Thus $N_+(y,E)$ and $N_-(y,E)$ are upward and downward radiance functions defined on the general plane X_y over the time interval E.

The sets of incident radiometric functions are enumerated as:

A_1 : all incident radiance functions like $N_+(z,E)$

A_2 : all incident radiance functions like $N_-(x,E)$

The sets of response functions of interest are $(x \le y \le z)$:

B_1 : all response radiance functions like $N_+(y,E)$

B_2 : all response radiance functions like $N_-(y,E)$

The interaction principle then asserts the existence of four interaction operators s_{ij}:

$$s_{11} \text{ -- } \mathcal{T}(z,y,x,E)$$

$$s_{12} \text{ -- } \mathcal{R}(z,y,x,E)$$

$$s_{21} \text{ -- } \mathcal{R}(x,y,z,E)$$

$$s_{22} \text{ -- } \mathcal{T}(x,y,z,E)$$

The corresponding interaction equations thus are:

$$N_+(y,E) = N_+(z,E)\mathcal{T}(z,y,x,E) + N_-(x,E)\mathcal{R}(x,y,z,E) \qquad (13)$$

$$N_-(y,E) = N_-(x,E)\mathcal{T}(x,y,z,E) + N_+(z,E)\mathcal{R}(z,y,x,E). \qquad (14)$$

The requisite invariant imbedding relation then is:

$$\boxed{(N_+(y,E),N_-(y,E)) = (N_+(z,E),N_-(x,E))\mathcal{M}(x,y,z,E)} \qquad (15)$$

where we have written:

$$\text{"}\mathcal{M}(x,y,z,E)\text{"} \quad \text{for} \quad \begin{bmatrix} \mathcal{T}(z,y,x,E) & \mathcal{R}(z,y,x,E) \\ \mathcal{R}(x,y,z,E) & \mathcal{T}(x,y,z,E) \end{bmatrix}. \qquad (16)$$

Integral Representation of Time Dependent $\mathcal{R}$ and $\mathcal{T}$ Operators

We currently need to go further than the preceding operator statement of the invariant imbedding relation. In particular we wish to obtain specific representations of the $\mathcal{R}$ and $\mathcal{T}$ operators as integral operators over the time domain E. Thus we must enter Stage II of the interaction method. To set Stage II in motion, we choose an arbitrary time t in E and hold it fixed until further notice. Next, consider operator $\mathcal{T}(x,y,z,E)$. Choose and fix a point p in X_y and fix a direction ξ in Ξ_-. Then, by (13), for every $N_-(x,E)$ in A_2:

$$N_-(p,\xi,t) = [N_-(x,E)\mathcal{T}(x,y,z,E)](p,\xi,t) \tag{17}$$

is a non negative number--indeed, it is the radiance $N_-(p,\xi,t)$ induced by $N_-(x,E)$. Thus in the present setting with fixed p,ξ,t, the operator $\mathcal{T}(x,y,z,E)$ is a positive linear functional on A_2. By Theorem A of Sec. 3.16 there is a measure $\mu(x,y,z,\cdot,p,\xi,t)$ on the set E such that:

$$\mathcal{T}(x,y,z,E) = \int_E [\ ]\, d\mu(x,y,z,\cdot,p,\xi,t) \tag{18}$$

(where p,ξ,t are implicit in the notation on the left) so that (17) may be represented as:

$$N_-(p,\xi,t) = \int_E N(x,t')\, d\mu(x,y,z,t',p,\xi,t)\ , \tag{19}$$

where $N(x,t')$ is the value of $N_-(x,E)$ for the variable t' in E. The next step is to observe that the measure $\mu(x,y,z,\cdot,p,\xi,t)$ is absolutely continuous with respect to the time measure on E. This simply amounts to the physically based assertion that: for every subinterval F of E, if F is of zero duration, then: $\mu(x,y,t,F,p,\xi,t) = 0$. In other words $\mathcal{T}(x,y,z,F)$ will not transmit any finite incident radiance $N_-(x,F)$ where F is of zero duration. Thus the measure μ has the AC property and Theorem B of Sec. 3.16 asserts the existence of a kernel function $\mathcal{T}(x,y,z,t',p,\xi,t)$ such that:

$$\mu(x,y,z,G,p,\xi,t) = \int_G \mathcal{T}(x,y,z,t',p,\xi,t)\, dt' \tag{20}$$

for every subset G of E. Theorem C of Sec. 3.16 now lets us write (18) as

$$\mathcal{T}(x,y,z,E) = \int_E [\ ]\,\mathcal{T}(x,y,z,t',t)\, dt' \tag{21}$$

where we have suppressed the p and ξ in going from (20) to (21), since they were arbitrary, and we now wish to work on the function level. In this way we arrive at the following integral representation of (17). For each t in E:

$$N_-(x,E)\,\mathcal{T}(x,y,z,E) = \int_E N_-(x,t')\,\mathcal{T}(x,y,z,t',t)\,dt'. \quad (22)$$

It should be noted that $\mathcal{T}(x,y,z,t',t)$ just found is the time dependent version of the complete transmittance operator defined in Sec. 3.7 and is itself an operator which, under suitable regularity conditions, can be represented as an integral operator over Ξ_- and over the upper boundary X_x of $X(x,z)$, and which acts on downward incident radiance distributions over X_x. Since this particular type of integral representation has been used in the steady state studies throughout this work, there are ample examples of such operators which the reader may turn to, so that we may go on with the main line of discussion.

In a similar way we define the remaining three complete $\mathcal{R}$ and $\mathcal{T}$ operator kernels and derive the remaining three integral forms of the operations in (13) and (14):

$$N_+(z,E)\,\mathcal{T}(z,y,x,E) = \int_E N_+(z,t')\,\mathcal{T}(z,y,x,t',t)\,dt' \quad (23)$$

$$N_-(x,E)\,\mathcal{R}(x,y,z,E) = \int_E N_-(x,t')\,\mathcal{R}(x,y,z,t',t)\,dt' \quad (24)$$

$$N_+(z,E)\,\mathcal{R}(z,y,x,E) = \int_E N_+(z,t')\,\mathcal{R}(z,y,x,t',t)\,dt'\,. \quad (25)$$

This type of integral of $N_\pm$ with the $\mathcal{R}$ and $\mathcal{T}$ operators occurs so often, let us agree to write:

$$\text{"}f(\bar{t})g(\bar{t})\text{"} \quad \text{for} \quad \int_E f(t)g(t)\,dt \quad (26)$$

where f and g are any functions or operators such that for every t the "product" f(t)g(t) is defined over E. The "product" could be the customary multiplicative numerical type, or matrix type, or general operator type. With this convention we may write the invariant imbedding relations (13) and (14) as:

$$N_+(y,t) = N_+(z,\bar{t}')\mathcal{T}(z,y,x,\bar{t}',t) + N_-(x,\bar{t}')\mathcal{R}(x,y,z,\bar{t}',t) \quad (27)$$

$$N_-(y,t) = N_-(x,\bar{t}')\mathcal{T}(x,y,z,\bar{t}',t) + N_+(z,\bar{t}')\mathcal{R}(z,y,x,\bar{t}',t) \quad (28)$$

and so (27), (28) can be written succinctly as:

$$\boxed{\left(N_+(y,t),N_-(y,t)\right) = \left(N_+(z,\bar{t}'),N_-(x,\bar{t}')\right)\mathcal{M}(x,y,z,\bar{t}',t)} \quad (29)$$

Time Dependent Principles of Invariance

From the time dependent invariant imbedding equations (27), (28), we can deduce the four time dependent principles of invariance for plane-parallel media. First, in analogy to (44)-(47) of Sec. 3.7, we write:

$$\text{"}T(x,z,t',t)\text{"} \quad \text{for} \quad \mathcal{T}(x,z,z,t',t) \tag{30}$$

$$\text{"}R(x,z,t',t)\text{"} \quad \text{for} \quad \mathcal{R}(x,x,z,t',t) \tag{31}$$

and require:

$$\mathcal{T}(x,x,z,t',t) = I \tag{32}$$

$$\mathcal{R}(x,z,z,t',t) = 0 \quad . \tag{33}$$

Definitions (30) and (31) define the time dependent versions of the standard reflectance and transmittance operators for $X(x,z)$. If we had derived these standard operators directly from the interaction principles (after the manner of Ex. 3, Sec. 3.7), then (30) and (31) would have become derived equality statements (as in the case of (44), (45) of Sec. 3.7). In the time dependent setting we impose two further conditions on R and T which are useful in numerical work, as well as theoretical manipulations, namely:

$$R(x,z,t',t) = 0 \quad \text{for} \quad t - t' \leq 0 \tag{34}$$

and

$$T(x,z,t',t) = 0 \quad \text{for} \quad t - t' \leq |z-x|/v \ . \tag{35}$$

Conditions (34) and (35) are *causality conditions*, whose physical significance is readily seen. For concreteness in the present formulations, we will specify the time interval E of the general derivations above, as the interval $(-\infty,t)$, where t is an arbitrary fixed time throughout a given discussion.

With these conventions and observations, we can write down the four time dependent principles of invariance for $X(x,z)$ and $X(a,b)$, $a \leq x \leq y \leq z \leq b$, after the manner of Ex. 3 and Ex. 4, Sec. 3.7, as follows:

Letting x = y in (27) and using (30), (31):

$$\boxed{\text{I.} \quad N_+(y,t) = N_+(z,\bar{t}')\, T(z,y,\bar{t}',t) + N_-(y,\bar{t}')R(y,z,\bar{t}',t)} \ .$$

Setting z = y in (28) and using (30), (31):

$$\boxed{\text{II.} \quad N_-(y,t) = N_-(x,\bar{t}')T(x,y,\bar{t}',t) + N_+(y,\bar{t}')R(y,x,\bar{t}',t)} \ .$$

Using I twice, first let $y = a$, $z = b$; then $y = a$, with z arbitrary:

$$\boxed{\begin{aligned} \text{III.}\quad N_+(a,t) &= N_+(b,\bar{t}')T(b,a,\bar{t}',t) + N_-(a,\bar{t}')R(a,b,\bar{t}',t) \\ &= N_+(z,\bar{t}')T(z,a,\bar{t}',t) + N_-(a,\bar{t}')R(a,z,\bar{t}',t) \end{aligned}} \quad .$$

Using II twice: first let $y = b$, $x = a$; then let $y = b$ with x arbitrary:

$$\boxed{\begin{aligned} \text{IV.}\quad N_-(b,t) &= N_-(a,\bar{t}')T(a,b,\bar{t}',t) + N_+(b,\bar{t}')R(b,a,\bar{t}',t) \\ &= N_-(x,\bar{t}')T(x,b,\bar{t}',t) + N_+(b,\bar{t}')R(b,x,\bar{t}',t) \end{aligned}} \quad .$$

Differential Equations for the Time Dependent R and T Operators

The differential equations for the time dependent R and T operators may be derived by imagining a powerful short pulse of light pumped into $X(a,b)$ at its upper boundary X_a. The directional structure of this incident radiance distribution may be arbitrary as also its dependence with location on X_a. We shall assume that this is the only source of flux in $X(a,b)$ and that $N_-(a,t)$ is such that $N_-(a,t) = N_-(a,t_o)\delta(t-t_o), t_o \leq t$, where t_o is the time at which the pulse is incident on X_a. The subsequent operations with Dirac-delta functions are governed by the usual conventions which may, e.g., be found in [95].

We begin by applying the operator D_-/Dy (for time variable t) to principle I in which we have set $z = b$, and have used the fact that $N_+(b,t') = 0$ for every $t' \leq t$:

$$\frac{D_-N_+(y,t)}{Dy} = \frac{D_-N_-(y,\bar{t}')}{Dy} R(y,b,\bar{t}',t) + N_-(y,\bar{t}') \frac{D_-R(y,b,\bar{t}',t)}{Dy} \tag{36}$$

Next the operation $\lim_{y \to a}$, is applied to each side of (36). The left side of (36) yields, by means of (5):

$$N_+(a,t)\tau(a,t) + N_-(a,t)\rho(a,t) =$$

$$= N_-(a,t_o)[R(a,b,t_o,t)\tau(a,t) + \delta(t-t_o)\rho(a,t)] \quad .$$

The second equality is derived from principle III and the adopted form of $N_-(a,\cdot)$. By (5) and (6) we have, on applying the operator D_-/Dy (for time variable t) to $N_-(y,t')$:

$$\frac{D_-N_-(y,t')}{Dy} = \frac{\partial N_-(y,t')}{\partial y}$$

$$= \frac{1}{\mu v}\frac{\partial N_-(y,t')}{\partial t'} - N_-(y,t')\tau(y,t') - N_+(y,t')\rho(y,t')$$

so that:

$$\lim_{y\to a}\frac{D_-N_-(y,t')}{Dy} = N_-(a,t_o)\left[\frac{1}{\mu v}\delta'(t'-t_o) - \delta(t'-t_o)\tau(a,t') - \right.$$
$$\left. - R(a,b,t_o,t')\rho(a,t')\right]$$

where $\delta'(t'-t_o)$ is the symbolic derivative of the Dirac delta function with respect to t'. Finally:

$$\lim_{y\to a}\frac{D_-R(y,b,t',t)}{Dy} = -\frac{\partial R(a,b,t',t)}{\partial a} + \frac{1}{\mu v}\frac{\partial R(a,b,t',t)}{\partial t} .$$

Assembling all these results and using them in (36), rearranging (36), and cancelling the arbitrary function $N_-(a,\cdot)$, the resultant operator equation is obtained:

$$\text{I}' \quad -\frac{\partial R(a,b,t_o,t)}{\partial a} + \frac{1}{\mu v}\left[-\frac{\partial R(a,b,t_o,t)}{\partial t_o} + \frac{\partial R(a,b,t_o,t)}{\partial t}\right] =$$
$$= \delta(t-t_o)\rho(a,t) + \tau(a,t_o)R(a,b,t_o,t) + R(a,b,t_o,t)\tau(a,t) +$$
$$+ R(a,b,t_o,\bar{t}')\rho(a,\bar{t}')R(a,b,\bar{t}',t)$$

Now starting with principle II and applying in turn the operation D_+/Dy and the limit operation $\lim_{y\to b}$, and then making use of (5), (6) and principle IV, we have in a similar way:

$$\text{II}' \quad \frac{\partial T(a,b,t_o,t)}{\partial b} + \frac{1}{\mu v}\frac{\partial T(a,b,t_o,t)}{\partial t} =$$
$$= T(a,b,t_o,t)\tau(b,t) + T(a,b,t_o,\bar{t}')\rho(b,\bar{t}')R(b,a,\bar{t}',t)$$

The third differential relation follows from principle III by applying the same general procedure used to establish I' above:

$$\text{III}' \quad \frac{\partial R(a,b,t_0,t)}{\partial b} = T(a,b,t_0,\bar{t}')\rho(b,\bar{t}')T(b,a,\bar{t}',t)$$

The fourth differential relation follows from principle IV by applying the same general procedure used to establish II' above:

$$\text{IV}' \quad -\frac{\partial T(a,b,t_0,t)}{\partial a} - \frac{1}{\mu v}\frac{\partial T(a,b,t_0 t)}{\partial t_0} =$$
$$= \tau(a,t_0)T(a,b,t_0,t) + R(a,b,t_0,\bar{t}')\rho(a,\bar{t}')T(a,b,\bar{t}',t)$$

Discussions of the Differential Equations

The set of equations I'-IV' above is the desired set of differential equations for the time dependent R and T operators for plane-parallel media. These operators are homogeneous with respect to time--i.e., they depend on the difference $|z-x|$ of depth parameters--only if the medium is separable, where "separability" by definition means that σ/α is independent of depth (so that α may be separated into two factors: one spatial, the other directional). When the medium is homogeneous--or more generally, separable--then there are precisely two R and T operators associated with X(a,b). However, when X(a,b) is not separable, then there generally are four R and T operators for X(a,b): a reflectance-transmittance pair for flux incident on X_a, and a pair for X_b. Thus in nonseparable media, R and T exhibit *polarity*, i.e., we have $R(a,b,t',t) \neq R(b,a,t',t)$ or $T(a,b,t',t) \neq T(b,a,t',t)$ for some t',t in E. The general order of solution of the preceding equations is the same as the steady state case; thus one may solve the above system in either the order I',IV',II',III' or I',IV',III',II'. Furthermore, the pair I',IV' is the autonomous pair of the set of four equations in the sense that they determine R(a,b,t',t) and T(a,b,t',t) for X(a,b); and by interchanging "a" and "b" throughout and wherever necessary, the algebraic signs of the spatial derivatives, they also determine R(b,a,t',t) and T(b,a,t',t). Hence I' and IV' may be used for the determination of all four R and T operators for X(a,b). Numerical procedures for the solutions of I'-IV' may be constructed for the set in either undecomposed form or in decomposed form (cf. Sec. 7.1) of the T operators.

7.3 Algebraic and Analytic Properties of the R and T Operators

Consider a thin layer of scattering material in a plane-parallel medium. Suppose that this layer and another layer twice as thick are irradiated with radiant flux in the same manner. Is the reflectance of the latter layer twice that of the first layer? Intuition seems to say yes. Another question we may ask concerns the transmittance of the doubly thick layer relative to that of the layer of half its thickness. Intuition says the transmittance is simply the square of the single thin layer. In certain special cases both these intuitive guesses are essentially correct. But what of the general relation between the reflectances and transmittances of a medium of arbitrary thickness with those of its parts, arising under a general partitioning of the medium? Is there a general formula which relates the reflectances and transmittances of the 'sum' of two parts with those of each 'summand'? In this section we answer such questions for the case of an arbitrary, stratified plane-parallel medium. The various formulas we shall find are characteristic of the general case, i.e., they are essentially unchanged if one makes the transition to more general geometries and asks the same questions there. Hence the derivations which take place below are algebraically representative of the derivations in the more general settings, but the details have the advantage of being intuitively and analytically simpler than the general case.

Partition Relations for R and T Operators

The setting for the present derivation is depicted in part (a) of Fig. 7.2 in which a plane-parallel optical medium X(a,c) has been conceptually partitioned into two parts X(a,b) and X(b,c). We ask: what is the connection between the operators R(a,c), T(a,c) of X(a,c) and the reflectance and transmittance properties of its parts X(a,b) and X(b,c)? Another way of looking at essentially the same problem is to imagine that a given medium X(a,b) is imbedded in a larger medium, X(a,c) by the adjunction of the given layer X(b,c), and it is required to find the properties of X(a,c) in terms of the imbedded medium X(a,b) and the added medium X(b,c). See (b) of Fig. 7.2. Throughout the present discussion in our quest for the answer to the preceding question, we will draw freely on the concepts and relations developed in Secs. 3.6 and 3.7, especially Examples 2, 3 of Sec. 3.7.

We begin by assuming that X(a,c) is irradiated by an arbitrary $N_-(a)$ and with $N_+(c) = 0$. Using principle of invariance III of Ex. 3 in Sec. 3.7 applied to X(a,b), we have:

$$N_+(a) = N_-(a)R(a,b) + N_+(b)T(b,a) \quad . \qquad (1)$$

Principle III is again applied, now to X(a,c), to yield:

$$N_+(a) = N_-(a)R(a,c) \qquad (2)$$

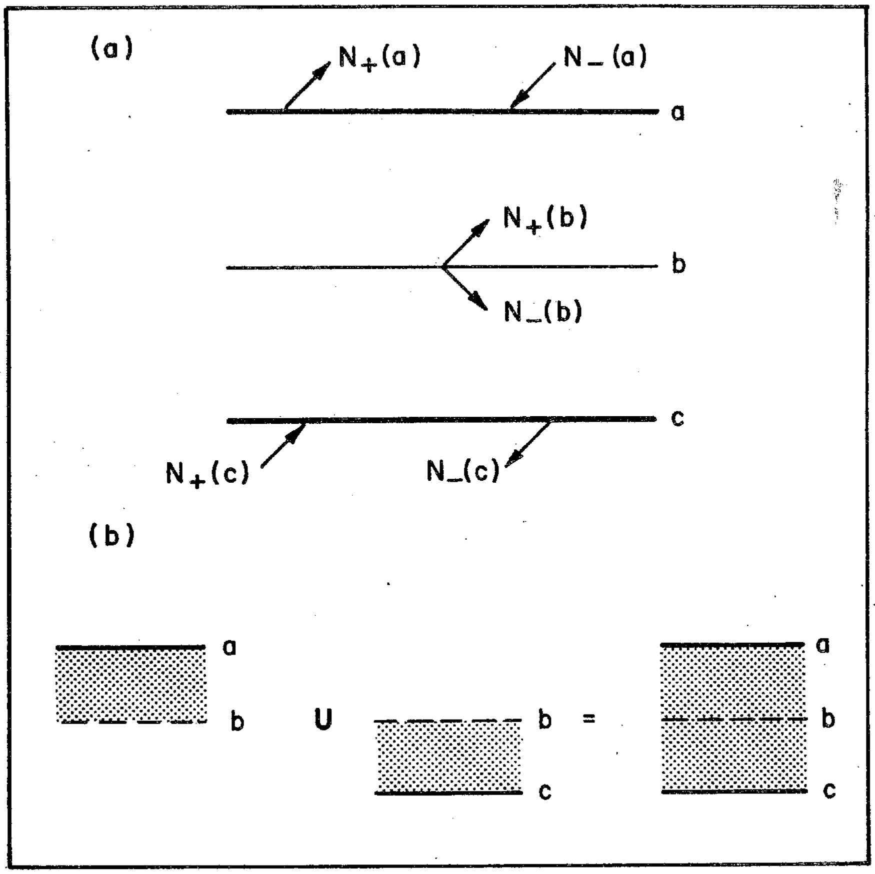


FIG. 7.2 Deriving the Partition Relations for R and T Operators.

by using the condition $N_+(c) = 0$.

Next, (25) of Sec. 3.7 is adapted to X(a,c) by replacing each "y" by "b" and each "b" by "c" in that equation. The result is:

$$N_+(b) = N_-(a)T(a,b)R(b,c)\left[I - R(b,a)R(b,c)\right]^{-1} \tag{3}$$

in which we have again used the condition $N_+(c) = 0$. Using (2) and (3) in (1), and noting that $N_-(a)$ is arbitrary, we obtain the first of the desired *partition relations*:

$$\boxed{R(a,c) = R(a,b) + T(a,b)R(b,c)\left[I - R(b,a)R(b,c)\right]^{-1}T(b,a)} \tag{4}$$

Before giving the physical interpretation of (4), we go on to find its companion formula for T(a,c). Using the same incident lighting conditions as before, we appeal to principle IV of Ex. 3 in Sec. 3.7 applied to X(b,c):

$$N_-(c) = N_-(b)T(b,c) \tag{5}$$

in which $N_+(c) = 0$ was used.

Principle IV is again applied, now to X(a,c):

$$N_-(c) = N_-(a)T(a,c) \tag{6}$$

using once again the condition $N_+(c) = 0$. Next we use (26) of Sec. 3.7 adapted to the present case by replacing each "y" by "b" and each "b" by "c" in that equation. The result is:

$$N_-(b) = N_-(a)T(a,b)\left[I - R(b,c)R(b,a)\right]^{-1} \tag{7}$$

using the condition $N_+(c) = 0$.

Using (7) in (5) and using (6) to represent the left side of (5), and also recalling that $N_-(a)$ is arbitrary, we obtain:

$$\boxed{T(a,c) = T(a,b)\left[I - R(b,c)R(b,a)\right]^{-1}T(b,c)} \tag{8}$$

which is the second of the two desired partition relations.

We now discuss some of the properties of (4) and (8). First of all, we see that our simple intuitive guesses about R(a,c) and T(a,c) given at the outset of this section are hardly correct for general media. The presence of the term $[I-R(b,c)R(b,a)]^{-1}$ in each equation (recall the observation on (28) of Sec. 3.7) represents the complex activity of interreflections between X(a,b) and X(b,c). However, to see that our intuitions are not wholly misleading, suppose this interreflection factor were absent from (4) and (8) or practically equal to I. This occurs when, e.g., the media X(a,b) and X(b,c) are optically thin so that R(b,c) and R(b,a) are very small. For example, to within first order of infinitesimals we have from (19) of Sec. 7.1:

$$R(a,b) = R(b,a) = \rho(b)|b-a| \tag{9}$$

$$R(b,c) = R(c,b) = \rho(b)|c-b| \tag{10}$$

where the numerical differences $|c-b|$ and $|b-a|$ of layer depths are small compared to the attenuation length $L_\alpha = 1/\alpha$. (This is what it means for X(a,b) and X(b,c) to be "*optically* thin".) Similarly, from (27) of Sec. 7.1 we have, retaining only the first order of infinitesimals:

$$T(a,b) = T(b,a) = I + \tau(b)|b-a| \tag{11}$$

$$T(b,c) = T(c,b) = I + \tau(b)|c-b| \tag{12}$$

where I is the identity operator and once again the absolute values $|c-b|$ and $|b-a|$ are small compared to L_α. It may be noted that the argument "b" of the ρ and τ operators may be replaced by "a" without changing the validity of the approximations.

With (9)-(12) in force, (4) becomes:

$$R(a,c) = R(a,b) + R(b,c) + o(|c-b|) \tag{13}$$

and (8) becomes:

$$T(a,c) = T(a,b)T(b,c) + o(|c-b|) \tag{14}$$

where "$o(|c-b|)$" denotes a quantity which goes to zero faster than the difference $|c-b|$, that is $\lim_{x\to 0} o(x)/x = 0$. Thus for practical work with optically thin media $X(a,b)$, $X(b,c)$, one may drop "$o(|b-c|)$" from (13) and (14); the result is a pair of equations which bears out the intuitive guesses stated in the introductory remarks above. *In brief, with respect to composite properties of optically thin media, their reflectances add and their transmittances multiply.*

We next cast (4) and (8) into alternative forms using the complete reflectance and transmittance operators associated with X(a,c). The advantages accrued from such a reformulation are both formal and intuitive. Thus from (40) of Sec. 3.7 now adapted to X(a,c) by replacing each "b" by "c" and each "y" by "b", we have as an alternate to (4):

$$\boxed{R(a,c) = R(a,b) + \mathcal{R}(a,b,c)T(b,a)} \tag{15}$$

and (8) becomes:

$$\boxed{T(a,c) = \mathcal{T}(a,b,c)T(b,c)} \tag{16}$$

using (42) of Sec. 3.7 suitably adapted to X(a,c). The relation of (16), to (51) of Sec. 3.7 should not escape notice. We see also that (15) is an important addition to the family of functional relations studied in Chapter 3, of the semigroup type for interaction operators over X(a,c). We shall see repeatedly below and in Sec. 7.4 the important uses to which (15) and (16) and their generalizations may be put. The relations (15) and (16) characterize the alternate point of view suggested in the introductory remarks; namely, that X(a,b) may be considered as imbedded in a larger medium X(a,c). The operators $\mathcal{R}(a,b,c)$ and $\mathcal{T}(a,b,c)$ point up this alternate view, being the $\mathcal{R}$ and $\mathcal{T}$ operators of the invariant imbedding relation.

Similar formulas hold for R(c,a) and T(c,a) associated with X(a,c). For purposes of reference these are given below:

$$\boxed{\begin{aligned} R(c,a) &= R(c,b) + \mathcal{R}(c,b,a)T(b,c) \\ T(c,a) &= \mathcal{T}(c,b,a)T(b,a) \end{aligned}} \qquad \begin{aligned} (17) \\ (18) \end{aligned}$$

Partition relations (15)-(18) will be generalized to the case of the $\mathcal{R}$ and $\mathcal{T}$ operators in Sec. 7.4 (see (76) of Sec. 7.4).

Alternate Derivations of the Differential Equations for R and T Operators

In Sec. 7.1 we derived the differential equations I'-IV' for the four R and T operators associated with X(a,b). One of the main ingredients of the derivations was the local forms of the principles of invariance (5), (6) of Sec. 7.1, i.e., the equation of transfer in operator form. The purpose of the present discussion is to show how one may derive the differential equations for the R and T operators without direct recourse to the equation of transfer. The knowledge we gain from such a tactic is of great theoretical importance: by deriving the differential equations for R and T directly from (15) and (16) above, we show that the theory of radiative transfer can be made to rest on the principles of invariance, the equation of transfer then being a law derived incidentally from the principles. This point of view of radiative transfer was explored in detail in Ref. [251]. We now present a simple exposition of this matter in the setting of plane-parallel media.

We begin with a derivation of the simplest of the four differential equations from (4), namely, (28) of Sec. 7.1. Rearrange (4) as follows:

$$\frac{R(a,c) - R(a,b)}{(c-b)} = T(a,b)\,\frac{R(b,c)}{(c-b)}\left[I - R(b,a)R(b,c)\right]^{-1}T(b,a) \tag{19}$$

We next assert that:

$$\lim_{c\to b}\frac{R(a,c) - R(a,b)}{(c-b)} = \frac{\partial R(a,b)}{\partial b} \tag{20}$$

$$\lim_{c\to b}\frac{R(b,c)}{(c-b)} = \rho(b) \tag{21}$$

$$\lim_{c\to b}\frac{T(b,c) - I}{(c-b)} = \tau(b) \tag{22}$$

for every

$$a,b,c:\quad a \le b \le c \quad . \tag{23}$$

Observe that in (20)-(22) we are boldly asserting the existence of these limits, and are not deducing them from such established results as, e.g., (9)-(12). In effect we are *defining* the members on the right sides of (20)-(22). This is in accordance with our present aim of starting with the principles of invariance as initial points of the derivation of the theory. In the present approach statements (20)-(22) are then called *regularity properties* of the R and T operators occurring in the statements of the principles of invariance. Any physical theory using the calculus and the concept of limit in particular must, somewhere along the path of its construction, in effect assume regularity properties of the principal functions of the theory under study. In the case of radiative transfer theory, a detailed and systematic enumeration of such properties was made in Ref. [216]. In the present work, commensurate with its different goals, these properties were for the most part implicitly assumed as each was needed (see, e.g., Secs. 3.10-3.15).

With the preceding assumptions in force, we let c approach b in (19). Physically, this amounts to letting the slab X(b,c) in Fig. 7.2 approach zero thickness. Mathematically, this results in the statement:

$$\frac{\partial R(a,b)}{\partial b} = T(a,b)\rho(b)T(b,a) \tag{24}$$

which is (28) of Sec. 7.1.

We next derive from (4) the most complicated of the four differential equations, namely (18) of Sec. 7.1. Subtracting R(b,c) from each side of (4) and using (28) of Sec. 3.7 to establish the fact that

$$[I - R(b,a)R(b,c)]^{-1} = I + R(b,a)R(b,c) + o(|b-a|) \tag{25}$$

(which follows from (21)) we can rewrite (4) as:

$$\frac{R(a,c) - R(b,c)}{b-a} = \frac{R(a,b)}{b-a} + \frac{[T(a,b)R(b,c)T(b,a) - R(b,c)]}{b-a} +$$

$$+ \frac{T(a,b)R(b,c)R(b,a)R(b,c)T(b,a)}{b-a} + \frac{o(|b-a|)}{b-a} . \tag{26}$$

Applying (22) to the operator T(a,b) and T(b,a) we have:

$$T(a,b) = I + \tau(a)|b-a| + o_1(|b-a|)$$

$$T(b,a) = I + \tau(a)|b-a| + o_2(|b-a|)$$

where $o_1(|b-a|)$ and $o_2(|b-a|)$ are analogous to $o(|b-a|)$ defined above. Hence:

$$\frac{T(a,b)R(b,c)T(b,a)-R(b,c)}{b-a} = R(b,c)\tau(a) + \tau(a)R(b,c) + \frac{o(|b-a|)}{b-a}$$

Postulating that:

$$\lim_{b \to a} \frac{R(a,c) - R(b,c)}{b-a} = -\frac{\partial R(a,c)}{\partial a} ,$$

we then see that equation (26), under the application of the limit operator $\lim_{b \to a}$, becomes:

$$-\frac{\partial R(a,c)}{\partial a} = \rho(a) + \tau(a)R(a,c) + R(a,c)\tau(a) + R(a,c)\rho(a)R(a,c)$$

which is (18) of Sec. 7.1 in the setting of $X(a,c)$. We may go on to deduce (27) and (29) of Sec. 7.1 from (8) in a similar manner. However, the point of the present derivation now seems well made and we leave such details to the interested reader and pass on to the next matter of the present section.

Asymptotic Properties of R and T Operators

How do the reflectance and transmittances of optically thick media behave with depth of the media? As in the case of optically thin media (cf. (13), (14)) our intuitions supply some rough answers to this question. In the case of reflectance, imagine an observer over a horizontally extensive homogeneous fog bank illuminated by the sun. The fog bank is optically very thick and is virtually blinding to the observer. Suppose now that, as the fog is under observation, it is noticeably decreasing its depth. However, the brilliantly reflected light does not seem to lose any of the intense magnitude until the final stages of dissipation. From common occurrences such as this we form the opinion that as the optical depth of a very deep homogeneous optical medium increases, there is eventually no appreciable change in its reflectance properties so that an upper limiting value of $R(a,b)$ is expected as $|b-a|$ increases without bound. On the other hand, to an observer on the ground below the great fog bank every bit of decrease in the thickness of the layer is noticeable as a corresponding increase in the general radiance distribution transmitted down to the observer. From recollections such as this, we form the expectation that the transmittance $T(a,b)$ should decrease rapidly to zero as $|b-a|$ increases without bound. We now show how these empirical facts are borne out by means of simple arguments using the differential equations for the R and T operators. A more detailed and rigorous analysis of the present ideas will be made in Chapter 10. For the present we simply pursue these ideas on a heuristic level. That is, we shall attempt to discover the requisite properties of $R(a,b)$ and $T(a,b)$ by treating them as if they were numerical magnitudes and the equations they satisfy as ordinary algebraic or differential equations of numerical valued functions.

Suppose a slab $X(a,b)$ is imbedded in an infinitely deep homogeneous plane-parallel medium $X(a,\infty)$. The homogeneity of $X(a,\infty)$ requires $\rho(y)$ and $\tau(y)$ to be independent of y for all y such that $a \leq y$. It follows from (18) of Sec. 7.1 that the differential equation for R(a,b) is:

$$-\frac{\partial R(a,b)}{\partial a} = \rho + \tau R(a,b) + R(a,b)\tau + R(a,b)\rho R(a,b). \quad (27)$$

From this we infer at once that R(a,b) = R(b,a) and the common value depends only on the difference $|b-a|$. Equation (28) of Sec. 7.1 implies that:

$$\frac{\partial R(a,b)}{\partial b} > 0 \quad (28)$$

since T(a,b), T(b,a) and ρ are analogous to positive valued functions. Thus we recover within the theory the empirical fact that for fixed a, R(a,b) increases as b, and hence $|b-a|$ increases. It is a bit more difficult to establish:

$$\frac{\partial T(a,b)}{\partial b} < 0 \quad , \quad (29)$$

i.e., the fact that for fixed a, T(a,b) decreases as $|b-a|$ increases. This can be made plausible by noting that the term τ+ R(b,a) in (27) of Sec. 7.1 is negative when there is absorption but no scattering in the medium, i.e., when $\sigma = 0$ and $a > 0$. Since T(a,b) is positive, (27) of Sec. 7.1 then implies (29) above. However, by slowly increasing σ from 0 to small positive values, the inequality (29) clearly persists for a while; and indeed, in all natural optical media, (29) can be shown to hold with only mild regularity properties imposed. From (29) and (28) of Sec. 7.1 we now can see that:

$$\lim_{b\to\infty} \frac{\partial R(a,b)}{\partial b} = 0$$

so that, by (27) above (i.e., since R(a,b) = R(b,a)):

$$\rho + \tau R(a,\infty) + R(a,\infty)\tau + R(a,\infty)\rho R(a,\infty) = 0 \quad (30)$$

where we have written:

$$\text{"}R(a,\infty)\text{"} \quad \text{for} \quad \lim_{b\to\infty} R(a,b) \quad . \quad (31)$$

Equation (30) shows that $R(a,\infty)$ is independent of a since ρ and τ are. This property was formally used by Ambarzumian in [1] to derive some of the earliest forms of the integral equations indigenous to the invariant imbedding point of view of transfer phenomena. When certain reciprocity conditions hold for the medium, we have:

$$R(a,\infty)\tau = \tau R(a,\infty)$$

and

$$R(a,\infty)\rho = \rho R(a,\infty)$$

i.e., we have *commutativity* of the T, R, τ and ρ operators.

Under such conditions (which hold, e.g., when scattering is isotropic) (30) becomes:

$$\rho + 2\tau R(a,\infty) + \rho R^2(a,\infty) = 0 \quad . \qquad (32)$$

The solutions of (30) or the special case (32) yield the form for $R(a,\infty)$. A numerical procedure leading to $R(a,b)$ for a range of finite b (from which $R(a,\infty)$ is estimable) will be given in Sec. 7.6 for spaces $X(a,b)$ in which scattering is isotropic. We shall return to these heuristic operations with the R and T operators in Sec. 8.7. The operations just performed can be redone in the irradiance context and can be made fully rigorous without the need for advanced mathematical techniques. See (35)-(38) and (39)-(42) of Sec. 8.7.

7.4 Algebraic Properties of the Invariant Imbedding Operators

The various invariant imbedding operators introduced in examples 4-7 of Sec. 3.7 will now be studied in greater detail. Our main purpose in the present section will be to demonstrate the fact that the collection $\Gamma_2(a,b)$ of operators of the form $\mathcal{M}(x,y)$, which we found in Example 4 of Sec. 3.7 to constitute a partial group, may be used as basic building blocks to systematically construct, via simple algebraic procedures, all other operators of the collections $\Gamma_3(a,b)$ and $\Gamma_4(a,b)$, and hence all R and T operators and their simple combinations. The net result of these possible constructions will be novel procedures for solving transfer problems in plane-parallel and, indeed, general optical media. In other words, we shall demonstrate that the operators $\mathcal{M}(x,y)$ can serve as the computational work horses on both theoretical and practical levels in the theory of radiative transfer and thereby have them earn their right to reside among the giants, the elements of $\Gamma_4(a,b)$, which in turn serve to unify the theory and to link the theory with the interaction principles.

Throughout this section, unless stated otherwise, we shall work with an arbitrarily source-free plane-parallel medium $X(a,b)$, $a \le b$, with arbitrary incident radiance distributions $N_-(a)$ and $N_+(b)$ over the upper boundary X_a and lower boundary X_b respectively. Generalizations of the indicated results to general one-parameter media are immediate; generalizations to arbitrary media can be patterned after the discussions of Sec. 25, Ref. [251]. Throughout the discussion all reference to various regularity properties required for inverse operations, differentiations, integrations, etc., has been avoided so as to bring out the highly intuitive flavor of the operator algebra.

The Operator M(x,z)

The simplest interaction operator associated with a general plane-parallel medium $X(a,b)$, $a \le b$ is that which maps (or transforms) the pair $(N_+(b),N_-(a))$ of incident radiance distributions on $X(a,b)$ into the pair $(N_+(a),N_-(b))$ of response radiance distributions for $X(a,b)$. It is a simple exercise in the use of the principles of invariance for $X(a,b)$ to determine this operator. Thus, from principle III, in Example 3 of Sec. 3.7, we have:

$$N_+(a) = N_+(b)T(b,a) + N_-(a)R(a,b) \quad ;$$

and from principle IV we have:

$$N_-(b) = N_-(a)T(a,b) + N_+(b)R(b,a) \quad .$$

The matricial form of this system of equations is:

$$\left(N_+(a),N_-(b)\right) = \left(N_+(b),N_-(a)\right)\begin{bmatrix} T(b,a) & R(b,a) \\ R(a,b) & T(a,b) \end{bmatrix} \quad .$$

The displayed matrix of R and T operators is the requisite interaction operator. More generally, let $X(x,z)$ be an arbitrary plane-parallel subset of $X(a,b)$, $a \le x \le z \le b$, and suppose $\left(N_+(z),\ N_-(x)\right)$ and $\left(N_+(x),\ N_-(z)\right)$ are, respectively, the incident and response radiance distributions on $X(x,z)$ as they exist in the medium $X(a,b)$ which is irradiated by an arbitrary set $N_+(b)$, $N_-(a)$ of radiance distributions on its lower and upper boundaries, respectively. (See Fig. 7.1.) Then principle I in Example 3 of Sec. 3.7 yields for the case $x = y$:

$$N_+(x) = N_+(z)T(z,x) + N_-(x)R(x,z) \quad .$$

Similarly, principle II yields: for the case $y = z$:

$$N_-(z) = N_-(x)T(x,z) + N_+(z)R(z,x) \quad .$$

The matricial form of this system of equations is:

$$\left(N_+(x),N_-(z)\right) = \left(N_+(z),N_-(x)\right)\begin{bmatrix} T(z,x) & R(z,x) \\ R(x,z) & T(x,z) \end{bmatrix} \quad . \tag{1}$$

Let us write:

$$\text{"}M(x,z)\text{"} \quad \text{for} \quad \begin{bmatrix} T(z,x) & R(z,x) \\ R(x,z) & T(x,z) \end{bmatrix} \quad , \tag{2}$$

where $a \le x \le z \le b$. Thus $M(x,z)$ is a 2×2 operator matrix which is defined for depth variables x, z such that the preceding equalities hold. Some experimentation with (1) will show why this restriction (namely $x \le z$) is necessary if we are to retain the useful convention of always writing radiance distribution pairs with the upward (+) distributions as the first member of the pair. Another advantage in preserving the fixed order of variables x,z in $M(x,z)$ shows up in the detailed computations below wherein it will always be clear whether an operator on an upward or downward flow in $X(a,b)$ is being represented. Thus in all that follows, $M(x,z)$ with $x \le z$ is a useful conceptual anchor whose components have simple physical significance. Let us denote by "$\Gamma_2(a,b)$" the set of all operators $M(x,z)$, $a \le x \le z \le b$.

The Connections Between $M(x,z)$, $\mathcal{M}(x,z)$, and $\mathcal{M}(z,x)$

We now establish the connections between the operator $M(x,z)$ and the operators $\mathcal{M}(x,z), \mathcal{M}(z,x)$ in the setting of an arbitrary sub-medium $X(x,z)$ in $X(a,b)$. (Recall (78) of Sec. 3.7.) Once this connection is established, we will have an effective means of computing $\mathcal{M}(x,z)$ and $\mathcal{M}(z,x)$ in terms of the standard R and T operators for $X(x,z)$: and conversely, the operator $M(x,z)$ will be directly representable in terms of the operators $\mathcal{M}(x,z)$, $\mathcal{M}(z,x)$. This latter representation will be a prototype of more general representations of the members of $\Gamma_3(a,b)$ and $\Gamma_4(a,b)$ to be derived subsequently, and will be instrumental in developing novel methods of solution of light fields in $X(a,b)$, later in this chapter.

To establish the requisite connections we require the partition of the identity operator I on $\Gamma_2(a,b)$:

$$I = C_+ + C_- \quad , \tag{3}$$

where we write:

$$\text{"}C_+\text{"} \quad \text{for} \quad \begin{bmatrix} I_+ & 0 \\ 0 & 0 \end{bmatrix} \tag{4}$$

and

$$\text{"}C_-\text{"} \quad \text{for} \quad \begin{bmatrix} 0 & 0 \\ 0 & I_- \end{bmatrix} \quad . \tag{5}$$

In C_+ and C_-, I_+ is the identity operator on the set of all* upward radiance distributions and I_- is the identity operator on the set of all downward radiance distributions associated with $X(a,b)$. No confusion will result if in the subsequent discussions we drop the signed subscripts from the identity operators (their positions in the matrices provide adequate identification). The general working properties of C_+ and C_- are obtained by direct computation:

$$\begin{gathered} C_+^2 = C_+ \\ C_-^2 = C_- \\ C_-C_+ = C_+C_- = 0 \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} C_+ = \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} C_- = \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix} \end{gathered} \tag{6}$$

*We drop +,- on $I_\pm$ when direction is clear.

$$C_+\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$$

$$C_-\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix}$$

$$(a,b)C_+ = (a,0)$$

$$(a,b)C_- = (0,b)$$

Hence, via suitable pre- and post-multiplications by C_+ or C_-, various elements of a matrix of operators or of a vector can be isolated as needed.

Now, equation (1) holds for all incident radiances $(N_+(z),N_-(x))$ on $X(x,z)$. From the definition of the operators $\mathcal{M}(x,z)$ and $\mathcal{M}(z,x)$ and the partition operators of I, we have:

$$\big(N_+(x),\ 0\big) = \big(N_+(z),N_-(z)\big)\mathcal{M}(z,x)C_+$$

$$\big(0,N_-(z)\big) = \big(N_+(z),N_-(z)\big)C_- \quad .$$

Adding, we have:

$$\big(N_+(x),N_-(z)\big) = \big(N_+(z),N_-(z)\big)\big[\mathcal{M}(z,x)C_+ + C_-\big] \quad . \tag{7}$$

Further:

$$\big(N_+(z),0\big) = \big(N_+(z),N_-(z)\big)C_+$$

$$\big(0,N_-(x)\big) = \big(N_+(z),N_-(z)\big)\mathcal{M}(z,x)C_- \quad .$$

Adding, we have:

$$\big(N_+(z),N_-(x)\big) = \big(N_+(z),N_-(z)\big)\big[C_+ + \mathcal{M}(z,x)C_-\big] \quad . \tag{8}$$

Combining (1), (7) and (8),

$$\big(N_+(z),N_-(z)\big)\big[C_+ + \mathcal{M}(z,x)C_-\big]M(x,z) = \big(N_+(z),N_-(z)\big)\big[\mathcal{M}(z,x)C_+ + C_-\big]$$

This holds for every incident light field on $X(x,z)$. Hence:

$$\big[C_+ + \mathcal{M}(z,x)C_-\big]M(x,z) = \big[\mathcal{M}(z,x)C_+ + C_-\big] \tag{9}$$

whence

$$\boxed{M(x,z) = \big[C_+ + \mathcal{M}(z,x)C_-\big]^{-1}\big[\mathcal{M}(z,x)C_+ + C_-\big]} \quad . \tag{10}$$

On the other hand, solving (9) for $\mathcal{M}(z,x)$, we have:

$$\mathcal{M}(z,x)\left[C_-M(x,z) - C_+\right] = \left[C_- - C_+M(x,z)\right]$$

whence:

$$\boxed{\mathcal{M}(z,x) = \left[C_- - C_+M(x,z)\right]\left[C_-M(x,z) - C_+\right]^{-1}} \quad . \quad (11)$$

On taking inverses of each side of (11):

$$\boxed{\mathcal{M}(x,z) = \left[C_-M(x,z) - C_+\right]\left[C_- - C_+M(x,z)\right]^{-1}} \quad . \quad (12)$$

This may be solved for $M(x,z)$ to yield a companion formula to (10):

$$\boxed{M(x,z) = \left[\mathcal{M}(x,z)C_+ + C_-\right]^{-1}\left[C_+ + \mathcal{M}(x,z)C_-\right]} \quad . \quad (13)$$

Equations (10)-(13) are the desired connections between the operators $M(x,z)$, $\mathcal{M}(x,z)$, and $\mathcal{M}(z,x)$ for levels x,z in $X(a,b)$ with $x \le z$.

Invertibility of Operators

The inverse operators in the preceding representations can be examined in detail so as to allow us to establish some conditions sufficient to insure their existence. The inverses generally encountered in computations with (10)-(12) are of the form:

$$\left[C_+ + AC_-\right]^{-1}$$

$$\left[AC_+ + C_-\right]^{-1}$$

$$\left[C_- - C_+A\right]^{-1}$$

$$\left[C_-A - C_+\right]^{-1}$$

where:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad .$$

"A" denotes either the $\mathcal{M}$ or M matrices so that a, b, c, and d are generally operators on radiance distributions. To evaluate these inverses consider for example the first; we require a 2×2 matrix with elements α, β, γ, δ such that:

$$\left[C_+ + AC_-\right]\begin{bmatrix}\alpha & \beta \\ \gamma & \delta\end{bmatrix} = \begin{bmatrix}I & b \\ 0 & d\end{bmatrix}\begin{bmatrix}\alpha & \beta \\ \gamma & \delta\end{bmatrix} = \begin{bmatrix}I & 0 \\ 0 & I\end{bmatrix} \quad .$$

From this are obtained the four equations:

$$\alpha + b\gamma = I$$

$$\beta + b\delta = 0$$

$$d\gamma = 0$$

$$d\delta = I$$

which in turn determine the elements of the inverse:

$$\alpha = I$$

$$\beta = -bd^{-1}$$

$$\gamma = 0$$

$$\delta = d^{-1} \quad .$$

Hence:

$$\left[C_+ + AC_-\right]^{-1} = \begin{bmatrix}I & -bd^{-1} \\ 0 & d^{-1}\end{bmatrix} \quad . \tag{14}$$

The remaining three inverses are obtained similarly:

$$\left[AC_+ + C_-\right]^{-1} = \begin{bmatrix}a^{-1} & 0 \\ -ca^{-1} & I\end{bmatrix} \tag{15}$$

$$\left[C_- - C_+A\right]^{-1} = \begin{bmatrix}-a^{-1} & -a^{-1}b \\ 0 & I\end{bmatrix} \tag{16}$$

$$\left[C_-A - C_+\right]^{-1} = \begin{bmatrix}-I & 0 \\ d^{-1}c & d^{-1}\end{bmatrix} \quad . \tag{17}$$

From an inspection of this collection of inverses it is clear that their existences depend in turn on the existences of the inverses of the component operators a and d in A. When A is $M(x,z)$, this requires the transmittance operators $T(x,z)$ and $T(z,x)$ to have inverses. In most natural optical media (oceans, atmosphere), the volume scattering function σ and volume absorption function a are positive throughout the media. This property of σ and a generally insures the norm contraction property of $I-T(x,z)$ or $I-T(z,x)$ so that under these conditions the inverses of $T(x,z)$ and $T(z,x)$ exist. Of course in any specific instance, it is good practice to have the invertibility of the transmittance operators verified in detail. In the present discussions our interest is solely in the algebraic structure of and interconnections between the various interaction operators, and the discussion proceeds on the assumption that all required regularity properties are in force.

Representations for the Components of $\mathcal{M}(x,z), \mathcal{M}(z,x)$

By means of the functional equations (11), (12) for $\mathcal{M}(z,x)$ and $\mathcal{M}(x,z)$ we can find explicit formulas for the components of these operators in terms of the four standard R and T operators for X(x,z). Thus let us write:

$$\text{"}\begin{bmatrix} \mathcal{M}_{++}(x,z) & \mathcal{M}_{+-}(x,z) \\ \mathcal{M}_{-+}(x,z) & \mathcal{M}_{--}(x,z) \end{bmatrix}\text{"} \quad \text{for} \quad \mathcal{M}(x,z) \tag{18}$$

thereby defining, in context, four operator components of $\mathcal{M}(x,z)$. A similar definition is made for $\mathcal{M}(z,x)$. Next we observe that the two factors comprising $\mathcal{M}(z,x)$ in (11) may be written:

$$\begin{bmatrix} C_- & - C_+M(x,z) \end{bmatrix} = \begin{bmatrix} -T(z,x) & -R(z,x) \\ 0 & I \end{bmatrix}$$

and, by (17):

$$\begin{bmatrix} C_-M(x,z) & - C_+ \end{bmatrix}^{-1} = \begin{bmatrix} -I & 0 \\ T^{-1}(x,z)R(x,z) & T^{-1}(x,z) \end{bmatrix} .$$

With these specific representations of the factors in (11), we have:

$$\mathcal{M}(z,x) = \begin{bmatrix} T(z,x)-R(z,x)T^{-1}(x,z)R(x,z) & -R(z,x)T^{-1}(x,z) \\ T^{-1}(x,z)R(x,z) & T^{-1}(x,z) \end{bmatrix}$$

whence

$$\mathcal{M}_{++}(z,x) = T(z,x) - R(z,x)T^{-1}(x,z)R(x,z) \tag{19}$$

$$\mathcal{M}_{+-}(z,x) = -R(z,x)T^{-1}(x,z) \tag{20}$$

$$\mathcal{M}_{-+}(z,x) = T^{-1}(x,z)R(x,z) \tag{21}$$

$$\mathcal{M}_{--}(z,x) = T^{-1}(x,z) \quad . \tag{22}$$

Next, we use (12) to find the component operators of $\mathcal{M}(x,z)$. The first factor in (12) is:

$$\left[C_- M(x,z) \ - \ C_+\right] = \begin{bmatrix} -I & 0 \\ R(x,z) & T(x,z) \end{bmatrix} .$$

The inverse operator is evaluated by means of (16):

$$\left[C_- \ - \ C_+ M(x,z)\right]^{-1} = \begin{bmatrix} -T^{-1}(z,x) & -T^{-1}(z,x)R(z,x) \\ 0 & I \end{bmatrix} .$$

Then (12) becomes:

$$\mathcal{M}(x,z) = \begin{bmatrix} T^{-1}(z,x) & T^{-1}(z,x)R(z,x) \\ -R(x,z)T^{-1}(z,x) & T(x,z)-R(x,z)T^{-1}(z,x)R(z,x) \end{bmatrix}$$

whence

$$\mathcal{M}_{++}(x,z) = T^{-1}(z,x) \tag{23}$$

$$\mathcal{M}_{+-}(x,z) = T^{-1}(z,x)R(z,x) \tag{24}$$

$$\mathcal{M}_{-+}(x,z) = -R(x,z)T^{-1}(z,x) \tag{25}$$

$$\mathcal{M}_{--}(x,z) = T(x,z) \ - \ R(x,z)T^{-1}(z,x)R(z,x) \quad . \tag{26}$$

The components of M(x,z) may be represented in two equivalent ways, depending on whether (10) or (13) is used. Using (10), the factors are, explicitly:

$$\left[\mathcal{M}(z,x)C_+ \ + \ C_-\right] = \begin{bmatrix} \mathcal{M}_{++}(z,x) & 0 \\ \mathcal{M}_{-+}(z,x) & I \end{bmatrix} ,$$

and from (14) with $A = \mathcal{M}(z,x)$:

$$\left[\mathcal{M}(z,x)C_+ \ + \ C_-\right]^{-1} = \begin{bmatrix} I & -\mathcal{M}_{+-}(z,x)\mathcal{M}_{--}^{-1}(z,x) \\ 0 & \mathcal{M}_{--}^{-1}(z,x) \end{bmatrix} .$$

Then

$$M(x,z) = \begin{bmatrix} \mathcal{M}_{++}(z,x)-\mathcal{M}_{-+}(z,x)\mathcal{M}_{+-}(z,x)\mathcal{M}_{--}^{-1}(z,x) & -\mathcal{M}_{+-}(z,x)\mathcal{M}_{--}^{-1}(z,x) \\ \mathcal{M}_{--}^{-1}(z,x)\mathcal{M}_{-+}(z,x) & \mathcal{M}_{--}^{-1}(z,x) \end{bmatrix} .$$

From this:

$$T(z,x) = \mathcal{M}_{++}(z,x) \ - \ \mathcal{M}_{-+}(z,x)\mathcal{M}_{+-}(z,x)\mathcal{M}_{--}^{-1}(z,x) \tag{27}$$

$$R(z,x) = -\mathcal{M}_{+-}(z,x)\,\mathcal{M}_{--}^{-1}(z,x) \tag{28}$$

$$R(x,z) = \mathcal{M}_{--}^{-1}(z,x)\,\mathcal{M}_{-+}(z,x) \tag{29}$$

$$T(x,z) = \mathcal{M}_{--}^{-1}(z,x) \quad . \tag{30}$$

Alternatively, the factors in (13) are:

$$\left[C_+ + \mathcal{M}(x,z)C_-\right] = \begin{bmatrix} I & \mathcal{M}_{+-}(x,z) \\ 0 & \mathcal{M}_{--}(x,z) \end{bmatrix}$$

and from (15):

$$\left[\mathcal{M}(x,z)C_+ + C_-\right]^{-1} = \begin{bmatrix} \mathcal{M}_{++}^{-1}(x,z) & 0 \\ -\mathcal{M}_{-+}(x,z)\,\mathcal{M}_{++}^{-1}(x,z) & I \end{bmatrix} \quad .$$

Then

$$M(x,z) = \begin{bmatrix} \mathcal{M}_{++}^{-1}(x,z) & \mathcal{M}_{++}^{-1}(x,z)\mathcal{M}_{+-}(x,z) \\ -\mathcal{M}_{-+}(x,z)\mathcal{M}_{++}^{-1}(x,z) & \mathcal{M}_{--}(x,z)-\mathcal{M}_{-+}(x,z)\mathcal{M}_{++}^{-1}(x,z)\mathcal{M}_{+-}(x,z) \end{bmatrix}$$

From this:

$$T(z,x) = \mathcal{M}_{++}^{-1}(x,z) \tag{31}$$

$$R(z,x) = \mathcal{M}_{++}^{-1}(x,z)\mathcal{M}_{+-}(x,z) \tag{32}$$

$$R(x,z) = -\mathcal{M}_{-+}(x,z)\mathcal{M}_{++}^{-1}(x,z) \tag{33}$$

$$T(x,z) = \mathcal{M}_{--}(x,z) - \mathcal{M}_{-+}(x,z)\mathcal{M}_{++}^{-1}(x,z)\mathcal{M}_{+-}(x,z) . \tag{34}$$

The connections between the two sets of representations (27)-(34) of $M(x,z)$ rest on the fact that $\mathcal{M}(x,z)$ and $\mathcal{M}(z,x)$ are mutual inverses. The four component equations harbored by:

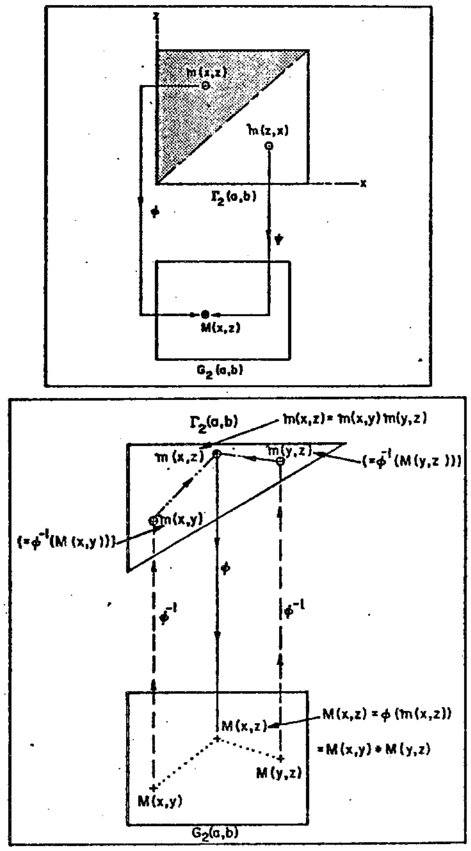
$$\mathcal{M}(x,z)\mathcal{M}(z,x) = I$$

provide the necessary explicit link between the two preceding sets of representations. It is interesting to observe that one may go from one set of representations to another by simultaneously interchanging the arguments "x" and "z" along with the subscripts "+" and "-". This interchange rule also works for the sets (19)-(22) and (23)-(26), and also for the functional equations (10)-(13) (leaving $M(x,z)$ inviolate). The physical basis of this rule is that such interchanges applied to the radiance vector $(N_+(z),N_-(x))$ and the matrices $\mathcal{M}(x,z)$, $\mathcal{M}(z,x)$, effectively reverse the incident and response radiances and the operators applied to them.

The Isomorphism ϕ Between $\Gamma_2(a,b)$ and $G_2(a,b)$

The algebraic links just established between the operators $M(x,z)$ and $\mathcal{M}(x,z)$ suggest a close overall structural resemblance between the members of the set $G_2(a,b)$ (i.e., all $M(x,z)$, with $a \le x \le z \le b$) and the members of the partial group $\Gamma_2(a,b)$ (i.e., all $\mathcal{M}(x,z)$, $a \le x \le b$, $a \le z \le b$). We can use this strong tie between the two sets to induce a means for multiplying together members of $G_2(a,b)$ in a way that faithfully mirrors the natural multiplication of elements of $\Gamma_2(a,b)$. The practical utility of the newly formed multiplication process will become clear as this discussion nears its close.

