Hybrid Coupled Finite–Boundary Element Methods for Elliptic Systems of Second Order

G.C. Hsiao[1], E. Schnack[2], W.L. Wendland[3]

[1]Department of Mathematical Sciences, University of Delaware,
Newark, Delaware, 19716–2553, USA

[2]Institute of Solid Mechanics, Karlsruhe University,
Kaiserstraße 12, D–76128 Karlsruhe, Fed. Rep. Germany

[3]Mathematisches Institut A, Universität Stuttgart,
Pfaffenwaldring 57, D–70569 Stuttgart, Fed. Rep. Germany

Summary. In this hybrid method, we consider, in addition to traditional finite elements, the Trefftz elements for which the governing equations of equilibrium are required to be satisfied a priori within the subdomain elements. If the Trefftz elements are modelled with boundary potentials supported by the individual element boundaries, this defines the so–called macro–elements. These allow one to handle in particular situations involving singular features such as cracks, inclusions, corners and notches providing a locally high resolution of the desired stress fields, in combination with a traditional global variational FEM analysis. The global stiffness matrix is here sparse as the one in conventional FEM. In addition, with slight modifications, the macro–elements can be incorporated into standard commercial FEM codes. The coupling between the elements is modelled by using a generalized compatibility condition in a weak sense with additional elements on the skeleton. The latter allows us to relax the continuity requirements for the global displacement field. In particular, the mesh points of the macro–elements can be chosen independently of the nodes of the FEM structure. This approach permits the combination of independent meshes and also the exploitation of modern parallel computing facilities. We present here the formulation of the method and its functional analytic setting as well as corresponding discretizations and asymptotic error estimates. For illustration, we include some computational results in two– and three–dimensional elasticity.

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Contents

1 Introduction

For the engineer it is necessary to have reliable and detailed information on stress peaks and gradients in order to make decisions on the design of machine parts for their guaranteed lifetime. Therefore one needs methods which are able to combine global characterization of the stress field with locally high resolution in some chosen subdomains. As will be seen, the proposed hybrid method in this paper will be one of those desired methods.

In recent years, combined methods of boundary and finite elements have received increasing attention in computational mechanics. Now there is a growing body of literature on these topics. However, in spite of many different formulations of the coupling procedures, conceptually there are only two fundamental approaches. In the first one, the domain under consideration is divided into a finite number of subdomains in which either the boundary element method (BEM) or the finite element method (FEM) will be employed to construct approximate solutions depending on their suitability (see e. g. [64], [65], [7], [18], [42], [62], [10]). The second one is merely a variant of the hybrid–element approach in the FEM analysis [41]. In this approach, by using Trefftz elements, the governing equations of equilibrium are satisfied a–priori within the subdomain elements, where the Trefftz elements are modelled with boundary potentials supported by the individual element boundaries, the so–called macro–elements. These macro–elements allow us to handle, in particular, situations involving singular features such as cracks, inclusions, corners and notches and provide a locally high resolution of the desired stress fields, in combination with a global FEM analysis. The corresponding global stiffness matrix is rather sparse and the number of degrees of freedom is smaller than that in conventional hybrid FEM. In addition, with some slight modifications, the macro–elements can be incorporated into standard commercial FEM codes. This is particularly desirable from the computational point of view, since the macro–elements can be treated efficiently on parallel multi–processor computers by taking advantage of modern computer architectures.

Incorporating Trefftz elements into FEM where its stiffness contributions are expressed in terms of local boundary integrals defining a Reissner functional dates back to contributions by Tong, Pian and Lasry [57], Schnack [45] and later by Atluri and Grannell [2], to name a few. (The approach in [52] is different.) In the present paper, following [61], we present some error and stability analysis for a macro–element approach. The macro–elements employed here are based on the hybrid–stress method with boundary elements developed by Schnack [46] for treating problems in solid mechanics with regions of high stress concentration. The precise formulation of our hybrid stress method is based on the variational approach of the coupling procedures in [18] and [62]. This method has been numerically implemented successfully for two– and three–dimensional problems by Schnack and his research group (see [12], [32], [47], [48], [49], [50], [51], [58]). The essential feature of this coupling procedure is the use of a generalized compatibility condition [47] which allows to relax the continuity requirements for the displacement field. In particular, the mesh points of the macro–elements can be chosen independently of the nodes of the finite element structure so that various independent meshes can easily be connected via mortar–like elements on the skeleton. Moreover, this method can also serve as a basic algorithm for coupled preconditioned iterative solution schemes in domain decompositions such as, e. g. the Glowinski–Wheeler algorithm via BEM in [25], [26], [19] or more general preconditioned iteration schemes (see e.g. [11], [53], [54]).

The paper is organized as follows: In Sections 3–5, we present the functional analytic formulation of the coupling procedure and consider two particular choices of the so–called geometric skeleton associated with the macro–elements. Various discrete forms of the method are presented in Section 8. In particular, for the macro–elements, we consider BEM–Galerkin, BEM–collocation, and following Türke [58], the Neumann series approximation of the Poincaré–Steklov operator, i. e. , the Neumann–Dirichlet map. It is worth mentioning that in the latter case the exact stiffness matrix corresponding to the Poincaré–Steklov operator is symmetric which can be controlled numerically in the discrete case (see [58]). Section 7 is devoted to error estimates, stability and convergence results. Finally, in Section 8, various numerical results in mechanics are presented which can be served as an illustration for the efficiency of the method.

2 The hybrid method for the Laplacian

To illustrate the main ideas we begin with the simple model problem for the Laplacian as a special scalar case.

Let us consider the boundary value problem

$$\Delta u = 0 \text{ in } \Omega \subset \mathbb{R}^n \quad (n = 2 \text{ or } 3), \tag{2.1}$$

$$Tu := \frac{\partial u}{\partial n} = \psi \text{ on } \Gamma_N, \tag{2.2}$$

$$u = \varphi \text{ on } \Gamma_D \tag{2.3}$$

where Ω is a bounded domain with a piece–wise smooth boundary $\partial\Omega = \Gamma_N \cup \Gamma_D$. The functions ψ and φ are given on the closed Neumann and the Dirichlet parts Γ_N and Γ_D, respectively, of the boundary (see Figure 1) where $\overset{\circ}{\Gamma}_N = \partial\Omega \setminus \Gamma_D$.

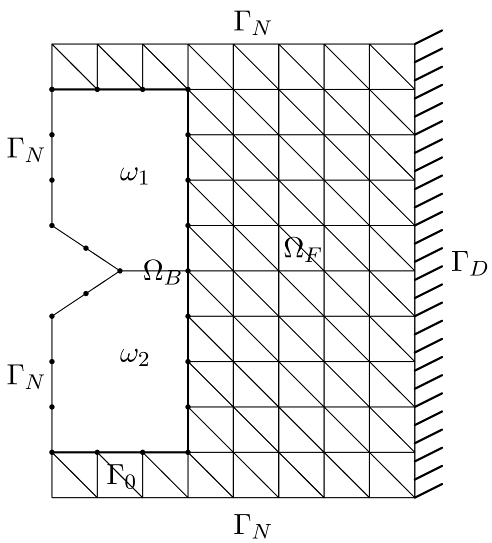

Figure 1: Finite and macro elements

To describe the procedure, we first decompose Ω into two subregions Ω_F and Ω_B with $\Omega_F \cup \Omega_B \cup \Gamma_0 = \Omega$ where $\Gamma_0 = \partial\Omega_F \cap \partial\Omega_B$ is the corresponding coupling boundary.

We further decompose Ω_B into two disjoint subdomains ω_1 and ω_2 as in Figure 1. The main idea here is to approximate the solution in Ω_F by finite elements, in ω_j by corresponding Trefftz elements modelled via boundary potentials, and an additional approximation for the trace of the solution on the so–called skeleton Υ which will be defined later on. For the motivation of the corresponding variational scheme we begin with the generalized total potential functional

$$
\begin{aligned}
&\Pi(u_F,\widetilde{u};u_1,u_2;\lambda_j,\lambda_2) := \\
&\quad \frac{1}{2}\int\limits_{\Omega_F}|\nabla u_F|^2dx + \frac{1}{2}\sum_{j=1}^{2}\int\limits_{\partial\omega_j} u_j\frac{\partial u_j}{\partial n}ds - \sum_{j=1}^{2}\int\limits_{\Gamma_N\cap\partial\omega_j}\psi\widetilde{u}ds - \int\limits_{\Gamma_N\cap\partial\Omega_F}\psi u_F ds \\
&\quad + \sum_{j=1}^{2}\int\limits_{\partial\omega_j}\lambda_j(\widetilde{u}-u_j)ds\,.
\end{aligned}
\tag{2.4}
$$

In this functional, as in elasticity, the first four terms resemble the standard total potential energy where in ω_j the characterization of Trefftz elements in the form

$$
\int\limits_{\omega_j}|\nabla u_j|^2dx = \int\limits_{\partial\omega_j}u_j\frac{\partial u_j}{\partial n}ds \ \text{ for } \Delta u_j = 0 \ \text{ in } \omega_j \tag{2.5}
$$

has been used. The λ_j in the last sum in (2.4) play the rôle of Lagrangian multipliers enforcing continuity for the solution at $\partial\omega_j$ in a weak form.

The additional variable $\widetilde{u}$ is defined by the trace of the solution only on the skeleton Υ which is presently chosen as $\Upsilon := \partial\omega_1\cup\partial\omega_2$. In addition, we enforce the pointwise continuity by the requirement $\widetilde{u} = u_F$ on Γ_0, The introduction of $\widetilde{u}$ corresponds to the so–called mortar elements in the work by Bernardi and Maday et al. [4], [5] (see also [6]).

As usual, we consider the first variation of Π in an admissible space $\mathcal{U}_{ad}$ and the corresponding test functions $(v_F,\widetilde{v};v_1,v_2;\chi_1,\chi_2)\in\mathcal{V}$ with $v_F=\widetilde{v}$ on Γ_0. The spaces $\mathcal{U}_{ad}$ and $\mathcal{V}$ need to be specified precisely later on in order to obtain a saddle–point formulation.

$$
\frac{\partial\Pi}{\partial u_F} = 0 : \int\limits_{\Omega_F}\nabla u_F\cdot\nabla v_F ds + \sum_{j=1}^{2}\int\limits_{\partial\omega_j\cap\Gamma_0}\lambda_j v_F ds = \int\limits_{\partial\Omega_F\cap\Gamma_N}\psi v_F ds\,, \tag{2.6}
$$

where we have tacitly used the relation $\widetilde{u}_{|\Gamma_0} = u_{F|\Gamma_0}$ in (2.4);

$$
\frac{\partial\Pi}{\partial\widetilde{u}} = 0 : \sum_{j=1}^{2}\int\limits_{\partial\omega_j}\lambda_j\widetilde{v}ds = \int\limits_{\Gamma_N\cap(\partial\omega_1\cup\partial\omega_2)}\psi\widetilde{v}ds \ \text{ where } \widetilde{v}_{|\Gamma_0}=0 \ \text{ since } \widetilde{u}_{|\Gamma_0} = u_{F|\Gamma_0}\,; \tag{2.7}
$$

$$
\frac{\partial\Pi}{\partial u_j} = 0 : \frac{1}{2}\int\limits_{\partial\omega_j}\left(\frac{\partial u_j}{\partial n}v_j + \frac{\partial v_j}{\partial n}u_j\right)ds - \int\limits_{\partial\omega_j}\lambda_j v_j ds = 0\,.
$$

Since for the Trefftz elements $\Delta u_j = 0$ and $\Delta v_j = 0$ in ω_j,

these equations take the form

$$
\int\limits_{\partial\omega_j}\left\{\frac{\partial u_j}{\partial n}-\lambda_j\right\}v_j ds = 0\,; \tag{2.8}
$$

$$
\frac{\partial\Pi}{\partial\lambda_j} = 0 : \int\limits_{\partial\omega_j}\chi_j(\widetilde{u}-u_j)ds = 0\,. \tag{2.9}
$$

The equations (2.6)–(2.9) are to be satisfied for all test functions $(v_F,\widetilde{v};v_1,v_2;\chi_1,\chi_2)\in\mathcal{V}$. They are the weak form of the Euler equations of the functional Π and define the mixed variational formulation for the transmission problem associated with the boundary value problem (2.1)–(2.3).

Before we present the rigorous justification and the details of our method, we first enforce the following simplification: To eliminate equation (2.8) we require

$$\lambda_j = \frac{\partial u_j}{\partial n} \text{ on } \partial\omega_j \tag{2.10}$$

which reduces the variational formulation to:
Find $(u_F, \widetilde{u}; \lambda_1, \lambda_2)$ *with* $\widetilde{u} = u_F$ *on* Γ_0 *and*

$$\int_{\partial\omega_j} \lambda_j ds = 0 \tag{2.11}$$

satisfying

$$\int_{\Omega_F} \nabla u_F \cdot \nabla v_F dx + \sum_{j=1}^{2} \int_{\partial\omega_j} \lambda_j \widetilde{v} ds = \int_{\partial\Omega_F \cap \Gamma_N} \psi v_F ds + \sum_{j=1}^{2} \int_{\Gamma_N \cap \partial\omega_j} \psi \widetilde{v} ds \tag{2.12}$$

for all v_F *and* $\widetilde{v}$ *with* $\widetilde{v} = v_F$ *on* Γ_0,

$$\int_{\partial\omega_j} \chi_j (\widetilde{u} - u_j) ds = 0 \ \textit{ for all } \chi_j\,. \tag{2.13}$$

In the last equations, u_j is related to λ_j by the solution of the local Neumann problem

$$\Delta u_j = 0 \ \text{ in } \omega_j \ \text{ and } \frac{\partial u_j}{\partial n} = \lambda_j \ \text{ on } \partial\omega_j\,, \tag{2.14}$$

which requires the necessary compatibility conditions (2.11) as normalization conditions for the desired solution $(u_F, \widetilde{u}; \lambda_1, \lambda_2)$ of (2.12)–(2.13).

The realization of (2.14) with (2.11) can be achieved by introducing the Poincaré–Steklov operator, i. e. the Neumann–Dirichlet map

$$U_j \,:\, \lambda_j \mapsto u_j|_{\partial\omega_j} = U_j\lambda_j \ \text{ with respect to } \Delta u_j = 0 \ \text{ in } \omega_j\,;\ j = 1, 2\,. \tag{2.15}$$

The Poincaré–Steklov operator U_j will be expressed explicitly by the use of boundary integral operators.

In terms of this operator U_j, the variational formulation reads:

Find $(u_F, \widetilde{u}; \lambda_j) \in \mathcal{U}_{ad}$ *satisfying* $u_F|_{\Gamma_D} = \varphi$, $\widetilde{u}_{|\Gamma_0} = u_{F|\Gamma_0}$ *such that the equations* (2.12)–(2.13) *are fulfilled with* $u_j = U_j\lambda_j$ *in* (2.13) *for all test functions* $(v_F, \widetilde{v}; \chi_j) \in \mathcal{V}$.

For the discretization we use two levels characterized by two meshsize parameters H and h. The parameter H is used for the global grids, i. e. for the finite element grid in Ω_F and for the grid on the skeleton Υ. The latter, h, is used for the local macro–element boundary discretizations on $\partial\omega_1$ and on $\partial\omega_2$, respectively. It is understood, that the boundary grids on $\partial\omega_1$ and on $\partial\omega_2$ may be chosen independently. Note that, geometrically, on $\partial\omega_1 \cap \partial\omega_2$ we then may find three different grids since $\partial\omega_1 \cap \partial\omega_2 \subset \Upsilon$ (see Figure 2).

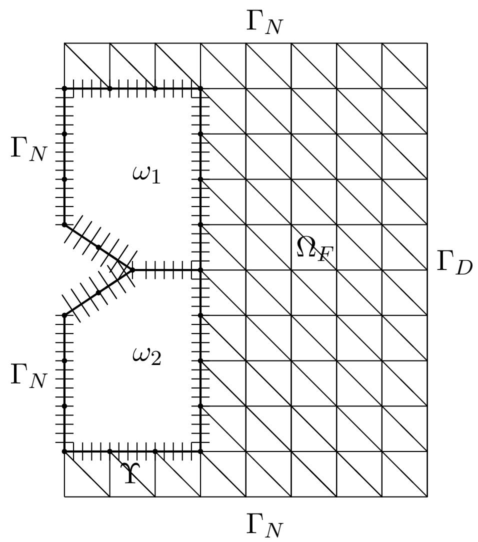

Figure 2: Finite, skeleton and macro element grids

We denote the finite–dimensional space of admissible functions by $(\mathcal{H}_H, \mathcal{H}_{1h}, \mathcal{H}_{2h})$. Here $\mathcal{H}_H = \{v^H = (v_F^H, \tilde{v}^H) \,|\, v_F^H = \tilde{v}^H \text{ on } \Gamma_0\}$ is a chosen finite element space on Ω_F extended to the skeleton Υ according to the chosen grid. The finite–dimensional spaces $\mathcal{H}_{jh} = \{\chi_j^h \,|\, \int_{\partial\omega_j} \chi_j^h ds = 0\}$ are chosen on $\partial\omega_j$ as boundary element spaces for the unknowns λ_j^h according to the boundary element grids. Now the Galerkin equations to (2.12), (2.13) read:

Find $(u^H; \lambda_1^h, \lambda_2^h) \in \mathcal{H}_H \times \mathcal{H}_{1h} \times \mathcal{H}_{2h}$ *such that*

$$a_{\Omega_F}(u^H, v^H) + \sum_{j=1}^{2} \int_{\partial\omega_j} \lambda_j^h v^H ds = \int_{\partial\Omega_F \cap \Gamma_N} \psi v^H ds + \sum_{j=1}^{2} \int_{\partial\omega_F \cap \Gamma_N} \psi v^H ds \qquad (2.16)$$

$$\text{for all } v^H \in \mathcal{H}_H ,$$

$$\int_{\partial\omega_j} \chi_j^h U_j^h \lambda_j^h ds = \int_{\partial\omega_j} \chi_j^h u^H ds \quad \text{for all } \chi_j^h \in \mathcal{H}_{jh} . \qquad (2.17)$$

Here $a_{\Omega_F}(u^H, v^H) := \int_{\Omega_F} \nabla u^H \cdot \nabla v^H dx$ and U_j^h is a suitable approximation of the Poincaré–Steklov operator which will be specified in terms of boundary integral operators and Neumann series on $\partial\omega_j$ below. To describe the algorithm, let us introduce bases of the approximating spaces: $\mathcal{H}_H = \text{span}\{\varphi_k\}_{k=1}^N$ and $\mathcal{H}_{jh} = \text{span}\{\nu_{j\kappa}\}_{\kappa=1}^{L_j}$, $j = 1, 2$. Then we seek the solution in the form

$$u^H = \sum_{k=1}^{N} \alpha_k \varphi_k \quad \text{and} \quad \lambda_j^h = \sum_{\kappa=1}^{L_j} \beta_{j\kappa} \nu_{j\kappa} .$$

To solve (2.16), (2.17), we use two levels and solve first the second equation, i. e. (2.17) for $\vec{\beta}_j = (\beta_{j\kappa})_{\kappa=1}^{L_j}$ in terms of $\vec{\alpha} = (\alpha_k)_{k=1}^N$. In matrix notation this amounts to solving the linear systems

$$\mathsf{U}_j \vec{\beta}_j = \mathsf{B}_j \vec{\alpha} \quad \text{for } \vec{\beta}_j \text{ where} \qquad (2.18)$$

$$\mathsf{U}_j := \Big(\!\Big(\int_{\partial\omega_j} \nu_{j\varrho} U_j^h \nu_{j\kappa} ds \Big)\!\Big)_{\varrho,\kappa=1}^{L_j} \text{ and } \mathsf{B}_j := \Big(\!\Big(\int_{\partial\omega_j} \nu_{j\varrho} \varphi_k ds \Big)\!\Big)_{\varrho=1,\dots,L_j; k=1,\dots,N} . \qquad (2.19)$$

Substituting the solution $\lambda_j^h = \sum_{\kappa=1}^{L_j} \beta_{j\kappa}\nu_{j\kappa}$ with $\beta_{j\kappa}$ known into (2.16) we obtain with the choice $v^H = \varphi_m$ the algebraic system

$$\mathsf{A}\vec{\alpha} + \sum_{j=1}^{2} \mathsf{B}_j^\top \mathsf{U}_j^{-1}\mathsf{B}_j\vec{\alpha} = \mathsf{f} \tag{2.20}$$

where $\mathsf{A} = ((\int_{\Omega_F} \nabla\varphi_k \cdot \nabla\varphi_m dx))_{k,m=1,\dots,N_F}$ denotes the stiffness matrix of the finite elements in Ω_F and f the vector of right–hand sides given by (2.16). The matrix U_j is the so–called flexibility matrix and $\mathsf{B}_j^\top \mathsf{U}_j^{-1}\mathsf{B}_j$ describes the stiffness matrix corresponding to the macro–element ω_j which is symmetric and positive semidefinit provided U_j^h is symmetric. These properties are controlled in our computational procedure as will be explained in detail lateron. The resulting algebraic system (2.20) can be solved by using a conventional finite element procedure.

Note that for resembling the matrix $(\mathsf{B}_j^\top \mathsf{U}_j^{-1}\mathsf{B}_j)$ one does not need to compute the inverse matrix U_j^{-1}: instead one computes $(\mathsf{U}_j^{-1}\mathsf{B}_j)$. This amounts to solve equations (2.18) only for the few right–hand sides $\vec{\alpha} = (\delta_{k,m})_{m=1}^N$ for those $k \in \{1,\dots,N\}$ for which $\overset{\circ}{\operatorname{supp}}\,\varphi_k \cap \partial\omega_j \neq \emptyset$ ($\circ$ stands for the interior and $\delta_{k,m}$ for the Kronecker symbol).

To conclude this section, we remark that for our model problem one may also choose the skeleton Υ by incorporating the given boundary conditions into the local macro–elements. In particular, one may require here $\lambda_j^h = \psi$ on $\Gamma_N \cap \partial\omega_j$ and choose $\Upsilon := (\partial\omega_1 \cup \partial\omega_2) \setminus \Gamma_N$ (see Figure 3). We will pursue this idea in following chapters.

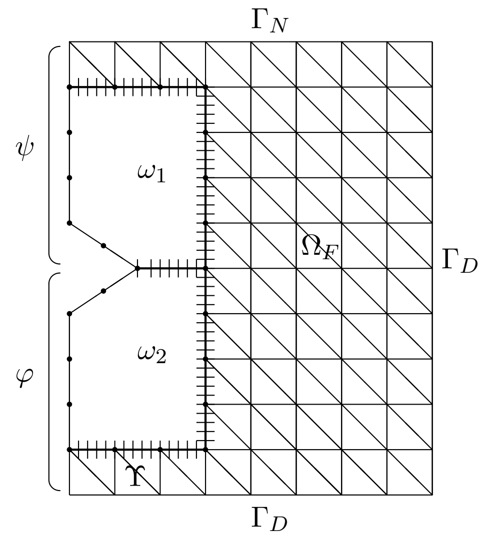

Figure 3: Skeleton without the exterior boundary Γ

3 The saddle point formulation for second order elliptic systems

Let $\Omega \subset \mathbb{R}^n (n = 3 \;\text{ or } 3)$ be a strong Lipschitz domain with strong Lipschitz boundary (see [38]). We consider the second order $p \times p$ system of strongly elliptic equations

$$\mathcal{P}\mathbf{u} := \sum_{\ell,k=1}^{n} \frac{\partial}{\partial x_\ell}\left(a_{\ell k}\frac{\partial \mathbf{u}}{\partial x_k}\right) + \sum_{\ell=1}^{n} b_\ell \frac{\partial \mathbf{u}}{\partial x_\ell} + c\mathbf{u} = \mathbf{0} \;\text{ in } \Omega\,. \tag{3.1}$$

The coefficients $a_{\ell k}, b_\ell$ and c are constant, real $p \times p$ matrices. Although the general scheme of our coupled method is applicable to systems with variable coefficients, we restrict ourselves to the self–adjoint constant coefficient case where $b_\ell = 0$ and $a_{\ell k} = a_{\ell k}^\top = a_{k\ell}, c^\top = c$. In addition to (3.1) we require the mixed boundary conditions

$$T\mathbf{u}_{|\Gamma_N} := \sum_{\ell,k=1}^{n} n_\ell a_{\ell k} \frac{\partial \mathbf{u}}{\partial x_k} = \boldsymbol{\psi} \text{ on } \Gamma_N \text{ and } \mathbf{u}_{|\Gamma_D} = \boldsymbol{\varphi} \text{ on } \Gamma_D \,. \tag{3.2}$$

Here, $\mathbf{n} = (n_\ell)_{\ell=1}^n$ is the exterior unit normal vector on $\Gamma = \partial\Omega$ and the vector fields $\boldsymbol{\psi}$ and $\boldsymbol{\varphi}$ are the given Neumann and Dirichlet boundary data on the corresponding closed boundary parts Γ_N and Γ_D where $\Gamma_N = \partial\Omega \setminus \overset{\circ}{\Gamma}_D$.

For the system (3.1) we will impose the following basic assumptions:

(A1) Strong ellipticity: There is $\gamma_0 > 0$ such that

$$\sum_{\ell,k=1}^{n} \zeta^\top a_{\ell k} \xi_\ell \xi_k \overline{\zeta} \geq \gamma_0 |\xi|^2 \, |\zeta|^2 \text{ for all } \xi \in \mathbb{R}^n \text{ and } \zeta \in \mathbb{C}^p \,. \tag{3.3}$$

(A2) Gårding's inequality:

$$\int_\Omega \left\{ \sum_{\ell,k=1}^{n} \frac{\partial \mathbf{v}}{\partial x_\ell}^\top a_{\ell k} \frac{\partial \mathbf{v}}{\partial x_k} - \mathbf{v}^\top c \mathbf{v} \right\} dx \geq \gamma_0 \|\mathbf{v}\|^2_{H^1(\Omega)} - \gamma_1 \|\mathbf{v}\|^2_{L^2(\Omega)} \text{ for all } \mathbf{v} \in H^1(\Omega) \,. \tag{3.4}$$

(A3) Definiteness:

$$\int_\Omega \left\{ \sum_{\ell,k=1}^{n} \frac{\partial \mathbf{v}}{\partial x_j}^\top a_{\ell k} \frac{\partial \mathbf{v}}{\partial x_k} - \mathbf{v}^\top c \mathbf{v} \right\} dx \geq 0 \text{ for all } \mathbf{v} \in H^1(\Omega) \,. \tag{3.5}$$

In what follows, we shall still write $H^1(\Omega)$ instead of $(H^1(\Omega))^p$ etc. .

Note that strong ellipticity (3.3) implies Gårding's inequality only on the subspace $H^1_{\partial\Omega}(\Omega) := \{\mathbf{v} \in H^1 \,|\, \mathbf{v}_{|\partial\Omega} = 0\} \overset{\bullet}{\subset} H^1(\Omega)$ [38, Theorem 7.3], whereas Gårding's inequality (3.4) on the *whole* space $H^1(\Omega)$ can only be guaranteed under additional assumptions such as formally positive ellipticity [38, Theorem 7.6].

Note that for any $\mathbf{u} \in H^1(\Omega)$ with $\mathcal{P}\mathbf{u} \in L^2(\Omega)$, the conormal derivative $T\mathbf{u}_{|\partial\Omega} \in H^{-\frac{1}{2}}(\partial\Omega)$ is well defined in the weak sense via the first Green's formula

$$\langle T\mathbf{u}, \mathbf{v} \rangle_{\partial\Omega} := \int_\Omega \left\{ \sum_{\ell,k=1}^{n} \frac{\partial \mathbf{u}^\top}{\partial x_\ell} a_{\ell k} \frac{\partial \mathbf{v}}{\partial x_k} - \mathbf{u}^\top c \mathbf{v} + (\mathcal{P}\mathbf{u})^\top \mathbf{v} \right\} dx \tag{3.6}$$

for all $\mathbf{v} \in H^1(\Omega)$ and $\mathbf{v}_{|\partial\Omega} \in H^{\frac{1}{2}}(\partial\Omega)$ due to the trace theorem.

We further assume negative semidefiniteness for the matrix coefficient $c \leq 0$; then the definiteness (3.5) is an obvious consequence of (3.3).

As in the introductory example, we decompose the given domain Ω into the subsets Ω_F and Ω_B with $\Omega = \Omega_F \cup \Omega_B \cup \Gamma_0$ where $\Gamma_0 = \partial\Omega_F \cap \partial\Omega_B$ is the global coupling boundary. The subset Ω_F will describe the finite element geometry while Ω_B will denote the macro–element part. In Ω_F, we are given some triangulation $\{\tau_\ell\}_{\ell=1}^N$ with $\overline{\Omega_F} = \bigcup_{\ell=1}^{N} \overline{\tau_\ell}$ where τ_ℓ denotes the individual finite element subdomains (which may be of rather general shape). The solution restricted to the finite element part will be denoted by $\mathbf{u}_F := \mathbf{u}_{|\Omega_F}$. The macro–element part Ω_B consists of M individual macro–element domains ω_j with $\overline{\Omega_B} = \bigcup_{j=1}^{M} \overline{\omega_j}$ and piecewise smooth, strong Lipschitz boundaries $\partial\omega_j$. Correspondingly, we denote the restriction of the solution to the macro–elements by $\mathbf{u}_j := \mathbf{u}_{|\omega_j}$.

Similar to the fundamental concept of FEM– and BEM–analysis, the coupling procedure is based on the weak formulation of a corresponding transmission problem in weak formulation. For this purpose we shall use the Sobolev space $H^1(\Omega)$ and introduce further appropriate function spaces. The energy test space corresponding to Ω_F will be denoted by

$$H^1_D(\Omega_F) := \{\mathbf{v}_F \in H^1(\Omega_F) \,|\, \mathbf{v}_{F|\partial\Omega_F \cap \Gamma_D} = \mathbf{0}\}\,. \tag{3.7}$$

Throughout the paper we assume the property $\partial\Omega_F \cap \overset{\circ}{\Gamma}_D \neq \emptyset$. Then, under assumptions (A1)–(A3), the bilinear form

$$a_F(\mathbf{u},\mathbf{v}) := \int\limits_{\Omega} \left\{ \sum_{\ell,k=1}^{n} \frac{\partial \mathbf{u}^\top}{\partial x_\ell} a_{\ell k} \frac{\partial \mathbf{v}^\top}{\partial x_k} - \mathbf{u}^\top c\, \mathbf{v} \right\} dx \tag{3.8}$$

is $H^1_D(\Omega_F)$–elliptic; i. e. there exists a constant $\alpha_0 > 0$ such that

$$a_F(\mathbf{v},\mathbf{v}) \geq \alpha_0 \|\mathbf{v}\|^2_{H^1(\Omega_F)} \;\text{ for all } \mathbf{v} \in H^1_D(\Omega_F)\,. \tag{3.9}$$

The trace spaces of the macro–element spaces $H^1(\omega_j)$ are denoted by $H^{\frac{1}{2}}(\partial\omega_j)$ and their dual spaces by $H^{-\frac{1}{2}}(\partial\omega_j)$. The corresponding duality pairing will be written as

$$\langle \boldsymbol{\chi}, \mathbf{v}\rangle_j := \int\limits_{\partial\omega_j} \boldsymbol{\chi}\cdot\mathbf{v} ds \;\text{ for } j = 1,\dots,M\,, \tag{3.10}$$

where $\boldsymbol{\chi}\cdot\mathbf{v}$ is the $\mathrm{I\!R}^n$–scalar product. We will also need the subspaces

$$H_0^{-\frac{1}{2}}(\partial\omega_j) := \{\boldsymbol{\chi} \in H^{-\frac{1}{2}}(\partial\omega_j) \,|\, \langle \boldsymbol{\chi}, \mathbf{r}_j\rangle_j = 0 \;\text{ for all } \mathbf{r}_j \in \Re_j\}$$

where

$$\Re_j := \{\mathbf{r}_j \in H^1(\omega_j) \,|\, \mathcal{P}\mathbf{r}_j = \mathbf{0} \wedge T_j \mathbf{r}_j = \mathbf{0} \;\text{ on } \partial\omega_j\} \tag{3.11}$$

denotes the solution space of the homogeneous local Neumann problem in ω_j and T_j denotes the conormal derivative on $\partial\omega_j$.

The basis assumptions (A1)–(A3) then also imply on every ω_j the coerciveness inequality

$$a_{\omega_j}(\mathbf{v}_j,\mathbf{v}_j) \geq \alpha_{0j} \|\mathbf{v}_j\|^2_{H^1(\omega_j)/\Re_j} \tag{3.12}$$

with the individual constants $\alpha_{0j} > 0$ where the $H^1(\omega_j)/\Re_j$–norm denotes the norm in the quotient space:

$$\|\mathbf{v}_j\|_{H^1(\omega_j)/\Re_j} = \inf_{\mathbf{r}_j \in \Re_j} \|\mathbf{v}_j + \mathbf{r}_j\|_{H^1(\omega_j)}\,.$$

In addition, we define the geometric skeleton Υ of the macro–elements. In the introductory example, we chose $\Upsilon = \bigcup_{j=1}^{M} \partial\omega_j$. In general, one may choose Υ as a connected, closed part of $\bigcup_{j=1}^{M} \partial\omega_j$ satisfying $\bigcup_{j=1}^{M} \overline{\partial\omega_j \setminus \partial\Omega} \subseteq \Upsilon \subseteq \bigcup_{j=1}^{M} \partial\omega_j$. On this skeleton Υ we introduce the mortar functions

$$\widetilde{\mathbf{u}} \in H^{\frac{1}{2}}(\Upsilon) := \{\widetilde{\mathbf{w}} = \mathbf{w}_{|\Upsilon} \,|\, \mathbf{w} \in H^1(\Omega) \text{ and } \|\widetilde{\mathbf{w}}\|_{H^{\frac{1}{2}}(\Upsilon)} := \inf \|\mathbf{w}\|_{H^1(\Omega)}\} \tag{3.13}$$

and the product space of pairs of restrictions

$$\mathcal{H} := \{(\mathbf{w}_F, \widetilde{\mathbf{w}}) \in H^1(\Omega_F) \times H^{\frac{1}{2}}(\Upsilon) \,|\, \widetilde{\mathbf{w}}_{|\Upsilon} = \mathbf{w}_{F|\Upsilon}\} \tag{3.14}$$

equipped with the norm

$$\|(\mathbf{w}_F, \widetilde{\mathbf{w}})\|_{\mathcal{H}} := \inf\{\|\mathbf{w}\|_{H^1(\Omega)} \,|\, \mathbf{w} \in H^1(\Omega)\,,\ \mathbf{w}_{|\Omega_F} = \mathbf{w}_F \text{ and } \mathbf{w}_{|\Upsilon} = \widetilde{\mathbf{w}}\}\,.$$

The test function space associated with Υ, will be denoted by

$$\mathcal{H}_D := \{(\mathbf{v}_F, \widetilde{\mathbf{v}}) \in \mathcal{H} \,|\, \mathbf{v}_F \in H^1_D(\Omega_F) \text{ and } \widetilde{\mathbf{v}}_{|\Gamma_D \cap \Upsilon} = \mathbf{0}\}\,. \tag{3.15}$$

In addition to the spaces $H^{\frac{1}{2}}(\partial\omega_j)$ and $H^{-\frac{1}{2}}(\partial\omega_j)$ we shall also need the subspaces

$$\widetilde{H}^{\frac{1}{2}}(\partial\omega_{jN}) := \{\mathbf{v} \in H^{\frac{1}{2}}(\partial\omega_j) \,|\, \operatorname{supp}(\mathbf{v}) \subset \partial\omega_{jN}\}$$

equipped with the $H^{\frac{1}{2}}(\partial\omega_j)$–norm where $\partial\omega_j = \partial\omega_{jD} \cup \partial\omega_{jN}$ with closed boundary parts $\partial\omega_{jD}$ and $\partial\omega_{jN} = \partial\omega_j \setminus \partial\mathring{\omega}_{jD}$ to be specified when needed, and

$$\widetilde{H}^{-\frac{1}{2}}(\partial\omega_{jD}) := \{\boldsymbol{\chi} \in H^{-\frac{1}{2}}(\partial\omega_j) \,|\, \operatorname{supp}(\boldsymbol{\chi}) \subset \partial\omega_{jD}\}$$

equipped with the norm of $H^{-\frac{1}{2}}(\partial\omega_j)$. Functions in these subspaces will also be considered as functions on all of $\partial\omega_j$.

