INVERSE SCATTERING FOR A PENETRABLE CAVITY AND THE TRANSMISSION EIGENVALUE PROBLEM

by

Shixu Meng

A dissertation submitted to the Faculty of the University of Delaware in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Applied Mathematics

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Shixu Meng

Approved: ______________________________

Louis Rossi, Ph.D.
Chair of the Department of Mathematical Sciences

Approved: ______________________________

George H. Watson , Ph.D.
Dean of the College of Art and Science

Approved: ______________________________

Ann L. Ardis, Ph.D.
Senior Vice Provost for Graduate and Professional Education

I certify that I have read this dissertation and that in my opinion it meets the academic and professional standard required by the University as a dissertation for the degree of Doctor of Philosophy.

Signed: ______________________________

David Colton, Ph.D.
Professor in charge of dissertation

I certify that I have read this dissertation and that in my opinion it meets the academic and professional standard required by the University as a dissertation for the degree of Doctor of Philosophy.

Signed: ______________________________

Fioralba Cakoni, Ph.D.
Member of dissertation committee

I certify that I have read this dissertation and that in my opinion it meets the academic and professional standard required by the University as a dissertation for the degree of Doctor of Philosophy.

Signed: ______________________________

Y. J. Leung, Ph.D.
Member of dissertation committee

I certify that I have read this dissertation and that in my opinion it meets the academic and professional standard required by the University as a dissertation for the degree of Doctor of Philosophy.

Signed: ______________________________

Rakesh, Ph.D.
Member of dissertation committee

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ABSTRACT

In this thesis, we consider the scattering of point sources inside a cavity surrounded by an inhomogeneous medium and its inverse problem of determining the boundary of the cavity from measurements of the scattered field inside the cavity. We apply the linear sampling method and factorization method to numerically reconstruct the boundary of the cavity. We prove that the linear sampling method works when the wave number is not an exterior transmission eigenvalue. We prove that the exterior transmission eigenvalues form a discrete set. We then consider both the exterior transmission eigenvalue problem and the interior transmission eigenvalue problem for a spherically stratified media and study the inverse spectral problem for the exterior transmission eigenvalue problem. Finally we consider the interior transmission eigenvalue problem for Maxwell's equations corresponding to non-magnetic inhomogeneities with contrast in electric permittivity that changes sign inside its support. We prove that the set of transmission eigenvalues is nonempty discrete, infinite and without finite accumulation points.

Chapter 1

INTRODUCTION

The field of inverse scattering theory plays an important role in non-destructive testing, medical imaging, geophysical exploration and numerous problems associated with target identification. Inverse scattering problems are complicated by the fact that such problems are both nonlinear and improperly posed in the sense that the solution does not depend continuously on the data. The literature on inverse scattering is huge and the discussion here is hence limited. *Qualitative methods* are reconstructive methods to retrieve information about the scattering media which require little or no a priori information. The *linear sampling method* as introduced by Colton and Kirsch [23], and Colton et at. [30] (see also the monograph [9]) and *factorization method* as introduced by Kirsch [53] (see also the monograph [55]) belong to the class of qualitative methods. Qualitative methods are concerned with locating and analyzing scatterers from measurements of the scattered field due to a known interrogating wave. Qualitative methods have been further developed by many others and have played an important role in non-destructive testing and imaging [9, 55]. For a more detailed introduction to qualitative methods we refer to the book [9].

There are several types of inverse scattering problems, for example inverse scattering for obstacles, inverse scattering for inhomogeneous media and inverse scattering for cracks, etc. We consider an inverse scattering problem due to sources placed on a surface C inside a cavity D surrounded by a penetrable, inhomogeneous medium D_1 with the aim of determining the shape of the cavity (see Figure 1.1 for an example of the geometry). Such problems arise in the use of nondestructive methods for determining the integrity of the interior boundary of a container that is inaccessible to direct observation (c.f. [44, 45, 65, 72–74, 84]).