Let us denote by "$\phi(\mathcal{M}(x,z))$" the operator $M(x,z)$ found from $\mathcal{M}(x,z)$ using (31)-(34), and let "$\phi^{-1}(M(x,z))$" denote the operator $\mathcal{M}(x,z)$ obtained from $M(x,z)$ using (23)-(26)



FIGS. 7.3 ,7.4 The meaning of the isomorphism between $\Gamma_2(a,b)$ and $G_2(a,b)$.

where, $a \le x \le z \le b$. In this manner we define in context a function ϕ on part of $\Gamma_2(a,b)$ (call it the *upper triangle* of $\Gamma_2(a,b)$ onto $G_2(a,b)$. (That is, we do not define ϕ for all pairs x,z, but only those such that $x \le z$.) This function is *one to one* in the sense that to each $M(x,z)$ in $G_2(a,b)$ there is assigned at most $\mathcal{M}(x,z)$ in the upper triangle of $\Gamma_2(a,b)$ for any choice of levels x,z in $X(a,b)$, where $x \le z$. The term "upper triangle" of $\Gamma_2(a,b)$ is suggested by the fact that in a cartesian coordinate plot of the pairs depths (x,z), those pairs such that $x \le z$, lie above the diagonal line. (See shaded region in Fig. 7.3.) An alternate one to one mapping ψ from the lower triangle of $\Gamma_2(a,b)$ onto $G_2(a,b)$ is possible using the systems (19)-(22) and (17)-(20). Either mapping ϕ or ψ will suffice for our present purposes. We choose to work with ϕ as far as possible. With this choice of (12), (13) may be rewritten as:

$$M(x,z) = \phi(\mathcal{M}(x,z)) = \left[\mathcal{M}(x,z)C_+ + C_-\right]^{-1}\left[C_+ + \mathcal{M}(x,z)C_-\right]$$

$$\mathcal{M}(x,z) = \phi^{-1}(M(x,z)) = \left[C_-M(x,z) - C_+\right]\left[C_- - C_+M(x,z)\right]^{-1} .$$

The induction of the multiplication process on $G_2(a,b)$ is now carried out as follows. Let $M(x,y)$ and $M(y,z)$ be any two elements of $G_2(a,b)$, provided that they have a depth level in common (e.g., y, as shown). It seems natural to require that their "product" be such that the usual matrix product of the corresponding operators $\phi^{-1}(M(x,y))$ and $\phi^{-1}(\mathcal{M}(y,z))$ in $\Gamma_2(a,b)$ maps back, under ϕ, to the required "product". (See Fig. 7.4). Thus we agree to write:

$$\text{"}M(x,y)*M(y,z)\text{"} \quad \text{for} \quad \phi\left[\phi^{-1}(M(x,y))\phi^{-1}(M(y,z))\right] . \tag{35}$$

By definition of ϕ^{-1} and the one to one properties of ϕ:

$$\mathcal{M}(x,y) = \phi^{-1}(M(x,y))$$

and:

$$\mathcal{M}(y,z) = \phi^{-1}(M(y,z)) .$$

Hence:

$$\phi(\mathcal{M}(x,y)) = M(x,y)$$

$$\phi(\mathcal{M}(y,z)) = M(y,z) .$$

Therefore an alternate way of expressing (35) is:

$$\boxed{\phi(\mathcal{M}(x,y)\mathcal{M}(y,z)) = \phi(\mathcal{M}(x,y))*\phi(\mathcal{M}(y,z))} \tag{36}$$

This alternate form of describing the *star product* of elements of $G_2(a,b)$ defined in (35) shows how the structure of multiplication in $G_2(a,b)$ mirrors that of $\Gamma_2(a,b)$. In modern algebra the function ϕ which induces operations such as the operation * is called an *isomorphism*, the etymology of the word in this physical case being most appropriate (*iso* = same;

morph = form). Under the introduction of the star product, $G_2(a,b)$ becomes a partial semigroup, with an identity operator of the form $M(x,x)$, and with the associativity property and inverse properties holding.

The Physical Interpretation of the Star Product

The star product on $G_2(a,b)$ introduced above has a most interesting physical interpretation. It is worthwhile to pursue this interpretation as it will permit us to tie together the territory covered so far in this section with that of Section 7.3. Since $M(x,y)$ describes the reflectance and transmittance properties of $X(x,y)$, and $M(y,z)$ describes those of $X(y,z)$, we ask: What physical description, relative to $X(x,z)$, does the star product $M(x,y)*M(y,z)$ represent? The clue to this description is given by examining (35). The right side of the definition is simply the image, under ϕ, of $\mathcal{M}(x,z)$. Hence we see that:

$$\boxed{M(x,z) = M(x,y)*M(y,z)} \quad . \tag{37}$$

Therefore the star product of $M(x,z)$ and $M(y,z)$ is the operator $M(x,z)$ associated with the union (the sum) of the two contiguous slabs $X(x,y)$ and $X(y,z)$ (as depicted e.g., in (b) of Fig. 7.2).

Let us find the components of the star product $M(x,z)*M(y,z)$ directly in terms of the components of the factors $M(x,z)$ and $M(y,z)$. We begin the derivation with (35). Thus, by (23)-(26):

$$\phi^{-1}\big(M(x,y)\big) = \begin{bmatrix} T^{-1}(y,x) & T^{-1}(y,x)R(y,x) \\ -R(x,y)T^{-1}(y,x) & T(x,y)-R(x,y)T^{-1}(y,x)R(y,x) \end{bmatrix} .$$

Similarly:

$$\phi^{-1}\big(M(y,z)\big) = \begin{bmatrix} T^{-1}(z,y) & T^{-1}(z,y)R(z,y) \\ -R(y,z)T^{-1}(z,y) & T(y,z)-R(y,z)T^{-1}(z,y)R(z,y) \end{bmatrix} .$$

The product of these matrices is:

$$\phi^{-1}\big(M(x,y)\big)\phi^{-1}\big(M(y,z)\big) = \mathcal{M}(x,z) \quad ,$$

and where:

$$\mathcal{M}_{++}(x,z) = T^{-1}(y,x)T^{-1}(z,y) - T^{-1}(y,x)R(y,x)R(y,z)T^{-1}(z,y)$$

$$\mathcal{M}_{+-}(x,z) = T^{-1}(y,x)T^{-1}(z,y)R(z,y) +$$
$$+T^{-1}(y,x)R(y,x)[T(y,z)-R(y,z)T^{-1}(z,y)R(z,y)]$$

$$\mathcal{M}_{-+}(x,z) = -R(x,y)T^{-1}(y,x)T^{-1}(z,y) -$$
$$-[T(x,y)-R(x,y)T^{-1}(y,x)R(y,x)][R(y,z)T^{-1}(z,y)]$$

$$\mathcal{M}_{--}(x,z) = -R(x,y)T^{-1}(y,x)T^{-1}(z,y)R(z,y) +$$
$$+[T(x,y)-R(x,y)T^{-1}(y,x)R(z,y)][T(y,x)-R(y,z)T^{-1}(z,y)R(z,y)]$$

Each of these may be reduced considerably if we use algebraic formulas developed earlier. For example:

$$\begin{aligned}\mathcal{M}_{++}(x,z) &= T^{-1}(y,x)[I - R(y,x)R(y,z)]T^{-1}(z,y)\\ &= T^{-1}(z,x) \quad ,\end{aligned}$$

when the last inequality is based on (18) of Sec. 7.3. (See also (8) of Sec. 7.3.) In a similar (but slightly more arduous) manner the remaining components may be reduced so that they may be used in (31)-(34). The net result of the mapping back to $M(x,z)$ from $\mathcal{M}(x,z)$ is:

$$M(x,z) = M(x,y)*M(y,z) =$$
$$= \begin{bmatrix} \mathcal{T}(z,y,x)T(y,x) & R(z,y)+\mathcal{R}(z,y,x)T(y,z) \\ R(x,y)+\mathcal{R}(x,y,z)T(y,x) & \mathcal{T}(x,y,z)T(y,z) \end{bmatrix} . \tag{38}$$

In this way the representation of the star product is rendered into a mathematically self-contained form by means of the partition relations developed in 7.3. The representation is made particularly meaningful physically by using the complete reflectance and transmittance operator for $X(x,z)$, so that each component of the product can be read directly in terms of reflectances and transmittances. We summarize (38) by saying that: *the star product of* $M(x,y)$ *and* $M(y,z)$ *is the mathematical form of the partition relations (15)-(18) of Sec. 7.3 for the medium* $X(x,z)$, *and therefore contains all the information for determining the standard reflectance and transmittance operators of the union* $X(x,y) \cup X(y,z)$ *of two contiguous media, knowing the respective operators of each component of the union.*

The Link Between $\mathcal{M}(a,x,b)$ and $\mathcal{M}(a,y,b)$

Two invariant imbedding operators for $X(a,b)$, such as $\mathcal{M}(a,x,b)$ and $\mathcal{M}(a,y,b)$, may be linked by the operator $\mathcal{M}(x,y)$ as follows. The definition of the invariant imbedding operator yields the equations:

$$\big(N_+(y),N_-(y)\big) = \big(N_+(b),N_-(a)\big)\mathcal{M}(a,y,b)$$
$$\big(N_+(x),N_-(x)\big) = \big(N_+(b),N_-(a)\big)\mathcal{M}(a,x,b) \quad .$$

Since

$$\big(N_+(y),N_-(y)\big) = \big(N_+(x),N_-(x)\big)\mathcal{M}(x,y) \quad ,$$

it follows at once from these three equations that:

$$\big(N_+(b),N_-(a)\big)\big(\mathcal{M}(a,x,b)\mathcal{M}(x,y)\big) = \big(N_+(b),N_-(a)\big)\mathcal{M}(a,y,b) \quad .$$

The incident radiance distributions being arbitrary, we have:

$$\boxed{\mathcal{M}(a,y,b) = \mathcal{M}(a,x,b)\,\mathcal{M}(x,y)} \tag{39}$$

for every x, y in (a,b). If the inverse of $\mathcal{M}(a,x,b)$ exists, we find:

$$\boxed{\mathcal{M}(x,y) = \mathcal{M}^{-1}(a,x,b)\mathcal{M}(a,y,b)} \tag{40}$$

which shows how $\mathcal{M}(x,y)$ is represented in terms of the third order invariant imbedding operators.

It is interesting to view (39) not as representing a static link between members of $\Gamma_3(a,b)$ but as depicting the transformation of the interval [a,b] into the set $\Gamma_3(a,b)$. This new view is obtained by first fixing level x, $a \le x \le b$. Then for each choice of y in the interval [a,b] equation (39) assigns to y the operator $\mathcal{M}(a,y,b)$ in $\Gamma_3(a,b)$. In this way $\mathcal{M}(x,\cdot)$ serves as a mapping or transformation from [a,b] to $\Gamma_3(a,b)$.

Building on the preceding viewpoint, equation (39) may be envisioned as stating four "principles of invariance" for the complete $\mathcal{R}$ and $\mathcal{T}$ operators. Thus, unfolding (39) component by component:

$$\mathcal{T}(b,y,a) = \mathcal{T}(b,x,a)\mathcal{M}_{++}(x,y) + \mathcal{R}(b,x,a)\mathcal{M}_{-+}(x,y)$$
$$\mathcal{R}(b,y,a) = \mathcal{T}(b,x,a)\,\mathcal{M}_{+-}(x,y) + \mathcal{R}(b,x,a)\,\mathcal{M}_{--}(x,y)$$
$$\mathcal{R}(a,y,b) = \mathcal{R}(a,x,b)\,\mathcal{M}_{++}(x,y) + \mathcal{T}(a,x,b)\,\mathcal{M}_{-+}(x,y)$$
$$\mathcal{T}(a,y,b) = \mathcal{R}(a,x,b)\,\mathcal{M}_{+-}(x,y) + \mathcal{T}(a,x,b)\,\mathcal{M}_{--}(x,y) \quad .$$

In the present point of view the $\mathcal{R}$ and $\mathcal{T}$ operators act the role the radiances did in the final statements (e.g., Ex. 3, Sec. 3.7) and the components of $\mathcal{M}(x,y)$ act like transmittance and reflectance operators: those with like signs are transmittance operators, those with unlike signs are reflectance operators. This analogy is exact in the sense that an operatorial theory for the $\mathcal{R}$ and $\mathcal{T}$ operators can be developed which is essentially parallel to the radiometric theory for $N_{\pm}(y)$. This and still other analogies (some of which are brought to light below), open up vistas in algebraic radiative transfer theory which are beyond the scope of this work but which are potential areas of basic research in the theory. See Problem X, Sec. 141, Ref. [251].

Representations of $\mathcal{M}(x,y,z)$ by Elements of $\Gamma_2(a,b)$

In view of the success in representing the basic operators $M(x,y)$ by means of the imbedding operator $\mathcal{M}(x,y)$ (See (10)-(13)) we are led to seek still further representations of interaction operators by members of the partial group $\Gamma_2(a,b)$. We shall find that the set $\Gamma_2(a,b)$ is an extremely powerful set of operators in the sense that virtually all operators in modern radiative transfer theory are representable by suitable algebraic combinations of members of $\Gamma_2(a,b)$. In the next few paragraphs we shall assemble some evidence in this direction. The formulas so gathered will be employed in Sec. 7.5 to find various differential equations governing the interaction operators, equations which should suggest novel solution procedures in radiative, neutron, and generally linear transport theory.

On the one hand the light field at level y in $X(a,b)$ is obtained from arbitrary incident light fields at levels x and z, $x \le y \le z$, by the relation:

$$\big(N_+(y),N_-(y)\big) = \big(N_+(z),N_-(x)\big)\mathcal{M}(x,y,z) \quad . \qquad (41)$$

On the other hand those on levels x and z are related by that on level y by using the following operators:

$$\big(N_+(z),0\big) = \big(N_+(y),N_-(y)\big)\mathcal{M}(y,z)C_+$$

$$\big(0,N_-(x)\big) = \big(N_+(y),N_-(y)\big)\mathcal{M}(y,x)C_- \quad .$$

Adding these equations:

$$\big(N_+(z),N_-(x)\big) = \big(N_+(y),N_-(y)\big)\big[\mathcal{M}(y,z)C_+ + \mathcal{M}(y,x)C_-\big]$$

and using (41):

$$\big(N_+(z),N_-(x)\big) = \big(N_+(z),N_-(x)\big)\mathcal{M}(x,y,z)\big[\mathcal{M}(y,z)C_+ + \mathcal{M}(y,x)C_-\big]$$

which, in view of the arbitrary nature of the incident distributions, yields the desired representation:

$$\boxed{\mathcal{M}(x,y,z) = \big[\mathcal{M}(y,z)C_+ + \mathcal{M}(y,x)C_-\big]^{-1}} \quad . \qquad (42)$$

It is interesting to speculate what would happen if we allowed the variables x,y,z in (42) to take on *any* three values in the depth interval [a,b]. The derivation of (42) by convention (but not essentially) is performed only for the depths x, y, z, in the usual order $x \leq y \leq z$ within X(a,b). But since the operators $\mathcal{M}(y,z)$ and $\mathcal{M}(y,x)$ are defined for all pairs of depths, and since the inverse of the indicated linear combination of these operators should exist just as often as those in more orthodox settings, there now is a way, as indicated by (42), of formally extending the domain of definition of the invariant imbedding operators.

A Constructive Extension of the Domain of $\mathcal{M}(x,y,z)$

The preceding observations of the potential extensibility of the domain of definition of the invariant imbedding operator $\mathcal{M}(x,y,z)$ is reinforced by recalling equation (39), in particular the interpretation of the equation as implicitly defining a mapping which, in effect assigned to each y in the interval [a,b] an operator $\mathcal{M}(a,y,b)$, as explained above. Suppose then we write, *ad hoc*:

$$"\overline{\mathcal{M}}(x,u,z)" \quad \text{for} \quad \mathcal{M}(x,y,z)\mathcal{M}(y,u) \quad . \qquad (43)$$

It follows that, as long as we have $x \leq u \leq z$, the operator $\overline{\mathcal{M}}(x,u,z)$ is, by (39), simply $\mathcal{M}(x,u,z)$. But the product of the operators in (43) is certainly compatible for any u, given each factor associated with that u. In this way, then, we can formally extend the domain of $\mathcal{M}(x,y,z)$ so that the parameters may fall outside of the subinterval [x,z] in [a,b]. Once the extension is fully and unambiguously made, the bar above "$\mathcal{M}$" in (43) may be dropped in practice.

The extension just made is a *constructive extension* of $\mathcal{M}(x,y,z)$ in the sense that, given $\mathcal{M}(x,y,z)$ and $\mathcal{M}(y,u)$ there is a definite construction procedure that may be followed in this case, a simple matrix product effecting the extension. It should be recalled, of course, that $\mathcal{M}(x,y,z)$ is in "already extended" form as it is cut directly from the more comprehensive mold of the generalized invariance imbedding relation. (See the discussion of (76) of Sec. 3.7.) Thus we may simply write:

$$"\mathcal{M}(x,y,z)" \quad \text{for} \quad \mathcal{M}(x,y;z,y) \quad , \qquad (44)$$

where x,y,z are any three levels in X(z,b), and study $\mathcal{M}(x,y,z)$, so formed, as a special instance of the generalized invariant imbedding operation. Thus (43) without the bar over $\mathcal{M}$ is in the last analysis simply a consequence of the semigroup property (84) of Sec. 3.7. Further, if one returns to the derivation of (42) or repeats its derivation, now using the definition (44) for $\mathcal{M}(x,y,z)$, the same functional relation (42) would be obtained, and the speculations on the extension of (42) to general parameters x,y,z, now have a solid affirmative basis.

Representation of $\mathcal{M}(v,z;u,y)$ by Elements of $\Gamma_2(a,b)$ and $\Gamma_3(a,b)$

We begin the derivations by representing $\mathcal{M}(v,z;u,y)$ as a product of two simpler operators by means of the semigroup relation (84) of Sec. 3.7.

$$\mathcal{M}(v,z;u,y) = \mathcal{M}(v,x;u,x)\,\mathcal{M}(x,z;x,y) \tag{45}$$

in which we have set $x = w$. With this simple identification of x and w we have managed to represent $\mathcal{M}(v,z;u,y)$ as a product of two operators of the extended type $\mathcal{M}(x,y,z)$. Thus, the first factor $\mathcal{M}(v,x;u,x)$ in (45)is simply an extended invariant imbedding operator $\mathcal{M}(v,x,u)$ as defined in (44). The other factor appears to be the inverse of such an extended operator. Indeed, using the semigroup relation (84) of Sec. 3.7 once again, it is clear that:

$$\mathcal{M}(x,z;x,y)\,\mathcal{M}(z,x;y,x) = I \quad . \tag{46}$$

Hence:

$$\begin{aligned}\mathcal{M}(x,z;x,y) &= \mathcal{M}^{-1}(z,x;y,x)\\ &= \mathcal{M}^{-1}(z,x,y) \quad .\end{aligned} \tag{47}$$

It remains only to return to (42) and make the appropriate substitution of variables to obtain the desired representation of $\mathcal{M}(v,z;u,y)$. Thus from (42):

$$\mathcal{M}(v,x;u,x) = \mathcal{M}(v,x,u) = \left[\mathcal{M}(x,u)C_+ + \mathcal{M}(x,v)C_-\right]^{-1}. \tag{48}$$

Once again from (42):

$$\mathcal{M}(z,x;y,x) = \mathcal{M}(z,x,y) = \left[\mathcal{M}(x,y)C_+ + \mathcal{M}(x,z)C_-\right]^{-1}. \tag{49}$$

In view of (47), equation (45) therefore becomes:

$$\boxed{\mathcal{M}(v,z;u,y) = \left[\mathcal{M}(x,u)C_+ + \mathcal{M}(x,v)C_-\right]^{-1}\left[\mathcal{M}(x,y)C_+ + \mathcal{M}(x,z)C_-\right]} \tag{50}$$

which is the requisite representation of an arbitrary member of $\Gamma_4(a,b)$ by members of $\Gamma_2(a,b)$, and which holds for every u,v,x,y,z in [a,b], provided, of course, that the inverse operator in (50) exists in a given setting. An alternate form of (50), using the generalized invariant imbedding operator, is:

$$\boxed{\mathcal{M}(v,z;u,y) = \mathcal{M}(v,x,u)\,\mathcal{M}^{-1}(z,x,y)} \quad . \tag{51}$$

In equations (50) and (51) the depth variable x is free to be chosen anywhere in [a,b]. Observe in (51) how the first factor, as a generalized invariant imbedding operator, maps

$(N_+(u),N_-(v))$ into $(N_+(x),N_-(x))$, and then how the inverse factor maps the latter radiance distribution into $(N_+(y),N_-(z))$. The composite mapping of these functions is precisely that performed by $\mathcal{M}(v,z;u,y)$. Thus one could almost write down (51) by sight if the various ranges and domains of the operator are kept in mind. It is of interest to compare (51) with (40) which yields a representation of $\mathcal{M}(x,y)$ in a similar vein to that of $\mathcal{M}(v,z;u,y)$ above in (51).

The representation (50) may be used to yield at once, under suitable confluence of the variables u,v,y,z, the entire family of interaction operators considered so far in this section. This is left as an exercise for the interested reader. The derivation of the following alternate representation of $\mathcal{M}(v,z;u,y)$ by extended members of $\Gamma_3(a,b)$ is also left to the reader:

$$\boxed{\mathcal{M}(v,z;u,y) = \mathcal{M}(v,y,u)C_+ + \mathcal{M}(v,z,u)C_-} \quad . \tag{52}$$

The Connection Between $\Psi(x,y)$ and $\mathcal{M}(s,y)$

The interaction operators for media with internally distributed sources of radiant flux differ fundamentally from those designed to describe radiative transfer in source-free media. The origin of this difference was pinpointed in the equation (31) of Sec. 3.9 for the operator $\Psi_{++}(s,y)$; and the subsequent discussion of this operator showed that it was discontinuous at the point (s,s) of its domain, a property not possessed by operators of the source-free kind. The operator $\Psi(s,y)$ introduced in Example 3 of Sec. 3.9 (of which $\Psi_{++}(s,s)$ is one of four components) is specifically designed to describe light fields in a media which have internally distributed sources. Since we have apparently reached in this section a culmination point in the discussion of source-free media, it would be of interest to relate the operator $\Psi(s,y)$ to the basic operator $\mathcal{M}(x,y)$ for source-free media.

We now momentarily abrogate the standing condition about source-free media X(a,b) made at the outset of this section. We postulate instead a source of flux arbitrarily distributed over level s in X(a,b), $a \le s \le b$. The source is represented as an arbitrary radiance distribution $N^o(s)$, where $N^o(s)$ is conceptually partitioned into the pair $(N_+^o(s),N_-^o(s))$ of upward (+) and downward (-) radiance distributions. Then the radiance distribution $N(y)$ $(= (N_+(y),N_-(y))$ at any level y in X(a,b) is given, according to (15) of Sec. 3.9, by:

$$\left(N_+(y),N_-(y)\right) = \left(N_+^o(s),N_-^o(s)\right)\Psi(s,y) \quad .$$

What we must do next is to use the operator $\mathcal{M}(s,y)$, which is designed for use in source-free contexts, to relate the radiance distribution at level s to that at level y. It is important, therefore, to renew acquaintance with the manner in which the source radiance function $N^o(s)$ is viewed in radiative transfer theory. A re-reading of the opening paragraph of Example 3, Sec. 3.9 will serve this purpose. We see that the source is pictured very much like a thin transparent layer of pure light sandwiched between the media $X(a,s)$ and $X(s,b)$. For true internal sources, we require $a < s < b$, and this is the condition used throughout the earlier and the present discussion. Furthermore, the presence of the source is detectable in practice by an effective discontinuity of the radiance readings of a radiance meter as the meter passes through the layer containing the sources. However, in levels y of $X(a,b)$ distinct from level s, the general properties of the light field are identical with those of any source-free medium. Therefore, to relate $N(y)$ to the light field at level $s \neq y$ we may use $\mathcal{M}(s,y)$ provided we feed into $\mathcal{M}(s,y)$ the *total* incident light field as it is measured at level s. This means that the input radiance distribution for $\mathcal{M}(s,y)$ is to be $N^o(s) + N(s)$, where $N(s)$ is the resultant light field at level s generated throughout $X(a,b)$ by the source $N^o(s)$. Therefore:

$$\big(N_+(y),N_-(y)\big) = \big(N_+^o(s) + N_+(s),N_-^o(s) + N_-(s)\big)\mathcal{M}(s,y) \quad (53)$$

and this equation holds only for $y \neq s$. By setting $y = s$ in (53) we obtain a contradiction. Herein, then, lies the salient difference between $\mathcal{M}(s,y)$ and $\Psi(s,y)$ in general media: $\mathcal{M}(s,s) = I$, but $\Psi(s,s) \neq I$; thus $\Psi(s,s)$ is that irreducible core of $\Psi(s,y)$ whose task it is to take specific cognizance of the presence of the obtrusive layer of light at level s.

With the metaphysics over, we can now proceed to the final steps that relate $\mathcal{M}(s,y)$ to $\Psi(s,y)$. Equation (53) may be written:

$$\big(N_+(y),N_-(y)\big) = \Big[\big(N_+^o(s),N_-^o(s)\big) + \big(N_+(s),N_-(s)\big)\Big]\mathcal{M}(s,y) \quad (54)$$

Setting $s = y$ in $\Psi(s,y)$:

$$\big(N_+(s),N_-(s)\big) = \big(N_+^o(s),N_-^o(s)\big)\Psi(s,s) \quad (55)$$

which, when used in (54) in conjunction with the equations (15) and (35) of Sec. 3.9, yields ($C_\pm$ for $[y-s] \gtrless 0$):

$$\big(N_+^o(s),N_-^o(s)\big)\Psi(s,y) = \big(N_+^o(s),N_-^o(s)\big)\big[C_\pm + \Psi(s,s)\big]\mathcal{M}(s,y) \quad .$$

Since $N^o(s)$ is arbitrary, we obtain the desired connection in the form:

$$\boxed{\Psi(s,y) = \big[C_\pm + \Psi(s,s)\big]\mathcal{M}(s,y)} \quad (56)$$

where we use C_+ for $s < y$ and C_- for $s > y$ in accordance with the jump property of $\Psi(s,y)$ at $s = y$. $C_\pm$ are defined in (4), (5).

A Star Product for the Operators $\mathcal{M}(x,y,z)$

We end the present section on three ascending general notes, of which the present discussion sounds the first: we wish to extend the concept of the star product of the operators $M(x,z)$, as developed in (35), to the invariant imbedding operators $\mathcal{M}(x,y,z)$. This product, which we found to be the algebraic essence of the partition relations (15)-(18) of Sec. 7.3, serves to show how to combine the interaction properties of two contiguous media $X(a,b)$ and $X(b,c)$ to find the corresponding interaction properties of their union $X(a,c)$. We now attempt to do the same for the complete reflectance and transmittance operators of any two adjacent media.

Figure 7.5 depicts the present setting. We imagine a plane-parallel optical medium $X(a,b)$ to be the union of two arbitrarily overlapping sub-media: $X(a,z)$ and $X(x,b)$. Let y be any level in $X(a,b)$ such that $x \le y \le z$. The problem before us is: to represent $\mathcal{R}(a,y,b)$, $\mathcal{T}(a,y,b)$, and $\mathcal{M}(a,y,b)$ in terms of a suitable algebraic combination of the complete $\mathcal{R}$ and $\mathcal{T}$ operators associated with $X(a,z)$ and $X(x,b)$. The present problem is geometrically slightly more general than its counterpart posed in Sec. 7.3 for the R and T operators in the sense that we require not contiguity of $X(a,z)$ and $X(x,b)$ (so that necessarily $x = z$), but merely intersection of the media (so that $x \le z$).

The incident radiance distributions $N_-(a)$ and $N_+(b)$ on $X(a,b)$ generate a light field at general levels x,y,z in $X(a,b)$ which may be computed several ways depending on which medium one envisions the levels to be in, i.e., as light fields in $X(a,b)$, or in $X(a,z)$ or in $X(x,b)$. Thus $N_\pm(y)$, as

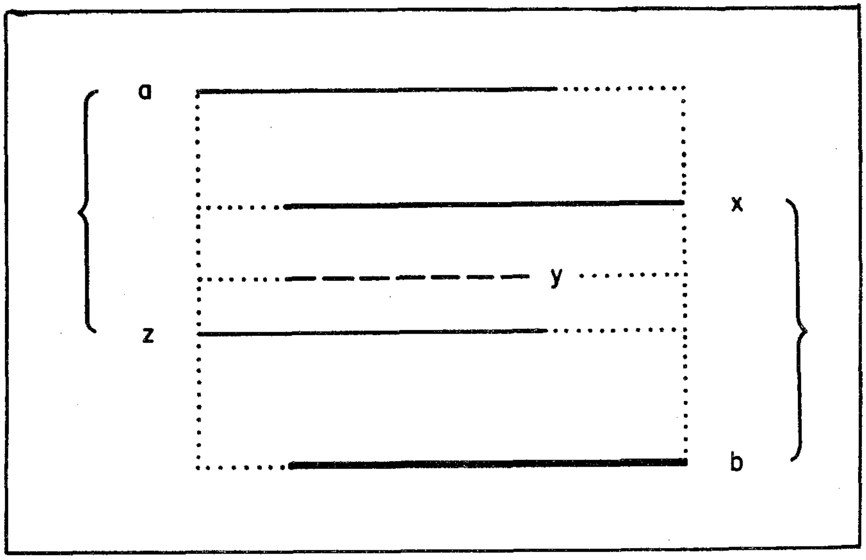


FIG. 7.5 The setting for the star product in $\Gamma_3(a,b)$.

radiance distributions in X(a,b) are given by:

$$N_+(y) = N_-(a)\mathcal{R}(a,y,b) + N_+(b)\mathcal{T}(b,y,a) \quad (57)$$

$$N_-(y) = N_-(a)\mathcal{T}(a,y,b) + N_+(b)\mathcal{R}(b,y,a) \quad (58)$$

which follows at once from the invariant imbedding relation for X(a,b). On the other hand, the distribution $N_+(y)$ considered as being in X(a,z), is given by:

$$N_+(y) = N_-(a)\mathcal{R}(a,y,z) + N_+(z)\mathcal{T}(z,y,a) \quad (59)$$

and $N_-(y)$, considered as being in X(x,b) is given by:

$$N_-(y) = N_-(x)\mathcal{T}(x,y,b) + N_+(b)\mathcal{R}(b,y,x) \quad (60)$$

which are the results of applying the invariant imbedding operators of X(a,z) and X(x,b), respectively. Now the distributions $N_+(z)$ and $N_-(x)$ appearing on (59) and (60) can be found by solving the system:

$$N_+(z) = N_-(x)\mathcal{R}(x,z,b) + N_+(b)\mathcal{T}(b,z,x) \quad (61)$$

$$N_-(x) = N_-(a)\mathcal{T}(a,x,z) + N_+(z)\mathcal{R}(z,x,a) \quad (62)$$

which is derived similarly to (59), (60) by considering level x as occurring in X(a,z) and level z as occurring in X(x,b). The solutions are:

$$N_+(z) =$$

$$= \left[N_-(a)\mathcal{T}(a,x,z)\mathcal{R}(x,z,b) + N_+(b)\mathcal{T}(b,z,x)\right]\left[I - \mathcal{R}(z,x,a)\mathcal{R}(x,z,b)\right]^{-1} \quad (63)$$

$$N_-(x) =$$

$$= \left[N_-(a)\mathcal{T}(a,x,z) + N_+(b)\mathcal{T}(b,z,x)\mathcal{R}(z,x,a)\right]\left[I - \mathcal{R}(x,z,b)\mathcal{R}(z,x,a)\right]^{-1} \quad (64)$$

These equations should be compared with (9), (10), (25) and (26) of Sec. 3.7 and (49), (50) of Sec. 3.9 for structural similarities.

Next consider the two alternative ways of describing $N_+(y)$ in (57) and (59). If these two expressions for $N_+(y)$ are equated and if $N_+(z)$ as given in (63) is used, then since $N_+(b)$ and $N_-(a)$ are arbitrary, we derive the following two operator equations as a result:

$$\mathcal{R}(a,y,b) =$$

$$= \mathcal{R}(a,y,z) + \mathcal{T}(a,x,z)\mathcal{R}(x,z,b)\left[I - \mathcal{R}(z,x,a)\mathcal{R}(x,z,b)\right]^{-1}\mathcal{T}(z,y,a) \quad (65)$$

$$\mathcal{T}(b,y,a) = \mathcal{T}(b,z,x)\left[I - \mathcal{R}(z,x,a)\mathcal{R}(x,z,b)\right]^{-1}\mathcal{T}(z,y,a) \quad (66)$$

These two equations constitute a rather interesting generalization of (15) and (18) of Sec. 7.3. For by letting y = z in (65) and (66) we have:

$$\mathcal{R}(a,z,b) = \mathcal{T}(a,x,z)\mathcal{R}(x,z,b)[I-\mathcal{R}(z,x,a)\mathcal{R}(x,z,b)]^{-1} \tag{67}$$

and

$$\mathcal{T}(b,z,a) = \mathcal{T}(b,z,x)[I-\mathcal{R}(z,x,a)\mathcal{R}(x,z,b)]^{-1} \tag{68}$$

and these representations may be plowed back into (65) and (66) to yield the following compact forms of (65),(66):

$$\mathcal{R}(a,y,b) = \mathcal{R}(a,y,z) + \mathcal{R}(a,z,b)\mathcal{T}(z,y,a) \tag{69}$$

$$\mathcal{T}(b,y,a) = \mathcal{T}(b,z,a)\mathcal{T}(z,y,a) \quad . \tag{70}$$

The latter equation is simply the semigroup property (52) of Sec. 3.7 for the $\mathcal{T}$ operator. However, (69) is a relatively novel equation, much in the way (15) of Sec. 7.3 was a newcomer to the semigroup scene in that setting. Equation (69) will be used at crucial points of the investigation in 7.13. The analogy between the present derivation and those leading to (15) and (16) of Sec. 7.3 appears to be a throughgoing one, on the strength of which we can write down the remaining two correspondents of (16) and (17) of Sec. 7.3:

$$\mathcal{T}(a,y,b) = \mathcal{T}(a,x,b)\mathcal{T}(x,y,b) \tag{71}$$

$$\mathcal{R}(b,y,a) = \mathcal{R}(b,y,x) + \mathcal{R}(b,x,a)\mathcal{T}(x,y,b) \tag{72}$$

where

$$\mathcal{T}(a,x,b) = \mathcal{T}(a,x,z)[I-\mathcal{R}(x,z,b)\mathcal{R}(z,x,a)]^{-1} \tag{73}$$

$$\mathcal{R}(b,x,a) = \mathcal{T}(b,z,x)\mathcal{R}(z,x,a)[I-\mathcal{R}(x,z,b)\mathcal{R}(z,x,a)]^{-1} \tag{74}$$

The requisite star product for the invariant imbedding operator $\mathcal{M}(a,y,z)$ and $\mathcal{M}(x,y,b)$ associated with the submedia X(a,z) and X(x,b) may then be defined as follows. We write:

$$\text{"}\mathcal{M}(a,y,z) * \mathcal{M}(x,y,b)\text{"} \quad \text{for} \quad \mathcal{M}(a,y,b) \tag{75}$$

where $\mathcal{M}(a,y,b)$ in (75) is constructed from the operators of $\mathcal{M}(a,y,z)$ and $\mathcal{M}(x,y,b)$ using (69)-(72) in which $\mathcal{R}(a,z,b)$, $\mathcal{T}(b,z,a)$, $\mathcal{T}(a,x,b)$, and $\mathcal{R}(b,x,a)$ are as given in (67),(68), (73) and (74), respectively. Thus:

$$\mathcal{M}(a,y,b) = \mathcal{M}(a,y,z) * \mathcal{M}(x,y,b) =$$

$$= \begin{bmatrix} \mathcal{T}(b,z,a)\mathcal{T}(z,y,a) & \mathcal{R}(b,y,x)+\mathcal{R}(b,x,a)\mathcal{T}(x,y,b) \\ \mathcal{R}(a,y,z)+\mathcal{R}(a,z,b)\mathcal{T}(z,y,a) & \mathcal{T}(a,x,b)\mathcal{T}(x,y,b) \end{bmatrix} \tag{76}$$

This star product will be used subsequently in the study of irradiance fields in interacting media (cf. (91) of Sec. 8.7).

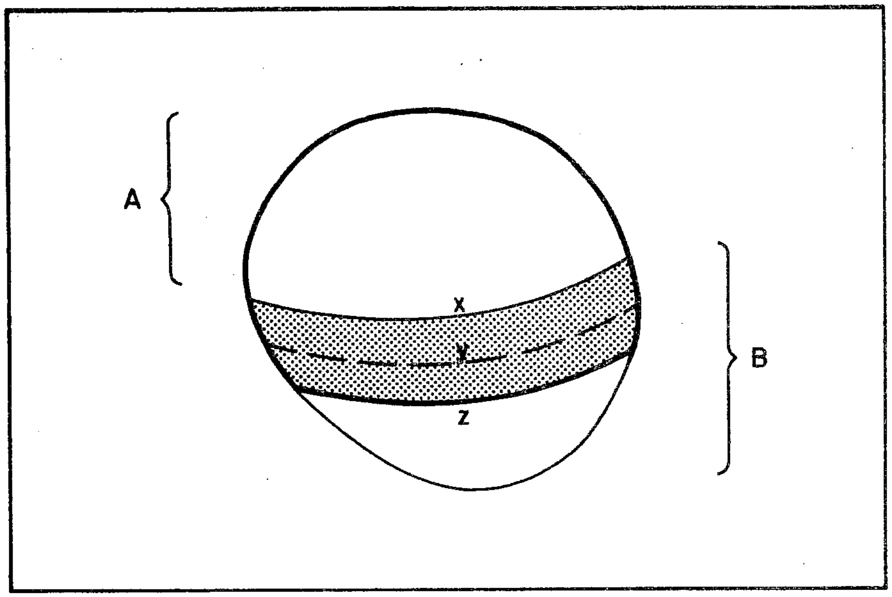


FIG. 7.6 The star product of invariant imbedding operators can be defined for arbitrary media.

Recall that the depth variables x and z in (75) are arbitrary, subject only to the condition $x \le y \le z$, i.e., that the media $X(a,z)$ and $X(z,b)$ overlap, and that y be chosen in the intersection of these media. Equation (76) is to be compared with (39).

The power of the present algebraic approach to radiative transfer theory can be appreciated in some detail if we now turn to the general invariant imbedding relation (51) of Sec. 3.9 and observe that all activity we have gone through to reach (76) can be repeated for the general medium X of examples 4 and 5 of Sec. 3.9. Thus if we have a medium X with two overlapping submedia A and B of a one-parameter medium as in Fig. 7.6, and more generally, if we have two media X and Y which intersect in a region z as in Fig. 7.7, then we can form a star product of the invariant imbedding operators of X and Y to obtain the invariant imbedding operator of their union $X \cup Y$, in exact analogy to (76).

In view of these observations, the possibilities for further exploration of the algebraic theory of radiative transfer are clearly mounting in number and in depth. The possibilities branch off into topological and algebraic directions which, if kept bound together by suitably defined concepts, will raise the theory of radiative transfer the remaining distance to its logical haven: a possible general theory of linear transport processes. Such a pursuit is unfortunately beyond the scope of the present work, and we rest the matter here.

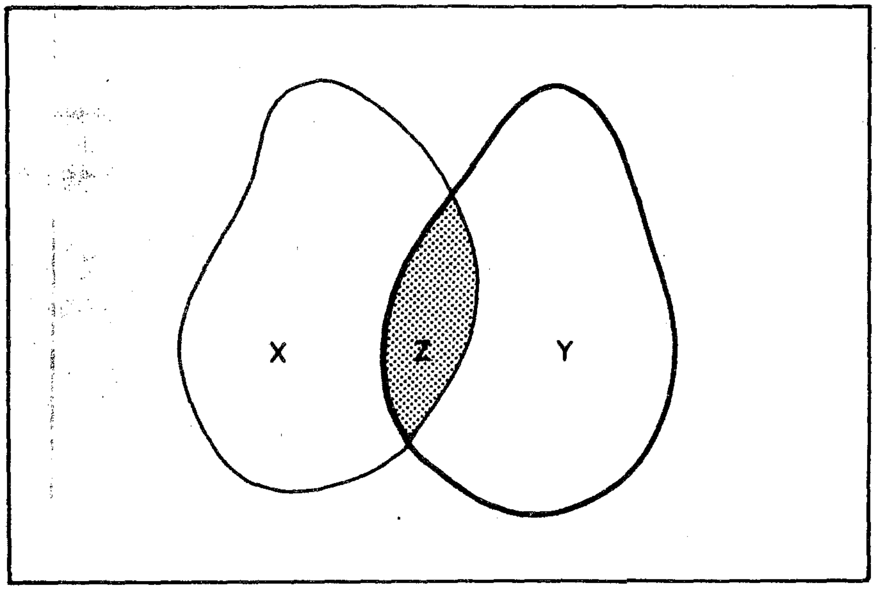


FIG. 7.7 A general setting for the star product of interaction operators in general optical media.

Possibilities Beyond $\mathcal{M}(v,x;u,w)$

In this the penultimate note of the present section, the possibility of operators more comprehensive than those in $\Gamma_*(a,b)$ will be considered. We shall show that such possibilities of arbitrarily great comprehensiveness are easily constructed. However, in a sense, such generality is no longer needed now that operators like $\mathcal{M}(x,y)$ harnessed in parallel have been shown to have sufficient computational power (cf. (50), (51) and (52)) to do everything $\mathcal{M}(v,x;u,w)$ can do. For simplicity, we shall remain in the setting of one-parameter media during the present discussion.

To see what direction we may take in generalizing $\mathcal{M}(v,x;u,w)$, let us return to its definition in (56) of Sec. 3.7. Recall that the primary motivation for $\mathcal{M}(v,x;u,w)$ was the need for an operator which would take as input the pair $(N_+(u),N_-(v))$ of radiance distributions on arbitrary levels of u and v in X(a,b) and yield as output the pair $(N_+(w),N_-(x))$ on still two more arbitrary levels w, x in X(a,b). In this way we achieved a comprehensive, symmetric setting for all classical operators. In particular, these choices of input and response distributions constitute the natural generalization of the classical type of inputs and responses of M(x,y) (cf. (2) above) and the general invariant imbedding operator

$\mathcal{M}(x,y,z)$. Having thus extended the input and output types to a reasonably general kind (there is still room beyond here too--consider, e.g., partitioning Ξ_+ and Ξ_- into many and sundry pieces) we turn to consider the effect of an increase in the number of levels. Thus, suppose we ask for an operator which takes as input the 2m component radiance vector:

$$\left(N_+(u_1),N_-(v_1);N_+(u_2),N_-(v_2);\ \ldots\ ;\ N_+(u_m),N_-(v_m)\right) \qquad (77)$$

and yields as output the 2n component radiance vector:

$$\left(N_+(w_1),N_-(x_1);N_+(w_2),N_-(x_2);\ \ldots\ ;\ N_+(w_n),N_-(x_n)\right) \qquad (78)$$

where $u_1,\ldots,u_m$ and $v_1,\ldots,v_m$ are 2m arbitrary levels in $X(a,b)$ and $w_1,\ldots,w_n$ and $x_1,\ldots,x_n$ are 2n arbitrary levels in $X(a,b)$. $N_+(u_i)$ is as usual the upward radiance distribution over level u_i. Similarly with the other radiances. Then the interaction principle supplies an operator

$$\mathcal{M}(v_m,x_m;\ \ldots\ ;\ v_1,x_1;u_n,w_n;\ \ldots\ ;\ u_1w_1) \qquad (79)$$

which is a 2m × 2n matrix of operators of which 2mn are $\mathcal{T}$-like and 2mn are $\mathcal{R}$-like (which we need not display here) and which clearly reduces to $\mathcal{M}(v,x;u,w)$ by setting m = n = 1. We shall now show that the operator (79) can be represented as a linear combination of generalized invariant imbedding operators of the form $\mathcal{M}(v_i,x_i;u_j,w_j)$. Then, in view of (50), the algebraic representation of (79) in terms of members of the partial group $\Gamma_2(a,b)$ will stand established.

The key to the desired representation of (79) rests in the following two partitions of identity operators:

$$I_m = \sum_{i=1}^{m} C_i[\mathrm{trn}\ C_i] \qquad (80)$$

$$I_n = \sum_{i=1}^{n} D_j[\mathrm{trn}\ D_j] \qquad (81)$$

where C_i is a 2m × 2 matrix and D_j is a 2n × 2 matrix of the form:

$$\begin{array}{c} \\ i \\ 2m \\ \\ \end{array}\begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ I \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} = C_i \qquad\qquad \begin{array}{c} \\ j \\ 2n \\ \\ \end{array}\begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ I \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} = D_j \qquad (82)$$

and where "0" denotes the 2 × 2 zero matrix and "I" the 2 × 2 identity matrix considered earlier (e.g., in (3)). The notation "trn C_i" denotes the transpose of C_i, i.e., $2 \times 2m$ matrix obtained by turning C_i on its side so that the identity operator I_m in C_i is the ith matrix counting from the left as usual. That (80) and (81) represent, respectively, the $2m \times 2m$ and the $2n \times 2n$ identity matrices is readily established, and is left as an exercise to the reader. Observe also that $[\text{trn } C_i]C_i$ is the 2×2 identity matrix I, for every i.

The operators C_i have the useful properties that:

$$\left.\begin{aligned} \left(N_+(u_i), N_-(v_i)\right) &= a\ C_i \\ \left(N_+(w_j), N_-(x_j)\right) &= b\ D_j \end{aligned}\right\} \tag{83}$$

for $1 \leq i \leq m$, $1 \leq j \leq n$, and where "a" and "b" denote (77) and (78), respectively. We shall continue to use these abbreviations "a" and "b" in the remainder of this discussion.

Now, we know how to relate $(N_+(u_i), N_-(v_i))$ and $(N_+(w_j), N_-(x_j))$. Such relating is the specific task of $\mathcal{M}(v_i,x_j;u_i,w_j)$. Thus:

$$\left(N_+(w_j),\ N_-(x_j)\right) = \left(N_+(u_i), N_-(v_i)\right)\mathcal{M}(v_i,x_j;u_i,w_j)\ . \tag{84}$$

In other words (84) states that:

$$b\ D_j = a\ C_i \mathcal{M}_{ij} \tag{85}$$

where we have written, *ad hoc*:

$$"\mathcal{M}_{ij}" \quad \text{for} \quad \mathcal{M}(v_i,x_j;u_i,w_j) \quad . \tag{86}$$

Equation (85) therefore suggests that we start with:

$$b = a\mathcal{M} \quad , \tag{87}$$

where "$\mathcal{M}$" at present denotes (79), and insert the $2m \times 2m$ identity operator I_m, in the form (80), between a and $\mathcal{M}$ in (87) to obtain:

$$b = a \sum_{i=1}^{m} C_i[\text{trn } C_i]\ \mathcal{M} \quad . \tag{88}$$

Once this is done we operate on each side of (88) with D_j to obtain:

$$b\ D_j = a \sum_{i=1} C_i\{[\text{trn } C_i]\ \mathcal{M}\ D_j\} \quad . \tag{89}$$

It is clear from (89) and (85) and the fact that these equations hold for every incident radiance vector a, that:

$$\boxed{\mathcal{M}_{ij} = [\text{trn } C_i]\ \mathcal{M}\ D_j} \quad . \tag{90}$$

In this way we see the 2×2 operator matrix $\mathcal{M}_{ij}$ is a special case of $\mathcal{M}$. Going on with the present analysis of (87), we operate on each side of (89) with trn D_j and sum over all j. Thus:

$$b = b\sum_{j=1}^{n} D_j[\text{trn } D_j] = a \sum_{j=1}^{n} \sum_{i=1}^{m} C_i \mathcal{M}_{ij}[\text{trn } D_j] \quad . \tag{91}$$

Again, since a is arbitrary, we have, from (87) and (91):

$$\boxed{\mathcal{M} = \sum_{j=1}^{n} \sum_{i=1}^{m} C_i \mathcal{M}_{ij}[\text{trn } D_j]} \tag{92}$$

By means of this equation, we see that $\mathcal{M}$ is representable as an m×n block matrix with $\mathcal{M}_{ij}$ as the element in the ith row and jth column.

Equation (92) is the desired representation of $\mathcal{M}$ in terms of $\mathcal{M}_{ij}$, i.e., in terms of the members of $\Gamma_4(a,b)$. Thus we see that $\mathcal{M}$ may be represented by a suitable algebraic combination of elements of $\Gamma_2(a,b)$, using (50).

Possibilities Beyond $\Gamma_2(a,b)$

We conclude this section with some observations on the possible direction in which the notion of the partial group $\Gamma_2(a,b)$ for a plane-parallel medium X(a,b) may be extended.

An immediate extension of $\Gamma_2(a,b)$ may be made to a one-parameter three-dimensional optical medium in which "a" and "b" are indices of the two-parameter surfaces bounding the general curvilinear medium X(a,b). The resultant algebraic structures are isomorphic, (i.e., algebraically identical) to that of the plane-parallel case and so will not be explicitly considered.

An extension of $\Gamma_2(a,b)$ beyond one-parameter media would be to an arbitrary connected medium X in which "x" and "y" in $\mathcal{M}(x,y)$ now denote two arbitrary points of X or possibly small subsets of X. We shall call x a *point* in either case in what follows. This extension is of great physical interest and we pause to examine it using formal operations in just enough detail to see how the generalization may go.

Let X be an optical medium in three-dimensional Euclidean space, i.e., the space which represents an ordinary everyday world. Within X we can simulate portions of the earth's atmosphere, or its seas and lakes. Let N^o be the incident radiance function on X and N the associated response radiance function on X. Then the interaction principle supplies an interaction operator $\mathcal{M}(x)$ which maps N^o into N:

$$N = N^o \mathcal{M}(X) \quad . \tag{93}$$

The reader will recall that N is a function which assigns to each x in X and ξ in Ξ the radiance $N(x,\xi)$ at x in the direction ξ. Thus, from N we can obtain the radiance distribution $N(x)$ at point x. Let $E(x)$ be the operator which assigns to point x in X the radiance distribution $N(x)$ at point x as induced by the radiance function N. Thus:

$$N(x) = NE(x) \quad . \tag{94}$$

In other words, $E(x)$ is a continuous (or generalized) version of C_i or D_i introduced in (82). Conversely, from knowledge of $N(x)$ at each point x of X, we can reconstruct N. Let "trn $E(x)$" be the operator such that:

$$I = \int_X [\ ]E(x)\ [\text{trn } E(x)]\ dV(x) \tag{95}$$

where I is the identity operator (transformation) on the vector space $V(X)$ of all radiance functions defined on X. (The use of vector space concepts was introduced in an earlier discussion; see Example 15 of Sec. 2.11.) We shall not go into the details of construction of the operator trn $E(x)$. It will suffice to note that it is intended to be analogous to the transpose operators discussed in (80) and (81) and may be constructed using theorems A,B,C of the interaction method in Sec. 3.16. Using this partition of I in (93) we have:

$$N = N^o I\, \mathcal{M}(X) = \int_X N^o E(x)\ [\text{trn } E(x)]\ \mathcal{M}(x)\ dV(x) \ . \tag{96}$$

Applying the operator $E(y)$ to each side of (96) we have:

$$N(y) = NE(y) = \int_X N^o E(x)\ [\text{trn } E(x)]\, \mathcal{M}(x)\ E(y)\ dV(x) \, . \tag{97}$$

Let us write:

$$\text{"}\mathcal{M}^o(X;x,y)\text{"} \quad \text{or} \quad \text{"}\mathcal{M}^o(x,y)\text{"} \quad \text{for}$$

$$\int_X [\ ]\ [\text{trn } E(x)]\, \mathcal{M}(x)\ E(y)\ dV(x) \quad . \tag{98}$$

Then (97) can be written as:

$$\boxed{N(y) = N^o(x)\, \mathcal{M}^o(x,y)} \tag{99}$$

where:

$$N^o(x) = N^o E(x) \quad . \tag{100}$$

We shall now assume that the operator $\mathcal{M}^o(x,y)$ is *one to one* for every pair (x,y) of points in X, in the sense that two distinct incident radiance distributions $N_1{}^o(x)$, $N_2{}^o(x)$ always are mapped into distinct corresponding response radiance distribution functions $N_1(y)$, $N_2(y)$ using (93). By *distinct* radiance distribution N_1, N_2 it shall be understood that for some set Ξ_0 of directions in Ξ, $\int_{\Xi_0} [N_1(x,\xi)-N_2(x,\xi)]\, d\Omega(\xi)>0$. Matters can usually be arranged so that an optical medium X can be partitioned into pieces X_i over each of which the operator $\mathcal{M}^o(X_i;x,y)$ is one to one. Hence no essential loss in generality will be engendered in what follows if we assume $\mathcal{M}^o(X;x,y)$ is one to one over an arbitrary optical medium X. The one to one property of $\mathcal{M}^o(x,y)$ is used to insure that the inverse $(\mathcal{M}^o(x,y))^{-1}$ of $\mathcal{M}^o(x,y)$ exists. For, once this inverse is available, we can directly relate any two radiance distributions in X. Thus, from (99) used twice:

$$N(y) = N^o(x)\,\mathcal{M}^o(x,y)$$

$$N(z) = N^o(x)\,\mathcal{M}^o(x,z)$$

whence:

$$N(y)\,(\mathcal{M}^o(x,y))^{-1} = N(z)\,(\mathcal{M}^o(x,z))^{-1}$$

whence again:

$$N(z) = N(y)\,[\mathcal{M}^o(x,y)]^{-1}\,\mathcal{M}^o(x,z)$$

which holds for every x in X, so that if we write:

$$\text{"}\mathcal{M}(y,z)\text{"} \quad \text{for} \quad [\mathcal{M}^o(x,y)]^{-1}\mathcal{M}^o(x,z) \tag{101}$$

we have:

$$\boxed{N(z) = N(y)\,\mathcal{M}(y,z)} \tag{102}$$

for every pair y,z of points in X. In this way we generalize the invariant imbedding operator $\mathcal{M}(u,x)$ of Sec. 3.7 to a wider geometric setting, i.e., to one in which x and y are not surfaces, but possibly points or subsets of X. We retain the notation "$\mathcal{M}(x,y)$" without fear of confusion with the simpler concept in the present discussion. Recall that x and y are now *points* or subsets of X rather than depth parameters for surfaces. We shall denote the set of all $\mathcal{M}(x,y)$ with x and y in X, by "$\Gamma_2(X)$."

It follows at once from (101) that the operators $\mathcal{M}(x,y)$ in $\Gamma_2(X)$ form a partial group in the sense explained in the discussion around equation (79) in Sec. 3.7. Hence $\Gamma_2(X)$ is a proper generalization of $\Gamma_2(a,b)$.

Several directions of further development of (102) are possible at this exploratory stage of the analysis. For example, using Stage II of the interaction method we can represent $\mathcal{M}(y,z)$ as an integral operator over Ξ. Alternatively, we could partition $\mathcal{M}(y,z)$ into a 2×2 matrix analogously to the partition in (18) for the plane-parallel case, and develop a theory for $\mathcal{M}(y,z)$ analogous in every detail to that between (18) and (92) above, but now for the general medium X. Since this is representative of a nontrivial extension of the invariant imbedding group $\Gamma_2(a,b)$ to more general settings, we shall now explore the initial details of such an extension.

In the plane-parallel case we had the terrestrially-based coordinate system as a frame of directional reference for the partition of $\mathcal{M}(x,z)$ as shown in (18). In the present case there is no preferred or pre-existing coordinate frame from which to launch the construction of the present counterparts to $\mathcal{M}_{++}(x,z)$, $\mathcal{M}_{+-}(x,z)$, $\mathcal{M}_{-+}(x,z)$, and $\mathcal{M}_{--}(x,z)$. Therefore for the first stage of the present extension we simply assign to each point x in X a partition $\Xi_1(x)$, $\Xi_2(x)$ of Ξ into two parts. This partition can follow any rule, so that $\Xi_i(x)$ need not be a hemisphere. Once the partition is specified at each x in X, the radiance distribution N(x) is restricted to $\Xi_1(x)$ and $\Xi_2(x)$ resulting in $N_1(x)$ and $N_2(x)$, respectively--in complete analogy to the $N_+(y)$ and $N_-(y)$ of the plane-parallel case. This partitioning of N(x) at each x in X into the pair $(N_1(x),N_2(x))$ in turn induces a cleavage of $\mathcal{M}(y,z)$ into a 2×2 operator matrix such that:

$$\mathcal{M}(y,z) = \begin{bmatrix} \mathcal{M}_{11}(y,z) & \mathcal{M}_{12}(y,z) \\ \mathcal{M}_{21}(y,z) & \mathcal{M}_{22}(y,z) \end{bmatrix} . \qquad (103)$$

The details of this partitioning of $\mathcal{M}(y,z)$ are very much like those used to establish $\mathcal{M}^o(y,z)$ from $\mathcal{M}(x)$ above or $\mathcal{M}_{ij}$ from $\mathcal{M}$ in (90), except now (95) is replaced by a formula like (80):

$$I = \sum_{i=1}^{2} C_i[\text{trn } C_i] \qquad (104)$$

where C_i, i = 1,2, is the operator which assigns $N_i(y)$ to N(y), so that the C_i are like C_+ and C_- in (4), (5). Indeed, the partition (103) is obtained in precise analogy to (92) for the case m = n = 2. Hence we may refer the reader to equations (80)-(92) for the general outline of the details.

With this decomposition (103) of $\mathcal{M}(y,z)$, equation (102) may be written:

$$\big(N_1(z),N_2(z)\big) = \big(N_1(y),N_2(y)\big) \begin{bmatrix} \mathcal{M}_{11}(y,z) & \mathcal{M}_{12}(y,z) \\ \mathcal{M}_{21}(y,z) & \mathcal{M}_{22}(y,z) \end{bmatrix} . \qquad (105)$$

As a specific instance of (105), let $\mathcal{P}(y,z)$ be a smooth directed path in X connecting point y to point z (in that order). Once $\mathcal{P}(y,z)$ is specified then any point x along it is located by a single parameter--the distance of x from y along the curve, and the tangent to the curve is given the usual sense at X. See Fig. 7.8. At each point x of $\mathcal{P}(y,z)$, let $\xi(x)$ be the tangent to the curve. Then let $\Xi_1(x)$ and $\Xi_2(x)$ of the general discussion above be $\Xi_+(\xi(x))$, $\Xi_-(\xi(x))$, respectively, where $\Xi_+(\xi(x))$, it will be recalled (Sec. 2.4), is the hemisphere of Ξ consisting of all directions ξ' such that $\xi' \cdot \xi(x) \geq 0$, and $\Xi_-(\xi(x))$ consists of all ξ' such that $\xi' \cdot \xi(x) \leq 0$. With these assignations of $\Xi_1(x)$, $\Xi_2(x)$, the formula (105) takes the form:

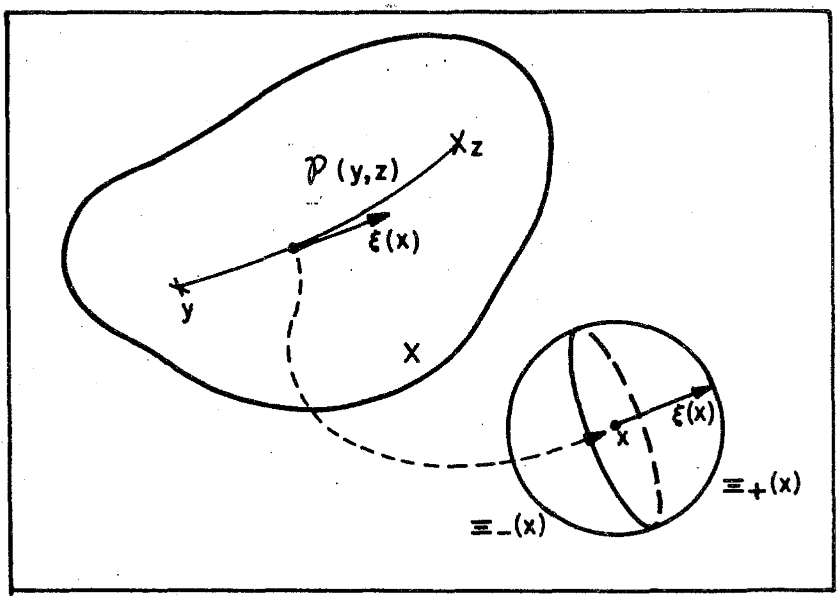


FIG. 7.8 Extending the $\mathcal{M}(x,y)$ operators to general geometries.

$$\left(N_+(z),N_-(z)\right) = \left(N_+(y),N_-(y)\right)\begin{bmatrix}\mathcal{M}_{++}(y,z) & \mathcal{M}_{+-}(y,z)\\ \mathcal{M}_{-+}(y,z) & \mathcal{M}_{--}(y,z)\end{bmatrix}, \quad (106)$$

where $N_+(x)$ and $N_-(x)$ are now the restrictions of $N(x)$ to $\Xi_+(\xi(x))$ and $\Xi_-(\xi(x))$, respectively. Thus (106) is formally indistinguishable from its plane-parallel counterpart; furthermore the algebraic properties of $\mathcal{M}(y,z)$ in (106) are identical to those of its algebraic counterpart and (106) reduces to the stratified plane-parallel case when $\mathcal{P}(y,z)$ is the straight path from level y to level z and such that $\mathcal{P}(y,z)$ is perpendicular to the parallel planes of $X(y,z)$ in $X(a,b)$. As we shall see in Sec. 7.11, (106) reduces the type of solution procedures used for light fields in a general medium X to those used in plane-parallel media, with arbitrary lighting and optical conditions.