The variational saddle point formulation now reads:

Find $(\mathbf{u}_F, \widetilde{\mathbf{u}}; \mathbf{u}_j; \boldsymbol{\lambda}_j) \in \mathcal{U}_{ad}$ *such that*

$$a_{\Omega_F}(\mathbf{u}_F, \mathbf{v}_F) + \sum_{j=1}^{M} \int_{\partial\omega_j \cap \Upsilon} \boldsymbol{\lambda}_j \cdot \widetilde{\mathbf{v}} ds = \int_{\partial\Omega_F \cap \Gamma_N \setminus \Upsilon} \boldsymbol{\psi} \cdot \mathbf{v}_F ds + \int_{\Gamma_N \cap \Upsilon} \boldsymbol{\psi} \cdot \widetilde{\mathbf{v}} ds \tag{3.16}$$

and the weak coupling conditions

$$\int_{\partial\omega_j \cap \Upsilon} \boldsymbol{\chi}_j \cdot (\widetilde{\mathbf{u}} - \mathbf{u}_j) ds \ = \ 0\,, \tag{3.17}$$

$$\int_{\partial\omega_j \cap \Upsilon} (T_j \mathbf{u}_j - \boldsymbol{\lambda}_j) \cdot \mathbf{v}_j ds \ = \ 0 \tag{3.18}$$

are satisfied for all test functions $(\mathbf{v}_F, \widetilde{\mathbf{v}}; \mathbf{v}_j; \boldsymbol{\chi}_j) \in \mathcal{V}$. *The admissible functions* $\mathcal{U}_{ad}$ *are given by*

$$(\mathbf{u}_F, \widetilde{\mathbf{u}}) \in \mathcal{H} \ \textit{with } \mathbf{u}_F = \boldsymbol{\varphi} \ \textit{on } \partial\Omega_F \cap \Gamma_D \ \textit{and } \widetilde{\mathbf{u}} = \boldsymbol{\varphi} \ \textit{on } \Upsilon \cap \Gamma_D\,; \tag{3.19}$$

$$\left.\begin{array}{l} \mathbf{u}_j \in H^1(\omega_j) \ \textit{satisfying } \mathcal{P}\mathbf{u}_j = 0 \ \textit{in } \omega_j \\ \quad \textit{with } \mathbf{u}_j = \boldsymbol{\varphi} \ \textit{on } \partial\omega_j \cap \Gamma_D \setminus \Upsilon \ \textit{and } T_j\mathbf{u}_j = \boldsymbol{\psi} \ \textit{on } \partial\omega_j \cap \Gamma_N \setminus \Upsilon\,; \end{array}\right\} \tag{3.20}$$

$$\left.\begin{array}{l} \boldsymbol{\lambda}_j \in H^{-\frac{1}{2}}(\partial\omega_j \cap \Upsilon) \ \textit{with the additional constraint that} \\ \boldsymbol{\lambda}_j^* := \left\{\begin{array}{l} \boldsymbol{\lambda}_j \ \textit{on } \partial\omega_j \cap \Upsilon \ \textit{and} \\ \boldsymbol{\psi} \ \textit{on } \partial\omega_j \cap \Gamma_N \setminus \Upsilon \end{array}\right\} \ \textit{belongs to } H^{-\frac{1}{2}}(\partial\omega_j \setminus (\Gamma_D \setminus \Upsilon))\,. \end{array}\right\} \tag{3.21}$$

The test functions $(\mathbf{v}_F, \widetilde{\mathbf{v}}; \mathbf{v}_j; \boldsymbol{\chi}_j) \in \mathcal{V}$ *are defined by* $(\mathbf{v}_F, \widetilde{\mathbf{v}}) \in \mathcal{H}_D$ *and*

$$\left.\begin{array}{rcl} \mathbf{v}_j & in & H^1(\omega_j) \ \text{subject to the constraints} \\ \mathcal{P}\mathbf{v}_j & = & 0 \ \text{in}\ \omega_j\,,\ \mathbf{v}_j = 0 \ \text{on}\ \partial\omega_j \cap \Gamma_D \setminus \Upsilon \ \text{and}\ T\mathbf{v}_j = \mathbf{0} \ \text{on}\ \partial\omega_j \cap \Gamma_N \setminus \Upsilon\,; \end{array}\right\} \tag{3.22}$$

and $\boldsymbol{\chi}_j \in \widehat{H}_0^{-\frac{1}{2}}(\partial\omega_j \cap \Upsilon)$

$$:= \{\boldsymbol{\chi} \in H_0^{-\frac{1}{2}}(\partial\omega_j) \ \text{there exists}\ \boldsymbol{\chi}^* \in \widetilde{H}_0^{-\frac{1}{2}}\Big(\partial\omega_j \setminus (\Gamma_N \setminus \Upsilon)\Big) \ \text{with}\ \boldsymbol{\chi} = \boldsymbol{\chi}^* \ \text{on}\ \partial\omega_j \cap \Upsilon\} \tag{3.23}$$

equipped with the norm

$$\|\boldsymbol{\chi}\|_{\widehat{H}_0^{-\frac{1}{2}}} := \inf_{\boldsymbol{\chi}^*} \|\boldsymbol{\chi}^*\|_{H_0^{-\frac{1}{2}}(\partial\omega_j)}\,. \tag{3.24}$$

In equation (3.14) we have cancelled the common term $\sum_{j=1}^M \int\limits_{\partial\omega_j \cap \Gamma \setminus \Upsilon} \boldsymbol{\psi} \cdot \widetilde{\mathbf{v}}^* ds$ on both sides of (3.16), where $\widetilde{\mathbf{v}}^* \in \widetilde{H}^{\frac{1}{2}}\Big(\partial\omega_j \setminus (\Gamma_D \setminus \Upsilon)\Big) \subset H^{\frac{1}{2}}(\partial\omega_j)$ is any extension of $\widetilde{\mathbf{v}}$.

In the following, in order to take into account the constraints (3.19)–(3.21) for $\mathcal{U}_{ad}$ and to satisfy (3.18) identically, we first identify

$$\boldsymbol{\lambda}_j = T_j \mathbf{u}_j \ \text{ and } \boldsymbol{\chi}_j = T_j \mathbf{v}_j \ \text{ on } \partial\omega_j \cap \Upsilon\,. \tag{3.25}$$

Next, we shall reduce the corresponding local boundary value problems for $\mathbf{u}_j$ and $\mathbf{v}_j$ in ω_j to boundary integral equations. This will simplify the variational formulation (3.16)–(3.18) from the four unknowns $(\mathbf{u}_F, \widetilde{\mathbf{u}}; \mathbf{u}_j; \boldsymbol{\lambda}_j)$ to the three unknowns either $(\mathbf{u}_F, \widetilde{\mathbf{u}}; \boldsymbol{\lambda}_j)$ or to $(\mathbf{u}_F, \widetilde{\mathbf{u}}; \mathbf{u}_j)$. For this purpose we need to introduce the local Neumann–Dirichlet mappings, i. e. the Poincaré–Steklov operators U_j — or the local Dirichlet–Neumann mappings , i. e. the Steklov–Poincaré operators S_j.

4 The variational formulation based on the local Neumann data

In this section we first introduce the abstract local Poincaré–Steklov operator

$$\widehat{U}_j \,:\, H^{\frac{1}{2}}(\partial\omega_{jD}) \times H^{-\frac{1}{2}}(\partial\omega_{jN}) \ni (\boldsymbol{\varphi}, \boldsymbol{\lambda}_j^*) \mapsto \mathbf{u}_{j|_{\partial\omega_j}} \in H^{\frac{1}{2}}(\partial\omega_j)$$

by solving the local mixed boundary value problem

$$\mathcal{P}\mathbf{u}_j = \mathbf{0} \ \text{ in } \omega_j\,,\ \mathbf{u}_j = \boldsymbol{\varphi} \ \text{ on } \partial\omega_{jD} := \partial\omega_j \cap \Gamma_D \setminus \mathring{\Upsilon}\,,\ T_j\mathbf{u}_j = \boldsymbol{\lambda}_j^* \ \text{ on } \partial\omega_{jN} := \partial\omega_j \setminus \partial\mathring{\omega}_{jD} \tag{4.1}$$

in $H^1(\omega_j)$. The configuration of $\partial\omega_j$ is described in Figure 4 below.

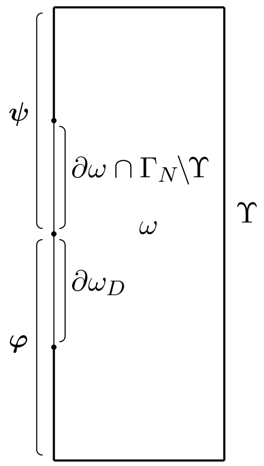

Figure 4: Macro element with $\partial\omega_D := \partial\omega \cap \Gamma_D \setminus \Upsilon$

Then the variational formulation (3.16)–(3.18) reads:

Find

$$(\mathbf{u}_F, \widetilde{\mathbf{u}}; \boldsymbol{\lambda}_j) \in \mathcal{H} \times \prod_{j=1}^{M} H^{-\frac{1}{2}}(\Upsilon \cap \partial\omega_j) \;\; \textit{with}\; \mathbf{u}_F = \boldsymbol{\varphi} \;\; \textit{on}\; \Gamma_D \cap \partial\Omega_F \;\; \textit{and}\; \widetilde{\mathbf{u}} = \boldsymbol{\varphi} \;\; \textit{on}\; \Gamma_D \cap \Upsilon$$

such that the global equations

$$a_{\Omega_F}(\mathbf{u}_F, \mathbf{v}_F) + \sum_{j=1}^{M} \int\limits_{\partial\omega_j \cap \Upsilon} \boldsymbol{\lambda}_j \cdot \widetilde{\mathbf{v}} ds = \int\limits_{\partial\Omega_F \cap \Gamma_N \setminus \Upsilon} \boldsymbol{\psi} \cdot \mathbf{v}_F ds + \int\limits_{\Gamma_N \cap \Upsilon} \boldsymbol{\psi} \cdot \widetilde{\mathbf{v}} ds \tag{4.2}$$

and the weak coupling conditions

$$\int\limits_{\partial\omega_j \cap \Upsilon} \boldsymbol{\chi}_j \cdot \Big(\widetilde{\mathbf{u}} - \widehat{U}_j(\boldsymbol{\varphi}, \boldsymbol{\lambda}_j^*)\Big) ds = 0 \tag{4.3}$$

are satisfied for all test functions $(\mathbf{v}_F, \widetilde{\mathbf{v}}) \in \mathcal{H}_D$ *and* $\boldsymbol{\chi}_j \in \widehat{H}_0^{-\frac{1}{2}}(\partial\omega_j \cap \Upsilon)$.

We recall here that $\boldsymbol{\lambda}_j^*$ is defined by $\boldsymbol{\lambda}_j^* = \boldsymbol{\psi}$ on $\partial\omega_j \cap \Gamma_N \setminus \Upsilon$ and $\boldsymbol{\lambda}_j^* = \boldsymbol{\lambda}_j$ on $\partial\omega_j \cap \Upsilon$. Moreover, the trial functions $\boldsymbol{\lambda}_j$ and $\widetilde{\mathbf{u}}$ have to satisfy pointwise continuity requirements at the end points of the skeleton in a weak sense, namely, $\boldsymbol{\lambda}_j^* \in H^{-\frac{1}{2}}(\partial\omega_j \cap (\Gamma_N \cup \Upsilon))$ and $\widetilde{\mathbf{u}}_j^* \in H^{\frac{1}{2}}(\partial\omega_j)$ where $\widetilde{\mathbf{u}}_j^*$ is defined by $\widetilde{\mathbf{u}}_j^* = \widetilde{\mathbf{u}}$ on $\partial\omega_j \cap \Upsilon$ and $\widetilde{\mathbf{u}}_j^* = \mathbf{u}_j$ on $\partial\omega_j \setminus \Upsilon$, where $\mathbf{u}_j = \widehat{U}_j(\boldsymbol{\varphi}, \boldsymbol{\lambda}_j^*)$.

In the special case $\partial\omega_j \cap \Gamma_D = \emptyset$ we set $\widehat{U}_j(\boldsymbol{\varphi}, \boldsymbol{\lambda}_j^*) := U_j(\boldsymbol{\lambda}_j^*)$. Since then the solution of (4.1) is only unique modulo $\Re_j$, i. e.

$$\mathbf{u}_j = U_j(\boldsymbol{\lambda}_j^*) + \mathbf{r}_j^* \;\; \text{on}\; \partial\omega_j \;\; \text{with some}\; \mathbf{r}_j^* \in \Re_j\,, \tag{4.4}$$

we require the compatibility conditions

$$\int\limits_{\partial\omega_j \cap \Upsilon} \mathbf{r}_j \cdot \Big(\widetilde{\mathbf{u}} - U_j(\boldsymbol{\lambda}_j^*) - \mathbf{r}_j^*\Big) ds = 0 \;\; \text{for all}\; \mathbf{r}_j \in \Re_j \tag{4.5}$$

determining $\mathbf{r}_j^*$ uniquely.

We now state the main theorem concerning the ellipticity property of the mixed variational form (4.2) which will be needed for existence, uniqueness as well as the stability and convergence analysis of the numerical scheme.

Theorem 1 *The bilinear form defined by (4.2) and (4.3) is continuous and $\mathcal{H}_D \times \prod_{j=1}^M \widehat{H}_0^{-\frac{1}{2}}(\partial\omega_j \cap \Upsilon)$–elliptic, i. e. there exists $\alpha_0 > 0$ such that*

$$a_{\Omega_F}(\mathbf{v}_F, \mathbf{v}_F) + \sum_{j=1}^{M} \int\limits_{\partial\omega_j\cap\Upsilon} \boldsymbol{\chi}_j \cdot \widetilde{\mathbf{v}} ds \geq \alpha_0 \left\{ \|\mathbf{v}_F\|^2_{H^1(\Omega_F)} + \sum_{j=1}^{M} \|\boldsymbol{\chi}_j\|^2_{\widehat{H}_0^{-\frac{1}{2}}(\partial\omega_j\cap\Upsilon)} \right\} \tag{4.6}$$

provided

$$\int\limits_{\partial\omega_j\cap\Upsilon} \boldsymbol{\chi}_j \cdot \Big(\widetilde{\mathbf{v}} - \widehat{U}_j(\mathbf{0}, \boldsymbol{\chi}_j)\Big) ds = 0\,. \tag{4.7}$$

Since the proof of this theorem depends on the solution of local problems in every macro–element ω_j we will postpone the proof to the end of this section.

4.1 Local problems based on local Neumann data

For ease of reading we suppress the index j when dealing with the local problem in ω_j in this section.

4.1.1 The local Neumann problem

We begin with the local problem (4.1) for the case $\partial\overset{\circ}{\omega}_D = \emptyset$. As is well known, the local Neumann problem is to find $u \in H^1(\omega)$ as the solution of

$$\mathcal{P}\mathbf{u} = \mathbf{0} \ \text{ in } \omega\,, \ T\mathbf{u} = \boldsymbol{\psi} \ \text{ on } \partial\omega \tag{4.8}$$

where the given datum $\boldsymbol{\psi} \in H_0^{-\frac{1}{2}}(\partial\omega)$ satisfies the compatibility conditions (viz. (3.10))

$$\langle \boldsymbol{\psi}, \mathbf{r} \rangle = 0 \ \text{ for all } \mathbf{r} \in \Re(\partial\omega)\,. \tag{4.9}$$

For uniqueness we require also the solution to satisfy

$$\langle \mathbf{u}, \mathbf{r} \rangle = 0 \ \text{ for all } \mathbf{r} \in \Re(\partial\omega)\,. \tag{4.10}$$

Clearly, the solution of the Neumann problem (4.8)–(4.10) is well defined and unique due to our assumptions **(A1)–(A3)**. Hence, the Poincaré–Steklov mapping

$$U \ : \ H_0^{-\frac{1}{2}}(\partial\omega) \to H_0^{\frac{1}{2}}(\partial\omega) \ \text{ with } \boldsymbol{\psi} \mapsto U\boldsymbol{\psi} := \mathbf{u}_{|\partial\omega} \tag{4.11}$$

is well defined where $H_0^{\frac{1}{2}}(\partial\omega) = \{\mathbf{u} \in H^{\frac{1}{2}}(\partial\omega) \,|\, \langle \mathbf{u} \cdot \mathbf{r} \rangle_{\partial\omega} = 0 \text{ for all } \mathbf{r} \in \Re(\partial\omega)\}$. Moreover, assumptions **(A1)–(A3)** imply the $H_0^{-\frac{1}{2}}(\partial\omega)$–ellipticity of U, i. e. there exists $\alpha_0 > 0$ such that

$$a_\omega(\mathbf{u}, \mathbf{u}) = \langle \boldsymbol{\psi}, U\boldsymbol{\psi} \rangle_{\partial\omega} \geq \alpha_0 \|\boldsymbol{\psi}\|^2_{H^{-\frac{1}{2}}(\partial\omega)} \ \text{ for all } \boldsymbol{\psi} \in H_0^{-\frac{1}{2}}\,. \tag{4.12}$$

Our aim here is to employ the boundary integral equation method for computing the solutions of the local problems. To this end, let $E(x, y)$ be the fundamental solution for the operator $\mathcal{P}$ in $\mathbb{R}^n$ which exists due to **(A1), (A2)** (see e. g. [31], [37]). Then the solution $\mathbf{u}$ admits the representation

$$\mathbf{u}(x) = \int\limits_{\partial\omega} E(x, y)\boldsymbol{\psi}(y) ds_y - \int\limits_{\partial\omega} \Big(T_y[E(x, y)]\Big)^\top \mathbf{u}(y) ds_y \ \text{ for all } x \in \omega\,. \tag{4.13}$$

The boundary integral equation of the second kind

By using in (4.13) the standard jump relations from potential theory for $x \to \partial\omega$ we arrive at the boundary integral equation of the second kind

$$(\tfrac{1}{2}I + K)\mathbf{u} = V\boldsymbol{\psi} \ \text{ on } \partial\omega\,. \tag{4.14}$$

In this equation, V and K are, respectively, the boundary integral operators of single and double layer potentials on the boundary $\partial\omega$, defined by

$$\begin{aligned} V\boldsymbol{\lambda}(x) &:= \int\limits_{\partial\omega} E(x,y)\boldsymbol{\lambda}(y)ds_y \ \text{ and} && (4.15)\\ K\mathbf{v}(x) &:= \text{p.v.}\int\limits_{\partial\omega} \Big(T_y E(x,y)\Big)^{\top}\mathbf{v}(y)ds_y \ \text{ for } x \in \partial\omega && (4.16)\end{aligned}$$

where p.v. stands for the Cauchy principal value integral.

Mapping properties of V and K

The following mapping properties can be established for V and K (see [15], [27]):

$$\begin{aligned} V \,:\, H^{-\frac{1}{2}}(\partial\omega) \to H^{\frac{1}{2}}(\partial\omega) &\qquad \text{is continuous and is } H_0^{-\frac{1}{2}}(\partial\omega)\text{–elliptic, i. e.} && (4.17)\\ \langle\boldsymbol{\lambda}, V\boldsymbol{\lambda}\rangle \ge \alpha_0\|\boldsymbol{\lambda}\|^2_{H_0^{-\frac{1}{2}}(\partial\omega)} &\qquad \text{holds for all } \boldsymbol{\lambda} \in H_0^{-\frac{1}{2}}(\partial\omega)\,. && (4.18)\end{aligned}$$

$$\text{The singular integral operator } K \,:\, H^{\frac{1}{2}}(\partial\omega) \to H^{\frac{1}{2}}(\partial\omega) \ \text{ is continuous } (see[13],[14],[15],[27]). \tag{4.19}$$

The solvability of (4.14)

The inverse $(\frac{1}{2}I + K)^{-\frac{1}{2}}$ exists only on the range $(\frac{1}{2}I + K)H^{\frac{1}{2}}(\partial\omega)$ since $\Re$ is the nullspace of $(\frac{1}{2}I + K)$. Moreover, the solution of (4.14) exists for $\boldsymbol{\psi}$ satisfying (4.9) and $V\boldsymbol{\psi}$ is in the range of $(\frac{1}{2}I + K)$ which is characterized by

$$\text{range } (\tfrac{1}{2}I + K) = \Big\{\boldsymbol{\varphi} \in H^{\frac{1}{2}}(\partial\omega) \,|\, \langle\boldsymbol{\varphi}, \mathbf{r}'\rangle = 0 \ \text{ for all } \mathbf{r}' \in \Re'\Big\} \tag{4.20}$$

where $\Re'$ is the finite–dimensional nullspace of $(\frac{1}{2}I + K')$ in $H^{-\frac{1}{2}}(\partial\omega)$. The solution of (4.14) is unique only up to $\Re$. Since the range (4.20), in general, does not coincide with $H_0^{\frac{1}{2}}(\partial\omega)$, and in order to express the Poincaré–Steklov mapping explicitly in terms of the operator $(\frac{1}{2}I + K)^{-1}$, we need to modify the equation (4.14) so that (4.11) holds. For this purpose we need to introduce the projection operator $P_{\Re} \,:\, H^{\frac{1}{2}}(\partial\omega) \to H_0^{\frac{1}{2}}(\partial\omega)$ defined by

$$\langle(P_{\Re}\mathbf{v}), \mathbf{r}\rangle = 0 \ \text{ for all } \mathbf{r} \in \Re\,. \tag{4.21}$$

Then, instead of (4.14), we solve

$$\mathbf{u} - P_{\Re}(\tfrac{1}{2}I - K)\mathbf{u} = P_{\Re}V\boldsymbol{\psi} \tag{4.22}$$

for $\mathbf{u} \in H_0^{\frac{1}{2}}(\partial\omega)$ and obtain

$$\mathbf{u} = U\boldsymbol{\psi} = \{I - P_{\Re}(\tfrac{1}{2}I - K)\}^{-1}P_{\Re}V\boldsymbol{\psi}\,, \tag{4.23}$$

where U is the Poincaré–Steklov operator (4.11).

The numerical solution of (4.14) or (4.22) and the corresponding approximation of U can be achieved by any appropriate boundary element method.

The Neumann series for (4.22)

From the practical point of view we want to compute the solution $\mathbf{u}$ in (4.23) by using a Neumann series for $\{I - P_{\Re}(\frac{1}{2}I - K)\}^{-1}$. For this purpose we require the following additional assumption for this particular method for treating the boundary integral equation (4.14).

$$(\mathbf{B1}) \qquad \text{The spectral radius of } (\tfrac{1}{2}I - K) \text{ on } H^{\frac{1}{2}}(\partial\omega)/\Re \text{ is smaller than } 1\,. \tag{4.24}$$

This implies that the spectral radius of $P_{\Re}(\frac{1}{2}I - K)$ on $H_0^{\frac{1}{2}}(\partial\omega)$ is also smaller than 1.

It is known that Condition **(B1)** is satisfied in the case of classical potential theory for the Laplacian as well as in elasticity for smooth $\partial\omega$ and for rather large classes of piecewise smooth boundaries (see [33], [35, pp. 362–364],[36, Chap. II], [39, Chap.1, Section 7], [58], [60]); for elasticity problems see in particular [34]. Under the condition **(B1)**, the operator U can be expressed in terms of the series

$$U = \sum_{\ell=0}^{\infty} \Big(P_{\Re}(\tfrac{1}{2}I - K)\Big)^{\ell} P_{\Re} V \tag{4.25}$$

which converges in the associated operator norm. In Section 8, the computational results in elasticity are based on (4.25).

The hypersingular integral equation

Alternatively, from the representation formula (4.13), applying T_x at $\partial\omega$ to (4.13), yields the hypersingular equation

$$D\mathbf{u} = (\tfrac{1}{2}I - K')\boldsymbol{\psi} \ \text{ on } \partial\omega \tag{4.26}$$

for $\mathbf{u} \in H_0^{\frac{1}{2}}(\partial\omega)$ where D and K' are, respectively, the hypersingular operator and the adjoint operator to K, defined by

$$D\mathbf{v}(x) \ := \ -T_x \left[\text{p.v.} \int_{\partial\omega} \Big(T_y[E(x,y)]\Big)^{\top} \mathbf{v}(y) ds_y \right] \ \text{ and } \tag{4.27}$$

$$K'\boldsymbol{\lambda}(x) \ := \ \text{p.v.} \int_{\partial\omega} \Big(T_x[E(x,y)]\Big) \boldsymbol{\lambda}(y) ds_y \ \text{ for } x \in \partial\omega\,. \tag{4.28}$$

Mapping properties of D and K'

Similar to V and K, the mapping properties for D and K' are available.

$D : H^{\frac{1}{2}}(\partial\omega) \to H_0^{-\frac{1}{2}}(\partial\omega)$ is continuous and $H_0^{\frac{1}{2}}(\partial\omega)$–elliptic, i. e. there exists $\alpha_0 > 0$ such that

$$\langle D\mathbf{v}, \mathbf{v}\rangle \geq \alpha_0 \|\mathbf{v}\|^2_{H^{\frac{1}{2}}(\partial\omega)} \quad \text{for all } \mathbf{v} \in H_0^{\frac{1}{2}}(\partial\omega)\,. \tag{4.29}$$

The singular integral operator $K' : H^{-\frac{1}{2}}(\partial\omega) \to H^{-\frac{1}{2}}(\partial\omega)$ is continuous. Moreover,

$$\langle K\mathbf{v}, \boldsymbol{\lambda}\rangle = \langle \mathbf{v}, K'\boldsymbol{\lambda}\rangle \ \text{ for all } \mathbf{v} \in H^{\frac{1}{2}}(\partial\omega) \ \text{ and } \boldsymbol{\lambda} \in H^{-\frac{1}{2}}(\partial\omega)\,,$$

(see [13], [14], [15], [27]).

Solvability of (4.26)

If $\boldsymbol{\psi}$ satisfies (4.9), i. e. $\boldsymbol{\psi} \in H_0^{-\frac{1}{2}}(\partial\omega)$, then equation (4.26) admits a unique solution $\mathbf{u} \in H_0^{\frac{1}{2}}(\partial\omega)$ characterized by the variational equation

$$\langle \mathbf{v}, D\mathbf{u}\rangle = \langle \mathbf{v}, (\tfrac{1}{2}I - K')\boldsymbol{\psi}\rangle \ \text{ for all } \mathbf{v} \in H_0^{\frac{1}{2}}(\partial\omega)\,. \tag{4.30}$$

Let $P_{\Re}{}'$ be the adjoint orthogonal mapping of $H^{-\frac{1}{2}}(\partial\omega)$ onto $H_0^{-\frac{1}{2}}(\partial\omega)$ with respect to the duality pairing $\langle \bullet, \bullet\rangle_{\partial\omega}$. Then it is easily to be seen that $P_{\Re}{}' = P_{\Re}$ on $H^{\frac{1}{2}}(\partial\omega) \subset H^{-\frac{1}{2}}(\partial\omega)$. Hence, $P_{\Re}{}'$ is the continuous extension of $P_{\Re}$ to $H^{-\frac{1}{2}}(\partial\omega)$ and is selfadjoint, in view of which we shall write $P_{\Re} = P_{\Re}{}'$ in what follows.

With $D_0 := P_{\Re} D P_{\Re}$ then (4.30) implies

$$U = D_0^{-1}(\tfrac{1}{2}I - K')P_{\Re}\,. \tag{4.31}$$

The symmetric formulation of U

Based on (4.22) together with (4.31), one may also represent U in the symmetric form

$$U = P_{\Re}(\tfrac{1}{2}I - K)D_0^{-1}(\tfrac{1}{2}I - K')P_{\Re} + V_0 \tag{4.32}$$

where $V_0 := P_{\Re} V P_{\Re}$.

For the numerical approximation of U, one may apply some appropriate boundary element method to any of the above representations and corresponding boundary integral equations such as (4.22), (4.25), (4.30), (4.22) and (4.31).

With $\mathbf{u}$ determined on $\partial\omega$ one may now use the representation formula (4.13) to find $\mathbf{u}(x)$ in ω.

Next we consider the case $\partial\mathring{\omega}_D \neq \emptyset$.

4.1.2 The local mixed boundary value problem

In the case $\partial\mathring{\omega}_D \neq \emptyset$, we have to solve (4.1), i. e.

$$\mathcal{P}\mathbf{u} = 0 \ \text{ in } \omega \ \text{ with } \mathbf{u} = \boldsymbol{\varphi} \ \text{ on } \partial\omega_D\,,\ T\mathbf{u} = \boldsymbol{\psi} \ \text{ on } \partial\omega_N = \partial\omega \setminus \partial\mathring{\omega}_D \tag{4.33}$$

which defines the operator $\widehat{U}(\boldsymbol{\varphi}, \boldsymbol{\psi}) := \mathbf{u}_{|\partial\omega}$.

Here, $\partial\omega_D = \partial\omega \cap \Gamma_D \setminus \mathring{\Upsilon}$ is the Dirichlet part of the boundary with $\partial\mathring{\omega}_D \neq \emptyset$. In order to reduce the boundary value problem (4.33) to boundary integral equations we first extend the given data as follows:

$$\begin{aligned} \boldsymbol{\varphi}^* &\in H^{\frac{1}{2}}(\partial\omega) && \text{is chosen with } \boldsymbol{\varphi}^* = \boldsymbol{\varphi} \ \text{ on } \partial\omega_D \ \text{ and} \\ \boldsymbol{\psi}^* &\in H_0^{-\frac{1}{2}}(\partial\omega) && \text{is chosen with } \boldsymbol{\psi}^* = \boldsymbol{\psi} \ \text{ on } \partial\omega_N = \partial\omega \setminus \partial\mathring{\omega}_D\,. \end{aligned} \tag{4.34}$$

The missing Cauchy data can be rewritten in the form

$$\begin{aligned} T\mathbf{u} &= \boldsymbol{\psi}^* + \boldsymbol{\lambda}_0 \quad \text{with} \quad \boldsymbol{\lambda}_0 \in \widetilde{H}_0^{-\frac{1}{2}}(\partial\omega_D) \quad \text{on } \partial\omega_D\,, \\ \mathbf{u} &= \boldsymbol{\varphi}^* + \mathbf{u}_0 \quad \text{with} \quad \mathbf{u}_0 \in \widetilde{H}^{\frac{1}{2}}(\partial\omega_N) \quad \text{on } \partial\omega_N\,. \end{aligned} \tag{4.35}$$

Here both $\boldsymbol{\lambda}_0 \in \widetilde{H}_0^{-\frac{1}{2}}(\partial\omega_D)$ and $\mathbf{u}_0 \in \widetilde{H}^{\frac{1}{2}}(\partial\omega_N)$ still need to be determined.

Reduction to the Poincaré–Steklov operator

With the help of the Poincaré–Steklov operator U of the pure Neumann problem given by (4.11), (4.23), (4.31) or (4.32) we have

$$\mathbf{u} = UT\mathbf{u} + \mathbf{r}^* \ \text{ on } \partial\omega \ \text{ with some } \mathbf{r}^* \in \Re\,. \tag{4.36}$$

By using (4.11) we have the equation for $\boldsymbol{\lambda}_0 \in \widetilde{H}_0^{-\frac{1}{2}}(\partial\omega_D)$,

$$U\boldsymbol{\lambda}_0 = -U\boldsymbol{\psi}^* - \mathbf{r}^* + \boldsymbol{\varphi}^* + \mathbf{u}_0 \ \text{ on } \partial\omega\,. \tag{4.37}$$

In weak form, (4.37) with test functions $\boldsymbol{\chi} \in \widetilde{H}_0^{-\frac{1}{2}}(\partial\omega_D)$ implies that $\boldsymbol{\lambda}_0$ can be found from the variational equation

$$\langle \boldsymbol{\chi}, U\boldsymbol{\lambda}_0 \rangle = \int\limits_{\partial\omega_D} \boldsymbol{\chi} \cdot U\boldsymbol{\lambda}_0 ds = \langle \boldsymbol{\chi}, \boldsymbol{\varphi}^* - U\boldsymbol{\psi}^* \rangle \ \text{ for all } \boldsymbol{\chi} \in \widetilde{H}_0^{-\frac{1}{2}}(\partial\omega_D)\,. \tag{4.38}$$

Since $\widetilde{H}_0^{-\frac{1}{2}}(\partial\omega_D)$ is defined as to be a closed subspace of $H_0^{-\frac{1}{2}}(\partial\omega)$ and because of the $H_0^{-\frac{1}{2}}(\partial\omega)$–ellipticity (4.12), there exists a unique solution $\boldsymbol{\lambda}_0 \in \widetilde{H}_0^{-\frac{1}{2}}(\partial\omega_D)$ of (4.38) due to the Lax–Milgram theorem. Once $\boldsymbol{\lambda}_0$ is known, $\mathbf{r}^* \in \Re$ can be found from the equations

$$\int\limits_{\partial\omega_D} \mathbf{r} \cdot \mathbf{r}^* ds = \int\limits_{\partial\omega_D} \mathbf{r} \cdot \Big(\boldsymbol{\varphi} - U(\boldsymbol{\psi}^* + \boldsymbol{\lambda}_0)\Big) ds \ \text{ for all } \mathbf{r} \in \Re\,. \tag{4.39}$$

This determines completely

$$\mathbf{u} = U(\boldsymbol{\psi}^* + \boldsymbol{\lambda}_0) + \mathbf{r}^* = \boldsymbol{\varphi}^* + \mathbf{u}_0 \ \text{ on } \partial\omega\,.$$

We remark that in practice, for the solution of (4.38), again the Neumann series (4.25) can be used for utilizing U.

With $\boldsymbol{\lambda}_0$ and $\mathbf{u}_0$ available, the solution $\mathbf{u}(x)$ in ω can be determined by Green's representation formula

$$\begin{aligned} \mathbf{u}(x) &= \int\limits_{\partial\omega} E(x,y)\boldsymbol{\psi}^*(y) ds_y - \int\limits_{\partial\omega} \Big(T_y[E(x,y)]\Big)^\top \boldsymbol{\varphi}^*(y) ds_y \\ &\quad + \int\limits_{\partial\omega_D} E(x,y)\boldsymbol{\lambda}_0(y) ds_y - \int\limits_{\partial\omega_N} \Big(T_y[E(x,y)]\Big)^\top \mathbf{u}_0(y) ds_y \ \text{ for all } x \in \omega\,. \end{aligned} \tag{4.40}$$

Reduction to an unsymmetric system of boundary integral equations

Alternatively, one may reduce the mixed boundary value problem to a system of boundary integral equations for $\boldsymbol{\lambda}_0 \in \widetilde{H}_0^{-\frac{1}{2}}(\partial\omega_D)$ and $\mathbf{u}_0 \in \widetilde{H}^{\frac{1}{2}}(\partial\omega_N)$. More precisely, from the representation formula (4.40) and the jump relation for the double layer potential, as $x \to \partial\omega$, we obtain the system

$$\begin{aligned} (\tfrac{1}{2}I + K)\mathbf{u}_0 - V\boldsymbol{\lambda}_0 &= V\boldsymbol{\psi}^* - (\frac{1}{2}I + K)\boldsymbol{\varphi}^* \ \text{ for } x \in \partial\omega_N\,, \\ -K\mathbf{u}_0 + V\boldsymbol{\lambda}_0 &= (\frac{1}{2}I + K)\boldsymbol{\varphi}^* - V\boldsymbol{\psi}^* \ \text{ for } x \in \partial\omega_D \end{aligned} \tag{4.41}$$

where we have tacitly taken into account $\mathbf{u}_0 = 0$ on $\partial\omega_D$ and $\boldsymbol{\lambda}_0 = 0$ on $\partial\omega_N$.

Clearly, this system (4.41) has a solution since our assumptions guarantee that the mixed boundary value problem (4.18) has a unique solution.

If we assume that the corresponding exterior mixed boundary value problem has at most one solution $\mathbf{u}^c$ in $H^1_{\text{loc}}(\omega^c)\,,\ \omega^c = \mathbb{R}^n \setminus \overline{\omega}$, with $a_{\omega^c}(\mathbf{u}^c, \mathbf{u}^c) < \infty$ then the uniqueness of the solution $\mathbf{u}_0, \boldsymbol{\lambda}_0$ to the system (4.41) can be established by using the arguments of classical potential theory. We note that in the special cases of the Laplacian and of elasticity, (4.41) is uniquely solvable (see e. g. [22], [30], [35]).

Reduction to a 'symmetric' system of boundary integral equations

Instead of the first equation in (4.41), one may also consider a boundary integral equation defined with the hypersingular boundary integral operator and solve the system

$$\begin{aligned} D\mathbf{u}_0 + K'\boldsymbol{\lambda}_0 &= (\tfrac{1}{2}I - K')\boldsymbol{\psi}^* - D\boldsymbol{\varphi}^* \quad \text{ on } \partial\omega_N\,, \\ -K\mathbf{u}_0 + V\boldsymbol{\lambda}_0 &= (\tfrac{1}{2}I + K)\boldsymbol{\varphi}^* - V\boldsymbol{\psi}^* \quad \text{ on } \partial\omega_D\,. \end{aligned} \tag{4.42}$$

This system (4.42) is uniquely solvable due to the Lax–Milgram theorem since both D and V are $H_0^{\frac{1}{2}}(\partial\omega)$–elliptic and $H_0^{-\frac{1}{2}}(\partial\omega)$–elliptic, respectively, which implies, with some constant $\alpha_0 > 0$,

$$\langle \mathbf{u}_0, D\mathbf{u}_0 + K'\boldsymbol{\lambda}_0\rangle + \langle \boldsymbol{\lambda}_0, -K\mathbf{u}_0 + V\boldsymbol{\lambda}_0\rangle \geq \alpha_0 \left\{ \|\mathbf{u}_0\|^2_{H^{\frac{1}{2}}(\partial\omega)/\Re} + \|\boldsymbol{\lambda}_0\|^2_{\widetilde{H}_0^{-\frac{1}{2}}(\partial\omega_D)} \right\} \tag{4.43}$$

for all $\mathbf{u}_0 \in \widetilde{H}^{\frac{1}{2}}(\partial\omega_N)$ and $\boldsymbol{\lambda}_0 \in \widetilde{H}_0^{-\frac{1}{2}}(\partial\omega_D)$ where $H^{\frac{1}{2}}(\partial\omega)/\Re$ denotes the quotient space.

(For the Laplacian and the elasticity equations see e. g. [59], [26]. For the second order systems see [27], [36].)

With the solutions of the local problems available, we now are able to define the operator $\widehat{U}$ associated with $\partial\omega = \partial\omega_D \cup \partial\omega_N$ which maps $(\boldsymbol{\varphi}, \boldsymbol{\psi})$ into the trace via

$$\widehat{U}(\boldsymbol{\varphi}, \boldsymbol{\psi}) := \mathbf{u}_{|\partial\omega} = \boldsymbol{\varphi}^* + \mathbf{u}_0\,. \tag{4.44}$$

In the special case $\partial\omega = \partial\omega_N$ we recover $U = \widehat{U}$ and for $\partial\omega = \partial\omega_D$ we define $\widehat{U} := I$.

4.2 Proof of Theorem 1

It is understood that we apply the solution procedures in ω to each macro–element ω_j individually; and corresponding operators will be appended with the index j when necessary.