The direct scattering problem we are considering is for frequencies in the resonance region and in a time harmonic setting. A point source located at y is defined by

$$\Phi(x,y) = \begin{cases} \frac{i}{4} H_0^{(1)}(k|x-y|) & \text{in } \mathbb{R}^2 \\ \\ \dfrac{e^{ik|x-y|}}{4\pi|x-y|} & \text{in } \mathbb{R}^3. \end{cases}$$

For any point source $\Phi(\cdot, y)$ located at $y \in D$, there is a corresponding scattered field in D, denoted by $u^s(\cdot, y)$, which satisfies

$$\Delta u^s(\cdot, y) + k^2 u^s(\cdot, y) = 0 \qquad \text{in} \quad D$$

where k is the wave number.

In the exterior of D the total field, defined by $w(\cdot, y) := \Phi(\cdot, y) + u^s(\cdot, y)$, satisfies

$$\nabla \cdot A \nabla w(\cdot, y) + k^2 n w(\cdot, y) = 0 \qquad \text{in} \quad \mathbb{R}^d \setminus \overline{D}$$

and the radiation condition

$$\lim_{r \to \infty} r^{\frac{d-1}{2}} \left(\frac{\partial w(\cdot, y)}{\partial r} - ikw(\cdot, y) \right) = 0$$

where $d = 2, 3$ denotes the dimension of D, and the matrix valued function A and scalar function n are functions that are determined by the material properties of the medium surrounding the cavity.

The fields inside and outside of the cavity D are connected by the boundary conditions

$$\begin{aligned} w(\cdot, y) - u^s(\cdot, y) &= \Phi(\cdot, y) \quad \text{on} \quad \partial D \\ A \nabla w(\cdot, y) \cdot \nu - \nabla u^s(\cdot, y) \cdot \nu &= \nabla \Phi(\cdot, y) \cdot \nu \quad \text{on} \quad \partial D \end{aligned}$$

where ν is the unit outward normal vector on ∂D. The direct scattering problem is well-posed according to Hadamard, i.e. the solution exists, is unique, and depends continuously on the initial data. We now formulate the corresponding inverse problem:

For any point source at $y \in C$, we measure the scattered field $u^s(x,y)$ at $x \in C$. We call $u^s(x,y)$ (for all x and y on C) the (full aperture) near field data. From this information we want to determine the unknown boundary ∂D of the cavity.

Contrary to the direct scattering problem, the inverse scattering problem is ill-posed. The *linear sampling method* for solving the above inverse problem is based on finding a regularized solution of the ill-posed integral equation

$$\int_C u^s(x,y) g_z(y) ds(y) = \Phi(x,z) \tag{1.1}$$

where $g_z \in L^2(C)$ and $\Phi(\cdot, z)$ is a point source at z. The boundary ∂D of the cavity is then determined by using regularization methods to solve (1.1) for g_z. In particular, ∂D is determined by examing the norm of g_z which, roughly speaking, is large outside D and relatively small inside D (a closely related method for doing this is the above mentioned *factorization method*). However this approach is only valid for values of k that are not exterior transmission eigenvalues, i.e. values of k such that there exists a non trivial solution (w, v) satisfying

$$\nabla \cdot A \nabla w + k^2 n(x) w = 0 \quad \text{and} \quad \Delta v + k^2 v = 0 \qquad \text{in} \qquad \mathbb{R}^d \setminus \overline{D} \tag{1.2}$$

$$w = v \quad \text{and} \quad \frac{\partial w}{\partial \nu_A} = \frac{\partial v}{\partial \nu} \qquad \text{on} \qquad \partial D \tag{1.3}$$

$$\lim_{r \to \infty} r^{\frac{d-1}{2}} \left(\frac{\partial w}{\partial r} - ikw\right) = 0 \quad \text{and} \quad \lim_{r \to \infty} r^{\frac{d-1}{2}} \left(\frac{\partial v}{\partial r} - ikv\right) = 0 \tag{1.4}$$