In the preceding explorations of the possibilities beyond $\Gamma_2(a,b)$ there is a general pattern forming for one such family of extensions. We conclude these explorations with a summary and review of the incipient pattern for the case of an arbitrary subset S of a medium X. The formation of the extensions begins with an invocation of the First Stage of the interaction method. This yields the generic equation:

$$N = N^{\circ}\mathcal{M}(S)$$

where S may be all of X or a proper part of X. Furthermore N may now be radiance functions for polarized light, and may depend explicitly on scattering with change in wavelength, etc. Hence X may be more than three-dimensional. Let us assume X is n-dimensional. (See opening remarks, Sec. 99 of Ref. [251].) Using the technique of decomposing the identity operator, as in (80), (81), (95), or (104), the basic equation (107) can be systematically taken apart leaving an operator which forms a member of a new partial semigroup $\Gamma_2(S)$. The ways in which (107) may be so analyzed are manifold. The examples cited above show that the partition of the identity operator may be over spatial variables (as in the case of (80), (81), and (95)) or over directional variables (as in the case of (104)). The work of Ref. [251] shows how the partition of the identity operator may in other contexts be over the location space of a *discrete* optical medium (Sec. 90, Ref. [251]) with the resultant generation of the local operator ψ^o analogous to $\mathscr{M}^o$ in (98). In addition, the technique of partitioning the identity operator is applicable to the polarized radiance context (Sec. 114, Ref. [251]) and also the heterochromatic radiative transfer and even the general Markov-process context of general radiative transfer of equation VII, (Sec. 119 of Ref.[251]). With these examples in mind let us assume a quite general partitioning of the identity operator I on the vector space of radiance functions on S, thus:

$$I = \int_S [\ ]C(x)\ [\mathrm{trn}\ C(x)]\ dV(x) \quad , \tag{108}$$

where now x is a point of the subset S of the n-dimensional space X, and V is the volume measure on X. (The various dimensions of X may arise from the various parameters needed to describe N--location variables, direction variables, polarization parameters, wavelength parameters, etc.). The operators C(x) are analogous to E(x) in (95). Therefore we write:

$$\text{"}N(x)\text{"} \quad \text{for} \quad NC(x) \tag{109}$$

in complete analogy to the earlier special cases of N(x).

Next we insert I, in the form given by (108), between N^o and $\mathscr{M}(S)$ in (107) to obtain:

$$N = N^o I\, \mathscr{M}(X) = \int_S N^o C(x)\ [\mathrm{trn}\ C(x)]\, \mathscr{M}(S)\ dV(x)\ . \tag{110}$$

By (109) we have:

$$N^o(x) = N^o C(x) \tag{111}$$

and in analogy to (98) we write:

$$\text{"}\mathscr{M}^o(S;x,y)\text{"} \quad \text{or} \quad \text{"}\mathscr{M}^o(x,y)\text{"} \quad \text{for}$$

$$\int_X [\ ]\ [\mathrm{trn}\ C(x)]\, \mathscr{M}(S) C(y)\ dV(x)\quad . \tag{112}$$

Therefore, upon operating on each side of (110) with C(y) we have:

$$\boxed{N(y) = N^{o}(x)\mathcal{M}^{o}(x,y)} \quad . \tag{113}$$

Assuming the integral operators $\mathcal{M}^{o}(x,y)$ to be one to one for every x and y in S we define:

$$"\mathcal{M}(y,z)" \quad \text{for} \quad [\mathcal{M}^{o}(x,y)]^{-1}\mathcal{M}^{o}(x,z) \tag{114}$$

in analogy to (101). The collection $\Gamma_2(S)$ of all operators $\mathcal{M}(y,z)$, with y,z in S is seen to be a partial group as in the earlier instances; so that for every y,z in S:

$$\boxed{N(z) = N(y)\mathcal{M}(y,z)} \tag{115}$$

which holds for an arbitrary radiance field N in S.

Further partitions of the identity can now be made on the vector spaces of radiance distributions with elements N(x), x fixed in S. For example, if x is simply the spatial variable then further partitioning of the direction space or wavelength space can be made if desired. Thus in general, let $D_\alpha(x)$ and $D_\beta(x)$ be operators such that:

$$I(x) = \int_A [\ ]\ D_\alpha(x)\ [\mathrm{trn}\ D_\alpha(x)]\ d\mu(\alpha) \tag{116}$$

is the partition of the identity operator on the vector space of functions N(x) at x in X. The space A is the space (either discrete or continuous) which is being partitioned and μ is the measure, and could be direction space or wavelength space, etc. Let us write:

$$"N_\alpha(x)" \quad \text{for} \quad N(x)D_\alpha(x) \tag{117}$$

and

$$"\mathcal{M}_{\alpha\beta}(S;s,y)" \quad \text{or} \quad "\mathcal{M}_{\alpha\beta}(x,y)" \quad \text{for} \quad [\mathrm{trn}\ D_\alpha(x)]\mathcal{M}(x,y)D_\beta(y) \ . \tag{118}$$

The functions $N_\alpha(x)$ with α in A; and $\mathcal{M}_{\alpha\beta}(x,y)$ are generalizations of $N_i(x)$, and $\mathcal{M}_{ij}(x,y)$ in (105), where now the space A is quite arbitrary. See also the discrete example (85) of [118]. Then to see how far these generalizations can go, we return to (115) and observe that we may write:

$$\begin{aligned} N(z) &= N(y)I(y)\mathcal{M}(y,z) \\ &= \int_A N(y)D_\alpha(y)\ [\mathrm{trn}\ D_\alpha(y)]\,\mathcal{M}(y,z)\ d\mu(\alpha) \quad . \end{aligned}$$

Operating on each side with $D_\beta(y)$:

$$N_\beta(z) = \int_A N(y)D_\alpha(y)\ [\mathrm{trn}\ D_\alpha(y)]\,\mathcal{M}(y,z)D_\beta(y)\ d\mu(\alpha)\quad .$$

Hence:

$$N_\beta(z) = \int_A N_\alpha(y)\,\mathcal{M}_{\alpha\beta}(y,z)\ d\mu(\alpha) \tag{119}$$

for every α,β in A. Operating on each side of (119) with $[\mathrm{trn}\ D_\beta(z)]$ and integrating over A:

$$N(z) = \int_A N_\beta(t)\ d\mu(\beta) =$$

$$= N(y)\int_A\int_A [\ ]D_\alpha(y)\,\mathcal{M}_{\alpha\beta}(y,z)[\mathrm{trn}\ D_\beta(z)]\ d\mu(\alpha)\ d\mu(\beta)\quad .$$

Since $N(y)$ is arbitrary, we have from this and (115):

$$\boxed{\mathcal{M}(y,z) = \int_A\int_A [\ ]D_\alpha(y)\,\mathcal{M}_{\alpha\beta}(y,z)[\mathrm{trn}\ D_\beta(z)]\ d\mu(\alpha)\ d\mu(\beta)} \tag{120}$$

which is one of the possible generalizations of the type to which (92) belongs. This concludes the summary and overview of a possible general method of constructing partial groups $\Gamma_2(S)$ of operators on the subset S of the optical medium X. The problem of generalizing $\Gamma_2(a,b)$ to $\Gamma_2(S)$ will be considered once again in Sec. 7.11.

7.5 Analytic Properties of the Invariant Imbedding Operators

We now continue the work, begun in Sec. 7.1, of deriving the differential equations governing the main invariant imbedding operators. In particular we shall derive the various functional differential equations governing the operators $\mathcal{M}(x,y)$, $\mathcal{M}(x,y,z)$, and $\mathcal{M}(v,x;u,w)$. Since these operators are in turn 2×2 matrices of operators, each such differential equation is a potential plethora of differential equations for its component operators. Such a superabundance of operator differential equations would constitute an embarrassment of riches for the theory were it not for the insight gained into such operators in the preceding section. Indeed, our studies there showed that the operators of the form $\mathcal{M}(v,x;u,w)$ could be studied in terms of those of the form $\mathcal{M}(x,y,z)$, and the latter in terms of those of the form $\mathcal{M}(x,y)$. Hence the operators $\mathcal{M}(x,y)$ emerge as the undisputed victors in any contest of conceptual simplicity and inherent power of representation. In summary, then, it was shown how the members of $\Gamma_2(a,b)$ could represent, via simple algebraic formulas, all the other invariant imbedding operators of $\Gamma_3(a,b)$ and $\Gamma_4(a,b)$, plus the operators of $G_2(a,b)$, and even the classical R and T operators. Hence the multitude of operators can be reduced, formally at least, to just those in $\Gamma_2(a,b)$. This power of representation of the members of $\Gamma_2(a,b)$ will be used again in the present section to derive the requisite differential equations for all invariant imbedding operators from those for $\mathcal{M}(x,y)$. From these differential equations, in turn, the operators may be systematically constructed by various solution procedures using the inherent optical properties of the appropriate media.

Throughout this section, unless otherwise stated, we shall work with an arbitrary source-free plane-parallel medium $X(a,b)$, $a \le b$, with arbitrary incident radiance distributions $N_-(a)$ and $N_+(b)$ over the upper boundary X_a and the lower boundary X_b, respectively. As in the case of Sec. 7.4, the present results are readily generalized to wider settings, namely general one-parameter settings and general unparameterized optical media. Also, as in the case of Sec. 7.4, the exposition is primarily heuristic, with rigorous developments left for future study.

Differential Equations for $\mathcal{M}(x,y)$

Starting with the basic equation concerning the operator $\mathcal{M}(x,y)$; namely:

$$N(y) = N(x)\mathcal{M}(x,y)$$

introduced and studied in Ex. 7 of Sec. 3.7, we apply the differential operator d/dy to each side of this equation, where the differential operator occurs in (1) of Sec. 7.1. Thus:

$$\frac{dN(y)}{dy} = \frac{\partial}{\partial y}\left(N(x)\mathcal{M}(x,y)\right) = N(x)\left(\frac{\partial \mathcal{M}(x,y)}{\partial y}\right) \quad .$$

For the reader unfamiliar with analytic (i.e., differential, integral, and general limit) operations on operators, we may note here that the rules governing these operations are the same in all essential respects as those for the everyday type of function encountered in the domain of elementary calculus. Hence for the purposes of the present discussion, the reader will require no more advanced techniques than those encountered in such a domain. Needless to add, however, the physical content of the ensuing statements are far from trivial and are worthy of further analysis and application.

Continuing now with the derivation, we use (9) of Sec. 7.1 to reduce the preceding result to:

$$\frac{dN(y)}{dy} = N(y)\mathcal{K}(y) = N(x)\,\frac{d\mathcal{M}(x,y)}{dy} \quad .$$

Using the basic equation for $\mathcal{M}(x,y)$ once again, we obtain:

$$N(y)\mathcal{K}(y) = \left(N(x)\mathcal{M}(x,y)\right)\mathcal{K}(y) = N(x)\,\frac{d\mathcal{M}(x,y)}{dy} \quad .$$

Under the present lighting conditions, N(x) is arbitrary so that:

$$\boxed{\frac{\partial \mathcal{M}(x,y)}{\partial y} = \mathcal{M}(x,y)\mathcal{K}(y)} \tag{1}$$

which is our first main result and which holds for arbitrary x,y in X(a,b). This differential equation for $\mathcal{M}(x,y)$ harbors the four differential equations for its four components, which are operator-valued functions. For future reference, these are:

$$-\frac{d\mathcal{M}_{++}(x,y)}{dy} = \mathcal{M}_{++}(x,y)\tau(y) + \mathcal{M}_{+-}(x,y)\rho(y) \tag{2}$$

$$-\frac{d\mathcal{M}_{-+}(x,y)}{dy} = \mathcal{M}_{-+}(x,y)\tau(y) + \mathcal{M}_{--}(x,y)\rho(y) \tag{3}$$

$$\frac{\partial\mathcal{M}_{+-}(x,y)}{\partial y} = \mathcal{M}_{++}(x,y)\rho(y) + \mathcal{M}_{+-}(x,y)\tau(y) \tag{4}$$

$$\frac{\partial\mathcal{M}_{--}(x,y)}{\partial y} = \mathcal{M}_{-+}(x,y)\rho(y) + \mathcal{M}_{--}(x,y)\tau(y) \quad . \tag{5}$$

Observe how (2),(4) and (3),(5) are autonomous and indeed are copies of (5),(6) in Sec. 7.1. Hence $(\mathcal{M}_{++},\mathcal{M}_{+-})$ pair with (N_+,N_-) as do also $(\mathcal{M}_{-+},\mathcal{M}_{--})$. The initial value of $(\mathcal{M}_{++},\mathcal{M}_{+-})$ is (I,0), while that of $(\mathcal{M}_{-+},\mathcal{M}_{--})$ is (0,I).

A companion equation to (1) (its adjoint) is obtained when the differentiation is performed with respect to x rather than y. To obtain the companion equation observe that:

$$0 = \frac{d}{dx} I = \frac{\partial}{\partial x}\left[\mathcal{M}(y,x)\mathcal{M}(x,y)\right]$$

$$= \frac{\partial\mathcal{M}(y,x)}{\partial x}\mathcal{M}(x,y) + \mathcal{M}(y,x)\frac{\partial\mathcal{M}(x,y)}{\partial x} \quad .$$

Hence:

$$\frac{\partial\mathcal{M}(x,y)}{\partial x} = -\mathcal{M}(x,y)\frac{\partial\mathcal{M}(y,x)}{\partial x}\mathcal{M}(x,y) \quad . \tag{6}$$

Applying (1) to the derivative on the right in (6) we have

$$\frac{\partial\mathcal{M}(x,y)}{\partial x} = -\mathcal{M}(x,y)[\mathcal{M}(y,x)\mathcal{K}(x)]\mathcal{M}(x,y) \quad ,$$

whence:

$$\boxed{\frac{\partial\mathcal{M}(x,y)}{\partial x} = -\mathcal{K}(x)\mathcal{M}(x,y)} \quad . \tag{7}$$

Equations (1) and (7) are reducible to n×n matrix equations using angular discretization techniques such as those to be described in Secs 7.7, 7.9 and 7.10. The initial values are of course in each case $\mathcal{M}(x,x) = I$, the identity matrix. The four functions taken in the above pairs comprising $\mathcal{M}$, are called the *fundamental solutions of the equation of transfer*. As we have already seen in 7.4, by judicious linear combinations of these solutions, we can obtain all the useful scattering properties (e.g., R, T, $\mathcal{R}$, $\mathcal{T}$) of an optical medium.

The tactic used above to find (6) is a special case of the general procedure for finding the derivative of the inverse A^{-1} of an operator, knowing the derivative of A. Thus, if A is differentiable and depends on x:

$$0 = \frac{d}{dx} I = \frac{d}{dx}(AA^{-1}) = \frac{dA}{dx} A^{-1} + A \frac{dA^{-1}}{dx} \quad ,$$

whence:

$$\frac{dA^{-1}}{dx} = -A^{-1} \frac{dA}{dx} A^{-1} \quad . \tag{8}$$

This formula is based on the standing assumption that the inverse of A exists so that the product AA^{-1} is defined, and that A and A^{-1} are in some sense differentiable. Equation (8) is a general form of the formula:

$$\frac{d(1/y)}{dx} = -\frac{1}{y^2}\frac{dy}{dx} = -y^{-1}\frac{dy}{dx}y^{-1}$$

encountered in elementary calculus for the derivative of the numerical valued function 1/y in terms of that of y. Now, however, we generally are not permitted to join together the two inverses A^{-1} in (8) since operator multiplication is generally not commutative.

Differential Equations for $\mathcal{M}(x,y,z)$

By means of the representation of $\mathcal{M}(x,y,z)$ in terms of $\mathcal{M}(y,z)$ and $\mathcal{M}(y,x)$, as given in (42) of Sec. 7.4, we can find the differential equations governing $\mathcal{M}(x,y,z)$. There are generally three such equations, one arising from differentiation of $\mathcal{M}(x,y,z)$ with respect to each of the three distinct depth variables x,y,z within [a,b]. Throughout the derivations, then, x,y,z will be distinct variables, unless specifically noted otherwise.

Thus, differentiating each side of:

$$\mathcal{M}(x,y,z) = \left[\mathcal{M}(y,z)C_+ + \mathcal{M}(y,x)C_-\right]^{-1} \tag{9}$$

with respect to x and using (8) and (1):

$$\frac{\partial \mathcal{M}(x,y,z)}{\partial x} = -\mathcal{M}(x,y,z)\ \frac{\partial}{\partial x}\ [\mathcal{M}(y,z)C_+ + \mathcal{M}(y,x)C_-]\mathcal{M}(x,y,z)$$

$$= -\mathcal{M}(x,y,z)\mathcal{M}(y,x)\ \mathcal{K}(x)C_-\mathcal{M}(x,y,z) \quad .$$

Now, the general relation:

$$\mathcal{M}(x,v,z) = \mathcal{M}(x,y,z)\mathcal{M}(y,v) \tag{10}$$

follows from the definition of $\mathcal{M}(y,v)$ and (44) of Sec. 7.4 along with the general relation (84) of Sec. 3.7. Compare (10) with (39) of Sec. 7.4. Using (10), we have, in particular:

$$\mathcal{M}(x,y,z)\mathcal{M}(y,x) = \mathcal{M}(x,x,z) \quad ,$$

which allows us to write:

$$\boxed{\frac{\partial \mathcal{M}(x,y,z)}{\partial x} = -\mathcal{M}(x,x,z)\,\mathcal{K}(x)C_-\,\mathcal{M}(x,y,z)} \quad . \tag{11}$$

We defer discussion of (11) until its two companions have been derived. Toward this end, differentiating each side of (9) with respect to y and this time using (8) along with (7):

$$\frac{\partial \mathcal{M}(x,y,z)}{\partial y} = -\mathcal{M}(x,y,z)\,[-\mathcal{K}(y)\mathcal{M}(y,z)C_+ - \mathcal{K}(y)\mathcal{M}(y,x)C_-]\mathcal{M}(x,y,z)$$

$$= \mathcal{M}(x,y,z)\mathcal{K}(y)\,[\mathcal{M}(y,z)C_+ + \mathcal{M}(y,x)C_-]\mathcal{M}(x,y,z)$$

$$= \mathcal{M}(x,y,z)\mathcal{K}(y)\,[\mathcal{M}(x,y,z)]^{-1}\mathcal{M}(x,y,z) \quad .$$

Hence:

$$\boxed{\frac{\partial \mathcal{M}(x,y,z)}{\partial y} = \mathcal{M}(x,y,z)\,\mathcal{K}(y)} \quad . \tag{12}$$

Finally, differentiating each side of (9) with respect to z:

$$\frac{\partial \mathcal{M}(x,y,z)}{\partial z} = -\mathcal{M}(x,y,z)\,[\mathcal{M}(y,z)\mathcal{K}(z)C_+]\mathcal{M}(x,y,z)$$

which may be simplified, using (10), to:

$$\boxed{\frac{\partial \mathcal{M}(x,y,z)}{\partial z} = -\mathcal{M}(x,z,z)\mathcal{K}(z)C_+\mathcal{M}(x,y,z)} \quad . \tag{13}$$

Now for a brief discussion of (11), (12), and (13). All three equations show how to construct $\mathcal{M}(x,y,z)$ given the relatively simpler operators $\mathcal{M}(x,z,z)$, $\mathcal{K}(z)$, C_+ (in the case of (13) or $\mathcal{M}(x,x,z)$, $\mathcal{K}(x)$, C_- (in the case of (11)). For example, in the case of (11), part (a) of Fig. 7.9 shows that by starting with the basic slab X(y,z) (shaded) and building it up to level x as shown, we can compute $\mathcal{M}(x,y,z)$ for every x, such that $x \leq y \leq z$. All that is needed to start the calculation is information on $\mathcal{M}(x,y,z)$ for the special case x=y. This information, in view of (44)-(47) of Sec. 3.7, is tantamount to knowledge of the standard operators R(y,z) and T(y,z). Of course, one must know in addition the local transmittance and reflectance operators for the required range of x above the level y. A similar observation holds for (13) whose geometric significance is depicted in part (c) of Fig. 7.9. Finally, Eq. (12) strikes the middle road between (11) and (13) and shows how $\mathcal{M}(x,y,z)$ can be obtained by working *inward* from either boundary of X(x,z), and initially knowing $\mathcal{M}(x,x,z)$ or $\mathcal{M}(x,z,z)$, as the case may be. The former of

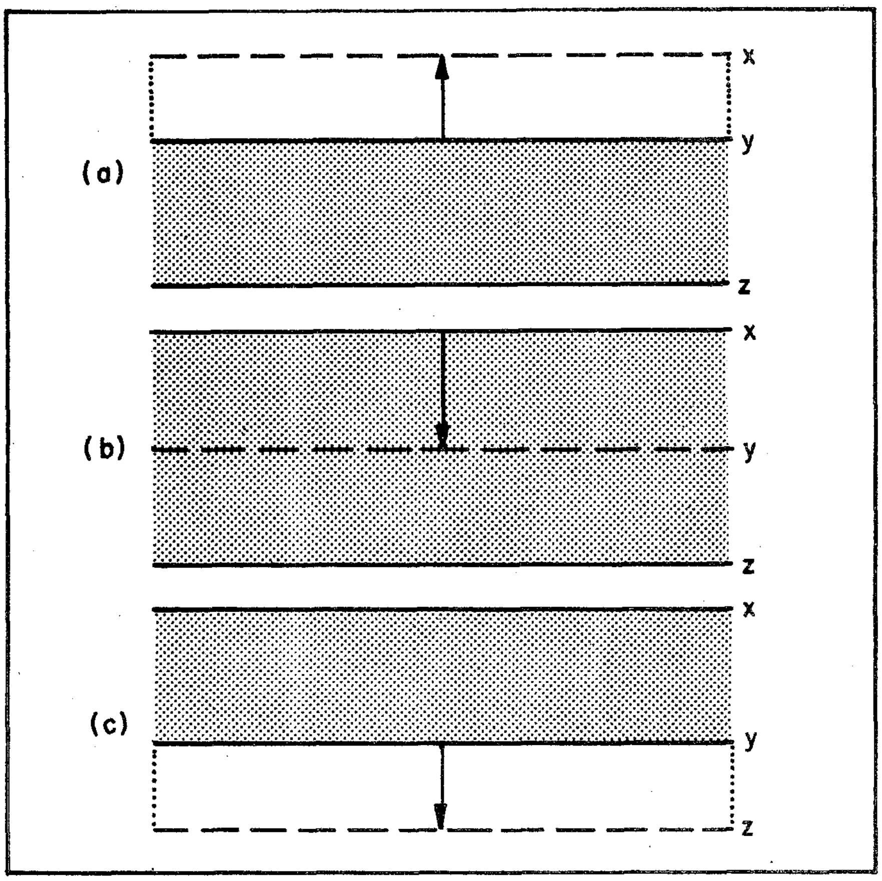


FIG. 7.9 Three ways in which to generate invariant imbedding operator $\mathcal{M}(x,y,z)$.

these cases is shown in (b) of Fig. 7.9.

It is instructive to unravel some of the information contained in these equations. We begin with (12) which yields the following four differential equations for the complete reflectance and transmittance operators:

$$-\frac{\partial \mathcal{T}(z,y,x)}{\partial y} = \mathcal{T}(z,y,x)\tau(y) + \mathcal{R}(z,y,x)\rho(y) \tag{14}$$

$$-\frac{\partial \mathcal{R}(x,y,z)}{\partial y} = \mathcal{R}(x,y,z)\tau(y) + \mathcal{T}(x,y,z)\rho(y) \tag{15}$$

$$\frac{\partial \mathcal{R}(z,y,x)}{\partial y} = \mathcal{T}(z,y,x)\rho(y) + \mathcal{R}(z,y,x)\tau(y) \tag{16}$$

$$\frac{\partial \mathcal{T}(x,y,z)}{\partial y} = \mathcal{R}(x,y,z)\rho(y) + \mathcal{T}(x,y,z)\tau(y) \quad . \tag{17}$$

Observe how (14), (16) are fundamentally similar to (5), (6) of Sec. 7.1, while (15), (17) are likewise autonomous and similar. Recall also the discussions of (1), (7) above. These earlier equations are no more fundamental than the present equations. Indeed, (14)-(17) may be used as the basis for all two-point boundary value problems by adopting the set of (two-point) fundamental solutions defined in (38)-(40) below.

Next, since the situations depicted in parts (a) and (c) of Fig. 7.9 are basically alike, we shall give only the details of unravelling Eq. (11), which goes with part (a) of the figure. The result is readily obtained by first noting that:

$$\begin{aligned}
-\mathcal{M}(x,x,z)\mathcal{K}(x)C_- &= -\begin{bmatrix} \mathcal{T}(z,x,x) & \mathcal{R}(z,x,x) \\ \mathcal{R}(x,x,z) & \mathcal{T}(x,x,z) \end{bmatrix} \begin{bmatrix} -\tau(x) & \rho(x) \\ -\rho(x) & \tau(x) \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\
&= -\begin{bmatrix} T(z,x) & 0 \\ R(x,z) & I \end{bmatrix} \begin{bmatrix} 0 & \rho(x) \\ 0 & \tau(x) \end{bmatrix} \\
&= -\begin{bmatrix} 0 & T(z,x)\rho(x) \\ 0 & R(x,z)\rho(x)+\tau(x) \end{bmatrix} \quad .
\end{aligned}$$

In view of this, (11) reduces to :

$$\frac{\partial \mathcal{M}(x,y,z)}{\partial x} = -\begin{bmatrix} 0 & T(z,x)\rho(x) \\ 0 & R(x,z)\rho(x)+\tau(x) \end{bmatrix} \begin{bmatrix} \mathcal{T}(z,y,x) & \mathcal{R}(z,y,x) \\ \mathcal{R}(x,y,z) & \mathcal{T}(x,y,z) \end{bmatrix}$$

whence:

$$-\frac{\partial \mathcal{T}(z,y,x)}{\partial x} = T(z,x)\rho(x)\mathcal{R}(x,y,z) \tag{18}$$

$$-\frac{\partial \mathcal{R}(z,y,x)}{\partial x} = T(z,x)\rho(x)\mathcal{T}(x,y,z) \tag{19}$$

$$-\frac{\partial \mathcal{R}(x,y,z)}{\partial x} = \big(R(x,z)\rho(x) + \tau(x)\big)\mathcal{R}(x,y,z) \tag{20}$$

$$-\frac{\partial \mathcal{T}(x,y,z)}{\partial x} = \big(R(x,z)\rho(x) + \tau(x)\big)\mathcal{T}(x,y,z) \quad . \tag{21}$$

The various physical interpretations of these equations are instructive and the reader may gain understanding of the dynamics of scattering problems by translating each of the preceding equations into words or appropriate mental images. For example, (18) describes how steadily flowing upward radiance (imagined incident at level z) changes at level y in X(x,z) when material is added to X(x,z) at its upper boundary X_x. Thus (refer to part (a) of Fig. 7.9) when a thin layer is added to X(x,z) at level x, the normally transmitted radiance (represented by T(z,x)) is now locally reflected in the new layer (represented by ρ(x)) and then globally reflected (as represented by $\mathcal{R}(x,y,z)$) down to layer y in X(x,z).

Further elucidation of the dynamics of scattering-absorbing media is forthcoming from the present differential equations for $\mathcal{M}(x,y,z)$ by observing how the differential equations for R and T, as derived in Sec. 7.1, may be derived anew in the present setting. As an example, consider Eq. (18) of Sec. 7.1. That equation describes, in essence, how reflected radiance at level a in X(a,b) changes when an incremental layer is added to X(a,b) at level a. In terms of the present equations such a change in R(a,b) is the sum of the changes in $\mathcal{R}(a,y,b)$ when a and y are simultaneously varied for the special instance when a = y, i.e., when the derivatives of $\mathcal{R}(a,y,b)$ with respect to y and a are added together for the case a = y. Thus, from (15) and (20):

$$-\left(\frac{\partial \mathcal{R}(x,y,z)}{\partial x} + \frac{\partial \mathcal{R}(x,y,z)}{\partial y}\right) =$$

$$= \mathcal{R}(x,y,z)\tau(y) + \mathcal{T}(x,y,z)\rho(y) + \big(R(x,z)\rho(x)+\tau(x)\big)\mathcal{R}(x,y,z) \tag{22}$$

Letting y approach x, the right side of this equation becomes, after rearrangements:

$$\rho(x) + \tau(x)R(x,z) + R(x,z)\tau(x) + R(x,z)\rho(x)R(x,z) \quad .$$

Furthermore, the left side of (22) is related to R(x,z) by the equation:

$$\frac{\partial R(x,z)}{\partial x} = \lim_{y \to x} \left[\frac{\partial \mathscr{R}(x,y,z)}{\partial x} + \frac{\partial \mathscr{R}(x,y,z)}{\partial y} \right] \tag{23}$$

which follows from:

$$R(x,z) = \lim_{y \to x} \mathscr{R}(x,y,z) \quad .$$

These limit equalities devolve on (45) of Sec. 3.7 and the usually available continuity of $\mathscr{R}(x,y,z)$ and its derivative. Combining these results, (18) of Sec. 7.1 is obtained from (22) but now as seen in the light of a superposition of changes of the complete reflectance function $\mathscr{R}(x,y,z)$.

The remaining three equations of Sec. 7.1 may also be viewed from the new vantage point of the invariant imbedding relation. For instance, Eq. (27) of Sec. 7.1 may be obtained, in essence, from (14) and (18) via the observation that:

$$\frac{\partial T(z,x)}{\partial x} = \lim_{y \to x} \left[\frac{\partial \mathscr{T}(z,y,x)}{\partial y} + \frac{\partial \mathscr{T}(z,y,x)}{\partial x} \right] \tag{24}$$

which follows from:

$$T(z,x) = \lim_{y \to x} \mathscr{T}(z,y,x) \quad .$$

These limit equalities devolve on (44) of Sec. 3.7 and the usually available continuity of $\mathscr{T}(z,y,x)$ and its derivatives. On the other hand, Eq. (28) of Sec. 7.1 is obtained directly from (19) after passing to the limit $y \to z$ and suitable rearrangement of coordinate variables. Finally, (29) of Sec. 7.1 follows directly from (21) in a similar way. The reason for the direct derivations in the latter two cases stems from the observation that:

$$R(z,x) = \lim_{y \to z} \mathscr{R}(z,y,x)$$

and

$$T(x,z) = \lim_{y \to z} \mathscr{T}(x,y,z)$$

and that the derivatives in (19) and (21) are with respect to x.

Differential Equations for $\mathscr{M}(v,x;u,w)$

Our starting point for the present derivations may be either (50), (51) or (52) of Sec. 7.4. We choose the representation (51) of $\mathscr{M}(v,x;u,w)$ so as to build directly on the results (11)-(13) just obtained and to gain some practice in the semigroup properties of the $\mathscr{M}$-operators. In the present notation, (51) becomes:

$$\mathcal{M}(v,x;u,w) = \mathcal{M}(v,y,u)\mathcal{M}^{-1}(x,y,w) \tag{25}$$

We generally expect four distinct differential equations to govern each member of $\Gamma_*(a,b)$. Thus, assuming u,v,w,x,y,z to be pairwise distinct variables, we have first of all:

$$\frac{\partial \mathcal{M}(v,x;u,w)}{\partial v} = \frac{\partial \mathcal{M}(v,y,u)}{\partial v}\mathcal{M}^{-1}(x,y,w) \quad .$$

By (11):

$$\frac{\partial \mathcal{M}(v,y,u)}{\partial v} = -\mathcal{M}(v,v,u)\mathcal{K}(v)C_-\mathcal{M}(v,y,u) \quad .$$

Hence:

$$\boxed{\frac{\partial \mathcal{M}(v,x;u,w)}{\partial v} = -\mathcal{M}(v,v,u)\mathcal{K}(v)C_-\mathcal{M}(v,x;u,w)} \quad . \tag{26}$$

This is the first of the requisite differential equations. Next, from (25):

$$\frac{\partial \mathcal{M}(v,x;u,w)}{\partial x} = \mathcal{M}(v,y,u)\frac{\partial \mathcal{M}^{-1}(x,y,w)}{\partial x} \quad .$$

But from (8) and (11):

$$\frac{\partial \mathcal{M}^{-1}(x,y,w)}{\partial x} = -\mathcal{M}^{-1}(x,y,w)[-\mathcal{M}(x,x,w)\mathcal{K}(x)C_-\mathcal{M}(x,y,w)]\mathcal{M}^{-1}(x,y,w) \quad .$$

Hence the second requisite differential equation is:

$$\boxed{\frac{\partial \mathcal{M}(v,x;u,w)}{\partial x} = \mathcal{M}(v,x;u,w)\mathcal{M}(x,x,w)\mathcal{K}(x)C_-} \tag{27}$$

This equation may be simplified by recalling (76) of Sec. 3.7 which states that:

$$\mathcal{M}(x,x,w) = \mathcal{M}(x,x;w,x)$$

and also the semigroup property (84) of Sec. 3.7:

$$\begin{aligned}\mathcal{M}(v,x;u,w)\mathcal{M}(x,x;w,x) &= \mathcal{M}(v,x;u,x)\\ &= \mathcal{M}(v,x,u)\end{aligned}$$

so that (27) becomes:

$$\frac{\partial \mathcal{M}(v,x;u,w)}{\partial x} = \mathcal{M}(v,x,u)\mathcal{K}(x)C_- = \frac{\partial \mathcal{M}(v,x,u)}{\partial x}C_- \quad . \tag{28}$$

Of the two preceding differential equations, (27) is the natural form of the requisite differential equation (the operator sought occurs explicitly on both sides of the equation). From a conceptual and computational point of view, (28) shows that, as far as dependence on the variable x is concerned, $\mathcal{M}(v,x;u,w)$ behaves essentially like the members of $\Gamma_2(a,b)$ or $\Gamma_3(a,b)$ (cf. (1) and (12)).

Next, from (25):

$$\frac{\partial \mathcal{M}(v,x;u,w)}{\partial u} = \frac{\partial \mathcal{M}(v,y,u)}{\partial u} \mathcal{M}^{-1}(x,y,w) \quad .$$

By (13):

$$\frac{\partial \mathcal{M}(v,y,u)}{\partial u} = -\mathcal{M}(v,u,u)\mathcal{K}(u)C_+\mathcal{M}(v,y,u)$$

so that:

$$\boxed{\frac{\partial \mathcal{M}(v,x;u,w)}{\partial u} = -\mathcal{M}(v,u,u)\mathcal{K}(u)C_+\mathcal{M}(v,x;u,w)} \quad . \tag{29}$$

Finally, from (25) once again:

$$\frac{\partial \mathcal{M}(v,x;u,w)}{\partial w} = \mathcal{M}(v,y,u) \frac{\partial \mathcal{M}^{-1}(x,y,w)}{\partial w} \quad .$$

From (8) and (13):

$$\frac{\partial \mathcal{M}^{-1}(x,y,w)}{\partial w} = -\mathcal{M}^{-1}(x,y,w)\{-\mathcal{M}(x,w,w)\mathcal{K}(w)C_+\mathcal{M}(x,y,w)\}\mathcal{M}^{-1}(x,y,w).$$

It then follows that:

$$\boxed{\frac{\partial \mathcal{M}(v,x;u,w)}{\partial w} = \mathcal{M}(v,x;u,w)\mathcal{M}(x,w,w)\mathcal{K}(w)C_+} \quad . \tag{30}$$

This equation, as (27), can be simplified slightly if we use the fact that:

$$\begin{aligned} \mathcal{M}(v,x;u,w)\mathcal{M}(x,w,w) &= \mathcal{M}(v,w;u,w) \\ &= \mathcal{M}(v,w,u) \quad . \end{aligned}$$

Hence (30) becomes:

$$\begin{aligned} \frac{\partial \mathcal{M}(v,x;u,w)}{\partial w} &= \mathcal{M}(v,w,u)\mathcal{K}(w)C_+ \\ &= \frac{\partial \mathcal{M}(v,w,u)}{\partial w} C_+ \quad . \end{aligned} \tag{31}$$

This equation and (28) show that, as far as the response variables x, w are concerned, $\mathcal{M}(v,x;u,w)$ behaves essentially like the members of $\Gamma_2(a,b)$ or $\Gamma_3(a,b)$ (see (1) and (12)). These observations could have been obtained directly using (52) of Sec. 7.4; however, the plausibility of (28) and (31) has now been reinforced by taking the preceding route.

Differential Equations for M(x,y) and Ψ(s,y)

It is interesting to derive the differential equations for the simplest of operators M(x,y) and the most complex of operators Ψ(s,y) encountered so far in our studies. Simplicity and complexity are measured here in terms of the ostensible algebraic structure of the components of M(x,y) and Ψ(s,y). As far as the simplicity and complexity of their differential equations are concerned, matters are reversed, as we shall now see. Thus for Ψ(s,y) we use (56) of Sec. 7.4 and find that:

$$\frac{\partial\Psi(s,y)}{\partial y} = [I + \Psi(s,s)]\,\frac{\partial\mathcal{M}(s,y)}{\partial y}$$

$$= [I + \Psi(s,s)]\,\mathcal{M}(s,y)\mathcal{K}(y) \quad , \qquad (32)$$

whence*:

$$\boxed{\frac{\partial\Psi(s,y)}{\partial y} = \Psi(s,y)\mathcal{K}(y)} \qquad s \neq y \qquad (33)$$

This result shows that the dependence of Ψ(s,y) on y is essentially that of $\mathcal{M}(x,y)$ on y. The integration of (33) starts from the initial given operator Ψ(s,s). The derivation of the differential equation showing how Ψ(s,y) varies with s is somewhat more complex and left to the reader. The differential equations for the Ψ-operators will be considered again in Sec. 7.12 wherein they will be represented in terms of complete reflectance and transmittance operators.

Turning now to the derivation of the differential equation for M(x,y), we use as a base the representation given by (10) of Sec. 7.4. From this we see that it is necessary to find:

$$\frac{\partial}{\partial z}\left[C_+ + \mathcal{M}(z,x)C_-\right]^{-1} =$$

$$= -\left[C_+ + \mathcal{M}(z,x)C_-\right]^{-1}\left(\frac{\partial\mathcal{M}(z,x)}{\partial z}\,C_-\right)\left[C_+ + \mathcal{M}(z,x)C_-\right]^{-1}$$

in which we have used (8). Hence, with the aid of (7) and (11):

---

*For all y we add $I_-\delta(s-y)$ where $I_- = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$, by (56) of Sec. 7.4.

$$\frac{\partial M(x,z)}{\partial z} = -[C_+ + \mathcal{M}(z,x)C_-]^{-1}(-\mathcal{X}(z)\mathcal{M}(z,x)C_-) \times$$

$$\times \quad [C_+ + \mathcal{M}(z,x)C_-]^{-1}[\mathcal{M}(z,x)C_+ + C_-]$$

$$+ \ [C_+ + \mathcal{M}(z,x)C_-]^{-1}(-\mathcal{X}(z)\mathcal{M}(z,x)C_+)$$

$$= [C_+ + \mathcal{M}(z,x)C_-]^{-1}\mathcal{X}(z)\mathcal{M}(z,x)[C_-M(x,z) - C_+] \ .$$

Hence

$$\frac{\partial M(x,z)}{\partial z} = [C_+ + \mathcal{M}(z,x)C_-]^{-1}\mathcal{X}(z)[C_- - C_+M(x,z)]. \quad (34)$$

Replacing $\mathcal{M}(z,x)$ in (34) by either of its representations, (11) or (12) of Sec. 7.4, the desired differential equation is obtained. The details are left to the interested reader.

Analysis of the Differential Equation for R(y,b)

The differential equation for R(a,b), as given in (18) of Sec. 7.1, was shown in the discussion of that section to be of central importance in evaluating the reflectance and transmittance operators associated with a plane-parallel medium X(a,b). In view of this importance, it is desirable to gain as much insight as possible into the structure of the differential equation governing R(a,b). We now analyze the equation for R(a,b), in two different manners, into a relatively simple pair of linear operator equations using the invariant imbedding relation. The result will perhaps shed some light on the methods of determining radiance fields within natural optical media.

We begin with the semigroup relation (53) of Sec. 3.7:

$$\mathcal{R}(a,z,b) = \mathcal{T}(a,y,b)\mathcal{R}(y,z,b) \quad .$$

By setting y = z in this relation, $\mathcal{R}(y,z,b)$ becomes R(y,b), so that:

$$\boxed{R(y,b) = \mathcal{T}^{-1}(a,y,b)\mathcal{R}(a,y,b)} \quad . \qquad (35)$$

This is the key representation for the reflectance operator R(y,b) in X(a,b) using the complete reflectance and transmittance operators $\mathcal{R}(a,y,b)$ and $\mathcal{T}(a,y,b)$. The inverse of the operator $\mathcal{T}(a,y,b)$ usually exists in most natural media, and so we shall proceed on the assumption of its availability, in order to see where it leads. Now, it follows from (15) and (17) that:

$$-\frac{\partial \mathcal{R}(a,y,b)}{\partial y} = \mathcal{R}(a,y,b)\tau(y) + \mathcal{T}(a,y,b)\rho(y) \qquad (36)$$

$$\frac{\partial \mathcal{T}(a,y,b)}{\partial y} = \mathcal{R}(a,y,b)\rho(y) + \mathcal{T}(a,y,b)\tau(y) \; . \qquad (37)$$

Therefore, on differentiating each side of (35) with respect to y:

$$\frac{\partial R(y,b)}{\partial y} =$$

$$= \frac{\partial \mathcal{T}^{-1}(a,y,b)}{\partial y}\mathcal{R}(a,y,b) + \mathcal{T}^{-1}(a,y,b)\frac{\partial \mathcal{R}(a,y,b)}{\partial y}$$

$$= -\mathcal{T}^{-1}(a,y,b)\left[\mathcal{R}(a,y,b)\rho(y) + \mathcal{T}(a,y,b)\tau(y)\right]\mathcal{T}^{-1}(a,y,b)\mathcal{R}(a,y,b)$$

$$+\mathcal{T}^{-1}(a,y,b)\left[-\mathcal{R}(a,y,b)\tau(y) - \mathcal{T}(a,y,b)\rho(y)\right] \quad .$$

Using (35) again, this may be simplified to :

$$\frac{\partial R(y,b)}{\partial y} = -R(y,b)\rho(y)R(y,b) - \tau(y)R(y,b) - R(y,b)\tau(y) - \rho(y) \quad .$$

On rearranging the preceding equation, we obtain (18) of Sec. 7.1. Equations (35), (36), (37) therefore constitute the required analysis of (18) of Sec. 7.1. We may summarize this finding alternatively as follows: *The system* (36), (37) *of linear differential equations for the complete* $\mathcal{R}$ *and* $\mathcal{T}$ *operators together with* (35) *uniquely determines* R(y,b) *in* X(a,b). The system (36), (37) may be represented succinctly by:

$$\boxed{\frac{d\,\mathcal{A}(y)}{dy} = \mathcal{A}(y)\,\mathcal{X}(y)} \qquad (38)$$

where we have written:

$$\text{"}\mathcal{A}(y)\text{"} \quad \text{for} \quad \left(\mathcal{R}(a,y,b),\mathcal{T}(a,y,b)\right) \quad .$$

Therefore the construction of $\mathcal{R}(a,y,b)$, $a < y < b$ is tantamount to solving (38) over the interval [a,b] with the initial condition:

$$\mathcal{A}(b) = \left(0,T(a,b)\right) \qquad (39)$$

and then working up from level b to level y in X(a,b); or with the initial condition:

$$\mathcal{A}(a) = \left(R(a,b),I\right) \qquad (40)$$

and then working down from level a to level y in X(a,b).

The physical significance of these observations is quite interesting. Suppose we are confronted with a plane-parallel optical medium X(a,b) (such as a portion of the atmosphere or the sea) and know only its overall transmittance T(a,b). *It is therefore in principle possible, via* (38) *and* (39), *to find* $\mathcal{R}(a,y,b)$, $\mathcal{T}(a,y,b)$ *for every intermediate level* y *within* X(a,b), *and hence* R(y,b) *for every intermediate level* y *within* X(a,b) *knowing the inherent optical properties of* X(a,b). A similar observation may be made using (40) and knowledge of R(a,b). Putting this in even more practical terms: one can essentially find the light field in the atmosphere at any altitude y knowing the overall transmittance T(a,b) of X(a,b) and $\mathcal{K}(y)$ throughout X(a,b); or the light field in the sea at any depth y can be obtained from R(a,b) and $\mathcal{K}(y)$ throughout X(a,b). The initial incident radiance distributions in each case are, of course, assumed given at levels a and b. Thus under suitable conditions the system (38) can yield knowledge of the radiometric situation inside a medium X(a,b), by knowing either the overall transmittance T(a,b) or overall reflectance R(a,b) of X(a,b). These observations are especially useful in the context of separable plane-parallel media, for in such media the R and T operators do not possess polarity (Sec. 7.1). Hence T(a,b) = T(b,a) and R(a,b) = R(b,a), so that the number of interaction operators for X(a,b) is cut in half.

We turn now to the second analysis of (18) of Sec. 7.1. The second manner of analyzing (18) of Sec. 7.1 is carried out by starting with the representation:

$$R(y,b) = \mathcal{M}_{--}^{-1}(b,y)\mathcal{M}_{-+}(b,y) \tag{41}$$

which is obtained from (29) of Sec. 7.4. The similarity of this representation with (35) is quite close: in each case the inverse operator is that of a transmittance-like operator, the remaining factor being a reflectance-like operator. In the present analysis we have, corresponding to (36) and (37), the following equations:

$$-\frac{\partial \mathcal{M}_{-+}(b,y)}{\partial y} = \mathcal{M}_{-+}(b,y)\tau(y) + \mathcal{M}_{--}(b,y)\rho(y) \tag{42}$$

$$\frac{\partial \mathcal{M}_{--}(b,y)}{\partial y} = \mathcal{M}_{-+}(b,y)\rho(y) + \mathcal{M}_{--}(b,y)\tau(y) \tag{43}$$

which are derived from (3) and (5). It should now be clear, without any further detailed discussion, that the system (42), (43), along with (41), determines R(y,b) for every level y in X(a,b). The parallel with the preceding analysis is completed by writing (42), (43) in matricial form and adducing the requisite initial condition, the present counterparts to (39) and (40). The only salient difference between the two analyses just given is that the concepts used in the first analysis are slightly more meaningful physically, and that the initial conditions in the second analysis are perhaps more convenient numerically. Thus, writing:

$$"\mathcal{B}(y)" \quad \text{for} \quad [\mathcal{M}_{-+}(b,y), \mathcal{M}_{--}(b,y)] \tag{44}$$

the system (42), (43) can be written:

$$\boxed{\frac{d\mathcal{B}(y)}{dy} = \mathcal{B}(y)\,\mathcal{X}(y)} \tag{45}$$

for $a \le y \le b$, and we have as initial condition:

$$\mathcal{B}(b) = (0, I) \quad . \tag{46}$$

7.6 Special Solution Procedures for R(a,b) and T(a,b) in Plane-Parallel Media

In this and the remaining sections of this chapter we shall discuss some of the solution procedures for light fields in natural media suggested by the theories of the preceding sections. The discussions will also serve to exhibit the inner analytic structure of the functional equations for the R and T operators. We begin with the differential equation (18) of Section 7.1 for the reflectance operator R(a,b) of a plane-parallel medium. In order to illustrate the procedure of reducing R(a,b) to the appropriate forms on a numerical level, we assume in this section that the medium X(a,b) is homogeneous and that its volume scattering function σ is isotropic, i.e., that:

$$\sigma(z;\xi';\xi) = s/4\pi \tag{1}$$

where s is the volume total scattering function for the medium.

Starting with (18) of Sec. 7.1, reproduced here for convenience; we have:

$$-\frac{\partial R(a,b)}{\partial a} = \rho(a)+\tau(a)R(a,b)+R(a,b)\tau(a)+R(a,b)\rho(a)R(a,b) \tag{2}$$

This is the differential equation for the reflectance operator R(a,b) for downward incident flux on X(a,b). Our immediate objective is to "shell" each term of (2) and to extract the kernel function of each of the indicated operators. It is the kernel function of R(a,b) which is to be evaluated in the present discussion, and we must somehow lay bare its presence in (2). An effective means towards this end is to postulate that the only radiance distribution on X(a,b) is incident on the upper boundary X_a of X(a,b) and is a radiance function $N_-(a)$ with Dirac-delta structure, i.e., for some positive radiance N^o and vector ξ^o in Ξ_-,

$$N_-(a)(x,\xi) = N^o\delta(\xi-\xi^o) \tag{3}$$

for every point x in X_a and direction ξ in Ξ_-. Once the response of X(a,b) to this arbitrary singular input is determined, the corresponding response to an arbitrary input is determinable by an integration over the direction set of the new input radiance distribution. The basis for this rests on the additivity and continuity properties of the function S defined in (7) of Sec. 3.6. This function, via the definitions (8)-(11) of Sec. 3.6, and (31),(32) of Sec. 3.7, yields the desired integral representations of R(a,b) and T(a,b) for X(a,b). Thus:

$$R(a,b) = \frac{1}{|\xi\cdot\mathbf{k}|}\int_{\Xi_-} [\ ]R(a,b;\xi';\xi)\ d\Omega(\xi') \tag{4}$$

$$T(a,b) = \frac{1}{|\xi\cdot\mathbf{k}|}\int_{\Xi_-} [\ ]T(a,b;\xi';\xi)\ d\Omega(\xi') \quad . \tag{5}$$

Our present goal is to describe a solution procedure for the reflectance *function* $R(a,b;\cdot;\cdot)$ on $\Xi_-\times\Xi_+$. Toward this end, we apply each of the five terms in (2) to the function $N_-(a)$ as defined in (3):

$$N_-(a)\ \frac{\partial R(a,b)}{\partial a} = \frac{1}{|\xi\cdot\mathbf{k}|}\int_{\Xi_-} N^o\delta(\xi'-\xi^o)\ \frac{\partial R(a,b;\xi';\xi)}{\partial a}\ d\Omega(\xi')$$

$$= \frac{N^o}{|\xi\cdot\mathbf{k}|}\ \frac{\partial R(a,b;\xi^o;\xi)}{\partial a} \tag{6}$$

$$N_-(a)\rho(a) = \frac{1}{|\xi\cdot\mathbf{k}|}\int_{\Xi_-} N^o\delta(\xi'-\xi^o)\sigma(a;\xi';\xi)\ d\Omega(\xi')$$

$$= \frac{N^o}{|\xi\cdot\mathbf{k}|}\ \sigma(a;\xi^o;\xi) \quad . \tag{7}$$

As for the term $\tau(a)R(a,b)$ in (2), we reduce it in two stages; first:

$$N_-(a)\tau(a) = \frac{1}{|\xi\cdot\mathbf{k}|}\int_{\Xi_-} N^o\delta(\xi'-\xi^o)\sigma(a;\xi';\xi)\ d\Omega(\xi') - \frac{N^o\delta(\xi-\xi^o)\alpha(a)}{|\xi\cdot\mathbf{k}|}$$

$$= \frac{N^o}{|\xi\cdot\mathbf{k}|}\ \Big[\sigma(a;\xi^o;\xi) - \delta(\xi-\xi^o)\alpha(a)\Big] \quad .$$

Second:

$$\left(N_-(a)\tau(a)\right)R(a,b) =$$

$$= \frac{N^o}{|\xi\cdot\mathbf{k}|}\int_{\Xi_-} \frac{1}{|\xi'\cdot\mathbf{k}|}\left[\sigma(a;\xi^o;\xi') - \delta(\xi'-\xi^o)\alpha(a)\right]R(a,b;\xi';\xi)\ d\Omega(\xi')$$

$$= \frac{N^o}{|\xi\cdot\mathbf{k}|}\int_{\Xi_-}\left[\frac{1}{|\xi'\cdot\mathbf{k}|}\right]\sigma(a;\xi^o;\xi')R(a,b;\xi';\xi)\ d\Omega(\xi')$$

$$- \frac{N^o\alpha(a)}{|\xi^o\cdot\mathbf{k}|\,|\xi\cdot\mathbf{k}|}\ R(a,b;\xi^o;\xi) \quad . \tag{8}$$

Next, the term $R(a,b)\tau(a)$ yields up, in turn:

$$N_-(a)R(a,b) = \frac{N^o}{|\xi\cdot\mathbf{k}|}\int_{\Xi_-}\delta(\xi'-\xi^o)R(a,b;\xi';\xi)\ d\Omega(\xi')$$

$$= \frac{N^o}{|\xi\cdot\mathbf{k}|}\ R(a,b;\xi^o;\xi) \tag{9}$$

$$\left(N_-(a)R(a,b)\right)\tau(a) = \frac{N^o}{|\xi\cdot\mathbf{k}|}\int_{\Xi_+}\left[\frac{1}{|\xi'\cdot\mathbf{k}|}R(a,b;\xi^o;\xi')\right]\sigma(a;\xi';\xi)\ d\Omega(\xi')$$

$$- \frac{N^o\alpha(a)}{|\xi\cdot\mathbf{k}|^2}\ R(a,b;\xi^o;\xi) \quad . \tag{10}$$

As for the term $R(a,b)\rho(a)R(a,b)$, we use (9) to obtain at once:

$$\left(N_-(a)R(a,b)\right)\rho(a) = \frac{N^o}{|\xi\cdot\mathbf{k}|}\int_{\Xi_+}\frac{N^o}{|\xi'\cdot\mathbf{k}|}R(a,b;\xi^o;\xi')\sigma(a;\xi';\xi)\ d\Omega(\xi')$$

Hence:

$$\left(N_-(a)R(a,b)\rho(a)\right)R(a,b) =$$

$$= \frac{N^o}{|\xi\cdot\mathbf{k}|}\int_{\Xi_-}\left[\int_{\Xi_+}\left[\frac{1}{|\xi'\cdot\mathbf{k}|}\cdot\frac{1}{|\xi''\cdot\mathbf{k}|}\right]R(a,b;\xi^o;\xi')\sigma(a;\xi';\xi'')R(a,b;\xi'';\xi)\,d\Omega(\xi')\right]d\Omega(\xi'') \tag{11}$$

The General Equation for $R(a,b;\xi';\xi)$

Assembling the results (6), (7), (8), (10) and (11) by means of (2), and rearranging terms, we have:

$$-\frac{\partial R(a,b;\xi^o;\xi)}{\partial a} + \alpha(a)\left(\frac{1}{\mu^o} + \frac{1}{\nu}\right)R(a,b;\xi^o;\xi) =$$

$$= \sigma(a;\xi^o;\xi) + \int_{\Xi_-}\sigma(a;\xi^o;\xi')R(a,b;\xi';\xi)\,\frac{d\Omega(\xi')}{\mu'}$$

$$+ \int_{\Xi_+}R(a,b;\xi^o;\xi')\sigma(a;\xi';\xi)\,\frac{d\Omega(\xi')}{\nu'}$$

$$+ \int_{\Xi_-}\left[\int_{\Xi_+}R(a,b;\xi^o;\xi')\sigma(a;\xi';\xi'')R(a,b;\xi'';\xi)\,\frac{d\Omega(\xi')}{\nu'}\right]\frac{d\Omega(\xi'')}{\mu''} \tag{12}$$

where we have written:

$$\text{"}\mu\text{"} \quad \text{for} \quad |\xi\cdot\mathbf{k}| \tag{13}$$

whenever ξ is in Ξ_- and:

$$\text{"}\nu\text{"} \quad \text{for} \quad |\xi\cdot\mathbf{k}| \tag{14}$$

whenever ξ is in Ξ_+. The minus sign before the derivative term in (12) rests on the physically based convention of measuring distance positively in the downward (or inward) direction in $X(a,b)$ from the boundary X_a. Equation (12) is the general integrodifferential equation for the reflectance function. Despite its apparent formidability, the equation is, in the last analysis, relatively tractable, since $R(a,b;\xi';\xi)$ is constructable from (12) starting with the initial condition:

$$R(b,b;\xi';\xi) = 0 \tag{15}$$

for every (ξ',ξ) in $\Xi_-\times\Xi_+$. In other words, (12) and (15) define the R function in terms of an initial value (or one-point boundary value) problem, a type of problem eminently suitable for the grist of modern electronic computer mills.

The Isotropic Scattering Case for R

A further simplification in the reduction of (2) is now possible, using the homogeneity and the isotropic scattering property (1) of $X(a,b)$. Writing:

$$\text{"}\rho\text{"} \quad \text{for} \quad s/\alpha$$

as usual, (12) reduces to:

$$-\frac{1}{\alpha}\frac{\partial R(a,b;\xi^o;\xi)}{\partial a} + \left(\frac{1}{\mu^o} + \frac{1}{\nu}\right) R(a,b;\xi^o;\xi) =$$

$$= \frac{\rho}{4\pi}\left[1 + \int_{\Xi_-} R(a,b;\xi';\xi)\,\frac{d\Omega(\xi')}{\mu'} + \int_{\Xi_+} R(a,b;\xi^o;\xi')\,\frac{d\Omega(\xi')}{\nu'}\right.$$

$$\left. + \int_{\Xi_-}\int_{\Xi_+} R(a,b;\xi^o;\xi')R(a,b;\xi'';\xi)\,\frac{d\Omega(\xi')}{\nu'}\cdot\frac{d\Omega(\xi'')}{\mu''}\right] \qquad (16)$$

A further reduction is possible in (16) by noting the following two facts. First, from an examination of (16) or the equation of transfer for the uniform scattering case (i.e., under the assumption (1) on σ) it follows that the radiance distribution $N(z,\cdot)$ at depth z in $X(a,b)$ is azimuth-independent, that is

$$N(z,\xi) = N(z,\xi')$$

whenever

$$\xi\cdot\mathbf{k} = \xi'\cdot\mathbf{k} \quad .$$

Indeed, the path function for such a medium as the present one is of the form:

$$\begin{aligned} N_*(z,\xi) &= \int_{\Xi} N(z,\xi')\sigma(z;\xi';\xi)\,d\Omega(\xi') \\ &= \frac{s}{4\pi}\int_{\Xi} N(z,\xi')\,d\Omega(\xi') \\ &= \frac{sh(z)}{4\pi} \quad . \end{aligned}$$

Hence N_* depends on depth only so that the path radiance of a path of sight with initial point of depth z_o, direction ξ, and length r is:

$$\begin{aligned} N_r^*(z,\xi) &= \int_0^r N_*(z',\xi)T_{r-r'}(z_o,\xi)\,dr' \\ &= \frac{s}{4\pi}\int_0^r h(z')e^{-\alpha(r-r')}\,dr' \end{aligned}$$

where

$$z' = z_o - r'\xi\cdot\mathbf{k}$$

This shows quite clearly that $N_r^*(z,\xi)$ is azimuth independent. For if the path is changed only in azimuth, (so that $\xi\cdot\mathbf{k}$ is unchanged) the result N_r^* of the preceding calculation is basically unchanged. This azimuthal independence of

N_r^* is then inherited by $N_r(z,\xi)$ for all ranges r, depths z, and all directions ξ in Ξ_+ (and, interestingly, for all ξ' in Ξ_-, except one, namely ξ^o, because of the singular residual radiance $N^o(z,\xi^o)$ at each depth z). Hence in particular $N_r(a,\xi)$ with $r = (b-a)/|\xi\cdot\mathbf{k}|$, namely the reflected radiance from X(a,b), is azimuth independent. In this way the values $R(a,b;\xi';\xi)$ themselves are seen to be azimuth independent of each direction ξ', and ξ. This independence serves to cut down the number of variables needed to describe $R(a,b;\xi';\xi)$. Indeed, we need henceforth only write:

$$\text{"}R(a,b;\mu',\nu)\text{"} \quad \text{for} \quad R(a,b;\xi';\xi) \tag{17}$$

in order to go on with the solution procedure, where μ' and ν are defined in (13), (14). (See Fig. 7.10.) Hence we reduce the number of directional variables from four (two real numbers each for ξ',ξ) to two, namely μ' and ν. This, then, is the first simplification (16) may undergo.

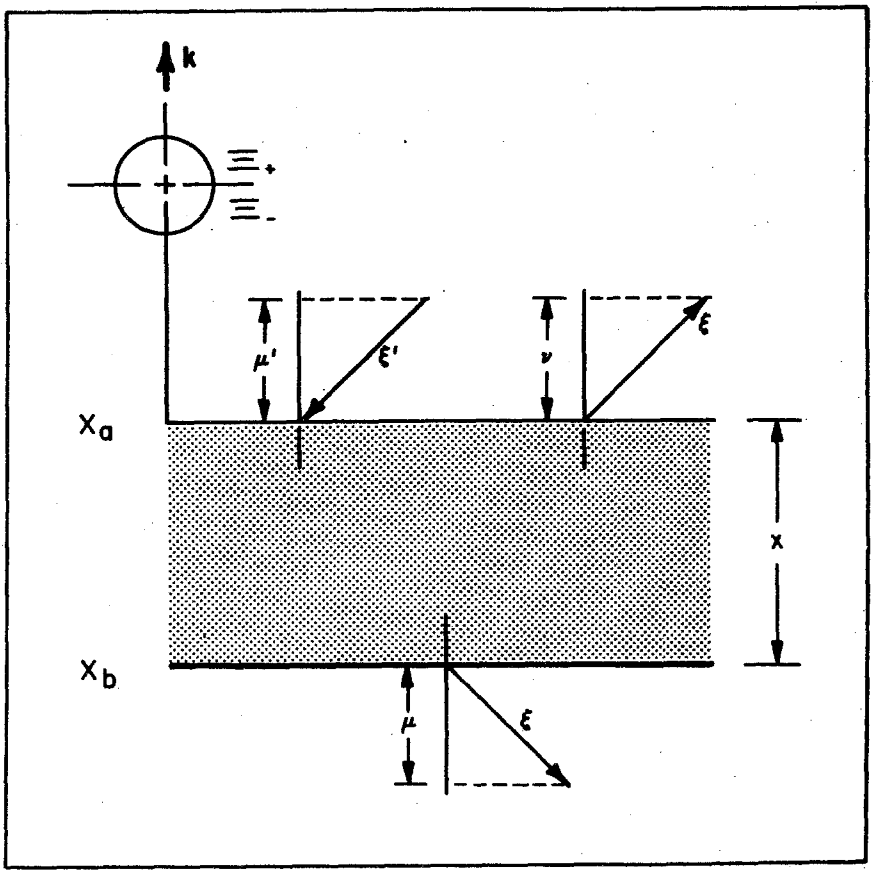


FIG. 7.10 Direction conventions for reflectance and transmittance operators on a slab X(a,b).