In Theorem 1, (4.7), $\widehat{U}_j$ corresponds to $\partial\omega_{jD} = \partial\omega_j \cap \Gamma_D \setminus \Upsilon$ and $\partial\omega_{jN} = \partial\omega_j \setminus \partial\mathring{\omega}_{jD}$ and the weak coupling condition (4.7) implies

$$\int\limits_{\partial\omega_j \cap \Upsilon} \boldsymbol{\chi}_j \cdot \widetilde{\mathbf{v}} ds = \int\limits_{\partial\omega_j \cap \Upsilon} \boldsymbol{\chi}_j \cdot \widehat{U}_j(\mathbf{0}, \boldsymbol{\chi}_j) ds\,.$$

Since the local test function $\mathbf{v}_j = \widehat{U}_j(\mathbf{0},\boldsymbol{\chi}_j) + \mathbf{r}_j^*$ satisfies the boundary conditions

$$T_j\mathbf{v}_j = \boldsymbol{\chi}_j \in \widehat{H}_0^{-\frac{1}{2}}(\partial\omega_j \cap \Upsilon) \text{ on } \partial\omega_j \cap \Upsilon \text{ and therefore } T_j\mathbf{v}_j = \mathbf{0} \text{ on } \partial\omega_N \setminus \Upsilon$$

whereas

$$\mathbf{v}_j = \widehat{U}_j(\mathbf{0},\boldsymbol{\chi}_j) + \mathbf{r}_j^* \text{ on } \partial\omega_j \text{ and } \widehat{U}_j(\mathbf{0},\boldsymbol{\chi}_j) = \mathbf{0} \text{ on } \partial\omega_{jD}\,;$$

then Green's theorem implies that

$$\begin{aligned} a_{\omega_j}(\mathbf{v}_j,\mathbf{v}_j) &= \int_{\partial\omega_j} (T_j\mathbf{v}_j)\cdot\mathbf{v}_j ds = \int_{\partial\omega_j} (T_j\mathbf{v}_j)\cdot\widehat{U}_j(\mathbf{0},\boldsymbol{\chi}_j) ds = \int_{\partial\omega_j} \boldsymbol{\chi}_j\cdot\widehat{U}_j(\mathbf{0},\boldsymbol{\chi}_j) ds \\ &= \int_{\partial\omega_j\cap\Upsilon} \boldsymbol{\chi}_j\cdot\widehat{U}_j(\mathbf{0},\boldsymbol{\chi}_j) ds = \int_{\partial\omega_j\cap\Upsilon} \boldsymbol{\chi}_j\cdot\tilde{\mathbf{v}} ds\,. \end{aligned}$$

Now, by the continuity properties of T_j it follows that

$$\|\boldsymbol{\chi}_j\|_{\widehat{H}_0^{-\frac{1}{2}}(\partial\omega_j\cap\Upsilon)} = \|T_j[\mathbf{v}_j+\mathbf{r}_j]\|_{\widehat{H}_0^{-\frac{1}{2}}(\partial\omega_j\cap\Upsilon)} \le \|T_j[\mathbf{v}_j+\mathbf{r}_j]\|_{H_0^{-\frac{1}{2}}(\partial\omega_j)} \le c_j\|\mathbf{v}_j+\mathbf{r}_j\|_{H^1(\omega_j)}$$

for all $\mathbf{r}_j \in \Re_j$. This implies

$$\|\boldsymbol{\chi}_j\|^2_{\widehat{H}_0^{-\frac{1}{2}}(\partial\omega_j\cap\Upsilon)} \le c_j^2\|\mathbf{v}_j\|^2_{H^1(\omega_j)/\Re_j} \le \frac{c_j^2}{\alpha_{0j}} a_{\omega_j}(\mathbf{v}_j,\mathbf{v}_j)\,,$$

where the last inequality follows from (3.12). Collecting terms, we get the proposed inequality (4.6),

$$\begin{aligned} a_{\Omega_F}(\mathbf{v}_F,\mathbf{v}_F) + \sum_{j=1}^M \int_{\partial\omega_j\cap\Upsilon} \boldsymbol{\chi}_j\tilde{\mathbf{v}} ds &= a_{\Omega_F}(\mathbf{v}_F,\mathbf{v}_F) + \sum_{j=1}^M a_{\omega_j}(\mathbf{v}_j,\mathbf{v}_j) \\ &\ge \alpha_0 \left\{ \|\mathbf{v}_F\|^2_{H^1(\Omega_F)} + \sum_{j=1}^M \|\boldsymbol{\chi}_j\|^2_{\widehat{H}_0^{-\frac{1}{2}}(\partial\omega_j\cap\Upsilon)} \right\} \end{aligned}$$

where we made use of $\mathbf{v}_F = \mathbf{0}$ on $\partial\Omega_F \cap \Gamma_D$ and $\tilde{\mathbf{u}} = \mathbf{0}$ on $\Gamma_D \cap \Upsilon$. □

5 The variational formulation based on the local Dirichlet data

In the conventional coupling formulation, the Neumann data $\boldsymbol{\lambda}_j$ are eliminated by using the Steklov–Poincaré operator

$$\widehat{S}_j \,:\, H^{\frac{1}{2}}(\partial\omega_{jD}) \times H^{-\frac{1}{2}}(\partial\omega_{jN}) \ni (\mathbf{u}_j^*,\boldsymbol{\psi}) \mapsto \boldsymbol{\lambda} = T_j\mathbf{u} \in H^{-\frac{1}{2}}(\partial\omega_j)\,. \tag{5.1}$$

This amounts to solving the local mixed boundary value problem

$$\mathcal{P}\mathbf{u}_j = 0 \text{ in } \omega_j\,,\ T_j\mathbf{u}_j = \boldsymbol{\psi} \text{ on } \partial\omega_{jN} := \partial\omega_j \cap \Gamma_N \setminus \mathring{\Upsilon} \text{ and } \mathbf{u}_j = \mathbf{u}_j^* \text{ on } \partial\omega_{jD} := \partial\omega_j \setminus \partial\mathring{\omega}_{jN} \tag{5.2}$$

in $H^1(\omega_j)$. Here $\mathbf{u}_j^*$ satisfies the constraint $\mathbf{u}_j^* = \boldsymbol{\varphi}$ on $\partial\omega_{jD}\setminus\Upsilon$. We notice that, in contrast to (4.1), here the definition of $\partial\omega_{jN}$ is inferred in terms of the given Neumann datum on $\partial\omega_j \cap \Gamma_N \setminus \mathring{\Upsilon}$ and $\partial\omega_{jD}$ is the complement.

If $\partial\omega_j = \partial\omega_{jD}$ then $\widehat{S}_j = S_j$ defines the standard Dirichlet–Neumann mapping

$$S_j \,:\, \mathbf{u}_j \mapsto T_j\mathbf{u}_j \;\text{ on } \partial\omega_j \tag{5.3}$$

and U_j is the right–inverse to S_j:

$$S_jU_j = I_j \;\text{ on } H_0^{-\frac{1}{2}}(\partial\omega_j) \;\text{ and } U_jS_j\boldsymbol{\varphi}_j = \boldsymbol{\varphi}_j + \mathbf{r}_j \;\text{ with } \mathbf{r}_j \in \Re_j \;\text{ for all } \boldsymbol{\varphi}_j \in H^{\frac{1}{2}}(\partial\omega_j)\,. \tag{5.4}$$

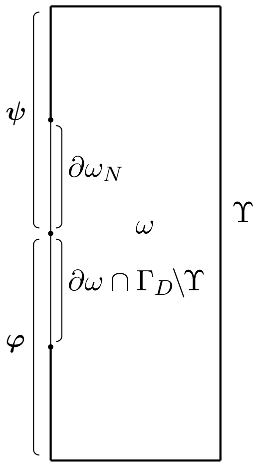

Figure 5: Macro element with $\partial\omega_N := \partial\omega \cap \Gamma_N \setminus \Upsilon$

In terms of $\widehat{S}_j$, the variational formulation (3.16)–(3.18) can be reduced to the following problem:

Find $(\mathbf{u}_F, \tilde{\mathbf{u}}; \mathbf{u}_j) \in \mathcal{H} \times \prod\limits_{j=1}^{M} H^{\frac{1}{2}}(\partial\omega_j)$ *satisfying the inhomogeneous boundary conditions*

$$\mathbf{u}_F = \boldsymbol{\varphi} \;\text{ on } \Gamma_D \cap \partial\Omega_F \;\text{ and } \tilde{\mathbf{u}} = \boldsymbol{\varphi} \;\text{ on } \Gamma_D \cap \Upsilon \;\text{ and } \mathbf{u}_j = \boldsymbol{\varphi} \;\text{ on } \partial\omega_j \cap (\Gamma_D \setminus \Upsilon)$$

such that

$$\begin{aligned} & a_{\Omega_F}(\mathbf{u}_F, \mathbf{v}_F) + \sum_{j=1}^{M} \int\limits_{\partial\omega_j \cap \Upsilon} \widehat{S}_j(\mathbf{u}_j, \mathbf{0}) \cdot \tilde{\mathbf{v}} ds \\ & = \int\limits_{\Gamma_N \cap \partial\Omega_F \backslash \Upsilon} \boldsymbol{\psi} \cdot \mathbf{v}_F ds + \int\limits_{\Gamma_N \cap \Upsilon} \boldsymbol{\psi} \cdot \tilde{\mathbf{v}} ds - \sum_{j=1}^{M} \int\limits_{\partial\omega_j \cap \Upsilon} \widehat{S}_j(\mathbf{0}, \boldsymbol{\psi}_j^*) \cdot \tilde{\mathbf{v}} ds \end{aligned} \tag{5.5}$$

for all $(\mathbf{v}_F, \tilde{\mathbf{v}}) \in \mathcal{H}_D$ *and, for* $j = 1, \ldots, M$*:*

$$\int\limits_{\partial\omega_j \cap \Upsilon} (\mathbf{u}_j - \tilde{\mathbf{u}}) \cdot \widehat{S}_j(\mathbf{v}_j, \mathbf{0}) ds = 0 \;\text{ for all } \mathbf{v}_j \in \widehat{H}^{\frac{1}{2}}(\partial\omega_j \cap \Upsilon)\,. \tag{5.6}$$

Here we define the space

$$\widehat{H}^{\frac{1}{2}}(\partial\omega_j \cap \Upsilon) := \{\mathbf{v}_j | \;\text{ there exists } \mathbf{v}_j^* \in \widetilde{H}^{\frac{1}{2}}(\partial\omega_j \setminus (\Gamma_D \setminus \Upsilon)) \;\text{ with } \mathbf{v}_j = \mathbf{v}_j^* \;\text{ on } \partial\omega_j \cap \Upsilon\} \tag{5.7}$$

equipped with the norm

$$\|\mathbf{v}_j\|_{\widehat{H}^{\frac{1}{2}}(\partial\omega_j \cap \Upsilon)} := \begin{cases} \inf_{\mathbf{v}_j^*} \|\mathbf{v}_j^*\|_{H^{\frac{1}{2}}(\partial\omega_j)} & \text{if } \partial\overset{\circ}{\omega}_j \cap \Gamma_D \setminus \Upsilon \neq \emptyset\,, \\ \|\mathbf{v}_j\|_{H^{\frac{1}{2}}(\partial\omega_j)/\Re_j} & \text{if } \partial\overset{\circ}{\omega}_j \cap \Gamma_D \setminus \Upsilon = \emptyset\,. \end{cases} \tag{5.8}$$

In the formulation of (5.5), (5.6), the solution of the local macro–element problems is hidden in the action of the operators $\widehat{S}_j$.

Similar to Theorem 1 we now have

Theorem 2 *The bilinear form defined by* (5.5) *and* (5.6) *is continuous and* $\mathcal{H}_D \times \prod_{j=1}^M \widehat{H}^{\frac{1}{2}}(\partial\omega_j \cap \Upsilon)$*–elliptic, i. e. there exists a constant* $\alpha_0 > 0$ *such that*

$$a_{\Omega_F}(\mathbf{v}_F, \mathbf{v}_F) + \sum_{j=1}^M \int\limits_{\partial\omega_j\cap\Upsilon} \widehat{S}_j(\mathbf{v}_j, \mathbf{0}) \cdot \tilde{\mathbf{v}} ds \geq \alpha_0 \left\{ \|\mathbf{v}_F\|^2_{H^1(\Omega_F)} + \sum_{j=1}^M \|\mathbf{v}_j\|^2_{\widehat{H}^{\frac{1}{2}}(\partial\omega_j\cap\Upsilon)} \right\} \tag{5.9}$$

provided

$$\int\limits_{\partial\omega_j\cap\Upsilon} (\mathbf{v}_j - \tilde{\mathbf{v}}) \cdot \widehat{S}_j(\mathbf{v}_j, \mathbf{0}) ds = 0\,. \tag{5.10}$$

The proof will be presented at the end of this section.

5.1 Local problems based on local Dirichlet data

Again, we first collect the results concerning the local problems in ω_j and, for ease of reading, suppress the subindex j.

5.1.1 The local Dirichlet problem

In this case, under assumptions **(A1)–(A3)**, the Dirichlet problem defined by

$$\mathcal{P}\mathbf{u} = 0 \ \text{ in } \omega \ \text{ with } \mathbf{u} = \boldsymbol{\varphi} \in H^{\frac{1}{2}}(\partial\omega) \ \text{ on } \partial\omega \tag{5.11}$$

admits exactly one solution $\mathbf{u} \in H^1(\omega)$ and the Steklov–Poincaré operator $S : H^{\frac{1}{2}}(\partial\omega) \to H_0^{-\frac{1}{2}}(\partial\omega)$ is given by

$$S\mathbf{u} := T\mathbf{u} \ \text{ on } \partial\omega$$

which is $H_0^{\frac{1}{2}}(\partial\omega)$–elliptic since

$$a_\omega(\mathbf{u}, \mathbf{u}) = \langle S\mathbf{u}, \mathbf{u} \rangle \geq \alpha_0 \|\mathbf{u}\|^2_{H^{\frac{1}{2}}(\partial\omega)} \ \text{ for all } \mathbf{u} \in H_0^{\frac{1}{2}}(\partial\omega) \tag{5.12}$$

with some $\alpha_0 > 0$. With this definition it is clear that the Poincaré–Steklov operator U in (4.12) is the right–inverse of S, i. e. we have (5.4).

To construct S, we may choose one of the following different possibilities:

The boundary integral equation of the second kind

If we use $\mathbf{u} = \boldsymbol{\varphi}$ on $\partial\omega$ in (4.26) then we obtain the boundary integral equation of the second kind

$$(\tfrac{1}{2}I - K')\boldsymbol{\lambda} = D\boldsymbol{\varphi} \ \text{ on } \partial\omega \tag{5.13}$$

for $\boldsymbol{\lambda} \in H_0^{-\frac{1}{2}}(\partial\omega)$ which can also be written in the form

$$\boldsymbol{\lambda} - P_{\Re}(\tfrac{1}{2}I + K')P_{\Re}\boldsymbol{\lambda} = D\boldsymbol{\varphi}\,. \tag{5.14}$$

The direct solution of (5.12) provides us with the Steklov–Poincaré operator S via

$$S\boldsymbol{\varphi} = \boldsymbol{\lambda} = \{I - P_{\Re}(\tfrac{1}{2}I + K')\}^{-1} D\boldsymbol{\varphi}\,. \tag{5.15}$$

The Neumann series for (5.18)

Corresponding to **(B1)**, for solving (5.15) with the Neumann series, we require here:

$$\text{(\textbf{B2}) The spectral radius of } (\tfrac{1}{2}I + K') \text{ on } H_0^{-\frac{1}{2}}(\partial\omega) \text{ is smaller than } 1\,. \tag{5.16}$$

In all cases when **(B1)** is available, one also has **(B2)** (see [33], [34], [35, pp. 362–364],[36, Chap. II], [39, Chap.1, Section 7], [58], [60]).

Under the condition **(B2)**, the operator S can be expressed in terms of the Neumann series

$$S = \sum_{\ell=0}^{\infty} \Big(P_{\Re}(\tfrac{1}{2}I + K') \Big)^{\ell} D \tag{5.17}$$

which converges in the associated operator norm. Note that, in contrast to (4.25), we have here $P_{\Re}D = D$.

The boundary integral equation of the first kind

One may solve the boundary integral equation of the first kind,

$$V\boldsymbol{\lambda} = (\tfrac{1}{2}I + K)\boldsymbol{\varphi} \ \text{ on } \partial\omega \tag{5.18}$$

(cf. (4.12)) for $\boldsymbol{\lambda} \in H_0^{-\frac{1}{2}}(\partial\omega)$ by inverting V on the subspace $H_0^{-\frac{1}{2}}(\partial\omega)$ since V is $H_0^{-\frac{1}{2}}(\partial\omega)$–elliptic (4.18). In variational form this amounts to determine $\boldsymbol{\lambda} \in H_0^{-\frac{1}{2}}(\partial\omega)$ from

$$\langle \boldsymbol{\chi}, V\boldsymbol{\lambda} \rangle = \langle \boldsymbol{\chi}, (\tfrac{1}{2}I + K)\boldsymbol{\varphi} \rangle \ \text{ for all } \boldsymbol{\chi} \in H_0^{-\frac{1}{2}}(\partial\omega)\,. \tag{5.19}$$

In terms of boundary integral operators we may express S explicitly via

$$S = V_0^{-1} P_{\Re}(\tfrac{1}{2}I + K) \tag{5.20}$$

where $P_{\Re}$ is the projection defined by (4.21) and $V_0 = P_{\Re} V P_{\Re}$.

The symmetric representation of S

Alternatively, one may define S in terms of the hypersingular operator D (4.27) by using (4.22) together with (4.26) in a symmetric form, i. e.

$$S = \Big((\tfrac{1}{2}I + K')P_{\Re} \Big) V_0^{-1} \Big(P_{\Re}(\tfrac{1}{2}I + K) \Big) + D\,. \tag{5.21}$$

All the above constructions can numerically be executed by solving corresponding boundary integral equations, in particular via Galerkin methods.

5.1.2 The local mixed boundary value problem

Since in Section 4.1.2 we already presented the details for mixed boundary value problems, we here collect only some relevant formulations in the connection with construction of

$$\widehat{S}(\boldsymbol{\varphi}, \boldsymbol{\psi}) = T\mathbf{u} \ \text{ on } \partial\omega = \partial\omega_D \cup \partial\omega_N\,.$$

Here, in contrast to (4.1), we have from (5.2) $\partial\omega_N = \partial\omega \cap \Gamma_N \setminus \overset{\circ}{\Upsilon}$ and $\partial\omega_D = \partial\omega \setminus \partial\overset{\circ}{\omega}_N$ and $\partial\overset{\circ}{\omega}_N \neq \emptyset$. As in (4.34) and (4.35), the missing Cauchy data are written in the form $\mathbf{u} = \boldsymbol{\varphi}^* + \mathbf{u}_0$ and $\boldsymbol{\lambda} = \boldsymbol{\psi}^* + \boldsymbol{\lambda}_0$ with the unknowns $\mathbf{u}_0 \in \widetilde{H}^{\frac{1}{2}}(\partial\omega_N)$ and $\boldsymbol{\lambda}_0 \in \widetilde{H}_0^{-\frac{1}{2}}(\partial\omega_D)$.

Reduction to the Steklov–Poincaré operator

Similar to (4.36), we use the Steklov–Poincaré operator S of the pure Dirichlet problem obtained with any of the constructions in the previous section. We write

$$\widehat{S}(\boldsymbol{\varphi},\boldsymbol{\psi}) = S(\boldsymbol{\varphi}^* + \mathbf{u}_0) = \boldsymbol{\psi}^* + \boldsymbol{\lambda}_0 \,. \tag{5.22}$$

Since we seek the solution $\mathbf{u}_0 \in \widetilde{H}^{\frac{1}{2}}(\partial\omega_N)$ and $\boldsymbol{\lambda}_0 = 0$ on $\partial\omega_N$, we obtain the variational equation

$$\langle \mathbf{v}_0, S\mathbf{u}_0\rangle = \int\limits_{\partial\omega_N} \mathbf{v}_0 \cdot S\mathbf{u}_0 ds = \langle \mathbf{v}_0, \boldsymbol{\psi}^* - S\boldsymbol{\varphi}^*\rangle \text{ for all } \mathbf{v}_0 \in \widetilde{H}^{\frac{1}{2}}(\partial\omega_N)\,. \tag{5.23}$$

The latter is equivalent to finding $\mathbf{u}_0^\perp := P_\Re \mathbf{u}_0 \in P_\Re \widetilde{H}^{\frac{1}{2}}(\partial\omega_N) \subset H_0^{\frac{1}{2}}(\partial\omega)$ from

$$\langle \mathbf{w}_0, S\mathbf{u}_0\rangle = \langle \mathbf{w}_0, S\mathbf{u}_0^\perp\rangle = \langle \mathbf{w}_0, \boldsymbol{\psi}^* - S\boldsymbol{\varphi}^*\rangle \text{ for all } \mathbf{w}_0 \in P_\Re\widetilde{H}^{\frac{1}{2}}(\partial\omega_N)\,. \tag{5.24}$$

The $H_0^{\frac{1}{2}}(\partial\omega)$–ellipticity of S implies the existence of a unique solution $\mathbf{u}_0^\perp$ in the subspace $P_\Re\widetilde{H}^{\frac{1}{2}}(\partial\omega_N) \subset H_0^{\frac{1}{2}}(\partial\omega_N)$. Finally, we obtain in terms of S on $\partial\omega$:

$$\widehat{S}(\boldsymbol{\varphi},\boldsymbol{\psi}) := S\mathbf{u}_0^\perp + S\boldsymbol{\varphi}^* \text{ on } \partial\omega\,. \tag{5.25}$$

Systems of boundary integral equations

First, we solve either the system of boundary integral equations (4.41) or the system (4.42) for $\mathbf{u}_0 \in \widetilde{H}^{\frac{1}{2}}(\partial\omega_N)$ and $\boldsymbol{\lambda}_0 \in \widetilde{H}_0^{-\frac{1}{2}}(\partial\omega_D)$. Then $\widehat{S}$ can be constructed via

$$\widehat{S}(\boldsymbol{\varphi},\boldsymbol{\psi}) := T\mathbf{u} = \boldsymbol{\psi}^* + \boldsymbol{\lambda}_0 \text{ on } \partial\omega\,. \tag{5.26}$$

5.2 The proof of Theorem 2

The proof resembles all the arguments of the proof to Theorem 1. It suffices to verify that with $\partial\omega_j = (\Gamma_D \cap \partial\omega_j \setminus \Upsilon) \cup (\partial\omega_j \cap \Upsilon) \cup (\Gamma_N \cap \partial\omega_j \setminus \Upsilon)$ we have with $\widehat{S}_j$ defined for $\partial\omega_{jN} = \Gamma_N \cap \partial\omega_j \setminus \overset{\circ}{\Upsilon}$ and $\partial\omega_{jD} = \partial\omega_j \setminus \partial\overset{\circ}{\omega}_{jN}$:

$$a_{\omega_j}(\mathbf{v}_j,\mathbf{v}_j) = \int\limits_{\partial\omega_j} \mathbf{v}_j \cdot T\mathbf{v}_j ds = \int\limits_{\partial\omega_j\cap\Upsilon} \tilde{\mathbf{v}} \cdot \widehat{S}_j(\mathbf{v}_j,\mathbf{0}) ds\,.$$

Indeed, this relation holds because of (5.10) and $\mathbf{v}_j = \mathbf{0}$ on $\Gamma_D \cap (\partial\omega_j \setminus \Upsilon)$ and $T\mathbf{v}_j = \mathbf{0}$ on $\partial\omega_{jN}$. Moreover, if $\gamma = \Gamma_D \cap \partial\omega_j \setminus \Upsilon \neq \emptyset$ then from $a_{\omega_j}(\mathbf{v}_j,\mathbf{v}_j) = 0$ one obtains with $\mathbf{v}_j = \mathbf{0}$ on γ and $\mathbf{v}_j \in \Re_j$ that $\mathbf{v}_j = \mathbf{0}$ in $H^1(\omega_j)$. Hence, (A2) with (A3) implies the existence of some constant $\alpha_{0j} > 0$ such that

$$a_{\omega_j}(\mathbf{v}_j,\mathbf{v}_j) \geq \alpha_{0j}\|\mathbf{v}_j\|^2_{H^1(\omega_j)}$$

which yields with $\|\mathbf{v}_j\|_{\widehat{H}^{\frac{1}{2}}(\partial\omega_j\cap\Upsilon)} \leq \|\mathbf{v}_j\|_{H^{\frac{1}{2}}(\partial\omega_j)}$ and the trace theorem the desired inequality (5.9). $\square$

6 The Discretization of the Hybrid Methods

Now we consider the approximation of the variational equations (4.2), (4.3) or (5.5), (5.6) by using combined FEM–BEM. These approximations are based on two families of meshes with parameters of meshwidth H and $h(= h_j)$ corresponding to the global domain and skeleton elements and the local boundary elements on $\partial\omega_j$, respectively.

We begin with the finite element approximation based on a triangulation $\{\tau_\ell\}_{\ell=1}^N$ of Ω_F satisfying diam $(\tau_\ell) \le cH$ for all $\ell = 1,\dots,N$. We always assume that the point set $\overline{\Gamma}_N \cap \overline{\Gamma}_D \cap \overline{\Omega}_F$ belongs to the FEM edges (for $n = 3$) or nodes (for $n = 2$) of $\partial\tau_\ell$. On $\{\tau_\ell\}_{\ell=1}^N$ we choose a conforming finite element space $H_H^1(\Omega_F) \subset H^1(\Omega_F)$ having the approximation degree $d \ge 2$. Then H_H^1 defines grid points of the FEM triangulation on Γ_0. In addition, on the remaining part of Υ, we introduce more global grid points such that the distance between any two neighboring grid points is less than cH. On the skeleton Υ we introduce there a familiy of finite–dimensional subspaces of continuous functions $B_H^{\frac12}(\Upsilon) \subset H^{\frac12}(\Upsilon)$ (e. g. for $n = 2$ one–dimensional splines on the macro–element boundary curves and for $n = 3$ finite elements on the macro–element boundary surfaces $\partial\omega_j$) and impose here the continuity requirements:

$$H_H^1(\Omega_F)_{|\Gamma_0} = B_H^{\frac12}(\Upsilon)_{|\Gamma_0}\,. \tag{6.1}$$

The elements in $B_H^{\frac12}(\Upsilon)$ will then be used as skeleton mortar elements for the global coupling. The elements $\mathbf{w}^H \in H_H^1(\Omega_F) \times B_H^{\frac12}(\Upsilon)$ defined by the pairs $(\mathbf{w}_F^H, \widetilde{\mathbf{w}}^H)$ with

$$\mathbf{w}_F^H = \widetilde{\mathbf{w}}^H \ \text{ on } \Gamma_0 \tag{6.2}$$

now define the finite–dimensional subspace of "global" approximations $\mathcal{H}_H \subset \mathcal{H} = (H^1(\Omega)_{|\Omega_F \cup \Upsilon})$.

For asymptotic analysis we shall consider a whole **family** of finite and skeleton mortar element spaces with $H \to 0$ and require further the following properties according to [3]:

approximation property:[1]

For every $\mathbf{v} \in H^t(\Omega)$ with $1 \le t \le d$ there exists an element family $w^H \in \mathcal{H}_H$ such that

$$\|(\mathbf{w}-\mathbf{w}^H)_{|\Omega_F}\|_{H^1(\Omega_F)} + \|(\mathbf{w}-\widetilde{\mathbf{w}}^H)_{|\Upsilon}\|_{H^{\frac12}(\Upsilon)} \le cH^{t-1}\|\mathbf{v}\|_{H^t(\Omega)}\,; \tag{6.3}$$

inverse assumption:

With some $\delta \in (0, \frac12)$ and for every $\mathbf{w}^H \in \mathcal{H}_H$ we have on Υ

$$\|\mathbf{w}^H\|_{H^{\frac12+\delta}(\Upsilon)} \le cH^{-\delta}\|\mathbf{w}^H\|_{H^{\frac12}(\Upsilon)}\,. \tag{6.4}$$

Here, the norms on the skeleton Υ are defined by

$$\|\mathbf{w}^H\|_{H^{\frac12+\delta}(\Upsilon)} = \inf\{\|\mathbf{v}\|_{H^{\frac12+\delta}(\Omega)} \,|\, \mathbf{v} \in H^{1+\delta}(\Omega) \ \text{ and } \mathbf{v}_{|\Upsilon} = \mathbf{w}^H\}\,.$$

On the individual macro–element boundaries $\partial\omega_j$ we define local quasi–regular boundary element grids with the mesh–size parameter h characterizing the largest distance between neighboring grid points on $\partial\omega_j$. Also here we assume that the point set $\overline{\Gamma}_N \cap \overline{\Gamma}_D \cap \partial\omega_j$ will lie in the set of nodal points of the BEM grid for $n = 2$ or on the BEM nodal lines for

[1] By c we shall denote a generic constant which may have different values in the analysis at different occasions but is independent of the mesh–sizes.

$n = 3$ which are defined by the BEM boundaries of the triangulation of the macro–element surface $\partial\omega_j$. For $n = 2$, the macro–element boundaries $\partial\omega_j$ are curves where on the local grids we introduce splines $\mathcal{S}^{d'}_{jh} \subset H^{\frac{1}{2}}(\partial\omega_j)$ and $\mathcal{S}^{d''}_{jh'} \subset H^{-\frac{1}{2}}(\partial\omega_j)$ of polynomial degrees $d' - 1 \geq d'' - 1 \geq 0$ and $d' > 1$. For $n = 3$, the boundary element functions are defined on (local) triangulations of each of the surfaces $\partial\omega_j$ as associated finite element spaces $\mathcal{S}^{d'}_{jh}$ and $\mathcal{S}^{d''}_{jh'}$, either by lifting via parametric surface representation in the parametric plane or by appropriate isoparametric elements. Similar to FEM analysis, the boundary elements are assumed to provide the

approximation property: For both approximations with $\widetilde{d} = d'$, and $\widetilde{d} = d''$ where $h = h'$, we require:

For every $\boldsymbol{\chi} \in H^s(\partial\omega_j)$ and $t \leq s \leq \widetilde{d}$ and $t < \widetilde{d} - 1/2$ for $n = 2$ or $t \leq \widetilde{d} - 1$ for $n = 3$, there exists an element family $\boldsymbol{\chi}^h \in \mathcal{S}^{\widetilde{d}}_{jh}(\partial\omega_j)$ such that

$$\|\boldsymbol{\chi} - \boldsymbol{\chi}^h\|_{H^t(\partial\omega_j)} \leq ch^{s-t}\|\boldsymbol{\chi}\|_{H^s(\partial\omega_j)} \,. \tag{6.5}$$

Moreover, the $L_2(\partial\omega_j)$–projection P_{jh} onto $\mathcal{S}^{\widetilde{d}}_{jh}(\partial\omega_j)$ has the uniform boundedness property

$$\|P_{jh}\boldsymbol{\chi}\|_{H^t(\partial\omega_j)} \leq c\|\boldsymbol{\chi}\|_{H^t(\partial\omega_j)} \,. \tag{6.6}$$

(See [3]). For quasi–uniform BEM grid families, the property (6.6) follows from the inverse assumption on the fine grids. For more general grids see [16].

inverse assumption (on the fine grids):

With some $\delta \in (0, \frac{1}{2})$ and for every $\boldsymbol{\chi}^h \in \mathcal{S}^{\widetilde{d}}_{jh}(\partial\omega_j)$ we have on $\partial\omega_j$

$$\|\boldsymbol{\chi}^h\|_{H^{\pm\frac{1}{2}+\delta}(\partial\omega_j)} \leq ch^{-\delta}\|\boldsymbol{\chi}^h\|_{H^{\pm\frac{1}{2}}(\partial\omega_j)} \tag{6.7}$$

with the $+$sign *for $\widetilde{d} = d'$ and the* $-$sign *for $\widetilde{d} = d''$.*

We remark that — without repetition — the mesh with h' instead of h should be used in the formulation of the approximation properties and inverse assumptions on $\partial\omega_j$ when the Neumann bases with $\widetilde{d} = d''$ will be used.

Note that the global meshes on Ω_F and Υ are different and independent of the two local grids h and h' on the various macro–element boundaries $\partial\omega_j$.

Before discussing the local discretizations on the macro–elements, let us first introduce the global finite element spaces on $\Omega_F \cup \Upsilon$ as follows:

$$\begin{aligned}
\mathcal{H}_H &= \operatorname{span}\{\varphi_k \,|\, k \in \mathcal{N}\}\,, \\
\mathcal{H}_{HD} &:= \mathcal{H}_H \cap \mathcal{H}_D = \operatorname{span}\{\varphi_k \,|\, k \in \mathcal{N}_D\}\,, \\
\mathcal{H}_{FD} &:= \mathcal{H}_{HD} \cap H^1_D(\Omega_F) = \operatorname{span}\{\varphi_k \,|\, k \in \mathcal{N}_F\}\,,
\end{aligned}$$

where the index sets satisfy $\mathcal{N}_F \subseteq \mathcal{N}_D \subseteq \mathcal{N}$. Since the discretizations on $\partial\omega_j$ will depend on the choice of the local bases, the corresponding local boundary element spaces will be presented correspondingly.

6.1 The discretization with local Neumann bases

We begin with the definition of the local discrete spaces on $\partial\omega_j$ in terms of Neumann bases. For ease of reading we suppress the index j whenever possible.

$$
\begin{aligned}
\mathcal{H}_h^{-\frac{1}{2}} &:= \mathcal{S}_h^{d''}(\partial\omega) \cap H_0^{-\frac{1}{2}}(\partial\omega) = \mathrm{span}\{\boldsymbol{\nu}_\iota \,|\, \iota \in \mathcal{I}_N\}\,,\\
\widetilde{\mathcal{H}}_h^{-\frac{1}{2}}(\partial\omega \setminus (\Gamma_N \setminus \Upsilon)) &:= \mathcal{H}_h^{-\frac{1}{2}} \cap \widetilde{H}^{-\frac{1}{2}}(\partial\omega \setminus (\Gamma_N \setminus \Upsilon)) = \mathrm{span}\{\boldsymbol{\nu}_\iota \,|\, \iota \in \widetilde{\mathcal{I}}_{N\Upsilon+}\}\,,\\
\widetilde{\mathcal{H}}_h^{-\frac{1}{2}}(\partial\omega_D) &:= \mathcal{H}_h^{-\frac{1}{2}} \cap \widetilde{H}^{-\frac{1}{2}}(\partial\omega_D) = \mathrm{span}\{\boldsymbol{\nu}_\iota \,|\, \iota \in \widetilde{\mathcal{I}}_N\}\,,\\
\mathcal{H}_{0h}^{\frac{1}{2}} &:= \{\mathbf{v}^h \in \mathcal{S}_h^{d'}(\partial\omega) \,|\, \langle \mathbf{v}^h, \mathbf{r}\rangle = 0 \;\forall \mathbf{r} \in \Re\} = \mathrm{span}\{\mathring{\boldsymbol{\mu}}_\iota \,|\, \iota \in \mathcal{I}_{0D}\}\,.
\end{aligned}
$$

Here, the index sets satisfy $\widetilde{\mathcal{I}}_N \subseteq \mathcal{I}_N$, $\widetilde{\mathcal{I}}_{N\Upsilon+} \subseteq \mathcal{I}_N$ according to the corresponding discrete function spaces. In addition, we define $\mathcal{I}_{N\Upsilon} := \{\iota \in \widetilde{\mathcal{I}}_{N\Upsilon+} \,|\, \mathrm{supp}\,\boldsymbol{\nu}_\iota \cap \mathring{\Upsilon} \neq \emptyset\}$. Then $\widetilde{\mathcal{I}}_{N\Upsilon+} = \mathcal{I}_{N\Upsilon} \cup \widetilde{\mathcal{I}}_N$ and $\{\boldsymbol{\nu}_\iota$ for $\iota \in \mathcal{I}_{N\Upsilon}\}$ form a basis of

$$
\widehat{\mathcal{H}}_h^{-\frac{1}{2}}(\partial\omega \cap \Upsilon) := \widetilde{\mathcal{H}}_h^{-\frac{1}{2}}\Big((\partial\omega \setminus (\Gamma_N \setminus \Upsilon)\Big) \cap \widehat{H}_0^{-\frac{1}{2}}(\partial\omega \cap \Upsilon)
$$

on $\partial\omega \cap \Upsilon$.

In connection with the mixed boundary value problems on ω_j for $\partial\omega_j \cap \Gamma_D \setminus \Upsilon \neq \emptyset$ we require the existence of a linear prolongation operator family

$$
\wp_\nu \;:\; \widehat{\mathcal{H}}_h^{-\frac{1}{2}}(\partial\omega \cap \Upsilon) \to \widetilde{\mathcal{H}}_h^{-\frac{1}{2}}\Big(\partial\omega \setminus (\Gamma_N \setminus \Upsilon)\Big)
$$

with the **prolongation properties**: For every $\boldsymbol{\sigma} \in \widehat{\mathcal{H}}_h^{-\frac{1}{2}}(\partial\omega \cap \Upsilon)$:

$$
\wp_\nu \boldsymbol{\sigma}^h|_{(\partial\omega\cap\Upsilon)} = \boldsymbol{\sigma}^h \;\text{ on } \partial\omega \cap \Upsilon \;\text{ and } \|\wp_\nu \boldsymbol{\sigma}^h\|_{H^{-\frac{1}{2}}(\partial\omega)} \le c\|\boldsymbol{\sigma}^h\|_{\widehat{H}^{-\frac{1}{2}}(\partial\omega\cap\Upsilon)}\,. \tag{6.8}
$$

with c independent of h.