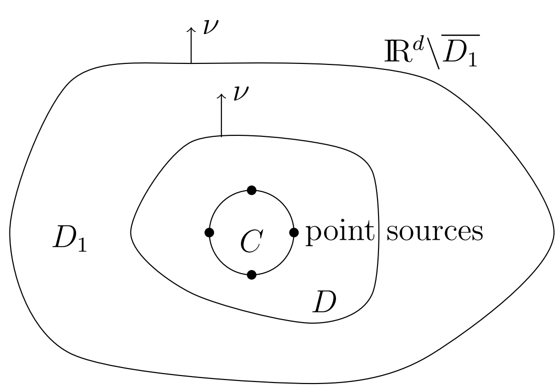


Figure 1.1: An example of the geometry

where the above partial differential equations are satisfied in the sense of distributions. The nonhomogeneous exterior transmission eigenvalue problem is, given $f \in H^{\frac{1}{2}}(\partial D)$, $h \in H^{-\frac{1}{2}}(\partial D)$, $\ell_1 \in L^2(B_R \setminus \overline{D})$ and $\ell_2 \in L^2(B_R \setminus \overline{D})$, find $w \in H^1_{loc}(\mathbb{R}^d \setminus \overline{D})$, $v \in H^1_{loc}(\mathbb{R}^d \setminus \overline{D})$ such that

$$\nabla \cdot A\nabla w + k^2 n w = \ell_1 \qquad \text{in} \quad \mathbb{R}^d \setminus \overline{D} \tag{1.5}$$

$$\Delta v + k^2 v = \ell_2 \qquad \text{in} \quad \mathbb{R}^d \setminus \overline{D} \tag{1.6}$$

$$w - v = f \qquad \text{on} \quad \partial D \tag{1.7}$$

$$\frac{\partial w}{\partial \nu_A} - \frac{\partial v}{\partial \nu} = h \qquad \text{on} \quad \partial D \tag{1.8}$$

$$\lim_{r\to\infty} r^{\frac{d-1}{2}}\left(\frac{\partial w}{\partial r} - ikw\right) = 0 \qquad \text{and} \qquad \lim_{r\to\infty} r^{\frac{d-1}{2}}\left(\frac{\partial v}{\partial r} - ikv\right) = 0, \tag{1.9}$$

where ℓ_1 and ℓ_2 vanish in $\mathbb{R}^d \setminus \overline{B_R}$ and R is the radius of the ball B_R outside of which $A = I$ and $n = 1$. In particular the well-posedness of the nonhomogeneous exterior transmission eigenvalue problem (1.5)-(1.9) allows us to prove that a knowledge of the exact scattering data on C uniquely determine the support of the cavity (see Chapter 3).

The exterior transmission eigenvalue problem (1.2)-(1.4) that appears in the above inverse scattering problem for a cavity is just one of a number of transmission eigenvalue problems that have appeared in inverse scattering. Here, in addition to (1.2)-(1.4), we have also considered several different transmission eigenvalue problem that have appreared in the literature and have examined the questions of the discreteness and existence of the transmission eigenvalues.

In particular, we have considered electromagnetic wave propagation in a non-absorbing isotropic non-magnetic medium. In particular, the transmission eigenvalues are values of k for which there exist non-trivial $(\mathbf{w}, \mathbf{v})$ such that

$$\text{curlcurl}\mathbf{w} - k^2 n(x)\mathbf{w} = 0 \quad \text{and} \quad \text{curlcurl}\mathbf{v} - k^2\mathbf{v} = 0 \qquad \text{in} \qquad D \tag{1.10}$$

$$\nu \times \mathbf{w} = \nu \times \mathbf{v} \quad \text{and} \quad \nu \times \text{curl}\mathbf{w} = \nu \times \text{curl}\mathbf{v} \qquad \text{on} \qquad \partial D \tag{1.11}$$

where the above partial differential equations are satisfied in the sense of distributions. For both (1.2)-(1.4) and (1.10)-(1.11) we have shown that transmission eigenvalues exist and form a discrete set (see Chapter 6 and Chapter 7). Note that the exterior transmission eigenvalue problem is defined in an exterior domain while the interior transmission eigenvalue problem is defined in an interior domain.