The remaining simplification of (16) is to note that the sum of the four terms on the right of (16) may be neatly factored into the product of two terms. Combining these two reductions, and noting that, in view of (17), we may write:

$$\int_{\Xi_-} R(a,b;\xi';\xi)\,\frac{d\Omega(\xi')}{\mu'} = \int_0^{2\pi}\int_0^1 R(a,b;\mu',\nu)\,\frac{d\mu' d\phi}{\mu'}$$

$$= 2\pi\int_0^1 R(a,b;\mu',\nu)\,\frac{d\mu'}{\mu'} \tag{18}$$

along with:

$$\int_{\Xi_+} R(a,b;\xi,\xi')\,\frac{d\Omega(\xi')}{\nu'} = 2\pi\int_0^1 R(a,b;\mu',\nu)\,\frac{d\nu}{\nu} \tag{19}$$

we have, at last:

$$-\frac{1}{\alpha}\frac{\partial R(a,b;\mu',\nu)}{\partial a} + \left(\frac{1}{\mu'} + \frac{1}{\nu}\right)R(a,b;\mu',\nu) =$$

$$= \frac{\rho}{4\pi}\left[1 + 2\pi\int_0^1 R(a,b;\mu',\nu)\,\frac{d\mu'}{\mu'}\right]\left[1 + 2\pi\int_0^1 R(a,b;\mu',\nu)\,\frac{d\nu}{\nu}\right] \tag{20}$$

This equation may be 'rationalized' if desired, by writing:

"$r(a;\mu',\nu)$" for $4\pi R(a,b;\mu',\nu)$,

thereby suppressing also the inactive fixed variable b and resulting in the equation:

$$\boxed{\begin{aligned}&-\frac{1}{\alpha}\frac{\partial r(a;\mu',\nu)}{\partial a} + \left(\frac{1}{\mu'} + \frac{1}{\nu}\right) r(a;\mu',\nu) =\\ &\quad= \rho\left[1 + \frac{1}{2}\int_0^1 r(a;\mu',\nu)\,\frac{d\mu'}{\mu'}\right]\left[1 + \frac{1}{2}\int_0^1 r(a;\mu',\nu)\,\frac{d\nu}{\nu}\right]\end{aligned}} \tag{21}$$

The variable a in (21) may be changed to one, say x which is the distance, in terms of attenuation lengths, measured positively upward from the lower boundary X_b of $X(a,b)$. See Fig. 7.10. Then the derivative term becomes positive, the initial condition becomes:

$$r(0;\mu',\nu) = 0$$

for $0 < \mu' \leq 1$, $0 < \nu \leq 1$, and (21) is thereby ready for numerical solution. The numerical solution is facilitated by using reciprocity, namely that $r(x;\mu',\nu) = r(x;\nu,\mu')$. This is readily established using the functional equations for $R(a,b;\xi',\xi)$ and the isotropy of the medium. Reciprocity will be discussed further in Sec. 7.12. Observe that the homogeneity of $X(a,b)$ is used in an essential way in reaching (21). Hence (21) may be used as it stands for separable media, i.e., media in which ρ is independent of depth. A slight generalization is possible by letting ρ vary with depth.

A Sample Numerical Solution for $r(x;\mu',\nu)$

A numerical solution of (21) was recently constructed using an electronic computer [15]. Figures 7.11, 7.12, 7.13, and 7.14 summarize some typical results of the computation. Reference [15] should be consulted for full details. However, the following basic information of radiative transfer interest may be noted here: for homogeneous media with scattering-absorption ratios $\rho \leq 0.5$ (called "λ" in the cited reference) a thickness of three optical depths is essentially equivalent to infinite thickness as far as reflectance is concerned, the agreement being to two or three decimal places. Thus $r(3,\mu',\nu)$ differs insignificantly from $r(\infty,\mu',\nu)$ for ρ values encountered in natural optical media. In general, as ρ increases (i.e., as s/α increases) toward 1, the infinite medium reflectance $r(\infty,\mu,\nu)$ is approached more slowly by $r(x;\mu',\nu)$. For example when $\rho = 0.9$, six optical depths are

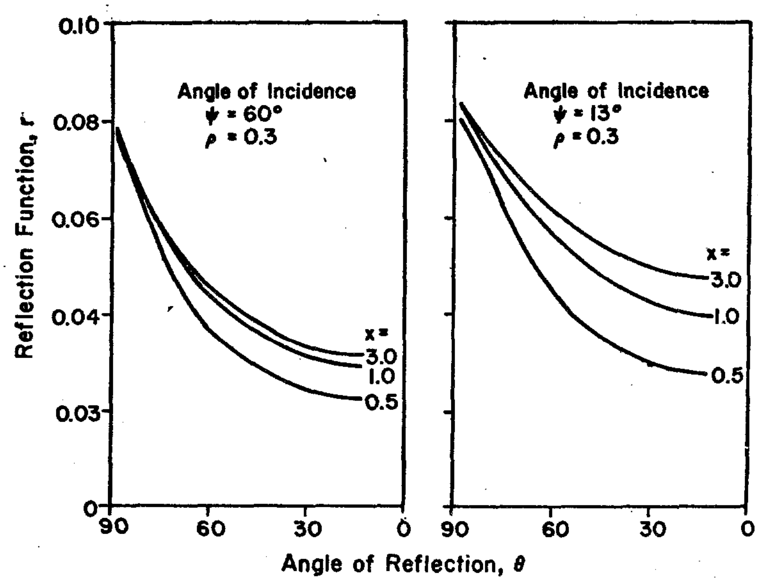


FIG. 7.11 Some typical curves with scattering-attenuation ratio $\rho = 0.3$. (From [15] by permission.)

needed to reasonably simulate infinite optical depth; and when $\rho = 1.0$, more than twenty attenuation lengths are needed to obtain agreement with $r(\infty,\mu',\nu)$ where three lengths sufficed above.

We observe in passing that the integrodifferential equation (21) yields immediately the equation governing $r(\infty,\mu',\nu)$. For, the derivative of $r(\infty,\mu',\nu)$ with respect to a is zero (cf. (30) of Sec. 7.3 and the comments below it), so that we obtain the following nonlinear integral equation for $r(\infty,\mu',\nu)$, (written, for brevity, as "$r(\mu',\nu)$"):

$$\left(\frac{1}{\mu'} + \frac{1}{\nu}\right) r(\mu',\nu) = \rho\left[1 + \frac{1}{2}\int_0^1 r(\mu',\nu)\,\frac{d\mu'}{\mu'}\right]\left[1 + \frac{1}{2}\int_0^1 r(\mu',\nu)\,\frac{d\nu}{\nu}\right]$$

This equation played an important part in the early phases of modern radiative transfer theory (cf. [1], [2], [43]).

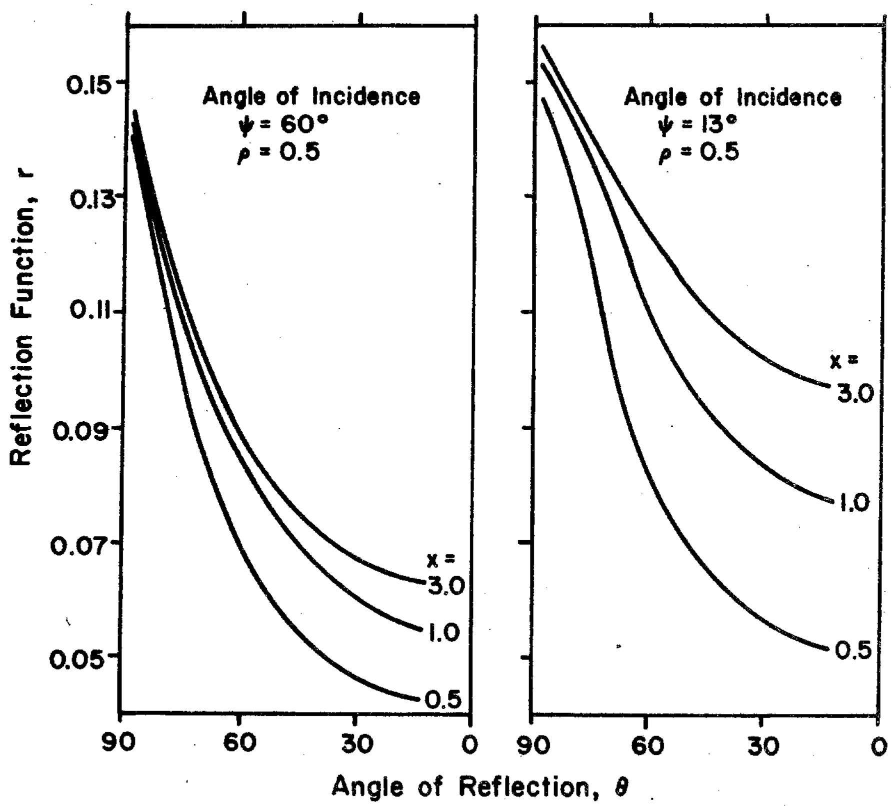


FIG. 7.12 Some typical curves with scattering-attenuation ratio $\rho = 0.5$. (From [15] by permission.)

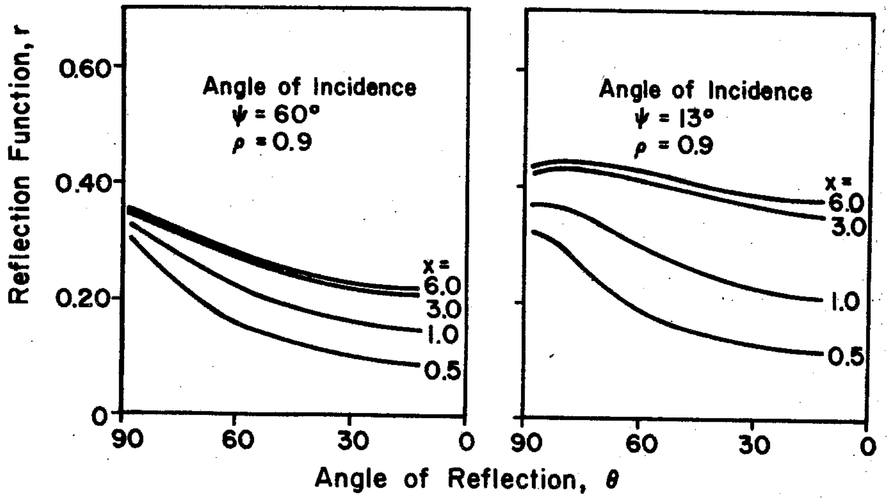


FIG. 7.13 Some typical curves with scattering-attenuation ratio ρ = 0.9. (From [15] by permission.)

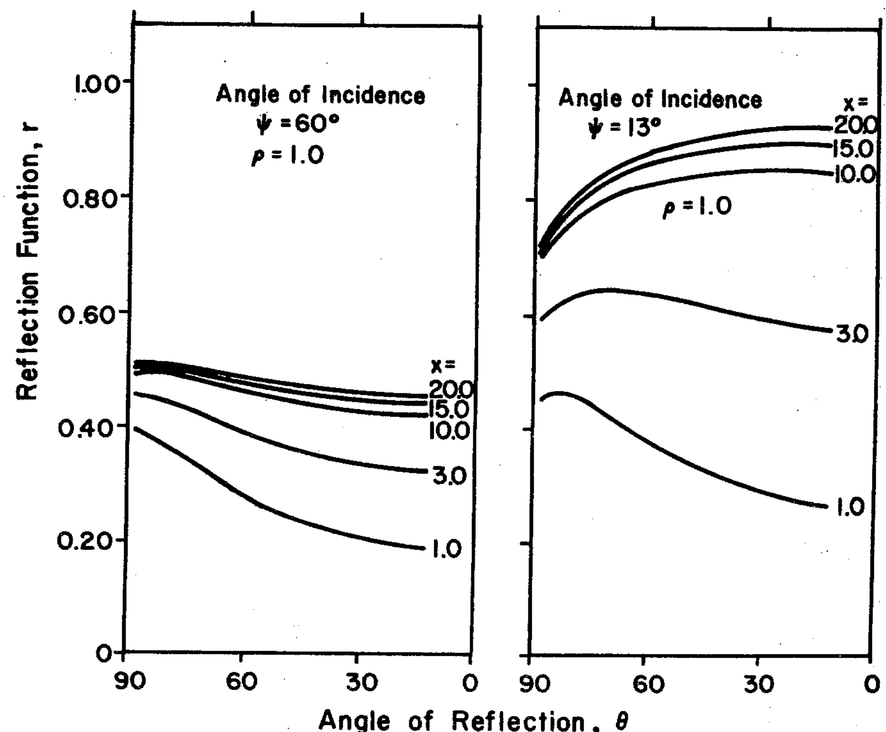


FIG. 7.14 Some typical curves with scattering-attenuation ratio ρ = 1.0. (From [15] by permission.)

The General Equation for $T^*(a,b;\xi';\xi)$

The tabulations of $R(x;\mu,\nu)$ in [16] may be built upon to obtain the companion transmittance function $T(x;\mu',\mu)$ by using equations (27) or (29) of Sec. 7.1, suitably reduced. Since the R and T operators for the present homogeneous space $X(a,b)$ do not possess polarity, i.e., since we have $R(a,b) = R(b,a)$ and $T(a,b) = T(b,a)$, $X(a,b)$ has associated with it only two operators, so that finding $T(x;\mu',\mu)$ will round out the basic information needed to determine the light field within $X(a,b)$ given the external incident radiances.

The reduction details of (29) of Sec. 7.1 to function form generally proceed as do those of the operator equation for $R(a,b)$. Since the resultant equation is of some importance, we now pause to sketch the details of the reduction. First we observe that the ultimate use of the reduced equation will be in a numerical procedure rather than a theoretical discussion; therefore it would be desirable to use (68) of 7.1 instead of (29) of Sec. 7.1, for the reason that the angular dependence of $T^*(a,b;\xi';\xi)$ is continuous while that of $T(a,b;\xi';\xi)$ is discontinuous. The basis for this fact is given in detail in the discussion on "Functional Relations for Decomposed Light Fields" in Sec. 7.1.

Starting with the derivative term in (68) of Sec. 7.1 we have:

$$N_-(a)\,\frac{\partial T^*(a,b)}{\partial a} = \frac{1}{|\xi\cdot\mathbf{k}|}\int_{\Xi_-} N^o\delta(\xi'-\xi^o)\,\frac{\partial T^*(a,b;\xi';\xi)}{\partial a}\,d\Omega(\xi')$$

$$= \frac{N^o}{|\xi\cdot\mathbf{k}|}\,\frac{\partial T^*(a,b;\xi^o;\xi)}{\partial a} \tag{22}$$

Then, analogously to the second stage of finding (8) of Sec. 7.6:

$$\big(N_-(a)\tau(a)\big)T^*(a,b) =$$

$$= \frac{N^o}{|\xi\cdot\mathbf{k}|}\int_{\Xi_-}\frac{1}{|\xi'\cdot\mathbf{k}|}\Big[\sigma(a;\xi^o;\xi')-\delta(\xi'-\xi^o)\alpha(a)\Big]T^*(a,b;\xi';\xi)\,d\Omega(\xi')$$

$$= \frac{N^o}{|\xi\cdot\mathbf{k}|}\int_{\Xi_-}\frac{1}{|\xi'\cdot\mathbf{k}|}\cdot\sigma(a;\xi^o;\xi')T^*(a,b;\xi';\xi)\,d\Omega(\xi')$$

$$- \frac{N^o\alpha(a)}{|\xi^o\cdot\mathbf{k}|\,|\xi\cdot\mathbf{k}|}\,T^*(a,b;\xi^o;\xi) \quad . \tag{23}$$

Using the expression for $N(a)R(a,b)\rho(a)$ found just prior to (11) of Sec. 7.6, we have:

$$(N_-(a)R(a,b)\rho(a))T^*(a,b) =$$

$$= \frac{N^o}{|\xi\cdot\mathbf{k}|}\int_{\Xi_-}\int_{\Xi_+}\left[\frac{1}{|\xi'\cdot\mathbf{k}|}\frac{1}{|\xi''\cdot\mathbf{k}|}\right]R(a,b;\xi^o;\xi')\sigma(a;\xi';\xi'')T^*(a,b;\xi'';\xi)\,d\Omega(\xi')\,d\Omega(\xi'') \tag{24}$$

Some entirely new terms are next forthcoming from (68) of Sec. 7.1:

$$N_-(a)\tau^*(a) = \frac{1}{|\xi\cdot\mathbf{k}|}\int_{\Xi_-} N^o\delta(\xi'-\xi^o)\sigma(a;\xi';\xi)\;d\Omega(\xi')$$

$$= \frac{N^o}{|\xi\cdot\mathbf{k}|}\,\sigma(a;\xi^o;\xi) \quad . \tag{25}$$

Next, using (32) of Sec. 7.1 on the result just obtained:

$$(N_-(a)\tau^*(a))T^o(a,b) = \int_{\Xi_-}\left[\frac{N^o}{|\xi'\cdot\mathbf{k}|}\sigma(a;\xi^o;\xi')\right]T_r(\mu',\xi')\delta(\xi-\xi')\;d\Omega(\xi')$$

$$= \frac{N^o\ e^{-\alpha r}}{|\xi\cdot\mathbf{k}|}\,\sigma(a;\xi^o;\xi) \tag{26}$$

where:

$$r = |b-a|/|\xi\cdot\mathbf{k}| \quad . \tag{27}$$

Finally, applying $T^o(a,b)$ now to $N_-(a)R(a,b)\rho(a)$:

$$(N_-(a)R(a,b)\rho(a))T^o(a,b) =$$

$$= \int_{\Xi_-}\left[\frac{N^o}{|\xi'\cdot\mathbf{k}|}\int_{\Xi_+}\frac{1}{|\xi''\cdot\mathbf{k}|}R(a,b;\xi^o;\xi'')\sigma(a;\xi'';\xi')\,d\Omega(\xi'')\right]T_r(\mu',\xi')\delta(\xi-\xi')\;d\Omega(\xi')$$

$$= \frac{N^o\ e^{-\alpha r}}{|\xi\cdot\mathbf{k}|}\int_{\Xi_+}\frac{1}{|\xi'\cdot\mathbf{k}|}\;R(a,b;\xi^o;\xi')\sigma(a;\xi';\xi)\;d\Omega(\xi') \tag{28}$$

where r is given in (27).

The preceding results are now ready for assembly in (68) of Sec. 7.1:

$$
\begin{aligned}
-\frac{\partial T^*(a,b;\xi^o;\xi)}{\partial a} &+ \frac{\alpha(a)}{\mu^o} T^*(a,b;\xi^o;\xi) = \\
&= \int_{\Xi_-} \sigma(a;\xi^o;\xi')T^*(a,b;\xi';\xi) \frac{d\Omega(\xi')}{\mu'} + \\
&\quad + \int_{\Xi_-}\int_{\Xi_+} R(a,b;\xi^o;\xi')\sigma(a;\xi';\xi'')T^*(a,b;\xi'';\xi) \frac{d\Omega(\xi')}{\nu'} \frac{d\Omega(\xi'')}{\mu''} + \\
&\quad + e^{-\alpha r}\left[\sigma(a;\xi^o;\xi) + \int_{\Xi_+} R(a,b;\xi^o;\xi')\sigma(a;\xi';\xi) \frac{d\Omega(\xi')}{\nu'}\right] \qquad (29)
\end{aligned}
$$

This is the requisite integrodifferential equation for $T^*(a,b;\xi';\xi)$ associated with the initial condition: (cf. (43) of Sec. 7.1):

$$T^*(b,b;\xi';\xi) = 0$$

for every ξ' in Ξ_- and ξ in Ξ_+.

The Isotropic Scattering Case for T^*

Under the conditions of homogeneity and isotropic scattering, (29) may be further reduced. Thus, analogously to the reduction of (12) to (16), (29) now goes over into:

$$
\begin{aligned}
-\frac{1}{\alpha}\frac{\partial T^*(a,b;\xi^o;\xi)}{\partial a} &+ \frac{1}{\mu^o} T^*(a,b;\xi^o;\xi) = \\
&= \frac{\rho}{4\pi}\int_{\Xi_-} T^*(a,b;\xi';\xi) \frac{d\Omega(\xi')}{\mu'} + \\
&\quad + \frac{\rho}{4\pi}\int_{\Xi_-}\int_{\Xi_+} R(a,b;\xi^o;\xi')T^*(a,b;\xi'';\xi) \frac{d\Omega(\xi')}{\mu'} \frac{d\Omega(\xi'')}{\nu''} + \\
&\quad + \frac{\rho}{4\pi} e^{-\alpha r}\left[1 + \int_{\Xi_+} R(a,b;\xi^o;\xi') \frac{d\Omega(\xi')}{\nu'}\right] .
\end{aligned}
$$

The preceding iterated integrals now uncouple, and, as before, the induced azimuthal symmetry encourages us to write:

$$"T^*(a,b;\mu',\mu)" \quad \text{for} \quad T^*(a,b;\xi';\xi) \quad , \tag{30}$$

so that we may use (19), and its present counterpart:

$$\int_{\Xi_-} T^*(a,b;\xi';\xi)\,\frac{d\Omega(\xi')}{\mu'} = 2\pi\int_0^1 T^*(a,b;\mu',\mu)\,\frac{d\mu'}{\mu'}$$

to come down to:

$$-\frac{1}{\alpha}\frac{\partial T^*(a,b;\mu',\mu)}{\partial a} + \frac{1}{\mu'}T^*(a,b;\mu',\mu) =$$

$$= \frac{\rho}{4\pi}\left[1 + 2\pi\int_0^1 R(a,b;\mu',\nu)\,\frac{d\nu}{\nu}\right]\left[2\pi\int_0^1 T^*(a,b;\mu',\mu)\,\frac{d\mu'}{\mu'}\right] +$$

$$+ \frac{\rho}{4\pi}\,e^{-\alpha r}\left[1 + 2\pi\int_0^1 R(a,b;\mu',\nu)\,\frac{d\nu}{\nu}\right] \quad . \tag{31}$$

This may be "rationalized" by writing:

$$"t^*(a;\mu',\mu)" \quad \text{for} \quad 4\pi T^*(a,b;\mu',\mu)$$

so that we have, after a final rearrangement:

$$\boxed{\begin{aligned} &-\frac{1}{\alpha}\frac{\partial t^*(a;\mu',\mu)}{\partial a} + \frac{1}{\mu'}t^*(a;\mu',\mu) = \\ &= \rho\left[e^{-\alpha r} + \frac{1}{2}\int_0^1 t^*(a;\mu',\mu)\,\frac{d\mu'}{\mu'}\right]\left[1 + \frac{1}{2}\int_0^1 r(a;\mu',\nu)\,\frac{d\nu}{\nu}\right] \end{aligned}} \tag{32}$$

This is the required integrodifferential equation for the diffuse transmittance function. In this equation, the function values:

$$\frac{1}{2}\int_0^1 r(a;\mu',\nu)\,\frac{d\nu}{\nu} \quad ,$$

which depend generally on a and μ', are known from the solution procedure for (21) and enter the solution procedure of (32) as given data, along with the initial condition:

$$t^*(0;\mu',\mu) = 0$$

for t^*, $0 < \mu' \le 1$, $0 < \mu \le 1$. Matters can be arranged so that the depth variable x of $t^*(x;\mu',\mu)$ is in attenuation lengths measured upward from the lower boundary X_b of X(a,b), thereby eliminating the minus sign in (32), and absorbing "α" into the derivative notation. That is, for numerical purposes, we can always change variables in (32) according to the equation $x = -y\alpha$, $a \le y \le b$. (See Figure 7.10.) The solution problem of (32) is now a straightforward initial value problem which may be reduced, by standard Gaussian quadrature procedures applied to the indicated integrals, to a finite system of simultaneous first order differential equations.

7.7 General Solution Procedures for R(a,b) and T(a,b) in Plane-Parallel Media

We return to the general integrodifferential equations for $R(a,b;\xi';\xi)$ and $T^*(a,b;\xi';\xi)$ as given in (12) and (29) of Sec. 7.6, and develop a general numerical procedure for their solution without the benefit of homogeneity and isotropic scattering within the medium X(a,b). The approach we shall follow is quite direct, one which requires a minimum of numerical preliminaries, thereby leaving such matters for choice in the individual programming procedure for the numerical solution. For example, with only minor changes, the following analysis may be repeated using Gaussian quadrature procedures. For the purposes of the present exposition, the determination problem for the operators R(a,b) and T(a,b) is considered solved when their correct functional equations have been found and suitably reduced to an initial value problem for some set of approximating (or occasionally exact) differential equations. Perhaps the greatest value of the following discussion is to allow students of the subject to come to grips with the inner workings of the integral operators R(a,b), and T(a,b). Once this is done, perhaps some efficient solution procedures will eventually come to mind.

We begin with a partition of Ξ_- and Ξ_+ into m and n sets of directions A_i and B_i, respectively (see Fig. 7.15); that is, we assume:

$$\Xi_- = \bigcup_{i=1}^{m} A_i \; ; \quad A_i \cap A_j = \phi \tag{1}$$

$$\Xi_+ = \bigcup_{i=1}^{n} B_i \; ; \quad B_i \cap B_j = \phi \quad . \tag{2}$$

These partitions of Ξ_- and Ξ_+, if sufficiently fine, let the integrals over them be reduced to simple numerical sums, as follows.

Consider, for example, the integral term occurring in (12) of Sec. 7.6:

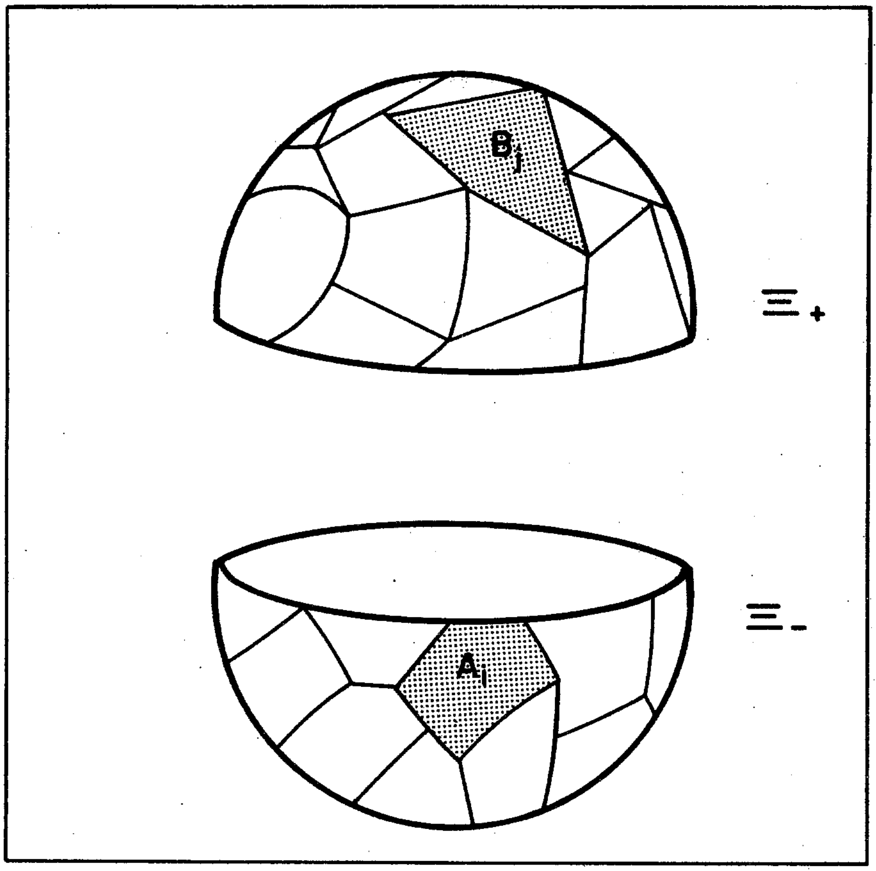


FIG. 7.15 General partitions of each half of the sphere of unit directions.

$$\int_{\Xi_-} \sigma(a;\xi^o;\xi')R(a,b;\xi';\xi)\ \frac{d\Omega(\xi')}{\mu'}$$

which with the partition of Ξ_- in force, may be written equivalently as the sum:

$$\sum_{i=1}^{m}\int_{A_i}\sigma(a;\xi^o;\xi')R(a,b;\xi';\xi)\ \frac{d\Omega(\xi')}{\mu'} \quad .$$

Now, if the partition of Ξ_- is sufficiently fine and σ and R, as functions of ξ' are continuous (a condition always available in geophysical settings) the integral over A_i may be represented arbitrarily closely by a term of the form:

$$\sigma(a;\xi^o;\xi_i')R(a,b;\xi_i';\xi)\ \frac{\Omega(A_i)}{\mu_i'}$$

where ξ_i' is a direction in A_i and μ_i' is $|\xi_i'\cdot\mathbf{k}|$.

Now ξ^o and ξ as they occur in the reduced term, happen also (for some j and k) to be in partition members A_j and B_k respectively, so that we can completely replace the functions σ and R, for the present purposes, by sets of numerical quantities, as follows. We write:

$$\text{"}R(a,b;i,j)\text{"} \quad \text{for} \quad R(a,b;\xi_i;\xi_j) \tag{3}$$

where ξ_i and ξ_j are selected fixed directions in A_i and B_j, respectively. Hence the function $R(a,b;\cdot;\cdot)$ is replaced by the mn numbers $R(a,b;i,j)$. These direction selections are fixed for the remainder of this discussion. Further, we write:

$$\text{"}\rho_-(a;i,j)\text{"} \quad \text{for} \quad \sigma(a;\xi_i;\xi_j) \tag{4}$$

whenever ξ_i is in A_i and ξ_j is in B_j:

$$\text{"}\rho_+(a;i,j)\text{"} \quad \text{for} \quad \sigma(a;\xi_i;\xi_j) \tag{5}$$

whenever ξ_i is in B_i and ξ_j is in A_j:

$$\text{"}\tau_-(a;i,j)\text{"} \quad \text{for} \quad \sigma(a;\xi_i;\xi_j) \tag{6}$$

whenever ξ_i is in A_i and ξ_j is in A_j:

$$\text{"}\tau_+(a;i,j)\text{"} \quad \text{for} \quad \sigma(a;\xi_i;\xi_j) \tag{7}$$

whenever ξ_i is in B_i and ξ_j is in B_j.

The values in each of the four cases just defined are readily determinable from the given volume scattering function σ. Observe that we are not assuming that the medium is isotropic, so that no use will be made of reciprocity principles (of which $r(a;\mu',\nu) = r(a;\nu,\mu')$ was an instance in Sec. 7.6). With these four definitions we may disassemble (12) of Sec. 7.6 into the following system of differential equations which can approximate (12) of 7.6 arbitrarily closely:

$$-\frac{\partial R(a,b;i,j)}{\partial a} + \alpha(a)\left(\frac{1}{\mu_i} + \frac{1}{\nu_j}\right)R(a,b;i,j) =$$

$$= \rho_-(a;i,j) + \sum_{k=1}^{m}\tau_-(a;i,k)R(a,b;k,j)\ \frac{\Omega(A_k)}{\mu_k'} +$$

$$+ \sum_{k=1}^{n}R(a,b;i,k)\tau_+(a;k,j)\ \frac{\Omega(B_k)}{\nu_k}$$

$$+ \sum_{\ell=1}^{m}\ \sum_{k=1}^{n}R(a,b;i,k)\rho_+(a;k,\ell)R(a,b;\ell,j)\ \frac{\Omega(B_k)}{\nu_k'}\ \frac{\Omega(A_\ell)}{\mu_\ell} \tag{8}$$

The unknowns in this system are the mn functions R(a,b;i,j), i = 1,..., m; j = 1,..., n. Holding b fixed, a is allowed to vary from b up to any arbitrary distance above b. Hence we have an initial value problem, which is got underway using the initial condition:

$$R(b,b;i,j) = 0$$

for every choice of the mn directional pairs. The $(m+n)^2$ dimensionless numbers such as:

$$\rho_-(a;i,j)\cdot\frac{1}{\alpha(a)}\quad,\quad \begin{matrix} i = 1,\dots, m \\ j = 1,\dots, n\end{matrix}$$

$$\tau_-(a;i,k)\ \frac{\Omega(A_k)}{\mu_k'}\cdot\frac{1}{\alpha(a)}\quad,\quad \begin{matrix} i = 1,\dots, m \\ k = 1,\dots, m\end{matrix}$$

$$\tau_+(a;k,j)\ \frac{\Omega(B_k)}{\nu_k}\cdot\frac{1}{\alpha(a)}\quad,\quad \begin{matrix} k = 1,\dots, n \\ j = 1,\dots, n\end{matrix}$$

and

$$\rho_+(a;k,l)\ \frac{\Omega(B_k)}{\nu_k'}\ \frac{\Omega(A_l)}{\mu_l}\ \frac{1}{\alpha(a)}\quad,\quad \begin{matrix} k = 1,\dots, n \\ l = 1,\dots, m\end{matrix}$$

can be assembled neatly into four matrices, denoted, say, by: "$\mathbf{r}_-(a)$", "$\mathbf{t}_-(a)$", "$\mathbf{t}_+(a)$", "$\mathbf{r}_+(a)$", respectively. (For an alternative procedure, see (17)-(21) below) Then writing "$\mathbf{R}(a,b)$" for the m×n matrix formed of the numbers R(a,b;i,j), (8) can be written succinctly as:

$$\boxed{\begin{aligned} &-\frac{1}{\alpha(a)}\frac{\partial \mathbf{R}(a,b)}{\partial a} + [\mathbf{D}\mathbf{R}(a,b) + \mathbf{R}(a,b)\mathbf{E}] = \\ &= \mathbf{r}_-(a) + \mathbf{t}_-(a)\mathbf{R}(a,b) + \mathbf{R}(a,b)\mathbf{t}_+(a) + \mathbf{R}(a,b)\mathbf{r}_+(a)\mathbf{R}(a,b) \\ &= [\mathbf{r}_-(a) + \mathbf{R}(a,b)\mathbf{t}_+(a)] + [\mathbf{t}_-(a) + \mathbf{R}(a,b)\mathbf{r}_+(a)]\mathbf{R}(a,b) \end{aligned}}\tag{9}$$

where we have written:

$$\text{"}\mathbf{D}\text{"}\quad \text{for}\quad \begin{bmatrix} 1/\mu_1 & & & 0 \\ & 1/\mu_2 & & \\ & & \ddots & \\ 0 & & & 1/\mu_m \end{bmatrix}\tag{10}$$

$$\text{"}\mathbf{E}\text{"}\quad \text{for}\quad \begin{bmatrix} 1/\nu_1 & 1/\nu_2 & & 0 \\ & & \ddots & \\ 0 & & & 1/\nu_n \end{bmatrix}\tag{11}$$

A check of the linear dimensions of the matrices shows that they are all properly commensurate. Thus, $\mathbf{r}_-(a)$ is m×n, $\mathbf{t}_-(a)$ is m×m, $\mathbf{t}_+(a)$ is n×n and $\mathbf{r}_+(a)$ is n×m. The choice of the partitions of Ξ_- and Ξ_+ governs the dimensions of these matrices. In some settings it is quite possible and, indeed, desirable to partition Ξ_- and Ξ_+ in essentially the same manner so that m = n and the μ_i and ν_i are equal for each i = 1,..., n. Hence we would have $\mathbf{D} = \mathbf{E}$ and all matrices would be n×n, and there would be n^2 equations in n^2 unknowns implicit in (9). As in the earlier reductions in Sec. 7.6, a transition to dimensionless depth parameters would be numerically convenient. Hence the change of variables given there should be adopted for numerical work. Equation (9) can be used to find both $R(a,b;\xi';\xi)$ and $R(b,a;\xi'';\xi')$ in the case of non-separable media, simply by integrating from level X_b to X_a in the first case and from X_a to X_b in the second case (cf. Fig. 7.10 in which the integration from X_b to X_a is depicted).

It remains to reduce Equation (29) of Sec. 7.6 to its approximating matricial counterpart. We retain the general partition of Ξ used for R(a,b) and write:

$$\text{"}T^*(a,b;i,j)\text{"} \quad \text{for} \quad T^*(a,b;\xi_i,\xi_j)$$

whenever ξ_i is in A_i and ξ_j is in A_j. In this way we generate the m×m matrix $\mathbf{T}^*(a,b)$. The system of differential equations approximate to (29) of Sec. 7.6 may then be written:

$$
\begin{aligned}
-\frac{\partial T^*(a,b;i,j)}{\partial a} + \frac{\alpha(a)}{\mu_i} T^*(a,b;i,j) &= \\
&= \sum_{k=1}^{m} \tau_-(a;i,k) T^*(a,b;k,j) \frac{\Omega(A_k)}{\mu_k'} + \\
&+ \sum_{\ell=1}^{m} \sum_{k=1}^{n} R(a,b;i,k)\rho_+(a;k,\ell) T^*(a,b;\ell,j) \frac{\Omega(B_k)}{\nu_k'} \frac{\Omega(A_\ell)}{\mu_\ell} + \\
&+ T_{r_j}(a,b)\left[\tau_-(a;i,j) + \sum_{k=1}^{n} R(a,b;i,k)\rho_+(a;k,j) \frac{\Omega(B_k)}{\nu_k'}\right] \qquad (12)
\end{aligned}
$$

where "$T_{r_j}(a,b)$" denotes the beam transmittance $T_{r_j}(a,\xi_j)$ and r_j is now the distance r, as defined in (27) of Sec. 7.6, associated with ξ_j. Using the battery of matrices defined above, this can be cast into the compact matricial form:

$$-\frac{1}{\alpha(a)}\frac{\partial \mathbf{T}^*(a,b)}{\partial a} + \mathbf{D}\mathbf{T}^*(a,b) =$$

$$= \mathbf{t}_-(a)\mathbf{T}^*(a,b) + \mathbf{R}(a,b)\mathbf{r}_+(a)\mathbf{T}^*(a,b) +$$

$$+ [\mathbf{t}_-(a) + \mathbf{R}(a,b)\mathbf{r}_+(a)]\mathbf{T}^o(a,b) \tag{13}$$

where we have written:

$$\text{"}\mathbf{T}^o(a,b)\text{"} \quad \text{for} \quad \begin{bmatrix} T_{r_1}(a,b) & & 0 \\ & T_{r_2}(a,b) & \\ & & \ddots \\ 0 & & T_{r_n}(a,b) \end{bmatrix} \tag{14}$$

The differential equation (13) may be written, compactly as:

$$\boxed{-\frac{1}{\alpha(a)}\frac{\partial \mathbf{T}^*(a,b)}{\partial a} + \mathbf{D}\mathbf{T}^*(a,b) = [\mathbf{t}_-(a)+\mathbf{R}(a,b)\mathbf{r}_+(a)]\mathbf{T}(a,b)} \tag{15}$$

where we have written:

$$\text{"}\mathbf{T}(a,b)\text{"} \quad \text{for} \quad \mathbf{T}^o(a,b) + \mathbf{T}^*(a,b) \quad . \tag{16}$$

The resemblance of the right side of (15) with that of (29) of Sec. 7.1 is striking. Similarly with (9) above and (18) of Sec. 7.1. The solution procedures for (9) and (15) can run along parallel to each other, for as (9) is solved for $\mathbf{R}(a,b)$, these values could be fed into (15) a fraction of a second later to help construct $\mathbf{T}^*(a,b)$. Indeed, a study of (9) and (15) shows that whole groups of terms of matrices are shared by both equations and that their simultaneous computation would help produce efficient computation programs.

A concluding word about the choice of definitions of the various matrices, such as $\mathbf{r}_-(a)$ and $\mathbf{R}(a,b)$ made above, is in order. These definitions are not unique and may be replaced by variants which, in the press of numerical work, may be found more amenable to the computational procedures than those exhibited above. For example, instead of $R(a,b;i,j)$, we could use as unknowns the terms:

$$R(a,b;i,j)/\nu_j \tag{17}$$

and then $\mathbf{R}(a,b)$ would be made up anew of such entries. Once this is done, it automatically dictates the following recombinations of terms which are guided by an examination of (8) and the list of terms following it. Thus, instead of $\rho_-(a;i,j)/\alpha(a)$ we would have:

$$\rho_-(a;i,j)\ \frac{1}{\nu_j}\ \frac{1}{\alpha(a)} \quad . \tag{18}$$

Further, we would adopt:

$$\tau_-(a;i,k)\ \frac{\Omega(A_k)}{\mu_k'}\ \frac{1}{\alpha(a)} \tag{19}$$

$$\tau_+(a;k,j)\ \frac{\Omega(B_k)}{\nu_k}\ \frac{1}{\alpha(a)} \tag{20}$$

as before. But now we would use:

$$\rho_+(a;k,1)\ \frac{\Omega(B_k)}{\nu_k'}\ \frac{\Omega(A_1)}{\alpha(a)} \quad . \tag{21}$$

The resulting equations would have the same gestalt as (9) and (15) so that the same general numerical procedure would be applicable to either system.

7.8 The Method of Modules for Deep Homogeneous Media

Now that we have some explicit computational procedures for finding the R and T operators for a plane-parallel medium (as given, for example, in Secs. 7.6 and 7.7), we turn to the task of finding the actual light fields within the medium. Finding $R(a,b)$ and $T(a,b)$ for a plane-parallel medium $X(a,b)$ allows one to determine the reflected and transmitted radiance at the boundaries of $X(a,b)$, but they do not directly supply the radiance distributions at internal depths y, $a < y < b$, in $X(a,b)$. The concept that allows the systematic determination of these internal radiance distributions, given the standard reflectance and transmittance operators $R(x,z)$ and $T(x,z)$, $a \le x \le z \le b$, within $X(a,b)$, is the invariant imbedding relation (36) of Sec. 3.7.

The general method of obtaining $N_+(y)$ and $N_-(y)$ for depths y, $a < y < b$, in $X(a,b)$ is delineated by the equations of Example 4 in Sec. 3.7 in which the invariant imbedding relation is derived. Our present purpose is to apply those general equations to a commonly encountered situation in hydrologic optics: the problem of the penetration of light into the sea, lakes, and other natural hydrosols. Now, some interesting features about such natural hydrosols is, first of all, that they are in many important instances homogeneous (or separable) and infinitely deep optically. This implies, among other facts, that the reflectances $R(a,y)$ reach an asymptotic value within a few attenuation lengths down from the surface X_a. That is, for all practical purposes, $R(a,y) = R(a,z)$ for y and z below some depth x. This phenomenon was touched upon in the discussions of the numerical solutions for $r(x;\mu',\nu)$ in Sec. 7.6. Another fact that may be of use in considering such deep natural hydrosols is that $R(y,\infty) = R(z,\infty)$ for all depths y and z below the surface X_a. This is the type of insight which arose in the study of (3) of Sec. 7.3; it now arises in a planetary oceanographic context rather than a stellar atmospheric context. Finally, the homogeneity (or more generally the separability) of $X(a,b)$ bars polarity from infecting the R and T operators, so that $R(x,z) = R(z,x)$ and $T(x,z) = T(z,x)$ for all depths x,z in $X(a,b)$. Hence these operators depend only on the geometrical thickness (or optical thickness if $X(a,b)$ is separable) of $X(a,b)$. We shall use these three major features of natural hydrosols in the discussions, now to begin.

The Invariant Imbedding Relation for Deep Hydrosols

Let us examine the invariant imbedding relation (36) of Sec. 3.7:

$$\big(N_+(y),N_-(y)\big) = \big(N_+(z),N_-(x)\big)\mathcal{M}(x,y,z) \qquad (1)$$

with the preceding physical observations in mind. To represent the fact that the medium $X(a,b)$ is a natural hydrosol and infinitely deep either optically or geometrically, we set $a = 0$ and $b = \infty$ in $X(a,b)$. This fact of infinite depth makes itself felt in (1) when we set $z = b = \infty$ and $x = a = 0$, in such a way that (1) becomes:

$$\big(N_+(y),N_-(y)\big) = \big(0,N_-(0)\big)\mathcal{M}(0,y,\infty) \quad . \qquad (2)$$

Note that our observations lead us to set $N_+(b) = N_+(\infty) = 0$; a physically obvious condition to impose at present. Equation (2) is the invariant imbedding relation for deep hydrosols (and for semi-infinite media in general) which are irradiated only at their upper boundaries X_a. The operator equations yielded by (2) are:

$$N_+(y) = N_-(0)\mathcal{R}(0,y,\infty) \quad . \qquad (3)$$

$$N_-(y) = N_-(0)\mathcal{T}(0,y,\infty) \qquad (4)$$

These equations, as simple as they are, can be made simpler by invoking the semigroup property for the complete reflectance operator $\mathcal{R}(a,y,\infty)$ as given in (53) of Sec. 3.7. When that property is adapted to the present case, we have:

$$\mathcal{R}(0,y,\infty) = \mathcal{T}(0,y,\infty)R(y,\infty) \quad . \qquad (5)$$

This was obtained by setting $y = z$ and $b = \infty$ in (53) of Sec. 3.7. Using this in (3), we go on to obtain:

$$N_+(y) = N_-(0)\mathcal{T}(0,y,\infty)R(y,\infty) \quad . \qquad (6)$$

Now our introductory discussion elicited the fact that $R(y,\infty)$ is independent of y in deep homogeneous media. In view of this, let us write:

$$\text{"}R_\infty\text{"} \quad \text{for} \quad R(y,\infty) \qquad (7)$$

for every $y \geq 0$. R_∞ is readily calculated using the techniques of Secs. 7.6 or 7.7.

The problem of describing the upwelling ($N_+(y)$) and downwelling ($N_-(y)$) radiance distributions in $X(a,\infty)$ has now been reduced to the determination of R_∞ for $X(0,\infty)$ and the complete transmittance operator $\mathcal{T}(0,y,\infty)$ for $X(a,\infty)$. On examining the representation of $\mathcal{T}(a,y,b)$, as given in (42) of Sec. 3.7 for the present case, we see that:

$$\mathcal{T}(0,y,\infty) = T(0,y)[I - R_\infty R(y,0)]^{-1} \quad . \tag{8}$$

In view of our observations about the effects of homogeneity of $X(0,\infty)$ on the imbedding relation, we see that $\mathcal{T}(0,y,\infty)$ depends only on the difference $|y-0| = y$ of the depths 0 and y. More generally, we may state that for any three depths x,y,z:

$$\mathcal{T}(x,y,\infty) = T(x,y)[I - R_\infty R(y,x)]^{-1} \tag{9}$$

and:

$$\mathcal{T}(y,z,\infty) = T(y,z)[I - R_\infty R(z,y)]^{-1} \tag{10}$$

in which $\mathcal{T}(x,y,\infty)$ and $\mathcal{T}(y,z,\infty)$ depend, respectively, only on $|y-x|$, and $|z-y|$. By multiplying these complete transmittance operators together we obtain, by virtue of the semigroup property (52) of Sec. 3.7:

$$\mathcal{T}(x,z,\infty) = \mathcal{T}(x,y,\infty)\,\mathcal{T}(y,z,\infty) \tag{11}$$

where $\mathcal{T}(x,z,\infty)$ again depends only on $|z-x|$. These observations suggest that we write:

$$\text{"}\mathcal{T}(s)\text{"} \quad \text{for} \quad \mathcal{T}(x,z,\infty)$$

whenever:

$$s = |z-x| \quad .$$

Hence (11) may be written more succinctly as:

$$\boxed{\mathcal{T}(r + s) = \mathcal{T}(r)\,\mathcal{T}(s)} \tag{12}$$

and equations (3) and (4) may be reduced to:

$$\boxed{\begin{aligned} N_+(y) &= N_-(0)\,\mathcal{T}(y)R_\infty \\ N_-(y) &= N_-(0)\,\mathcal{T}(y) \end{aligned}} \quad . \tag{13}$$

These are the requisite invariant imbedding equations for the light field at depth y in an infinitely deep homogeneous hydrosol irradiated at its upper boundary by an arbitrary given radiance distribution $N_-(0)$. The equations (13) may be interpreted in either the integral operator form (the general, exact interpretation) or in matrix form (the approximate interpretation) where the matrices are built up from those in Sec. 7.7. When the matrix interpretation of (13) is intended in the discussions below, bold face type will be used.

It is Equation (12) which brings the name "semigroup" into the present discussion. For by considering the collection $\{\mathcal{T}(r)\}$ of all complete transmittance operators $\mathcal{T}(r)$, $r \geq 0$, we see that this collection is closed under composition $\mathcal{T}(r)\mathcal{T}(s)$ of every two operators $\mathcal{T}(r)$ and $\mathcal{T}(s)$ (the composition, by (12), is $\mathcal{T}(r+s)$). Further, the associativity law holds:

$$\mathcal{T}(r)[\mathcal{T}(s)\mathcal{T}(t)] = [\mathcal{T}(r)\mathcal{T}(s)]\mathcal{T}(t)$$

and finally, $\mathcal{T}(0) = I$, the identity property holds. The collection $\{\mathcal{T}(r)\}$, so endowed, is called a *semigroup*, with unit, and is an instance of a more general concept of the same name in advanced functional analysis.

The Module Equations

A practical numerical procedure for determining the light field at depths y in $X(0,\infty)$ is suggested by the system (13) and (8). Suppose we agree to partition the natural hydrosol into layers of equal thickness d (in either meters or attenuation lengths). For example, a practical choice of d may be between 1/2 to 1 attenuation length. Once d is fixed, we compute the operators $T(0,d)$ or the matrices $\mathbf{T}(0,d)$,$\mathbf{R}(0,d)$, and $\mathbf{R}_\infty$, according to the procedures in, say, Sec. 7.7. Then, by (8), we find $\mathcal{T}(d)$. It follows from (12) and (13) that we can go on to obtain $N_+(jd)$ and $N_-(jd)$ for any integer $j \geq 0$ by means of the equations:

$$\boxed{\begin{aligned} N_+(jd) &= N_-(0)\mathcal{T}^j(d)R_\infty \\ N_-(jd) &= N_-(0)\mathcal{T}^j(d) \end{aligned}} \qquad (14)$$

Hence the problem of finding $N_\pm(y)$ in the sea or in lakes or other deep natural optical media has been reduced to the problem of raising a fixed matrix or integral operator $\mathcal{T}(d)$ to an integral power, a relatively simple operation in this day of electronic computers. A slab in $X(0,\infty)$ of thickness d is called a *module* of $X(0,\infty)$, and once this thickness is fixed, the determination of $N_\pm(jd)$ is a mere mechanical detail of computation from the *module equations* (14). Linear interpolation procedures should be sufficient to determine $N_\pm(y)$ for $jd \leq y < (j+1)(d)$.

Empirical Bases for the Use of the Module Equations

The module equations (14), which have been deduced from the invariant imbedding relation, are basically theoretical equations whose lineage can be traced all the way back to the interaction principle of Chapter 3. Be this as it may, the system (14) nevertheless suggests the intriguing possibility of computing the light fields in natural optical media by knowing just two bits of empirical information about the media, namely $\mathbf{R}_\infty$ and $\mathcal{T}(d)$. To explain the idea behind the measurements of these quantities consider the following ideal experiment. We calm the surface of the sea and remove the atmosphere of the earth, and position the sun in the sky so that its parallel rays irradiate the sea, in turn, in each of the directions ξ_i associated with the m parts A_i of Ξ_-, as defined in (1) of Sec. 7.7. For each incident direction ξ_i, we then measure the radiance reflected from the sea in each of the response directions ξ_j associated with the n parts B_j of Ξ_+, as defined in (2) of Sec. 7.7. Thus, in effect, we can determine the mn numbers $R(0,\infty;i,j)$ as defined in (3) of Sec. 7.7. Analogously to $R(a,b;\xi';\xi)$, $\mathcal{T}(d)$ can be given a matricial form, that of a m×m matrix to be exact. By going down d units of depth in the sea and measuring $N(d,\xi_j)$ when the sun is irradiating the surface in the direction ξ_i, we, in effect, find the entry $\mathcal{T}_{ij}(d)$ of $\mathcal{T}(d)$. (The sun's radiance must be normalized to unity for each irradiation.)

The procedure just sketched for determining the matrices $\mathcal{T}(d)$ and R_∞ is, of course, not to be taken seriously --at least not literally. It does, however, contain the germ of a possibly workable procedure for finding $\mathcal{T}(d)$. Our observations, in the form of (9) and (10), show that $\mathcal{T}(d)$ is in principle determinable if we measure a sufficient number of $N_-(x)$ and $N_-(x+d)$ values at some convenient depths $x \geq 0$ below the surface. For then:

$$N_-(x+d) = N_-(x)\,\mathcal{T}(d) \tag{15}$$

and converting the measured radiances $N_-(x+d)$ and $N_-(x)$ into m-component vectors (on the basis of the partition of Ξ_- into the parts A_i) and $\mathcal{T}(d)$ into an m×m matrix of unknown parts, we have a set of m equations in m^2 unknowns $\mathcal{T}_{ij}(d)$. What is needed, then, is a set of m measured vectors $\mathbf{N}_-(x)$ and their m measured correspondents $\mathbf{N}_-(x+d)$, and obtained in such a way that the set of vectors $\mathbf{N}_-(x)$ is linearly independent.

To see this in more detail, let us denote the jth column of the matrix $\mathcal{T}(d)$ by "$\mathcal{T}_j(d)$", and let us denote the ith measured radiance vector $N_-(x)$ by "$\mathbf{N}_-^i(x)$" for every depth x in $X(0,\infty)$. Further, let "$N_j^i(x)$" denote the jth component of $\mathbf{N}_-^i(x)$. Then the expanded matrix form of (15) leads to the following relation: for every i, j=1,...,m:

$$N_j^i(x+d) = \mathbf{N}_-^i(x) \cdot \mathcal{T}_j(d) \tag{16}$$

where the dot denotes the dot product for vectors.

Suppose we write:

$$\text{"}\mathbf{N}_-(x)\text{"} \quad \text{for} \quad \begin{bmatrix} \mathbf{N}_-^1(x) \\ \mathbf{N}_-^2(x) \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{N}_-^m(x) \end{bmatrix} \quad . \tag{17}$$

Then assembling the m^2 equations of (16), we have:

$$\mathbf{N}_-(x+d) = \mathbf{N}_-(x)\,\mathcal{T}(d) \quad . \tag{18}$$

The linear independence of the vectors $\mathbf{N}_-^i(x)$, i=1,...,m, implies the existence of the inverse $\mathbf{N}^{-1}(x)$ of the m×m matrix $\mathbf{N}(x)$. Equation (18) then yields:

$$\boxed{\mathcal{T}(d) = N_-^{-1}(x)N_-(x+d)} \qquad (19)$$

which holds for every depth x and module $X(x,x+d)$ of $X(0,\infty)$. In this way we find an empirical basis for the complete transmittance matrix for a module in a natural hydrosol.

It remains to determine the matrix R_∞ empirically. However, this determination is exactly analogous to that of $\mathcal{T}(d)$ and if "$N_-(x)$" and "$N_+(x)$" denote the incident and reflected m×m matrices of radiances measured at any depth x such that $N_-(x)$ is invertible, then:

$$\boxed{R_\infty = N_-^{-1}(x)N_+(x)} \qquad (20)$$

In (20) we have assumed that Ξ_+ and Ξ_- are both partitioned similarly into m pieces. Once measurements within a natural hydrosol have been made so that $\mathcal{T}(d)$ and R_∞ are obtained, then the module equations yield the light field at any integral depth jd in the medium knowing $N_-(0)$. These observations show that the radiative transfer problem of the penetration of light into the sea can be solved on either an empirical level or a theoretical level knowing the basic reflectance operator R_∞ for $X(0,\infty)$ and complete transmittance operator $\mathcal{T}(d)$ for a module $X(x, x+d)$ of $X(0,\infty)$.

The operators R_∞ and $\mathcal{T}(d)$ as used above are inherent optical properties of the hydrosol in the sense that they are independent of the light fields within the medium and that they depend only on the intrinsic physical makeup of the medium (cf. closing remarks of Sec. 3.12, and also Chap. 11 for definitions and discussions of inherent optical properties).

One of the significant features of the module equations is that they may be formulated, solved, and applied completely on the global level within the medium $X(0,\infty)$, and need make no appeal either directly or indirectly to the local properties of the medium such as the volume attenuation and scattering functions α and σ of the medium. Further discussion of the problem of determining the global optical properties of a medium using measured radiometric data is made in Sec. 13.10.

7.9 The Method of Semigroups for Deep Homogeneous Media

The results of the preceding section, in the form of the module method of solution of radiative transfer problems in the sea and the air, were so simple and direct that we are encouraged to explore the method in more detail, with an eye toward obtaining a general method applicable to all media. Thus our purpose in this section is to begin with the basis for the module equations, namely the system (13) of Sec. 7.8, and study the effect on the module equations when the module thickness is allowed to go to zero but with the depth z (=jd) held fixed. The resultant equations will reveal a general pattern which suggests the requisite generalization, namely the *method of semigroups*.

The Semigroup Equations for $\mathcal{T}(z)$

Consider (14) of Sec. 7.8, which describes the downward radiance distribution at depth jd, $j \geq 0$. and where d is the thickness of the module X(0,d) for the homogeneous infinitely deep medium $X(0,\infty)$. If we halve the depth of the module, then we must double the powers to be used to find $N_-(jd)$, i.e., we are observing that:

$$\mathcal{T}^{2j}(d/2) = \mathcal{T}^{j}(d) \quad , \tag{1}$$

which follows from (12) of Sec. 7.8. More generally, for every positive integer n:

$$\mathcal{T}^{nj}(d/n) = \mathcal{T}^{j}(d) \tag{2}$$

which follows by induction on n, starting with (1). Setting j = 1 in (2) we see that the resulting equation, namely

$$\mathcal{T}^{n}(d/n) = \mathcal{T}(d) \quad ,$$

demonstrates quite graphically that the module transmittance is the product of an arbitrarily large number of transmittances of 'submodules' of thickness d/n. Now, (12) of Sec. 7.8 shows that $\mathcal{T}(0) = I$, the identity operator on the set $\mathcal{N}_-$ of all downward radiance distributions ($\mathcal{N}_-$ is defined in the invariant imbedding statement Sec. 3.9). The continuity of $\mathcal{T}(s)$, which holds for the most part in all natural optical media, then implies that the transmittance operators for the submodules X(0,d/n) approach the identity operator I. That is:

$$\lim_{s \to 0} \mathcal{T}(s) = I \quad .$$

All this is quite clear when one reflects on the definition of $\mathcal{T}(s)$, being an instance of a complete transmittance operator. But now, in the light of the present approach, wherein the analogy of $\mathcal{T}(s)$ and beam transmittance T_s just waiting to appear explicitly upon the scene, in this light we are moved to consider next the limit:

$$\lim_{s \to 0} \frac{I - \mathcal{T}(s)}{s}$$

which is motivated by the defining equation for volume attenuation function α (Sec. 3.11):

$$\alpha = \lim_{s \to 0} \frac{1 - T_r}{r} \quad .$$

The limit involving $\mathcal{T}(s)$ will, of course be an operator of some kind rather than a number, as is α; however, the analogy now a-building seems so suggestive that we are next moved to write:

$$\text{"A"} \quad \text{for} \quad \lim_{s \to 0} \frac{I - \mathcal{T}(s)}{s} \tag{3}$$

Hence, for any depth differences s, we have directly from (3):

$$A + \varepsilon(s) = \frac{I - \mathcal{T}(s)}{s} \tag{4}$$

where "$\varepsilon(\cdot)$" denotes an operator (actually defined implicitly by (4)) which goes to zero as its argument goes to zero. Solving (4) for $\mathcal{T}(s)$:

$$\boxed{\mathcal{T}(s) = (I - sA) + o(s)} \tag{5}$$

where "$o(\cdot)$" denotes $\varepsilon(\cdot)s$. The closer s is to zero, the closer $o(s)$ is to the zero operator. This equation is analogous to

$$T_s = (1 - s\alpha) + o(s)$$

for beam transmittance (where "$o(\cdot)$" in the latter equation is of course distinct from that in (5)).