In terms of these bases we introduce corresponding matrices as the finite element 'stiffness matrix'

$$
\mathsf{A} := ((a_{\Omega_F}(\boldsymbol{\varphi}_k, \boldsymbol{\varphi}_\ell))) \;\text{ where } k, \ell \in \mathcal{N}_F
$$

and the matrices generated by the corresponding boundary integral operators on the macro–element boundaries:

$$
\begin{aligned}
\mathsf{V} &:= ((\langle \boldsymbol{\nu}_\kappa, V\boldsymbol{\nu}_\iota\rangle))\,, & \mathring{\mathsf{M}} &:= ((\langle \boldsymbol{\nu}_\iota, \mathring{\boldsymbol{\mu}}_\alpha\rangle)) & &\text{where } \kappa, \iota \in \mathcal{I}_N\,;\\
\mathring{\mathsf{K}} &:= ((\langle \boldsymbol{\nu}_\kappa, K\mathring{\boldsymbol{\mu}}_\alpha\rangle))\,, & \mathring{\mathsf{D}} &:= ((\langle \mathring{\boldsymbol{\mu}}_\alpha, D\mathring{\boldsymbol{\mu}}_\varsigma\rangle)) & &\text{where } \alpha, \varsigma \in \mathcal{I}_{0D}\,.
\end{aligned} \tag{6.9}
$$

We now consider the discretized version of the variational equations (4.2), (4.3) in discrete spaces where $(\mathbf{u}^H, \boldsymbol{\lambda}_j^h) \in \mathcal{H}_H \times \prod_{j=1}^M \mathcal{H}_{jh}^{-\frac{1}{2}}$. The trial functions must satisfy $\mathbf{u}_H = \boldsymbol{\varphi}$ on $(\partial\Omega_F \cup \Upsilon) \cap \Gamma_D$ and $\boldsymbol{\lambda}_j^h = \boldsymbol{\psi}$ on $\partial\omega_j \cap \Gamma_N \setminus \Upsilon$. For simplifying the presentation we again extend $\boldsymbol{\varphi}$ to $\boldsymbol{\varphi}^{*H} \in \mathcal{H}_H$ onto the whole set $\partial\Omega_F \cup \Upsilon$ and $\boldsymbol{\psi}$ to $\boldsymbol{\psi}_j^{*h} \in \mathcal{H}_{jh}^{-\frac{1}{2}}$ onto each of $\partial\omega_j$ requiring $\langle \boldsymbol{\psi}_j^{*h}, \mathbf{r}_j\rangle = 0$ for all $\mathbf{r}_j \in \Re_j$. We now write

$$
\mathbf{u}^H = \boldsymbol{\varphi}^{*H} + \mathbf{u}_0^H \;\text{ and } \boldsymbol{\lambda}_j^h = \boldsymbol{\psi}_j^{*h} + \boldsymbol{\lambda}_{j0}^h\,. \tag{6.10}
$$

Here

$$
\mathbf{u}_0^H = \sum_k \alpha_k \boldsymbol{\varphi}_k \in \mathcal{H}_{HD}\,,\; k \in \mathcal{N}_D\,, \quad \text{and } \boldsymbol{\lambda}_{j0}^h = \sum_\kappa \beta_{j\kappa} \boldsymbol{\nu}_{j\kappa} \in \widehat{H}_{jh}^{-\frac{1}{2}}(\partial\omega \cap \Upsilon)\,,\; \kappa \in \mathcal{I}_{jN\Upsilon}\,, \tag{6.11}
$$

are the unknown approximations satisfying

$$\mathbf{u}_0^H = \mathbf{0} \text{ on } (\partial\Omega_F \cup \Upsilon) \cap \Gamma_D \text{ and } \boldsymbol{\lambda}_{j0}^h = \mathbf{0} \text{ on } \partial\omega_j \cap \Gamma_N \setminus \Upsilon\,, \text{ correspondingly.} \tag{6.12}$$

In terms of the bases, Galerkin's formulation of the variational equations (4.2), (4.3) is equivalent to finding the coefficients $\vec{\alpha} = (\alpha_k)$ with $k \in \mathcal{N}_D$ and $\vec{\beta}_j = (\beta_{j\kappa})$ with $\kappa \in \mathcal{I}_{jN\Upsilon}$ from the global equations

$$\begin{aligned} &\sum_k a_F(\boldsymbol{\varphi}_k, \boldsymbol{\varphi}_\ell)\alpha_k + \sum_{j=1}^M \sum_\kappa \Big(\int\limits_{\partial\omega_j\cap\Upsilon} \boldsymbol{\nu}_{j\kappa} \cdot \boldsymbol{\varphi}_\ell \, ds \Big) \beta_{j\kappa} \\ &= \int\limits_{(\partial\Omega_F\cup\Upsilon)\cap\Gamma_N} \boldsymbol{\psi} \cdot \boldsymbol{\varphi}_\ell \, ds - a_F(\boldsymbol{\varphi}^{*H}, \boldsymbol{\varphi}_\ell) - \sum_{j=1}^M \int\limits_{\partial\omega_j\cap\Upsilon} \boldsymbol{\psi}_j^{*h} \cdot \boldsymbol{\varphi}_\ell \, ds \end{aligned} \tag{6.13}$$

where $k \in \mathcal{N}_F$, $\kappa \in \mathcal{I}_{jN\Upsilon}$ and $\ell \in \mathcal{N}_D$, and with $a_F(\boldsymbol{\varphi}_k, \boldsymbol{\varphi}_\ell) = 0$ for $\ell \in \mathcal{N}_D \setminus \mathcal{N}_F$;
and the local equations

$$\begin{aligned} &\sum_k \Big(\int\limits_{\partial\omega_j\cap\Upsilon} \boldsymbol{\nu}_{j\varrho} \cdot \boldsymbol{\varphi}_k ds \Big) \alpha_k - \sum_\kappa \Big(\int\limits_{\partial\omega_j\cap\Upsilon} \boldsymbol{\nu}_{j\varrho} \cdot \widehat{U}_j^h(\mathbf{0}, \boldsymbol{\nu}_{j\kappa}) ds \Big) \beta_{j\kappa} \\ &= - \int\limits_{\partial\omega_j\cap\Upsilon} \boldsymbol{\nu}_{j\varrho} \cdot \Big(\boldsymbol{\varphi}^{*H} - \widehat{U}_j^h(\boldsymbol{\varphi}, \boldsymbol{\psi}_j^{*h}) \Big) ds \end{aligned} \tag{6.14}$$

$$\text{where } k \in \mathcal{N}_{jD}\,;\ \kappa, \varrho \in \mathcal{I}_{jN\Upsilon} \text{ and } j = 1, \ldots, M\,.$$

Here $\mathcal{N}_{jD} := \{k \in \mathcal{N}_D \mid \overset{\circ}{\text{supp}}\, (\boldsymbol{\varphi}_k) \cap \partial\omega_j \neq \emptyset\}$ and $\widehat{U}_j^h$ denotes one of the approximations of the Poincaré–Steklov operators $\widehat{U}_j$ which will be specified in what follows.

In terms of matrix and vector notation, the equations (6.13), (6.14) are

$$\mathsf{A}\vec{\alpha} + \sum_{j=1}^M \widehat{\mathsf{B}}_j^\top \vec{\beta}_j = \mathsf{f}\,, \tag{6.15}$$

$$\widehat{\mathsf{B}}_j \vec{\alpha} - \widehat{\mathsf{U}}_j \vec{\beta}_j = \widehat{\mathsf{g}}_j \text{ for } j = 1, \ldots, M\,. \tag{6.16}$$

Here the vector and matrix elements in (6.15) and (6.16) are defined in an obvious way from (6.13) and (6.14). In (6.16), the matrix $\widehat{\mathsf{U}}_j$ describes the local macro–element and is invertible. However, in (6.16), one only needs to solve the equations on $\partial\omega_j$ with the few right–hand sides $\widehat{\mathsf{B}}_j \vec{\mathsf{e}}_\ell - \widehat{\mathsf{g}}_j$ with $\vec{\mathsf{e}}_\ell = (\delta_{\ell,k})$ for $\ell, k \in \mathcal{N}_{jD}$. The determination of

$$\vec{\beta}_j = \widehat{\mathsf{U}}_j^{-1} \widehat{\mathsf{B}}_j \vec{\alpha} - \widehat{\mathsf{U}}_j^{-1} \widehat{\mathsf{g}}_j \text{ for } j = 1, \ldots, M \tag{6.17}$$

can completely be executed in parallel. This then yields the global system for $\vec{\alpha}$:

$$\mathsf{A}\vec{\alpha} + \sum_{j=1}^M \widehat{\mathsf{B}}_j^\top \widehat{\mathsf{U}}_j^{-1} \widehat{\mathsf{B}}_j \vec{\alpha} = \mathsf{f} + \sum_{j=1}^M \widehat{\mathsf{B}}_j^\top \widehat{\mathsf{U}}_j^{-1} \widehat{\mathsf{g}}_j\,. \tag{6.18}$$

Here A is the so–called 'stiffness matrix' for Ω_F whereas $\widehat{\mathsf{U}}_j$ is the 'flexibility matrix' and $\widehat{\mathsf{B}}_j^\top \widehat{\mathsf{U}}_j^{-1} \widehat{\mathsf{B}}_j$ is the 'stiffness matrix' for the individual macro–element ω_j.

6.1.1 The discrete local Neumann problem

In the following, we again suppress the index j.

Since in the case of the local Neumann problem we have $\partial\omega_D = \partial\omega \cap \Gamma_D \setminus \Upsilon$ where $\partial\overset{\circ}{\omega}_D = \emptyset$, the operator $\widehat{U}^h = U^h$ is the associated discrete Poincaré–Steklov operator for the whole boundary $\partial\omega$ and we omit $\widehat{\cdot}$. The Neumann datum in (6.10) has the form

$$\boldsymbol{\lambda}^h = \boldsymbol{\psi}^{*h} + \boldsymbol{\lambda}_0^h \tag{6.19}$$

where

$$\boldsymbol{\lambda}_0^h = \sum_\kappa \beta_\kappa \boldsymbol{\nu}_\kappa \ \text{ with } U\boldsymbol{\lambda}_0^h = \sum_\kappa \beta_\kappa U(\boldsymbol{\nu}_\kappa) \ \text{ and } \kappa \in \mathcal{I}_{N\Upsilon} = \widetilde{\mathcal{I}}_{N\Upsilon+}$$

is the unkwon datum in (6.19).

The central effort here is to compute the coefficient matrix U and its submatrix $\widehat{\mathsf{U}}$ and the right–hand side vector $\widehat{\mathsf{g}}$ in (6.16) for the individual ω. This reduces to compute approximations of

$$\begin{aligned}
&\mathsf{U}_{\varrho\kappa} = \int\limits_{\partial\omega\cap\Upsilon} \boldsymbol{\nu}_\varrho \cdot U(\boldsymbol{\nu}_\kappa) ds \ \text{ and } \mathsf{g} = \overset{1}{\mathsf{g}} + \overset{2}{\mathsf{g}} \ \text{ where} \\
&\overset{1}{\mathsf{g}} := \int\limits_{\partial\omega\cap\Upsilon} \boldsymbol{\nu}_\varrho \cdot U(\boldsymbol{\psi}^{*h}) ds \ \text{ and } \overset{2}{\mathsf{g}} := -\int\limits_{\partial\omega\cap\Upsilon} \boldsymbol{\nu}_\varrho \cdot \boldsymbol{\varphi}^{*H} ds \ \text{ for } \varrho,\kappa \in \mathcal{I}_{N\Upsilon} \subseteq \mathcal{I}_N\,.
\end{aligned} \tag{6.20}$$

Depending on the local boundary element implementation chosen for the approximation of the operator U defined in (4.11) we present four different approaches in terms of one of the boundary integral equations (4.26) or (4.34), respectively:

$$U(\boldsymbol{\lambda}) - P_{\Re}(\tfrac{1}{2}I - K)P_{\Re}U(\boldsymbol{\lambda}) = V_0 P_{\Re}\boldsymbol{\lambda}\,, \tag{6.21}$$

$$D_0 U(\boldsymbol{\lambda}) = (\tfrac{1}{2}I - K')P_{\Re}\boldsymbol{\lambda} \tag{6.22}$$

as if $\boldsymbol{\lambda}$ is given.

Direct inversion of the discrete boundary integral equation of the second kind

For the integral equation (4.22) of the second kind, we define the approximation

$$P_{\Re}^h \mathbf{v} := \sum_{\alpha\iota} \overset{\circ}{\mathsf{M}}{}^{+}_{\alpha\iota} \langle \boldsymbol{\nu}_\iota, \mathbf{v}\rangle \overset{\circ}{\boldsymbol{\mu}}_\alpha \ \text{ where } \alpha \in \mathcal{I}_{0D}\,,\ \iota \in \mathcal{I}_N \tag{6.23}$$

of $P_{\Re}\mathbf{v}$ for any $\mathbf{v} \in H^{\frac{1}{2}}(\partial\omega)$ where $\overset{\circ}{\mathsf{M}}{}^{+}$ is the pseudoinverse to $\overset{\circ}{\mathsf{M}}$ [56]. Equation (6.21) with $\boldsymbol{\lambda} = \boldsymbol{\nu}_\kappa$, tested with $\boldsymbol{\nu}_\varrho$ and substituting (6.23) for $P_{\Re}U(\boldsymbol{\nu}_\kappa)$ yields

$$\mathsf{U}_{\varrho\kappa} = \langle \boldsymbol{\nu}_\varrho, U^h(\boldsymbol{\nu}_\kappa)\rangle := \sum_{\alpha,\iota} \overset{\circ}{\mathsf{M}}{}^{+}_{\alpha\iota} \langle \boldsymbol{\nu}_\iota, U^h(\boldsymbol{\nu}_\kappa)\rangle \langle \boldsymbol{\nu}_\varrho, (\tfrac{1}{2}I - K)\overset{\circ}{\boldsymbol{\mu}}_\alpha\rangle + \mathsf{V}_{\varrho\kappa} \tag{6.24}$$

where the summation runs for $\alpha \in \mathcal{I}_{0D}$ and $\iota \in \mathcal{I}_N$ and the indices $\varrho,\kappa \in \mathcal{I}_N$ are fixed.

In matrix form, the latter implies

$$\{\mathsf{I} - (\tfrac{1}{2}\overset{\circ}{\mathsf{M}} - \overset{\circ}{\mathsf{K}})\overset{\circ}{\mathsf{M}}{}^{+}\}\mathsf{U} = \mathsf{V}\,, \tag{6.25}$$

from which one can obtain U by using either direct or indirect inversion of the $\mathcal{I}_N \times \mathcal{I}_N$–matrix $\{\mathsf{I} - (\tfrac{1}{2}\overset{\circ}{\mathsf{M}} - \overset{\circ}{\mathsf{K}})\overset{\circ}{\mathsf{M}}{}^{+}\}$.

Similarly, we find for the approximate $\overset{1}{\mathsf{g}}$ in the right–hand side g in (6.20) the linear equations

$$\{\mathsf{I} - (\tfrac{1}{2}\overset{\circ}{\mathsf{M}} - \overset{\circ}{\mathsf{K}})\overset{\circ}{\mathsf{M}}^{+}\}\overset{1}{\mathsf{g}} = \mathsf{p} \;\text{ where } \mathsf{p}_\iota = \langle \boldsymbol{\nu}_\iota, V_0 \boldsymbol{\psi}^{*h}\rangle\,,\; \iota \in \mathcal{I}_N\,. \tag{6.26}$$

With the $\mathcal{I}_N \times \mathcal{I}_N$ matrix U available, $\widehat{\mathsf{U}}$ in (6.16)–(6.18) is then the $\mathcal{I}_{N\Upsilon} \times \mathcal{I}_{N\Upsilon}$ submatrix of U and $\widehat{\mathsf{g}}$ is there the $\mathcal{I}_{N\Upsilon}$ subvector.

The discrete Neumann series

If the additional assumption **(B1)** in (4.24) is satisfied, then one can use the Neumann series also for the discrete equations (6.25), i. e.

$$\mathsf{U} = \sum_{\ell=0}^{\infty}\Big((\tfrac{1}{2}\overset{\circ}{\mathsf{M}} - \overset{\circ}{\mathsf{K}})\overset{\circ}{\mathsf{M}}^{+}\Big)^{\ell}\mathsf{V} \;\text{ and }\; \overset{1}{\mathsf{g}} = \sum_{\ell=0}^{\infty}\Big((\tfrac{1}{2}\overset{\circ}{\mathsf{M}} - \overset{\circ}{\mathsf{K}})\overset{\circ}{\mathsf{M}}^{+}\Big)^{\ell}\mathsf{p} \tag{6.27}$$

with p given in (6.26). For iteration, one may define the usual recurrence sequence of matrices

$$\mathsf{U}^{(r)} := \Big(\tfrac{1}{2}\overset{\circ}{\mathsf{M}} - \overset{\circ}{\mathsf{K}}\Big)\overset{\circ}{\mathsf{M}}^{+}\mathsf{U}^{(r-1)} + \mathsf{V} \;\text{ with }\; \mathsf{U}^{(0)} := 0 \tag{6.28}$$

for $r = 1, 2, \ldots$. In practice, one may control the convergence of $\mathsf{U}^{(r)}$ numerically.

Remark: A very important special case of (6.25), respectively (6.27), (6.28) arises when $\overset{\circ}{\mathsf{M}}$ given in (6.9) is **invertible**; then $\overset{\circ}{\mathsf{M}}^{+} = \overset{\circ}{\mathsf{M}}^{-1}$. This can be achieved by special combinations of the bases $\{\boldsymbol{\nu}_\iota\}$ and $\{\overset{\circ}{\boldsymbol{\mu}}_\alpha\}$. An important special choice ist $\overset{\circ}{\boldsymbol{\mu}}_\alpha = V\boldsymbol{\nu}_\alpha$; then $\overset{\circ}{\mathsf{M}} = \mathsf{V}$ is invertible due to (4.14). Another important choice is $\overset{\circ}{\boldsymbol{\mu}}_\alpha = \boldsymbol{\nu}_\alpha$ where one needs $1 < d' = d''$.

The discrete hypersingular integral equation

For equation (6.22), we first approximate $U(\boldsymbol{\lambda})$ in $\mathcal{H}^{\frac{1}{2}}_{0h}$, test the equation with $\overset{\circ}{\boldsymbol{\mu}}_\alpha$ for $\alpha \in \mathcal{I}_{0D}$ and invert $\overset{\circ}{\mathsf{D}}$. Then we use (6.23) and finally obtain

$$\begin{aligned} \mathsf{U} &= \overset{\circ}{\mathsf{M}}\overset{\circ}{\mathsf{D}}^{-1}(\tfrac{1}{2}\overset{\circ}{\mathsf{M}} - \overset{\circ}{\mathsf{K}})^{\top}\,, \\ \overset{1}{\mathsf{g}} &= \overset{\circ}{\mathsf{M}}\overset{\circ}{\mathsf{D}}^{-1}\mathsf{q} \;\text{ with } \mathsf{q}_\beta = \langle (\tfrac{1}{2}\overset{\circ}{\boldsymbol{\mu}}_\beta - K\overset{\circ}{\boldsymbol{\mu}}_\beta), \boldsymbol{\psi}^{*h}\rangle\,,\; \beta \in \mathcal{I}_{0D}\,. \end{aligned} \tag{6.29}$$

We note that, although the operator U is symmetric, its approximations in the form of U in (6.29), in general, are not.

The discrete symmetric formulation

The discrete and symmetric approximation of the symmetric formulation (4.32) is given by

$$\mathsf{U} = (\tfrac{1}{2}\overset{\circ}{\mathsf{M}} - \overset{\circ}{\mathsf{K}})\overset{\circ}{\mathsf{D}}^{-1}(\tfrac{1}{2}\overset{\circ}{\mathsf{M}} - \overset{\circ}{\mathsf{K}})^{\top} + \mathsf{V}\,. \tag{6.30}$$

The use of this matrix representation is equivalent to solving the system of Galerkin equations for U via $\widetilde{\mathbf{U}}^h$,

$$\begin{aligned} \langle U^h\boldsymbol{\lambda}^h, \boldsymbol{\chi}^h\rangle &= \langle (\tfrac{1}{2}I - K)\widetilde{\mathbf{U}}^h\,,\; \boldsymbol{\chi}^h\rangle + \langle V_0\boldsymbol{\lambda}^h, \boldsymbol{\chi}^h\rangle\,, \\ \langle \overset{\circ}{\boldsymbol{\mu}}{}^h, D\widetilde{\mathbf{U}}^h\rangle &= \langle (\tfrac{1}{2}I - K')\boldsymbol{\lambda}^h\,,\; \overset{\circ}{\boldsymbol{\mu}}{}^h\rangle \end{aligned} \tag{6.31}$$

where $\boldsymbol{\lambda}^h, \boldsymbol{\chi}^h \in \mathcal{H}_h^{-\frac{1}{2}}(\partial\omega)$ and $\overset{\circ}{\boldsymbol{\mu}}{}^h \in \mathcal{H}_{0h}^{\frac{1}{2}}(\partial\omega)$.

The forcing term can be expressed correspondingly as

$$\overset{1}{\mathsf{g}} = (\tfrac{1}{2}\overset{\circ}{\mathsf{M}} - \overset{\circ}{\mathsf{K}})\overset{\circ}{\mathsf{D}}{}^{-1}\mathsf{q} + \mathsf{p} \tag{6.32}$$

with q given by (6.29) and p given by (6.26).

6.1.2 The discrete local mixed boundary value problem

As in Section 4.1.2, now the boundary $\partial\omega$ is decomposed in the form $\partial\omega = \partial\omega_D \cap \partial\omega_N$ where $\partial\omega_D = \partial\omega \cap \Gamma_D \setminus \overset{\circ}{\Upsilon}$ with $\partial\overset{\circ}{\omega}_D \neq \emptyset$ for equations (4.37). Let

$$\boldsymbol{\psi}^{*h} = \sum_{\iota} \psi_\iota^* \boldsymbol{\nu}_\iota \in \mathcal{H}_h^{-\frac{1}{2}}\,,\ \iota \in \mathcal{I}_N \tag{6.33}$$

be an approximation of the extension $\boldsymbol{\psi}^*$. Then we seek the solution in the form

$$\boldsymbol{\lambda}^h = \boldsymbol{\psi}^{*h} + \sum_{\kappa} \vartheta_\kappa \boldsymbol{\nu}_\kappa \ \text{ with } \kappa \in \widetilde{\mathcal{I}}_N \tag{6.34}$$

and an approximation for $\mathbf{u} = \boldsymbol{\varphi}^* + \mathbf{u}_0$ where $\mathbf{u}_0 \in \widetilde{H}^{\frac{1}{2}}(\partial\omega_N)$. Note that for (6.14) we need to solve the local mixed boundary value problem also for $\boldsymbol{\varphi}^* = \mathbf{0}$ and $\boldsymbol{\psi}^{*h} = \boldsymbol{\nu}_\varrho$ where $\varrho \in \mathcal{I}_{N\Upsilon}$.

The use of the discrete Poincaré–Steklov operator U

In this approach we solve the mixed boundary value problem (4.33) in two steps. In the first step we determine the full matrix U as for the pure Neumann problem (4.8) in ω, by any of the methods in the previous section. With U now avaiblable, in step two we solve the equation(4.38) approximately where U is approximated by U in order to find the solution of the mixed boundary value problems needed in (6.14). With (6.34) this amounts to determine $\vec{\vartheta} = (\vartheta_\kappa)$ by solving the linear system

$$\sum_\kappa \mathsf{U}_{\varrho\kappa}\vartheta_\kappa = \langle \boldsymbol{\nu}_\varrho, \boldsymbol{\varphi}^* \rangle - \sum_\iota \mathsf{U}_{\varrho\iota}\psi_\iota^* \ \text{ where } \kappa \in \widetilde{\mathcal{I}}_N\,,\ \iota \in \mathcal{I}_N\,,\ \text{ for } \varrho \in \widetilde{\mathcal{I}}_N \tag{6.35}$$

with given $\boldsymbol{\varphi}^*$ and with ψ_ι^* in the right–hand side given via (6.33). We denote by $\widetilde{\mathsf{U}}$ the submatrix $((\mathsf{U}_{\varrho\kappa}))$ of U for $\varrho, \kappa \in \widetilde{\mathcal{I}}_N$. With $\vec{\vartheta}$ known, we find

$$\widehat{S}^h(\boldsymbol{\varphi}, \boldsymbol{\psi}^h) := \boldsymbol{\lambda}^h = \sum_\iota \psi_\iota^* \boldsymbol{\nu}_\iota - \sum_{\iota,\kappa} (\widetilde{\mathsf{U}}^{-1}\mathsf{U})_{\kappa,\iota}\psi_\iota^* \boldsymbol{\nu}_\kappa + \sum_{\varrho,\kappa} (\widetilde{\mathsf{U}}^{-1})_{\varrho,\kappa} \langle \boldsymbol{\nu}_\varrho, \boldsymbol{\varphi}^* \rangle \boldsymbol{\nu}_\kappa \tag{6.36}$$

where $\iota \in \mathcal{I}_N$ and $\varrho, \kappa \in \widetilde{\mathcal{I}}_N$.

For the matrix $\widehat{\mathsf{U}}$ and the right–hand side $\widehat{\mathsf{g}}$ we obtain

$$\begin{aligned} &\int\limits_{\partial\omega\cap\Upsilon} \widehat{U}^h(\mathbf{0}, \boldsymbol{\nu}_\kappa) \cdot \boldsymbol{\nu}_\varrho ds = \widehat{\mathsf{U}}_{\kappa\varrho} = \mathsf{U}_{\kappa\varrho} - (\mathsf{U}\widetilde{\mathsf{U}}^{-1}\mathsf{U})_{\kappa,\varrho} \ \text{ with } \kappa, \varrho \in \mathcal{I}_{N\Upsilon} \\ &\widehat{\mathsf{g}}_\varrho = \sum_\iota \Big(\mathsf{U}_{\varrho\iota} - (\mathsf{U}\widetilde{\mathsf{U}}^{-1}\mathsf{U})_{\varrho,\iota}\Big)\psi_\iota^* + \sum_\kappa (\mathsf{U}\widetilde{\mathsf{U}}^{-1})_{\varrho,\kappa} \langle \boldsymbol{\nu}_\kappa, \boldsymbol{\varphi}^* \rangle - \int\limits_{\partial\omega\cap\Upsilon} \boldsymbol{\varphi}^{*H} \cdot \boldsymbol{\nu}_\varrho ds \end{aligned} \tag{6.37}$$

where $\iota \in \mathcal{I}_N$ and $\varrho, \kappa \in \mathcal{I}_{N\Upsilon}$.

We remark, for the discrete local boundary value problem, one may also discretize (4.41) or (4.42). This will involve the Neumann as well as the Dirichlet bases. We shall pursue this idea after having discussed the case involving only the local Dirichlet bases.

With the computed coefficient matrices and right–hand sides in (6.15) and (6.16) we can solve the system (6.15), (6.16) for $\vec{\beta}_j$ and $\vec{\alpha}$ which provides $\mathbf{u}^H$ and $\boldsymbol{\lambda}_j^h$ for $j = 1, \dots, M$. To compute

$$\overset{\circ}{\mathbf{u}}{}^h = \sum_\alpha \gamma_\alpha \overset{\circ}{\boldsymbol{\mu}}_\alpha = \widehat{U}^h(\boldsymbol{\varphi}^h, \boldsymbol{\psi}^h)\,,\ \alpha \in \mathcal{I}_{0D} \tag{6.38}$$

from $\boldsymbol{\lambda}^h = \sum_\iota \lambda_\iota \boldsymbol{\nu}_\iota\,,\ \iota \in \mathcal{I}_N$, one may use (6.23) to recover (6.38) from the weights $\langle \widehat{U}(\boldsymbol{\varphi}, \boldsymbol{\psi}^h), \boldsymbol{\nu}_\iota \rangle\,,\ \iota \in \mathcal{I}_N$. Then we find

$$\mathbf{u}^h = \sum_{\alpha,\iota} (\overset{\circ}{\mathsf{M}}{}^+ \mathsf{U})_{\alpha,\iota} \lambda_\iota \overset{\circ}{\boldsymbol{\mu}}_\alpha + \mathbf{r}^* = \sum_{\alpha,\iota} (\overset{\circ}{\mathsf{M}}{}^+ \mathsf{U})_{\alpha,\iota} (\psi_\iota^* + \vartheta_\iota) \overset{\circ}{\boldsymbol{\mu}}_\alpha + \mathbf{r}^* \tag{6.39}$$

where $\alpha \in \mathcal{I}_{0D}$ and $\iota \in \mathcal{I}_N$ with $\vartheta_\iota = 0$ for $\iota \in \mathcal{I}_N \setminus \widetilde{\mathcal{I}}_N$.

The rigid motion $\mathbf{r}^*$ can be determined explicitly from

$$\int\limits_{\partial\omega_D} \mathbf{r} \cdot \mathbf{r}^* ds = \int\limits_{\partial\omega_D} \mathbf{r} \cdot \left(\boldsymbol{\varphi}^h - \sum_{\alpha,\tau,\iota} \overset{\circ}{\mathsf{M}}{}^+_{\alpha\tau} \mathsf{U}_{\tau\iota} (\psi_\iota^* + \vartheta_\iota) \overset{\circ}{\boldsymbol{\mu}}_\alpha \right) ds \ \text{ for all } \mathbf{r} \in \Re \tag{6.40}$$

with the indices as in (6.39) and $\tau \in \mathcal{I}_N$.

6.2 The discretization with local Dirichlet bases

For the discretization with local Dirichlet bases, the following function spaces are needed:

$$\begin{aligned} \mathcal{H}_h^{\frac{1}{2}} &:= \mathcal{S}_h^{d'}(\partial\omega) = \operatorname{span}\{\boldsymbol{\mu}_\iota \,|\, \iota \in \mathcal{I}_D\}\,, \\ \widetilde{\mathcal{H}}_h^{\frac{1}{2}}\Big(\partial\omega \setminus (\Gamma_D \setminus \Upsilon)\Big) &= \mathcal{H}_h^{\frac{1}{2}} \cap \widetilde{H}^{\frac{1}{2}}\Big(\partial\omega \setminus (\Gamma_D \setminus \Upsilon)\Big) = \operatorname{span}\{\boldsymbol{\mu}_\iota \,|\, \iota \in \widetilde{\mathcal{I}}_{D\Upsilon}\}\,, \\ \widetilde{\mathcal{H}}_h^{\frac{1}{2}}(\partial\omega_N) &:= \mathcal{H}_h^{\frac{1}{2}} \cap \widetilde{H}^{\frac{1}{2}}(\partial\omega_N) = \operatorname{span}\{\boldsymbol{\mu}_\iota \,|\, \iota \in \widetilde{\mathcal{I}}_D\}\,. \end{aligned}$$

We use the index sets $\widetilde{\mathcal{I}}_D \subseteq \mathcal{I}_D$, $\widetilde{\mathcal{I}}_{D\Upsilon} \subseteq \mathcal{I}_D$ and $\mathcal{I}_{0D} \subset \mathcal{I}_D$ according to the corresponding discrete function spaces. Moreover, we define $\mathcal{I}_{D\Upsilon} := \{\iota \in \widetilde{\mathcal{I}}_{D\Upsilon+} \,|\, \operatorname{supp} \boldsymbol{\mu}_\iota \cap \overset{\circ}{\Upsilon} \neq \emptyset\}$. Then $\widetilde{\mathcal{I}}_{D\Upsilon+} = \mathcal{I}_{D\Upsilon} \cup \widetilde{\mathcal{I}}_D$ and $\{\boldsymbol{\mu}_\alpha$ for $\alpha \in \mathcal{I}_{D\Upsilon}\}$ form a basis of the discrete space

$$\widehat{\mathcal{H}}_h^{\frac{1}{2}}(\partial\omega \cap \Upsilon) := \widetilde{\mathcal{H}}_h^{\frac{1}{2}}(\partial\omega \setminus (\Gamma_D \setminus \Upsilon)) \cap \widehat{H}^{\frac{1}{2}}(\partial\omega \cap \Upsilon)$$

on $\partial\omega \cap \Upsilon$.

Again, we require for the mixed boundary value problems on ω_j for $\partial\omega_j \cap \Gamma_N \setminus \Upsilon \neq \emptyset$ in terms of the Dirichlet bases the existence of a linear prolongation operator family

$$\wp_\mu \,:\, \widehat{\mathcal{H}}_h^{\frac{1}{2}}(\partial\omega \cap \Upsilon) \to \widetilde{H}_h^{\frac{1}{2}}\Big(\partial\omega \setminus (\Gamma_D \setminus \Upsilon)\Big)$$

with the **prolongation property:** For every $\mathbf{v}^h \in \widehat{\mathcal{H}}_h^{\frac{1}{2}}(\partial\omega \cap \Upsilon)$:

$$\wp_\mu \mathbf{v}^h|_{\partial\omega\cap\Upsilon} = \mathbf{v}^h \ \text{ on } \partial\omega \cap \Upsilon \ \text{ and } \|\wp_\mu \mathbf{v}^h\|_{H^{\frac{1}{2}}(\partial\omega)} \leq c\|\mathbf{v}^h\|_{\widehat{H}_h^{\frac{1}{2}}(\partial\omega\cap\Upsilon)} \tag{6.41}$$

with c independent of h.

The discrete version of the variational equations (5.5), (5.6) with Dirichlet bases is defined by Galerkin's scheme for $(\mathbf{u}^H, \mathbf{u}_j^h) \in \mathcal{H}_H \times \prod_{j=1}^{M} \mathcal{H}_{jh}^{\frac{1}{2}}$ where the trial functions must satisfy

$$\mathbf{u}^H = \boldsymbol{\varphi} \text{ on } (\partial\Omega_F \cup \Upsilon) \cap \Gamma_D \text{ and } \mathbf{u}_j^h = \boldsymbol{\varphi} \text{ on } \partial\omega_j \cap \Gamma_D \setminus \Upsilon\,.$$

As before, we extend $\boldsymbol{\varphi}$ to $\boldsymbol{\varphi}^{*H} \in \mathcal{H}_H$ on $\partial\Omega_F \cup \Upsilon$ and to $\boldsymbol{\varphi}_j^{*h} \in \mathcal{H}_{jh}^{\frac{1}{2}}$ onto each of $\partial\omega_j$, respectively. Now we write

$$\mathbf{u}^H = \boldsymbol{\varphi}^{*H} + \mathbf{u}_0^H \text{ and } \mathbf{u}_j^h = \boldsymbol{\varphi}_j^{*h} + \mathbf{u}_{j0}^h \tag{6.42}$$

with the unknown approximations

$$\begin{aligned} \mathbf{u}_0^H &= \sum_k \alpha_k \boldsymbol{\varphi}_k \in \mathcal{H}_{HD}\,,\ k \in \mathcal{N}_D \text{ and} \\ \mathbf{u}_{j0}^h &= \sum_\beta \gamma_{j\beta} \boldsymbol{\mu}_{j\beta} \in \widehat{\mathcal{H}}_h^{\frac{1}{2}}(\partial\omega_j \cap \Upsilon) \text{ on } \partial\omega_j \cap \Upsilon \text{ where } \beta \in \mathcal{I}_{D\Upsilon}\,, \end{aligned} \tag{6.43}$$

satisfying

$$\mathbf{u}_0^H = \mathbf{0} \text{ on } (\partial\Omega_F \cup \Upsilon) \cap \Gamma_D \text{ and } \mathbf{u}_{j0}^h = \mathbf{0} \text{ on } \partial\omega_j \cap \Gamma_D \setminus \Upsilon\,, \tag{6.44}$$

correspondingly.

By using these representations (6.43), the Galerkin approximation of the variational equations (5.5), (5.6) consists of the global equations

$$\begin{aligned} &\sum_k a_F(\boldsymbol{\varphi}_k, \boldsymbol{\varphi}_\ell)\alpha_k + \sum_{j=1}^{M} \sum_\beta \Big(\int\limits_{\partial\omega_j \cap \Upsilon} \widehat{S}_j^h(\boldsymbol{\mu}_{j\beta}, \mathbf{0}) \cdot \boldsymbol{\varphi}_\ell \, ds \Big) \gamma_{j\beta} \\ &= \int\limits_{(\partial\Omega_F \cup \Upsilon) \cap \Gamma_N} \boldsymbol{\psi} \cdot \boldsymbol{\varphi}_\ell ds - a_F(\boldsymbol{\varphi}^{*H}, \boldsymbol{\varphi}_\ell) - \sum_{j=1}^{M} \Big(\int\limits_{\partial\omega_j \cap \Upsilon} \widehat{S}_j^h(\boldsymbol{\varphi}_j^{*h}, \boldsymbol{\psi}_j) \cdot \boldsymbol{\varphi}_\ell \, ds \Big) \end{aligned} \tag{6.45}$$

where $k \in \mathcal{N}_F$; $\beta \in \mathcal{I}_{D\Upsilon}$ and $\ell \in \mathcal{N}_D$;

and the local equations

$$\begin{aligned} &\sum_k \Big(\int\limits_{\partial\omega_j \cap \Upsilon} \widehat{S}_j^h(\boldsymbol{\mu}_{j\beta}, \mathbf{0}) \cdot \boldsymbol{\varphi}_k ds \Big) \alpha_k - \Big(\sum_\varsigma \int\limits_{\partial\omega_j \cap \Upsilon} \widehat{S}_j^h(\boldsymbol{\mu}_{j\beta}, \mathbf{0}) \cdot \boldsymbol{\mu}_{j\varsigma} ds \Big) \gamma_{j\varsigma} \\ &= \int\limits_{\partial\omega_j \cap \Upsilon} (\boldsymbol{\varphi}_j^{*h} - \boldsymbol{\varphi}^{*H}) \cdot \widehat{S}_j^h(\boldsymbol{\mu}_{j\beta}, \mathbf{0}) ds \text{ where } k \in \mathcal{N}_D \text{ and } \varsigma, \beta \in \mathcal{I}_{D\Upsilon}\,. \end{aligned} \tag{6.46}$$

Here $\widehat{S}_j$ is associated with $\partial\omega_{jN} = \partial\omega_j \cap \Gamma_N \setminus \overset{\circ}{\Upsilon}$ and $\partial\omega_{jD} = \partial\omega_j \setminus \partial\overset{\circ}{\omega}_{jN}$, which implies that $(\partial\omega_j \cap \Upsilon) \subseteq \partial\omega_{jD}$. Note that here the boundary parts $\partial\omega_{jN}$ and $\partial\omega_{jD}$ are different from those in Section 6.1.