In the case of stratified media where the eigenfunctions are spherically stratified, the exterior transmission eigenvalue problem (1.2)-(1.4) is reduced to finding $y(r)$ and $y_0(r)$ such that

$$y'' + k^2 n(r) y = 0 \qquad \text{and} \qquad y_0'' + k^2 y_0 = 0 \quad \text{in} \quad [a, \infty), \tag{1.12}$$

$$a_0 y(a) = b_0 y_0(a) \qquad \text{and} \qquad a_0 y'(a) = b_0 y_0'(a). \tag{1.13}$$

y and y_0 are normalized such that

$$y(r) = y_0(r) = e^{ikr}, \quad r > b.$$

Then k is a transmission eigenvalue if and only if

$$D(k) := \det \begin{vmatrix} y(a) & e^{ika} \\ y'(a) & ike^{ika} \end{vmatrix} = 0$$

i.e. the transmission eigenvalues are the zeros of the entire function $D(k)$. Using the theory of entire function of a complex variable we have shown that there exist both real and complex transmission eigenvalues to (1.12)-(1.13) (see Chapter 5).

The scalar interior transmission eigenvalues are values of k such that there exists a non trivial solution (w, v) satisfying

$$\Delta w + k^2 n(x) w = 0 \quad \text{and} \quad \Delta v + k^2 v = 0 \qquad \text{in} \qquad D \tag{1.14}$$

$$w = v \quad \text{and} \quad \frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} \qquad \text{on} \qquad \partial D \tag{1.15}$$

where the above partial differential equations are satisfied in the sense of distributions. In the case of a spherically stratified scattering media the interior transmission eigenvalue problem (1.14)-(1.15) is reduced to finding $y(r)$ and $y_0(r)$ such that

$$y'' + k^2 n(r) y = 0 \qquad \text{and} \qquad y_0 = \sin kr \quad \text{in} \quad [0, a], \tag{1.16}$$

$$y(0) = 0 \qquad \text{and} \qquad y'(0) = 1. \tag{1.17}$$

In Chapter 5 we consider the existence and distribution of eigenvalues to (1.16)-(1.17).

The thesis is organized as following.

In Chapter 2, we begin by considering the scattering of time harmonic acoustic and electromagnetic waves by a point source $\Phi(\cdot, y)$ inside a cavity. We show that the direct scattering problem is well-posed and consider the inverse problem to determine the boundary ∂D from a knowledge of internal measurements inside the cavity. Relevant paper is [12] *F. Cakoni, D. Colton, S. Meng, The inverse scattering problem for a penetrable cavity with internal measurements, AMS Contemporary Mathematics, 615, 71-88(2014).*

In Chapter 3, we prove that the exterior transmission eigenvalues (corresponding to the inverse scattering problem in Chapter 2 form a discrete set. We use variational methods for anisotropic media [12] and integral equation methods [28] for isotropic media. Relevant papers are [12] *F. Cakoni, D. Colton, S. Meng, The inverse scattering problem for a penetrable cavity with internal measurements, AMS Contemporary Mathematics, 615, 71-88(2014)* and [28] *D. Colton and S. Meng, Spectral properties of the exterior transmission eigenvalue problem, Inverse Problems* **30***, no. 10, 105010 (2014).*

In Chapter 4, we apply the linear sampling method [12] and factorization method [68] to the inverse scattering problem in Chapter 2. We use the results from Chapter 3 to show that the data on a surface inside the cavity uniquely determines the boundary of the cavity. We then prove that the linear sampling method works when the wave number is not a transmission eigenvalue [12] and provide preliminary numerical examples. We adapt the factorization method developed in [61] and prove that we can avoid the transmission eigenvalues provided the sampling region is well chosen [68]. Relevant papers are [12] *F. Cakoni, D. Colton, S. Meng, The inverse scattering problem for a penetrable cavity with internal measurements, AMS Contemporary Mathematics, 615, 71-88(2014)* and [68] *S. Meng, H. Haddar, F. Cakoni, The factorization method for a cavity in an inhomogeneous medium, Inverse Problems, 30, 045008 (2014).*

In Chapter 5, we first consider the exterior transmission eigenvalue problem (1.12)-(1.13) for a spherically stratified media. We determine conditions on the index of refraction which guarantee the existence of infinitely many complex eigenvalues or infinitely many real eigenvalues. We then show that if two sets of spectral data are known, then under appropriate conditions, the index of refraction is uniquely determined [26]. Moreover we show that the refractive index can be uniquely determined from the knowledge of all the transmission eigenvalues [28]. We then consider the distribution of transmission eigenvalues for the transmission eigenvalue problem (1.16)-(1.17). In particular, we show that under smoothness condition on the index of refraction that there exist an infinite number of complex eigenvalues and situations when there are no real eigenvalues. We also consider the case when absorption is present and show that under appropriate conditions the eigenvalues accumulate near the real axis [27]. Relevant papers are [26] *D. Colton, Y.J. Leung and S. Meng, The inverse spectral problem for exterior transmission eigenvalues, Inverse Problems, 30, 055010 (2014)* and [27] *D. Colton, Y.J. Leung and S. Meng, Distribution of complex transmission eigenvalues for spherically stratified media, Inverse Problems, 31, 035006 (2015).*