The momentum of these definitions, and discoveries of analogy carry us on to consider the present analogous structure to the differential equation for beam transmittance:

$$\frac{dT_r}{dr} = -\alpha T_r$$

(cf. (2) of Sec. 3.11). Thus, we are led to form the difference quotient:

$$\frac{\mathcal{T}(r+s) - \mathcal{T}(r)}{s}$$

and obtain its limit as s goes to zero. In preparation for this, we write:

$$\frac{\mathcal{T}(r+s) - \mathcal{T}(r)}{s} = \frac{\mathcal{T}(s) - I}{s}\,\mathcal{T}(r)$$

$$= -\left(A + \varepsilon(s)\right)\mathcal{T}(r)$$

which follows on use of (12) of Sec. 7.8 and (4) above. Therefore we have:

$$\boxed{\frac{d\mathcal{T}(r)}{dr} = -A\mathcal{T}(r)} \tag{6}$$

wherein we have written

$$\text{"}\frac{d\mathcal{T}(r)}{dr}\text{"} \quad \text{for} \quad \lim_{s\to 0} \frac{\mathcal{T}(r+s) - \mathcal{T}(r)}{s}$$

We pause now to restate the purpose of the present discussion. We wish to find the continuous counterparts to the module equations (14) of Sec. 7.8. The first step, just completed, makes clear the structure of $\mathcal{T}(s)$ when s is allowed to approach zero. This structure is shown in (5) and (6). The next step is to find the continuous counterpart to $\mathcal{T}^j(d)$ as d goes to zero but such that jd is some fixed depth z. In view of the analogy between $\mathcal{T}_r$ and T_r which has guided the developments so far, it is clear that this next step should be equivalent to finding the operator version of:

$$T_r = \exp\{-\alpha r\} \quad .$$

This observation requires us to find the operator analog of $\exp\{-\alpha r\}$. At this point we recall that the Maclaurin series development of $\exp\{-\alpha r\}$ shows promise of being extendable to the operator context, especially since we have the basic derivative formula (6) to work from. Therefore by means of (6), taking all the integral derivatives of $\mathcal{T}(r)$ in succession, we obtain:

$$\frac{d^2\mathcal{T}(r)}{dr^2} = -\frac{d}{dr}\left(A\,\mathcal{T}(r)\right) = -A\,\frac{d\mathcal{T}(r)}{dr} = A^2\,\mathcal{T}(r)$$

$$\frac{d^3\mathcal{T}(r)}{dr^3} = \frac{d}{dr}\left(A^2\,\mathcal{T}(r)\right) = A^2\,\frac{d\mathcal{T}(r)}{dr} = -A^3\,\mathcal{T}(r)$$

and in general:

$$\frac{d^j\mathcal{T}(r)}{dr^j} = (-1)^j A^j\,\mathcal{T}(r) \quad . \tag{7}$$

Using the identity property of $\mathcal{T}(r)$, namely that $\mathcal{T}(0) = I$, (7) yields:

$$\left.\frac{d^j\mathcal{T}(r)}{dr^j}\right|_{r=0} = (-1)^j A^j \quad .$$

Following through on the Maclaurin series analogy we then write:

$$\text{"}\exp\{-Ar\}\text{"} \quad \text{for} \quad \sum_{j=0}^{\infty} \frac{(-1)^j (Ar)^j}{j!} \quad . \tag{8}$$

This definition makes sense from a strictly operational point of view. For we can perform, at least in principle, the iterations of the operator A to find A^j for every integer j. Furthermore we can multiply A^j by the real number $(-1)^j r^j/j!$

in each case; and we can add together any finite number n of such combinations to end up with

$$\sum_{j=0}^{n} \frac{(-1)^j (Ar)^j}{j!}$$

as a well-defined operator. Granted all this, it appears that, at least on a numerical or empirical level, the exponential characterization of $\mathcal{T}(r)$ is settled. The mathematical reader, however, will wish to dwell on the convergence problem entailed in the definition (8). Such considerations are quite simple and are readily characterized in terms of the radiometric norm (Ex. 15 of Sec. 2.11) which supplies the necessary machinery in algebraic radiative transfer theory to handle problems of convergence of operator sequences. Such a digression is not pertinent in the present discussion, and we can safely pass it by without serious effect on the remainder of our study. Interested readers wishing to study such matters in more detail are referred to Ref. [110], a book devoted almost exclusively to the extension of the ideas, inherent in $e^{-\alpha r}$, to their most general settings.

With the definition (8) and on the basis of the analogy of (8) with the scalar (numerical) context, we see that:

$$\boxed{\mathcal{T}(r) = \exp\{-Ar\}} \quad . \tag{9}$$

With this representation, $\mathcal{T}(r)$ exhibits directly and succinctly all its important properties ((12) of Sec. 7.8, and (3), (5), (6) above). We now may write (13) of Sec. 7.8 as:

$$\boxed{\begin{aligned} N_+(y) &= N_-(0) \exp\{-Ay\}R_\infty \\ N_-(y) &= N_-(0) \exp\{-Ay\} \end{aligned}} \tag{10}$$

Equations (9), (10) are the requisite semigroup equations for $\mathcal{T}(r)$ as they are applied to the determination of $N_\pm(y)$ in an infinitely deep homogeneous plane-parallel medium. The operator A is called the *infinitesimal generator* of the semigroup formed by the transmittance operators $\mathcal{T}(r)$. (The *semigroup* structure stems primarily from the property (12) of Sec. 7.8.) Readers acquainted with the theory of stochastic processes (in continuous time, say) will observe via (6) or (9) that a radiative transfer process in a deep homogeneous optical medium may be viewed as a Markov process which evolves continuously with depth in that medium.

The Infinitesimal Generator A

One final point remains in the preparation of the system (10) for actual numerical application, or in further theoretical work, and that is in determining the explicit dependence of A on the inherent optical properties of the medium. Thus we have the task of finding how A depends on the volume attenuation and scattering functions α and σ.

The key to the required answer rests in the equation (37) of Sec. 7.5. For, setting a = 0 and b = ∞, in that equation, we have:

$$\frac{\partial \mathcal{T}(0,y,a)}{\partial y} = \mathcal{R}(0,y,\infty)\rho(y) + \mathcal{T}(0,y,\infty)\tau(y)$$

or, in the contracted notation presently in use, this is:

$$\frac{d\mathcal{T}(y)}{dy} = \mathcal{R}(0,y,\infty)\rho + \mathcal{T}(y)\tau \quad .$$

By (5) of Sec. 7.8 this can be reduced to:

$$\frac{d\mathcal{T}(y)}{dy} = \mathcal{T}(y)[R_\infty\rho + \tau] \quad . \qquad (11)$$

Comparison of (11) with (6) leads us to the required representation:

$$\boxed{A = -(\tau + R_\infty\rho)} \quad . \qquad (12)$$

Equation (9) can now be written as:

$$\mathcal{T}(r) = \exp \ (\tau + R_\infty\rho)r \quad . \qquad (13)$$

Thus, the infinitesimal generator A of the semigroup $\{\mathcal{T}(r)\}$ of complete transmittance operators is characterizable as the sum of two operators: τ the local transmittance operator, and $R_\infty\rho$, the product of the (global) reflectance operator R_∞ of $X(0,\infty)$ and the local reflectance operator ρ. Observe that, by the homogeneity of $X(0,\infty)$, all three operators comprising A are independent of depth in $X(0,\infty)$, so that A has the same property. Further, observe that:

$$\mathcal{T}(r)A = A\mathcal{T}(r) \quad ,$$

by virtue of the discussion leading to (6), and in particular the semigroup relation (12) of Sec. 7.8. The approximate matricial form of A is readily forthcoming from those of τ, ρ and R_∞ as given by the discussions in Sec. 7.7 and 7.8. In particular, we use ρ_+ and τ_- in place of ρ and τ.

The role of the infinitesimal generator A as compared to that of the volume attenuation function α, as these roles are viewed from the theory of radiative transfer as a whole, is characterizable succinctly as follows: A is to N as α is to N^o. That is, A is the logarithmic derivative of downwelling observable radiance distributions in a plane-parallel medium, while α is the logarithmic derivative of directly transmitted (i.e., residual) radiance along a path. Putting it still another way, A is the 'volume attenuation function' for the *natural* (undecomposed) light field in deep homogeneous media such as the seas, lakes, and optically dense atmospheric media. This may be seen by returning to (13) of Sec. 7.8 and taking the derivative of $N_-(y)$ with respect to y. Thus:

$$\frac{dN_-(y)}{dy} = N_-(0)\,\frac{d\mathcal{T}(y)}{dy}$$

$$= N_-(0)\{-A\,\mathcal{T}(y)\} = N_-(0)\,\mathcal{T}(y)(-A)$$

$$= -N_-(y)A \quad .$$

When we apply the derivative operator d/dy to the upwelling radiance distributions as given in (13) of Sec. 7.8, we have:

$$\frac{dN_+(y)}{dy} = N_-(0)\,\frac{d\mathcal{T}(y)}{dy}\,R_\infty = N_-(0)\{-A\,\mathcal{T}(y)\}R_\infty$$

$$= -N_-(y)AR_\infty \quad .$$

This representation of the depth rate of change of $N_+(y)$, while not as direct as that for $N_-(y)$, still shows that the logarithmic depth rate of change of $N_+(y)$ is essentially A. Since commutativity of A and R_∞ need not generally hold, we cannot generally place "R_∞" next to "$N_-(y)$" in the preceding equation to get $N_+(y)$ as a result. This asymmetry in the local behaviour of $N_+(y)$ and $N_-(y)$ is a slight and inessential notational irregularity in the otherwise conceptually pleasing and powerful formulations of the method of modules and the method of semigroups. A search for a more symmetric treatment of the depth rates of change of $N_+(y)$ and $N_-(y)$ leads to the method of groups to be considered in the following section.

7.10 The Method of Groups for Deep Homogeneous Media

Once the flush of discovery of the semigroup equations (10) of Sec. 7.9 has passed and the critical eye runs over their asymmetric forms, one is moved to search for a new set of equations which incorporates both the conceptual and computational power of that set with a more pleasing symmetry of form. In this section we embark on such a search and are rewarded with a set of equations which fulfills all these requirements and more. The additional dividend is a novel perspective of Chandrasekhar's classical method of solution of the transfer equation in plane-parallel homogeneous media [43] from the heights of group theory and the modern theory of differential equations. As a result, we can view Chandrasekhar's classical method as but one of a large family of possible solution procedures unified from the viewpoint of invariant imbedding theory. This insight then unites with that encountered in Secs. 6.1-6.4, in which novel views of the spherical harmonic method were developed, to give an overview of all the classical solution techniques in radiative and neutron transport theory, and, indeed, all linear transport theories.

The setting for the present section is once again (as in Sections 7.7-7.9) an infinitely deep homogeneous source-free plane-parallel optical medium $X(a,b)$ with $a = 0$, $b = \infty$. $X(0,\infty)$ is irradiated at each point of its upper boundary by a given arbitrary incident radiance distribution $N_-(0)$, and has an arbitrary volume scattering function σ, and scattering-attenuation ratio s/α.

The Return of the Group $\Gamma_2(0,\infty)$

The natural candidate for the task of symmetrizing the semigroup relations (10) of Sec. 7.9 is the group $\Gamma_2(0,\infty)$ introduced in its general form in Sec. 3.7 (see, in particular (79)-(82) of that section) and studied at some length in Secs. 7.4 and 7.5. Toward this end, we direct some attention to the specific form of $\Gamma_2(0,\infty)$.

Now that we have a particularly simple physical setting, the structure of $\Gamma_2(0,\infty)$ takes on some rather interesting properties. For example, the homogeneity of $X(0,\infty)$ makes each member $\mathcal{M}(x,z)$ of $\Gamma_2(0,\infty)$ depend only on the difference $z-x$, where x and z are any two depths in $X(0,\infty)$. This fact may readily be seen by an inspection of equations (19)-(26) of Sec. 7.4. Consider, for example (19) of Sec. 7.4. Since $X(0,\infty)$ is homogeneous and isotropic our findings (of Sec. 7.7, e.g.) show that $T(x,z)$ and $R(x,z)$ depend only on the absolute difference $|x-z|$. Further, from (23) of Sec. 7.4 we see that $\mathcal{M}_{++}(x,z)$ is not generally the same as $\mathcal{M}_{++}(z,x)$, but still $\mathcal{M}_{++}(x,z)$ depends only on the magnitude of the difference $z-x$. Hence the operator matrix $\mathcal{M}(x,z)$ depends only on $z-x$ for which we shall write "s" for brevity, so that $\mathcal{M}(x,z)$ is written as "$\mathcal{M}(s)$" whenever $s = z-x$. The general group closure property of $\Gamma_2(0,\infty)$, namely:

$$\mathcal{M}(x,y)\mathcal{M}(y,z) = \mathcal{M}(x,z)$$

now takes the form:

$$\boxed{\mathcal{M}(r+s) = \mathcal{M}(r)\mathcal{M}(s)} \tag{1}$$

This should be compared with (12) of Sec. 7.8. We see that there is an important difference in the range of parameters s in $\mathcal{M}(s)$ and those of $\mathcal{T}(s)$. Whereas $s \geq 0$ in (12) of Sec. 7.8, we have $-\infty \leq s \leq \infty$ for s in (1). We may summarize these differences as follows: *The set $\Gamma_2(0,\infty)$ forms a group which is isomorphic to the additive group of real numbers.* Thus, to each pair of real numbers r,s there correspond operators $\mathcal{M}(r)$, $\mathcal{M}(s)$ of $\Gamma_2(0,\infty)$, and to the sum $r+s$ corresponds the operator $\mathcal{M}(r+s)$, such that (1) holds. On the other hand *the set $\{\mathcal{T}(r)\}$ of complete transmittance operators discussed in Sec. 7.9 forms a semigroup which is isomorphic to the additive semigroup of non negative real numbers.* Thus to each pair r,s

of non negative real numbers, there correspond operators $\mathcal{T}(r)$, $\mathcal{T}(s)$ of $\{\mathcal{T}(r)\}$ and to the sum r+s corresponds $\mathcal{T}(r+s)$ such that (12) of Sec. 7.9 holds. *In this way, by means of (1), we can view the theory of radiative transfer in homogeneous infinite plane-parallel media as an instance of the theory of continuous groups on the real line.* (That all of radiative transfer theory is essentially attainable via $\Gamma_2(a,b)$ was demonstrated in Sec. 7.4. See also the remarks leading to (19) of Sec. 7.3.)

The Infinitesimal Generator of $\Gamma_2(0,\infty)$

The concluding insight arrived at in the paragraph just above can be put into quite concrete terms. One way of putting it is to say that, conceptually, the theory of determining the radiance distribution N(y) at depth y in $X(0,\infty)$ is as simple as determining the reduced radiance N_r^0 of a beam a distance r from the source, for both quantities are governed by the exponential law. To see this in the case of N(y), recall the operator forms of the equation of transfer (9) of Sec. 7.1:

$$\frac{dN(y)}{dy} = N(y)\mathcal{K}(y) \qquad (2)$$

where $\mathcal{K}(y)$ is defined in (7) of Sec. 7.1, and "N(y) as usual denotes $(N_+(y), N_-(y))$". Next recall the functional equation governing $\mathcal{M}(x,y)$ as given in (1) of Sec. 7.5:

$$\frac{d\mathcal{M}(x,y)}{dy} = \mathcal{M}(x,y)\,\mathcal{K}(y) \qquad (3)$$

where $\mathcal{K}(y)$ *is the same operator as in (2).* In view of the homogeneity properties of $X(0,\infty)$ we can write (3) simply as:

$$\frac{d\mathcal{M}(y)}{dy} = \mathcal{M}(y)\,\mathcal{K} \qquad (4)$$

with the initial condition:

$$\mathcal{M}(0) = I \qquad (5)$$

and where:

$$\mathcal{K} = \begin{bmatrix} -\tau & \rho \\ -\rho & \tau \end{bmatrix} \quad . \qquad (6)$$

Here ρ and τ are the local reflectance and transmittance operators for $X(0,\infty)$. They are independent of depth y in $X(0,\infty)$. $\mathcal{K}$ is the *infinitesimal generator* of the group $\Gamma_2(0,\infty)$.

The Exponential Representation of $\mathcal{M}(y)$ and N(y)

By following the same motivations as those leading to (8) of Sec. 7.9, we write:

$$\text{"exp } \{\mathcal{X} y\}\text{"} \quad \text{for} \quad \sum_{j=0}^{\infty} \frac{(\mathcal{X} y)^j}{j!} \tag{7}$$

in which $-\infty \le y \le \infty$. This operator is a function of y and satisfies the same differential equation as $\mathcal{M}(y)$ in (4). Further, $\exp\{\mathcal{X} 0\} = I = \mathcal{M}(0)$. Hence:

$$\boxed{\mathcal{M}(y) = \exp\{\mathcal{X} y\}} \tag{8}$$

The exponential representation of N(y) at any depth y in $X(0,\infty)$ follows immediately from (2) using the same reasoning which yielded (8); or one may use the fact that:

$$N(y) = N(0)\mathcal{M}(y) \quad ,$$

which with (8) implies:

$$\boxed{N(y) = N(0) \exp\{\mathcal{X} y\}} \quad . \tag{9}$$

From this, we also have:

$$N(z) = N(y) \exp\{\mathcal{X}(z-y)\} \tag{10}$$

for every pair z,y of depth in $X(0,\infty)$. Equation (9) is the requisite symmetric rendition of (13) of Sec. 7.8.

The Exponential Representation of $\mathcal{A}(y)$

In Sec. 7.5 it was noted how close the connection was between the operators $\mathcal{M}(x,y)$ and the pair of complete operators $\mathcal{R}(a,x,b)$, $\mathcal{T}(a,x,b)$. The basis for this connection is summarized in (38) of Sec. 7.5. We pause to explore this similarity in the light of the present developments.

The similarity between equation (38) of Sec. 7.5 and (3) above shows that the operator $\mathcal{A}(y)$ has the same depth behavior as $\mathcal{M}(y)$, though their initial values differ. Thus, (38) of Sec. 7.5, adapted to $X(0,\infty)$, becomes:

$$\frac{d\mathcal{A}(y)}{dy} = \mathcal{A}(y)\mathcal{X}(y)$$

in which:

$$\mathcal{A}(y) = [\mathcal{R}(y), \mathcal{T}(y)]$$

where we have written:

$$\text{"}\mathcal{R}(y)\text{"} \quad \text{for} \quad \mathcal{R}(0,y,\infty)$$

and

$$\text{"}\mathcal{T}(y)\text{"} \quad \text{for} \quad \mathcal{T}(0,y,\infty) \quad .$$

Hence, while $\mathcal{A}(y)$ and $\mathcal{M}(y)$ satisfy the same differential equation, the initial condition for $\mathcal{A}(y)$ is:

$$\begin{aligned}\mathcal{A}(0) &= (\mathcal{R}(0), \mathcal{T}(0)) \\ &= (R_\infty, I)\end{aligned}$$

which follows from (40) of Sec. 7.5. Therefore, analogously to (8) we have:

$$\boxed{\mathcal{A}(y) = \mathcal{A}(0) \exp\{\mathcal{K}y\}} \quad . \qquad (11)$$

It is interesting to note the effect of the presence of "$\mathcal{A}(0)$" in (11) on the multiplication law of the operator $\mathcal{A}(y)$. It turns out that the set $\{\mathcal{A}(y)\}$ does not form a semigroup under ordinary operator composition. Indeed, from (11), used three times as follows:

$$\begin{aligned}\mathcal{A}(r) &= \mathcal{A}(0) \exp\{\mathcal{K} r\} \\ \mathcal{A}(s) &= \mathcal{A}(0) \exp\{\mathcal{K} s\} \\ \mathcal{A}(r+s) &= \mathcal{A}(0) \exp\{\mathcal{K}(r+s)\}\end{aligned}$$

we find that, at least formally:

$$\boxed{\mathcal{A}(r+s) = \mathcal{A}(r)\mathcal{A}^{-1}(0)\mathcal{A}(s)} \qquad (12)$$

This shows that we have gained the symmetry of (9) at the expense of the simple semigroup property for the set $\{\mathcal{A}(y)\}$. However, the loss is not essential. For, by defining the following star product of members of $\{\mathcal{A}(y)\}$, we establish a group structure for $\{\mathcal{A}(y)\}$. In view of the semigroup properties (52) and (53) of Sec. 3.7, let us write:

$$\text{"}\mathcal{A}(r) * \mathcal{A}(s)\text{"} \quad \text{for} \quad (\mathcal{T}(r)\mathcal{R}(s), \mathcal{T}(r)\mathcal{T}(s)) . \qquad (13)$$

Then it follows immediately that:

$$\boxed{\mathcal{A}(r) * \mathcal{A}(s) = \mathcal{A}(r+s)} \qquad (14)$$

The set $\{\mathcal{A}(y)\}$, with the preceding star product defined for its elements in the manner shown in (13), becomes a group isomorphic to $\Gamma_2(0,\infty)$, once the definitions of $\mathcal{T}(r)$ and $\mathcal{R}(r)$ are extended to negative values of r. This can be done directly through (11) by simply computing $\exp\{\mathcal{K}y\}$ for negative values of y. Further, the semigroup relation

$$\mathcal{R}(r+s) = \mathcal{T}(r)\mathcal{R}(s) \tag{15}$$

which is a special case of (53) of Sec. 3.7, is formally extended, for this purpose, to the domain of negative arguments. This extension can be rigorously included in the theory deducible from the interaction principle by adapting the extension of the group $\Gamma_3(a,b)$ (now for the special case $a = 0$, $b = \infty$) suggested in (44) of Sec. 7.4.

Numerical Procedures for N(y): The Exponential Technique

Equation (9) for the radiance field N(y), as already noted, is the primary goal for the present section. In its symmetric form rests the solution of the problem of the penetration of light into the sea, atmosphere, and other plane-parallel media. There are several ways of coaxing numbers and general information from its terse mathematical form, and we shall study such ways in this and the following paragraphs. Each technique to be considered is based on a preliminary reduction of (9) to an approximating matrix statement. This reduction is quite analogous to those developed in Sec. 7.7 for the differential equations of R and T. Hence we may pass through this preliminary reduction stage with relatively little explanation.

The reductions center principally on the operators ρ and τ making up the exponent operator $\mathcal{K}$ in (9). The form of $\mathcal{K}$ is given by (6), and ρ and τ in turn are defined in (3) and (4) of Sec. 7.1. For the purposes of the present reduction we may drop references to the depth variable; however, the directional variable ξ in the integral form of ρ and τ, must be explicitly exhibited:

$$\rho = \frac{1}{|\xi\cdot\mathbf{k}|}\int_{\Xi_\pm}[\ ]\sigma(\xi';\xi)\,d\Omega(\xi') \tag{16}$$

$$\tau = \frac{1}{|\xi\cdot\mathbf{k}|}\int_{\Xi_\pm}[\ ]\sigma(\xi';\xi)\,d\Omega(\xi') - \frac{1}{|\xi\cdot\mathbf{k}|}\,\alpha \tag{17}$$

Thus, the homogeneity of $X(0,\infty)$ allows a convenient suppression of the depth variable y in α,ρ,σ, and τ. When ρ is applied for example to the upward radiance distribution $N_+(y)$, we use Ξ_+ in (16) along with ξ in Ξ_-. By means of the general partitions of Ξ_+ and Ξ_- established in (1), (2) of Sec. 7.7 we can replace ρ by an $n\times m$ matrix whose general element in the ith row and jth column is:

$$\frac{\sigma(\xi_i;\xi_j)}{\alpha} \cdot \frac{\Omega(B_i)}{\nu_i} \tag{18}$$

in which ξ_i is in B_i ($\subset \Xi_+$) and ξ_j is in A_j ($\subset \Xi_-$) and "ν_i" denotes $|\xi_i \cdot \mathbf{k}|$. $\Omega(B_i)$ is the solid angle content of B_i. We denote this matrix by "ρ_+". A similar matrix ρ_- can be manufactured such that it has elements of the form:

$$\frac{\sigma(\xi_i;\xi_j)}{\alpha} \cdot \frac{\Omega(A_i)}{\mu_i} \tag{19}$$

where ξ_i is in A_i ($\subset \Xi_-$) and ξ_j is in B_j ($\subset \Xi_+$), and where "μ_i" denotes $|\xi_i \cdot \mathbf{k}|$. Hence ρ_- has dimension m×n. Further, we use:

$$\frac{\sigma(\xi_i;\xi_j)}{\alpha} \cdot \frac{\Omega(A_i)}{\mu_i} - \frac{1}{\mu_i} \tag{20}$$

as the ij-th element in an m×m matrix denoted by "τ_-". An n×n matrix τ_+ is constructed in a similar manner for upward radiance, its ij-th element being:

$$\frac{\sigma(\xi_i;\xi_j)}{\alpha} \cdot \frac{\Omega(B_i)}{\nu_i} - \frac{1}{\nu_i} \quad . \tag{21}$$

The preceding mode of reducing the operators ρ and τ is the most simple and direct mode. Alternate modes of a more sophisticated type (such as those using various quadrature formulas for σ) are possible; however, the structure of the main formula (25) below is independent of the choice of such modes. Reassembling these matrices into one grand matrix $\mathcal{K}$ where we have written:

$$\text{"}\mathcal{K}\text{"} \quad \text{for} \quad \begin{bmatrix} -\tau_+ & \rho_+ \\ -\rho_- & \tau_- \end{bmatrix} \tag{22}$$

and writing:

$$\text{"}N_+(y)\text{"} \quad \text{for} \quad [N(y,\xi_1), \ldots ,N(y,\xi_n)] \tag{23}$$

when,

$$\xi_i \text{ in } B_i \quad ,$$

and writing:

$$\text{"}N_-(y)\text{"} \quad \text{for} \quad [N(y,\xi_1), \ldots ,N(y,\xi_m)] \tag{24}$$

when $\qquad \xi_i$ in A_i ,

the approximating counterpart to (9) is seen to be:

$$\boxed{N(y) = N(0) \exp \{\mathcal{X} y\}} \tag{25}$$

and which shall stand as our base of operations for the remainder of this section.

Clearly N(y) is an (m+n)-component vector and $\mathcal{X}$ is a square matrix of order (m+n). As noted above, the general form of (25) is invariant under the choice of mode of reduction of the operators ρ,τ and radiance functions $N_+(y)$, $N_-(y)$. Therefore what we have to say about (25) below will hold above and beyond the details of the reduction procedure leading from (9) to (25).

Equation (25) as it stands can form the basis of perhaps the simplest and most direct of all techniques of solution of radiative transfer problems in homogeneous plane-parallel media with stratified light fields. For by simply raising the matrix $\mathcal{X}$ to the first p integral powers and constructing the sum:

$$\sum_{j=0}^{p} \frac{\mathcal{X}^j y^j}{j!}$$

where p is, perhaps as small as 5 or 6, one obtains a reasonable approximation to exp $\{\mathcal{X} y\}$, so that when applied to N(0), we have:

$$N(0) \sum_{j=0}^{p} \frac{\mathcal{X}^j y^j}{j!}$$

as a correspondingly reasonable estimate of N(y). Observe that knowing N(0) means knowing all m+n components of N(0). Hence we can predict N(y) once N(0), the surface or boundary lighting conditions are known. More generally, in view of (10), N(z) *is computable whenever* N(x) *is known, where* z *and* x *are any two depths.* This most remarkable fact points up in sharp clear detail our rather general assertions about the "strong inner structure" of natural light fields discussed in Sec. 3.7 (cf. Ex. 7 of Sec. 3.7).

An alternate scheme to that just discussed is based on the matricial counterpart to (2):

$$\frac{dN(y)}{dy} = N(y)\,\mathcal{X}(y) \tag{26}$$

where $\mathcal{X}(y)$ may now depend on y (hence $X(0,\infty)$ may be non homogeneous but stratified). Thus we work with (26) directly and integrate that system of linear ordinary differential equations on a general purpose computer. The initial condition on N(y), namely N(0) is assumed known.

It is unlikely that any computation techniques could be simpler in concept or in execution than those based on (25) or (26) (or on the decomposed versions of (26)) using N in the manner just explained. This points up one of the earmarks of invariant imbedding techniques, i.e., the ability to replace some of the classical and somewhat numerically cumbersome eigenvalue techniques by relatively simple initial value or one-point boundary value techniques, and which may be handled generally by the tools of semi-group theory.

The Characteristic Representation of N(y)

In deep homogeneous media with stratified light fields, such as those we are studying in this section, the exponential law (25) for radiance distributions N(y) can be cast into a particularly instructive form using the Jordan canonical form of $\mathcal{K}$. The Jordan canonical form of a matrix is defined in most works on modern algebra, and in some texts on ordinary differential equations such as [47], and we therefore need not digress to discuss the details of its computation. However, we shall define the canonical form and discuss its physical interpretations in the radiative transfer context. Our purpose in casting $\mathcal{K}$ into its Jordan canonical form is two-fold. First, we shall be able thereby to fulfill our promise, made at the outset, to show the special place of Chandrasekhar's theory of solution of the equation of transfer within the general theory of solutions as given by the invariant imbedding and interaction principles of radiative transfer. Second, the *characteristic representation of* N(y), as we shall call the resultant equation obtained below, deepens our understanding of the exponential structure of light fields in natural optical media by showing explicitly the delicate interplay of the various streams of radiant flux as they penetrate the body of an extensive optical medium, each stream with a characteristic mode of decay. In particular, we shall be able to explicitly observe the eventual dominance of a characteristic radiance distribution at great depth within the medium, the shape of the characteristic distribution being determined solely by the volume scattering function σ of $X(0,\infty)$ and being independent of the directional structure of N(0), the radiance at the boundary of the medium. All of this knowledge is possible without explicitly solving the equation of transfer for $X(0,\infty)$, as we shall now see.

To begin, we recall from the theory of linear algebra that the Jordan canonical form of the $(m+n)\times(m+n)$ matrix $\mathcal{K}$ can be obtained by the construction of a suitable $(m+n)\times(m+n)$ invertible matrix P and performing the operation:

$$P^{-1}\mathcal{K}P \quad .$$

Let us denote this resultant matrix by "$\mathcal{J}$". It follows immediately that:

$$\mathcal{K} = P\mathcal{J}P^{-1}$$

and that:

$$\exp\{\mathcal{K}y\} = \exp\{P\mathcal{J}P^{-1}y\}$$
$$= P\exp\{\mathcal{J}y\}P^{-1} \quad .$$

The latter equality may be verified by using the definition of $\exp\{\mathcal{J}y\}$.

Now, the general gestalt of the Jordan canonical matrix is as follows:

$$\mathcal{J} = \begin{pmatrix} \mathcal{J}_0 & & & 0 \\ & \mathcal{J}_1 & & \\ & & \ddots & \\ 0 & & & \mathcal{J}_s \end{pmatrix}$$

where:

$$\mathcal{J}_0 = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_q \end{bmatrix}$$

and where:

$$\mathcal{J}_i = \lambda_{q+i}\begin{bmatrix} 1 & \cdots & \cdots & \cdots \\ & 1 & & \vdots \\ & & 1 & \vdots \\ & & & \ddots & \vdots \\ & 0 & & & 1 & \vdots \\ & & & & & 1 \end{bmatrix}$$

for $i = 1, \ldots, s$ and, where $\mathcal{J}_i$ is an $r_i \times r_i$ matrix, so that $\left(\sum_{i=1}^{s} r_i\right) + q = m + n$. $\mathcal{J}_i$ is constructed so that all elements below the main diagonal (which has all 1^s displayed in it) are zero. Further, the elements in the upper jth diagonal, counting the main diagonal as the zero-th, are all the same, and of the common form: $y^j/j!$, $j = 0,\ldots, r_i-1$. These upper main diagonals are marked off by the inclined straight lines in the matrix symbol. The numbers λ_j, $j = 1,\ldots,q,q+1,\ldots,q+s$ are the distinct characteristic (or eigen) values associated with $\mathcal{K}$. The λ_j from $j = 1$ to $j = q$ have multiplicity 1, those of the form λ_{q+i}, $0 < i \leq s$, have multiplicity r_i. Hence, altogether, counting multiplicities, there are m+n characteristic values λ_i, as expected. So much for the abstract algebra of Jordan canonical forms.

Let us turn now to the particular matrix at hand, namely $\mathcal{K}$, and attempt to block out the salient structure of its canonical Jordan form. Imagine the operator $\mathcal{K}$, as given in (6) for the present context, to be replaced by its matricial approximant. The main outlines of (6) will persist and we will have, according to (22):

$$\mathcal{K} = \begin{bmatrix} -\tau_+ & \rho_+ \\ \\ -\rho_- & \tau_- \end{bmatrix}$$

For the purpose at hand, namely to deduce the special form of Chandrasekhar's equations, we adopt a special partition of Ξ, as follows: Let Ξ_- be partitioned in an arbitrary manner. Then reflect Ξ_-, in its partitioned form, in a horizontal plane. The result is a partitioning of Ξ_+ which will be a mirror image of that of Ξ_-. In particular, we number the partition elements A_i, B_i such that A_i and B_i are mirror images of one another. The effect of this type of partitioning on the Jordan canonical form of the resultant matrix $\mathcal{K}$ can be seen by examining typical entries of $\mathcal{K}$ as given in (18)-(21). Thus we find that, under the mirror image partition of Ξ:

$$\tau_+ = \tau_-$$

$$\rho_+ = \rho_-$$

and $\mathcal{K}$ becomes:

$$\mathcal{K} = \begin{bmatrix} -\tau & \rho \\ \\ -\rho & \tau \end{bmatrix}$$

where each indicated block matrix is an $m \times m$ matrix (since $m = n$, by virtue of the mirror partition). Now it is an elementary fact of matrix theory that a matrix such as $\mathcal{K}$, in its newly obtained form, has eigenvalues which come in signed pairs. Thus, if λ is an eigenvalue of $\mathcal{K}$, then so is $-\lambda$. For example, consider the following 2×2 matrix made up of the numbers a,b:

$$\begin{bmatrix} -a & b \\ -b & a \end{bmatrix} \quad .$$

The characteristic equation for this matrix is:

$$\det \begin{pmatrix} -a-\lambda & b \\ \\ -b & a-\lambda \end{pmatrix} = 0 \quad .$$

The λ's which satisfy this equation are the required characteristic values. The preceding equation simplifies to:

$$-(a+\lambda)(a-\lambda) + b^2 = 0$$

so that, λ is required to be: λ_+ or λ_- where

$$\lambda_\pm = \pm(a^2-b^2) \quad ,$$

that is $\lambda_+ = -\lambda_-$. Furthermore, it may be shown, on physical grounds, that the component matrices $\mathcal{J}_i$ of $\mathcal{J}$ (other than $\mathcal{J}_o$) do not occur in the case of $\mathcal{K}$. Briefly, for $\mathcal{J}_i$, $i > 0$, to appear in $\mathcal{J}$ it is necessary that there exist components of $\mathbf{N}(y)$ such that they can have scattering orders of *at most finite order* r_i. No component of $\mathbf{N}(y)$ has this property, so that the $\mathcal{J}_i$ do not occur in $\mathcal{J}$. Hence the Jordan canonical form of $\mathcal{K}$ must then be such that $s = 0$, i.e., $\mathcal{J}$ consists only of $\mathcal{J}_o$. The resultant form of (25) is then quite simple:

$$\begin{aligned}\mathbf{N}(y) &= \mathbf{N}(0) \exp \{\mathbf{P} \mathcal{J}_o \mathbf{P}^{-1} y\} \\ &= \mathbf{N}(0)\mathbf{P} \exp \{\mathcal{J}_o y\}\mathbf{P}^{-1} \quad .\end{aligned}$$

It is easy to see that:

$$\exp \{\mathcal{J}_o y\} = \begin{pmatrix} e^{\lambda_1 y} & & & \\ & e^{\lambda_2 y} & & \\ & & \ddots & \\ & & & e^{\lambda_{2m} y} \end{pmatrix} .$$

We define the *characteristic radiance vector* by writing:

$$\text{"}\tilde{\mathbf{N}}(y)\text{"} \quad \text{for} \quad \mathbf{N}(y)\mathbf{P} \quad .$$

Then (25) can be written:

$$\boxed{\tilde{\mathbf{N}}(y) = \tilde{\mathbf{N}}(0) \begin{pmatrix} e^{\lambda_1 y} & & & \\ & e^{\lambda_2 y} & & 0 \\ & & \ddots & \\ & 0 & & e^{\lambda_{2m} y} \end{pmatrix}} \quad . \tag{27}$$

Equation (27) is the requisite *equation for the characteristic radiance* vector $\mathbf{N}(y)$. Observe that if $\tilde{N}_j(y)$ is the jth component of $\tilde{\mathbf{N}}(y)$, then we have:

$$\boxed{\tilde{N}_j(y) = \tilde{N}_j(0)\, e^{\lambda_j y}} \quad . \tag{28}$$

Hence each component $\tilde{N}_j(y)$ of the characteristic radiance vector has a specific rate of growth (if $\lambda_j > 0$) or decay (if $\lambda_j < 0$). In infinitely deep media such as $X(0,\infty)$, wherein there are no internal sources and the only incident radiance is at the upper boundary, the components $\tilde{N}_j(0)$ associated with the positive valued eigenvalues are set to zero. The eigenvalues and components can be renumbered so that $\tilde{N}_j(0) = 0$ for $m+1 \le j \le 2m$. To see the effect of this on the physical radiance vectors $\mathbf{N}(y)$, let the elements of $\mathbf{P}$ be of the form a_{ij}, and those of $\mathbf{P}^{-1}$ be of the form b_{ij}, then from (27):

$$\tilde{N}(y) = (\tilde{N}_1(0)e^{\lambda_1 y}, \tilde{N}_2(0)e^{\lambda_2 y}, \ldots, \tilde{N}_m(0)e^{\lambda_m y}, 0, 0, \ldots, 0)$$

$$= N(y)P \quad .$$

Hence:

$$N(y) = (\tilde{N}_1(0)e^{\lambda_1 y}, \ldots, \tilde{N}_m(0)e^{\lambda_m y}, 0, \ldots, 0)P^{-1} \quad .$$

Therefore:

$$N_j(y) = \sum_{i=1}^{m} \tilde{N}_i(0) \cdot b_{ij} e^{\lambda_i y}$$

which holds for $j = 1,\ldots,2m$. We observe from the definition of $\tilde{N}(0)$ that:

$$\tilde{N}_i(0) = \sum_{k=1}^{2m} N_k(0)\ a_{ki} \quad .$$

Hence:

$$\boxed{\begin{array}{l} N_j(y) = \displaystyle\sum_{i=1}^{m} \left(\sum_{k=1}^{2m} N_k(0)\ a_{ki} \right) b_{ij}\ e^{\lambda_i y} \\ j = 1,\ldots,2m \end{array}} \quad . \qquad (29)$$

This is the desired *characteristic representation* of $N(y)$. Each λ_i is non positive, i.e., $\lambda_i \leq 0$ for $i = 1,\ldots,m$. Observe that each of the 2m quantities $N_j(y)$ is completely determinable, knowing the 2m quantities $N_k(0)$, the entries a_{ki} and b_{ij} of the matrices P and P^{-1}, and of course the m eigenvalues λ_i. By retracing the steps leading to (29) and assuming $X(0,\infty)$ to be replaced by a finitely deep homogeneous medium $X(0,d)$, $d < \infty$, we see that (29) changes only slightly: the upper limit of the i-sum becomes 2m and the non negative eigenvalues λ_i, $m+1 \leq i \leq 2m$ can enter the representation. Equation (29) or its counterpart for $X(0,d)$ is representative of the general form of Chandrasekhar's equations in his classical work [43]. The salient difference between them rests in the manner of representing N and σ over Ξ, thereby fixing the associated values of a_{ki}, b_{ij} and λ_i. Chandrasekhar uses Gauss' method of representing the N and σ functions by Legendre polynomials, whereas the present method appeals directly to the observable partition of the radiance function as given in (1), (2) of Sec. 7.7 and (23), (24). In this way we have arrived at the first goal of the present discussion, namely, the illustration of the place of Chandrasekhar's mode of solution of the equation of transfer in the general scheme of radiative transfer theory, as seen from the invariant imbedding point of view.

Asymptotic Property of N(y)

The final topic for discussion in this section is the matter of the asymptotic property of radiance distributions in deep homogeneous media. The property states that the shape of the radiance distribution $N(y,\cdot)$ approaches a limit as $y\to\infty$ in $X(0,\infty)$, and that this limit is determined solely by the structure of the volume scattering function σ on $X(0,\infty)$; and so, in particular, this limiting form of $N(y,\cdot)$ is independent of the radiance distriubtions at the surface of $X(0,\infty)$. We shall discuss this matter in detail in Chapter 13. However, there exists a simple instructive proof of the asymptotic radiance property using the general system of equations (29), i.e., the characteristic representation of $N(y)$, and while the momentum of the present discussion is still high, we shall give a demonstration of the asymptotic radiance property using (29) as a base.

Our present goal, therefore, is to show that the 2m-component vector $\mathbf{N}(y)$, whose jth component is given by (29), approaches a 2m-component vector $\hat{\mathbf{N}}(\infty)$ as a limit, that $\hat{\mathbf{N}}(\infty)$ is determined only by σ, and that $\hat{\mathbf{N}}(\infty)$ is independent of $\mathbf{N}(0)$. Now, the first thing to notice is that the m numbers λ_i are, in real media, all negative, so that $\mathbf{N}(y)$ generally goes to the zero vector $\mathbf{0}$ (i.e., the 2m-component vector with all components zero). This, of course is not the vector $\hat{\mathbf{N}}(\infty)$ we are seeking. The decrease in size of $\mathbf{N}(y)$ as $y\to\infty$ is distracting as one seeks its asymptotic shape, and this decrease can be erased by normalizing $\mathbf{N}(y)$ with respect to some factor which decreases to zero with y at the same rate as $\mathbf{N}(y)$. The graphical interpretation of this normalization is quite simple: the radiance distribution at each depth y is magnified in size so that one of the radiance components, say that representing vertically downward radiance, is of unit magnitude. Then all other components arrange themselves in size relative to this unit component. If $\mathbf{N}(y)$, so plotted, approaches a fixed vector, as $y\to\infty$, then we say that the limit $\hat{\mathbf{N}}(\infty)$ exists.

In the present case the 'normalization factor' may conveniently be chosen as e^{ky} where k is the smallest of the numbers $-\lambda_i$, $i = 1,\ldots,m$. Specifically, we write; *ad hoc*:

$$\text{"}k\text{"} \quad \text{for} \quad \min\{-\lambda_1,-\lambda_2,\ldots,-\lambda_m\} \quad .$$

This implies that e^{-ky} goes to zero with the least speed of all the factors $e^{\lambda_i y}$. In particular $e^{(\lambda_i+k)y}$ goes to 0 as y goes to ∞ for every i, except for when $\lambda_i = -k$. To be specific suppose $\lambda_\ell = -k$. Armed with this factor, we multiply each side of (29) by e^{ky} and let $y\to\infty$:

$$\lim_{y\to\infty} N_j(y)\, e^{ky} = \left(\sum_{k=1}^{2m} N_k(0)\, a_{k\ell}\right) b_{\ell j} \tag{30}$$

for $j = 1,\ldots, 2m$.

Let us write "a_l" for the th column of P, and "b^l" for the lth row of P^{-1}. Then the system of 2m limits (29) can be written:

$$\boxed{\hat{N}(\infty) = (N(0)\cdot a_l)b^l} \tag{31}$$

where in turn we have written:

$$"\hat{N}(\infty)" \quad \text{for} \quad (\hat{N}_1(\infty), \ldots, \hat{N}_{2m}(\infty))$$

and:

$$"\hat{N}_j(\infty)" \quad \text{for} \quad \lim_{y\to\infty} N_j(y)e^{ky} \quad .$$

Equation (31) shows clearly that the directional structure of $\hat{N}(\infty)$ is simply that of the lth row of P^{-1}. The lth row of P^{-1} is determined solely by the matrices ρ and τ which are manufactured from σ. Observe that the directional structure of $N(0)$ is wiped out by the taking of the dot product of $N(0)$ and the lth column of P. Hence the asymptotic directional structure of N(y) can be so determined solely by computing P^{-1}, and this is independent of $N(0)$. Looking back on the trail we have travelled, we recall that P is the matrix which maps $\mathcal{K}$ into its Jordan canonical form. Thus we can find $\hat{N}(\infty)$ by purely algebraic operations on $\mathcal{K}$ which, as we have seen, is the infinitesimal generator of the group $\Gamma_2(0,\infty)$ of invariant imbedding operators associated with the medium $X(0,\infty)$.

Asymptotic Properties of Polarized Radiance Fields

We conclude the discussion of the characteristic form of the radiance solution by noting that the techniques just used for the unpolarized context can equally well be applied to polarized radiance distributions. This means, in particular, that the *theoretical* questions of the asymptotic properties of polarized radiance fields raised in Sec. 4.6 and still earlier in Chapter 1 can be fully resolved using the preceding technique. Equation (31), as it stands, has the gestalt of the corresponding equation for polarized radiance, differing from the polarized version only in the dimensions of the vectors and matrices involved. This difference is precisely determinable: all vectors in the unpolarized context go over into the polarized context with a four-fold increase in components, and all matrices go over with a corresponding four-fold increase in their linear dimensions. However, beyond these quantitative differences, the two theories of polarized and unpolarized radiance distributions are algebraically alike. (See, e.g., Section 114 of Ref. [251].) Some experimental work on the asymptotic polarized light field has been done by Herman and Lenoble [107]. Otherwise, there exists at present very little experimental study of the asymptotic polarized light field.

7.11 Method of Groups for General Optical Media

The various methods of solution of the equation of transfer, such as the method of modules (Sec. 7.8), the method of semigroups (Sec. 7.9), and the method of groups in the preceding section hold within them a common core which, if extracted, can guide the construction of a method of solution of the radiance field in arbitrary optical media. This section is devoted to the isolation of the common conceptual kernel of those methods and to a brief exposition of the general method of solution it suggests.

Analysis of the Group Method: Initial Data

We begin with a recapitulation of the ground-forms for the two basic methods. The semigroup method rests on the semigroup relation (12) of Sec. 7.8 for the complete transmittance operator $\mathcal{T}(r)$ (i.e., $\mathcal{T}(x,z,\infty)$ where $r = |z-x|$). The fundamental equations for the light field in this method are given by the system (10) of Sec. 7.9 or the system (14) of Sec. 7.8, depending on whether the continuous variable y or the discrete variable $y = jd$ is used. The method of groups rests on (1) of Sec. 7.10 for the invariant imbedding operator $\mathcal{M}(r)$ (i.e., $\mathcal{M}(x,z)$ where $r = z-x$), which holds for all real numbers r and s. The equations (9) or (25) of Sec. 7.10 may be used to find the light field at any depth y in $X(0,\infty)$.

What are the basic data needed in the computational applications of each method? The data needed are: (a) α,σ throughout $X(0,\infty)$ and either (b): $N(0)$, the complete radiance distribution at level 0; or (c) $N_-(0)$ and R_∞, i.e., the downward incident radiance $N_-(0)$ at level 0, and the reflectance operator R_∞ for $X(0,\infty)$. Thus, the inherent optical properties α and σ are indispensable in finding $N(y)$ using either method. However, we clearly have an option on the initial radiance data. Alternative (b) requires the full radiance distribution at level 0. Alternative (c) requires only the downward incident radiance on level 0, but along with the reflectance operator for $X(0,\infty)$. Alternative (b) is possible when preliminary empirical estimates of $N(0)$ are available. As a result of having both $N_-(0)$ and $N_+(0)$ available, we then obviate the need of R_∞. However, in theoretical studies only $N_-(0)$ is generally available for use. The remaining part of $N(0)$, namely, $N_+(0)$, is simply some more unknown data to be sought along with $N(y)$, $y > 0$. Clearly, for deep homogeneous media $X(0,\infty)$, having to find $N_+(0)$ is tantamount to finding R_∞ for $X(0,\infty)$. We thus come to the first conclusion in our analysis of the group and semigroup methods: *Each method requires as given data either alternatives (a) and (b); or (a) and (c).* The first alternative is the *empirical alternative*; the second, the *theoretical alternative*. In discussion the extension of the method of groups to more general media, the theoretical alternative demands more attention than the empirical alternative. Hence when the extension is made below, it will be made with an eye to the adoption of the theoretical alternative, thereby resulting in a more powerful method of solution in the sense that it does not depend on basically superfluous preliminary empirical measurements.

Analysis of the Group Method: Limitations of the Equation of Transfer

Having settled the matter of what kind of initial data shall be required in the general method, we seek the theoretical equations which may be the basis of the new method. Now, both equations (12) of Sec. 7.8 and (1) of Sec. 7.10 use one-dimensional parameters, namely the depths r and s in $X(0,\infty)$. Physically, the interpretation of these equations is that, given the operators $\mathcal{T}(r)$ and $\mathcal{T}(s)$ (or $\mathcal{M}(r)$ and $\mathcal{M}(s)$) for two contiguous segments of a vertical path in $X(0,\infty)$, one knows how to find the operator $\mathcal{T}(r+s)$ (or $\mathcal{M}(r+s)$) associated with the union of the two path segments. What is the analogous case in general media? To fix ideas, suppose we still have $X(0,\infty)$, but that $X(0,\infty)$ is no longer homogeneous, nor even stratified: $X(0,\infty)$ is a natural chaos of variations in α,σ and initial incident radiances over the upper boundary X_0. It is now clear that the light field can vary markedly over planes X_y at depth y in $X(0,y)$. Hence it is no longer sufficient to simply give the depth in $X(0,\infty)$ in the description of the light field in $X(0,\infty)$; a full specification of the point in question must be given.

In this more general setting what then is the counterpart to the simple vertical path used in the stratified plane-parallel case? Figure 7.16 depicts a possible candidate in the form of a general path $\mathcal{P}$ with initial point x_0 at the boundary X_0 of $X(0,\infty)$ and terminal point x in $X(0,\infty)$. Here "x_0" and "x" denote ordered triples of real numbers giving the coordinates of x_0 and x with respect to some terrestrial frame of reference. It should be noted that x_0 need not be on X_0 for what follows. We have simply placed it there to fix ideas. In the homogeneous stratified case, $\mathcal{P}$ can be vertical and, given N(0) at x_0, we can find N(y) at any distance y along $\mathcal{P}$ using (9) of Sec. 7.10 (in the method of groups) or using (10) of Sec. 7.9 (using the method of semigroups). Alternatively, we can integrate directly along $\mathcal{P}$ using (26) of Sec. 7.10 or (38) of Sec. 7.5 to find N(y). This then suggests that we merely need to specify $\mathcal{P}$ as in Fig. 7.16 and, with the initial radiance N(0) given at x_0 integrate methodically along $\mathcal{P}$. But what of the curvilinear structure of $\mathcal{P}$? This appears to present no obstacles, at least in principle. For, let "**t**" denote the unit vector to $\mathcal{P}$ at x, and let Ξ be given a fixed partition as in Fig. 7.15 (see also (1), (2) of Sec. 7.7). Thus no matter where x is in $X(0,\infty)$, Ξ has the given fixed partition. Then for ξ in Ξ the equation of transfer at x may be written:

$$\xi\cdot\mathbf{t}\,\frac{dN(x,\xi)}{dr} = -\alpha(x)N(x,\xi) + N_*(x,\xi) \qquad (1)$$

where r is distance measured along $\mathcal{P}$ at x in the direction of the tangent **t**. If the partition of Ξ is now introduced, an approximating system to (1) can be formed using the techniques explained, e.g., in Sec. 7.7 or in (25) of Sec. 7.10. As a result, at each point x of $\mathcal{P}$ a system of ordinary differential equations just like (26) of Sec. 7.10 describes the light

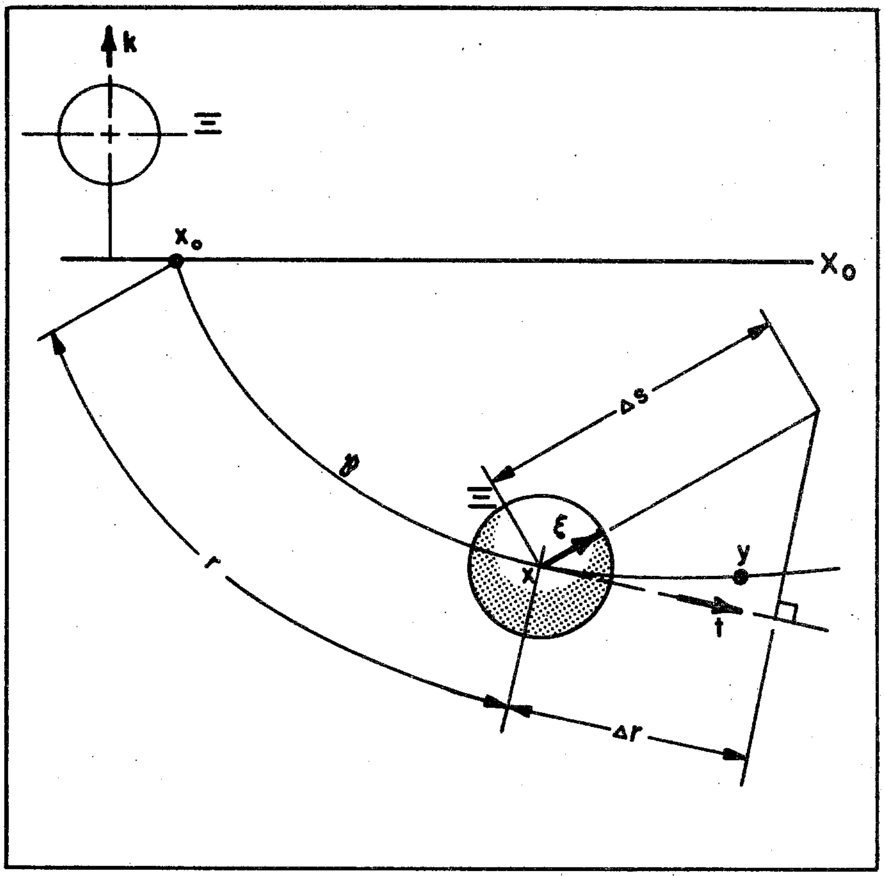


FIG. 7.16 Analysis of the group method; limitations of the equation of transfer.

field at x. Thus, if we write:

"$N_i(r)$" for $N(x,\xi_i)$

"$N_{*i}(r)$" for $N_*(x,\xi_i)$

"$\alpha(r)$" for $\alpha(x)$

where ξ_i is a fixed representative in the partition of Ξ and x is at a distance r from X_0, then (1) becomes:

$$\xi_i \cdot t \frac{dN_i(r)}{dr} = -\alpha(r)N_i(r) + N_{*i}(r) , \qquad (2)$$

$i = 1,\ldots, p$ where $p = m+n$, and we no longer explicitly distinguish between the members A_i, B_j of the partition. It appears that by knowing all p components $N_i(r)$ at x, we can compute $dN_i(r)/dr$ for each i from the right side of (2), and then use this derivative value to estimate each of the p values $N_i(r+\Delta r)$ for some reasonable incremental distance Δr along the path in the direction $\mathbf{t}$. In this way we can perhaps computationally inch our way along $\mathcal{P}$ and find $N(x,\xi_i)$, at least in principle, at any point x in $X(0,\infty)$ and for any of the p directions ξ_i!

Encouraged by the seemingly successful generalization of the homogeneous stratified case to the nonhomogeneous case as outlined above, we go on to see whether the preceding computational scheme can be phrased succinctly in group-theoretic terms. Granted the system (2) can be integrated along a given path $\mathcal{P}$ starting with the initial radiance distribution $N(x_o)$, we can then find $N(x_1)$ at x_1, a point a distance r along $\mathcal{P}$ from x_o. Then for the same reasons we may go on to find $N(x_2)$ at point x_2 of $\mathcal{P}$. Suppose we summarize the construction activity over the segment between x_o and x_1 by means of an operator $\mathcal{N}(x_o,x_1)$, and similarly let $\mathcal{N}(x_1,x_2)$ map $N(x_1)$ into $N(x_2)$. Then we could say,

$$\mathcal{N}(x_o,x_2) = \mathcal{N}(x_o,x_1)\,\mathcal{N}(x_1,x_2) \tag{3}$$

in analogy to the group relation which holds for the operator $\mathcal{M}(x,y)$ of $\Gamma_2(0,\infty)$. We thus appear to have arrived at the requisite group-theoretic relations in the form of (3) for the general case.

Before going any further and before we develop specific numerical schemes on (3) as a base, it would be well to test the validity of that scheme on some easily visualized case for which we know the answers. Toward this end, we suppose $X(0,\infty)$ is homogeneous once again. Now, however, we assume the incident radiance distributions on the upper boundary X_o of $X(0,\infty)$ to vary with location on X_o. To fix ideas, let $N_-(0)$ be vertical collimated radiance and let it undulate sinusoidally in magnitude along the direction from left to right with period r_o, as in Fig. 7.17, and be constant along directions normal to the Figure. Thus the light field in $X(0,\infty)$ is quite clearly *not* stratified, even though the inherent optical properties of $X(0,\infty)$ are about as innocuous as can be without being trivial. Since the argumentation leading to (3) was for an arbitrary path $\mathcal{P}$ in $X(0,\infty)$, let us now choose $\mathcal{P}$ to be a horizontal infinitely long path going from left to right just below X_o, as in Fig. 7.17. With this arrangement fixed we now turn to the system (2) and observe that the associated matrix operator $\mathcal{K}$ is, by the homogeneity of $X(0,\infty)$, independent of distance r along $\mathcal{P}$. It follows that, as far as the intrinsic structure of (2) is concerned, we have reverted to the full group-theoretic context of Sec. 7.10. In particular, the asymptotic radiance theorem states that $\mathbf{N}(r)e^{kr}$ should have a limit as $r\to\infty$, i.e., there should be a fixed radiance distribution toward which the p-component vector $\mathbf{N}(r)$ goes. Now, under the conditions just defined this conclusion is patently false! Quite obviously the radiance distributions along $\mathcal{P}$ vary sinusoidally in dependence with distance r along $\mathcal{P}$. Observe next that while we do not know exactly what $\mathbf{N}(r)$ is at r from x_o, we do know that $\mathbf{N}(r) = \mathbf{N}(r+r_o)$, i.e., that the

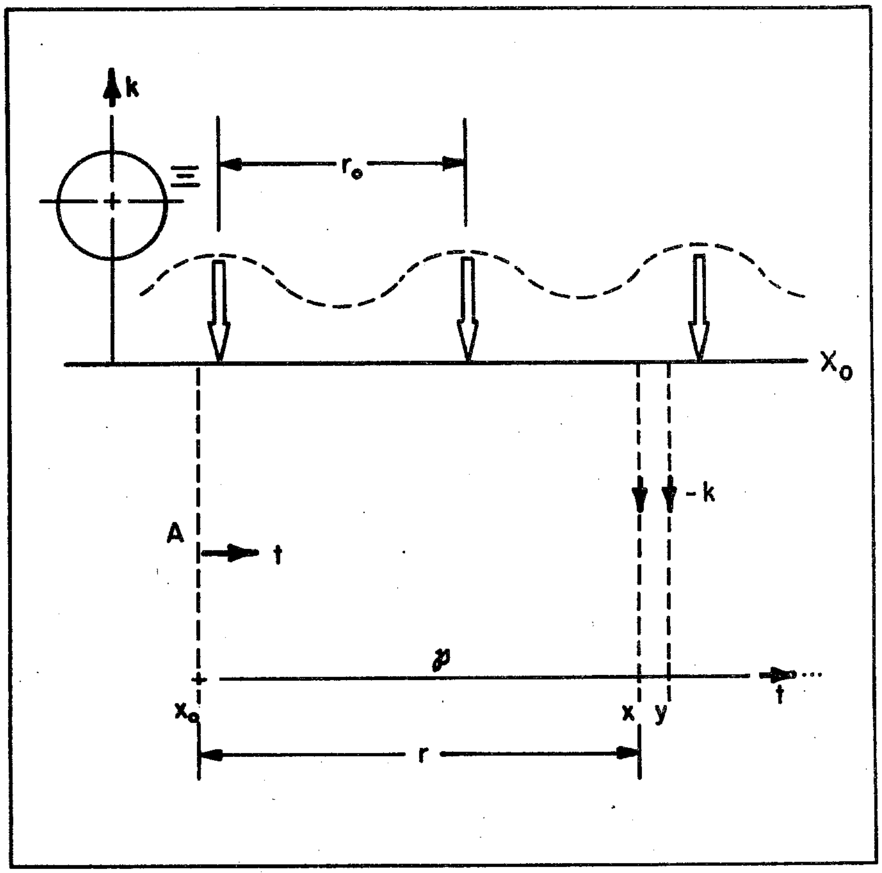


FIG. 7.17 The equation of transfer cannot directly relate radiances along parallel paths.

light field varies periodically with the same period r_0 as the incident radiance field on X_0. We have thus come to a contradiction, and the next task is to understand just where the reasoning leading to the general integration scheme along $\mathscr{P}$ was fallacious.