In terms of matrix and vector notation, the equations (6.45), (6.46) are written as

$$\mathsf{A}\vec{\alpha} + \sum_{j=1}^{M} \widehat{\mathsf{C}}_j^\top \vec{\gamma}_j = \mathsf{b}\,, \tag{6.47}$$

$$\widehat{\mathsf{C}}_j \vec{\alpha} - \widehat{\mathsf{S}}_j \vec{\gamma}_j = \widehat{\mathsf{d}}_j \text{ for } j = 1, \ldots, M \tag{6.48}$$

where the coefficient matrices and right–hand sides are defined in (6.45) and (6.46).

We note that, in contrast to the matrix $\widehat{\mathsf{B}}_j$ in (6.15), (6.16), here the matrix $\widehat{\mathsf{C}}_j$ can only be evaluated via the solution of the local problems. In principle, we can use the local equations (6.48) to eliminate $\vec{\gamma}$ in (6.47) to obtain global equations as for the case of local Neumann bases (cf. (6.16)). However, we observe that the matrix $\widehat{\mathsf{S}}_j$ is not always invertible, e. g. if $\partial\omega_{jN} = \emptyset$, i. e. for the pure Dirichlet problem in Section 5.1.1. On the other hand, due to the special form of $\widehat{\mathsf{C}}_j\vec{\alpha} - \widehat{\mathsf{d}}_j$ defined in (6.46), one can verify that the latter always belongs to the range of $\widehat{\mathsf{S}}_j$ and, hence, each $\vec{\gamma}_j$ can still be eliminated, even if it is nonunique. Hence, this leads to the global system for $\vec{\alpha}$:

$$\mathsf{A}\vec{\alpha} + \sum_{j=1}^{M} \widehat{\mathsf{C}}_j^\top \widehat{\mathsf{S}}_j^+ \widehat{\mathsf{C}}_j \vec{\alpha} = \mathsf{b} + \sum_{j=1}^{M} \widehat{\mathsf{C}}_j^\top \widehat{\mathsf{S}}_j^+ \widehat{\mathsf{d}}_j \tag{6.49}$$

where we denote by $\widehat{\mathsf{S}}_j^+$ the pseudoinverse of $\widehat{\mathsf{S}}_j$. (See e. g. [56].) We emphasize that the macroelement stiffness matrix $\widehat{\mathsf{C}}_j^\top \widehat{\mathsf{S}}_j^+ \widehat{\mathsf{C}}_j$ is unique. We remark that $\widehat{\mathsf{S}}_j^+ = \widehat{\mathsf{S}}_j^{-1}$ when the latter exists which is the case for the mixed boundary value problem with $\partial\omega_{jN} \neq \emptyset$. System (6.49) now corresponds completely to the system (6.18).

6.2.1 The local discrete Dirichlet problem

In the following, we suppress the index j.

If we have $\partial\omega \cap \Gamma_N \setminus \Upsilon = \emptyset$ then $\widehat{S}^h = S^h$ will be the associated discrete Steklov–Poincaré operator for the pure Dirichlet problem in ω and $\mathcal{I}_{D\Upsilon} = \widetilde{\mathcal{I}}_{D\Upsilon+}$. Hence, we suppress here $\widehat{\ }$. The Dirichlet datum in (6.42) is now given by $\mathbf{u}^h = \boldsymbol{\varphi}^{*h} + \mathbf{u}_0^h$ where

$$\mathbf{u}_0^h = \sum_\beta \gamma_\beta \boldsymbol{\mu}_\beta \ \text{ and } S\mathbf{u}_0^h = \sum_\beta \gamma_\beta S(\boldsymbol{\mu}_\beta) \ \text{ with } \beta \in \mathcal{I}_{D\Upsilon} \tag{6.50}$$

is the unknown datum in the equations (6.45), (6.46). For the local problem, we need to evaluate the approximations

$$\mathsf{S}_{\beta\varsigma} = \int\limits_{\partial\omega\cap\Upsilon} (S^h\boldsymbol{\mu}_\beta)\cdot\boldsymbol{\mu}_\varsigma ds\,,\ \mathsf{C}_{\beta\ell} = \int\limits_{\partial\omega\cap\Upsilon} (S^h\boldsymbol{\mu}_\beta)\cdot\boldsymbol{\varphi}_\ell\, ds \tag{6.51}$$

$$\mathsf{d}_\beta = \int\limits_{\partial\omega\cap\Upsilon} (S^h\boldsymbol{\mu}_\beta)\cdot(\boldsymbol{\varphi}^{*h} - \boldsymbol{\varphi}^{*H})ds \ \text{ for } \beta,\varsigma \in \mathcal{I}_{D\Upsilon} \ \text{ and } \ell \in \mathcal{N}_D \tag{6.52}$$

and $\mathsf{b}_\ell = \overset{1}{\mathsf{b}}_\ell + \overset{2}{\mathsf{b}}_\ell$ where

$$\overset{1}{\mathsf{b}}_\ell := -\int\limits_{\partial\omega\cap\Upsilon} (S^h\boldsymbol{\varphi}^{*h})\cdot\boldsymbol{\varphi}_\ell\, ds\,,\ \overset{2}{\mathsf{b}}_\ell := \int\limits_{(\partial\omega_F\cup\Upsilon)\cap\Gamma_N} \boldsymbol{\psi}\cdot\boldsymbol{\varphi}_\ell ds - a_F(\boldsymbol{\varphi}^{*H}, \boldsymbol{\varphi}_\ell)\,. \tag{6.53}$$

Depending on the implementation of boundary integral operators available, one may use one of the following three approximation schemes. The above matrices are defined for all indices $\beta,\varsigma \in \mathcal{I}_D$ whereas $\widehat{\mathsf{S}}, \widehat{\mathsf{C}}$ and $\widehat{\mathsf{d}}$ in (6.45)–(6.49) are given by the $\mathcal{I}_{D\Upsilon} \times \mathcal{I}_{D\Upsilon}$–submatrix of S in (6.51) and the submatrix of C and subvector of d, respectively.

The direct inversion of the boundary integral equation of the second kind

The idea here is to represent the action of the local Steklov–Poincaré operator in terms of the local Dirichlet basis $\{\boldsymbol{\mu}_\alpha\}, \alpha \in \mathcal{I}_D$ by utilizing the integral equation of the second kind (5.14). We need the matrices generated by the corresponding boundary integral operators on the macro–element boundaries as follows:

$$\mathsf{K} := (\!(\langle \boldsymbol{\nu}_\kappa, K\boldsymbol{\mu}_\beta\rangle)\!)\,,\ \mathsf{D} := (\!(\langle \boldsymbol{\mu}_\beta, D\boldsymbol{\mu}_\varsigma\rangle)\!)\ \text{ where } \kappa\in\mathcal{I}_N\,,\ \beta,\varsigma\in\mathcal{I}_D\,. \tag{6.54}$$

We will also need the mass matrix

$$\mathsf{M} := (\!(\langle \boldsymbol{\nu}_\iota, \boldsymbol{\mu}_\beta\rangle)\!)\ \text{ where } \iota\in\mathcal{I}_N \text{ and } \beta\in\mathcal{I}_D\,. \tag{6.55}$$

We now define the approximation

$$\mathsf{P}^h_{\Re}\boldsymbol{\lambda} = \sum_{\alpha,\iota}\mathsf{M}^+_{\alpha\iota}\langle\boldsymbol{\mu}_\alpha,\boldsymbol{\lambda}\rangle\boldsymbol{\nu}_\iota \ \text{ where } \alpha\in\mathcal{I}_D\,,\ \iota\in\mathcal{I}_N\,, \tag{6.56}$$

of $P_\Re$ on $H^{-\frac12}(\partial\omega)$ in terms of the Neumann basis with Dirichlet weights. By inserting (6.56) for $P_\Re\boldsymbol{\lambda} = P_\Re S\boldsymbol{\mu}_\beta$ into (5.14) we obtain the approximate equation

$$S^h\boldsymbol{\mu}_\beta = \sum_{\iota,\alpha}\mathsf{M}^+_{\alpha\iota}\langle\boldsymbol{\mu}_\alpha, S\boldsymbol{\mu}_\beta\rangle P_\Re(\tfrac12 I + K')\boldsymbol{\nu}_\iota + D\boldsymbol{\mu}_\beta\,. \tag{6.57}$$

Testing with $\boldsymbol{\mu}_\eta$ and replacing the left–hand side yields the discrete system

$$\mathsf{S}_{\eta\beta} := \langle\boldsymbol{\mu}_\eta, S^h\boldsymbol{\mu}_\beta\rangle = \sum_{\alpha,\iota}\mathsf{M}^+_{\alpha\iota}\mathsf{S}_{\alpha\beta}\langle\boldsymbol{\mu}_\eta, P_\Re(\tfrac12 I + K')\boldsymbol{\nu}_\iota\rangle + \mathsf{D}_{\eta\beta}\,. \tag{6.58}$$

By decomposing $\boldsymbol{\mu}_\eta = P_\Re\boldsymbol{\mu}_\eta + \mathbf{r}_\eta$ with $\mathbf{r}_\eta\in\Re$ we obtain

$$\begin{aligned}\langle\boldsymbol{\mu}_\eta, P_\Re(\tfrac12 I + K')\boldsymbol{\nu}_\iota\rangle &= \langle(\tfrac12 I + K)P_\Re\boldsymbol{\mu}_\eta, \boldsymbol{\nu}_\iota\rangle\\ &= \langle(\tfrac12 I + K)\boldsymbol{\mu}_\eta, \boldsymbol{\nu}_\iota\rangle - \langle(\tfrac12 I + K)\mathbf{r}_\eta, \boldsymbol{\nu}_\iota\rangle = \tfrac12\mathsf{M}_{\iota\eta} + \mathsf{K}_{\iota\eta}\,.\end{aligned}$$

Inserting this relation into (6.58) we obtain

$$\mathsf{S}_{\eta\beta} = \sum_\alpha\Big(\mathsf{M}^+(\tfrac12\mathsf{M}+\mathsf{K})\Big)^\top_{\eta,\alpha}\mathsf{S}_{\alpha\beta} + \mathsf{D}_{\eta\beta}\ \text{ for } \alpha,\beta\in\mathcal{I}_D\,. \tag{6.59}$$

In matrix form, the equations read

$$\Big\{\mathsf{I} - (\mathsf{M}^+(\tfrac12\mathsf{M}+\mathsf{K}))^\top\Big\}\mathsf{S} = \mathsf{D}\,. \tag{6.60}$$

If $\{\mathsf{I} - (\mathsf{M}^+((\frac12\mathsf{M}+\mathsf{K}))^\top\}$ is invertible, S can be determined, e. g. by direct inversion.

When S is available then $\widehat{\mathsf{S}}$ is the $\mathcal{I}_{D\Upsilon}\times\mathcal{I}_{D\Upsilon}$ submatrix of S. Moreover, we find C and d in (6.51) and (6.52) from (6.57), correspondingly, i. e.

$$\begin{aligned}\mathsf{C}_{\beta\ell} &= \int\limits_{\partial\omega\cap\Upsilon}\boldsymbol{\varphi}_\ell\cdot(S^h\boldsymbol{\mu}_\beta)ds\\ &= \sum_{\alpha,\iota}\mathsf{S}_{\beta\alpha}\mathsf{M}^+_{\alpha\iota}\int\limits_{\partial\omega\cap\Upsilon}\boldsymbol{\varphi}_\ell\cdot(\tfrac12 I + K')\boldsymbol{\nu}_\iota ds + \int\limits_{\partial\omega\cap\Upsilon}\boldsymbol{\varphi}_\ell\cdot D\boldsymbol{\mu}_\beta ds\end{aligned} \tag{6.61}$$

and

$$\mathsf{d}_\beta = \sum_{\alpha,\iota}\mathsf{S}_{\beta\alpha}\mathsf{M}^+_{\alpha\iota}\int\limits_{\partial\omega\cap\Upsilon}(\boldsymbol{\varphi}^{*h}-\boldsymbol{\varphi}^{*H})\cdot(\tfrac12 I + K')\boldsymbol{\nu}_\iota ds + \int\limits_{\partial\omega\cap\Upsilon}(\boldsymbol{\varphi}^{*h}-\boldsymbol{\varphi}^{*H})\cdot D\boldsymbol{\mu}_\beta ds$$

$$\text{where } \iota\in\mathcal{I}_N\,;\ \alpha,\beta\in\mathcal{I}_D \text{ and } \ell\in\mathcal{N}_D\,. \tag{6.62}$$

The matrix $\widehat{\mathsf{C}}$ and $\widehat{\mathsf{d}}$ are obtained for $\beta \in \mathcal{I}_{D\Upsilon}$ in (6.61) and (6.62).

For computing $\overset{1}{\mathsf{b}}_\ell$ we write

$$\boldsymbol{\varphi}^{*h} = \sum_\alpha \varphi^*_\alpha \boldsymbol{\mu}_\alpha$$

and obtain

$$\overset{1}{\mathsf{b}}_\ell = -\sum_\alpha \varphi^*_\alpha \int\limits_{\partial\omega\cap\Upsilon} \boldsymbol{\varphi}_\ell \cdot (S^h \boldsymbol{\mu}_\alpha) ds = -\sum_\alpha \mathsf{C}_{\alpha\ell}\varphi^*_\alpha \text{ with } \alpha \in \mathcal{I}_D\,. \tag{6.63}$$

The discrete Neumann series

Under the additional assumption **(B2)**, equation(6.60) can also be inverted by using the Neumann series. One obtains

$$\mathsf{S} \;=\; \sum_{\ell=0}^{\infty} \Big((\mathsf{M}^+(\tfrac{1}{2}\mathsf{M}+\mathsf{K}))^\top \Big)^\ell \mathsf{D} \tag{6.64}$$

or the recurrence sequence of matrices

$$\mathsf{S}^{(r)} \;:=\; \Big(\mathsf{M}^+(\tfrac{1}{2}\mathsf{M}+\mathsf{K})\Big)^\top \mathsf{S}^{(r-1)} + \mathsf{D} \text{ for } r = 1,2,\ldots \text{ with } \mathsf{S}^{(0)} := 0 \tag{6.65}$$

In terms of $\mathsf{S}^{(r)}$, corresponding $\mathsf{C}^{(r)}$, and $\overset{1}{\mathsf{b}}{}^{(r)}$ may be obtained by inserting $\mathsf{S}^{(r)}$ for S into (6.61), (6.62), and $\mathsf{C}^{(r)}$ into (6.63).

The discrete boundary integral equation of the first kind

The Galerkin method applied to (5.18) finally yields with $\mathsf{C} = \widehat{\mathsf{C}}$ from (6.45)–(6.48) and $\widehat{\mathsf{B}} = \mathsf{B}$ from (6.13)–(6.16) the approximations:

$$\begin{aligned}
\mathsf{S}_{\alpha\varsigma} &= \Big(\mathsf{M}^\top\mathsf{V}^{-1}(\tfrac{1}{2}\mathsf{M}+\mathsf{K})\Big)_{\alpha,\zeta}\,,\ \mathsf{C}_{\alpha\ell} = \Big(\mathsf{B}^\top\mathsf{V}^{-1}((\tfrac{1}{2}\mathsf{M}+\mathsf{K})\Big)_{\ell,\alpha}\,,\ \alpha\in\mathcal{I}_D\,,\ \zeta\in\mathcal{I}_{D\Upsilon} \\
&\text{and} \\
\mathsf{d}_\alpha &= \sum_\iota \Big((\tfrac{1}{2}\mathsf{M}^\top + \mathsf{K}^\top)\mathsf{V}^{-1}\Big)_{\alpha,\iota} \int\limits_{\partial\omega_j\cap\Upsilon} \boldsymbol{\nu}_\iota \cdot (\boldsymbol{\varphi}^{*h} - \boldsymbol{\varphi}^{*H}) ds\,,\ \iota\in\mathcal{I}_N \\
\overset{1}{\mathsf{b}}_\ell &= -\sum_\alpha \mathsf{C}_{\alpha\ell}\varphi^*_\alpha \text{ for } \alpha\in\mathcal{I}_D \text{ and } \ell\in\mathcal{N}_D\,.
\end{aligned} \tag{6.66}$$

The use of the symmetric discrete representation of S

With the Galerkin approximation applied to equations (4.26) and (4.14) we obtain the discrete version of (5.21) in the form

$$\mathsf{S}_{\beta\varsigma} = (\tfrac{1}{2}\mathsf{M}+\mathsf{K})^\top\mathsf{V}^{-1}(\tfrac{1}{2}\mathsf{M}+\mathsf{K})_{\beta,\varsigma} + \mathsf{D}_{\beta\varsigma} \text{ with } \beta,\varsigma\in\mathcal{I}_D \tag{6.67}$$

and $\widehat{\mathsf{S}}$ is the $\mathcal{I}_{D\Upsilon}\times\mathcal{I}_{D\Upsilon}$–submatrix. For computing $\widehat{\mathsf{C}}$ and $\widehat{\mathsf{d}}$ we introduce the Gram matrix

$$\mathsf{G}_D := ((\langle\boldsymbol{\mu}_\beta,\boldsymbol{\mu}_\varsigma\rangle)) \text{ for } \beta,\varsigma\in\mathcal{I}_D$$

and its inverse G_D^{-1}. In terms of this matrix, we have

$$
\begin{aligned}
\widehat{\mathsf{C}}_{\beta\ell} &= \sum_{\varsigma}(\mathsf{S}\mathsf{G}_D^{-1})_{\beta,\varsigma} \int\limits_{\partial\omega_j\cap\Upsilon} \boldsymbol{\mu}_\varsigma \cdot \boldsymbol{\varphi}_\ell \, ds \ \text{ for } \beta \in \mathcal{I}_{D\Upsilon}\,,\ \varsigma \in \mathcal{I}_D \ \text{ and } \ell \in \mathcal{N}_D\,, \\
\widehat{\mathsf{d}}_\beta &= \sum_{\varsigma}(\mathsf{S}\mathsf{G}_D^{-1})_{\beta,\varsigma} \int\limits_{\partial\omega_j\cap\Upsilon} \boldsymbol{\mu}_\varsigma \cdot (\boldsymbol{\varphi}^{*h} - \boldsymbol{\varphi}^{*H}) ds \ \text{ for } \beta \in \mathcal{I}_{D\Upsilon}\,,\ \varsigma \in \mathcal{I}_D\,,
\end{aligned}
\tag{6.68}
$$

and

$$
\overset{1}{\mathsf{b}}_\ell = -\sum_\alpha \mathsf{C}_{\alpha\ell}\varphi^*_\alpha \ \text{ for } \alpha \in \mathcal{I}_D\,,\ \ell \in \mathcal{N}_D\,.
$$

6.2.2 The discrete local mixed boundary value problem

Now we consider the case $\partial\omega_N = \partial\omega \cap \Gamma_N \setminus \overset{\circ}{\Upsilon}$ with $\partial\omega_D = \partial\omega \setminus \partial\overset{\circ}{\omega}_N$ and $\partial\overset{\circ}{\omega}_N \neq \emptyset$ as in Section 5.1.2. Let

$$
\boldsymbol{\varphi}^{*h} = \sum_\tau \varphi^*_\tau \boldsymbol{\mu}_\tau \in \mathcal{H}_h^{\frac{1}{2}}\,,\ \tau \in \mathcal{I}_D \ \text{ and } \boldsymbol{\psi}^{*h} = \sum_\iota \psi^*_\iota \boldsymbol{\nu}_\iota \in \mathcal{H}_h^{-\frac{1}{2}}\,,\ \iota \in \mathcal{I}_N
$$

be the appropriate extensions of the given data $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$. Then we seek the solution pair $\mathbf{u}^h$ and $\boldsymbol{\lambda}^h$ in the form $\mathbf{u}^h = \boldsymbol{\varphi}^{*h} + \mathbf{u}_0^h$ and $\boldsymbol{\lambda}^h = \boldsymbol{\psi}^{*h} + \boldsymbol{\lambda}_0^h$. Then here

$$
\begin{aligned}
\mathbf{u}_0^h &= \sum_\alpha \xi_\alpha \boldsymbol{\mu}_\alpha \in \widetilde{\mathcal{H}}_h^{\frac{1}{2}}(\partial\omega_N) \qquad \text{with } \alpha \in \widetilde{\mathcal{I}}_D \ \text{ and} \\
\boldsymbol{\lambda}_0^h &= \sum_\varrho \vartheta_\varrho \boldsymbol{\nu}_\varrho \in \widetilde{\mathcal{H}}_h^{-\frac{1}{2}}(\partial\omega_D) \qquad \text{with } \varrho \in \widetilde{\mathcal{I}}_N.
\end{aligned}
\tag{6.69}
$$

are to be determined.

The use of the discrete Steklov–Poincaré operator S

If $\partial\overset{\circ}{\omega}_{jD} = \emptyset$ we have $\partial\omega_{jN} = \partial\omega$ and $\widehat{S}_j\boldsymbol{\mu}_{j\beta}, \mathbf{0}) = \mathbf{0}$ and $\widehat{S}_j(\boldsymbol{\varphi}_j^*, \boldsymbol{\psi}_j) = \boldsymbol{\psi}_j$ in the corresponding terms of (6.45) and (6.46). In this case, the local pure Neumann problem in ω_j is still to be solved but is not necessary for the global equations (6.49).

Hence, let us now confine to the case where both $\partial\overset{\circ}{\omega}_D \neq \emptyset$ and $\partial\overset{\circ}{\omega}_N \neq \emptyset$ and drop j. Again, we solve the problem in two steps. First compute the matrix S by using one of the procedures in Section 6.2.1. By using S, the discrete Galerkin equations for (5.23) become

$$
\sum_\alpha \mathsf{S}_{\alpha\beta}\xi_\alpha = \sum_\iota \mathsf{M}_{\iota\beta}\psi^*_\iota - \sum_\tau \mathsf{S}_{\beta\tau}\varphi^*_\tau \ \text{ with } \alpha, \beta \in \widetilde{\mathcal{I}}_D\,,\ \iota \in \mathcal{I}_N\,,\ \tau \in \mathcal{I}_D\,. \tag{6.70}
$$

Now we define $\widetilde{\mathsf{S}} = ((\mathsf{S}_{\beta\alpha}))$ as the corresponding $\widetilde{\mathcal{I}}_D \times \widetilde{\mathcal{I}}_D$ submatrix of the $\mathcal{I} \times \mathcal{I}$ matrix S. Note that $\widetilde{\mathsf{S}}$ is invertible for $\partial\overset{\circ}{\omega}_D \neq \emptyset$. This follows from the fact that any eigensolution in the kernel of $\widetilde{\mathsf{S}}$ will also be in the kernel of S, i. e. a rigid motion which, however, cannot vanish identically on $\partial\omega_D$. By solving equations (6.70) for $\vec{\xi}$, the approximate solution $\mathbf{u}^h$ can be written as

$$
\mathbf{u}^h = \boldsymbol{\varphi}^{*h} + \sum_\beta \left\{ \sum_\iota (\widetilde{\mathsf{S}}^{-1}\mathsf{M}^\top)_{\beta,\iota}\psi^*_\iota - \sum_\tau (\widetilde{\mathsf{S}}^{-1}\mathsf{S})_{\beta,\tau}\varphi^*_\tau \right\} \boldsymbol{\mu}_\beta \tag{6.71}
$$

where $\beta \in \widetilde{\mathcal{I}}_D$, $\tau \in \mathcal{I}_D$, $\iota \in \mathcal{I}_N$. Here, it is understood that $\widetilde{\mathsf{S}}^{-1}$ is extended by zero–entries for the indices $\mathcal{I}_D \setminus \widetilde{\mathcal{I}}_D$. Then

$$\begin{aligned}\overset{1}{\mathsf{b}}_\ell &= -\int\limits_{\partial\omega\cap\Upsilon} \widehat{S}^h(\boldsymbol{\varphi}^{*h},\boldsymbol{\psi})\cdot\boldsymbol{\varphi}_\ell ds = -\int\limits_{\partial\omega\cap\Upsilon}(S^h\mathbf{u}^h)\cdot\boldsymbol{\varphi}_\ell ds \\ &= -\sum_\tau\Big(\mathsf{C}_{\tau\ell}-(\mathsf{C}^\top\widetilde{\mathsf{S}}^{-1}\mathsf{S})_{\ell,\tau}\Big)\varphi^*_\tau - \sum_\iota(\mathsf{S}\widetilde{\mathsf{S}}^{-1}\mathsf{M}^\top)_{\ell,\iota}\psi^*_\iota \ \text{ with } \ell\in\mathcal{N}_D\end{aligned}$$

for the macro–element contribution to the right–hand side in (6.45). Correspondingly, with $\boldsymbol{\varphi}^{*h} = \boldsymbol{\mu}_\beta$ and $\boldsymbol{\psi}^{*h} = \mathbf{0}$ in (6.71) we find

$$\widehat{U}^h(\boldsymbol{\mu}_\beta,\mathbf{0}) = \boldsymbol{\mu}_\beta - \sum_\varsigma(\mathsf{S}\widetilde{\mathsf{S}}^{-1})_{\beta,\varsigma}\boldsymbol{\mu}_\varsigma \ \text{ where } \beta\in\mathcal{I}_{D\Upsilon} \ \text{ and } \varsigma\in\widetilde{\mathcal{I}}_D\,.$$

This yields for the matrices in (6.45) and (6.46) the approximations

$$\begin{aligned}\widehat{\mathsf{C}}_{\beta\ell} &= \int\limits_{\partial\omega\cap\Upsilon}\widehat{S}(\boldsymbol{\mu}_\beta,\mathbf{0})\cdot\boldsymbol{\varphi}_\ell ds &&= \mathsf{C}_{\beta,\ell}-(\mathsf{S}\widetilde{\mathsf{S}}^{-1}\mathsf{C})_{\beta,\ell}\,, \\ \widehat{\mathsf{S}}_{\beta,\tau} &= \int\limits_{\partial\omega\cap\Upsilon}\widehat{S}(\boldsymbol{\mu}_\beta,\mathbf{0})\cdot\boldsymbol{\mu}_\tau ds &&= \mathsf{S}_{\beta,\tau}-(\mathsf{S}\widetilde{\mathsf{S}}^{-1}\mathsf{S})_{\beta,\tau} \ \text{ where } \beta,\tau\in\mathcal{I}_{D\Upsilon}\,,\ \ell\in\mathcal{N}_D\,.\end{aligned} \tag{6.72}$$

The coefficients d_β for the pure Dirichlet problem are defined by (6.52) and they are computable via (6.62), (6.66). With these coefficients we obtain here

$$\begin{aligned}\widehat{\mathsf{d}}_\beta &= \int\limits_{\partial\omega\cap\Upsilon}(\boldsymbol{\varphi}^{*h}-\boldsymbol{\varphi}^{*H})\cdot\widehat{S}(\boldsymbol{\mu}_\beta,\mathbf{0})ds \\ &= \mathsf{d}_\beta - \sum_\varsigma(\mathsf{S}\widetilde{\mathsf{S}}^{-1})_{\beta,\varsigma}\mathsf{d}_\varsigma \ \text{ where } \beta\in\mathcal{I}_{D\Upsilon} \ \text{ and } \varsigma\in\widetilde{\mathcal{I}}_D\,.\end{aligned}$$

Similar to the Neumann bases approach, we now are in the position to compute $\vec{\gamma}_j$ and $\vec{\alpha}$ from (6.48) and (6.49) providing $\mathbf{u}^H$ and $\mathbf{u}^h_j$ for $j = 1,\ldots,M$. To compute $\boldsymbol{\lambda}^h_j = \sum_\iota \vartheta_\iota\boldsymbol{\nu}_\iota$, $\iota\in\mathcal{I}_N$, we write $\mathbf{u}^h$ in (6.71) in short as

$$\mathbf{u}^h = \sum_\alpha \eta_\alpha\boldsymbol{\mu}_\alpha\,,\ \alpha\in\mathcal{I}_D$$

and define here the approximation of local $\boldsymbol{\lambda}$ by

$$\boldsymbol{\lambda}^h = \sum_\alpha\eta_\alpha S^h(\boldsymbol{\mu}_\alpha) = \sum_\alpha\eta_\alpha P^h_{\Re}S^h(\boldsymbol{\nu}_\alpha)$$

and, by using (6.56), obtain

$$\boldsymbol{\lambda}^h = \sum_\alpha\eta_\alpha\sum_{\gamma,\iota}\mathsf{M}^+_{\gamma\iota}\langle\boldsymbol{\mu}_\gamma, S\boldsymbol{\mu}_\alpha\rangle\boldsymbol{\nu}_\iota = \sum_{\alpha,\iota}(\mathsf{S}\mathsf{M}^+)_{\alpha,\iota}\eta_\alpha\boldsymbol{\nu}_\iota\,. \tag{6.73}$$

6.3 A simplified construction of the macro–stiffness matrix

If the Poincaré–Steklov operator S^h in symmetric form–based on the Dirichlet bases is available (viz (6.67))then the global algebraic equations (6.18) or (6.49) can be simplified without explicit inversion of the matrices $\widehat{\mathsf{U}}_j$ and $\widehat{\mathsf{S}}_j$, respectively, provided the grids satisfy additional **mesh restrictions**:

$$(\mathbf{R1})\ P_{\Re_j}(\widehat{\mathbf{u}}^H_{|\partial\omega_j}) \subset \mathcal{H}^{\frac{1}{2}}_{0h}(\partial\omega_j)\,; \qquad (\mathbf{R2})\,\mathrm{rank}(\overset{\circ}{\mathsf{M}}_j) = \dim \mathcal{H}^{\frac{1}{2}}_{0h}(\partial\omega_j)\,. \tag{6.74}$$

As we shall see, these assumptions allow the identification of $P_{\Re_j}\mathbf{u}^H$ and $P_{\Re_j}\mathbf{u}^h_j$ on Υ. Then the local $\boldsymbol{\lambda}^h_j$ can be directly associated with $S^h_j\mathbf{u}^H$.

We begin with the representation (6.43) of $\mathbf{u}^H$ on the skeleton:

$$\mathbf{u}^H = \boldsymbol{\varphi}^{*H} + \mathbf{u}^H_0 = \boldsymbol{\varphi}^{*H} + \sum_k \alpha_k \boldsymbol{\varphi}_k\,,\ k \in \mathcal{N}_D\,. \tag{6.75}$$

The basis function $\boldsymbol{\varphi}_k$ is only defined on Υ which will be extended to $\partial\omega_j$. Let

$$\widehat{\boldsymbol{\varphi}}_{jk} := \boldsymbol{\varphi}^{h*}_{jk} + \mathbf{v}^h_{0jk} \ \text{ on } \partial\omega_j$$

where $\boldsymbol{\varphi}^{h*}_{jk}$ is a continuous fixed extension of $\boldsymbol{\varphi}_{jk}$ from $\widehat{H}^{\frac{1}{2}}_h(\partial\omega_j \cap \Upsilon)$ to $\widetilde{\mathcal{H}}^{\frac{1}{2}}_h\Big(\partial\omega_j \setminus (\Gamma_D \setminus \Upsilon)\Big)$ and $\mathbf{v}^h_{0jk} \in \widetilde{\mathcal{H}}^{\frac{1}{2}}_h(\partial\omega_j \cap \Gamma_N \setminus \Upsilon)$.

To approximate $S_j\widehat{\boldsymbol{\varphi}}_{jk} = 0$ on $\partial\omega_{jN} = \partial\omega_j \cap \Gamma_N \setminus \overset{\circ}{\Upsilon}$ we require that $\mathbf{v}^h_{0jk}$ is the solution of

$$\langle S^h_j \mathbf{v}^h_{0jk}, \mathbf{w}^h_0\rangle = -\langle S^h_j \boldsymbol{\varphi}^{h*}_{jk}, \mathbf{w}^h_0\rangle \ \text{ for all } \mathbf{w}^h_0 \in \widetilde{\mathcal{H}}^{\frac{1}{2}}(\partial\omega_j \cap \Gamma_N \setminus \Upsilon)\,.$$

With $\mathbf{v}^h_{0jk}$ available, the extended basis functions $\widehat{\boldsymbol{\varphi}}_{jk}$ are known on the whole $\partial\omega_j$. In terms of the extended basis functions we will also express an extension of $\mathbf{u}^H$ in the form

$$\widehat{\mathbf{u}}^H_j = \boldsymbol{\varphi}^{*H}_j + \mathbf{u}^h_{j\psi} + \sum_k \alpha_k \widehat{\boldsymbol{\varphi}}_{kj} \ \text{ on } \partial\omega_j \tag{6.76}$$

where $\boldsymbol{\varphi}^{*H}_j \in \mathcal{H}^{\frac{1}{2}}_h(\partial\omega_j)$ is a fixed extension of $\boldsymbol{\varphi}^{*H}$ onto $\partial\omega_j$ and $\mathbf{u}^h_{j\psi} \in \mathcal{H}^{\frac{1}{2}}(\partial\omega \cap \Gamma_N \setminus \Upsilon)$ is the local solution of

$$\langle S^h_j \mathbf{u}^h_{j\psi}, \mathbf{w}^h_0\rangle_j = \langle \boldsymbol{\psi} - S^h_j \boldsymbol{\varphi}^{*H}_j, \mathbf{w}^h_0\rangle \ \text{ for all } \mathbf{w}^h_0 \in \widetilde{\mathcal{H}}^{\frac{1}{2}}(\partial\omega \cap \Gamma_N \setminus \Upsilon)\,.$$

Now we consider the weak coupling

$$\langle \mathbf{u}^H - \mathbf{u}^h_j, \boldsymbol{\chi}^{h'}\rangle_j = 0 \ \text{ for all } \boldsymbol{\chi}^{h'} \in \widehat{\mathcal{H}}^{-\frac{1}{2}}(\partial\omega_j \cap \Gamma_N \setminus \Upsilon)$$

which becomes

$$\langle \widehat{\mathbf{u}}^H - \mathbf{u}^h_j, \boldsymbol{\chi}^{h'}\rangle_j = 0 \ \text{ for all } \boldsymbol{\chi}^{h'} \in \mathcal{H}^{-\frac{1}{2}}_{h'}(\partial\omega_j)\,. \tag{6.77}$$

These equations are the basis of the following lemma.

Lemma 3 *Assume the mesh restrictions* **(R1)**, **(R2)** *in* (6.74)*. Then*

$$P_{\Re_j}(\widehat{\mathbf{u}}^H_{|\partial\omega_j}) = P^h_{\Re_j}\widehat{\mathbf{u}}^H = P_{\Re_j}\mathbf{u}^h_j \ \text{ on } \partial\omega_j\,. \tag{6.78}$$

Proof: Again, we drop the index j. From the definition of $P^h_{\Re}$ in (6.23) we have for any $\mathbf{w}_h = \sum_\beta w_\beta \overset{\circ}{\boldsymbol{\mu}}_\beta \in \mathcal{H}^{\frac{1}{2}}_{0h}(\partial\omega_j)$

$$P^h_{\Re}\mathbf{w}_h = \sum_{\alpha,\iota} \overset{\circ}{\mathsf{M}}{}^+_{\alpha\iota}\langle \boldsymbol{\nu}_\iota, \mathbf{w}_h\rangle \overset{\circ}{\boldsymbol{\mu}}_\alpha = \sum_{\alpha,\beta,\iota} \overset{\circ}{\mathsf{M}}{}^+_{\alpha\iota}\overset{\circ}{\mathsf{M}}_{\iota\beta} w_\beta \overset{\circ}{\boldsymbol{\mu}}_\alpha\,.$$

Hence, with the properties of a pseudo–inverse,

$$\langle \boldsymbol{\nu}_\varrho, P^h_{\Re}\mathbf{w}_h\rangle = \sum_{\alpha,\beta,\iota} \overset{\circ}{\mathsf{M}}{}^{+}_{\alpha\iota}\overset{\circ}{\mathsf{M}}_{\iota\beta} w_\beta \langle \boldsymbol{\nu}_\varrho, \overset{\circ}{\boldsymbol{\mu}}_\alpha\rangle = \sum_{\alpha,\beta,\iota} \overset{\circ}{\mathsf{M}}_{\varrho\alpha}\overset{\circ}{\mathsf{M}}{}^{+}_{\alpha\iota}\overset{\circ}{\mathsf{M}}_{\iota\beta} w_\beta = \sum_\beta \overset{\circ}{\mathsf{M}}_{\varrho\beta} w_\beta = \langle \boldsymbol{\nu}_\varrho, \mathbf{w}_h\rangle$$

or

$$\langle \boldsymbol{\chi}^{h\,\prime}, P^h_{\Re}\mathbf{w}_h\rangle = \langle \boldsymbol{\chi}^{h\,\prime}, \mathbf{w}_h\rangle \text{ for all } \boldsymbol{\chi}^{h\,\prime} \in \mathcal{H}_{h\,\prime}^{-\frac{1}{2}}(\partial\omega) \text{ and } \mathbf{w}_h \in \mathcal{H}^{\frac{1}{2}}_{0h}(\partial\omega)\,. \tag{6.79}$$

The rank condition **(R2)** implies the inequality

$$\|\mathbf{w}_h\|_{H^{\frac{1}{2}}(\partial\omega)} \le \gamma_0(h) \sup_{\|\boldsymbol{\chi}^{h\,\prime}\|_{H_0^{-\frac{1}{2}}(\partial\omega)}=1} |\langle \boldsymbol{\chi}^{h\,\prime}, \mathbf{w}_h\rangle|$$

with some $\gamma_0(h) > 0$ which yields with (6.79)

$$\|\mathbf{w}_h - P^h_{\Re}\mathbf{w}_h\|_{H^{\frac{1}{2}}(\partial\omega_j)} = 0\,;$$

i. e.

$$P^h_{\Re}\mathbf{w}_h = \mathbf{w}_h \text{ for every } \mathbf{w}_h \in \mathcal{H}^{\frac{1}{2}}_{0h}(\partial\omega)$$

and (6.78) with $\mathbf{w}_h := \left(P_{\Re}\widehat{\mathbf{u}}^H\right)_{|\partial\omega} \in \mathcal{H}^{\frac{1}{2}}_{0h}(\partial\omega)$ due to **(R1)**. □

Now the global equations corresponding to (6.45) take the form

$$a_F(\mathbf{u}^H, \boldsymbol{\varphi}_\ell) + \sum_{j=1}^M \oint_{\partial\omega_j} \boldsymbol{\lambda}^{h\,\prime}_j \cdot \widehat{\boldsymbol{\varphi}}_{j\ell} ds = \int_{\Gamma_N} \boldsymbol{\psi}\cdot\boldsymbol{\varphi}_\ell ds \text{ for all } \ell \in \mathcal{N}_D\,.$$

With

$$\boldsymbol{\lambda}^{h\,\prime}_j = S^h_j \widehat{\mathbf{u}}^H_j = S^h_j \boldsymbol{\varphi}^{*H}_j + S^h_j \mathbf{u}^h_{j\psi} + \sum_k \alpha_k S^h_j \widehat{\boldsymbol{\varphi}}_{jk} \tag{6.80}$$

we obtain the global equations

$$\begin{aligned} &\sum_{k\in\mathcal{N}_F} \alpha_k a_F(\boldsymbol{\varphi}_k, \boldsymbol{\varphi}_\ell) + \sum_{j=1}^M \alpha_k \langle S^h_j \widehat{\boldsymbol{\varphi}}_{jk}, \widehat{\boldsymbol{\varphi}}_{j\ell}\rangle_j \\ &= \int_{\Gamma_N} \boldsymbol{\psi}\cdot\boldsymbol{\varphi}_\ell ds - a_F(\boldsymbol{\varphi}^{*H}, \boldsymbol{\varphi}_\ell) - \sum_{j=1}^M \langle S^h_j \boldsymbol{\varphi}^{*H} + S^h_j \mathbf{u}^h_{j\psi}, \widehat{\boldsymbol{\varphi}}_{j\ell}\rangle_j \end{aligned} \tag{6.81}$$

or, in terms of matrix and vector notation

$$\mathsf{A}\vec{\alpha} + \sum_{j=1}^M \mathsf{A}_j\vec{\alpha} = \mathsf{z}\,. \tag{6.82}$$

where $\mathsf{A}_j = ((\langle S^h_j \widehat{\boldsymbol{\varphi}}_{jk}, \widehat{\boldsymbol{\varphi}}_{j\ell}\rangle_j))$ is the stiffness matrix contribution from the macro element ω_j and the vector z is defined from the right–hand side of the global equation (6.81). Note that in this formulation the local Steklov–Poincaré operators S^h_j are used directly without inversions as in (6.49).

Finally, from (6.78), the local solutions $\mathbf{u}^h_j$ can be determined up to rigid motions from (6.76) once $\vec{\alpha}$ is computed from solving (6.82). Correspondingly, $\boldsymbol{\lambda}^{h\,\prime}_j$ is given by (6.80).

7 Stability and Convergence

The stability and convergence results depend on how the local Poincaré–Steklov or Steklov–Poincaré operators, respectively, are approximated. In case the approximations preserve the corresponding strong ellipticity properties of the original mappings, one obtains stability which implies that the well known Babuška–Brezzi inf–sup conditions for the mixed formulation are satisfied and we do not need to corrolate the meshes of the finite– and skeleton elements to the local boundary element meshes. Then Cea's convergence lemma follows in the standard manner.

On the other hand, if coercivity is not preserved on the discrete spaces, our proof of the stability is more involved and requires additional restrictions such as inverse assumptions and mesh restrictions.

We begin with the error analysis of the discretizations with local Neumann bases.

7.1 Stability and convergence with local Neumann bases

7.1.1 Coercive approximate Poincaré–Steklov mappings

We first show that the construction of $\widehat{\mathsf{U}}^{h'}$ in Section 6.1 with the discrete symmetric formulation (6.30) provides the discrete $\widehat{H}_0^{-\frac{1}{2}}(\partial\omega \cap \Upsilon)$–ellipticity **without any restrictions on the meshes**. For ease of reading, the macro–element index j is occasionally suppressed.

Lemma 4 *If* U *is defined by the discrete symmetric approximation* (6.30) *and* $\widehat{U}^{h'}$ *is the corresponding approximation in* (6.37) *then* $\widehat{U}^{h'}$ *is* $\widehat{H}_0^{-\frac{1}{2}}(\partial\omega \cap \Upsilon)$*–elliptic on* $\widehat{\mathcal{H}}_{h'}^{-\frac{1}{2}}$*, i. e.*

$$\int\limits_{\partial\omega} \widehat{U}^{h'}(\mathbf{0}, \boldsymbol{\chi}^{h'}) \cdot \boldsymbol{\chi}^{h'} ds \geq \gamma_0 \|\boldsymbol{\chi}^{h'}\|^2_{\widehat{H}^{-\frac{1}{2}}(\partial\omega\cap\Upsilon)} \;\; \textit{for all}\; \boldsymbol{\chi}^{h'} \in \widehat{\mathcal{H}}_{h'}^{-\frac{1}{2}}(\partial\omega \cap \Upsilon)\,. \tag{7.1}$$

Moreover, $U^{h'}$ *as well as* $\widehat{U}^{h'}$ *are uniformely bounded, i. e.*

$$\|U^{h'}\boldsymbol{\chi}\|_{H^{\frac{1}{2}}(\partial\omega)} \;\leq\; c\|\boldsymbol{\chi}\|_{H^{-\frac{1}{2}}(\partial\omega)} \tag{7.2}$$

or

$$\|\widehat{U}^{h'}(\boldsymbol{\varphi}, \boldsymbol{\chi})\|_{H^{\frac{1}{2}}(\partial\omega)} \;\leq\; c\|\boldsymbol{\chi}\|_{\widehat{H}^{-\frac{1}{2}}(\partial\omega\cap\Upsilon)} + c\|\boldsymbol{\varphi}\|_{H^{\frac{1}{2}}(\partial\omega\cap\Gamma_D\setminus\mathring{\Upsilon})} \;\; \textit{for}\; \boldsymbol{\chi} \in H_0^{-\frac{1}{2}}(\partial\omega)\,, \tag{7.3}$$

respectively, where the constant c *is independent of* h'.

Proof: For any given $\boldsymbol{\chi}^{h'} \in \widehat{\mathcal{H}}_{h'}^{-\frac{1}{2}}(\partial\omega \cap \Upsilon)$, the definition of $\boldsymbol{\lambda}_0^{h'} \in \widetilde{\mathcal{H}}_{h'}^{-\frac{1}{2}}(\partial\omega_D)$ via (6.36), (6.37) is equivalent to the Galerkin approximation of (4.38), i. e.

$$\langle \boldsymbol{\nu}^{h'}, U^{h'}\boldsymbol{\lambda}_0^{h'}\rangle = -\langle \boldsymbol{\nu}^{h'}, U^{h'}\boldsymbol{\chi}^{h'}\rangle \;\text{ for all}\; \boldsymbol{\nu}^{h'} \in \widetilde{\mathcal{H}}_{h'}^{-\frac{1}{2}}(\partial\omega_D)\,,$$

which is particularly true for $\boldsymbol{\nu}^{h'} = \boldsymbol{\lambda}_0^{h'}$. From $\widehat{U}^{h'}(\mathbf{0}, \boldsymbol{\chi}^{h'}) = U^{h'}(\boldsymbol{\chi}^{h'} + \boldsymbol{\lambda}_0^{h'})$, it follows that

$$\langle \widehat{U}^{h'}(\mathbf{0}, \boldsymbol{\chi}^{h'}), \boldsymbol{\chi}^{h'}\rangle = \langle U^{h'}(\boldsymbol{\chi}^{h'} + \boldsymbol{\lambda}_0^{h'}), \boldsymbol{\chi}^{h'}\rangle = \langle U^{h'}(\boldsymbol{\chi}^{h'} + \boldsymbol{\lambda}_0^{h'}), \boldsymbol{\chi}^{h'} + \boldsymbol{\lambda}_0^{h'}\rangle\,.$$

Since the symmetric approximation $U^{h'}$ is $H_0^{-\frac{1}{2}}(\partial\omega)$–elliptic on $\mathcal{H}_{h'}^{-\frac{1}{2}}$ due to (4.18), this implies that

$$\begin{aligned}\langle \widehat{U}^{h'}(\mathbf{0},\boldsymbol{\chi}^{h'}),\boldsymbol{\chi}^{h'}\rangle &\geq \gamma_0\|\boldsymbol{\chi}^{h'}+\boldsymbol{\lambda}_0^{h'}\|^2_{H^{-\frac{1}{2}}(\partial\omega)}\\ &\geq \gamma_0\|\boldsymbol{\chi}^{h'}+\boldsymbol{\lambda}_0^{h'}\|^2_{H^{-\frac{1}{2}}(\partial\omega_N)} = \gamma_0\|\boldsymbol{\chi}^{h'}\|^2_{H^{-\frac{1}{2}}(\partial\omega_N)} \geq \gamma_0\|\boldsymbol{\chi}^{h'}\|^2_{\widehat{H}^{-\frac{1}{2}}(\partial\omega\cap\Upsilon)}.\end{aligned}$$

□

For the uniform boundedness of $U^{h'}$ we first consider the case $\partial\overset{\circ}{\omega}_D = \partial\omega\cap\Gamma_D\setminus\Upsilon=\emptyset$. Let $\widetilde{\mathbf{u}}^h \in \mathcal{H}_{0h}^{\frac{1}{2}}(\partial\omega)$ be given by the Galerkin solution of

$$\langle \overset{\circ}{\mathbf{u}}^h, D\widetilde{\mathbf{u}}^h\rangle = \langle \overset{\circ}{\mathbf{u}}^h, (\tfrac{1}{2}I-K')\boldsymbol{\chi}\rangle \text{ for all } \overset{\circ}{\boldsymbol{\mu}}^h \in \mathcal{H}_{0h}^{\frac{1}{2}}(\partial\omega)$$

for any given $\boldsymbol{\chi}\in H_0^{-\frac{1}{2}}(\omega)$. Since $U\boldsymbol{\chi}$ satisfies the equations

$$\langle \overset{\circ}{\mathbf{u}}^h, DU\boldsymbol{\chi}\rangle = \langle \overset{\circ}{\mathbf{u}}^h, (\tfrac{1}{2}I-K')\boldsymbol{\chi}\rangle \text{ for all } \overset{\circ}{\boldsymbol{\mu}}^h \in \mathcal{H}_{0h}^{\frac{1}{2}}(\partial\omega),$$

it follows from the $H_0^{\frac{1}{2}}(\partial\omega)$–ellipticity of D in (4.29) that

$$\|\widetilde{\mathbf{u}}^h - U\boldsymbol{\chi}\|_{H^{\frac{1}{2}}(\partial\omega)} \leq c\|U\boldsymbol{\chi}\|_{H^{\frac{1}{2}}(\partial\omega)} \leq c'\,\|\boldsymbol{\chi}\|_{H^{-\frac{1}{2}}(\partial\omega)}$$

where c and c' are independent of h. By the definition of $U^{h'}\boldsymbol{\chi}$ via (6.30) we have

$$U^{h'}\boldsymbol{\chi} = P_{h'}(\tfrac{1}{2}I-K)\widetilde{\mathbf{u}}^h + P_{h'}V\boldsymbol{\chi}.$$

Hence, with (6.6) and the triangle inequality, we obtain

$$\|U^{h'}\boldsymbol{\chi}\|_{H^{\frac{1}{2}}(\partial\omega)} \leq c\|\widetilde{\mathbf{u}}^h\|_{H^{\frac{1}{2}}(\partial\omega)} \leq c'\,\|\boldsymbol{\chi}\|_{H^{-\frac{1}{2}}(\partial\omega)} \leq c''\|\boldsymbol{\chi}\|_{H^{-\frac{1}{2}}(\partial\omega)},$$

where the constant c'' is independent of h and h', i. e. (7.2).

For the case $\partial\overset{\circ}{\omega}_D\neq\emptyset$ and for any $\boldsymbol{\chi}\in\widehat{H}^{-\frac{1}{2}}(\partial\omega\cap\Upsilon)$ the define the prolongation

$$\boldsymbol{\chi}^{*h'} := \wp_\nu P_{h'}\boldsymbol{\chi}\in\mathcal{H}_{h'}^{-\frac{1}{2}}(\partial\omega)$$

and decompose

$$\boldsymbol{\chi}^{h'} = \boldsymbol{\chi}^{*h'} + \boldsymbol{\lambda}_0^{h'} \text{ with } \boldsymbol{\lambda}_0^{h'}\in\widetilde{\mathcal{H}}_h^{-\frac{1}{2}}(\partial\omega_D).$$

The definition of $\widehat{U}^{h'}(\boldsymbol{\varphi},\boldsymbol{\chi})$ implies

$$\widehat{U}^{h'}(\boldsymbol{\varphi},\boldsymbol{\chi}) = U^{h'}\boldsymbol{\chi}^{h'} = U^{h'}\boldsymbol{\chi}^{*h'} + U^{h'}\boldsymbol{\lambda}_0^{h'}$$

where $\boldsymbol{\lambda}_0^{h'}$ is the solution of the Galerkin equations

$$\langle U^{h'}\boldsymbol{\lambda}_0^{h'},\boldsymbol{\nu}^{h'}\rangle = \langle\boldsymbol{\varphi}-U^{h'}\boldsymbol{\chi}^{*h'},\boldsymbol{\nu}^{h'}\rangle \text{ for all } \boldsymbol{\nu}^{h'}\in\widetilde{\mathcal{H}}_{h'}^{-\frac{1}{2}}(\partial\omega).$$

The coerciveness of $U^{h'}$ implies the estimate

$$\|\boldsymbol{\lambda}_0^{h'}\|_{H^{-\frac{1}{2}}(\partial\omega)} \leq c\left\{\|\boldsymbol{\varphi}\|_{H^{\frac{1}{2}}(\partial\omega_D)} + \|\boldsymbol{\chi}^{*h'}\|_{H^{-\frac{1}{2}}(\partial\omega)}\right\} \tag{7.4}$$

where c is independent of h'.

Recalling the definition of $\widehat{U}^{h'}(\boldsymbol{\varphi}, \boldsymbol{\chi})$, we find

$$\|\widehat{U}^{h'}(\boldsymbol{\varphi}, \boldsymbol{\chi})\|_{H^{\frac{1}{2}}(\partial\omega_D)} \leq c\Big\{\|\boldsymbol{\chi}^{*h'}\|_{H^{-\frac{1}{2}}(\partial\omega)} + \|\boldsymbol{\lambda}_0^{h'}\|_{H^{-\frac{1}{2}}(\partial\omega)}\Big\}$$

with the help of (7.2). Inserting (7.4) and using

$$\|\boldsymbol{\chi}^{*h'}\|_{H^{-\frac{1}{2}}(\partial\omega)} \leq c\|\boldsymbol{\chi}\|_{\widehat{H}^{-\frac{1}{2}}(\partial\omega\cap\Upsilon)}$$

which is obtained from (6.6) and (6.8), we find the desired uniform boundedness of $\widehat{U}^{h'}$.

Clearly, the system (6.13), (6.14) is equivalent to the variational formulation:

Find

$$(\mathbf{u}_0^H, \boldsymbol{\lambda}_{j0}^{h'}) \in \mathcal{H}_{h'}^N := \mathcal{H}_{HD} \times \prod_{j=1}^{M} \widehat{\mathcal{H}}_{h'}^{-\frac{1}{2}}(\partial\omega_j \cap \Upsilon)$$

such that

$$\begin{aligned}
\mathcal{A}(\mathbf{u}_0^H, \boldsymbol{\lambda}_{j0}^{h'}; \mathbf{v}^H, \boldsymbol{\chi}_j^{h'}) \;&:=\; a_F(\mathbf{u}_0^H, \mathbf{v}^H) + \sum_{j=1}^{M}\{\langle\boldsymbol{\lambda}_{j0}^{h'}, \mathbf{v}^H\rangle_j - \langle\boldsymbol{\chi}_j^{h'}, \mathbf{u}_0^H\rangle_j + \langle\boldsymbol{\chi}_j^{h'}, \widehat{U}_j^{h'}(\mathbf{0}, \boldsymbol{\lambda}_{j0}^{h'})\rangle_j\} \\
&=\; \mathcal{L}(\mathbf{v}^H, \boldsymbol{\chi}_j^{h'}) \;\; \textit{for all } \mathbf{v}^H \in \mathcal{H}_{HD}, \boldsymbol{\chi}_j^{h'} \in \widehat{\mathcal{H}}^{-\frac{1}{2}}(\partial\omega_j \cap \Upsilon) \qquad (7.5)
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{L}(\mathbf{v}^H, \boldsymbol{\chi}_j^{h'}) \;&:=\; \int\limits_{(\partial\Omega_F\cup\Upsilon)\cap\Gamma_N} \boldsymbol{\psi}\cdot\mathbf{v}^H ds - a_F(\boldsymbol{\varphi}^{*H}, \mathbf{v}^H) \\
&\quad + \sum_{j=1}^{M}\{\langle\boldsymbol{\chi}_j^{h'}, \boldsymbol{\varphi}^{*H}\rangle_j - \langle\boldsymbol{\psi}_j^{*h'}, \mathbf{v}^H\rangle_j - \langle\boldsymbol{\chi}_j^{h'}, \widehat{U}_j^{h'}(\boldsymbol{\varphi}, \boldsymbol{\psi}_j^{*h'})\rangle_j\}\,. \qquad (7.6)
\end{aligned}$$

Based on the property (7.1), it follows easily that $\mathcal{A}$ satisfies the BBL–condition. (For the BBL–condition and its extensions see [8, II.1.2].)

Theorem 5 *Under condition* (7.1)*, the bilinear from* $\mathcal{A}$ *in* (7.5) *satisfies the BBL–condition*

$$\begin{aligned}
\sup_{(\mathbf{v}^H, \boldsymbol{\chi}_j^{h'})\in\mathcal{H}_{h'}^N} |\mathcal{A}(\mathbf{u}_0^H, \boldsymbol{\lambda}_{j0}^{h'}; \mathbf{v}^H, \boldsymbol{\chi}_j^{h'})| \Big/ \Big\{\|\mathbf{v}^H\|_{\mathcal{H}_{HD}} \;&+\; \sum_{j=1}^{M}\|\boldsymbol{\chi}_j^{h'}\|_{\widehat{H}^{-\frac{1}{2}}(\partial\omega_j\cap\Upsilon)}\Big\} \\
\geq \gamma_0\Big\{\|\mathbf{u}_0^H\|_{\mathcal{H}_{HD}} \;&+\; \sum_{j=1}^{M}\|\boldsymbol{\lambda}_{0j}^{h'}\|_{\widehat{H}^{-\frac{1}{2}}(\partial\omega_j\cap\Upsilon)}\Big\}\,,
\end{aligned} \qquad (7.7)$$

where $\gamma_0 > 0$ *is a constant independent of* H *and* h'.

Proof: The result follows immediately by choosing $(\mathbf{v}^H, \boldsymbol{\chi}_j^{h'}) = (\mathbf{u}_0^H, \boldsymbol{\lambda}_{0j}^{h'})$ with the help of the coercivity condition (3.9) for a_F, assumption (7.1) for $\widehat{U}^{h'}$ and the uniform boundedness (7.2) or (7.3) . □

As a consequence of the stability result (7.4) given in Theorem 5, Céa's lemma is valid [8, II.2.4]. This together with the approximation properties (6.3) and (6.5) yields the asymptotic error estimate:

Corollary 6 *There exist constants c_1, c_2 which are independent of H and h' such that*

$$
\begin{aligned}
&\|\mathbf{u}-\mathbf{u}^H\|_{\mathcal{H}} + \sum_{j=1}^{M} \|T_j\mathbf{u} - \boldsymbol{\lambda}_j^{h'}\|_{\widehat{H}^{-\frac{1}{2}}(\partial\omega_j\cap\Upsilon)} \\
\leq\ & c_1 \inf_{\mathbf{v}^H\in\mathcal{H}_H} \|\mathbf{u}-\mathbf{v}^H\|_{\mathcal{H}} + c_2 \inf_{\boldsymbol{\chi}^{h'}\in\mathcal{H}_{h'}^{-\frac{1}{2}}} \sum_{j=1}^{M} \|T_j\mathbf{u} - \boldsymbol{\chi}^{h'}\|_{H^{-\frac{1}{2}}(\partial\omega_j)} \\
\leq\ & c_1{}'\, H^{t-1}\|\mathbf{u}\|_{H^t(\Omega)} + c_2{}'\, h'^{s+\frac{1}{2}}\|\mathbf{u}\|_{H^{s+3/2}(\Omega)}
\end{aligned}
\tag{7.8}
$$

where $1 \leq t \leq d$ and $-\frac{1}{2} \leq s \leq d''$.

We note that in the estimate (7.8), the error contribution $(T_j\mathbf{u} - \boldsymbol{\lambda}_j^{h'})$ is only available on $\partial\omega_j \cap \Upsilon$. For the estimate of the remaining part and for the estimate of $\|\mathbf{u}_j^h - \mathbf{u}\|_{H^{\frac{1}{2}}(\partial\omega_j)}$ we present the following results.

Corollary 7 *There exist constants c_1, c_2, c_3 which are independent of H, h and h', such that*

$$
\begin{aligned}
&\|\mathbf{u}-\mathbf{u}^H\|_{\mathcal{H}} + \sum_{j=1}^{M} \Big(\|T_j\mathbf{u} - \boldsymbol{\lambda}_j^{h'}\|_{H^{-\frac{1}{2}}(\partial\omega_j)} + \|\mathbf{u}-\mathbf{u}_j^h\|_{H^{\frac{1}{2}}(\partial\omega_j)} \Big) \\
\leq\ & c_1 \inf_{\mathbf{v}^H\in\mathcal{H}_H} \|\mathbf{u}-\mathbf{v}^H\|_{\mathcal{H}} + c_2 \sum_{j=1}^{M} \inf_{\boldsymbol{\mu}^h\in\mathcal{H}_h^{\frac{1}{2}}} \|\mathbf{u}-\boldsymbol{\mu}^h\|_{H^{\frac{1}{2}}(\partial\omega_j)} \\
&\quad + c_3 \sum_{j=1}^{M} \inf_{\boldsymbol{\chi}^{h'}\in\mathcal{H}_{h'}^{-\frac{1}{2}}} \|T_j\mathbf{u} - \boldsymbol{\chi}^{h'}\|_{H^{-\frac{1}{2}}(\partial\omega_j)} \\
\leq\ & c_1{}'\, H^{t-1}\|\mathbf{u}\|_{H^t(\Omega)+} + c_2{}'\, h'^{s'-\frac{1}{2}}\|\mathbf{u}\|_{H^{s'+\frac{1}{2}}(\Omega)} + c_3{}'\, h^{s+\frac{1}{2}}\|\mathbf{u}\|_{H^{s+\frac{3}{2}}(\Omega)}
\end{aligned}
\tag{7.9}
$$

where $1 \leq t \leq d$, $\frac{1}{2} \leq s' \leq d'$ and $-\frac{1}{2} \leq s \leq d''$, which becomes the right–hand side of (7.8) *if we can choose $d' = d''+1$ and $s' = s+1$.*

Proof: The proposed estimate will be a consequence of the stability of $\boldsymbol{\lambda}_j^h$ on the whole $\partial\omega_j$. For simplicity, in what follows, the subscript j will be suppressed. We recall that $\boldsymbol{\lambda}^h$ is defined by (6.31), (6.35)–(6.37) where U^h is given in matrix form by (6.30). This is equivalent to the Galerkin equations

$$
\langle U^{h'}\boldsymbol{\lambda}^{h'}, \boldsymbol{\chi}_0^{h'}\rangle = \langle \boldsymbol{\varphi}, \boldsymbol{\chi}_0^{h'}\rangle \text{ for all } \boldsymbol{\chi}_0^{h} \in \widetilde{H}_{h'}^{-\frac{1}{2}}(\partial\omega_D)\,.
$$

From $\boldsymbol{\lambda}^{h'} \in \widehat{\mathcal{H}}_{h'}^{-\frac{1}{2}}(\partial\omega\cap\Upsilon)$ we now take the prolongation

$$
\wp_\nu(\boldsymbol{\lambda}^{h'}|_{\partial\omega\cap\Upsilon}) \in \widetilde{\mathcal{H}}_{h'}^{-\frac{1}{2}}\Big(\partial\omega \setminus (\Gamma_N\setminus\Upsilon)\Big) \subset \mathcal{H}_{h'}^{-\frac{1}{2}}(\partial\omega)
$$

and obtain for

$$
\boldsymbol{\lambda}_0^{h'} := \boldsymbol{\lambda}^{h'} - \wp_\nu(\boldsymbol{\lambda}^{h'}|_{\partial\omega\cap\Upsilon}) \in \widetilde{\mathcal{H}}_{h'}^{-\frac{1}{2}}(\partial\omega_D)
$$

the Galerkin equations

$$
\langle U^{h'}\boldsymbol{\lambda}_0^{h'}, \boldsymbol{\chi}_0^{h'}\rangle = \langle \boldsymbol{\varphi} - U^{h'}\wp_\nu(\boldsymbol{\lambda}^{h'}|_{\partial\omega\cap\Upsilon}), \boldsymbol{\chi}_0^{h'}\rangle\,.
$$

The definition of $U^{h'}$ via (6.30), (6.31) and based on (4.32), together with the coerciveness of V in (4.18) yields with $\boldsymbol{\chi}_0^{h'} = \boldsymbol{\lambda}_0^{h'}$ the inequality

$$\alpha_0 \|\boldsymbol{\lambda}_0^{h'}\|^2_{H_0^{-\frac{1}{2}}(\partial\omega)} \le c \left\{ \|\boldsymbol{\varphi}\|_{H^{\frac{1}{2}}(\partial\omega_D)} + \|U^{h'} \wp_\nu (\boldsymbol{\lambda}^{h'}|_{\partial\omega\cap\Upsilon})\|_{H^{\frac{1}{2}}(\partial\omega)} \right\} \|\boldsymbol{\lambda}_0^{h'}\|_{H_0^{-\frac{1}{2}}(\partial\omega)} .$$

Hence, with the realtion between $\boldsymbol{\lambda}_0^{h'}$ and $\boldsymbol{\lambda}^{h'}$ and the prolongation assumption (6.8) we find the stability estimate

$$\|\boldsymbol{\lambda}^{h'}\|_{H^{-\frac{1}{2}}(\partial\omega)} \le c \left(\|\boldsymbol{\varphi}\|_{H^{\frac{1}{2}}(\partial\omega_D)} + \|\boldsymbol{\lambda}^{h'}\|_{\widehat{H}^{-\frac{1}{2}}(\partial\omega\cap\Upsilon)} \right) . \tag{7.10}$$

Based on this estimate, we also obtain stability for $\mathbf{u}^h = \overset{\circ}{\mathbf{u}}{}^h + \mathbf{r}^*$ defined by (6.38)–(6.40):

$$\begin{aligned}
\|\overset{\circ}{\mathbf{u}}{}^h\|_{H^{\frac{1}{2}}(\partial\omega)} &= \|\overset{\circ}{P}_h U^{h'} \boldsymbol{\lambda}^{h'}\|_{H^{\frac{1}{2}}(\partial\omega)} \\
&\le c\|\boldsymbol{\lambda}^{h'}\|_{H^{-\frac{1}{2}}(\partial\omega)} \le c \left(\|\boldsymbol{\varphi}\|_{H^{\frac{1}{2}}(\partial\omega_D)} + \|\boldsymbol{\lambda}^{h'}\|_{\widehat{H}^{\frac{1}{2}}(\partial\omega\cap\Upsilon)} \right) , \qquad (7.11) \\
\|\mathbf{r}^*\|_{H^{\frac{1}{2}}(\partial\omega)} &\le c \left\{ \|\boldsymbol{\varphi}\|_{H^{\frac{1}{2}}(\partial\omega)} + \|\boldsymbol{\psi}\|_{H^{-\frac{1}{2}}(\partial\omega)} + \|\overset{\circ}{\mathbf{u}}{}^h\|_{H^{\frac{1}{2}}(\partial\omega)} \right\}
\end{aligned}$$

Here $\overset{\circ}{P}_h$ denotes the L^2–projection onto $\mathcal{H}_{0h}^{\frac{1}{2}}(\partial\omega)$ satisfying (6.6). Hence, we finally obtain from (7.10) and (7.11) the stability estimate

$$\|\mathbf{u}^h\|_{H^{\frac{1}{2}}(\partial\omega)} \le c\|\mathbf{u}\|_{H^1(\Omega)} .$$

Collecting the stability estimates for $\mathbf{u}^H, \boldsymbol{\lambda}^{h'}$ and $\mathbf{u}^h$, we see that the family of linear Galerkin projectors $H^1(\Omega) \ni \mathbf{u} \mapsto (\mathbf{u}^H, \mathbf{u}_j^h, \boldsymbol{\lambda}_j^{h'}) \in \mathcal{H}_H \times \prod_{j=1}^M \left(\mathcal{H}_{jh}^{\frac{1}{2}}(\partial\omega_j) \times \mathcal{H}_{jh'}^{-\frac{1}{2}}(\partial\omega_j) \right)$ is uniformly bounded. Then Cea's lemma follows and the appoximation properties give the desired asymptotic estimates [7,II.2.4]. □

Remark 7.1: For the computations we have employed the Neumann series approach which, in general, does not provide stability and convergence without additional restrictions on the relation between the mesh sizes H, h and h'.

In addition, in our actual computations and also in [58], the convergence of the Neumann series is controlled numerically for fixed H. Then, by using the spectrum of the matrix $((\langle \widehat{S}_j^k(\varphi_\ell), \varphi_k \rangle_j))$ or of $\mathsf{B}_j^\top \widehat{\mathsf{U}}_j^{-1} \widehat{\mathsf{B}}_j$ as an indicator, the mesh size h is adapted in order to ensure the condition (7.1).

7.1.2 Neumann bases on restricted grids

In what follows, we give a rigorous proof of asymptotic stability and convergence in case of utilizing the boundary integral equation of the second kind (6.21) by (6.25) or the Neumann series approach (6.27) if the grids satisfy appropriate **additional assumptions**. In this case, our analysis is based on the coerciveness condition (7.1).

More precisely, as we shall see, our several **additional mesh restrictions** are the

(CF) coarse–fine grid relation

$$h \le h' \le c_0 H , \tag{7.12}$$

(with a constant c_0 to be chosen small enough) the inverse assumption (6.4), the restrictions **(R1)** and **(R2)** (6.74) and the condition **(B1)** (4.24) for the Neumann series.

In this case, as in [61], the asymptotic error analysis is based on an auxiliary problem associated with the coarse grid approximation $\mathbf{u}^H$ given on $\Omega_F \cup \Upsilon$ with $\mathbf{u}^H = \boldsymbol{\varphi}$ on $\Gamma_D \cap \Upsilon$. We define an auxiliary function $\widehat{\mathbf{u}}^H$ in the following way:

$$\widehat{\mathbf{u}}^H := \mathbf{u}^H \ \text{ on } \Omega_F$$

whereas on each of the macroelements ω_j we determine $\widehat{\mathbf{u}}^H$ by solving the mixed boundary value problem

$$\begin{array}{rcll} \mathcal{P}\widehat{\mathbf{u}}_j^H & = & 0 & \text{in } \omega_j\,; \\ \widehat{\mathbf{u}}_j^H & = & \mathbf{u}^H & \text{on } \partial\omega_j \cap \Upsilon \ \text{ and } \widehat{\mathbf{u}}_j^H = \boldsymbol{\varphi} \ \text{ on } \partial\omega_{jD}\,; \\ T\widehat{\mathbf{u}}_j^H & = & \boldsymbol{\psi} & \text{on } \partial\omega_{jN} \setminus \Upsilon\,; \end{array} \tag{7.13}$$

i. e. $\widehat{\mathbf{u}}_j^H$ is the $\mathcal{P}$–harmonic extension into ω_j.

Then $\widehat{\mathbf{u}}^H \in H^1(\Omega) \cap C^0(\overline{\Omega})$ and the familiy of auxiliary functions form a family of generalized conforming finite elements associated with the coarse grid approximation. We denote this function space by $\widehat{\mathcal{H}}_H$. The corresponding test function space is then defined by

$$\begin{aligned} \widehat{\mathcal{H}}_{HD} \ := \ & \Big\{ \widehat{\mathbf{w}}_0^H \in H^1(\Omega) \cap C^0(\overline{\Omega}) \Big| \widehat{\mathbf{w}}_0^H = 0 \ \text{ on } \Gamma_D \cap \Upsilon\,,\ \widehat{\mathbf{w}}^H_{0|_{\Omega_F}} = \mathbf{w}_0^H \in H^1_H(\Omega_F)\,, \\ & \mathcal{P}\widehat{\mathbf{w}}_0^H = \mathbf{0} \ \text{ in } \omega_j\,,\ \widehat{\mathbf{w}}_0^H|_{\partial\omega_j} = \mathbf{w}^H_{0\,|_{\partial\omega_j}}\,, \widehat{\mathbf{w}}_0^H|_{\partial\omega_{jD}} = \mathbf{0}\,,\ T_j\widehat{\mathbf{w}}_0^H = \mathbf{0} \ \text{ on } \partial\omega_{jN} \setminus \Upsilon\,. \Big\} \end{aligned} \tag{7.14}$$

By using $\widehat{\mathbf{u}}^H$ and the true solution $\mathbf{u}$ we now may rewrite equations (6.13) and (6.14) as modified standard Galerkin FE equations:

$$\begin{aligned} a_\Omega(\widehat{\mathbf{u}}^H, \widehat{\mathbf{v}}_0^H) &= a_F(\widehat{\mathbf{u}}^H, \widehat{\mathbf{v}}_0^H) + \sum_{j=1}^M a_{\omega_j}(\widehat{\mathbf{u}}^H, \widehat{\mathbf{v}}_0^H) \\ &= a_F(\mathbf{u}^H, \mathbf{v}_0^H) + \sum_{j=1}^M \oint_{\partial\omega_j} \widehat{\mathbf{v}}_0^H \cdot T_j\widehat{\mathbf{u}}^H ds \\ &= a_F(\mathbf{u}^H, \mathbf{v}_0^H) + \sum_{j=1}^M \oint_{\partial\omega_j} \widehat{\mathbf{v}}_0^H \cdot \boldsymbol{\lambda}_j^{h\,\prime} ds + \sum_{j=1}^M \oint_{\partial\omega_j} \widehat{\mathbf{v}}_0^H \cdot (T_j\widehat{\mathbf{u}}^H - \boldsymbol{\lambda}_j^{h\,\prime}) ds \\ &= a_F(\mathbf{u}, \mathbf{v}_0^H) + \sum_{j=1}^M \oint_{\partial\omega_j} \widehat{\mathbf{v}}_0^H \cdot T_j\mathbf{u} ds + \sum_{j=1}^M \oint_{\partial\omega_j} \widehat{\mathbf{v}}_0^H \cdot (T_j\widehat{\mathbf{u}}^H - \boldsymbol{\lambda}_j^{h\,\prime}) ds\,, \end{aligned}$$

i. e.

$$a_\Omega(\widehat{\mathbf{u}}^H, \widehat{\mathbf{v}}_0^H) = a_\Omega(\mathbf{u}, \widehat{\mathbf{v}}_0^H) + \sum_{j=1}^M \oint_{\partial\omega_j} \widehat{\mathbf{v}}_0^H \cdot (T_j\widehat{\mathbf{u}}^H - \boldsymbol{\lambda}_j^{h\,\prime}) ds \ \text{ for all } \widehat{\mathbf{v}}_0^H \in \widehat{\mathcal{H}}_{HD}\,. \tag{7.15}$$

With $\widehat{\mathbf{u}}^H = \widehat{\boldsymbol{\varphi}}^{*H} + \widehat{\mathbf{u}}_0^H$ where $\widehat{\mathbf{u}}_0^H \in \widehat{\mathcal{H}}_{HD}$, the $H^1_D(\Omega)$–ellipticity of a_Ω yields for $\widehat{\mathbf{v}}_0^H = \widehat{\mathbf{u}}_0^H$ the inequality

$$\begin{aligned} \alpha_0 \|\widehat{\mathbf{u}}_0^H\|^2_{H^1(\Omega)} \ \leq \ & c_1(\|\widehat{\boldsymbol{\varphi}}^{*H}\|_{H^1(\Omega)} + \|\mathbf{u}\|_{H^1(\Omega)})\|\widehat{\mathbf{u}}_0^H\|_{H^1(\Omega)} \\ & + \sum_{j=1}^M c_{2j} \|T_j\widehat{\mathbf{u}}^H - \boldsymbol{\lambda}_j^{h\,\prime}\|_{H^{-\frac12}(\partial\omega_j)} \|\widehat{\mathbf{u}}_0^H\|_{H^{\frac12}(\partial\omega_j)}\,. \end{aligned} \tag{7.16}$$

Next, we need to estimate the last terms on the right–hand side.