To detect the fallacy at hand, we quickly can dismiss equation (2) itself as the epicenter of difficulty; similarly, the asymptotic radiance theorem based on the characteristic representation (28) of Sec. 7.10 cannot be the trouble center. We therefore descend on the remaining possibility; namely, the equation of transfer itself, and this, understandably, is done with a measure of trepidation. The trouble appears to stem from the use of the equation of transfer in the setting depicted in Fig. 7.16. We therefore review with care the meaning of the terms in (1). The only strange aspect of (1) is the derivative term. But this has been correctly translated from the general directional derivative term $\xi \cdot \nabla N(x,\xi)$ that customarily appears in the equation of transfer. Indeed, let s be distance measured along direction ξ, as in Fig. 7.16. Then:

$$\xi \cdot \nabla N(x,\xi) = \frac{dN(x,\xi)}{ds} = \xi \cdot t \, \frac{dN(x,\xi)}{dr}$$

by virtue of the simple geometric fact that:

$$\xi \cdot t \Delta s = \Delta r$$

where Δs and Δr are two arbitrary distances along directions ξ and **t** but related as shown in Fig. 7.16. Therefore, whatever the source of difficulty, it is not one born of simple algebraic errors. It remains, therefore, to consider conceptual errors of application of the equation of transfer.

In going from (2) to the integration scheme over path $\mathcal{P}$ some error of interpretation of (1) was committed. The error must center on the intended meaning, i.e., the intended physical interpretation of the derivative term of (1). It was hoped that knowledge of the value of $\xi \cdot t dN(x,\xi)/dr$ would permit an estimate of $N(y,\xi)$ at the neighboring point $y = x + t\Delta r$ on $\mathcal{P}$. When phrased in this way (rather than in the abbreviated notation of (2)) the difficulty starts to resolve: *the derivative term* $\xi \cdot \nabla N(x,\xi)$ *of the equation of transfer is intended to give the rate of change of* $N(x,\xi)$ *at* x *in the direction* ξ *and in no other direction. Hence an attempt at extrapolating the value* $N(x,\xi)$ *at* x *in some direction* ξ', *using the equation of transfer, is permissible only when* $\xi' = \xi$. Therefore the integration scheme of (2) along $\mathcal{P}$ holds only when $\mathcal{P}$ is a straight line with direction ξ.

The preceding italicized observation stands out in bold relief when, now forewarned, we consider the following quite simple test situation. Let $X(0,\infty)$ be a purely absorbing medium. Thus $\sigma = 0$ throughout $X(0,\infty)$. Let $X(0,\infty)$ be irradiated by vertical collimated light as in Fig. 7.16, but now the spatial dependence over the upper boundary need not even be periodic, but simply some arbitrary given form. The equation of transfer can readily predict $N(x,-k)$, the downward radiance at any point x along any vertical path $\mathcal{P}$. However, given $N(x,-k)$, we cannot use the equation to predict $N(y,-k)$ where y is a point the same depth from X_0 and just next to x. From this we infer that in any optical medium *the equation of transfer is generally powerless to describe or interrelate directly the radiance flow at two neighboring points* x *and* y *which are directed along parallel paths containing* x *and* y.

In the simple case of a purely absorbing medium, it is clear that to know $N(x,\cdot)$ at each point of plane X_y in $X(0,\infty)$, it is necessary to know $N(0,\cdot)$ at each point of plane X_0 for the directions ξ in Ξ_-. It seems reasonable that this is indeed the case also for media with arbitrary scattering mechanisms extant within them. It shall turn out that this is so. Thus, in the counterexample of Fig. 7.17, the initial data at point x_0 is generally inadequate to predict radiance parallel to $\mathcal{P}$ at points above and below $\mathcal{P}$. What is needed is initial data over a *whole* vertical plane A (seen dashed, end on) throughout $X(0,\infty)$ which then permits a methodical computational march away from the data plane in the direction of its normal **t**, a march which eventually can in principle sweep through all of $X(0,\infty)$.

We have deliberately travelled the route just taken, i.e., from (1) to the preceding fallacy, and then to the resolution of the fallacy just above, principally to uncover the resultant insight into the nature of the equation of transfer enunciated above. Thus, while it is quite obvious from following any of the classical derivations of the equation of transfer, just what the equation *can do* in a given optical medium, it does not seem to have been emphasized what the equation of transfer *cannot do* by way of direct interrelation of the light fields at two neighboring points in the medium.

Analysis of the Group Method: Summarized

These observations on the limitations of the equation of transfer complete our analysis of the semigroup and group methods by showing the minimum number of necessary steps that must be taken in generalizing the methods to arbitrary optical media. *In particular, we have learned that the general operators* $\mathcal{M}(x,z)$ *of the simple homogeneous case, which worked well in predicting* $N(z)$ *knowing* $N(x)$ *along a vertical path on a stratified plane-parallel* $X(0,\infty)$, *must now be replaced by operators* $\mathcal{M}(x,z)$ *which relate the radiance distributions over all of plane* X_x *to the radiance distributions over all of plane* X_z. In short, we must develop the generalization of the group $\Gamma_2(0,\infty)$ for $X(0,\infty)$ in the context of a one-parameter representation of the space $X(0,\infty)$, i.e., a representation of $X(0,\infty)$ which conceives of $X(0,\infty)$ as a full three-dimensional body comprised of a one-parameter set of parallel planes. In this way we return to the general concept of a one-parameter optical medium as given in Example 2 of Sec. 3.9. Furthermore, the discussions following (93) of Sec 7.4 may now be restudied with profit. In the light of the preceding analysis, that discussion now takes on a deeper meaning which can be developed in concrete terms as follows.

The General Method of Groups

The geometric setting for the general method of groups is a rectangular parallelepiped $X(a,b,c)$ of dimensions a,b,c, and which is oriented and defined with the help of Fig. 7.18. The unit vectors **i**,**j**,**k** of the usual right-hand cartesian coordinate frame are shown. The standard hydrologic optics coordinate system measures depth z as positive, increasing in the direction **-k**. Analogously, the x and y measurements are positive, increasing in the directions **-i**,**-j**, respectively. This measuring convention is simply a logical extension of the useful plane-parallel convention of measuring z positive along **-k**. If desired, the **i** and **j** unit vectors may be reversed to obtain a right-hand coordinate system of more familiar appearance. Also the **k** unit vector may be reversed with **i** and **j**

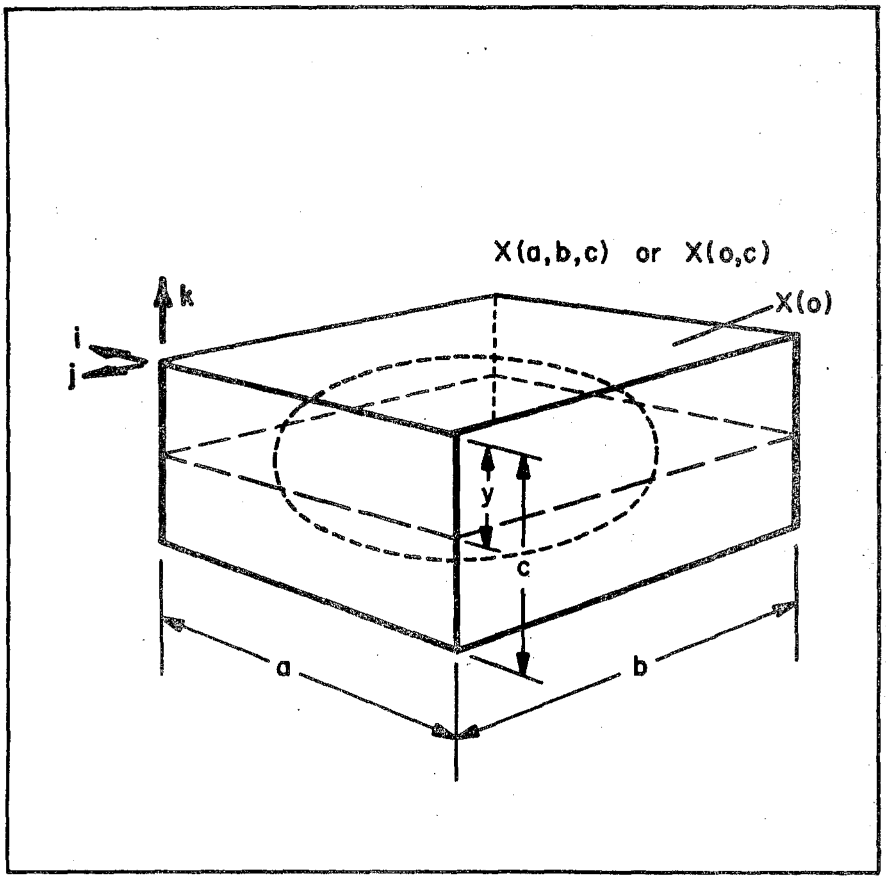


FIG. 7.18 The parallelepiped within which an arbitrary radiative transfer process can evolve and be studied.

suitably adjusted. We shall not adopt this latter reversal, as it will necessitate a massive revision of all direction conventions developed so far, and will cause difficulties in treating the unified planetary radiative transfer problems in which the atmosphere above the top plane boundary $X(0)$ of $X(a,b,c)$ is allowed to interact with $X(a,b,c)$. We call $X(a,b,c)$ a *monobloc*: it is the general version of a plane-parallel medium. The latter type of medium is the special case, of $X(a,b,c)$ for which $a = b = \infty$. We assume that an incident radiance distribution is defined over $X(0)$ and that there are no further sources on or within $X(a,b,c)$. We assume also that α and σ are specified throughout $X(a,b,c)$. A one-parameter representation of the monobloc is fixed by writing:

$$X(a,b,c) = \bigcup_{0 \le z \le c} X(z) \tag{4}$$

where $X(z)$ is the plane section of $X(a,b,c)$ normal to the direction **k** and at depth z below the plane $X(0)$. Hence each $X(z)$ is a plane of fixed dimensions a by b. Having fixed the parametrization direction of $X(a,b,c)$ as being parallel to **k**, we can suppress the dimensions a and b of $X(a,b,c)$ and write simply:

$$\text{"}X(0,c)\text{"} \quad \text{for} \quad X(a,b,c) \quad . \tag{5}$$

More generally, an arbitrary subslab of $X(0,c)$ between levels x and z is denoted by "$X(x,z)$" in complete analogy to the plane-parallel context discussed earlier in this chapter.

We note in passing that the choice of the direction of parametrization need not be along **k**. It can, for example, be along **i** or **j**, i.e., we could slice up $X(a,b,c)$ by planes normal to **i** (in which case $X(a,b,c)$ is denoted by "$X(0,a)$" in analogy to (5)) or normal to **j** (so that $X(a,b,c)$ is denoted by "$X(0,b)$"). For the general theory developed below we could even slice up $X(a,b,c)$ by parallel planes cocked at some outlandish angle, and being normal to an arbitrary direction ξ_0. Finally, the parametrization could even be accomplished with non-plane surfaces. However, as we shall presently see, the apparently special monobloc $X(a,b,c)$, the orthodox-looking parametrization (4), and the special lighting conditions are of sufficient generality to subsume *all* cases encountered in practice.

The basic equation of the group method is the operator form of the equation of transfer:

$$\frac{dN(y)}{dy} = N(y)\mathcal{X}(y) \tag{6}$$

as developed in Sec. 7.1. Under the present lighting conditions we have, from principle of invariance I of Example 2 in Sec. 3.9:

$$N_+(y) = N_-(y)R(y,c) \quad . \tag{7}$$

This was obtained by setting $z = c$ in principle I and using the fact that $N_+(c) = 0$. Finally, for convenience, we repeat (18) of Sec. 7.1 here (now adapted to $X(y,c)$):

$$-\frac{\partial R(y,c)}{dy} = \rho(y) + \tau(y)R(y,c) + R(y,c)\tau(y) + R(y,c)\rho(y)R(y,c) . \tag{8}$$

It is clear that the derivation of (8), originally performed in a stratified plane-parallel setting, holds also for the present monobloc setting. Indeed, as shown in Equation I' of Sec. 25 of Ref. [251], the gestalt of (8) persists in arbitrary optical media in euclidean three space. Equations (6), (7) and (8) are the basic equations of the general method of groups, and are used in numerical procedures as follows.

Stage One. Discretize the directional variables of equations (6), (7), (8) by partitioning Ξ after the manner of Secs. 7.9, 7.10. The end results are matricial versions of the three basic equations. The matrix version of each term will be written below in boldface type. Thus the function N(y) becomes the vector **N**(y) and the functional components of N(y), namely $N_+(y)$ and $N_-(y)$ become vectors $\mathbf{N}_+(y)$, $\mathbf{N}_-(y)$ as illustrated earlier in this chapter.

Stage Two. Solve (8) for all reflectance matrices **R**(y,c), $0 \leq y \leq c$ given the initial condition **R**(c,c) = 0 (the zero matrix). Integration proceeds from R(c,c) through R(y,c) to R(0,c). Thus, we in effect build up X(0,c) layer by layer and compute **R**(y,c) at each intermediate stage X(y,c) of construction.

Stage Three. Solve (6) for all radiance vectors $\mathbf{N}_-(y)$, $0 \leq y \leq c$ given the initial radiance $\mathbf{N}_-(0)$. Toward this end, use (6) in expanded form:

$$\frac{d\mathbf{N}_-(y)}{dy} = \mathbf{N}_-(y)\tau(y) + \mathbf{N}_+(y)\rho(y)$$

in which (7) has been substituted in the equation for $\mathbf{N}_-(y)$:

$$\frac{d\mathbf{N}_-(y)}{dy} = \mathbf{N}_-(y)[\tau(y) + \mathbf{R}(y,c)\rho(y)] \quad . \tag{9}$$

Equation (9) is solved, starting at level y = 0, and is used to work down through X(0,c) to level c. At each level y, $0 \leq y \leq c$, R(y,c) is used, as indicated, and is taken from the result of *Stage Two*. At each level y, $\mathbf{N}_+(y)$ is obtained from the matricial version of (7). Equation (9) governs $\mathbf{N}_-(y)$; the latter is an m-component vector (cf. (23), (24) of Sec. 7.10). Equations (7) and (9) are therefore used to burrow methodically from one layer in X(0,c) down to the next, gathering up new values of $\mathbf{N}_\pm(y)$ along the way. Observe how knowledge of $\mathbf{N}_-(y)$ over *all of* level y permits the derivatives of the components of $\mathbf{N}_-(y)$ to be computed, from which estimates of the components $\mathbf{N}_-(y+\Delta y)$ are obtained. Equation (7) then yields $\mathbf{N}_+(y+\Delta y)$.

Observations on the Method of Groups

A comparison of the preceding three stages of computation, especially the last two, shows that we are generalizing the method of semigroups as summarized in the system (13) of Sec. 7.8. In the present case R_∞ is replaced by R(y,c) and the infinitesimal generator A (as in (12) of Sec. 7.9) now uses depth-variable operators ρ, τ, and R.

A further examination of the three stages outlined above shows that in any radiative transfer problem, of all the global properties of an extended medium, only its standard reflectance is really indispensable along with the complete transmittance operator $\mathcal{T}(0,y,c)$:

$$N_-(y) = N_-(0)\,\mathcal{T}(0,y,c)$$

which implies (9) upon differentiation of each side with respect to y. (Use (17) of Sec. 7.5 and (52), (53) of Sec. 3.7.) These two operators and their governing laws are exhibited in general in (52) and (53) of Sec. 3.7. These were used in Sec. 7.8 to develop the semigroup method in the homogeneous plane-parallel setting. The group-theoretic structure residing just below the surface activity of *Stages One, Two,* and *Three* is latent in (14) of Sec. 7.10 and may be summarized as follows. Write:

$$"\mathcal{A}(x,z)" \quad \text{for} \quad \{\mathcal{R}(x,z,c), \mathcal{T}(x,z,c)\} \tag{10}$$

where $\mathcal{R}(x,z,c)$ and $\mathcal{T}(x,z,c)$ are the complete reflectance and transmittance operators associated with $X(0,c)$ and x,z are arbitrary levels in $X(0,c)$ such that $x \le z$. Observe that $\mathcal{A}(x,z)$ operates on $N_-(x)$ to yield $(N_+(z), N_-(z))$. For any two such operator pairs as $\mathcal{A}(x,y)$ and $\mathcal{A}(y,z)$, write:

$$"\mathcal{A}(x,y) * \mathcal{A}(y,z)" \quad \text{for} \quad \{\mathcal{T}(x,y,c)\mathcal{R}(y,z,c), \mathcal{T}(x,y,c)\mathcal{T}(y,z,c)\} \tag{11}$$

By (52) and (53) of Sec. 3.7 we see immediately that:

$$\boxed{\mathcal{A}(x,y) * \mathcal{A}(y,z) = \mathcal{A}(x,z)} \quad . \tag{12}$$

Furthermore, the binary operation $*$ defined in (11) is associative and $\mathcal{A}(x,x)$ for every x clearly serves as the identity element in the sense that:

$$\mathcal{A}(x,x) * \mathcal{A}(x,y) = \mathcal{A}(x,y) \quad . \tag{13}$$

By extending the meaning of $\mathcal{T}(x,y,c)$ and $\mathcal{R}(x,y,c)$ to the case where x and y are not restricted to the relation $x \le y$, the set $\{\mathcal{A}(x,y): \ 0 \le x, \ y \le c\}$ becomes a partial group. This extension can be made by following the suggestions given around (44) of Sec. 7.4. The partial group $\{\mathcal{A}(x,y): \ 0 \le x, \ y \le c\}$, which we denote by "$A_2(0,c)$", is clearly isomorphic to $\Gamma_2(0,c)$ introduced in Example 7 of Sec. 3.7.

It may be well to also make some observations of a practical nature concerning the integration of equations (8) and (9). Consider (9) first. The ith component of the radiance vector function $\mathbf{N}_-(z)$ at level z is of the form $N(x,y,z,\xi_i)$ where ξ_i is in the ith partition element A_i of Ξ_-. Here x,y,z now are the three coordinates of a point in the monobloc (see Fig. 7.18). Let us write "$N_i(x,y,z)$" or "$N_i(p)$" for this ith component of $\mathbf{N}_-(z)$, where "p" stands for "(x,y,z)". Equation (9) gives the rate of change of $N_i(p)$ at point p from which we may estimate $N_i(p+\xi_i\Delta r)$ where Δr is an increment of path length along the direction ξ_i. Our detailed analysis of the equation of transfer earlier in this section shows that this is the only type of extrapolation that the equation permits. However, now that $N_i(p)$ is known for every $i = 1,\dots,m$, and for every p over the plane $X(z)$, this limited mode of extrapolation is clearly adequate to propagate the

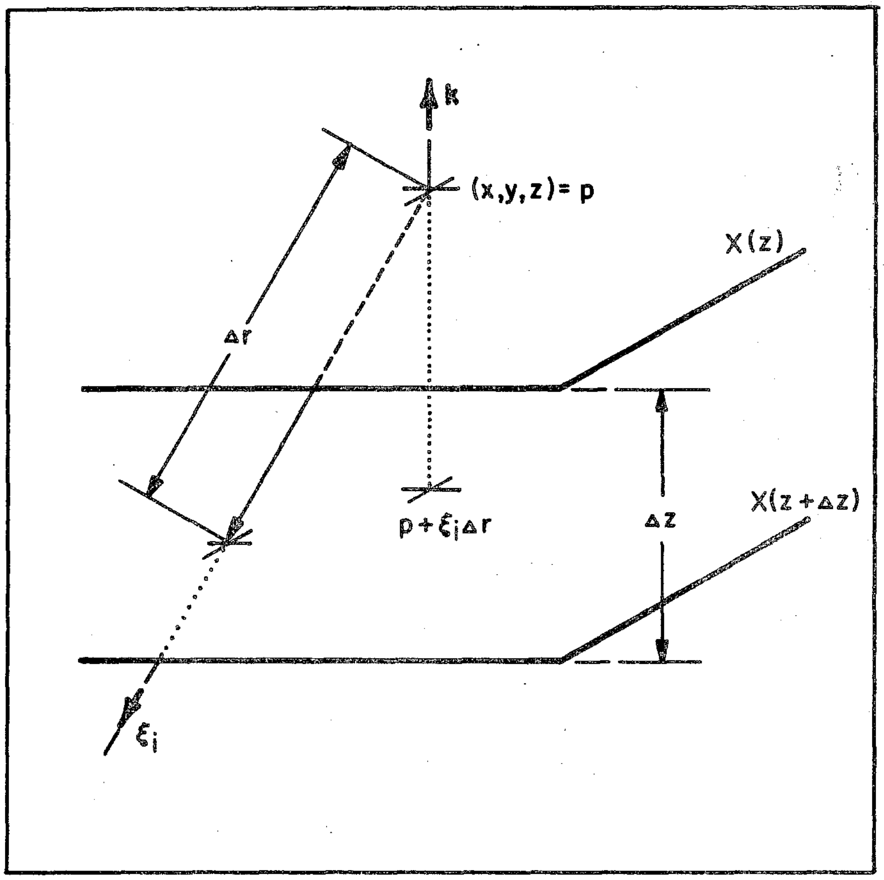


FIG. 7.19 How to propagate the radiance computations from one parameter surface to the next.

radiance field from plane X(z) to plane X(z+Δz) for suitably chosen Δz. Fig. 7.19 helps show how the function $\mathbf{N}_-(z+\Delta z)$ over X(z+Δz) is obtained from $\mathbf{N}_-(z)$ known over X(z). In particular we have for each p (= (x,y,z)) in X(z) and ξ_i in Ξ_-:

$$N_i(p+\xi_i\Delta r) = N_i(p) + \frac{dN_i(p)}{dz}\Delta z \tag{14}$$

where:

$$\Delta z = -\xi_i \cdot \mathbf{k}\Delta r \quad . \tag{15}$$

Once $\mathbf{N}_-(z+\Delta z)$ has been obtained, we use (7) to find $\mathbf{N}_+(z+\Delta z)$ by means of the formula:

$$\mathbf{N}_+(z+\Delta z) = \mathbf{N}_-(z+\Delta z)\mathbf{R}(z+\Delta z,c) \quad . \tag{16}$$

Now that (14) and (15) have been displayed, and the mode of propagation of $\mathbf{N}_-(z)$ to $\mathbf{N}_-(z+\Delta z)$ has been made clear, it may be well to observe that there is no royal road to the solution of the radiative transfer problem in a general monobloc such as $X(a,b,c)$. The sheer number of dimensions of $X(a,b,c)$ and Ξ_- must always combine to dampen the enthusiasm of the most intrepid computer. At least now that the invariant imbedding techniques have shown the fundamental structure of the present type of transfer problem (as outlined in *Stages One* to *Three* above), we can rest in the knowledge that the theory has progressed as far as it can go on the phenomenological level, and that what remains is the development of more adequate numerical procedures to use on (8) and (9) for the general monobloc $X(a,b,c)$. Of course this is not meant to discount the use of other procedures such as those based on the classical techniques of Chapter 6, or on the natural mode of solution (Chapter 5), or the canonical mode (Chapter 4), or their equivalents. As far as Eq. (8) is concerned, one such attempt has been made using invariant imbedding techniques in Ref. [251] wherein the solution of (8) is carried out on a monobloc using the approach of discrete-space theory developed in that reference (see, in particular, Chapter X).

The Method of Groups and the Inner Structure of Natural Light Fields

We now round out our discussion of the method of groups and also bring to a close some matters raised in Example 7 of Sec. 3.7 by outlining a proof of the general group-theoretic structure of light fields in natural optical media.

Let X be an arbitrary connected source-free subset of euclidean three-space. Let α,σ be given throughout X and let X be irradiated arbitrarily on its boundary. A parametrization of X is introduced so that:

$$X = \bigcup_{a \le z \le b} X(z) \quad . \tag{17}$$

This decomposition of X into a family of two-dimensional surfaces $X(z)$ is illustrated in (a) of Fig. 7.20. In this way X becomes a one-parameter optical medium.

To each subslab $X(x,z)$ of X, shaded in (a) of Fig. 7.20, we can assign reflectance and transmittance operators after the manner explained in Examples 2, 4, and 5 of Sec. 3.9 so that the invariant imbedding relation holds for X. Hence equations (6), (7), (8) can be suitably extended to the setting in X so that the general counterpart to (12) holds, and a partial group $A_2(a,b)$ can be assigned to X. In particular a computation procedure for $\mathbf{N}_-(y)$ can be initiated and sustained that will propagate $N_-(z)$ across each parameter surface $X(z)$ within X in a manner completely analogous to that based on (14)-(16).

The parametrization (17), being quite general, leads to an instructive mode of description of the inner structure of the light field. As an interesting special case of (17),

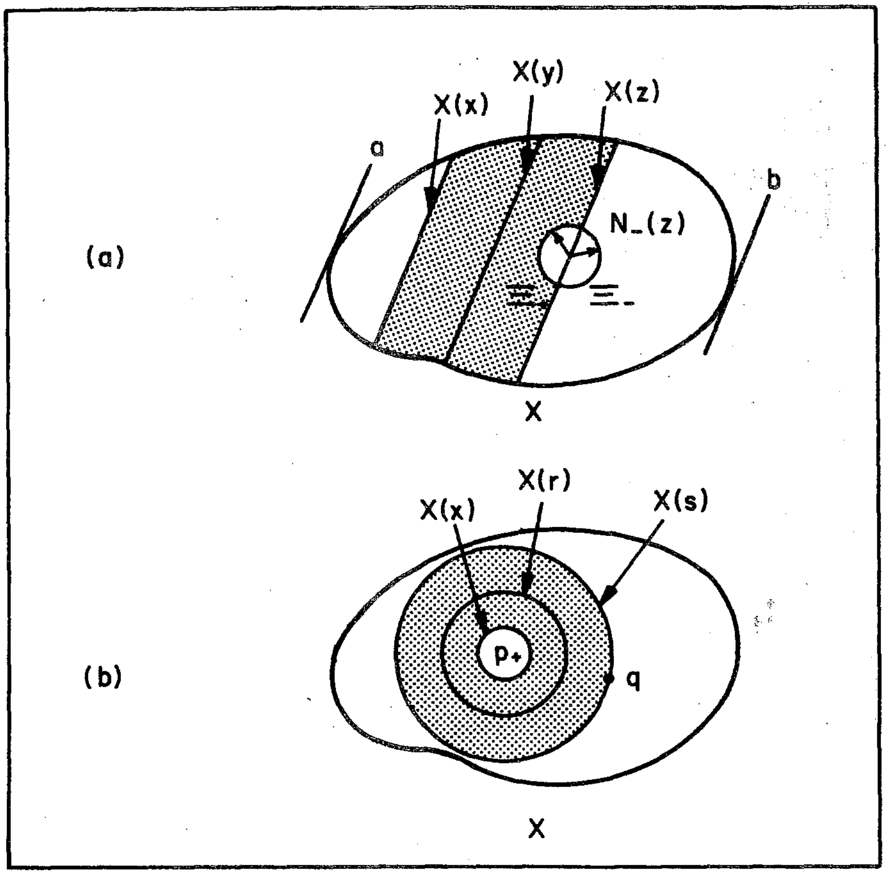


FIG. 7.20 There are many ways in which an optical medium can be made over into a one-parameter medium.

consider the parametrization of X by spherical shaped surfaces X(r) within X of radius r about an internal point p of X. We define X(r) as the intersection of a sphere of radius r with X. See (b) of Fig. 7.20. Suppose the light field is given on arbitrarily small spherical surface X(r). Then using the general one-parameter versions of (12), the radiance field can be computed at any point q in X, where q lies on X(s) for some radius s. Conversely, knowledge of the light field on some sphere about q as center could lead in principle to the determination of the light field at p after re-parametrization of X about q. This then is the most general description of the inner structure of natural light fields in an arbitrary optical medium X as defined above.

7.12 Homogeneity, Isotropy and Related Properties of Optical Media

In this section we collect together some special knowledge that has been gathering during the development of this and earlier chapters, knowledge concerning the properties of homogeneity, isotropy, polarity, and related concepts associated with optical media. This accumulation of facts is timely in that it will play an important role in rounding out the theory of internal-source generated light fields in natural optical media to be considered in the following section, and in Example 10 of Sec. 8.7.

As we shall see, the problem of internal sources in optical media requires for its solution no new concepts beyond those presented in Example 3 of Sec. 3.9. However, this battery of concepts gives rise to some relatively complex (but highly instructive) operations with the standard reflectance and transmittance operators for optical media. Any insights into the reduction of the number of the participating operators and their assemblies in the final formulations will correspondingly reduce the amount of labor required to effect specific numerical or theoretical answers to the source problems.

One of the classical means of simplifying radiative transfer formulations is the use of "symmetry principles", chief among which are various reciprocity principles governing the R and T functions. It is one of the purposes of the present section to define and discuss these symmetry properties, outline their extensions to general media, and to indicate when the extensions are or are not helpful. Perhaps the most important outcome of this discussion, at least from a practical point of view, is the unpleasant fact that most of the "symmetry principles" of the classical theory no longer hold in the general settings of arbitrary optical media. In other words, many of the "symmetries" that arose in the classical settings arose because the settings themselves were symmetrical and generally quite idealized, and not because there subsisted some inherent invariant character of the symmetry.

For example, by graduating from the use of irradiance or from scalar irradiance (or radiant density) within infinite or semi-infinite homogeneous isotropic media, to the use of radiance in such media, at least one important reciprocity theorem falls by the wayside. By making the space inhomogeneous, but still isotropic, an important symmetry property vanishes into the void. By making the space finite, inhomogeneous and irregular in geometric structure essentially all but one of the classical symmetry properties (reciprocity for radiant density) leave the investigator with handfuls of functional equations whose associated analytic difficulties must be squarely faced without any essential help forthcoming from the lone surviving symmetry principle. In short, the moment one steps from the nice one-dimensional spaces with their nice one-dimensional radiometric concepts and enters the representer of the real world, namely euclidean three-space, and attempts to describe radiant flux in that setting in terms of radiance rather than radiant density, then, except in the most singular cases, the classical symmetries no longer subsist and hence are no longer available to facilitate numerical and theoretical activity.

Despite the predominantly negative features of the following discussion it will still be instructive for the reader to have consolidated and clarified some of the more frequently used "symmetries" and "uniformities" in the processes of constructing models of natural optical media. To this task we now turn.

Throughout this section let X be an optical medium in euclidean three-space E_3 to which is associated a volume attenuation function α, a volume scattering function σ and an index of refraction n. The domain of n is X, the domain of α is $X \times \Xi$, that of σ is $X \times \Xi \times \Xi$, the values $\alpha(x,\xi)$ and $\sigma(x;\xi';\xi)$ are non negative real numbers. It seems best to proceed by making formal definitions and following them with appropriate comments that illustrate their physical meanings and interrelate them. We begin with the local concepts, i.e., concepts associated with the *points* of X. Following this the global concepts are introduced (those associated with subsets of X) and an attempt will be made to define the global concepts analogously to the local concepts whenever possible. One of the main problems of the present area of radiative transfer theory is to determine whether a valid local concept in a given medium X carries over to the global context. We shall indicate, by theorem and example, some instances of this problem as the discussion proceeds.

Local Concepts

Definition 1. X is said to be *homogeneous* if the values n(x), $\alpha(x,\xi)$ and $\sigma(x;\xi';\xi)$ are independent of x for every ξ',ξ in Ξ.

Since α depends generally on x and ξ, the values $\alpha(x,\xi)$ in a homogeneous space, while independent by definition of x, may possibly depend on ξ. Thus, e.g., while $\alpha(x,\xi) = \alpha(x',\xi)$ for every x, x' in X, this common value may depend on ξ. Perhaps this is an academic point in the sense that homogeneity is rarely found in such a general form in nature. Be that as it may, the present definition, being necessarily framed with n, α and σ as the basic concepts at hand, some decision must be made as to the ξ-dependence of n, α and σ in the homogeneous case. The decision adopted above imposes the least restrictions on the functions while capturing the basic idea behind homogeneity: the *uniformity in the spatial domain* of the values of n,α and σ.

Homogeneity helps simplify the equations of radiative transfer in many ways. The most immediate effect is in the structure of the beam transmittance function. In general, for a path $\mathcal{P}_r(x_0,\xi)$ we have:

$$T_r(x_0,\xi) = [n^2(x)/n^2(x_0)] \exp\{-\int_0^r \alpha(x',\xi)\, dr'\} , \qquad (1)$$

$n(x')$ is the index of refraction at $x' = x_0 + r'\xi$, a distance r' along $\mathscr{P}_r(x_0,\xi)$ from the initial point x_0. When X is homogeneous, the index of refraction function n is independent of location in X so that in particular $\mathscr{P}_r(x_0,\xi)$ is a straight line segment with direction ξ. Further, $n(x) = n(x_0)$, along with $\alpha(x',\xi) = \alpha(x_0,\xi)$. Thus under homogeneity, $T_r(x_0,\xi)$ becomes:

$$T_r(x_0,\xi) = e^{-\alpha(x_0,\xi)r} \tag{2}$$

where $\alpha(x_0,\xi)$ is the fixed value of α in X associated with the direction ξ, and r is the length of $\mathscr{P}_r(x_0,\xi)$.

It may be possible to have the index of refraction essentially constant on X without having α or σ independent of location. When this is the case we have *restricted inhomogeneity of* X. Such inhomogeneity is ideal for the theorist in radiative transfer: he has the opportunity of studying the main problems of radiative transfer without the annoying and distracting possibility of curved or broken paths $\mathscr{P}_r(x,\xi)$, and of varying radiance values in an otherwise clear medium (cf. Sec. 21 of Ref. [251]), or in the beam transmittance function (cf. (1)). Therefore, throughout this section, when we consider X to be inhomogeneous it will be understood to be a restricted inhomogeneity of X.

We conclude this discussion of homogeneity by rephrasing the definition in terms of the notion of a *displacement transformation* on X. A function D on X with values in X is a *displacement transformation* if, and only if, there exists a fixed point y such that:

$$D(x) = x + y \quad .$$

Here we are using the fact that the points of E_3 (and hence those of X) are ordered triples of numbers, as in analytic geometry, so that there is an algebraic basis for adding them together. The main part of Definition 1 may now be phrased analytically as follows. "X is *homogeneous*" means:

$$\text{whenever } D(x) \text{ is in X, then } n\big(D(x)\big) = n(x)$$

$$\text{and } \alpha\big(D(x),\xi\big) = \alpha(x,\xi) \text{ and } \sigma\big(D(x);\xi';\xi\big) = \sigma(x;\xi';\xi) \quad .$$

A less restrictive notion than homogeneity but one that still permits all the analytic blessings of homogeneity to be enjoyed by the theorist is the notion of separability of X:

Definition 2. X is said to be *separable* if the index of refraction function is constant and $\alpha(x,\xi)$ is independent of ξ and if $\sigma(x;\xi';\xi)/\alpha(x)$ is independent of x for every ξ',ξ in Ξ.

The reason for the name "separable" becomes clear on fixing x in X and writing:

$$\text{"}p(x;\xi';\xi)\text{"} \quad \text{for} \quad 4\pi\sigma(x;\xi';\xi)/\alpha(x) \quad . \tag{3}$$

The function p on $X \times \Xi \times \Xi$ so defined is called the *phase function* in astrophysical optics (cf. Ref. [43]) and by means of it $\sigma(x;\xi';\xi)$ may be written:

$$\sigma(x;\xi';\xi) = \frac{1}{4\pi}\,\alpha(x)p(x;\xi';\xi) \quad . \tag{4}$$

Hence in separable media $\sigma(x;\xi';\xi)$ may be written as the product of two functions: one which is free of x and the other which depends on x--so that the spatial dependence is uncoupled or separated from the main directional dependence. In particular, in a separable medium the spatial dependence of σ is carried by α, while the directional dependence of σ is carried by p. The utility of the separability assumption becomes clear on examining, e.g., the definitions of the matrices $\mathbf{r}_-(a)$, $\mathbf{t}_-(a)$, etc., occurring in (9) of Sec. 7.7. If the medium were assumed separable, then $\mathbf{r}_-(a)$, $\mathbf{t}_-(a)$, etc. would be independent of a, while still allowing a measure of inhomogeneity of the medium to be present.

In separable optical media, the natural measure of distance is not geometric distance but optical distance, in the following sense: If $\mathcal{P}_r(x,\xi)$ is a path in a separable medium, then its *optical length* is the number $\int_0^r \alpha(x')\,dr'$, the integration being taken along the path. This number is usually designated by "$\tau(r)$" and enters into the theory via the equation of transfer when a transition from r to $\tau(r)$ is made. Thus the equation:

$$\frac{dN}{dr} = -\alpha N + N_*$$

becomes:

$$\frac{1}{\alpha}\,\frac{dN}{dr} = -N + (N_*/\alpha) \quad .$$

Since:

$$\frac{d\tau}{dr} = \alpha$$

we have:

$$\boxed{\frac{dN}{d\tau} = (N_q - N)} \quad . \tag{5}$$

Beam transmittance in separable media becomes:

$$T_r(x,\xi) = e^{-\tau(r)} \quad . \tag{6}$$

If the dependence of τ on r is suppressed and τ is made the basic measure of distance, then the medium X is homogeneous, in the sense of Definition 1, with respect to the distance measure τ. Furthermore, the volume attenuation function in such a separable medium with optical distance τ is replaceable by a unit-valued function at all points of X. In other words, in a separable optical medium one can *normalize* the volume attenuation function and effectively remove it from the scene, and the volume scattering function is replaceable by the phase function.

Definition 3. An optical medium X is said to be *isotropic at* x if the values $\alpha(x,\xi)$ are independent of ξ and the values $\sigma(x;\xi';\xi)$ depend only on the scalar product $\xi'\cdot\xi$ of ξ' and ξ. X is *isotropic* if it is isotropic at every point.

From this we see, first of all, that while homogeneity of X is constancy of α and σ on X, isotropy of X is a constancy of α on Ξ along with a certain special constancy of σ on $\Xi \times \Xi$. Specifically, "X is isotropic at x" means

$$\text{for every } \xi',\xi \text{ in } \Xi,\ \alpha(x,\xi') = \alpha(x,\xi), \text{ and} \tag{7}$$

$$\text{for every } \xi_1,\xi_2,\xi_3,\xi_4 \text{ in } \Xi, \text{ if } \xi_1\cdot\xi_2 = \xi_3\cdot\xi_4, \text{ then}$$

$$\sigma(x;\xi_1;\xi_2) = \sigma(x;\xi_3;\xi_4) \quad . \tag{8}$$

Isotropy of X can be characterized by means of rotation transformations of E_3. Let T be a rotation of E_3 at x. Then the preceding isotropy conditions may be rendered as:

$$\alpha(x,T(\xi)) = \alpha(x,\xi)$$

and

$$\sigma(x,T(\xi),T(\xi')) = T(x;\xi';\xi)$$

for every ξ',ξ in Ξ and every rotation T at x.

Definition 4. A scattering process (or σ) is said to be *isotropic at* x in X if $\sigma(x;\xi';\xi)$ is independent of ξ',ξ in Ξ. An attenuation process (or α) is said to be *isotropic at* x in X if $\alpha(x,\xi)$ is independent of ξ in Ξ. A scattering or attenuation process is *isotropic* if it is isotropic at every x in X.

The distinction between the medium X being isotropic and the scattering process on X being isotropic is thus clear. The connections between the two ideas are as follows: *If* α *and* σ *are isotropic, then* X *is isotropic.* On the other hand, *if* X *is isotropic then* α *is isotropic, but* σ *need not be isotropic.* This anomaly of symmetry in the isotropy properties stems from the fact that σ has two spatial variables while α has only one. Hence nailing down isotropy of X fixes that of α but leaves σ a margin of variability, a margin, incidentally, which has been found most useful in the classical theory.

Observe that if σ is isotropic at x, then:

$$\sigma(x;\xi';\xi) = s(x)/4\pi \tag{9}$$

where s(x) is the value of the volume total scattering function of x. Furthermore, if X is separable and σ is isotropic, (4) and (9) combine to yield:

$$s(x) = \alpha(x)p(\xi';\xi)$$

so that:

$$p(\xi';\xi) = s(x)/\alpha(x) \quad . \tag{10}$$

From this we see that the phase function value $p(\xi';\xi)$ is independent of ξ' and ξ and is a real dimensionless number between 0 and 1, a number which we have called the *scattering-attenuation* ratio.

Definition 5. A scattering process (or σ) is said to be *reversible at* x in X if the following property of σ holds: for every ξ',ξ in Ξ, $\sigma(x;\xi';\xi) = \sigma(x;-\xi;-\xi')$. An attenuation process (or α) is said to be *reversible at* x in X if $\alpha(x,\xi) = \alpha(x,-\xi)$ for every ξ in Ξ. A scattering (or attenuation) process is *reversible* if it is reversible at every x in X.

It is clear that if a medium is isotropic at a point x, then σ is reversible at x, for, indeed, since $\xi'\cdot\xi = (-\xi)\cdot(-\xi')$ the reversibility follows from (8). However, the converse need not be true: reversibility of σ at x does not logically imply isotropy of X at x, and the reader may devise theoretical examples which show this.

We summarize the four main local properties of an optical medium in Fig. 7.21 which shows the class of all optical media in E_3 grouped into families which are homogeneous, separable, isotropic, and reversible. Observe how the class of homogeneous spaces is included in the class of separable spaces, and of how the class of reversible spaces (i.e., spaces with reversible σ) includes the isotropic spaces as special cases. The classes partially overlap in the Figure, showing that generally a space may have several, one, or none of the four general uniformities.

Global Concepts

We shall now show that the local concepts of homogeneity and isotropy can be carried over, after suitable modifications, to the global description of the scattering properties of extended media. To keep the introduction to these ideas simple and intuitively meaningful we shall at first consider only stratified plane-parallel media, i.e., media whose α and σ are independent of location on planes parallel to the boundaries. Later in the discussion more general media will be briefly discussed.

Now the counterpart to σ in the global context is the reflectance function $R(a,b;\xi';\xi)$ and the transmittance function $T(a,b;\xi';\xi)$ associated with a plane-parallel medium $X(a,b)$. These pairings are intuitive and not to be taken in a formal sense. They suggest various analogous properties of the global functions that one may seek. For example, the analogous global property to homogeneity is the condition that $R(x,z;\xi';\xi)$ and $T(x,z;\xi';\xi)$ depend only on the difference z-x, where, e.g., ξ' is in Ξ_-, and ξ is in Ξ_+, as the case may be. It is easy to see that, if and only if $X(a,b)$ is homogeneous or separable, then this property holds for R and T, either directly, or after shifting over to the optical length parameter.

The next concept which may be profitably extended to the global setting is that of isotropy of the medium X at a point x.

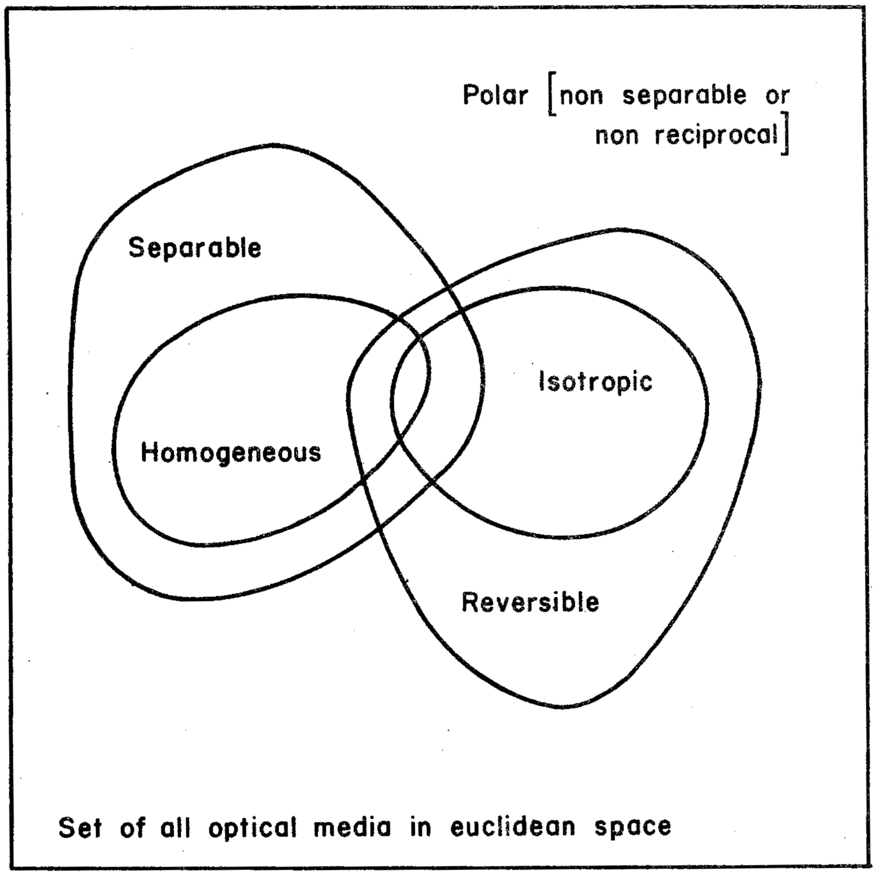


FIG. 7.21 The four principal categories for local properties of optical media and their general logical interdependence.

Instead of a point we now have a general subslab $X(x,z)$ in $X(a,b)$, and instead of the condition that $\xi'\cdot\xi$ be fixed in magnitude, we require that the directions be related by means of a reflection in a plane parallel to X_a thus:

Definition 6. A stratified plane-parallel medium $X(a,b)$ is said to be *symmetric* if the following properties hold for every subslab $X(x,z)$ of $X(a,b)$:

$$R(x,z;\xi';\xi) = R\big(z,x;M(\xi');M(\xi)\big) \tag{11}$$

and:

$$T(x,z;\xi';\xi) = T\big(z,x;M(\xi');M(\xi)\big) \tag{12}$$

for every reflection transformation M of Ξ in a plane parallel to X_a.

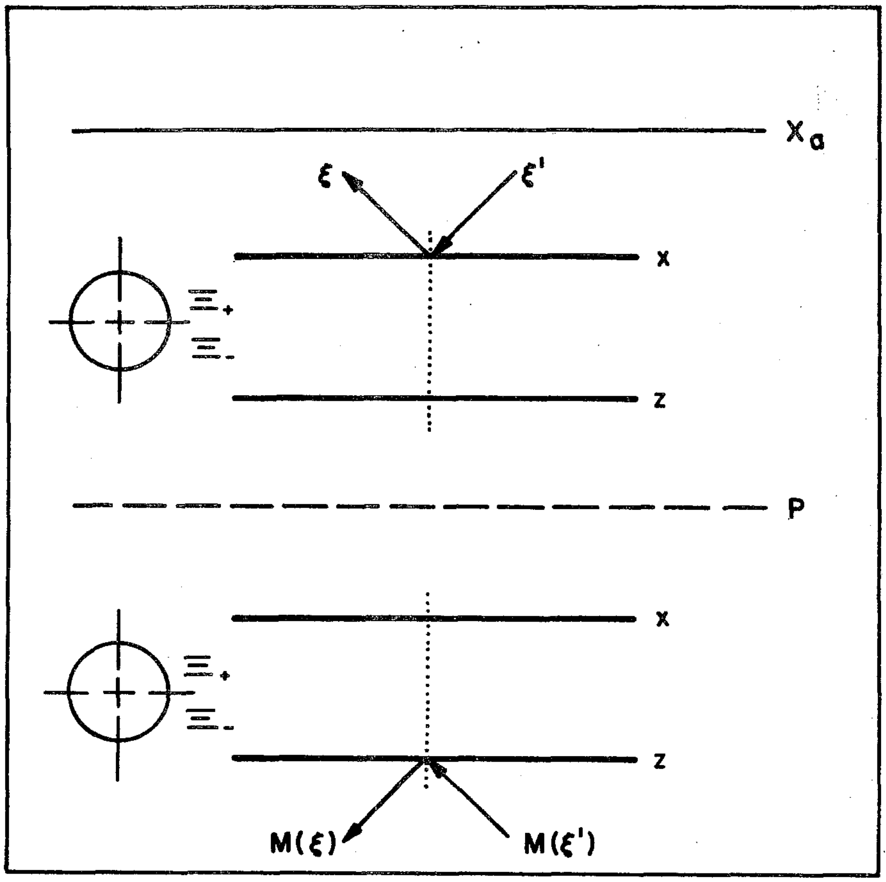


FIG. 7.22 The reflection of directions ξ, ξ' in plane P, as used in describing polarity of an optical medium.

The opposite notion to that of symmetry in the present context is polarity. Thus X(a,b) is *polar* or *exhibits polarity* if it is not symmetric; and this, by Definition 6, means that there exists a subslab X(x,z) of X(a,b) such that either

$$R(x,z;\xi';\xi) \neq R(z,x;M(\xi');M(\xi))$$

or:

$$T(x,z;\xi';\xi) \neq T(z,x;M(\xi');M(\xi))$$

for some reflection transformation M of Ξ in a plane P parallel to X_a (see Fig. 7.22 for the case of reflectance). The main theorem about polarity is the following:

Polarity Theorem: Let X(a,b) be a stratified plane-parallel medium. *(a) If* X(a,b) *is separable and isotropic, then* X(a,b) *is symmetric; (b) If* X(a,b) *is non separable and isotropic, then* X(a,b) *is polar.*

The proof of the theorem may be made to devolve on the differential equations for $R(a,b)$ and $T(a,b)$ in Sec. 7.3, but will be omitted here. The main point of the theorem is that symmetry of $X(a,b)$ may be lost by the presence of essential inhomogeneities in $X(a,b)$; by "essential inhomogeneity" is meant that the medium is not just separable, but rather such that $\sigma(z;\xi';\xi)/\alpha(z)$ depends on depth z in $X(a,b)$. A proof of the polarity theorem along with examples for discrete spaces is given in some detail in Sec. 57 of Ref.[251]

Going on now to the global counterpart of reversibility, we have:

Definition 7. A stratified plane-parallel medium $X(a,b)$ is said to be *reciprocal* if the following properties hold for every subslab $X(x,z)$ of $X(a,b)$:

$$R(x,z;\xi';\xi) = R(x,z;-\xi;-\xi') \tag{13}$$

and:

$$T(x,z;\xi';\xi) = T(z,x;-\xi;-\xi') \tag{14}$$

and:

$$R(z,x;\xi';\xi) = R(z,x;-\xi;-\xi') \tag{15}$$

and:

$$T(z,x;\xi';\xi) = T(x,z;-\xi;-\xi') \quad . \tag{16}$$

Examples can be given which show that symmetry and reciprocity of $X(a,b)$ are generally independent notions. Thus $X(a,b)$ may be symmetric but not reciprocal; and conversely, $X(a,b)$ may be reciprocal but exhibit polarity. That this is plausible may be seen without too much preliminary work by letting $X(x,z)$ approach zero thickness so that symmetry of $X(x,z)$ becomes a manifestation of isotropy of σ; and reciprocity of $X(x,z)$ reduces nearly to reversibility of σ. Since reversibility and isotropy of σ are partially independent, this independence can be inherited at least by very thin slabs $X(x,z)$. The main theorem on reciprocity is the following:

Reciprocity Theorem. Let $X(a,b)$ *be a stratified plane-parallel medium. If* σ *on* $X(a,b)$ *is reversible, then* $X(a,b)$ *is reciprocal.*

Observe that this theorem, which can be proved using the differential equations for $R(a,b)$ and $T(a,b)$ in Sec. 7.3, holds in particular for nonseparable media. The theorem was first stated and proved for separable plane-parallel media $X(a,b)$ by Chandrasekhar in Ref. [43]. A proof of the reciprocity theorem for general isotropic media is sketched in Ref. [40].

Summary

To summarize the main results of this section so far we may say that in going from the local to the global level in stratified plane-parallel media one generally can carry over the concept of reciprocity but not symmetry. More precisely, and in terms of the defined concepts above, a locally reversible medium is always reciprocal, but a locally isotropic medium may exhibit polarity.

The loss of symmetry (where we use the term in the sense of Definition 6) is a phenomenon that arises because of the adoption of radiance as the basic radiometric concept rather than irradiance or alternatively, radiant density. Had we used the latter concept, then symmetry would hold (in the scalar irradiance context) for inhomogeneous isotropic plane-parallel media. Symmetry would be lost in such a context only when isotropy was lost. By adopting radiance over irradiance we reap the benefits of a more detailed description of the light field at the expense of the classical symmetries possessed by irradiance. Furthermore, the reflectance and transmittance functions R and T in the scalar irradiance context, being scalars, commute; i.e., symbolically, RT = TR. By adopting radiance, R and T become integral operators or matrices, and these objects are notoriously noncommutative, thus blocking still further the passage of certain symmetries of the scalar formulations to the field of operator formulations.

Conclusion

In conclusion, then, the elevation of the local notions of homogeneity, separability, isotropy, and reversibility to the global settings in plane-parallel media is quite possible. However, only the local concept of reversibility is generally inherited by the space on the global level (in the form of reciprocity). But this inheritance is precarious and can conceivably vanish on graduation to arbitrarily shaped anisotropic media in which the radiometric concept used is radiance rather than irradiance or scalar irradiance. Thus all the classical symmetries are in principle left behind in the search for general invariant properties of scattering-absorbing media. The general principles of invariance, the invariant imbedding relations and their various semigroup properties are important examples of general properties of optical media which are invariant under the transition from local to global formulations within those media. This has been shown in detail in Chapter VI of Ref. [251], for general discrete spaces.

Further study of the problem of the extension of local symmetries to the global level are best handled by means of the standard $\mathscr{S}$-operator $\mathscr{S}(X;a,b)$. A detailed study of such extensions has yet to be made. It would be of interest to formulate the appropriate counterparts to homogeneity, and isotropy for general media using $\mathscr{S}(X;a,b)$, and then to find theorems, if possible, which are the appropriate generalization of the Polarity and Reciprocity theorems.

7.13 Functional Relations for Media with Internal Sources

In this section we return to the problem of internal sources in optical media introduced in Example 3 of Sec. 3.9 and reconsidered in Sec. 6.7. From a theoretical point of view the problem was completely solved in Sec. 3.9 and, in view of the methods of determination of the R and T operator discussed in Sec. 7.7, we may say that the practical numerical means of solving the internal-source problem are also well in hand. However, there remain several most interesting questions on the conceptual level, questions that arise when one examines the functional relations (35) and (36) of Sec. 6.7 with an eye toward the intuitive meaning of the equations and of their connection with the invariant imbedding relations that may be written down for the same medium. Specifically, we are confronted with two equations (35), (36) of Sec. 6.7, derived directly from the equation of transfer, and which are ostensibly statements of a certain type of invariance for scalar irradiance h(x). Their physical meanings as given by Elliott [88], are, however, occluded by the fact that their main terms f_0 and f_c are Fourier transforms of the scalar irradiance function $\hat{h}(x)$ (cf. (23) of Sec. 6.7) rather than h(x) itself. Therefore, one of the principal goals in this section is the development of a systematic method of derivation of the counterparts to (35), (36) of Sec. 6.7 for the case of radiance in a general one-parameter optical medium X(a,b) with an arbitrary set of sources on various levels within X(a,b), using only the concepts inherent in the invariant imbedding relation for the medium. We therebye shall establish intuitively meaningful generalizations of the Elliott equations and also extend their domain of validity. An additional dividend is accrued throughout in the form of further insight into the interconnections among the Ψ-operators and the invariant imbedding operators. These connections arise as a matter of course during the derivations. Throughout this section, let "X(a,b)" denote a one-parameter optical medium with artibrary α,σ. In particular X(a,b) will not be assumed isotropic, so that there are generally four local operators $\rho_\pm(t)$, $\tau_\pm(t)$ (cf. Sec. 7.1). Throughout this section sources shall be confined, for simplicity and without any serious loss of generality, to single depths s within the slab X(a,b), $a \le s \le b$. For sources at several discrete levels, superposition of the results below will yield the desired field expression. By passing to the limit of numbers of discrete sources, the theoretical way to continuously distributed sources is opened. These generalizations are left to the reader. For helpful hints in this direction see (36) of Sec. 3.9 and its discussion. Also see the paragraph on Two-D Models for Internal Sources in Sec. 8.5, and consult Example 10 of Sec. 8.7.

Preliminary Relations

One important dividend of the present efforts is a collection of auxiliary functional relations between the $\mathcal{R},\mathcal{T}$ operators and the Ψ-operator of Sec. 3.9. These equations place the interrelations of the Ψ-operator into a deeper perspective than is available from (31)-(34) of Sec. 3.9. Of particular interest at present are the connections between $\mathcal{R}(a,s,b)$, $\mathcal{T}(a,s,b)$, and $\Psi(s,s)$, where $a \le s \le b$. It follows from (20)-(23) of Sec. 3.9 and (40)-(43) of Sec. 3.7 that:

$$\mathcal{R}(a,s,b) = T(a,s)\,\Psi_{-+}(s,s:a,b) \tag{1}$$

$$\mathcal{R}(b,s,a) = T(b,s)\,\Psi_{+-}(s,s:a,b) \tag{2}$$

$$\mathcal{T}(a,s,b) = T(a,s)\{I + \Psi_{--}(s,s:a,b)\} \tag{3}$$

$$\mathcal{T}(b,s,a) = T(b,s)\{I + \Psi_{++}(s,s:a,b)\} \tag{4}$$

where we have written; for $a \le s \le b$, $a \le y \le b$:

$$"\Psi(s,y:a,b)" \quad \text{for} \quad \Psi(s,y) \tag{5}$$

to point up specifically the fact that $\Psi(a,y)$ belongs to $X(a,b)$. (See the remarks following the applications of the interaction method in Example 3 of Sec. 3.9.)

Equations (1)-(4) show clearly the interrelation between the complete reflectance operators and the local Ψ-operator for $X(a,b)$. These relations will be helpful in constructing a dual class of $\mathcal{R}$ or $\mathcal{T}$ operators needed subsequently.

Another set of functional relations, needed in the derivations below, is the following, which again is based on (20)-(23) of Sec. 3.9:

$$\Psi_{++}(s,s:a,b) = \Psi_{+-}(s,s:a,b)R(s,b) \tag{6}$$

$$= R(s,a)\,\Psi_{-+}(s,s:a,b) \tag{7}$$

$$I + \Psi_{++}(s,s:a,b) = [I - R(s,a)R(s,b)]^{-1} \tag{8}$$

$$\Psi_{--}(s,s:a,b) = \Psi_{-+}(s,s:a,b)R(s,a) \tag{9}$$

$$= R(s,b)\,\Psi_{+-}(s,s:a,b) \tag{10}$$

$$I + \Psi_{--}(s,s:a,b) = [I - R(s,b)R(s,a)]^{-1} \tag{11}$$

$$\Psi_{+-}(s,s:a,b) = R(s,a)[I + \Psi_{--}(s,s:a,b)] \tag{12}$$

$$= [I + \Psi_{++}(s,s:a,b)]R(s,a) \tag{13}$$

$$\Psi_{-+}(s,s:a,b) = R(s,b)[I + \Psi_{++}(s,s:a,b)] \tag{14}$$

$$= [I + \Psi_{--}(s,s:a,b)]R(s,b) \tag{15}$$

Integral Representations of the Local Ψ-Operators

We are now ready for the derivations of the representations of the local Ψ-operators in terms of the simpler $\mathcal{R}$ and $\mathcal{T}$ operators of the invariant imbedding relation. We fix attention at first on the setting within $X(a,b)$ depicted in Fig. 7.23.

An internal source is at level s in $X(a,b)$. The source may be a point, or some arbitrary discrete or continuous set of points on level s, and of arbitrary directional structure at each point of the set. We consider first the upward

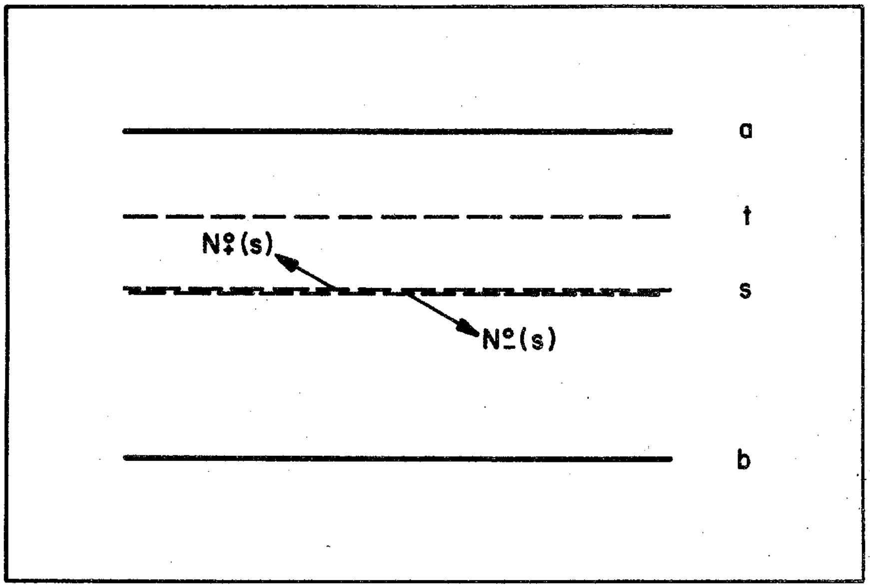


FIG. 7.23 An internal source situation in medium X(a,b). The first main case: source level (s) below observation level (t).

components $N_+^o(s)$ of the source $N^o(s)$. The resultant radiance field at level s generated throughout X(a,b) by this source component is, as explained in Sec. 3.9, given by the local Ψ-operator components $\Psi_{++}(s,s:a,b)$ and $\Psi_{+-}(s,s:a,b)$. The first of these gives the resultant upward field, the second gives the resultant downward field. Now consider $\Psi_{++}(s,s:a,b)$. We wish to study the dependence of $\Psi_{++}(s,s:a,b)$ on a, holding s and b fixed. It will turn out that knowledge of this dependence will lead directly to the requisite integral representation of $\Psi(s,s:a,b)$, in terms of a family of invariant imbedding operators for X(a,b).

Thus, imagine a family $\{X(t,b);a\leq t\leq b\}$ of optical media in which the original medium X(a,b) is imbedded (i.e., X(a,b) is a member of the family). The associated family of the local Ψ-operator is $\{\Psi_{++}(s,s:t,b): a\leq t\leq b\}$. The effect of varying t in $\Psi_{++}(s,s:t,b)$ can be determined by taking the derivative of $\Psi_{++}(s,s:t,b)$ in the following way:

$$\frac{\partial \Psi_{++}(s,s:t,b)}{\partial t} = \frac{\partial \Psi_{+-}(s,s:t,b)}{\partial t} R(s,b)$$

which is suggested by (6). Furthermore, (13) suggests that we write:

$$\frac{\partial \psi_{+-}(s,s:t,b)}{\partial t} = \frac{\partial}{\partial t}\{[I + \Psi_{++}(s,s:t,b)]R(s,t)\}$$

$$= \frac{\partial \Psi_{++}(s,s;t,b)}{\partial t} R(s,t)$$

$$+ [I + \Psi_{++}(s,s:t,b)] \frac{\partial R(s,t)}{\partial t}$$

$$= \frac{\partial \Psi_{++}(s,s:t,b)}{\partial t} R(s,t)$$

$$-[I + \Psi_{++}(s,s:t,b)]T(s,t)\rho_+(t)T(t,s) \quad .$$

The last equality comes from (28) of Sec. 7.1, applied to X(t,s). The minus sign comes from the fact that depth is measured positive from a to b. Returning to the original equation, we have:

$$\frac{\partial \Psi_{++}(s,s:t,b)}{\partial t} = \frac{\partial \Psi_{++}(s,s:t,b)}{\partial t} R(s,t)T(s,b)$$

$$-[I + \Psi_{++}(s,s:t,b)]T(s,t)\rho_+(t)T(t,s)R(s,b).$$

The derivative term may therefore be solved for and found to be of the form:

$$-\frac{\partial \Psi_{++}(s,s:t,b)}{\partial t} =$$

$$= [I + \Psi_{++}(s,s;t,b)]T(s,t)\rho_+(t)T(t,s)R(s,b)[I - R(s,t)R(s,b)]^{-1}$$

Upon examination, this seemingly complex representation collapses into the composition of three highly intuitive forms. One of these is $\mathscr{R}(t,s,b)$, since, by (42) of Sec. 3.7:

$$\mathscr{R}(t,s,b) = T(t,s)R(s,b)[I - R(s,t)R(s,b)]^{-1} \quad .$$

The next term to consider is:

$$[I + \Psi_{++}(s,s:t,b)]T(s,t) \quad . \tag{16}$$

This is an unfamiliar combination of operators. It has not arisen in our work as yet. However, there is a tantalizing asymmetry between it and (3) above. When the variables t,s,b are placed in (3) we have:

$$\mathscr{T}(t,s,b) = T(t,s)[I + \Psi_{--}(s,s:t,b)] \quad .$$

The interpretation of $\mathcal{T}(t,s,b)$ is at this stage of our studies well understood: downward incident radiance at level t generates a light field in X(t,b) and $\mathcal{T}(t,s,b)$ gives the downward component of that field of level s in X(t,b). (See Figure 7.23) The term (16) seems to give a dual interpretation to that of $\mathcal{T}(t,s,b)$. Thus, it says that upward source radiance at level s generates a light field within X(t,b) and (16) gives the upward component of that light field at level t in X(t,b). Hence (16) acts like a complete transmittance operator, but one whose input level (s) and output level (t) are exactly reversed from their customary relative orientations within X(t,b). With these interpretations and the dual kinship of (16) and $\mathcal{T}(t,s,b)$ in mind, let us write "$\mathcal{T}^{\dagger}(s,t,b)$" for (16). Then the differential equation for $\Psi_{++}(s,s:t,b)$ becomes:

— a
— t
— s
— b

$$\boxed{-\frac{\partial \Psi_{++}(s,s:t,b)}{\partial t} = \mathcal{T}^{\dagger}(s,t,b)\rho_{+}(t)\mathcal{R}(t,s,b)} \qquad (17)$$

Integrating each side of (17) over the interval [a,s], and using the fact that:

$$\Psi_{++}(s,s:s,b) = 0 \quad ,$$

we have:

— a
— t
— s
— b

$$\boxed{\begin{array}{l} \Psi_{++}(s,s:a,b) = \displaystyle\int_{a}^{s} \mathcal{T}^{\dagger}(s,t,b)\rho_{+}(t)\mathcal{R}(t,s,b)\, dt \\ a \le t \le s \le b \end{array}} \qquad (18)$$

The simple physical interpretation of (18) should not escape notice. Consider X(t,b). Imagine the source $N^{o}_{+}(s)$ at level s giving rise to the upward emergent radiance at level t in the space X(t,b). Then imagine a thin incremental layer added to X(t,b) at level t. This thin layer reflects some of the emergent flux via ($\rho_{+}(t)$) back down into X(t,b). This reflected flux sets up a light field in X(t,b), the upward component of which at level s being given by $\mathcal{R}(t,s,b)$. By letting X(t,b) grow another thin layer at level t, still another incremental light field is added to that at level s. By adding up all such increments, starting from level s and working up to level a, we obtain the total field at level s induced by the upward source component $N^{o}_{+}(s)$. The analytical representation (18) summarizes all this compactly, as shown. The little ideograph next to (17) and (18) serves to depict the relative positions of the depth variables in X(a,b).

The representation (18) is one of a pair of representations for $\Psi_{++}(s,s:a,b)$, the other arising when we imbed X(a,b) in the family $\{X(a,t): a \le t \le b\}$ of spaces. (See Figure 7.24) Then we consider:

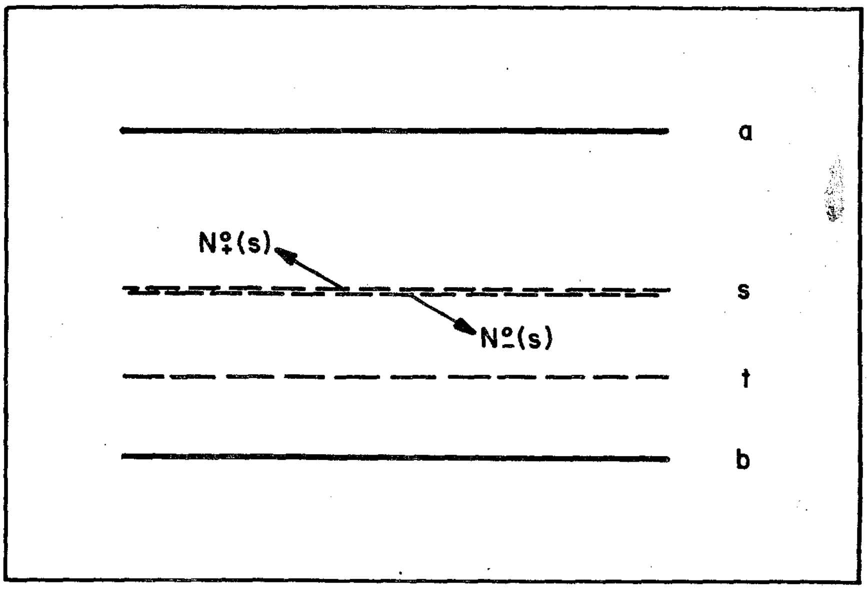


FIG. 7.24 An internal source situation in medium X(a,b). The second main case: source level (s) above observation level (t).

$$\frac{\partial \Psi_{++}(s,s:a,t)}{\partial t} = R(s,a) \frac{\partial \Psi_{-+}(s,s:a,t)}{\partial t} ,$$

using (7). This is analyzed further using (14), which yields:

$$\frac{\partial \Psi_{-+}(s,s:a,t)}{\partial t} = \frac{\partial R(s,t)}{\partial t} [I + \Psi_{++}(s,s:a,t)] + $$
$$+ R(s,t) \frac{\partial \Psi_{++}(s,s:a,t)}{\partial t} .$$

Once again a complex group of terms can be collapsed into a composition of three physically meaningful groups of terms. The operator $\rho_-(t)$ separates the two remaining terms. The last group of terms is simply $\mathcal{T}(t,s,a)$ (cf. (43) of Sec. 3.7). The first group of terms is one of those tantalizing duals to the invariant imbedding operators. This time, by studying (41) of Sec. 3.7 and (1) above, we see that we should write "$\mathcal{R}^\dagger(s,t,a)$" (see (26) below) so that the preceding differential equation for $\Psi_{++}(s,s:a,t)$ becomes:

— a
— s
— t
— b

$$\boxed{\frac{\partial \Psi_{++}(s,s:a,t)}{\partial t} = \mathcal{R}^\dagger(s,t,a)\rho_-(t)\, \mathcal{T}(t,s,a)} \qquad (19)$$

Integrating (19) over the interval [s,b] and using the fact that:

$$\Psi_{++}(s,s:a,s) = 0 \quad ,$$

we have:

— a
— s
— t
— b

$$\boxed{\Psi_{++}(s,s:a,b) = \int_s^b \mathcal{R}^{\dagger}(s,t,a)\rho_-(t)\mathcal{T}(t,s,a)\,dt \qquad a \le s \le t \le b} \tag{20}$$

The physical interpretation of (20) is as follows: consider the medium X(a,t) with upward source $N_+^o(s)$ at level s (Fig. 7.24). The light field in X(a,t) generated by this source has a downward component at level t given by $\mathcal{R}^{\dagger}(s,t,a)$. (If source flux is upward directed at s, then, since $\mathcal{R}^{\dagger}$ is a ***reflector***, response flux is downward directed at t.) Adding a thin increment to X(a,t) at level t causes a corresponding increment of reflected radiance (via $\rho_-(t)$) to re-enter X(a,t) and to be completely transmitted by $\mathcal{T}(t,s,a)$ to level s. By adding all such incremental layers on X(a,t) from t = s to t = b, the representation of $\Psi_{++}(s,s:a,b)$ is obtained.

The pattern emerging in these derivations should now be clear. On the basis of this emerging pattern the differential equations for $\Psi_{--}(s,s:t,b)$ and $\Psi_{--}(s,s:a,t)$ and other corresponding integrals can be written down directly without any further detailed derivation. However, the interested reader should verify the formulas so obtained:

— a
— t
— s
— b

$$\boxed{-\frac{\partial \Psi_{--}(s,s:t,b)}{\partial t} = \mathcal{R}^{\dagger}(s,t,b)\rho_+(t)\mathcal{T}(t,s,b)} \tag{21}$$

— a
— t
— s
— b

$$\boxed{\Psi_{--}(s,s:a,b) = \int_a^s \mathcal{R}^{\dagger}(s,t,b)\rho_+(t)\mathcal{T}(t,s,b)\,dt \qquad a \le t \le s \le b} \tag{22}$$

These equations are companions to (17) and (18) and go with Fig. 7.23. The following equations are companions to (19) and (20) and go with Fig. 7.24.

— a
— s
— t
— b

$$\boxed{\frac{\partial \Psi_{--}(s,s:a,t)}{\partial t} = \mathcal{T}^{\dagger}(s,t,a)\rho_-(t)\mathcal{R}(t,s,a)} \tag{23}$$

— a
— s
— t
— b

$$\boxed{\Psi_{--}(s,s:a,b) = \int_s^b \mathcal{T}^{\dagger}(s,t,a)\rho_-(t)\mathcal{R}(t,s,a)\,dt \qquad a \le s \le t \le b} \tag{24}$$

We now consolidate the definitions of the dual invariant imbedding operators. For this purpose, we use a general one-parameter setting X(a,b) and an arbitrary level s in X(a,b). The resultant definitions then will be completely dual to their counterparts in (1)-(4). Thus, for $a \le s \le b$ we write:

$$\text{"}\mathcal{R}^\dagger(s,a,b)\text{"} \quad \text{for} \quad \Psi_{-+}(s,s:a,b)T(s,a) \tag{25}$$

$$\text{"}\mathcal{R}^\dagger(s,b,a)\text{"} \quad \text{for} \quad \Psi_{+-}(s,s:a,b)T(s,b) \tag{26}$$

$$\text{"}\mathcal{T}^\dagger(s,a,b)\text{"} \quad \text{for} \quad [I + \Psi_{++}(s,s:a,b)]T(s,a) \tag{27}$$

$$\text{"}\mathcal{T}^\dagger(s,b,a)\text{"} \quad \text{for} \quad [I + \Psi_{--}(s,s:a,b)]T(s,b) \ . \tag{28}$$

The reader should study (1)-(4) and (25)-(28) to discover the rhyme and rule which bridges the gap between each of the pairs (1) and (25), (2) and (26), (3) and (27), (4) and (28). We call the four operators in (21)-(25) the *dual invariant imbedding operators*; $\mathcal{R}^\dagger$ being a *dual complete reflectance operator*, $\mathcal{T}^\dagger$ a *dual complete transmittance operator.*

The set of representations of the local Ψ-operator can be completed without any further derivative operations. To find the representation for $\Psi_{+-}(s,s:a,b)$ we may use either (12) or (13). For example, using (12) and (24) we have:

$$\Psi_{+-}(s,s:a,b) = R(s,a)\left[I + \int_s^b \mathcal{T}^\dagger(s,t,a)\rho_-(t)\mathcal{R}(t,s,a)\,dt\right] \ .$$

From this, we have at once:

—a
—s
—t
—b

$$\boxed{\begin{array}{l}\Psi_{+-}(s,s:a,b) = R(s,a) + \displaystyle\int_s^b \mathcal{R}^\dagger(s,t,a)\rho_-(t)\mathcal{R}(t,s,a)\,dt \\ a \le s \le t \le b\end{array}} \tag{29}$$

in which we have used the readily verified fact that:

$$\mathcal{R}^\dagger(s,t,a) = R(s,a)\mathcal{T}^\dagger(s,t,a) \quad . \tag{30}$$

On the other hand, using (13) and (20), we arrive at the same equation (29), as may be verified by the reader. (The dual relation to (30) is now used, namely (53) of Sec. 3.7.)

Finally $\Psi_{-+}(s,s:a,b)$ is found using either (14) or (15). For example, (14) and (18) yield:

$$\Psi_{-+}(s,s:a,b) = R(s,b)\left[I + \int_a^s \mathcal{T}^\dagger(s,t,b)\rho_+(t)\mathcal{R}(t,s,b)\,dt\right] \ .$$

From this and the fact that:

$$\mathcal{R}^\dagger(s,t,b) = R(s,b)\mathcal{T}^\dagger(s,t,b) \quad , \tag{31}$$

we have:

—a
—t
—s
—b

$$\boxed{\begin{array}{l} \Psi_{-+}(s,s:a,b) = R(s,b) + \displaystyle\int_a^s \mathscr{R}^\dagger(s,t,b)\rho_+(t)\mathscr{R}(t,s,b)\ dt \\ a \le t \le s \le b \end{array}} \tag{32}$$

Our goal requires us to find *all* possible representations of the local and global Ψ-operators. Thus, having found $\Psi_{+-}(s,s:a,b)$ in terms of an integral over $[s,b]$, we are led to seek the representation of $\Psi_{+-}(s,s:a,b)$ in terms of an integral over $[a,s]$. Equation (6) provides the clue: in (18) we should factor $R(s,b)$ from $\mathscr{R}(t,s,b)$ by means of (53) of Sec. 3.7. The result is:

$$\Psi_{++}(s,s:a,b) = \int_a^s \left[\mathcal{T}^\dagger(s,t,b)\rho_+(t)\mathcal{T}(t,s,b)\ dt\right]R(s,b) \quad .$$

By (6) we conclude that:

— a
— t
— s
— b

$$\boxed{\begin{array}{l} \Psi_{+-}(s,s:a,b) = \displaystyle\int_a^s \mathcal{T}^\dagger(s,t,b)\rho_+(t)\mathcal{T}(t,s,b)\ dt \\ a \le t \le s \le b \end{array}} \tag{33}$$

Equation (32) in turn spurs a search for a representation of $\Psi_{-+}(s,s:a,b)$ in terms of an integral over $[s,b]$. This time (7) makes it quite clear that, by factoring $R(s,a)$ from (20), in this manner:

$$\Psi_{++}(s,s:a,b) = R(s,a)\int_a^b \mathcal{T}^\dagger(s,t,a)\rho_-(t)\mathcal{T}(t,s,a)\ dt \quad ,$$

which is possible by (30), we must end up with:

— a
— s
— t
— b

$$\boxed{\begin{array}{l} \Psi_{-+}(s,s:a,b) = \displaystyle\int_s^b \mathcal{T}^\dagger(s,t,a)\rho_-(t)\mathcal{T}(t,s,a)\ dt \\ a \le s \le t \le b \end{array}} \tag{34}$$

Integral Representations of the Global Ψ-operators

It remains to derive the integral representations of the operators $\Psi(s,y:a,b)$. The set of relations connecting the global and local Ψ-operators, given in (31)-(34) of Sec. 3.9 are now put to work. Thus we consider first the case $a \le y < s \le b$ using (31) of Sec. 3.9 and (20). We have at once:

— a
— y
— s
— b

$$\boxed{\begin{array}{l}\Psi_{++}(s,y:a,b) = \mathcal{T}(s,y,a) + \displaystyle\int_s^b \mathcal{R}^{\dagger}(s,t,a)\rho_-(t)\,\mathcal{T}(t,y,a)\,dt \\ a \le y < s \le b\end{array}} \tag{35}$$

in which we have used the semigroup relation (52) of Sec. 3.7. Using (32) of Sec. 3.9 and (20), the result is:

— a
— y
— s
— b

$$\boxed{\begin{array}{l}\Psi_{+-}(s,y:a,b) = \mathcal{R}(s,y,a) + \displaystyle\int_s^b \mathcal{R}^{\dagger}(s,t,a)\rho_-(t)\mathcal{R}(t,y,a)\,dt \\ a \le y < s \le b\end{array}} \tag{36}$$

in which the semigroup relation (53) of Sec. 3.7 was used. Using (33) of Sec. 3.9 with (34) above, we have:

— a
— y
— s
— b

$$\boxed{\begin{array}{l}\Psi_{-+}(s,y:a,b) = \displaystyle\int_s^b \mathcal{T}^{\dagger}(s,t,a)\rho_-(t)\,\mathcal{T}(t,y,a)\,dt \\ a \le y < s \le b\end{array}} \tag{37}$$

Finally, according to (34) of Sec. 3.9 and (34) above:

— a
— y
— s
— b

$$\boxed{\begin{array}{l}\Psi_{--}(s,y:a,b) = \displaystyle\int_s^b \mathcal{T}^{\dagger}(s,t,a)\rho_-(t)\mathcal{R}(t,y,a)\,dt \\ a \le y < s \le b\end{array}} \tag{38}$$

The relations (35)-(38) constitute the set of functional relations for the case where the source level s is below the observation level y within X(a,b). For the case where the source level s is above the observation level y within X(a,b) we use the following readily verified set of dual equations to (31)-(34) of Sec. 3.9:

$$\Psi_{--}(s,y:a,b) = [I + \Psi_{--}(s,s:a,b)]\,\mathcal{T}(s,y,b) \tag{39}$$

$$\Psi_{-+}(s,y:a,b) = [I + \Psi_{--}(s,s:a,b)]\,\mathcal{R}(s,y,b) \tag{40}$$

$$\Psi_{+-}(s,y:a,b) = \Psi_{+-}(s,s:a,b)\,\mathcal{T}(s,y,b) \tag{41}$$

$$\Psi_{++}(s,y:a,b) = \Psi_{+-}(s,s:a,b)\,\mathcal{R}(s,y,b) \tag{42}$$

The preceding equations hold for $a \le s < y \le b$. To see how the derivations go in the present case, we use (22) and (39) to find:

— a
— s
— y
— b

$$\boxed{\Psi_{--}(s,y:a,b) = \mathcal{T}(s,y,b) + \int_a^s \mathcal{R}^{\dagger}(s,t,b)\rho_+(t)\,\mathcal{T}(t,y,b)\,dt \qquad a \le s < y \le b} \tag{43}$$

Equation (22) and (40) yield:

— a
— s
— y
— b

$$\boxed{\Psi_{-+}(s,y:a,b) = \mathcal{R}(s,y,b) + \int_a^s \mathcal{R}^{\dagger}(s,t,b)\rho_+(t)\,\mathcal{R}(t,y,b)\,dt \qquad a \le s < y \le b} \tag{44}$$

Equation (33) and (41) yield:

— a
— s
— y
— b

$$\boxed{\Psi_{+-}(s,y:a,b) = \int_a^s \mathcal{T}^{\dagger}(s,t,b)\rho_+(t)\,\mathcal{T}(t,y,b)\,dt \qquad a \le s < y \le b} \tag{45}$$

Finally, (33) and (42) yield:

— a
— s
— y
— b

$$\boxed{\Psi_{++}(s,y:a,b) = \int_a^s \mathcal{T}^{\dagger}(s,t,b)\rho_+(t)\,\mathcal{R}(t,y,b)\,dt \qquad a \le s < y \le b} \tag{46}$$

Incipient Patterns and Nascent Methods

The preceding development of the eight integral representations of the local Ψ-operator and the eight of the global Ψ-operator, followed fairly closely the actual sequence of discovery of the representations. This sequence was reproduced because it seems the most natural didactic path into the present subject matter. We have thus progressed far enough into the forest of integral representations to become acquainted with some of the important individual "trees". It is time, however, to rise above the trees and obtain a glimpse of the entire forest. By doing so we can discern the general pattern of the derivations given so far and thereby organize efficient methods of derivation of the remaining functional relations.

The principal observation to make on all the foregoing activity is on the manner of construction of the Ψ-operators in terms of simpler components; namely, the invariant imbedding operators and their duals and, of course, the local operators $\rho_\pm$ and $\tau_\pm$. We note in particular the way in which a layer in X(a,b) is made to grow from an imbedded core X(s,y) to the entire slab X(a,b), and how during this growth the original simple operators (standard R and T operators) are built up continuously and in a corresponding manner to obtain the Ψ-operator. This mode of construction of X(a,b) is closely related to the Categorical Synthesis Method developed in Ref. [251] for discrete spaces. Figure 7.25 helps describe the synthesis in the present continuous setting.

We begin with a degenerate case X(s,s), i.e., a single parameter surface X_s of X(a,b) which is irradiated by a source radiance $N^o(s)(=(N_+^o(s),N_-^o(s))$. Then a layer X(y,s), $y \le s$, is produced by letting X_y move upward away from X_s. The standard operators R(s,y), T(s,y) are shown in the figure as associated with X(s,y) or X(y,s) ("R(s,y)" denoting a reflectance for upward or downward radiance distribution, as the case may be). The response of X(s,y) or X(y,s) the source $N^o(s)$ at level s is given by these standard operators, since the source is external to the slabs. This completes the first stage of growth: we began with a single surface X_s and continuously built up a slab X(s,y) or X(y,s) from it keeping the irradiation $N^o(s)$ constant, and fixing attention on the response at level y.

The second stage of growth gives rise to four possibilities, as shown in Fig. 7.25. For example, we could start with the upper slab X(y,s) and let it grow to be X(a,s), $a \le y$. During this growth process, the source $N^o(s)$ is retained and we still want to know what the response at layer y is. We have just the conceptual tools to give us the answer, namely the complete reflectance and transmittance operators $\mathcal{R}(s,y,a)$, $\mathcal{T}(s,y,a)$. Alternatively, we could let X(y,s) grow into X(y,b), $s \le b$, with the source $N^o(s)$ remaining at level s. Now the source becomes "submerged" or imbedded in X(y,b) and the conceptual tools which will give us the response at level y are the *dual* invariant imbedding operators defined in (25) and (27) (with y = a). The two other remaining cases in the

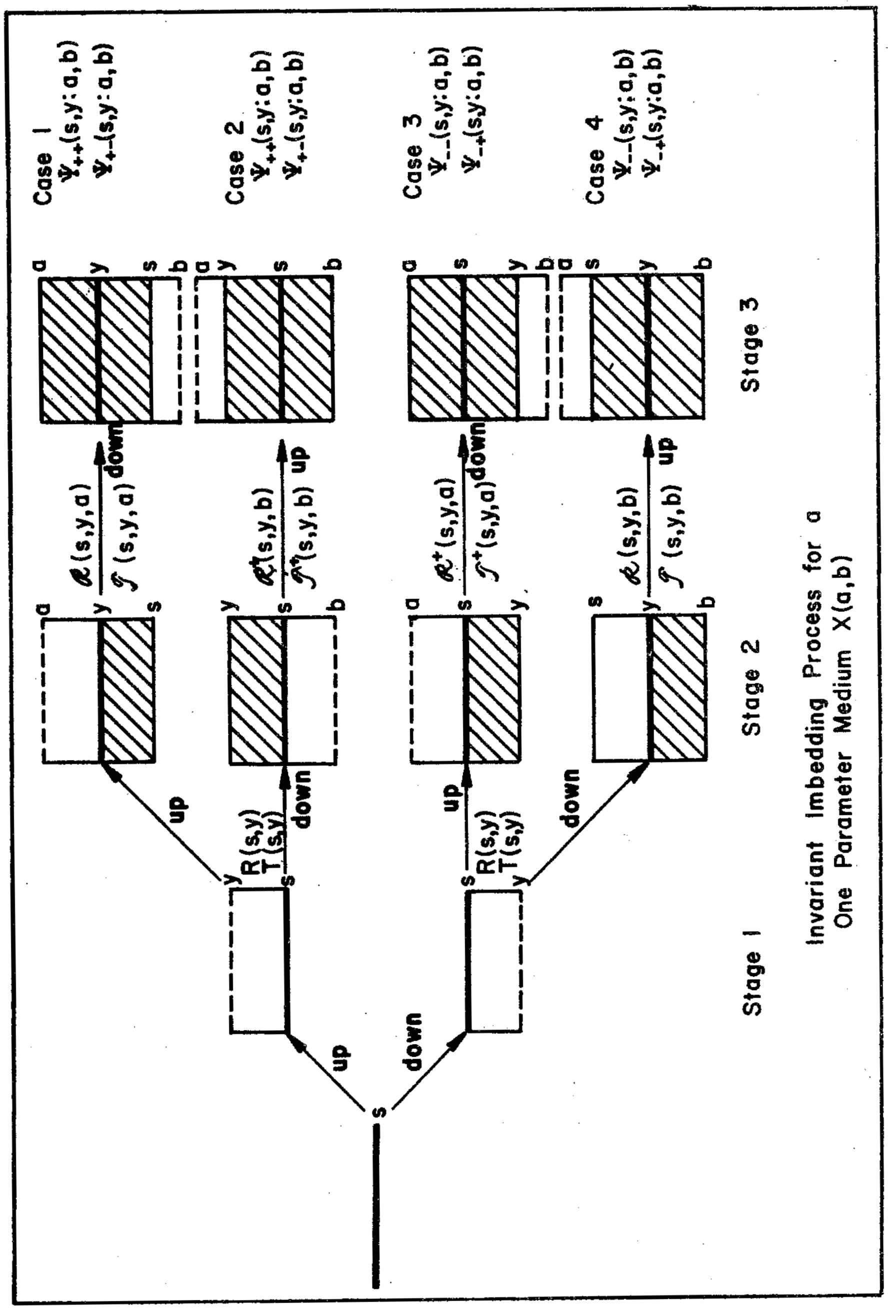


FIG. 7.25 A complete classification of the invariant imbedding process for a one-parameter medium X(a,b). Starting with the surface at level s, successive possibilities of growth of slabs are depicted in stages 1, 2, and 3.

second stage of growth are explained similarly.

The third stage of growth also gives rise to four distinct possibilities. For example, in case 1, starting with X(a,s) we let it grow downward to become X(a,b), $s \leq b$, still retaining the source at level s and still inquiring as to the response at level y. The interaction operator that describes this situation is the indicated global Ψ-operator. The remaining three cases are described similarly.

The results of the third stage of growth fall uniformly within the scope of the global Ψ-operator and thereby the most general source-response irradiation configuration within X(a,b) can be described using $\Psi(s,y:a,b)$. Thus, under the suitable confluence of the variables s,y,a,b, the invariant imbedding operators, their duals, and the standard slab operators are all forthcoming from $\Psi(s,y:a,b)$. This will be shown in detail later.

The overview of the preceding derivations of the eight integral representations of the local Ψ-operator is now before us. The settings of the derivations initially took place in the third stage of growth of Fig. 7.25 for the degenerate instance $s = y$. For example, Fig. 7.23, which goes with the derivation of (18), (22), (32), (33), falls under the degenerate instances of cases 2 and 4 in stage three. Finally, Fig. 7.24, which goes with the derivation of (20), (24), (29), (34), falls under the degenerate instances of cases 1 and 3 in stage 3.

The overview of the derivations of the global Ψ-operators in (35)-(46) given by Fig. 7.25 is quite interesting. Had we not made the systematic analysis of all growth possibilities, we might have missed the eight remaining possibilities beyond (35)-(46). To see this in detail, first note that representations (35)-(38) all fall under case 1 of stage three, and that representatives (43)-(46) all fall under case 4. Note further that cases 1 and 4 spring from the two associated invariant imbedding cases in stage 2. The two remaining cases yet to be derived spring from the *dual invariant imbedding contexts* of stage 2 and are depicted as cases 2 and 3 in Fig. 7.25. This will be done below.

A final facet of the overview obtained by means of Fig. 7.25 is that we should expect to find somewhere in the forest of functional relations currently under study a set of functional relations for the dual invariant imbedding operators $\mathcal{R}^\dagger$ and $\mathcal{T}^\dagger$ analogous in all respects to those for the invariant imbedding operators $\mathcal{R}$, $\mathcal{T}$, and $\mathcal{M}$ obtained in Sec. 7.5. The dual invariant imbedding operators encountered during the original versions of the derivations above appear to exist as intimate neighbors of the original operators and we should expect every property of the invariant imbedding operators $\mathcal{R}$ and $\mathcal{T}$ to have some 'dual' property in the other camp made up of the operators $\mathcal{R}^\dagger$ and $\mathcal{T}^\dagger$. Some of these relations for the dual operators will be derived subsequently.

These observations permit us to see the incipient patterns of similarity, in the functional relations at all stages of construction, forming during the invariant imbedding process on X(a,b), and also permit us to become aware of the possibility of a systematic method of derivation of the various integral representations of $\Psi(s,y:a,b)$. In the remaining space of this section we shall round out the family of integral and differential representations obtained so far by working on cases 2 and 3 in the third growth stage of Fig. 7.25. Furthermore, we shall look briefly into the matter of the functional relations for the dual invariant imbedding operators. Finally, we shall be able to make a thorough critique of the equations (35), (36) of Sec. 6.7, the equations which inspired the research leading to the results of the present section.

For a systematic imbedding procedure developed in complete detail in the discrete space context analogous to that depicted in Fig. 7.25, the reader may consult the Categorical Analysis method, Chapter X, Ref. [251].

Dual Integral Representations of the Global Ψ-operators

We now derive the dual integral representations to (43)-(46). Recall from our discussion of Fig. 7.25 that (43)-(46) are covered by case 4 in stage 3 of the growth pattern of $X(a,b)$. The dual case to this is case 2. Therefore we are to consider the situation where the source level is below the observation level, i.e., we have $a \le y < s \le b$. Equations (31)-(34) of Sec. 3.9 are therefore called up for use. Consider the component $\Psi_{++}(s,y:a,b)$. During the third stage of growth we have for case 2, with the help of (31) of Sec. 3.9:

$$-\frac{\partial \Psi_{++}(s,y:t,b)}{\partial t} = -\frac{\partial \Psi_{++}(s,s:t,b)}{\partial t}\,\mathcal{T}(s,y,t) - [I + \Psi(s,s:t,b)]\,\frac{\partial \mathcal{T}(s,y,t)}{\partial t}\;.$$

The derivative in the first term is given by (17); the derivative in the second term is given by (18) of Sec. 7.5. The resultant rate of change equation is:

$$\begin{aligned}-\frac{\partial \Psi_{++}(s,y:t,b)}{\partial t} &= \mathcal{T}^{\dagger}(s,t,b)\rho_{+}(t)\mathcal{R}(t,s,b)\,\mathcal{T}(s,y,t) + \\ &\quad + [I + \Psi_{++}(s,s:t,b)]T(s,t)\rho_{+}(t)\mathcal{R}(t,y,s) \\ &= \mathcal{T}^{\dagger}(s,t,b)\rho_{+}(t)[\mathcal{R}(t,s,b)\,\mathcal{T}(s,y,t)+\mathcal{R}(t,y,s)]\end{aligned}$$

The latter equation follows by use of (27). The final step uses (69) of Sec. 7.4, and we have the desired result:

$$\begin{array}{l} \text{— } a \\ \text{— } t \\ \text{— } y \\ \text{— } s \\ \text{— } b \end{array} \quad \boxed{\begin{array}{l} -\dfrac{\partial \Psi_{++}(s,y:t,b)}{\partial t} = \mathcal{T}^{\dagger}(s,t,b)\rho_{+}(t)\mathcal{R}(t,y,b) \\ a \leq t \leq y < s \leq b \end{array}} \tag{47}$$

Observe that (47) is identical to (17) in all respects save one: (47) has "y" in place of "s" in Ψ_{++} *and* in $\mathcal{R}$. This shows that the differential equation governing $\Psi_{++}(s,y:t,b)$ as a function of t holds for all y, $t \leq y \leq b$. This fact also holds for the differential equations governing the remaining three components of $\Psi(s,y:t,b)$. The requisite integral representation now follows from (47) by integrating from a to y and using the fact that:

$$\Psi_{++}(s,y:y,b) = \mathcal{T}^{\dagger}(s,y,b) \quad .$$

The result is:

$$\begin{array}{l} \text{—} a \\ \text{—} y \\ \text{—} s \\ \text{—} b \end{array} \quad \boxed{\begin{array}{l} \Psi_{++}(s,y:a,b) = \mathcal{T}^{\dagger}(s,y,b) + \displaystyle\int_a^y \mathcal{T}^{\dagger}(s,t,b)\rho_{+}(t)\mathcal{R}(t,y,b)\, dt \\ a \leq y < s \leq b \end{array}} \tag{48}$$

This equation is the dual to (43). The dual to (44) is based on the differential equation:

$$\begin{array}{l} \text{— } a \\ \text{— } t \\ \text{— } y \\ \text{— } s \\ \text{— } b \end{array} \quad \boxed{\begin{array}{l} -\dfrac{\partial \Psi_{+-}(s,y:t,b)}{\partial t} = \mathcal{T}^{\dagger}(s,t,b)\rho_{+}(t)\mathcal{T}(t,y,b) \\ a \leq t \leq y < s \leq b \end{array}} \tag{49}$$

which is derived analogously to (47), using (32) of Sec. 3.9. The corresponding integral representation is:

$$\begin{array}{l} \text{— } a \\ \text{— } y \\ \text{— } s \\ \text{— } b \end{array} \quad \boxed{\begin{array}{l} \Psi_{+-}(s,y:a,b) = \displaystyle\int_a^y \mathcal{T}^{\dagger}(s,t,b)\rho_{+}(t)\mathcal{T}(t,y,b)\, dt \\ a \leq y < s \leq b \end{array}} \tag{50}$$

in which we have used the fact that:

$$\Psi_{+-}(s,y:y,b) = 0 \quad .$$

The dual representation to (45) is based on the differential equation:

— a
— t
— y
— s
— b

$$\boxed{-\frac{\partial \Psi_{-+}(s,y:t,b)}{\partial t} = \mathcal{R}^{\dagger}(s,t,b)\rho_{+}(t)\mathcal{R}(t,y,b) \qquad a \le t \le y < s \le b} \tag{51}$$

The corresponding integral representation is:

— a
— y
— s
— b

$$\boxed{\Psi_{-+}(s,y:a,b) = \mathcal{R}^{\dagger}(s,y,b) + \int_a^y \mathcal{R}^{\dagger}(s,t,b)\rho_{+}(t)\mathcal{R}(t,y,b)\, dt \qquad a \le y < s \le b} \tag{52}$$

using the fact that:

$$\Psi_{-+}(s,y:y,b) = \mathcal{R}^{\dagger}(s,y,b) \quad .$$

Finally, the dual to (46) is based on:

— a
— t
— y
— s
— b

$$\boxed{-\frac{\partial \Psi_{--}(s,y:t,b)}{\partial t} = \mathcal{R}^{\dagger}(s,t,b)\rho_{+}(t)\mathcal{T}(t,y,b) \qquad a \le t \le y < s \le b} \tag{53}$$

Whence:

— a
— y
— s
— b

$$\boxed{\Psi_{--}(s,y:a,b) = \int_a^y \mathcal{R}^{\dagger}(s,t,b)\rho_{+}(t)\mathcal{T}(t,y,b)\, dt \qquad a \le y < s \le b} \tag{54}$$

in which was used the fact that:

$$\Psi_{--}(s,y:y,b) = 0 \quad .$$

It remains to derive four pairs of differential-integral representations of the components of $\Psi(s,y:a,b)$ for case 3 of stage 3, as depicted in Fig. 7.25. However, the details will be left as an instructive exercise for the interested student. The results should be dual to (35)-(38) (which is case 1 of stage 3) in the same general way that the preceding integral relations were dual to (43)-(46). The derivations must be carried out for the case $s < y$, so that one begins with (39)-(42). It should be observed that the sets (43)-(46) and their duals just derived are sufficient to completely describe the internal-source problem within X(a,b). Hence the derivations of the dual relations to (35)-(38) is an academic matter. Nevertheless a full understanding of the present method of derivation of the integral representations of $\Psi(s,y:a,b)$ is contingent on a complete list of the dual relations; for this reason they are appended below:

From:

$$\frac{\partial\Psi_{--}(s,y:a,t)}{\partial t} = \mathcal{T}^{\dagger}(s,t,a)\rho_{-}(t)\mathcal{R}(t,y,a) \quad , \qquad (55)$$

we have the dual to (35):

$$\Psi_{--}(s,y:a,b) = \mathcal{T}^{\dagger}(s,y,a) + \int_{y}^{b}\mathcal{T}^{\dagger}(s,t,a)\rho_{-}(t)\mathcal{R}(t,y,a)\,dt \quad . \qquad (56)$$

From:

$$\frac{\partial\Psi_{-+}(s,y:a,t)}{\partial t} = \mathcal{T}^{\dagger}(s,t,a)\rho_{-}(t)\mathcal{T}(t,y,a) \quad , \qquad (57)$$

we have the dual to (36):

$$\Psi_{-+}(s,y:a,b) = \int_{y}^{b}\mathcal{T}^{\dagger}(s,t,a)\rho_{-}(t)\mathcal{T}(t,y,a)\,dt \quad . \qquad (58)$$

From:

$$\frac{\partial\Psi_{+-}(s,y:a,t)}{\partial t} = \mathcal{R}^{\dagger}(s,t,a)\rho_{-}(t)\mathcal{R}(t,y,a) \quad , \qquad (59)$$

we have the dual to (37):

$$\Psi_{+-}(s,y:a,b) = \mathcal{R}^{\dagger}(s,y,a) + \int_{y}^{b}\mathcal{R}^{\dagger}(s,t,a)\rho_{-}(t)\mathcal{R}(t,y,a)\,dt \quad . \qquad (60)$$

From:

$$\frac{\partial\Psi_{++}(s,y:a,t)}{\partial t} = \mathcal{R}^{\dagger}(s,t,a)\rho_{-}(t)\mathcal{T}(t,y,a) \quad , \qquad (61)$$

we have the dual to (38):

$$\Psi_{++}(s,y:a,b) = \int_{y}^{b}\mathcal{R}^{\dagger}(s,t,a)\rho_{-}(t)\mathcal{T}(t,y,a)\,dt \quad . \qquad (62)$$

All these preceding equations (55)-(62) are valid for $a \le s < y \le t \le b$, and so may be envisioned by means of the ideograph:

— a
— s
— y
— t
— b

Logical Descendents of $\Psi(s,y:a,b)$

In our survey of the dynamics of the internal-source problem, as depicted in Fig. 7.25, we encountered during the building up of the medium X(a,b) through three stages, all the various interaction operators ranging from the standard operator for a slab (stage 1) through the invariant imbedding operators and their duals (stage 2), and culminating finally with the Ψ-operators. It follows that, given a general Ψ-operator, all these special interaction operators should be recoverable from $\Psi(s,y:a,b)$ under suitable choice of the parameters s,y,a,b. Various special cases were already encountered in the preceding work of this section. We now list these special instances of Ψ for convenient reference. One immediate use of the list is to reexamine the preceding representations and see how the Ψ-operator is built up from its most rudimentary special cases--an operation which on first view is reminiscent of pulling one's self up by one's bootstraps.

The setting for the present discussion is a general one-parameter space X(a,b) with an arbitrary source on level s, $a \le s \le b$. We shall consider two cases: first the case summarized as ($s=a$ or $s=b$), and then the case summarized as ($a<s<b$).

The invariant imbedding operators are, by their definitions, concerned with media X(a,b) which are source free. To simulate this, we set $s = a$ or $s = b$ in $\Psi(s,y:a,b)$. If we use (31)-(34) of Sec. 3.9 or (39)-(42) above, then the results are:

$$\Psi(a,y:a,b) = \begin{bmatrix} \Psi_{++}(a,y:a,b) & \Psi_{+-}(a,y:a,b) \\ \Psi_{-+}(a,y:a,b) & \Psi_{--}(a,y:a,b) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \mathcal{R}(a,y,b) & \mathcal{T}(a,y,b) \end{bmatrix} \tag{63}$$

$$\Psi(b,y:a,b) = \begin{bmatrix} \Psi_{++}(b,y:a,b) & \Psi_{+-}(b,y:a,b) \\ \Psi_{-+}(b,y:a,b) & \Psi_{--}(b,y:a,b) \end{bmatrix} = \begin{bmatrix} \mathcal{T}(b,y,a) & \mathcal{R}(b,y,a) \\ 0 & 0 \end{bmatrix} \tag{64}$$

Setting y = a in (63), we have:*

$$\Psi(a,a:a,b) = \begin{bmatrix} \Psi_{++}(a,a:a,b) & \Psi_{+-}(a,a:a,b) \\ \Psi_{-+}(a,a:a,b) & \Psi_{--}(a,a:a,b) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ R(a,b) & 0 \end{bmatrix} \quad . \tag{65}$$

Further y = a in (64) yields:

$$\Psi(b,a:a,b) = \begin{bmatrix} \Psi_{++}(b,a:a,b) & \Psi_{+-}(b,a:a,b) \\ \Psi_{-+}(b,a:a,b) & \Psi_{--}(b,a:a,b) \end{bmatrix}$$

$$= \begin{bmatrix} T(b,a) & 0 \\ 0 & 0 \end{bmatrix} \quad . \tag{66}$$

Setting y = b in (63):

$$\Psi(a,b:a,b) = \begin{bmatrix} \Psi_{++}(a,b:a,b) & \Psi_{+-}(a,b:a,b) \\ \Psi_{-+}(a,b:a,b) & \Psi_{--}(a,b:a,b) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & T(a,b) \end{bmatrix} \quad . \tag{67}$$

Setting y = b in (64):*

$$\Psi(b,b:a,b) = \begin{bmatrix} \Psi_{++}(b,b:a,b) & \Psi_{+-}(b,b:a,b) \\ \Psi_{-+}(b,b:a,b) & \Psi_{--}(b,b:a,b) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & R(b,a) \\ 0 & 0 \end{bmatrix} \quad . \tag{68}$$

A dual collection follows from $\Psi(s,a:a,b)$ by letting the source level s be inside X(a,b) (i.e., $a<s<b$) and limiting the response level y to be a or b. Thus we have:

*The operators Ψ_{++} and Ψ_{--} in these cases by convention (Sec. 3.9) must be interpreted as local Ψ-operators; hence the presence of the zero entries where formally one would have expected identity entries (cf. (20), (23) of Sec. 3.9).

$$\Psi(s,a:a,b) = \begin{bmatrix} \Psi_{++}(s,a:a,b) & \Psi_{+-}(s,a:a,b) \\ \Psi_{-+}(s,a:a,b) & \Psi_{--}(s,a:a,b) \end{bmatrix}$$

$$= \begin{bmatrix} \mathcal{T}^{\dagger}(s,a,b) & 0 \\ \mathcal{R}^{\dagger}(s,a,b) & 0 \end{bmatrix} . \tag{69}$$

$$\Psi(s,b:a,b) = \begin{bmatrix} \Psi_{++}(s,b:a,b) & \Psi_{+-}(s,b:a,b) \\ \Psi_{-+}(s,b:a,b) & \Psi_{--}(s,b:a,b) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \mathcal{R}^{\dagger}(s,b,a) \\ 0 & \mathcal{T}^{\dagger}(s,b,a) \end{bmatrix} . \tag{70}$$

The remaining four possibilities dual to (65)-(68) cannot be obtained directly from (69) and (70) by letting $s = a$ or $s = b$.

However, it is easy to see directly from the definitions (26)-(28) that:

$$\mathcal{T}^{\dagger}(a,a,b) = I \tag{71}$$

$$\mathcal{R}^{\dagger}(a,a,b) = R(a,b) \tag{72}$$

$$\mathcal{T}^{\dagger}(b,a,b) = T(b,a) \tag{73}$$

$$\mathcal{R}^{\dagger}(b,a,b) = 0 \tag{74}$$

$$\mathcal{R}^{\dagger}(a,b,a) = 0 \tag{75}$$

$$\mathcal{T}^{\dagger}(a,b,a) = T(a,b) \tag{76}$$

$$\mathcal{R}^{\dagger}(b,b,a) = R(b,a) \tag{77}$$

$$\mathcal{T}^{\dagger}(b,b,a) = I \tag{78}$$

Differential Equations for the Dual Operators

Our analysis of the three stages of the invariant imbedding process as depicted in Fig. 7.25 showed that certain dual operators to the invariant imbedding operators arose in stage two. The physical properties displayed by these dual operators throughout the discussions of this section indicated that we should expect the duality to be a thorough-going one--one that went deeper than their defining forms (26)-(28) or their physical interpretations pictorially summarized in Fig. 7.25. We now embark on a verification of that expectation by deriving the differential equations for the $\mathcal{R}^{\dagger}$ and $\mathcal{T}^{\dagger}$ operators arising in case 3 of stage 2 of the imbedding process. Case 2 of that stage, which proceeds analogously, will be left as an exercise for the reader who wishes to fix the method of derivation firmly in mind. The net result of the present activity will be a set of differential equations for the operators $\mathcal{R}^\dagger$ and $\mathcal{T}^\dagger$ which parallel in all essential respects those of the kind exemplified by (8)-(21) of Sec. 7.5 for the invariant imbedding operators and which can be used in practical numerical computations leading to approximate matricial forms of $\mathcal{R}^\dagger$ and $\mathcal{T}^\dagger$. The model for the reduction of the ensuing differential equations to matricial forms, amenable to numerical computations, was developed in Sec. 7.7.

In view of the introductory remarks, we fix attention on case 3 of stage 2, depicted in Fig. 7.25. In that case the slab $X(s,y)$ is allowed to grow upward or, in the terminology of invariant imbedding theory, we imbed $X(s,y)$ in the family $\{X(t,y): a \leq t \leq s\}$ of spaces, and then consider the rates of change of $\mathcal{T}^\dagger(s,t,y)$, and $\mathcal{R}^\dagger(s,t,y)$. From (27), on setting $b = y$, $a = t$, and on differentiating with respect to t, we have:

$$-\frac{\partial \mathcal{T}^\dagger(s,t,y)}{\partial t} = -\frac{\partial \Psi_{++}(s,s:t,y)}{\partial t} T(s,t) -$$

$$-[I + \Psi_{++}(s,s:t,y)] \frac{T(s,t)}{\partial t} \quad .$$

Using (17) (in which we set $b = y$) and using (27) of Sec. 7.1, suitably adapted to the present setting, we have:

$$\begin{aligned}-\frac{\partial \mathcal{T}^\dagger(s,t,y)}{\partial t} &= \mathcal{T}^\dagger(s,t,y)\rho_+(t)\mathcal{R}(t,s,y)T(s,t) + \\ &+[I+\Psi_{++}(s,s:t,y)]T(s,t)[\tau_+(t)+\rho_+(t)R(t,s)] \\ &= \mathcal{T}^\dagger(s,t,y)[\rho_+(t)\mathcal{R}(t,s,y)T(s,t)+\rho_+(t)R(t,s)+\tau_+(t)]\end{aligned}$$

Using (69) of Sec. 7.4 (in which we set $a = y = t$, $z = s$, $b = y$) we have:

$$R(t,y) = R(t,s) + \mathcal{R}(t,s,y)T(s,t) \quad .$$

In this way we arrive at:

— a
— t
— s
— y
— b

$$\boxed{-\frac{\partial \mathcal{T}^\dagger(s,t,y)}{\partial t} = \mathcal{T}^\dagger(s,t,y)[\tau_+(t) + \rho_+(t)R(t,y)] \qquad a \leq t \leq s \leq y \leq b} \tag{79}$$

The initial condition for $\mathcal{T}^\dagger(s,t,y)$ is (according to (71)):

$$\mathcal{T}^\dagger(s,s,y) = I \quad . \tag{80}$$

Thus, when integrating (79), we start with $t = s$. The value of the derivative at that initial point is known, being:

$$-\frac{\partial \mathcal{T}^{\dagger}(s,s,y)}{\partial t} = [\tau_{+}(s) + \rho_{+}(s)R(s,y)] \quad ,$$

where R(s,y) is known from the constructions of stage 1, depicted in Fig. 7.25.

Next, we return to (25), set b = y, a = t, and differentiate with respect to t:

$$-\frac{\partial \mathcal{R}^{\dagger}(s,t,y)}{\partial t} = -\frac{\partial \Psi_{-+}(s,s:t,y)}{\partial t} T(s,t) - \Psi_{-+}(s,s:t,y) \frac{\partial T(s,t)}{\partial t}$$

Using (32), we can find the derivative of $\Psi_{-+}(s,s:a,t)$ using the fundamental theorem of calculus (now applied to operator integrals). The remaining derivative is the same as before. The net result is:

$$-\frac{\partial \mathcal{R}^{\dagger}(s,t,y)}{\partial t} =$$

$$= \mathcal{R}^{\dagger}(s,t,y)\rho_{+}(t)\mathcal{R}(t,s,y)T(s,t) +$$

$$+ \Psi_{-+}(s,s:t,y)T(s,t)[\tau_{+}(t) + \rho_{+}(t)R(t,s)]$$

$$= \mathcal{R}^{\dagger}(s,t,y)[\rho_{+}(t)\mathcal{R}(t,s,y)T(s,t) + \rho_{+}(t)R(t,s) + \tau_{+}(t)]$$

Whence:

$$\begin{array}{l} \text{— a} \\ \text{— t} \\ \text{— s} \\ \text{— y} \\ \text{— b} \end{array} \quad \boxed{\begin{array}{l} -\dfrac{\partial \mathcal{R}^{\dagger}(s,t,y)}{\partial t} = \mathcal{R}^{\dagger}(s,t,y)[\tau_{+}(t) + \rho_{+}(t)R(t,y)] \\ a \le t \le s \le y \le b \end{array}} \quad . \quad (81)$$

The initial condition for $\mathcal{R}^{\dagger}(s,t,y)$ is (according to (72)):

$$\mathcal{R}^{\dagger}(s,s,y) = R(s,y) \quad .$$

Thus when integrating (81), we start with t = s. The value of the derivative at the initial point is known, being:

$$-\frac{\partial \mathcal{R}^{\dagger}(s,s,y)}{\partial s} = R(s,y)[\tau_{+}(s) + \rho_{+}(s)R(s,y)]$$

where R(s,y) is known from the constructions of stage 1, depicted in Fig. 7.25.

Together, (79) and (81) determine the dual invariant imbedding operators for case 3, stage 2 of the imbedding process on X(a,b). Yet the complete set of equations for that case requires four equations, as a perusal of the originals (18)-(21) of Sec. 7.5 would indicate. Evidently (81) and (79) are, respectively, the dual counterparts to (20) and (21) of Sec. 7.5. Recall that the operators ρ(t) and τ(t) in Sec. 7.5

were constructed in an isotropic medium, so that subscript signatures were not needed there, since $\rho_+(t) = \rho_-(t)$, and $\tau_+(t) = \tau_-(t)$. In the present setting isotropy was not assumed, principally to point up the interesting and thoroughgoing dualities under study in this section. For, without isotropy, we have generally distinct pairs of local operators $\rho_+(t)$, $\rho_-(t)$ and $\tau_+(t)$, $\tau_-(t)$. The subscripts on the local operators also help one to more readily write and read equations associated with upward (+) and downward (-) radiant fluxes. The remaining two equations for case 3, stage 2 are obtained from (26) and (28) on setting a = t, b = y and differentiating with respect to t:

$$-\frac{\partial \mathcal{R}^{\dagger}(s,y,t)}{\partial t} = -\frac{\partial \Psi_{+-}(s,s:t,y)}{\partial t}\,T(s,y)$$

$$= \mathcal{T}^{\dagger}(s,t,y)\,\rho_+(t)\,\mathcal{T}(t,s,y)\,T(s,y)$$

whence:

$$\begin{array}{l} \text{— a} \\ \text{— t} \\ \text{— s} \\ \text{— y} \\ \text{— b} \end{array} \quad \boxed{\begin{array}{l} -\dfrac{\partial \mathcal{R}^{\dagger}(s,y,t)}{\partial t} = \mathcal{T}^{\dagger}(s,t,y)\,\rho_+(t)\,T(t,y) \\ a \le t \le s \le y \le b \end{array}} \quad . \qquad (82)$$

Similarly:

$$\begin{array}{l} \text{— a} \\ \text{— t} \\ \text{— s} \\ \text{— y} \\ \text{— b} \end{array} \quad \boxed{\begin{array}{l} -\dfrac{\partial \mathcal{T}^{\dagger}(s,y,t)}{\partial t} = \mathcal{R}^{\dagger}(s,t,y)\,\rho_+(t)\,T(t,y) \\ a \le t \le s \le y \le b \end{array}} \quad . \qquad (83)$$

These equations are subordinate to (79) and (81) in the sense that they are powerless to support computations for their respective operators. Equations (79) and (81) are the *autonomous* equations for the present case. Once these are solved, (82) and (83) may be used to find $\mathcal{R}^{\dagger}(s,y,t)$ and $\mathcal{T}^{\dagger}(s,y,t)$.

A Colligation of the Component Ψ-operator Equations

We now reach one of the goals of this section by means of a study of equations (35), (36) of Sec. 6.7, as derived by Elliott in the neutron transport context in Ref. [88]. Specifically, our goal is to place the equations into their proper perspective within the domain of invariant imbedding techniques, to see their domain of validity, and to indicate their proper generalizations. In order to do this efficiently we must bind together the relatively massive collection of integral representations for the components of the global Ψ-operator obtained so far. To this task we now turn.

First of all we observe that equations (35), (36) of Sec. 6.7 are associated with two cases of location of the source level c relative to the observation level z. We can see these two cases in perspective by means of Fig. 7.25. Thus we find that the proper setting of Elliott's equations is in Stage 3 of the invariant imbedding process for the semi-infinite medium $X(0,\infty)$ ($a = 0$, $b = \infty$ in $X(a,b)$). In particular cases 2 and 3 correspond, respectively, to the source level below the observation level ($y<s$) and the source level above the observation level ($s<y$).

In case 2 the medium $X(y,b)$ with internal source at level s, $y<s\leq b$ is imbedded in the family $\{X(t,b): a\leq t\leq y\}$ of spaces. The source at level s is described by the incident source radiance distribution $N^o(s)$ which has an upward component $N^o_+(s)$ and a downward component $N^o_-(s)$ defined at each point of the parameter surface X_s. We are interested in the response radiance distribution $N(y)$ over X_y, where $N(y) = (N_+(y),N_-(y))$. By (15) of Sec. 3.9 we have:

$$N(y) = N^o(s)\Psi(s,y:a,b) \tag{84}$$

where now, by (48), (50), (52), and (54), $\Psi(s,y:a,b)$ can be given a specific representation in terms of integral operations on the invariant imbedding operators and their duals:

$$\Psi(s,y:a,b) =$$

$$= \begin{bmatrix} \mathcal{T}^{\dagger}(s,y,b) + \int_a^y \mathcal{T}^{\dagger}(s,t,b)\rho_+(t)\mathcal{R}(t,y,b)\,dt & \int_a^y \mathcal{T}^{\dagger}(s,t,b)\rho_+(t)\mathcal{T}(t,y,b)\,dt \\ \mathcal{R}^{\dagger}(s,y,b) + \int_a^y \mathcal{R}^{\dagger}(s,t,b)\rho_+(t)\mathcal{R}(t,y,b)\,dt & \int_a^y \mathcal{R}^{\dagger}(s,t,b)\rho_+(t)\mathcal{T}(t,y,b)\,dt \end{bmatrix}$$

This formidable structure reduces to a rather simple and intuitively interesting integral expression by means of (63) and (69). With these two equations as a base, it is easy to cast the preceding matrix into the form:

—a
—y
—s
—b

$$\boxed{\begin{gathered}\Psi(s,y:a,b) = \Psi(s,y:y,b) + \int_a^y \Psi(s,t:t,b)\,\mathcal{K}(t)\,\Psi(t,y:t,b)\,dt \\ a\leq y<s\leq b \qquad \text{case 2, stage 3, Fig. 7.25}\end{gathered}} \tag{85}$$

Here $\mathcal{K}(t)$ is the local interaction operator defined in (7) of Sec. 7.1. Equation (85) is the desired generalization of (35) of Sec. 6.7. We can pair off the corresponding functions in (35) of Sec. 6.7 and (85) as follows: $\Psi(s,y:0,\infty)$ pairs off with $f_s(y,\omega)$ (replacing "c" by "s" and "z" by "y" in (35) of Sec. 6.7). Further, $\Psi(s,y:y,\infty)$ pairs off with $f_o(s-y;\omega)$. This pairing is understood more clearly by noting at this point a certain reciprocity property of $f_o(z,\omega)$ proved in Ref. [88]. This property is the following:

$$f_o(z,\omega) = f_z(0,\omega)$$

for every depth z in $X(0,\infty)$.

In words, this states that the Fourier transform of the scalar irradiance field at depth 0 produced by a point source at depth s (represented by "$f_o(s,\omega)$") is equal to that at depth s produced by a point source at depth 0 (represented by "$f_s(0,\omega)$"). In this way we see how $\Psi(s,y:y,\infty)$ pairs off with $f_o(s-y,\omega)$ ($= f_{s-y}(0,\omega)$). Further, the operator $\Psi(s,t:t,\infty)$ pairs off with $f_o(s-t,\omega)$, and $\Psi(t,y:t,\infty)$ pairs off with $f_o((y-s)+(s-t),\omega) = f_o(y-t,\omega)$. Finally, $\mathcal{X}(t)$ pairs off with $s/h\eta$. The pairings between the terms of (85), and (35) of Sec. 6.7, of course, cannot be exact, for the obvious reason that (35) of Sec. 6.7 is a vastly simpler equation than (85). Yet, remarkably, the *general forms* of the two equations are identical and this is an attestation of the *invariant* nature of the semigroup formulation of transport processes with respect to both the changes in the superficial geometric structure of the media within which they evolve, and also with respect to the type of radiometric concept used in the formulation.

Having dispatched case 2, we now consider case 4 in stage 3 of Fig. 7.25, according to our present purposes. Equations (43)-(46) may be used to obtain the following matricial form of $\Psi(s,y:a,b)$, $s<y$:

$$\Psi(s,y:a,b) =$$

$$= \begin{bmatrix} \displaystyle\int_a^s \mathcal{T}^{\dagger}(s,t,b)\rho_+(t)\mathcal{R}(t,y,b)\,dt & \displaystyle\int_a^s \mathcal{T}^{\dagger}(s,t,b)\rho_+(t)\mathcal{T}(t,y,b)\,dt \\ \mathcal{R}(s,y,b) + \displaystyle\int_a^s \mathcal{R}^{\dagger}(s,t,b)\rho_+(t)\mathcal{R}(t,y,b)\,dt & \mathcal{T}(s,y,b) + \displaystyle\int_a^s \mathcal{R}^{\dagger}(s,t,b)\rho_+(t)\mathcal{T}(t,y,b)\,dt \end{bmatrix}$$

By means of (63) and (69) we can reduce this to the intuitively meaningful form:

$$\begin{array}{l} - a \\ - s \\ - y \\ - b \end{array} \boxed{\begin{array}{c} \Psi(s,y:a,b) = \Psi(s,y:s,b) + \displaystyle\int_a^s \Psi(s,t:t,b)\,\mathcal{X}(t)\,\Psi(t,y:t,b)\,dt \\ a \le s < y \le b \qquad \text{Case 4, stage 3, Fig. 7.25} \end{array}} \tag{86}$$

Equation (86) is the general correspondent to (36) of Sec. 6.7. We have now reached the first of our main goals of this section, in the form of (85) and (86), and we pause to make some observations on their structure and further observations on their physical interpretations.