Lemma 8 *Let* $\mathbf{u}^H$ *and* $\boldsymbol{\lambda}_j^{h'}$ *be the solution of* (6.13), (6.14) *with* $\widehat{\mathsf{U}}_j$ *realized via* (6.25) *and* (6.37). *Let* δ *be chosen with* $0<\delta<\frac{1}{2}$. *Then*

$$\|T_j\widehat{\mathbf{u}}^H-\boldsymbol{\lambda}_j^{h'}\|_{H^{-\frac{1}{2}}(\partial\omega_j)}\le c_j\|P_{\Re_j}\widehat{\mathbf{u}}^H-P_{\Re_j}^h\widehat{\mathbf{u}}^H\|_{H^{\frac{1}{2}}(\partial\omega_j)}+c_j{}'\,h'^{\delta}\|T_j\widehat{\mathbf{u}}^H\|_{H^{-\frac{1}{2}+\delta}(\partial\omega_j)}\,. \tag{7.17}$$

Proof: Since the proof concerns only ω_j, we suppress the index j. We begin with the case $\partial\overset{\circ}{\omega}_D=\emptyset$. Similar to (7.15) replacing (6.13), the equations of weak coupling (6.14) are equivalent to

$$\langle\boldsymbol{\chi}^{h'},\widehat{\mathbf{u}}^H-\mathbf{u}^h\rangle=0\ \text{ for all }\boldsymbol{\chi}^{h'}\in\mathcal{H}_{h'}^{-\frac{1}{2}}(\partial\omega)$$

with $\mathbf{u}^h\in\operatorname{span}\{\overset{\circ}{\boldsymbol{\mu}}_\alpha\}$, $\alpha\in\mathcal{I}_{0D}$. Hence, with (6.23),

$$\mathbf{u}^h=P_{\Re}^h\widehat{\mathbf{u}}^H\,.$$

Since $\widehat{\mathbf{u}}^H$ in ω is the $\mathcal{P}$–harmonic extension, the boundary integral equation

$$P_{\Re}(\tfrac{1}{2}I+K)P_{\Re}\widehat{\mathbf{u}}^H=P_{\Re}V(T\widehat{\mathbf{u}}^H)\ \text{ on }\partial\omega$$

is satisfied, whereas $\mathbf{u}^h$ satisfies the Galerkin equations

$$\langle\boldsymbol{\chi}^{h'},\mathbf{u}^h-P_{\Re}(\tfrac{1}{2}I-K)\mathbf{u}^h\rangle=\langle\boldsymbol{\chi}^{h'},V\boldsymbol{\lambda}^{h'}\rangle$$

corresponding to (6.21) and (6.25). Hence,

$$\langle\boldsymbol{\chi}^{h'},\{(\mathbf{u}^h-P_{\Re}\widehat{\mathbf{u}}^H)-(\tfrac{1}{2}I-K)(\mathbf{u}^h-P_{\Re}\widehat{\mathbf{u}}^H)\}\rangle=\langle\boldsymbol{\chi}^{h'},V(\boldsymbol{\lambda}^{h'}-T\widehat{\mathbf{u}}^H)\rangle\,, \tag{7.18}$$

which is equivalent to

$$\langle\boldsymbol{\chi}^{h'},V\boldsymbol{\lambda}^{h'}\rangle=\langle\boldsymbol{\chi}^{h'},V\{T\widehat{\mathbf{u}}^H-V^{-1}\{(\mathbf{u}^h-P_{\Re}\widehat{\mathbf{u}}^H)-(\tfrac{1}{2}I-K)(\mathbf{u}^h-P_{\Re}\widehat{\mathbf{u}}^H)\}\}\rangle\,. \tag{7.19}$$

Then the asymptotic error estimates for Galerkin's method with V on $\mathcal{H}_{h'}^{-\frac{1}{2}}$ yield the estimate (7.17).

If $\partial\overset{\circ}{\omega}_D\neq\emptyset$ then $(\mathbf{u}^h-P_{\Re}^h\widehat{\mathbf{u}}^H)_{|\partial\omega_D}=\mathbf{0}$ and $\mathbf{u}^h,\boldsymbol{\lambda}^{h'}$ are related by the Galerkin solution of the mixed boundary value problem. Therefore, the equations (7.18) are still satisfied, not only on $\partial\omega_N$ but also on $\partial\omega_D$. Then (7.17) follows in the same manner. □

Corollary 9 *If under condition* **(B1)** *the Neumann series* (6.27) *with* (6.28) *is used for utilizing* $\widehat{\mathsf{U}}$ *then there exists* $h_0>0$ *and to every* $\varepsilon>0$ *there exists* $r_0(\varepsilon)\in\mathbb{N}$ *such that*

$$\|(U_j^{h'(r)}-U_j^{h'})\boldsymbol{\lambda}_j^{h'}\|_{H^{\frac{1}{2}}(\partial\omega_j)}\le\varepsilon\|\boldsymbol{\lambda}_j^{h'}\|_{H^{-\frac{1}{2}}(\partial\omega_j)} \tag{7.20}$$

for all $r\ge r_0$ *and all* $0<h'\le h_0$. *Moreover, then, with* $\delta(0,\frac{1}{2})$ *from Lemma 8,*

$$\begin{aligned}&\|T_j\widehat{\mathbf{u}}^H-\boldsymbol{\lambda}_j^{h'}\|_{H_0^{-\frac{1}{2}}(\partial\omega_j)} \\ \le\ & c_j\|P_{\Re_j}\widehat{\mathbf{u}}^H-P_{\Re_j}^h\widehat{\mathbf{u}}^H\|_{H^{\frac{1}{2}}(\partial\omega_j)}+c_jh'^{\delta}\|T_j\widehat{\mathbf{u}}^H\|_{H^{-\frac{1}{2}+\delta}(\partial\omega_j)}+c_j{}'\,\varepsilon\|P_{\Re_j}^h\widehat{\mathbf{u}}^H\|_{H^{\frac{1}{2}}(\partial\omega_j)}\end{aligned} \tag{7.21}$$

where c_j *and* $c_j{}'$ *are independent of* h,h',H,ε *(but may depend on* ω_j*).*

Proof: Again, we omit j. First we consider the case $\partial\overset{\circ}{\omega}_D = \emptyset$. Let $\varrho < 1$ be the spectral radius of $(\frac{1}{2}I - K)$. We first show, that the Galerkin equations (6.24), (6.25) define a so–called a–proper approximation family $U^{h'}$ to U.

Definition: The family of operators $A^h : X \to Y$ approximating $A : X \to Y$ is called **a–proper** (approximation–proper) if every bounded sequence $\{\Lambda^h\} \subset X$ with $\lim_{h\to 0} A^h\Lambda^h = v \in Y$ contains a convergent subsequence $\Lambda^{\widetilde{h}} \to \Lambda$ and the limit Λ satisfies $A\Lambda = v$.

If $U^{h'}\boldsymbol{\lambda}^{h'} = \mathbf{w}^h \in \mathcal{H}_h^{\frac{1}{2}}$ we choose to $\mathbf{w} = \lim_{h'\to 0}\mathbf{w}^h$ in $H_0^{\frac{1}{2}}(\partial\omega)$ the element $\boldsymbol{\lambda} := S\mathbf{w}$. Then, by definition of $U^{h'}$, we have the Galerkin equations

$$\langle \mathbf{v}_\iota, \mathbf{w}^h - (\tfrac{1}{2}I - K)\mathbf{w}^h\rangle \;=\; \langle \mathbf{v}_\iota, V\boldsymbol{\lambda}^{h'}\rangle \text{ for all } \iota \in \mathcal{I}_N$$

and

$$\mathbf{w} - (\tfrac{1}{2}I - K)\mathbf{w} \;=\; V\boldsymbol{\lambda}\,.$$

Hence, from the Galerkin approximation with V we obtain

$$\|\boldsymbol{\lambda}^{h'} - \boldsymbol{\lambda}\|_{H^{-\frac{1}{2}}(\partial\omega)} \le c_1\|(I - (P_{h'}VP_{h'})^{-1}P_{h'}V)\boldsymbol{\lambda}\|_{H^{-\frac{1}{2}}(\partial\omega)} + c_2\|\mathbf{w} - \mathbf{w}^h\|_{H^{\frac{1}{2}}(\partial\omega)}$$

and $\boldsymbol{\lambda}^{h'} \to \boldsymbol{\lambda}$ for $h' \to 0$. Then, the family $U^{h'}$ is here a–proper for U. In this case it is known that for $h' \to 0$ the spectral sets converge [29]. Hence, there exists $h_0 > 0$ and $0 < \varrho_0 < 1$, such that for all $0 < h' \le h_0$ the spectral radii of $\{(\frac{1}{2}\overset{\circ}{\mathsf{M}} - \overset{\circ}{\mathsf{K}})\overset{\circ}{\mathsf{M}}^{+}\}$ can uniformly be bounded by ϱ_0 [29]. As a consequence,

$$\|U^{h'(r)} - U^{h'}\|_{H^{-\frac{1}{2}}(\partial\omega_j)\to H^{\frac{1}{2}}(\partial\omega_j)} \le c\varrho_0^r \tag{7.22}$$

where c is independent of r and h' and h. Since U is invertible on $H^{-\frac{1}{2}}(\partial\omega)/\Re$, so is $U^{(r)}$ for $r \ge r_0$ and (7.22) implies a uniform bound

$$\|(U^{h'(r)})^{-1}\|_{H^{\frac{1}{2}}(\partial\omega_j)\to H^{-\frac{1}{2}}(\partial\omega_j)} \le M\,.$$

Since $U^{h'(r)}$ is used instead of $U^{h'}$, here the weak coupling equation (6.14) reads

$$U^{h'(r)}\boldsymbol{\lambda}^{h'} = P_\Re^h\widehat{\mathbf{u}}^H\,;\quad \text{and } U^{h'}\boldsymbol{\lambda}^{h'} = P_\Re^h\widehat{\mathbf{u}}^H + (U^{h'} - U^{h'(r)})\boldsymbol{\lambda}^{h'}$$

implies

$$\langle \boldsymbol{\chi}^{h'}, V(\boldsymbol{\lambda}^{h'} - P_\Re^h T_j\widehat{\mathbf{u}}^H)\rangle = \langle \boldsymbol{\chi}^{h'}, (\tfrac{1}{2}I + K)(P_\Re^h - P_\Re)\widehat{\mathbf{u}}^H + (U^{h'} - U^{h'(r)})\boldsymbol{\lambda}^{h'}\rangle$$

for all $\boldsymbol{\lambda}^{h'} \in \mathcal{H}_{h'}^{-\frac{1}{2}}(\partial\omega)$. These Galerkin equations, together with continuity yield the estimate

$$\begin{aligned} &\|T\widehat{\mathbf{u}}^H - \boldsymbol{\lambda}^h\|_{H^{-\frac{1}{2}}(\partial\omega)} \\ &\le\; c\|P_\Re\widehat{\mathbf{u}}^H - P_\Re^h\widehat{\mathbf{u}}^H\|_{H^{\frac{1}{2}}(\partial\omega)} + ch'^{\delta}\|T\widehat{\mathbf{u}}^H\|_{H^{-\frac{1}{2}+\delta}(\partial\omega)} + c'\,\varrho_0^r\|\boldsymbol{\lambda}^{h'}\|_{H^{-\frac{1}{2}}(\partial\omega)}\,. \end{aligned}$$

Choose r_0 such that $\varrho_0^{r_0}\cdot M < \varepsilon$ and use $\|\boldsymbol{\lambda}^{h'}\|_{H^{-\frac{1}{2}}(\partial\omega)} \le M\|P_\Re^h\widehat{\mathbf{u}}\|_{H^{\frac{1}{2}}(\partial\omega)}$ to obtain (7.21).

For $\partial\overset{\circ}{\omega}_D \ne \emptyset$, the proposition follows in the same manner as in Lemma 8. □

In the Lemma 8 and the Corollary 9 we see that the term $P_{\Re_j}\widehat{\mathbf{u}}^H - P_{\Re_j}^h\widehat{\mathbf{u}}^H$ appears in all the estimates. But as we have seen in Lemma 3 this term vanishes under the rather mild mesh restrictions **(R1)**, **(R2)** in (6.74).

Theorem 10 *Assume that the assumptions of Lemma 8 (and Corollary 9 for the Neumann series approach) and, in addition, the mesh restrictions* **(R1)**, **(R2)** *are satisfied. Then we have the asymptotic stability estimate*

$$\|\mathbf{u}^H\|_{\mathcal{H}} + \sum_{j=1}^{M} \left(\|\boldsymbol{\lambda}_j^{h'}\|_{H^{-\frac{1}{2}}(\partial\omega_j)} + \|\mathbf{u}_j^h\|_{H^{\frac{1}{2}}(\partial\omega_j)} \right) \le c\|\mathbf{u}\|_{H^1(\Omega)} \tag{7.23}$$

for all $0 < h' \le h_0$ *with a constant* c *independent of* h', H *provided the constant* $c_0 > 0$ *in the coarse–fine grid relation* **(CF)** *in* (7.12) *is sufficiently small.*

Under the above conditions, the asymptotic error estimates in the Corollaries 6 and 7 are valid again.

Proof: We begin with the estimate (7.16) for the $\mathcal{P}$–harmonic extension $\widehat{\mathbf{u}}_0^H = \widehat{\mathbf{u}}^H - \widehat{\boldsymbol{\varphi}}^{*H}$ and make use of (7.17) (or (7.21) in the case of the Neumann series–approach). This yields

$$\begin{aligned}\alpha_0\|\widehat{\mathbf{u}}_0^H\|^2_{H^1(\Omega)} \le \; & c_1 \left(\|\widehat{\boldsymbol{\varphi}}^{*H}\|_{H^1(\Omega)} + \|\mathbf{u}\|_{H^1(\Omega)} \right) \|\widehat{\mathbf{u}}_0^H\|_{H^1(\Omega)} \\ & + \sum_{j=1}^{M} c_j ({h'}^{\delta} \|T_j\widehat{\mathbf{u}}^H\|_{H^{-\frac{1}{2}+\delta}(\partial\omega_j)} + \varepsilon\|P_{\Re}\widehat{\mathbf{u}}^H\|_{H^{\frac{1}{2}}(\partial\omega_j)})\|\widehat{\mathbf{u}}_0^H|_{\partial\omega_j}\|_{H^{\frac{1}{2}}(\partial\omega_j)} \,.\end{aligned} \tag{7.24}$$

By using the inverse assumption (6.4) on $\partial\omega_j$ together with the coarse–fine grid relation **(CF)** in (7.12), we obtain the estimate

$${h'}^{\delta}\|T_j\widehat{\mathbf{u}}^H\|_{H^{-\frac{1}{2}+\delta}(\partial\omega_j)} \le c_j c_0^{\delta}\|\widehat{\mathbf{u}}^H\|_{H^1(\partial\omega_j)} \,. \tag{7.25}$$

From (7.25) and (7.24) we now conclude

$$\begin{aligned}&\alpha_0\|\widehat{\mathbf{u}}_0^H\|^2_{H^1(\Omega)} \le c_1 \left(\|\widehat{\boldsymbol{\varphi}}^{*H}\|_{H^1(\Omega)} + \|\mathbf{u}\|_{H^1(\Omega)} \right) \|\widehat{\mathbf{u}}_0^H\|_{H^1(\Omega)} \\ &+ (c_0^{\delta} + \varepsilon){c_1}' \sum_{j=1}^{M} \|\widehat{\mathbf{u}}_0^H\|^2_{H^1(\partial\omega_j)} + c_2(c_0^{\delta} + \varepsilon)\|\widehat{\boldsymbol{\varphi}}^{*H}\|_{H^1(\Omega)}\|\widehat{\mathbf{u}}_0^H\|_{H^1(\Omega)} \,.\end{aligned}$$

We now require c_0 and ε sufficiently small so that $c_0 + \varepsilon \le \frac{1}{2}\alpha_0$. Then we obtain the desired stability result for $\|\widehat{\mathbf{u}}_0^H\|_{H^1(\Omega)}$ and, hence, for $\|\widehat{\mathbf{u}}^H\|_{H^1(\Omega)}$. Since $\widehat{\mathbf{u}}^H$ is the harmonic extension of $\mathbf{u}^H$, we obtain from (7.13) the stability estimate (7.23) for $\|\mathbf{u}^H\|_{\mathcal{H}}$,i. e.

$$\|\mathbf{u}^H\|_{\mathcal{H}} \le c\|\mathbf{u}\|_{H^1(\Omega)} \,.$$

For the local $\boldsymbol{\lambda}_j^{h'}$ we use (7.19) where $\mathbf{u}_j^h = P_{\Re_j}\widehat{\mathbf{u}}^H$ due to **(R1)**. This implies

$$\|\boldsymbol{\lambda}_j^{h'}\|_{H^{-\frac{1}{2}}(\partial\omega_j)} \le c_j\|T_j\widehat{\mathbf{u}}^H\|_{H^{-\frac{1}{2}}(\partial\omega_j)} \le {c_j}'\|\widehat{\mathbf{u}}^H\|_{H^1(\omega_j)}$$

and

$$\|\mathbf{u}_j^h\|_{H^{\frac{1}{2}}(\partial\omega_j)} \le c_j\|\widehat{\mathbf{u}}^H\|_{H^1(\omega_j)} \,.$$

Collecting these inequalities yields the final estimate (7.23). □

7.2 Stability and convergence with local Dirichlet bases

7.2.1 Coercive approximate Steklov–Poincaré mappings

In contrast to Section 7.1.1 we show now that the construction of $\widehat{S}^h$ in (6.67) and (6.72) provides discrete $H^{\frac{1}{2}}(\partial\omega_j)/\Re$ or $\widehat{H}^{\frac{1}{2}}(\partial\omega_j \cap \Upsilon)$–ellipticity. Again, we do **not need any additional restriction** on the grids — as in Section 7.1.1.

Lemma 11 *If* S *is defined by the discrete symmetric approximation* (6.67) *then* S^h *is* $H^{\frac{1}{2}}(\partial\omega)/\Re$*–elliptic,i. e.*

$$\int\limits_{\partial\omega} S^h(\mathbf{w}^h)\cdot\mathbf{w}^h ds \geq \gamma_0 \|\mathbf{w}^h\|^2_{H^{\frac{1}{2}}(\partial\omega)/\Re} \quad \textit{for all}\, \mathbf{w}^h \in \mathcal{H}_h^{\frac{1}{2}}(\partial\omega)\,. \tag{7.26}$$

If, in addition, $\partial\overset{\circ}{\omega}_{jN} \neq \emptyset$ *and* $\widehat{S}^h$ *is defined via* (6.72) *then* $\widehat{S}^h$ *is* $\widehat{H}^{\frac{1}{2}}(\partial\omega\cap\Upsilon)$*–elliptic, i. e.*

$$\int\limits_{\partial\omega} \widehat{S}^h(\mathbf{w}^h,\mathbf{0})\cdot\mathbf{w}^h ds \geq \gamma_0 \|\mathbf{w}^h\|^2_{\widehat{H}^{\frac{1}{2}}(\partial\omega\cap\Upsilon)}\,. \tag{7.27}$$

Moreover, S^h *as well as* $\widehat{S}^h$ *are uniformely bounded, i. e.*

$$\|S^h\mathbf{w}\|_{H^{-\frac{1}{2}}(\partial\omega)} \leq c\|\mathbf{w}\|_{H^{\frac{1}{2}}(\partial\omega)} \tag{7.28}$$

or

$$\|\widehat{S}^h(\mathbf{w},\boldsymbol{\psi})\|_{H^{-\frac{1}{2}}(\partial\omega)} \leq c\left\{\|\mathbf{w}\|_{H^{\frac{1}{2}}(\partial\omega\cap\Upsilon)} + \|\boldsymbol{\psi}\|_{H^{-\frac{1}{2}}(\partial\omega\cap\Gamma_N\setminus\Upsilon)}\right\} \tag{7.29}$$

respectively, where the constant c *is independent of* h.

Proof: In the case $\partial\overset{\circ}{\omega}_N \neq \emptyset$, for (7.26), it follows from (6.67) that

$$\int\limits_{\partial\omega} S^h(\mathbf{w}^h)\cdot\mathbf{w}^h ds \geq \int\limits_{\partial\omega} \mathbf{w}^h\cdot D\mathbf{w}^h ds \geq \gamma_0\|\mathbf{w}^h\|^2_{H^{\frac{1}{2}}(\partial\omega)/\Re}\,,$$

because of (4.29).

Let $\boldsymbol{\lambda}^{h'}$ be given by the solution of

$$\langle \boldsymbol{\chi}^{h'}, V\boldsymbol{\lambda}^{h'}\rangle = \langle \boldsymbol{\chi}^{h'}, (\tfrac{1}{2}I + K)\mathbf{w}\rangle\,.$$

Since $S\mathbf{w}$ satisfies

$$\langle \boldsymbol{\chi}^{h'}, VS\mathbf{w}\rangle = \langle \boldsymbol{\chi}^{h'}, (\tfrac{1}{2}I + K)\mathbf{w}\rangle\,,$$

and the Galerkin method for the operator V is stable (cf. (4.18)) we find

$$\|\boldsymbol{\lambda}^{h'} - S\mathbf{w}\|_{H^{-\frac{1}{2}}(\partial\omega)} \leq c\|S\mathbf{w}\|_{H^{-\frac{1}{2}}(\partial\omega)} \leq c'\,\|\mathbf{w}\|_{H^{\frac{1}{2}}(\partial\omega)}$$

where c' is independent of h'. The definition of S^h via (6.67) means

$$S^h\mathbf{w} = P_h(I + K')\boldsymbol{\lambda}^{h'} + P_h D\mathbf{w}\,.$$

Hence, with (6.6) and the triangle inequality, we obtain

$$\begin{aligned}\|S^h\mathbf{w}\|_{H^{-\frac{1}{2}}(\partial\omega)} &\leq c\|\boldsymbol{\lambda}^{h'}\|_{H^{-\frac{1}{2}}(\partial\omega)} + c'\,\|\mathbf{w}\|_{H^{\frac{1}{2}}(\partial\omega)}\\ &\leq c''\|S^h\mathbf{w}\|_{H^{-\frac{1}{2}}(\partial\omega)} + c'\,\|\mathbf{w}\|_{H^{\frac{1}{2}}(\partial\omega)} \leq c'''\|\mathbf{w}\|_{H^{\frac{1}{2}}(\partial\omega)}\,.\end{aligned}$$

In the case $\partial\overset{\circ}{\omega}_N \neq \emptyset$ let us first recall the construction of $\widehat{S}^h(\mathbf{w}, \mathbf{0})$ from (6.70)–(6.72).

Let $\widehat{\boldsymbol{\lambda}}^{h\prime} := \widehat{S}^h(\mathbf{w}, \boldsymbol{\psi}) \in \mathcal{H}_h^{-\frac{1}{2}}{}'(\partial\omega \setminus \partial\omega_N)$ and $\mathbf{w}^{*h} := \wp_\mu P_h \mathbf{w}$ be the prolongation of $P_h\mathbf{w}$. The construction of $\widehat{S}^h$ is now equivalent to finding

$$\mathbf{w}^h = \mathbf{w}^{*h} + \mathbf{w}_0^h \in \widetilde{\mathcal{H}}_h^{\frac{1}{2}}(\partial\omega \setminus (\partial\omega_D \setminus \Upsilon))$$

satisfying the Galerkin equations

$$\int\limits_{\partial\omega} (S^h \mathbf{w}^h) \cdot \mathbf{v}_0^h ds = \int\limits_{\partial\omega} \boldsymbol{\psi} \cdot \mathbf{v}_0^h ds$$

or

$$\int\limits_{\partial\omega} (S^h \mathbf{w}_0^h) \cdot \mathbf{v}_0^h ds = \int\limits_{\partial\omega} (\boldsymbol{\psi} - S^h \mathbf{w}^{*h}) \cdot \mathbf{v}_0^h ds \text{ for all } \mathbf{v}_0^h \in \widetilde{\mathcal{H}}_h^{\frac{1}{2}}(\partial\omega_N)\,. \tag{7.30}$$

Here $\mathbf{w}_0^h \in \widetilde{\mathcal{H}}_h^{\frac{1}{2}}(\partial\omega_N)$ is to be determined such that $\widehat{S}^h(\mathbf{w}^h, \boldsymbol{\psi}) = S^h(\mathbf{w}^h)$. We therefore define

$$\widehat{S}^h(\mathbf{w}, \boldsymbol{\psi}) = S^h \mathbf{w}^{*h} + S^h \mathbf{w}_0^h\,.$$

Note that (7.30) has a unique solution $\mathbf{w}_0^h$ if $\partial\omega_N \neq \partial\omega$ and is unique up to a rigid motion if $\partial\overset{\circ}{\omega}_j \cap \Gamma_D \setminus \Upsilon = \emptyset$, i. e. $\partial\omega_N = \partial\omega$. With $\mathbf{w}^h$ determined, we have in the special case $\boldsymbol{\psi} = \mathbf{0}$, from (7.26)

$$\int\limits_{\partial\omega} \widehat{S}^h(\mathbf{w}^h, \mathbf{0}) \cdot \mathbf{w}^h ds = \int\limits_{\partial\omega} S^h(\mathbf{w}^h) \cdot \mathbf{w}^h ds \geq \gamma_0 \|\mathbf{w}^h\|^2_{H^{\frac{1}{2}}(\partial\omega)/\Re}\,.$$

For the remaining case in (7.27) with $\partial\omega_D \setminus \Upsilon \neq \emptyset$ we proceed as follows:

We note that $\|\mathbf{v}\|_{H^{\frac{1}{2}}(\partial\omega)/\Re}$ is equivalent to $a_\omega(\mathbf{v}, \mathbf{v})$ where $\mathbf{v}$ is the $\mathcal{P}$–harmonic extension of $\mathbf{v} \in H^{\frac{1}{2}}(\partial\omega)$ to $\mathbf{v} \in H^1(\omega)$. As in the proof of Theorem 2,

$$a_\omega(\mathbf{v}, \mathbf{v}) \geq \alpha \|\mathbf{v}\|^2_{\widehat{H}^{\frac{1}{2}}(\partial\omega\cap\Upsilon)}$$

as in the case $\partial\omega_D \setminus \Upsilon \neq \emptyset$.

With the previous uniform boundedness (7.28) of S^h and (7.26), (7.30) it follows that

$$\begin{aligned}
\|\mathbf{w}_0\|_{H^{\frac{1}{2}}(\partial\omega)} &\leq c\|S^h \mathbf{w}^{*h}\|_{H^{-\frac{1}{2}}(\partial\omega)} + c\|\boldsymbol{\psi}\|_{H^{-\frac{1}{2}}(\partial\omega\cap\Gamma_N\setminus\overset{\circ}{\Upsilon})} \\
&\leq c'\,\|\wp_\mu P_h \mathbf{w}\|_{H^{\frac{1}{2}}(\partial\omega)} + c\|\boldsymbol{\psi}\|_{H^{-\frac{1}{2}}(\partial\omega\cap\Gamma_N\setminus\overset{\circ}{\Upsilon})} \\
&\leq c\|\mathbf{w}\|_{\widehat{H}^{\frac{1}{2}}(\partial\omega\cap\Upsilon)} + c\|\boldsymbol{\psi}\|_{H^{-\frac{1}{2}}(\partial\omega\cap\Gamma_N\setminus\overset{\circ}{\Upsilon})}
\end{aligned}$$

and, correspondingly,

$$\begin{aligned}
\|\widehat{S}^h(\mathbf{w}, \boldsymbol{\psi})\|_{H^{-\frac{1}{2}}(\partial\omega)} &\leq c\left\{\|\mathbf{w}^{*h}\|_{H^{\frac{1}{2}}(\partial\omega)} + \|\mathbf{v}_0\|_{H^{\frac{1}{2}}(\partial\omega)}) + c\|\boldsymbol{\psi}\|_{H^{-\frac{1}{2}}(\partial\omega\cap\Gamma_N\setminus\overset{\circ}{\Upsilon})}\right\} \\
&\leq c\|\mathbf{w}\|_{\widehat{H}^{\frac{1}{2}}(\partial\omega\cap\Upsilon)} + c\|\boldsymbol{\psi}\|_{H^{-\frac{1}{2}}(\partial\omega\cap\Gamma_N\setminus\overset{\circ}{\Upsilon})}\,.
\end{aligned}$$

□

Similar to Section 7.1.1, the discrete system (6.45), (6.46) is equivalent to the variational formulation:

Find

$$(\mathbf{u}^H, \mathbf{u}_{j0}^h) \in \mathcal{H}_h^D := \mathcal{H}_{HD} \times \prod_{j=1}^{M} \widehat{\mathcal{H}}_h^{\frac{1}{2}}(\partial\omega_j \cap \Upsilon)$$

such that

$$\begin{aligned}
\mathcal{B}(\mathbf{u}_0^H, \mathbf{u}_{j0}^h; \mathbf{v}^H, \mathbf{w}_j^h) &:= a_F(\mathbf{u}_0^H, \mathbf{v}^H) \\
&\quad + \sum_{j=1}^{M} \Big(\langle \widehat{S}_j^h(\mathbf{u}_{j0}^h, \mathbf{0}); \mathbf{v}^H \rangle_j - \langle \widehat{S}_j^h(\mathbf{w}_j^h, \mathbf{0}), \mathbf{u}_0^H \rangle_j + \langle \widehat{S}_j^h(\mathbf{w}_j^h, \mathbf{0}), \mathbf{u}_{j0}^h \rangle_j \Big) \\
&= \mathcal{M}(\mathbf{v}^H, \mathbf{w}_j^h) \ \ \textit{for all } (\mathbf{v}^H, \mathbf{w}_j^h) \in \mathcal{H}_h^D
\end{aligned} \tag{7.31}$$

where

$$\begin{aligned}
\mathcal{M}(\mathbf{v}^H, \mathbf{w}_j^h) &= \int\limits_{\Gamma_N \cap \partial\Omega_F \cup \Upsilon} \boldsymbol{\psi} \cdot \mathbf{v}^H ds - \sum_{j=1}^{M} \int\limits_{\partial\omega_j \cap \Upsilon} \widehat{S}_j^h(\boldsymbol{\varphi}_j^*, \boldsymbol{\psi}_j^*) \cdot \mathbf{v}^H ds \\
&\quad + \sum_{j=1}^{M} \int\limits_{\partial\omega_j \cap \Upsilon} (\boldsymbol{\varphi}_j^{*H} - \boldsymbol{\varphi}_j^{*h}) \cdot \widehat{S}_j^h(\mathbf{w}_j^h, \mathbf{0}) ds
\end{aligned} \tag{7.32}$$

is the given linear functional defined by (6.45) *and* (6.46)*.*

Based on Lemma 11, Theorem 5 and Corollary 6 now are replaced by:

Theorem 12 *If* (7.26) *and* (7.27) *are satisfied then the bilinear form $\mathcal{B}$ in* (7.31) *satisfies the BBL–condition*

$$\begin{aligned}
\sup_{(\mathbf{v}^H, \mathbf{w}_j^h) \in \mathcal{H}_h^D} & |\mathcal{B}(\mathbf{u}_0^H, \mathbf{u}_{j0}^h; \mathbf{v}^H, \mathbf{w}_j^h)| \Big/ \{ \|\mathbf{v}^H\|_{\mathcal{H}_{HD}} + \sum_{j=1}^{M} \|\mathbf{w}_j^h\|_{\widehat{H}^{\frac{1}{2}}(\partial\omega_j \cap \Upsilon)} \} \\
&\geq \gamma_0 \Big\{ \|\mathbf{u}_0^H\|_{\mathcal{H}_{HD}} + \sum_{j=1}^{M} \|\mathbf{u}_{j0}^h\|_{\widehat{H}^{\frac{1}{2}}(\partial\omega_j \cap \Upsilon)} \Big\}
\end{aligned} \tag{7.33}$$

where $\gamma_0 > 0$ is a constant independent of H and h.

Proof: Take $(\mathbf{v}^H, \mathbf{w}_j^h) = (\mathbf{u}_0^H, \mathbf{u}_{j0}^h)$ in (7.31), then (7.26) and/or (7.27) together with the uniform boundedness (7.28), (7.29) guarantee the proposed inequality (7.33). □

Again, (7.33) together with the approximation properties (6.3) and (6.5) implies the asymptotic convergence in the form of Céa's lemma:

Corollary 13 *There exist constants c_1 and c_2 which are independent of H and h such that*

$$\begin{aligned}
&\|\mathbf{u} - \mathbf{u}^H\|_{\mathcal{H}} + \sum_{j=1}^{M} \|\mathbf{u} - \mathbf{u}_j^h\|_{\widehat{H}^{\frac{1}{2}}(\partial\omega_j \cap \Upsilon)} \\
&\leq c_1 \inf_{\mathbf{v}^H \in \mathcal{H}_H} \|\mathbf{u} - \mathbf{v}^H\|_{\mathcal{H}} + c_2 \inf_{\mathbf{w}_j^h \in \mathcal{H}_h^{\frac{1}{2}}} \sum_{j=1}^{M} \|\mathbf{u} - \mathbf{w}_j^h\|_{H^{\frac{1}{2}}(\partial\omega_j)} \\
&\leq c_1{}' \, H^{t-1} \|\mathbf{u}\|_{H^t(\Omega)} + c_2{}' \, h^{s-\frac{1}{2}} \|\mathbf{u}\|_{H^{s+\frac{1}{2}}(\Omega)}
\end{aligned} \tag{7.34}$$

where $1 \leq t \leq d\,,\ \frac{1}{2} \leq s \leq d'$.

In a similar manner as in Section 7.1.1 with the Neumann bases, we now present the complete asymptotic error estimate in the case of the Dirichlet bases.

Corollary 14 *There exist constants c_1, c_2, c_3 which are independent of H, and h' such that*

$$
\begin{aligned}
\|\mathbf{u}-\mathbf{u}^H\|_{\mathcal{H}} &+ \sum_{j=1}^{M}\left(\|\mathbf{u}-\mathbf{u}_j^h\|_{H^{\frac{1}{2}}(\partial\omega_j)/\Re_j} + \|T_j\mathbf{u}-\boldsymbol{\lambda}_j^{h'}\|_{H^{-\frac{1}{2}}(\partial\omega_j)}\right) \\
\leq\;& c_1 \inf_{\mathbf{v}^H\in\mathcal{H}_H}\|\mathbf{v}-\mathbf{v}^H\|_{\mathcal{H}} + c_2\sum_{j=1}^{M}\inf_{\mathbf{w}_j^h\in\mathcal{H}_h^{\frac{1}{2}}}\|\mathbf{u}-\mathbf{w}_j^h\|_{H^{\frac{1}{2}}(\partial\omega_j)} \\
&+ c_3\sum_{j=1}^{M}\inf_{\boldsymbol{\chi}^h\in\mathcal{H}_{h'}^{-\frac{1}{2}}}\|T_j\mathbf{u}-\boldsymbol{\lambda}_j^{h'}\|_{H^{-\frac{1}{2}}(\partial\omega_j)} \\
\leq\;& c_1{}'\,H^{t-1}\|\mathbf{u}\|_{H^t(\Omega)} + c_2{}'(h^{s-\frac{1}{2}} + h'^{s-\frac{1}{2}})\|\mathbf{u}\|_{H^{s+\frac{1}{2}}(\Omega)}
\end{aligned}
\tag{7.35}
$$

where $0 \leq s \leq \min\{d', d''+1\}$.

Remark 7.2: For $\partial\omega_j \cap \Upsilon \neq \partial\omega_j$, the term $\|\mathbf{u}-\mathbf{u}_j^h\|_{H^{\frac{1}{2}}(\partial\omega_j)/\Re_j}$ on the left–hand side in (7.35) can be replaced by $\|\mathbf{u}-\mathbf{u}_j^h\|_{H^{\frac{1}{2}}(\partial\omega_j)}$.

Proof: We note that the equation (6.72) together with (6.70) corresponds to the discrete version of (5.23), namely

$$\langle S^h\mathbf{u}^h, \mathbf{v}_0^h\rangle = \langle\boldsymbol{\psi}, \mathbf{v}_0^h\rangle \;\text{ for all } \mathbf{v}_0^h \in \widetilde{\mathcal{H}}_h^{\frac{1}{2}}(\partial\omega_N)\,.$$

(Here, the subscript j is again suppressed.) Now we write

$$\mathbf{u}^h = \wp_\mu(\mathbf{u}^h|_{\partial\omega\cap\Upsilon}) + \mathbf{u}_0^h \tag{7.36}$$

in terms of the prolongation operator defined in (6.8) with $\mathbf{u}_0^h \in \widetilde{\mathcal{H}}_h^{\frac{1}{2}}(\partial\omega_N)$. Then, with $\mathbf{v}_0^h = \mathbf{u}_0^h$, the definition (6.67) of S^h and the coerciveness (4.29) of D, we find

$$
\begin{aligned}
\alpha_0\|\mathbf{u}_0^h\|^2_{\widetilde{H}^{\frac{1}{2}}(\partial\omega_N)} &\leq \langle S^h\mathbf{u}_0^h, \mathbf{u}_0^h\rangle = \langle\boldsymbol{\psi} - S^h\Big(\wp_\mu(\mathbf{u}^h|_{\partial\omega\cap\Upsilon})\Big)\mathbf{u}_0^h\rangle \\
&\leq c\left\{\|\boldsymbol{\psi}\|_{H^{-\frac{1}{2}}(\partial\omega_N)} + \|S^h\Big(\wp_\mu(\mathbf{u}^h|_{\partial\omega\cap\Upsilon})\Big)\|_{H^{-\frac{1}{2}}(\partial\omega)}\right\}\|\mathbf{u}_0^h\|_{\widetilde{H}^{\frac{1}{2}}(\partial\omega_N)} \\
&\leq c'\left\{\|\boldsymbol{\varphi}\|_{H^{-\frac{1}{2}}(\partial\omega_N)} + \|\mathbf{u}^h\|_{\widehat{H}^{\frac{1}{2}}(\partial\omega\cap\Upsilon)}\right\}\|\mathbf{u}_0^h\|_{\widetilde{H}^{\frac{1}{2}}(\partial\omega_N)}
\end{aligned}
$$

where c' is independent of H and h. This, together with (7.36) and (7.34) yields the stability estimate

$$\|\mathbf{u}^h\|_{H^{\frac{1}{2}}(\partial\omega)} \leq c\|\mathbf{u}\|_{H^1(\Omega)}\,. \tag{7.37}$$

With the stability of $\mathbf{u}^h$ on the whole $\partial\omega$ in (7.37) available, we find from the definition (6.73) of $\boldsymbol{\lambda}^{h'}$ on $\partial\omega$ and the uniform boundedness of S^h (7.28) the stability estimate

$$\|\boldsymbol{\lambda}^{h'}\|_{H^{-\frac{1}{2}}(\partial\omega)} = \|P_{h'}S^h\mathbf{u}^h\|_{H^{-\frac{1}{2}}(\partial\omega)} \leq c\|\mathbf{u}^h\|_{H^{\frac{1}{2}}(\partial\omega)} \leq c'\,\|\mathbf{u}^h\|_{H^1(\Omega)}\,.$$

Here $P_{h'}$ denotes the L^2–projection onto $\mathcal{H}_{h'}^{-\frac{1}{2}}(\partial\omega)$ satisfying (6.6). With stability of $\mathbf{u}^H, \mathbf{u}^h$ and $\boldsymbol{\lambda}^{h'}$ available, uniform boundedness of the corresponding family of Galerkin projections implies Cea's lemma and the proposed asymptotic error estimates. □

7.2.2 Dirichlet bases on restricted grids

In an similar manner as in the case of Neumann bases we consider the asymptotic stability and convergence without using the coerciveness property (7.26), when the Steklov–Poincaré operator is constructed based on (5.20) under **additional restrictions on the grids**.

Specifically, we need the additional mesh restrictions **(R1)** in (6.74) and the **mesh restriction**

$$(\mathbf{R3}) \qquad h' \leq c_{02} h \tag{7.38}$$

with a constant $c_{02} > 0$ to be chosen small enough, and the inverse assumption (6.7). We note that because of **(R1)**, $h \leq H$ is still satisfied.

Again, let $\widehat{\mathbf{u}}^H$ denote the $\mathcal{P}$–harmonic extension defined in (7.13). Then, from (6.45) we obtain the variational equation

$$\begin{aligned}
a_\Omega(\widehat{\mathbf{u}}^H, \widehat{\mathbf{v}}_0^H) &= a_F(\mathbf{u}^H, \mathbf{v}_0^H) + \sum_{j=1}^M \oint_{\partial\omega_j} \widehat{\mathbf{v}}_0^H \cdot T_j \widehat{\mathbf{u}}^H ds \\
&= a_F(\mathbf{u}^H, \mathbf{v}_0^H) + \sum_{j=1}^M \oint_{\partial\omega_j} \widehat{\mathbf{v}}_0^H \cdot S_j^h \mathbf{u}_j^h ds + \sum_{j=1}^M \oint_{\partial\omega_j} \widehat{\mathbf{v}}_0^H \cdot (S_j^h \widehat{\mathbf{u}}^H - S_j^h \mathbf{u}_j^h) ds\,, \\
a_\Omega(\widehat{\mathbf{u}}^H, \widehat{\mathbf{v}}_0^H) &= a_F(\mathbf{u}, \mathbf{v}_0^H) + \sum_{j=1}^M \oint_{\partial\omega_j} \widehat{\mathbf{v}}_0^H \cdot T_j \mathbf{u} ds + \sum_{j=1}^M \oint_{\partial\omega_j} \widehat{\mathbf{v}}_0^H \cdot (S_j^H \widehat{\mathbf{u}}^H - S_j^h \mathbf{u}_j^h) ds \\
&\qquad \text{for all test functions } \mathbf{v}_0^H \in \widehat{\mathcal{H}}_{HD}\,.
\end{aligned} \tag{7.39}$$

With $\widehat{\mathbf{u}}^H = \widehat{\boldsymbol{\varphi}}^{*H} + \widehat{\mathbf{u}}_0^H$ where $\widehat{\mathbf{u}}_0^H \in \widehat{\mathcal{H}}_{HD}$, the $H_D^1(\Omega)$–ellipticity of a_Ω yields for $\mathbf{v}_0^H = \widehat{\mathbf{u}}_0^H$ the inequality

$$\alpha_0 \|\widehat{\mathbf{u}}_0^H\|_{H^1(\Omega)}^2 \leq c_1(\|\widehat{\boldsymbol{\varphi}}^{*H}\|_{H^1(\Omega)} + \|\mathbf{u}\|_{H^1(\Omega)})\|\widehat{\mathbf{u}}_0^H\|_{H^1(\Omega)} + \sum_{j=1}^M |\oint_{\partial\omega_j} \widehat{\mathbf{u}}_0^H \cdot (S_j \widehat{\mathbf{u}}^H - S_j^h \mathbf{u}_j^h) ds| \tag{7.40}$$

To analyze the last term we have to rewrite these integrals in such a way that we can make use of the consistency $(S_j - S_j^h)$. For this purpose we need the weak coupling conditions (6.46) which we write in the form

$$\langle \widehat{\mathbf{u}}^H - \mathbf{u}_j^h, S_j^h \mathbf{v}_j^h \rangle_j = 0 \text{ for all } \mathbf{v}_j^h \in \mathcal{H}_h^{\frac{1}{2}}(\partial\omega_j)\,. \tag{7.41}$$

Due to assumption the mesh restriction **(R1)** in (6.74) we may choose in (7.41) $\mathbf{v}_j^h = \widehat{\mathbf{u}}^H - \mathbf{u}_j^h$ to obtain

$$\langle \widehat{\mathbf{u}}^H - \mathbf{u}_j^h, S_j(\widehat{\mathbf{u}}^H - \mathbf{u}_j^h)\rangle_j = \langle \widehat{\mathbf{u}}^H - \mathbf{u}_j^h, (S_j - S_j^h)(\widehat{\mathbf{u}}^H - \mathbf{u}_j^h)\rangle_j$$

or

$$\langle P_\Re(\widehat{\mathbf{u}}^H - \mathbf{u}_j^h), S_j P_\Re(\widehat{\mathbf{u}}^H - \mathbf{u}_j^h)\rangle_j = \langle P_\Re(\widehat{\mathbf{u}}^H - \mathbf{u}_j^h), (S_j - S_j^h)(\widehat{\mathbf{u}}^H - \mathbf{u}_j^h)\rangle_j\,.$$

With the coerciveness of S_j on $H_0^{\frac{1}{2}}(\partial\omega_j)$ this yields

$$\alpha_0 \|P_\Re(\widehat{\mathbf{u}}^H - \mathbf{u}_j^h)\|_{H^{\frac{1}{2}}(\partial\omega)}^2 \leq c\|P_\Re(\widehat{\mathbf{u}}^H - \mathbf{u}_j^h)\|_{H^{\frac{1}{2}}(\partial\omega)} \|(S_j - S_j^h)(\widehat{\mathbf{u}}^H - \mathbf{u}_j^h)\|_{H^{-\frac{1}{2}}(\partial\omega_j)}\,. \tag{7.42}$$

Lemma 15 *Under the mesh restrictions* **(R1)***, the inverse assumption for* $\mathcal{H}_h^{\frac{1}{2}}$ *and* **(R3)** *where* $c_{02} > 0$ *is independent of* h *and is to be chosen sufficiently small we have the consistency estimate*

$$\|(S_j - S_j^h)\mathbf{w}^h\|_{H^{-\frac{1}{2}}(\partial\omega)} \le c_\delta c_{02}^\delta \|\mathbf{w}^h\|_{H^{\frac{1}{2}}(\partial\omega)} \quad \text{for every } \mathbf{w}^h \in \mathcal{H}_h^{\frac{1}{2}}(\partial\omega)\,. \tag{7.43}$$

The constant c_δ *is independent of* h.

Proof: Here, we suppress the subscript j. Let $\boldsymbol{\sigma} := S\mathbf{w}^h$ and $\boldsymbol{\sigma}^{h'} := S^h\mathbf{w}^h$. Then with (5.20) we have

$$\langle \boldsymbol{\chi}^{h'}, V\boldsymbol{\sigma}^{h'}\rangle = \langle \boldsymbol{\chi}^{h'}, (\tfrac{1}{2}I + K)\mathbf{w}^h\rangle = \langle \boldsymbol{\chi}^{h'}, V\boldsymbol{\sigma}\rangle \text{ for all } \boldsymbol{\chi}^{h'} \in \mathcal{H}_{h'}^{-\frac{1}{2}}(\partial\omega)\,.$$

Hence, $\boldsymbol{\sigma}^{h'}$ is the V–Galerkin projection of $\boldsymbol{\sigma}$ providing the standard asymptotic estimate

$$\|\boldsymbol{\sigma}^{h'} - \boldsymbol{\sigma}\|_{H^{-\frac{1}{2}}(\partial\omega)} \le ch'^\delta \|V^{-1}\mathbf{w}^h\|_{H^{-\frac{1}{2}+\delta}(\partial\omega)} \le c'\, h'^\delta \|\mathbf{w}^h\|_{H^{\frac{1}{2}+\delta}(\partial\omega)}\,.$$

By the inverse assumption (6.7) for $\mathcal{S}_h^{d'}(\partial\omega)$ we finally obtain

$$\|S\mathbf{w}^h - S^h\mathbf{w}^h\|_{H^{-\frac{1}{2}}(\partial\omega)} \le c\Big(\tfrac{h'}{h}\Big)^\delta \|\mathbf{w}^h\|_{H^{\frac{1}{2}}(\partial\omega)} \le c\cdot (c_{02})^\delta \|\mathbf{w}^h\|_{H^{\frac{1}{2}}(\partial\omega)}\,.$$

□

With the consistency (7.43) available, we now return to (7.42) and obtain

$$\begin{aligned}
\|P_{\Re}(\widehat{\mathbf{u}}^H - \mathbf{u}^h)\|_{H^{\frac{1}{2}}(\partial\omega)} &\le \|(S - S^h)(\widehat{\mathbf{u}}^H - \mathbf{u}^h)\|_{H^{-\frac{1}{2}}(\partial\omega)} \\
&\le c(c_{02})^\delta \|\widehat{\mathbf{u}}^H - \mathbf{u}^h\|_{H^{\frac{1}{2}}(\partial\omega)} \\
&\le c(c_{02})^\delta \left\{ \|\widehat{\mathbf{u}}^H\|_{H^{\frac{1}{2}}(\partial\omega)} + \|\widehat{\mathbf{u}}^h\|_{H^{\frac{1}{2}}(\partial\omega)} \right\}.
\end{aligned} \tag{7.44}$$

Equation (7.41) with $\mathbf{v}^h = \mathbf{u}^h$ implies

$$\begin{aligned}
\alpha_0 \|\mathbf{u}^h\|^2_{H^{\frac{1}{2}}(\partial\omega)} &\le \langle \mathbf{u}^h, S\mathbf{u}^h\rangle = \langle \widehat{\mathbf{u}}^H, S^h\mathbf{u}^h\rangle + \langle \mathbf{u}^h, (S - S^h)\mathbf{u}^h\rangle \\
&\le c\left\{ \|\widehat{\mathbf{u}}^H\|_{H^{\frac{1}{2}}(\partial\omega)} \|S^h\mathbf{u}^h\|_{H^{-\frac{1}{2}}(\partial\omega)} + (c_{02})^\delta \|\mathbf{u}^h\|^2_{H^{\frac{1}{2}}(\partial\omega)} \right\}.
\end{aligned} \tag{7.45}$$

From (5.19) we have

$$\langle \boldsymbol{\chi}^{h'}, VS^h\mathbf{u}^h\rangle = \langle \boldsymbol{\chi}^{h'}, (\tfrac{1}{2}I + K)\mathbf{u}^h\rangle \text{ for all } \boldsymbol{\chi}^{h'} \in \mathcal{H}_h^{-\frac{1}{2}}(\partial\omega)$$

from which the coerciveness of V implies that

$$\|S^h\mathbf{u}^h\|_{H^{-\frac{1}{2}}(\partial\omega)} \le c\|\mathbf{u}^h\|_{H^{\frac{1}{2}}(\partial\omega)} \tag{7.46}$$

where c is independent of h'. Insert this inequality into (7.45) to obtain

$$(\alpha_0 - cc_{02}^\delta)\|\mathbf{u}^h\|_{H^{\frac{1}{2}}(\partial\omega)} \le c\|\widehat{\mathbf{u}}^H\|_{H^{\frac{1}{2}}(\partial\omega)}\,. \tag{7.47}$$

Then, by choosing c_{02} sufficiently small, we obtain from (7.44) the estimate

$$\|P_{\Re}(\widehat{\mathbf{u}}^H - \mathbf{u}^h)\|_{H^{\frac{1}{2}}(\partial\omega)} \le c'(c_{02})^\delta \|\widehat{\mathbf{u}}^H)\|_{H^{\frac{1}{2}}(\partial\omega)}\,. \tag{7.48}$$

Now we are in the position to estimate the last terms in (7.40).

Lemma 16 *Under the assumptions of Lemma* 15 *with* c_{02} *chosen sufficiently small, the following estimate holds:*

$$|\oint_{\partial\omega_j} \widehat{\mathbf{u}}_0^H \cdot (S_j\widehat{\mathbf{u}}^H - S_j^h\mathbf{u}_j^h)ds| \leq cc_{02}^\delta \|\widehat{\mathbf{u}}_0^H\|_{H^{\frac{1}{2}}(\partial\omega_j)}\|\widehat{\mathbf{u}}^H\|_{H^{\frac{1}{2}}(\partial\omega_j)}\,. \tag{7.49}$$

Proof: By the triangle inequality we have

$$|\oint_{\partial\omega} \widehat{\mathbf{u}}_0^H \cdot (S_j\widehat{\mathbf{u}}^H - S_j^h\mathbf{u}_j^h)ds| \leq |\oint_{\partial\omega_j} \widehat{\mathbf{u}}_0^H \cdot (S_j - S_j^h)\widehat{\mathbf{u}}^H ds| + |\oint_{\partial\omega_j} \widehat{\mathbf{u}}_0^H \cdot S_j^h(\widehat{\mathbf{u}}^H - \mathbf{u}_j^h)ds|\,.$$

The first term can be estimated by the consistency (7.43) with $\mathbf{w}^h = \widehat{\mathbf{u}}^H$ due to **(R1)**; this gives

$$|\oint_{\partial\omega_j} \widehat{\mathbf{u}}_0^H \cdot (S_j - S_j^h)\widehat{\mathbf{u}}^H ds| \leq c\cdot c_{02}^\delta \|\widehat{\mathbf{u}}_0^H\|_{H^{\frac{1}{2}}(\partial\omega_j)}\|\widehat{\mathbf{u}}^H\|_{H^{\frac{1}{2}}(\partial\omega_j)}\,.$$

For the second term we use (7.48) and the uniform boundedness of S_j^h which can be seen in the same manner as in (7.46), to obtain

$$|\oint_{\partial\omega_j} \widehat{\mathbf{u}}_0^H \cdot S_j(\widehat{\mathbf{u}}^H - \mathbf{u}_j^h)ds| = |\oint_{\partial\omega_j} \widehat{\mathbf{u}}_0^H \cdot S_j P_\Re(\widehat{\mathbf{u}}^H - \mathbf{u}_j^h)ds| \leq c\cdot c_{02}^\delta \|\widehat{\mathbf{u}}_0^H\|_{H^{\frac{1}{2}}(\partial\omega_j)}\|\widehat{\mathbf{u}}^H\|_{H^{\frac{1}{2}}(\partial\omega_j)}\,.$$

These two estimates give (7.49). □

Now it is clear that for c_{02} sufficiently small, (7.40) will provide the asymptotic stability of $\|\widehat{\mathbf{u}}^H\|_{H^1(\Omega)}$.

Collecting the inequalities (7.40), (7.46) and (7.47), these results can be summarized in the following theorem.

Theorem 17 *Assume that the assumptions of Lemma* 15, *i. e. the mesh restrictions* **(R1)** *and* **(R3)** *are satisfied. Then we have the asymptotic stability estimate*

$$\|\mathbf{u}^H\|_{\mathcal{H}} + \sum_{j=1}^M \left(\|\boldsymbol{\lambda}_j^{h'}\|_{H^{-\frac{1}{2}}(\partial\omega_j)} + \|\mathbf{u}_j^h\|_{H^{\frac{1}{2}}(\partial\omega_j)}\right) \leq c\|\mathbf{u}\|_{H^1(\Omega)}\,. \tag{7.50}$$

for all $0 < h \leq h_0$ *with a constant* c *independent of* h', h *and* H, *provided the constant* c_{02} *in* **(R3)** *is chosen sufficiently small.*

As a consequence of the stability (7.50) *and the approximation properties, the asymptotic error estimates in Corollary* 14 *remain valid.*

7.3 Stability and convergence for the simplified macro–stiffness matrix.

If the mesh restrictions (**R1)** and (**R2)** hold and the local Steklov–Poincaré operators S_j are approximated by the symmetric form (6.67) then the global equations in (6.81) take the form (6.81) which is already stable **without further restrictions on the grids**.

We begin with the stability of the approximate solution of $\widehat{\mathbf{u}}^H$ of the global equations (6.79) which are equivalent to

$$\begin{aligned} a_F(\mathbf{u}_0^H, \mathbf{v}_0^H) &+ \sum_{j=1}^M \oint_{\partial\omega_j} (S_j^h \widehat{\mathbf{u}}_{0j}^H)\cdot \mathbf{v}_0^H ds \\ &= \int_{\Gamma_N} \boldsymbol{\psi}\cdot\mathbf{v}_0^H ds - a_F(\boldsymbol{\varphi}^{*H}, \mathbf{v}_0^H) - \sum_{j=1}^M \langle S_j^h\boldsymbol{\varphi}_j^{*h} + S_j^h\mathbf{u}_{j\psi}^h, \widehat{\mathbf{v}}_0^H\rangle_j\,, \end{aligned} \tag{7.51}$$

(see also (7.37)). Now choose $\widehat{\mathbf{v}}_0^H = \widehat{\mathbf{u}}_0^H$ and in view of the ellipticity (3.9) and (7.51) one obtains the following stability result.

Lemma 18 *Under the mesh restrictions* **(R1)** *and* **(R2)***, the solutions of* (7.51) *are uniformly bouded:*

$$\|\widehat{\mathbf{u}}_0^H\|_{H^1(\Omega_F)} + \sum_{j=1}^{M} \|\widehat{\mathbf{u}}_{0j}^H\|_{H^{\frac{1}{2}}(\partial\omega_j)/\Re_j} \le c\left\{\|\boldsymbol{\varphi}\|_{H^{\frac{1}{2}}(\Gamma_D)} + \|\boldsymbol{\psi}\|_{H^{-\frac{1}{2}}(\Gamma_N)}\right\} \tag{7.52}$$

where the constant c is independent of h, h' and H.

Remark 7.3: It follows from (7.52), the definition (6.75) of $\mathbf{u}_0^H$ and $\widehat{\mathbf{u}}_0^H$ and from (7.51) that $\mathbf{u}^H$ and $\mathbf{u}_j^H$ are also uniformly bounded, i. e.

$$\|\mathbf{u}^H\|_{H^1(\Omega_F)} + \sum_{j=1}^{M} \|\widehat{\mathbf{u}}_j^H\|_{H^{\frac{1}{2}}(\partial\omega_j)/\Re_j} \le c\|\mathbf{u}\|_{H^1(\Omega)}$$

where the constant c is independent of h, h' and H.

By using Lemma 3 with $P_{\Re_j}\widehat{\mathbf{u}}_j^H = P_{\Re_j}\mathbf{u}_j^h$ and the definition of $\boldsymbol{\lambda}_j^{h'}$ in (6.78), we obtain from Lemma 18 the stability result:

Lemma 19 *Under the assumptions of Lemma* 18 *there holds the following stability estimate:*

$$\|\mathbf{u}^H\|_{H^1(\Omega)} + \sum_{j=1}^{M}\left(\|\boldsymbol{\lambda}_j^{h'}\|_{H^{-\frac{1}{2}}(\partial\omega_j)} + \|\mathbf{u}_j^h\|_{H^{\frac{1}{2}}(\partial\omega_j)/\Re_j}\right) \le c\|\mathbf{u}\|_{H^1(\Omega)}$$

where the constant c is independent of h, h' and H.

As a consequence, Céa's Lemma is valid:

Corollary 20 *Under the assumptions of Lemma* 18*, i. e. the mesh restriction* **(R1)***,* **(R2)***, Corollary* 14 *remains valid.*

8 Numerical Results

In this section we present numerical results for the notch problems in two and three dimensions, where the macro–elements are in the near field of the notch. The examples are based on the local Neumann bases approach given in Section 6. The governing equations (3.1) are now the Lamé equations in linear elasticity:

$$\mathcal{P}\mathbf{u} = \Delta^*\boldsymbol{u} := \mu\Delta\boldsymbol{u} + (\lambda+\mu)\,\mathbf{grad}\,(\operatorname{div}\boldsymbol{u}) = \mathbf{0} \text{ in } \Omega \subset \mathrm{IR}^n\,,\; n = 2,3\,. \tag{8.1}$$

with the Lamé constants satisfying $\lambda > \frac{2}{n}\mu$ and $\mu > 0$. The mixed boundary conditions (3.2) now consist of the tractions

$$T[\boldsymbol{u}]_{|\Gamma_N} = \lambda(\operatorname{div}\boldsymbol{u})\boldsymbol{n} + 2\mu\frac{\partial\boldsymbol{u}}{\partial\boldsymbol{n}} + \mu\boldsymbol{n}\times\mathbf{curl}\,\boldsymbol{u}_{|\Gamma_N} = \boldsymbol{\psi} \text{ on } \Gamma_N \tag{8.2}$$

and the displacement field

$$\boldsymbol{u}_{|\Gamma_D} = \boldsymbol{\varphi} \text{ on } \Gamma_D. \tag{8.3}$$

The given stress and displacement fields $\boldsymbol{\psi}$ and $\boldsymbol{\varphi}$ will be specified explicitly according to the examples.

8.1 The two-dimensional example

Figure 6 shows a quadratic plate under uniform symmetric tension $\sigma_0 = 1N/mm^2$ in direction of the y-axis. In the center of the plate an elliptic cutout is located as the notch configuration.

In the numerical treatment the problem is analyzed as a pure Neumann problem. The discretization is based on the local Neumann bases introduced in Section 6.1 where also the discrete Neumann series is employed. The domain decomposition is also shown in Figure 6. Two macro–elements ω_1 and ω_2 are placed in the center near the elliptical cutout. The skeleton Υ is chosen as the union of the boundaries $\partial\omega_1$ and $\partial\omega_2$, $\Upsilon = \partial\omega_1 \cup \partial\omega_2$. The boundaries $\partial\omega_1$ and $\partial\omega_2$ of the macro–elements are discretized with elements on one fine grid, where $h' = h$, while the far field is discretized with triangular finite elements by using quadratic trial functions for the displacement field $\mathbf{u}^H$. In order to obtain a positive definite global system for the Neumann problem under consideration, we have to eliminate the three degrees of freedom of the rigid body motion. This is achieved by fixing the following finite element nodes at A, B and C:

$$u_x^A = 0\,, \quad u_x^C = 0\,, \quad u_y^B = 0$$

We are working with a rather coarse grid of meshwidth H for the global finite elements in Ω_F as well as on Υ, while the density of the fine grids for the macro–elements will be refined adaptively. The constructions of these grids are totally independent from each other. With this strategy it is possible to resolve the high stress peaks and high stress gradients which occur in the near field of the elliptical cutout. In comparison with conventional finite element techniques it is here not necessary to work with an adaptive scheme concerning the refinement of the coarse grid (H). In our method the numerical properties of the problem can be controlled adaptively by the macro–element operators. This is simpler since here adaptivity involves only the local boundary element method.

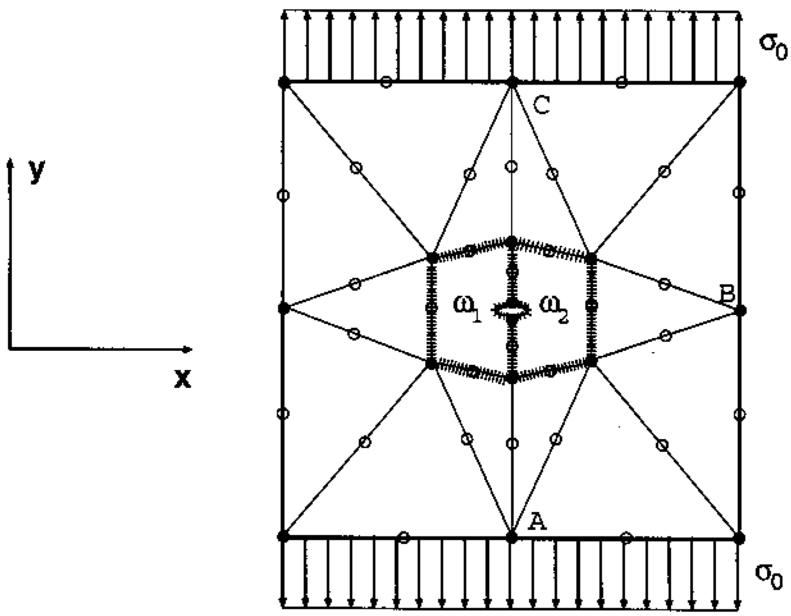

Figure 6: Discretization of a notch problem, two macro–elements

For the determination of the constant c_o in the coarse–fine grid relation **(CF)** in (7.12), in order to achieve stability and convergence, we use an indirect method by introducing error indicators for the discrete operators. These error indicators control the main properties of the macro–element stiffness matrix $\mathsf{H}_j = \widehat{\mathsf{B}}_j^\top \widehat{\mathsf{U}}_j^{-1} \widehat{\mathsf{B}}_j$ approximated by the partial sums (6.28) of the Neumann series as symmetry and positive definiteness. The exact macro–element stiffness matrix is symmetric and positive definite. Hence, its minimum eigenvalue λ_{min} has to be bounded away from zero. We define an averaged symmetry defect for the quadratic $M_j \times M_j$ matrix H_j with the elements $\mathsf{H}_{k\ell} = \sum\limits_{\iota,\rho} \widehat{\mathsf{B}}_{\iota k} \left(\widehat{\mathsf{U}}^{-1}\right)_{\iota,\rho} \widehat{\mathsf{B}}_{\rho\ell}$ by

$$MSD = \frac{\sum\limits_{k=1}^{M_j} \sum\limits_{\ell=1}^{k-1} |\mathsf{H}_{\ell k} - \mathsf{H}_{k\ell}|}{\sum\limits_{k=1}^{M_j} \sum\limits_{\ell=1}^{k-1} |\mathsf{H}_{\ell k} - \mathsf{H}_{k\ell}|}$$

and introduce error bounds ε_1 and ε_2 with

$$0 < MSD < \varepsilon_1 \quad \text{and} \quad \lambda_{min} > \varepsilon_2 > 0.$$

The control of these parameters for any given positive ε_1, ε_2 proved to be very efficient when using the Neumann series (4.25) as utilized by Türke in [58]. He also incorporated smoothness at the artificial interior cross points in terms of corresponding exact compatibility conditions for $\mathbf{u}_j^h$ and $\boldsymbol{\sigma}_j^h$ for all macro elements ω_j at common cross–point corners in the interior of Ω.

In the following numerical experiments for the given model problem in Figure 6, we determine the stress peak value $\alpha_{num} = \sigma_{max}/\boldsymbol{\sigma}_o$. For the same notch problem, but with an infinitely long plate, the Kolosov–Muskhelishvili representation provides an analytic solution which yields

$$\alpha_{exact} = 4.7632 \quad .$$

To compare the numerical and the exact values we use the relative error

$$\varepsilon_\alpha = \frac{|\alpha_{num} - \alpha_{exact}|}{\alpha_{exact}} \quad .$$

We start the numerical investigations with the coarse grid (H) as shown in Figure 6 with 14 finite elements ($n_{FE} = 14$). The fine grid in each macro-element ω_1 and ω_2 is refined in two steps with $n_{BE} = 135$ and $n_{BE} = 177$ boundary elements. Table 1 shows the results obtained for the deviation of the stress peak ε_α, the minimum eigenvalue λ_{min} and the symmetry defect MSD of the macro–element stiffness matrix H_j depending on the fine grid h and the number of iteration steps *iter* used in the Neumann series.

	iter	1	10	20	30	40	50
$n_{BE} = 135$	ε_α [%]	51.7	10.6	3.80	1.66	0.86	0.53
	MSD $[10^{-3}]$	1.903	2.144	2.269	2.317	2.336	2.344
	$\lambda_{\min}$ $[10^{-6}]$	1.1048	1.4325	1.4316	1.4318	1.4319	1.4320
$n_{BE} = 177$	ε_α [%]	51.7	10.6	3.80	1.66	0.86	0.55
	MSD $[10^{-3}]$	1.900	2.140	2.249	2.291	2.307	2.314
	$\lambda_{\min}$ $[10^{-6}]$	1.105	1.4330	1.4317	1.4320	1.4321	1.4321

Table 1: Results for coarse grid (H) with $n_{FE} = 14$

In Table 2 below, the results for a refined coarse grid with 46 finite elements are listed.

	iter	1	10	20	30	40	50
$n_{BE} = 135$	ε_α [%]	51.9	10.3	3.36	1.15	0.34	0.025
	MSD $[10^{-3}]$	1.903	2.144	2.269	2.317	2.336	2.344
	$\lambda_{\min}$ $[10^{-6}]$	1.1048	1.4325	1.4316	1.4318	1.4319	1.4320
$n_{BE} = 177$	ε_α [%]	51.9	10.3	3.36	1.17	0.36	0.046
	MSD $[10^{-3}]$	1.900	2.140	2.249	2.291	2.307	2.314
	$\lambda_{\min}$ $[10^{-6}]$	1.105	1.4330	1.4317	1.4320	1.4321	1.4321

Table 2: Results for refined coarse grid (H) with $n_{FE} = 46$

In all runs we observe an increasing accuracy for the stress peak with increasing numbers of Neumann iterations. The most accurate value is obtained for the discretization $n_{FE} = 46$ and $n_{FE} = 135$ with a deviation from the analytical value by 0.025%. The refinement of the coarse grid (H) in Ω_F and on Υ leads to an improvement of the calculated stress concentration.

In all cases we obtain a relative symmetry defect between 0.0019 and 0.0023 and positive values for λ_{min}. A refinement of the fine grid (h) leads to a more accurate stiffness matrix H with decreasing MSD and increasing λ_{min}. This motivates the a priori choice of an upper bound for MSD and a lower bound for λ_{min}. Then the algorithm automatically reduces adaptively the fine grid meshwidth h and the number of iteration steps for the Neumann series, until the given requirements of symmetry and stability are fulfilled.

Finally Figure 7 gives a visualization of the normal stress distribution in direction of the loading for the whole plate. In the far field of the notch we have a constant stress field with $\sigma_y = \sigma_o = 1N/mm^2$, while the high stress gradients in the near field of the notch are very accurately approximated within each of the macro-elements.

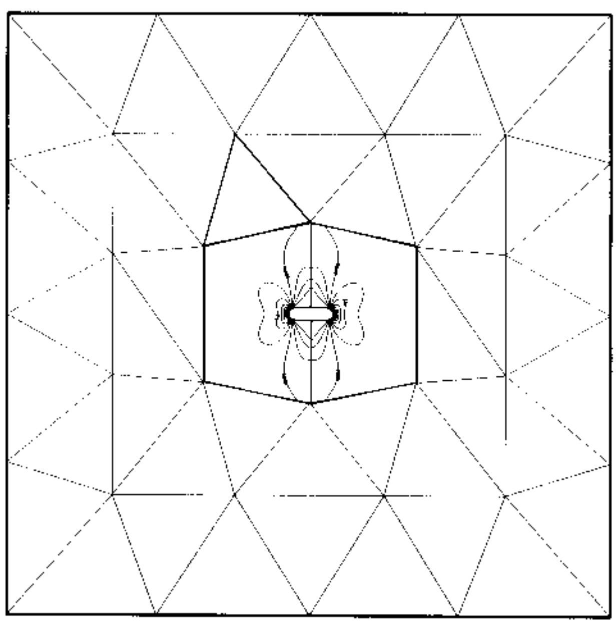

Figure 7: Stress isolines of the component σ_y^h of σ^h in ω_1 and ω_2

8.2 The three–dimensional example

For the numerical realization of the macro-element technique in three-dimensional problems we are using two non-conforming grids on the surface $\partial\omega_j$ of the macro-elements. On the finite–element side we have the coarse grid (H) belonging to the FE-discretized substructure and on the macro–element side the fine grid (h) for solving the boundary integral equation on $\partial\omega_j$.

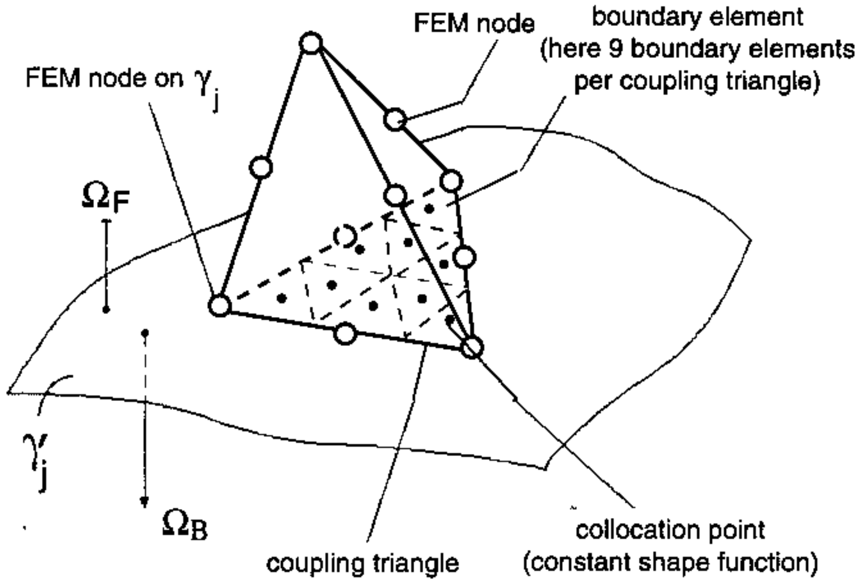

Figure 8: Discretization of the macro–element surface $\partial\omega_j$

The scheme for the numerical realization is shown in Figure 8. For the fine grid triangulation of the macro–element boundary $\partial\omega_j$ we are using the BEM with point collocation. Since here piecewise constant trial functions are chosen on the BEM side, the collocation points are placed at the centers of gravity of the collocation triangles. In comparison with the number of collocation points on the fine grid (h) we have only a small number of coupling nodes on the coarse grid (H). The coarse grid belongs to classical finite elements. Here in this example we use tetrahedral elements and quadratic trial functions with nodes at the corners and at the midpoints of the element-edges.

We present here the computational results from [51] for a thick plate under tension with an elliptical cutout as shown in Figure 9. The first three–dimensional numerical results for that coupling procedure were published in [32]. One macro–element ω is defined near the notch region with the largest curvature of the cutout, where the static field shows high stress peaks and high stress gradients.

In the numerical computation we study the influence of the parameter ratio b/w and

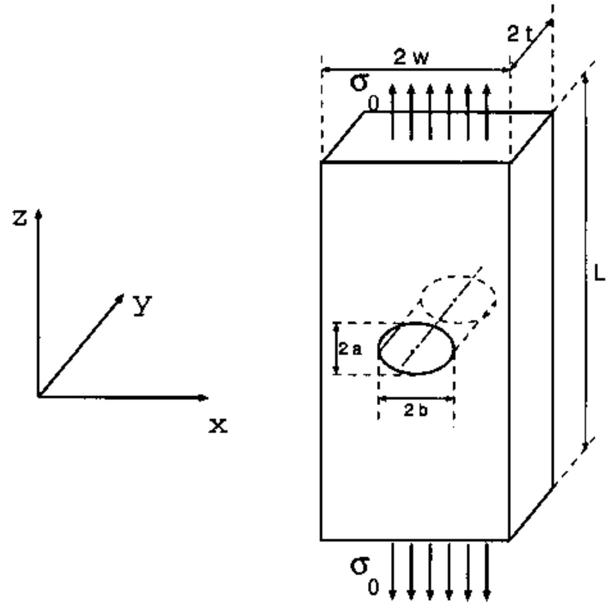

Figure 9: Plate with an elliptical cutout under tension

compare the stress peaks

$$\alpha_{\text{num}} = \frac{(\sigma_z^h)_{max}}{\sigma_0}$$

with analytical results presented by Peterson [40] and Isida [28].

Ratio b/w	α_{num}	ε_α [%]
0.33	5.60	0.18
0.4	5.83	0.18
0.5	6.50	0.15
0.7	9.42	0.21
0.8	12.70	3.05

Table 3: Numerical calculation of stress peaks on an IBM 3090

As shown in Table 3, the parameters vary from 0.33 to 0.8. We are using about 160 classical tetrahedral elements and have 16 coupling triangles on the interface $\partial\omega_j \cap \Upsilon$ which are discretized on the boundary element side in 144 boundary elements. The CPU time is about 100 seconds on the IBM 3090 computer. It turns out that the numerical results for the stress peaks are rather close to the analytical values and the relative errors vary between 0.2 and 3%. For the ratio 0.8 we have a relative error for the stress peak of about 3 %, and for the corresponding large stress peak $\alpha_{\text{num}} = 12.7$, plasticity effects will certainly occur which are not allowd by our linear model.

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