The first things to observe about (85) and (86) are some of their common features: they both use the same integrand in their integral terms; however, the limits differ and this difference reflects the two cases of source-observation levels in X(a,b). Both equations yield an expression for $\Psi(s,y:a,b)$ using invariant imbedding operators and their duals (cf., (63), (69)). These operators are found in stage 2 of the invariant imbedding process on X(a,b). Hence (85) and (86), in the framework of Fig. 7.25, are simply instructions on how to construct the operators of stage 3 from those of stage 2.

Asymmetries of the Ψ-operator

Next we observe a most interesting dissimilarity between (85) and (86). This occurs in the form of the term added to the integral. In (86) the added term is made up of the invariant imbedding operators $\mathscr{R}(s,y,b)$ and $\mathscr{T}(s,y,b)$. In (85) the added term is made up of the *dual* invariant imbedding operators $\mathscr{R}^\dagger(s,y,b)$, $\mathscr{T}^\dagger(s,y,b)$. The diagrammatic representations of X(a,b) next to the equations helps to immediately recall the physical interpretations of these various operators. Now, in the scalar irradiance context, one form of counterpart to $\Psi(s,y:s,b)$ is $f_o(y-s,\omega)$ and that of $\Psi(s,y:y,b)$ is, as we have seen, $f_{s-y}(0,\omega)$. Elliott has shown, in a homogeneous isotropic half space with scalar irradiance as the radiometric quantity described by $f_s(y,\omega)$, that the reciprocity relation:

$$f_o(y-s,\omega) = f_{y-s}(0,\omega) \tag{87}$$

holds when y>s, say. This being so, our attention turns immediately to $\Psi(s,y:s,b)$ and $\Psi(s,y:y,b)$ and we ask: under what conditions on the geometry of X(a,b) and its inherent optical properties do we have:

$$\Psi(s,y:s,b) = \Psi(y,s:s,b) \tag{88}$$

— s
— y
— b

as a valid equation?

The diagrammatic insert under the equation shows the geometric context in which the question is asked. It should be observed that this is equivalent to the equation:

$$\Psi(s,y:s,b) = \Psi(s,y:y,b) \tag{89}$$

— s	— y
— y	— s
— b	— b

in the geometric context where names of the levels s and y are interchanged in the same medium as shown by the diagrams below (89).

To see the conditions under which (88) holds, it is sufficient to examine each of the four operator equations within (88). Thus, consider, for example, the equation arising from the "++" components of (88):

$$\Psi_{++}(s,y:s,b) = \Psi_{++}(y,s:s,b) \quad . \tag{90}$$

From this we see at once that (88) cannot generally hold on the operator level since the left side is always zero, while the right is generally not zero. This establishes the fact that the simple scalar condition (87) has no *exact* counterpart in the general operational transport formulations we are now considering. However, we still may inquire as to the other pairs of components in (88). Those pairs that are not zero--are they ever equal? Or: are the *sums* of the components of the left side equal to the *sums* of the components on the right side of (88)? The latter question is prompted by energy conservation considerations. The latter question will be considered subsequently in Chapter 8 in a setting where the question makes physical sense (Example 10, Sec. 8.7). For the present we examine the former question out of simple curiosity.

The diagram below (88) suggests that if we are to find a corresponding pair of nonzero components in (88), it would be those with the signature "-+". (Cf. (63), (69).) Consider then, for possible validity, the statement:

$$\Psi_{-+}(s,y:s,b) = \Psi_{-+}(y,s:s,b)$$

which is equivalent to:

$$\mathcal{R}(s,y,b) = \mathcal{R}^{\dagger}(y,s,b) \quad .$$

By (1) and (25) this is equivalent to:

$$R(y,b)[I-R(y,s)R(y,b)]^{-1}T(y,s) = T(s,y)[I-R(y,b)R(y,s)]^{-1}R(y,b)$$

which in turn is equivalent to:

$$[I-R(y,s)R(y,b)]T(s,y)R(y,b) = R(y,b)T(y,s)[I-R(y,s)R(y,b)] \quad .$$

For this to be valid, it is sufficient to have commutation freely possible between R(y,s),R(y,b) and T(s,y), T(y,s) along with

$$T(s,y) = T(y,s) \tag{91}$$

and, among other things:

$$R(y,s)R(y,b)T(s,y) = T(y,s)R(y,s)R(y,b) \quad . \tag{92}$$

At this point, our studies of Sec. 7.12 may be used to help clear the air of present question. The polarity theorem asserts that a *plane-parallel medium* X(a,b) must be isotropic and separable in order that (91) hold. This is not too stringent a requirement on the medium and its inherent optical properties. However, if X(a,b) is not plane-parallel, it is generally the case that (91) no longer holds, no matter how regular its inherent optical properties. That commutativity and condition (92) are also to hold--i.e., to have a reciprocity condition--is hopeless in general. One exception occurs in the scalar context, i.e., when the $R(y,b)$, $R(y,s)$ and $T(s,y)$ are real valued functions of s, y, b and not matrices or integral operators (as in the present discussion).

In this way we see that (35) and (36) of Sec. 6.7 cannot be directly generalized to the operator level without loss of the rather special reciprocity condition (87). This is a small loss in view of the fact that (85) and (86) are capable, as they stand, of solving in principle the most general point source problems on continuous one-parameter optical media. Their complementary counterparts associated with cases 1 and 3 in stage 3 of Fig. 7.25 are also capable of performing this service. The derivation of the associated equations are left to the reader as an important exercise (cf., (108)-(111) below.

A Royal Road to the Internal-Source Functional Relations

It was perhaps somewhat of an anticlimax for the attentive reader to see the four operator equations of case 2, stage 3 (in Fig. 7.25), so hard-won through the early portions of this section, unceremoniously collapsed into the simple operator equation (85). Still another such revelation may have occurred when (86) was reached. Be that as it may, the relative simplicity of (85) and (86), compared with the system of their progenitors, attests to the correctness of the deductions and to the power of the invariant imbedding approach which gave us the general Ψ-operator concept. But yet the very simplicity of these results invites an attempt of a correspondingly simple derivation of (85) and (86). We shall now indicate the outlines of such a derivation. We shall be very careful not to add all the rigorous details or else we shall simply retrace the work of this section. Thus we shall embark on a 'royal road' to (85) and (86), in the sense that it is ostensibly well-paved with no long steep grades, and along which the analytic and algebraic pitfalls have been filled and smoothed with rhetoric.

We choose as a setting case 2 of stage 3 in Fig. 7.25. The present derivation begins with a partition of $X(a,b)$ by the internal surface X_t, $a \leq t \leq y$. The only source on or in $X(a,b)$ is at level s, $y < s \leq b$; and this is of an arbitrary nature. Having partitioned $X(a,b)$ into two parts $X(a,t)$, $X(t,b)$, we isolate $X(t,b)$ and consider all sources incident on it. There are two incident sources on $X(t,b)$: the hypothesized internal source at level s, and the externally incident flux at level t coming from $X(a,t)$, and which is part of the integral light field set up throughout $X(a,b)$ by the source at level s. Therefore, by the interaction principle (cf. (38), Sec. 3.9), we have two operators $\Psi(s,y{:}t,b)$ and $\Psi(t,y{:}t,b)$ associated with $X(t,b)$ such that the response field $N(y)$ at level y in $X(t,b)$ is given by:

$$N(y) = N^{o}(s)\Psi(s,y{:}t,b) + N(t)\Psi(t,y{:}t,b) \quad . \qquad (93)$$

A detailed analysis by means of the principles of invariance on $X(a,b)$ shows that the radiance distribution over X_t is the result of two activities: the overall transmission of the effects of $N^o(s)$ within $X(t,b)$ up to X_t *and* the response at level t of the total interaction of this transmitted flux as it oscillates between $X(a,b)$ and $X(t,b)$. For the first of these we have:

$$N^o(s)\Psi(s,t:t,b) \quad ,$$

and accounting for the second of these we have:

$$N(t) = [N^o(s)\Psi(s,t:t,b)]\Psi(t,t:a,b) \quad . \tag{94}$$

Using this in (93) we see that:

$$N(y) = N^o(s)[\Psi(s,y:t,b) + \Psi(s,t:t,b)\Psi(t,t:a,b)\Psi(t,y:t,b)] \tag{95}$$

Since we have also:

$$N(y) = N^o(s)\Psi(s,y:a,b) \quad ,$$

the conclusion is:

$$\boxed{\begin{aligned}&\Psi(s,y:a,b) = \Psi(s,y:t,b) + \Psi(s,t:t,b)\Psi(t,t:a,b)\Psi(t,y:t,b)\\ &\quad a\le t\le y, \quad t\le s\le b\end{aligned}} \tag{96}$$

Equation (96) is the desired functional relation for case 2. An examination of its derivation shows that it is actually quite general, holding also for case 4 and within an arbitrary one-parameter space $X(a,b)$ with a source at level s. Equation (96) is a *finite algebraic counterpart* to (85) and (86). Observe in particular how $\Psi(s,y:a,b)$ can be calculated from knowledge of $\Psi(s,y:t,b)$ (the operator for a smaller medium $X(t,b)$ within $X(a,b)$) and the invariant imbedding operators $\Psi(s,t:t,b)$, $\Psi(t,y:t,b)$, and the local Ψ-operator $\Psi(t,t:a,b)$.

The next step is to use (96) to form the difference quotient:

$$\frac{\Psi(s,y:a,b) - \Psi(s,y:t,b)}{t-a} = \Psi(s,t:t,b)\,\frac{\Psi(t,t:a,b)}{t-a}\,\Psi(t,y:t,b)$$

and go to the limit as $t\to a$. The left side becomes $-\partial\Psi(s,y:a,b)/\partial a$; the right side can be reduced with the aid of (63), (69) and the set (20)-(23) of Sec. 3.9. The result is:

$$\boxed{-\frac{\partial\Psi(s,y:a,b)}{\partial a} = \Psi(s,a:a,b)\mathcal{K}(a)\Psi(a,y:a,b)} \tag{97}$$

where $\mathcal{X}(a)$ is defined in (7) of Sec. 7.1. The reader may also find (21) of Sec. 7.3 helpful in the verification of (97).

Equation (85) is now readily forthcoming from (97): in (97) replace "a" by "t" and integrate over all t from a to y. To obtain (86), integrate the modified (97) from a to s. Once the integral expressions have been obtained, they may be dissected to release their associated quartets of operator equations. In this way eight of the earlier integral expressions are obtained--namely those for cases 2 and 4 in stage 3 of Fig. 7.25.

The preceding derivation has demonstrated the rich analytic harvest that can be yielded up by (97). However, the potentialities of (97) have by no means been exhausted. Suppose we return to Fig. 7.25 and now move over to stage 2 in the invariant imbedding process for X(a,b). Equation (97) will now be associated with cases 2 and 4 in stage 2. For example, in case 2 we have a = y and we are therefore led to consider derivatives of the form:

$$\frac{\partial \Psi(s,y:y,b)}{\partial y} \quad .$$

It will be helpful to note that $\Psi(s,y:y,b)C_+ = \Psi(s,y:y,b)$. By the rules of elementary calculus, the preceding derivative is to be interpreted as:

$$\lim_{a\to y} \frac{\partial \Psi(s,a:y,b)}{\partial a} C_+ + \lim_{a\to y} \frac{\partial \Psi(s,y:a,b)}{\partial a} C_+ \quad .$$

The matrix C_+ serves to keep the second columns of the two matrices zero, as required. This type of derivative operation has been considered earlier in connection with the standard reflectance and transmittance operators (cf. (23), (24) of Sec. 7.5). The first of these derivatives has been studied earlier ((33) of Sec. 7.5) and we have:

$$\begin{aligned}\lim_{a\to y} \frac{\partial \Psi(s,y:a,b)}{\partial y} &= \lim_{a\to y} \Psi(s,y:a,b)\mathcal{X}(a) \\ &= \Psi(s,y:y,b)\mathcal{X}(y) \quad .\end{aligned}$$

The second derivative is obtained from (97):

$$\lim_{a\to y} \frac{\partial \Psi(s,y:a,b)}{\partial a} = -\Psi(s,y:y,b)\mathcal{X}(y)\Psi(y,y:y,b) \quad .$$

Hence:

$$\boxed{-\frac{\partial \Psi(s,y:y,b)}{\partial y} = \Psi(s,y:y,b)\mathcal{X}(y)\left[\Psi(y,y:y,b) - C_+\right]} \qquad (98)$$

by noting that $\Psi(y,y:y,b)C_+ = \Psi(y,y:y,b)$. Equation (98) is the generic differential equation for the invariant imbedding operators in case 2 of stage 2 in Fig. 7.25. These operators were already obtained piecemeal in (79)-(83). In like manner, the operators of stage 1 of Fig. 7.25 may be obtained from (98) (and its complement associated with cases 1 and 3 of stage 2) by suitable confluence of variables (namely $b \to s$). This is left to the reader as an important exercise.

It is of interest to note that (97) and (98) may be cast into a form which explicitly exhibits the invariant imbedding operators. First recall that we have written:

$$"\mathcal{M}(a,y,b)" \quad \text{for} \quad \begin{bmatrix} \mathcal{T}(b,y,a) & \mathcal{R}(b,y,a) \\ \mathcal{R}(a,y,b) & \mathcal{T}(a,y,b) \end{bmatrix} . \tag{99}$$

With this and (69) and (70) as guides, we write:

$$"\mathcal{M}^\dagger(s,a,b)" \quad \text{for} \quad \begin{bmatrix} \mathcal{T}^\dagger(s,a,b) & \mathcal{R}^\dagger(s,b,a) \\ \mathcal{R}^\dagger(s,a,b) & \mathcal{T}^\dagger(s,b,a) \end{bmatrix} . \tag{100}$$

Then (97) becomes:

$$\boxed{-\frac{\partial \Psi(s,y:a,b)}{\partial a} = \mathcal{M}^\dagger(s,a,b)C_+ \mathcal{X}(a)C_- \mathcal{M}(a,y,b)} \tag{101}$$

where C_+ and C_- are defined in (4), (5) of Sec. 7.4. Turning now to (98) for the purpose of converting it into invariant imbedding operator form, we note that:

$$\Psi(s,y:y,b) + \Psi(s,b:y,b) = \mathcal{M}^\dagger(s,y,b) \tag{102}$$

for $y \le s \le b$. This may be checked by recalling (100), (69), and (70). To find the derivative of $\mathcal{M}^\dagger(s,y,b)$ with respect to y, we need only find those of $\Psi(s,y:y,b)$ and $\Psi(s,b:y,b)$ with respect to y. Equation (98) gives us one of these; and (97) gives us the other on making the permissible substitutions: $(y \to b)$, $(a \to y)$ in (97). With these observations, we have:

$$\begin{aligned} -\frac{\partial \mathcal{M}^\dagger(s,y,b)}{\partial y} &= -\left[\frac{\partial \Psi(s,y:y,b)}{\partial y} + \frac{\partial \Psi(s,b:y,b)}{\partial y}\right] \\ &\quad + \Psi(s,y:y,b)\mathcal{X}(y)[\Psi(y,y:y,b) - C_+] \\ &\quad + \Psi(s,y:y,b)\mathcal{X}(y)[\Psi(y,b:y,b)] \\ &= \Psi(s,y:y,b)\mathcal{X}(y)[\Psi(y,y:y,b) + \Psi(y,b:y,b) - C_+] \end{aligned}$$

Recall (102) and note that:

$$\mathcal{M}^{\dagger}(s,y,b)C_{+} = \Psi(s,y:y,b) \quad .$$

Then the net result is:

$$\boxed{-\frac{\partial \mathcal{M}^{\dagger}(s,y,b)}{\partial y} = \mathcal{M}^{\dagger}(s,y,b)C_{+}\,\mathcal{X}(y)[\mathcal{M}^{\dagger}(y,y,b) - C_{+}] \quad y \le s \le b} \tag{103}$$

If this operator equation is opened up, the four resultant component equations have precisely the forms of (79), (81), (82), (83). The dual to (103) is (11) of Sec. 7.5, as a perusal of cases 1, 2 of stage 2 in Fig. 7.25 would indicate.

It is a relatively simple matter to derive the complementary functional relations to (96) and (97). Toward this end, we partition X(a,b) into pieces X(a,t), X(t,b) such that $a \le s \le t$, $y \le t \le b$. This partition goes with cases 1 and 3 of stage 3 in Fig. 7.25. Using the preceding derivation of (96) as a pattern, the reader may show that:

$$\boxed{\Psi(s,y:a,b) = \Psi(s,y:a,t) + \Psi(s,t:a,t)\Psi(t,t:a,b)\Psi(t,y:a,t) \quad a \le s \le t,\ y \le t \le b} \tag{104}$$

Forming the difference quotient via (98):

$$\frac{\Psi(s,y:a,b)-\Psi(s,y:a,t)}{b-t} = \Psi(s,t:a,t)\,\frac{\Psi(t,t:a,b)}{b-t}\,\Psi(t,y:a,t)$$

we go to the limit as $t \to b$. The result is:

$$\boxed{\begin{aligned}\frac{\partial \Psi(s,y:a,b)}{\partial b} &= \Psi(s,b:a,b)\,\mathcal{X}(b)\Psi(b,y:a,b)\\ &= \mathcal{M}^{\dagger}(s,b,a)C_{-}\,\mathcal{X}(b)C_{+}\mathcal{M}(b,y,a)\end{aligned}} \quad . \tag{105}$$

Equation (105) can be used to generate the eight functional equations governing cases 1, 3 of stage 3 in Fig. 7.25. To do so, replace "b" by "t" in (105) and integrate over all t from s to b for case 1, and from y to b for case 3. Two integral expressions will result which are the complements of (85) and (86). These results will be summarized below.

Summary and Prospectus

The problem of internal sources in an arbitrary optical medium X has presented the opportunity to develop in the present section the full strength of the invariant imbedding technique as applied to radiative transfer--or generally, linear transport--phenomena. We shall now summarize this technique.

Let X be an arbitrary optical medium with internal sources. Let "X(a,b)" denote the parametrization of X, i.e., the representation of X as partitioned into a family $\{X_y: a \le y \le b\}$ of surfaces X_y, each indexed by a real number y drawn from a closed interval [a,b] of real numbers. In the case of plane-parallel media the X_y are planes parallel to the boundary planes. In the case of an arbitrary X, any slicing up of X by a one-parameter family of surfaces will do for the present summary (re: (17) of Sec. 7.11). For simplicity of exposition (and without any attendant loss of generality), we let the sources in X be confined to a single surface X_s. For if the sources are several discrete sources or are continuous and are confined to an interval of depths, the resultant light field is obtained by a suitable superposition operation, since the theory is completely linear.

The general invariant imbedding process begins with X_s and imbeds it in a family $\{X_z: y \le z \le s\}$ of surfaces for the *upward case* or in the family $\{X_z: s \le z \le y\}$ for the *downward case*. This is stage 1 of the imbedding process and serves to develop the four standard R and T operators associated with the slabs X(y,s) and X(s,y). The theory of these operators has been the primary concern of various earlier sections of this chapter and in Chapter 3.

Stage 2 of the invariant imbedding process gives rise to the advent of the invariant imbedding operators $\mathcal{R}$, $\mathcal{T}$ and their duals $\mathcal{R}^{\dagger}$, $\mathcal{T}^{\dagger}$. The operators $\mathcal{R}$ and $\mathcal{T}$ have been studied at some length in this chapter and Chapter 3. The dual operators are newcomers to the scene and fulfill the role of completing with $\mathcal{R}$ and $\mathcal{T}$ the full description of Stage 2 of the invariant imbedding process in X(a,b). The theory of the dual operators was shown in the present section to be parallel in all essential respects to that of the original invariant imbedding operators.

Stage 3 culminates the imbedding process and is the setting for the derivation of the functional relations for the global Ψ-operators. There are two generic differential equations which go with Stage 3. The first is (97), repeated here for convenience:

$$\boxed{-\frac{\partial \Psi(s,y:a,b)}{\partial a} = \Psi(s,a:a,b)\,\mathcal{K}(a)\,\Psi(a,y:a,b)} \qquad (106)$$

and which governs cases 2 and 4 of Stage 3. The second equation is (105):

$$\boxed{\frac{\partial \Psi(s,y:a,b)}{\partial b} = \Psi(s,b:a,b)\mathcal{X}(b)\Psi(b,y:a,b)} \tag{107}$$

and which governs cases 1 and 3 of stage 3.

Equations (106) and (107) are perhaps two of the most productive equations in the invariant imbedding theory of radiative transfer phenomena, in the sense that they yield differential equations for the various R, T, $\mathcal{R}$, $\mathcal{T}$, and $\mathcal{M}$ operators (cf., (63)-(78)). Thus, as was shown at great length above, they hold within themselves the means toward the differential functional relations of all three stages of an invariant imbedding process on X(a,b), and this includes in particular the differential equations for the operators $\mathcal{M}(a,y,b)$ and $\mathcal{M}^\dagger(y,a,b)$, and hence the differential equations for the standard R and T operators R(a,b), R(b,a), T(a,b), T(b,a). Their algebraic progenitors are (96) and (104).

Each equation (106) and (107) gives rise, by means of an integration, to a pair of integral representations, depending on whether s<y or y<s. Going down the four cases in stage 3, as depicted in Fig. 7.25, the associated integral representations are:

$\Psi(s,y:a,b) = \Psi(s,y:a,s) + \int_s^b \Psi(s,t:a,t)\mathcal{X}(t)\Psi(t,y:a,t)\,dt$	From (107) Case 1 y<s (108)
$\Psi(s,y:a,b) = \Psi(s,y:y,b) + \int_a^y \Psi(s,t:t,b)\mathcal{X}(t)\Psi(t,y:t,b)\,dt$	From (106) Case 2 y<s (109)
$\Psi(s,y:a,b) = \Psi(s,y:a,y) + \int_y^b \Psi(s,t:a,t)\mathcal{X}(t)\Psi(t,y:a,t)\,dt$	From (107) Case 3 s<y (110)
$\Psi(s,y:a,b) = \Psi(s,y:s,b) + \int_a^s \Psi(s,t:t,b)\mathcal{X}(t)\Psi(t,y:t,b)\,dt$	From (106) Case 4 s<y (111)

in which $a \le s \le b$, $a \le y \le b$. These four operator equations blossom into the sixteen operator equations scattered here and there throughout this section and which completely describe how to find $\Psi(s,y:a,b)$ in every case of stage 3 using the operators ($\mathcal{R}, \mathcal{R}^\dagger; \mathcal{T}, \mathcal{T}^\dagger$) which belong to the results of stage 2.

The invariant imbedding process for X(a,b), as depicted in Fig. 7.25, is closely related to the *Categorical Analysis Method* described in detail in Ref. [251]. In that work the geometric setting is a discrete space rather than a continuous space, and the analysis is thereby permitted to descend directly to the point level in the medium. Together, the invariant imbedding process for one-parameter continuous media $X(a,b)$ as summarized in Fig. 7.25, and the Categorical Analysis Method for discrete media X_n as given in Ref. [251] present a potentially complete means of solving all steady state internal source problems in radiative transfer theory. Be that as it may, much work remains yet to be done in exploring the many special cases arising in particular geometries and physical settings. Thus many opportunities for original research lie in the relatively unexplored new territory of radiative transfer theory surveyed in this section.

Final Observations on the Relations Between the Operators $\mathcal{M}(v,x:u,w)$ and $\Psi(s,y:a,b)$

The two most general radiative transfer operators considered in this work are the $\mathcal{M}$-operator $\mathcal{M}(v,x:u,w)$ introduced for the purpose of formulating the generalized invariant imbedding relation (51) of Sec. 3.7, and the internal source operator Ψ in the form $\Psi(s,y:a,b)$, introduced in (15) of Sec. 3.9 for the study of internal sources. The latter operator we have discussed at length in this section; the former operator was discussed in Sec. 3.7 and at length in Secs. 7.4 and 7.5. Here, by way of summary, is the manner in which one may view these operators conceptually and analytically: the $\mathcal{M}$-operator is to be used in source-free settings; the Ψ-operator is to be used in settings with sources. The $\mathcal{M}$-operator is sufficiently fundamental so that Ψ may be characterized in terms of it, as in (20)-(23) and (31)-(34) of Sec. 3.9. On the other hand, the $\mathcal{M}$-operator can also be represented by and built up from special cases of Ψ, as shown in this section (cf., (63)-(78), in particular). An important relation between them is summarized in (56) of Sec. 7.4. The $\mathcal{M}$-operator enjoys deep algebraic and differential properties, as shown in Secs. 7.4, 7.5; the operator Ψ enjoys a sweeping analytical power as shown in this section and summarized in (106)-(111). Therefore each operator, $\mathcal{M}$ or Ψ, is sufficient to carry radiative transfer theory by itself; and each has algebraic and analytic properties worthy of independent mathematical study.

7.14 Invariant Imbedding and Integral Transform Techniques

In this the final section of the present chapter on invariant imbedding techniques we shall briefly consider one of the more serious types of problems which, if left unchecked, may keep invariant imbedding techniques from reaching their full practical utility. This is the problem of the exploding variable-population. To see what is meant, consider the following observations.

There was a time when the radiative transferist was content with solving the Wiener-Hopf equation which described energy density in a homogeneous source-free infinitely deep medium. (See, e.g., (1) of Sec. 6.7 in which X is one-dimensional, i.e., the real line, and $h_\eta = 0$.) This was a highly idealized problem, deliberately idealized so that the Wiener-Hopf equation was a singular integral equation of the first kind in one dimension and eventually dispatched by a technique which has now become classical. There was also a time when it was a major breakthrough to have solved the radiative transfer problem on homogeneous plane-parallel media with isotropic scattering and stratified light fields (re: Sec. 6.4). The breakthrough was possible because of some judicious application of the spherical harmonic method to reduce the integro-differential equation of transfer to a set of coupled differential equations.

However, when these victories on the beachhead were over, there remained the more difficult high ground to take, and progress was correspondingly slower: the spaces arising in practice became odd-shaped and inhomogeneous, scattering became unmanageably anisotropic and heterochromatic, sources were encountered in the hitherto inviolate interiors of media, and matters were made worse by giving all physical quantities rapid temporal variations. The number of variables needed to fully describe the new radiometric environments grew from *one* (the depth location for scalar irradiance or radiant density) to *five* (steady-state monochromatic radiance transfer) to *seven* (time-dependent heterochromatic radiance transfer) to *twenty-eight* (when polarized light fields were considered). Clearly, the halcyon days of the subject were in the past and further progress with new real problems seemed to require new techniques and concepts, not only to solve them but to formulate them in the first place!

The advent of the principles of invariance (circa 1943) helped further high ground to be taken on the island of radiative transfer theory. The work of Ambarzumian and Chandrasekhar showed the potentialities of the concept of the global approach to transfer problems. This approach was subsequently considerably extended with the advent of the principles of invariance (Ref. [43]). The remaining high ground was surveyed using the invariant imbedding relation, Ref. [233], and its generalizations attained in Ref. [248], culminating in the interaction principle of Ref. [251] and the further results developed in the present work; the net result being compact formulations of transfer problems by means of one-point boundary value settings. The latter type settings are, as we have seen repeatedly throughout this chapter, simply elaborations of the basic integration problem:

$$\frac{dy(x)}{dx} \begin{cases} = f(x)y(x) + g(x) \\ = g\left(y(x),x\right) \end{cases} \qquad (*)$$

over an interval [a,b] given f and g on [a,b] and given the value y(a). The first equation is for linear, the second for nonlinear problems. In this way the invariant imbedding approach, the logical outgrowth to the principle of invariance approach, reduces radiative transfer problems to their simplest conceivable mathematical form.

Now the theoretical bases of the solution procedures of equations of the kind (*) and small finite systems of such equations are straightforward and usually dispatched in an introductory course in ordinary differential equations. The only serious difficulty such a system can present is on a practical and not a conceptual or theoretical level: the functions f, g, and y may no longer be scalar-valued, but matrix-valued, whose entries themselves are operators in integral or matricial form, and where the combinations $f(x)y(x)$ or $g(y(x),x)$ are no longer simple products but compositions of operators or functions $f(x)$, $y(x)$. In short, an uncontrollable, almost explosive increase in the number of variables needed to describe the domains and ranges of f, g, and y and their combinations could render (*) worthless from a practical point of view.

Since formulations of the kind (*) are quite clearly the simplest analytic forms into which the manifold problems of radiative transfer theory can be cast, the next major task on the practical front that faces radiative transfer theory is the successful handling of the variable-population explosion associated with (*). One immediate measure that can be taken is the judicious application of Laplace or Fourier transform techniques, or more generally, the application of integral transform techniques to the transport equations and the various principles. These integral transform methods have as their primary purpose in applied mathematics the reduction of the number of variables in a given physical formulation. For example, time derivatives can be transformed away and spatial or frequency convolution integrals can be transformed away resulting in two very important variable-reducing operations which can be effected by suitably chosen integral kernel transforms. In the case of the equation of transfer in radiative transfer theory, these two measures take on quite practical significance: the possibility of transforming away time derivatives means that a time-dependent problem can be reduced to a steady-state problem, solved in that context, and the solution so found, transformed back to the original setting. The possibility of transforming away convolution integrals could mean that various special heterochromatic transfer problems (in which scattering takes place from one wavelength to another) can be reduced to monochromatic transfer problems, solved in that context, and the solution transformed back to the original setting. Furthermore, since the multidimensional spatial settings used in the statement of the invariant imbedding relations (and their special principle of invariance forms) employ convolution integrals, relatively complex three-dimensional settings can occasionally be reduced to more tractable one-dimensional settings for a solution interim.

We now illustrate under what conditions integral kernel transforms can be used to reduce the number of variables in transport problems. We shall choose three examples for this purpose: the case of time-dependent radiative transfer; the case of heterochromatic radiative transfer; and the case of multidimensional radiative transfer on a plane-parallel medium with a non-stratified light field. For the benefit of readers not acquainted with the notions of integral kernel transforms, we precede the illustrations with a few introductory comments on this subject.

An Integral-Transform Primer

To introduce the notion of an integral kernel transform in sufficient detail for our present purposes requires remarkably little mathematical machinery. From the welter of formulas and theorems of transform theory we prescind the idea of a real or complex valued integral operation:

$$\int_X [\ ]\ K(x,\omega)\ dx \tag{1}$$

which can act on a real valued function f defined on a set X. Thus:

$$\int_X f(x)K(x,\omega)\ dx$$

is a number, denoted by "$\hat{f}(\omega)$" or by "$\mathcal{F}[f,\omega]$" which is the result of integrating the product of the functions f and $K(\cdot,\omega)$ over X. The function K determines the form of the integral transform (1). The numbers ω are drawn from some set Ω; the significance of Ω is immaterial for the present discussion. On the other hand X will take various familiar forms: the real line R extending from $-\infty$ to ∞, or the half line R^+ from 0 to ∞, or the xy-plane (i.e., the cartesian product $R \times R$), etc. The real or complex valued function K on $X \times \Omega$ is the *kernel* of (1). One may think of $X \times \Omega$, i.e., the set of all pairs (x,ω) with x in X and ω in Ω, as a 'plane' with X and Ω as axes. We shall assume K to be continuous on $X \times \Omega$.

It is remarkable that the only formal property needed for K in the present discussion is that, for some x_0, we have $K(x_0,\omega) \neq 0$; and that:

$$K(x+y,\omega) = K(x,\omega)K(y,\omega) \tag{2}$$

which is the *group property* of K. When x,y are drawn from the set R of real numbers, we can formally deduce from (2) the differential property:

$$\frac{\partial K(x,\omega)}{\partial x} = K(x,\omega)g(\omega) \tag{3}$$

much in the way we deduced the differential property of beam transmittance in (2) of Sec. 3.11. Indeed, the similarity is not accidental since $K(x,\omega)$ is an exponential function over X. For example, $K(x,\omega)$ can be $\exp\{-i\omega x\}$, or $\exp\{\omega x\}$, or $\exp\{1/2\ x\ (\omega-1/\omega)\}$ for the cases of Fourier, Laplace, and Hankel-type transforms, respectively. There are other kernels K used in practice which do not have the group property (2) (e.g., Mellin and Euler kernels) but we shall consider only kernels with property (2).

Now the group property of K endows the theory of the integral kernel transform (1) with a most useful theorem, which is attained as follows. Suppose we write:

$$\text{"}f*g(y)\text{"} \quad \text{for} \quad \int_X f(x)g(y-x)\,dx \tag{4}$$

where f and g are functions on X for which the integral exists. The function $f*g$ is the *convolution* of f and g. The value of $f*g$ at y in X is denoted by "$f*g(y)$", as shown. Now let f and g be defined on X, and zero elsewhere. Then observe that if X is either R^+, R, $R^+ \times R^+$, $R \times R$, or $R \times \ldots \times R$ to n factors:

$$\begin{aligned}\int_X f*g(y)K(y,\omega)\,dy &= \int_X \left[\int_X f(x)g(y-x)\,dx\right] K(y,\omega)\,dy \\ &= \int_X f(x)\left[\int_X g(y-x)K(y,\omega)\,dy\right] dx \\ &= \int_X f(x)\left[\int_X g(u)K(x+u,\omega)\,du\right] dx \\ &= \int_X f(x)K(x,\omega)\left[\int_X g(u)K(u,\omega)\,du\right] dx \\ &= \hat{f}(\omega)\hat{g}(\omega) \quad .\end{aligned} \tag{5}$$

That is:

$$\boxed{\mathcal{F}[f*g;\omega] = \int_X (f*g)(y)K(y,\omega)\,dy = \hat{f}(\omega)\hat{g}(\omega) = \mathcal{F}[f;\omega]\,\mathcal{F}[g;\omega]} \tag{6}$$

In words: *the transform of the convolution* $f*g$ of two functions on X is equal to the *numerical product of their transforms.* In this way a complicated operation (the functional product (4)) is seen to be replaceable by a vastly simpler operation (the numerical product (5)) by means of integral transform operations. Statement (6) is the *convolution theorem* for the operator (1). It is by far the single most important property of integral kernel transforms whose kernels obey (2).

The second most important property of integral kernel transforms whose kernels obey (2) is the *derivative property*. We shall need this property only for the case where X is R^+ or some interval [a,b] of R^+ (which could be all of R^+). Thus consider the transform of the derivative f' of a function f on [a,b]:

$$\int_a^b f'(x)K(x,\omega)\ dx$$

which via integration by parts and (3) becomes:

$$f(x)K(x,\omega)\Big|_a^b - \int_a^b f(x)K'(x,\omega)\ dx =$$

$$= [f(b)K(b,\omega)-f(a)K(a,\omega)] - g(\omega)\int_a^b f(x)K(x,\omega)\ dx \quad .$$

From this we conclude that:

$$\mathcal{F}[f';\omega] = [f(b)K(b,\omega)-f(a)K(a,\omega)] - \mathcal{F}[f;\omega]g(\omega) \ . \qquad (7)$$

For example, if $a = 0$ and $b = \infty$, then:

$$\boxed{\mathcal{F}[f';\omega] = -[f(0)+\hat{f}(\omega)g(\omega)]} \quad . \qquad (8)$$

This follows from the necessity of having $\lim_x f(x)K(x,\omega) = 0$ as $x\to\infty$ for an integrable function f, and from (2) which implies:

$$K(0,\omega) = 1$$

for every ω in Ω. To see this, set $y = 0$ in (2) and note that for every x, $K(x,\omega)$ cannot be zero. Equation (8) shows how the integral transform can do away with derivatives. It is true that the price one pays for ridding the scene of f' is the linear combination of $\hat{f}(\omega)$ and $f(0)$, but this *algebraic* combination is usually more tractable than the generally *transcendental* object f' (i.e., one whose definition requires, in addition to the usual algebraic operations, the operation of limit).

One final matter, and that is the explication of the concept of the inverse of $\mathcal{F}$. We need only remark here that the inverse $\mathcal{F}^{-1}$ of $\mathcal{F}$ is generally of the form:

$$\int_Y [\ \ ]H(x,\omega)\ d\omega$$

where x is in X, and Y is some subset of Ω. Matters can often be arranged so that $\mathcal{F}^{-1}$ exists in the mathematical sense. For example, in the Fourier transform case, if $K(x,\omega) = 1/\sqrt{2\pi}\ \exp\{-i\omega x\}$, then $H(x,\omega) = 1/\sqrt{2\pi}\ \exp\{i\omega x\}$, or if $K(x,\omega) = \exp\{-i\omega x\}$, then $H(x,\omega) = 1/2\pi\ \exp\{i\omega x\}$, and in either case $Y = X = R$. Further, in the Laplace transform case, if $K(x,\omega) = \exp\{-x\omega\}$, then $H(x,\omega) = K(-x,\omega)/2\pi i$; and if $X = R$, then for some real γ, $Y = \{\gamma + i\omega : \omega\varepsilon R\}$. The inverse operation

$\mathcal{F}^{-1}$ undoes what $\mathcal{F}$ does:

$$\text{If } \mathcal{F}[f;\omega] = \hat{f}(\omega), \text{ then } \mathcal{F}^{-1}[\hat{f};x] = f(x) \quad .$$

For example, in the one-dimensional Fourier Integral setting:

$$\hat{f}(\omega) = \mathcal{F}[f;\omega] = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx$$

$$\mathcal{F}^{-1}[\hat{f};x] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega x}\,d\omega \quad . \tag{9}$$

In the two-dimensional Fourier Integral setting:

$$\hat{f}(\omega_1,\omega_2) = \mathcal{F}[f;\omega_1,\omega_2] = \int_{-\infty}^{\infty}\int f(x_1,x_2)e^{-i[\omega_1 x_1+\omega_2 x_2]}\,dx_1 dx_2$$

$$f(x_1,x_2) = \mathcal{F}^{-1}[\hat{f};x_1,x_2] = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int \hat{f}(\omega_1,\omega_2)e^{i[\omega_1 x_1+\omega_2 x_2]}\,d\omega_1 d\omega_2 \tag{10}$$

Time-Dependent Radiative Transfer

Let us begin our studies of integral transform techniques with the time-dependent equation of transfer (4) of Sec. 3.15, to which is appended a source term N_η:

$$\frac{1}{v}\frac{\partial N(x,\xi,t)}{\partial t} + \frac{dN(x,\xi,t)}{dr} =$$

$$= -\alpha(x,\xi,t)N(x,\xi,t)+\int_{\Xi} N(x,\xi',t)\sigma(x;\xi';\xi,t)\,d\Omega(\xi')+N_\eta(x,\xi,t) \tag{11}$$

We require for the present discussion that $N(x,\xi,t) = 0$ for $t \leq 0$. We choose the following form of the integral operator (1):

$$X = R^+ = [0,\infty]$$

$$K(t,\omega) = e^{-t\omega}, \text{ so that } g(\omega) = -\omega$$

$$\Omega \text{ is the set of complex numbers}$$

with ω a positive real number in R^+. Let us write:

$$\text{"}\hat{N}(x,\xi,\omega)\text{"} \quad \text{for} \quad \int_0^{\infty} N(x,\xi,t)e^{-t\omega}\,dt \quad .$$

Applying the operator:

$$\int_0^{\infty} [\ ]e^{-t\omega}dt$$

to each side of (11) and using its linearity property, namely:

$$\int_0^{\infty} [cf(t) + dg(t)]e^{-t\omega}dt = c\hat{f}(\omega) + d\hat{g}(\omega)$$

we see that by (8) the time derivative term becomes:

$$\frac{1}{v}\int_0^{\infty} \frac{\partial N(x,\xi,t)}{\partial t}\, e^{-t\omega}dt = \frac{\omega}{v}\,\hat{N}(x,\xi,\omega) \quad .$$

The spatial derivative term becomes:

$$\int_0^{\infty} \frac{dN(x,\xi,t)}{dr}\, e^{-t\omega}dt = \frac{d\hat{N}(x,\xi,\omega)}{dr} \quad .$$

On the right side we have, for the first term:

$$-\int_0^{\infty} \alpha(x,\xi,t)N(x,\xi,t)e^{-t\omega}\, dt \quad .$$

At this point we realize that, for the Laplace transform to be effective in the present case, we require $\alpha(x,\xi,\cdot)$ be a constant function of time (i.e., constant on R^+). Hence we may study time-dependent transfer problems in which the radiance field is truly time-varying but this variation is of a transient nature traceable to the finite speed v of propagation of radiant flux throughout the medium and not to the time-dependence of the inherent optical properties of the medium. We therefore assume for the remainder of this discussion that α and σ are independent of time, and shall write "$\alpha(x,\xi)$" for $\alpha(x,\xi,0)$ and "$\sigma(x;\xi';\xi)$" for $\sigma(x;\xi';\xi;0)$. With this agreement the preceding transformed term becomes:

$$-\alpha(x,\xi)\hat{N}(x,\xi,\omega) \quad .$$

Altogether, the transformed terms of (11) become:

$$\boxed{\begin{aligned}&\frac{d\hat{N}(x,\xi,\omega)}{dr} = \\ &= -\left(\alpha(x,\xi)+\frac{\omega}{v}\right)\hat{N}(x,\xi,\omega) + \int_{\Xi}\hat{N}(x,\xi,\omega)\sigma(x;\xi';\xi)\,d\Omega(\xi') + \hat{N}_{\eta}(x,\xi,\omega)\end{aligned}} \tag{12}$$

Equation (12) is the Laplace-transformed equation of transfer. It has clearly the gestalt of a steady state equation of transfer with source term, and with a slightly-odd volume attenuation function. It seems that under transformation from transient to steady state, the volume attenuation function has been altered artificially by a fixed amount ω/v. Inspection of (12) thus yields the following observation: *The entire theory of the steady state field (all classical and invariant imbedding techniques) can be applied to (12)*--the variable ω is a fixed, passive complex variable, dangling throughout all the subsequent steady state proceedings like a useless appendix, but ready to play its role in the final movements of the solution procedure. The return to physical setting is made by means of the inverse operation:

$$N(x,\xi,t) = \lim_{\beta\to\infty} \frac{1}{2\pi i} \int_{\gamma-i\beta}^{\gamma+i\beta} \hat{N}(x,\xi,\omega)e^{t\omega}\, d\omega \quad . \qquad (13)$$

This inversion operation can be written as a real integral and performed numerically on computers. The details of the real integral representation may be found, e.g., in Ref. [46].

The reader can obtain practice with the Laplace transformation procedure by transforming (5), (6) of Sec. 7.2 directly into their 'steady state' forms. The results should agree with the local forms derived from (12).

The time-dependent invariant imbedding relation in its forms (27), (28) of Sec. 7.2 can be transformed directly by means of the Laplace integral operation. A typical term of the relation is:

$$\int_E N_+(z,t')\, \mathcal{T}(z,y,x,t',t)\, dt' \qquad (14)$$

where we now choose E to be R^+ and require $N_+(z,t') = 0$ for $t \le 0$. We assume that:

$$\mathcal{T}(z,y,x,t',t) = \mathcal{T}(z,y,x,u',u)$$

whenever $t-t' = u-u'$. This is tantamount to assuming that α and σ are time independent, and is the corresponding assumption on the global level of radiative transfer that one must make before the Laplace transform method can be invoked. With this agreement, we shall write:

$$"\mathcal{T}(z,y,x,t-t')" \quad \text{for} \quad \mathcal{T}(z,y,x,t',t)$$

so that (14) becomes:

$$\int_{R^+} N_+(z,t')\, \mathcal{T}(z,y,x,t-t')\, dt' = N_+(z) * \mathcal{T}(z,y,x) \quad . \qquad (15)$$

Therefore, operating on this convolution of $N_+(z)$ and the complete transmittance operator $\mathcal{T}(z,y,x)$, we have, by (6):

$$\mathcal{F}[N_+(z) * \mathcal{T}(z,y,x;\omega] = \hat{N}_+(z,\omega)\,\hat{\mathcal{T}}(z,y,x,\omega) \quad . \tag{16}$$

By our remarks in Sec. 7.2, especially those after (22) of Sec. 7.2, we realize that $\hat{\mathcal{T}}(z,y,x,\omega)$ is itself an integral operator over Ξ_+ and the parameter surface X_z, so that we are back again in the steady state context. Once $\hat{N}_+(z,\omega)$ has been found, the physically meaningful radiance N(z,t) can be recovered using (13). Or, again, $\mathcal{T}(z,y,x,t-t')$ can be recovered using (13). This is all new territory and is free to be explored by interested students of the subject.

The complete set of transformed invariant imbedding relations (27),(28) of Sec. 7.2 is given as:

$$\hat{N}_+(y,\omega) = \hat{N}_+(z,\omega)\,\hat{\mathcal{T}}(z,y,x,\omega) + \hat{N}_-(x,\omega)\,\hat{\mathcal{R}}(x,y,z,\omega) \tag{17}$$

$$\hat{N}_-(y,\omega) = \hat{N}_-(x,\omega)\,\hat{\mathcal{T}}(x,y,z,\omega) + \hat{N}_+(z,\omega)\,\hat{\mathcal{R}}(z,y,x,\omega)\,. \tag{18}$$

From these we can find the Laplace-transformed principles of invariance in the usual manner and all the functional relations for the transformed R and T operators paralleling those of Sec. 7.1. Or then, again, these functional relations may be obtained directly by transforming principles I'-IV' of Sec. 7.2.

Heterochromatic Radiative Transfer

When radiative transfer takes place across the spectrum of frequencies in addition to spatial and directional transfer we have heterochromatic radiative transfer. The volume transspectral scattering function $\hat{\sigma}$ is designed to do for the heterochromatic context what σ does in the monochromatic setting. The function $\hat{\sigma}$ is defined in Ex. 3, Sec. 3.17, and discussed in detail in Sec. 19 of Ref. [251]. Suppose the radiance field is of frequency ν and is in steady state (either real or the pseudo steady state, in ω-space, of a Laplace transformed N-field). For our present purposes we need only note that in addition to the term:

$$N_*(x,\xi,\nu) = \int_{\Xi} N(x,\xi',\nu)\sigma(x;\xi';\xi,\nu)\ d\Omega(\xi')$$

we have the term:

$$N_s(x,\xi,\nu) = \int_{\Xi}\int_{\Lambda} N(x,\xi',\nu')\hat{\sigma}(x;\xi';\xi;\nu',\nu)\ d\Omega(\xi')dl(\nu')$$

where "Λ" denotes the spectrum of frequencies, and is mathematically simply another name for R^+. Now if it is possible to find a mapping T on R^+ onto itself so that:

$$\hat{\sigma}\big(x;\xi';\xi;T(\nu'),T(\nu)\big) = \hat{\sigma}\big(x;\xi';\xi;T(\mu'),T(\mu)\big)$$

whenever

$$T(\nu) - T(\nu') = T(\mu) - T(\mu') \quad ,$$

then we may view:

$$\int_{\Lambda} N(x,\xi',\nu')\hat{\sigma}(x;\xi';\xi;T(\nu'),T(\nu)) \; dl(\nu')$$

as a convolution of $N(x,\xi',\cdot)$ and $\hat{\sigma}$. However, even if it is possible to find such a mapping T (in neutron transport theory such a mapping exists and can be used to develop the so-called 'Fermi-age theory') some reflection would show that the Laplace transform method will not be applicable in the present case. Thus, unless very limited regions in $\Lambda \times \Lambda$ are considered, outside of which $\hat{\sigma}$ is zero and inside which the following translation condition holds:

$$\hat{\sigma}(x;\xi';\xi;\nu',\nu) = \hat{\sigma}(x;\xi';\xi;\mu',\mu) \qquad (19)$$

whenever $\nu-\nu' = \mu-\mu'$, the Laplace transform method fails to simplify the heterochromatic radiative transfer problem. It is also physically unlikely that (19) will hold in the usual settings. Nevertheless, assuming (19) holds, then the convolution theorem yields:

$$\hat{N}_s(x,\xi,\omega) = \int_{\Xi} \hat{N}(x,\xi,\omega)\hat{\hat{\sigma}}(x;\xi';\xi;\omega) \; d\Omega(\xi')$$

and the transformed N_*-term is:

$$\hat{N}_*(x,\xi,\omega) = \int_{R^+} \left[\int_{\Xi} N(x,\xi',\nu)\sigma(x;\xi';\xi,\nu) \; d\Omega(\xi') \right] e^{-\nu\omega} \, d\nu \quad .$$

Since σ generally depends on ν, we encounter a difficulty similar to that with the term $\alpha(x,\xi,t)N(x,\xi,t)$ in (11) when the time-dependent case is considered. It is simply too much to ask σ to be generally independent of ν (whereas it was not too much of a sacrifice in accuracy to ask σ to be independent of time during the course of a given transfer process). Hence we conclude that except in the most special of settings, the integral transform method is generally of no use in the solution of heterochromatic radiative transfer problems. We shall instead rely on such methods as given in Refs. [136], [137] or more generally in Ref. [288] or Ref. [251] to solve the general heterochromatic radiative transfer problem.

Multidimensional Radiative Transfer

Multidimensional radiative transfer problems arise most frequently in practice in plane-parallel media in which the optical properties are all well behaved--stratified with depth or altitude or simply constant--but in which the light field is not stratified with depth and which is generally variable laterally over the plane surfaces parallel to the boundaries. The practical instances of these problems arise when clouds induce a checkerboard pattern of light and dark over a horizontal plane in an otherwise homogeneous body of air, sea, or lake, or when an isotropic point source or narrow beam of flux is present within or near these natural media. The "multidimensional" aspect of these settings consists in the full three spatial variables being required in addition to the two direction variables to describe the radiance field in such media.

To see how the integral transform methods are of use in such radiometric situations as just described, recall first of all the discussion of the point source problem for scalar irradiance in Sec. 6.7. Then, consider a general internal or external source problem on a stratified plane-parallel medium X(a,b). The theory of Example 3 of Sec. 3.9 and the work of Sec. 7.13 showed how this problem can be solved using invariant imbedding techniques, and of how the operators in the solution procedure could always be reduced to suitable assemblies of the standard R and T operators. Therefore we are to consider the R and T operators in their full generality as given in (8)-(11) of Sec. 3.6. To fix ideas, consider the operator R(a,b) acting on the incident radiance function $N_-(a)$ over the surface X_a. Then by (12) of Sec. 3.6:

$$N_-(a)R(a,b) = \int_{\Xi}\int_{X_a} N(x',\xi')S(X;x',\xi';x,\xi)\; dA(x')d\Omega(\xi')$$

where "X" denotes X(a,b). If the medium X(a,b) is stratified over horizontal planes, then:

$$S(X;x',\xi';x,\xi) = S(X;y',\xi';y,\xi)$$

whenever $x-x' = y-y'$, where x, x', y, y' are points in X_a. Thus, e.g., x' is an ordered triple of the form (x_1,x_2,x_3) with $x_3 = a$. Assuming stratification, we can write:

$$\text{"}S(X;x-x';\xi';\xi)\text{"} \quad \text{for} \quad S(X;x',\xi';x,\xi)$$

so that:

$$N_+(a) = N_-(a)R(a,b) = \int_{\Xi}\left[\int_{X_a} N(x',\xi')S(X;x-x';\xi';\xi)\; dA(x')\right] d\Omega(\xi') \tag{20}$$

Next we choose the following form of the integral operator (1):

$$X = R \times R$$

$$K(x,\omega) = \exp\{-i(ix_1+jx_2)\cdot(i\omega_1+j\omega_2)\}$$

$$\Omega = R \times R$$

where now clearly we need $x = (ix_1+jx_2)$ and $\omega = (i\omega_1+j\omega_2)$. See, e.g., (23) of Sec. 6.7. Hence (1) now becomes a two-dimensional Fourier transform. Applying the resultant integral transform to (20), we have by (6):

$$\hat{N}_+(a,\omega) = \int_{\Xi_-} \hat{N}(a,\xi',\omega)\hat{S}(X;\omega;\xi';\xi)\, d\Omega(\xi') \tag{21}$$

$$= \hat{N}(a,\omega)R(a,b;\omega)$$

which is vastly simpler to deal with than (20). In (21), "X" stands for X(a,b). Comparing the S-operator in equation (21) with the corresponding operator for stratified plane-parallel media with *stratified light field* ((31), (32) of Sec. 3.7), we see that we have returned to the fully stratified context and can apply the theory of stratified light fields to the following set of Fourier-transformed principles of invariance for X(a,b) (obtained from I, II of Example 3, Sec. 3.7 by applying the present Fourier transform operator):

$$\hat{N}_+(y,\omega) = \hat{N}_+(z,\omega)\hat{T}(z,y;\omega) + \hat{N}_-(y,\omega)\hat{R}(y,z;\omega) \tag{22}$$

$$\hat{N}_-(y,\omega) = \hat{N}_-(x,\omega)\hat{T}(x,y;\omega) + \hat{N}_+(y,\omega)\hat{R}(y,x;\omega) \quad . \tag{23}$$

Of course in actual practice we can drop the carets and the omegas so as to work with simpler notation. Equations (22) and (23) serve to show that the general structure of Fourier-transformed principles of invariance in the nonstratified case are the same as those of the stratified case, under the present assumptions.

Conclusion

Sufficient examples have now been given to show some of the power and the limitations of the integral transform method in radiative transfer theory in general, and particularly in conjunction with the operator equations of the invariant imbedding technique. Spatio-temporal inhomogeneities of the medium and heterochromatic radiative transfer severely limit the applicability of the integral transform techniques. Much work therefore remains to be done in the time-dependent and multidimensional problems.

7.15 Bibliographic Notes for Chapter 7.

The steady state functional relations for the standard R and T operators in Sec. 7.1 are based on the work in Ref. [234]. The time-dependent functional relations for R and T in Sec. 7.2 are drawn from Ref. [235]. The partition relations of Sec. 7.3 are continuous-operator versions of similar matricial relations developed in Ref. [251]. The algebraic studies of Sec. 7.4 grew out of Refs. [248] and [249]. We draw attention to some interesting related results in electrical network theory and diffusion theory found independently by Redheffer in Refs. [252]-[259]. Also the work of Reid is of interest in the present invariant imbedding studies [261], [262]. The analytic properties of the invariant imbedding operators in Sec. 7.5 appear to be new. The examples of numerical solutions for R(a,b) given in Sec. 7.6 are based on the work of Bellman, Kalaba and Prestrud in Ref. [15]. The general solution procedures of Secs. 7.11 are new, along with the developments of Sec. 7.13 concerned with the general internal source problem.

For a study of the internal source problem in the context of neutron transport theory, see the work of Elliott [88], and that of Bellman, Kalaba, and Wing [17].

Further discussion of the theory of polarized radiance fields as developed by Chandrasekhar and applied in natural hydrosols may be found in [108], [157]. Also the work of Sekera [284], although applied to the atmosphere, illustrates further the applications of Chandrasekhar's approach to the theory of polarized light fields.

The general functional equation approach of this chapter may be divided into the integral and differential approaches, and the hybrid integro-differential approach. The tap root of integral equation formulations of radiative transfer theory lies in the work of King [138], and that of the differential approach is the work of Schuster [279]. A brief, readable account of these two approaches, which places them in historical perspective, was given by Duntley [70]. Important future developments of the theory rest in using functional analysis along the lines developed throughout this chapter, particularly using the notions of semigroup theory. See, e.g., the discussion of the equation of evolution in [326], and recall the closing remarks of [216].

BIBLIOGRAPHY FOR VOLUME IV

1. Ambarzumian, V.A., "Diffuse reflection of light by a foggy medium," Compt. rend. (Doklady) Acad. Sci. U.R.S.S. *38*, 229 (1943).

2. Ambarzumian, V.A., "On the problem of the diffuse reflection of light," J. Phys. Acad. Sci. U.S.S.R. *8*, 65 (1944).

15. Bellman, R.E., Kalaba, R.E., and Prestrud, M.C., *Invariant Imbedding and Radiative Transfer on Slabs of Finite Thickness* (American Elsevier Pub. Co., New York, 1963).

40. Case, K.M., de Hoffman, F., and Placzek, G., *Introduction to the Theory of Neutron Diffusion* (Los Alamos Scientific Laboratory, Los Alamos, New Mexico, June 1953), vol. I.

43. Chandrasekhar, S., *Radiative Transfer* (Oxford, 1950).

46. Churchill, R.V., *Operational Mathematics* (McGraw-Hill, New York, 1958).

47. Coddington, E.N., and Levinson, N., *The Theory of Ordinary Differential Equations* (McGraw-Hill, New York, 1955).

70. Duntley, S.Q., "The mathematics of turbid media," J. Opt. Soc. Am. *33*, 252 (1943).

88. Elliott, J.P.,"Milne's problem with a point source," Proc. Roy. Soc., A, *228*, 424 (1955).

107. Herman, M., and Lenoble, J., "Asymptotic Radiation in a Scattering and Absorbing Medium," *Proceedings of the Symposium on Interdisciplinary Aspects of Radiative Energy Transfer*, 24-26 February 1966 (Cosponsors: General Electric Co., Joint Inst. for Lab. Astrophys. of Univ. of Col., and Off. of Nav. Res.).

108. Herman, M., and Lenoble, J., "Etude du Régime Asymptotique dans une Milieu Diffusant et absorbant," Revue d'Optique *43*, 555 (1964).

110. Hille, E., and Phillips, R.S., *Functional Analysis and Semi-groups* (Amer. Math. Soc., Providence, R.I., 1957).

118. Ivanoff, A., and Waterman, T.H.,"Factors, mainly depth and wavelength, affecting the degree of underwater polarization," J. Mar. Res. *16*, 283 (1958).

136. King, J.I.F., "Radiative equilibrium of a line-absorbing atmosphere, I," Astrophys. J. *121*, 711 (1955).

137. King, J.I.F., "Radiative equilibrium of a line-absorbing atmosphere, II," Astrophys. J. *124*, 272 (1956).

138. King, L.V., "On the scattering and absorption of light in gaseous media, with applications to the intensity of sky radiation," Phil. Trans. A *212*, 375 (1913).

157. Lenoble, J., "Theoretical study of transfer of radiation in the sea and verification on a reduced model," *Symposium on Radiant Energy in the Sea*, International Union of Geodesy and Geophysics, Helsinki Meeting, August 1960 (L'Institute Géographique National, Monograph No. 10, Paris, 1961).

216. Preisendorfer, R.W., "A mathematical foundation for radiative transfer theory," J. Math. and Mech. *6*, 685 (1957).

233. Preisendorfer, R.W., "Invariant imbedding relation for the principles of invariance," Proc. Nat. Acad. Sci. *44*, 320 (1958).

234. Preisendorfer, R.W., "Functional relations for the R and T operators on plane-parallel media," Proc. Nat. Acad. Sci. *44*, 323 (1958).

235. Preisendorfer, R.W., "Time-dependent principles of invariance," Proc. Nat. Acad. Sci. *44*, 328 (1958).

248. Preisendorfer, R.W., "Generalized invariant imbedding relation," Proc. Nat. Acad. Sci. *47*, 591 (1961).

249. Preisendorfer, R.W., *Spatial Semi-groups in Neutron Transport Theory* (General Atomic Report GA-2057, John Jay Hopkins Laboratory of Pure and Applied Science, San Diego, California, 1961).

251. Preisendorfer, R.W., *Radiative Transfer on Discrete Spaces* (Pergamon Press, New York, 1965).

252. Redheffer, R., "Remarks on the basis of network theory," J. Math. and Phys. *28*, 237 (1950).

253. Redheffer, R., "Novel uses of functional equations," J. Rat. Mech. and Anal. *3*, 271 (1954).

254. Redheffer, R., "On solutions of Riccati's equation as functions of the initial values," J. Rat. Mech. and Anal. *5*, 835 (1956).

255. Redheffer, R., "The Riccati equation: initial values and inequalities," Math. Ann. *133*, 235 (1957).

256. Redheffer, R., "Inequalities for a matrix Riccati equation," J. Math. and Mech. *8*, 349 (1959).

257. Redheffer, R., "Supplementary note on matrix Riccati equations," J. Math. and Mech. *9*, 745 (1960).

258. Redheffer, R., "The Myciolski-Pazkowski diffusion problem," J. Rat. Mech. and Anal. *9*, 607 (1960).

261. Reid, W.T., "Solutions of a Riccati matrix differential equation as functions of initial values," J. Math. and Mech. *8*, 221 (1959).

262. Reid, W.T., "Properties of solutions of a Riccati matrix differential equation," J. Math. and Mech. *9*, 749 (1960).

279. Schuster, A., "Radiation through a foggy atmosphere," Astrophys. J. *21*, 1 (1905).

284. Sekera, Z., *Radiative Transfer in a Planetary Atmosphere* (Rand Corp. Report R-413-PR, June 1963).

288. Sobolev, V., *A Treatise on Radiative Transfer* (Van Nostrand, New York, 1963).

326. Yosida, K., *Functional Analysis* (Academic Press, New York, 1965).

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