In Chapter 6, we consider the transmission eigenvalue problem for Maxwell's equations corresponding to non-magnetic inhomogeneities with contrast in electric permittivity that possibly changes sign inside its support. We formulate the transmission eigenvalue problem as an equivalent homogeneous system of boundary integral equation and prove that if the contrast is constant near the boundary of the support of the inhomogeneity, then the set of transmission eigenvalues is discrete without finite accumulation points [19]. Relevant paper is [19] *F. Cakoni, H. Haddar and S. Meng, Boundary Integral Equations for the Transmission Eigenvalue Problem for Maxwell Equations, J. Integral Equations and Applications, 27, No.3, 375-406, 2015.*

In Chapter 7, we continue our study of the problem considered in Chapter 6. We study this problem in the framework of semiclassical analysis and relate the transmission eigenvalues to the spectrum of a Hilbert-Schmidt operator. Under the additional assumption that the contrast is constant in a neighborhood of the boundary, we prove that the set of transmission eigenvalues is nonempty, discrete, infinite and without finite accumulation points. A notion of generalized eigenfunctions is introduced and a denseness result is obtained in an appropriate solution space [38]. Relevant paper is [38] *H. Haddar and S. Meng, Spectral analysis of the transmission eigenvalue problem for Maxwell's equations, submitted.*

Chapter 2

THE DIRECT AND INVERSE SCATTERING PROBLEM

2.1 Formulation of the Direct Problem

We begin by considering the propagation of sound wave in three dimensions viewed as a problem in fluid dynamics. Let $v(x,t)$ be the velocity vector of a fluid particle in an inviscid fluid and $p(x,t), \rho(x,t), S(x,t)$ denote the pressure, density, specific entropy, respectively, of the fluid. If no external forces are acting on the fluid, then we have the equations

$$\frac{\partial v}{\partial t} + (v \cdot \nabla)v + \frac{1}{\rho}\nabla p = 0 \qquad \text{(Euler's equation)} \tag{2.1}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0 \qquad \text{(equation of continuity)} \tag{2.2}$$

$$p = f(\rho, S) \qquad \text{(equation of state)} \tag{2.3}$$

$$\frac{\partial S}{\partial t} + v \cdot \nabla S = 0 \qquad \text{(adiabatic hypothesis)} \tag{2.4}$$

where f is a function depending on the fluid. Assuming $v(x,t), p(x,t), \rho(x,t), S(x,t)$ are small, we perturb these quantities around the static state $v = 0$, $p = p_0$ =constant, $\rho = \rho_0(x)$, $S = S_0(x)$ with $p_0 = f(p_0, S_0)$ and write their asymptotic expansion

$$v(x,t) = \epsilon v_1(x,t) + \cdots \tag{2.5}$$

$$p(x,t) = p_0 + \epsilon p_1(x,t) + \cdots \tag{2.6}$$

$$\rho(x,t) = \rho_0(x,t) + \epsilon \rho_1(x,t) + \cdots \tag{2.7}$$

$$S(x,t) = S_0(x,t) + \epsilon S_1(x,t) + \cdots \tag{2.8}$$

where $0 < \epsilon \ll 1$ and the dots refer to higher order terms in ϵ. We now substitute (2.1)-(2.4) into (2.5)-(2.8), retaining only the terms of order ϵ. Doing this gives us the linearized equations

$$\begin{aligned}\frac{\partial v_1}{\partial t} + \frac{1}{\rho_0}\nabla p_1 &= 0\\ \frac{\partial \rho_1}{\partial t} + \nabla\cdot(\rho_0 v_1) &= 0\\ c^2(x)\big(\frac{\partial \rho_1}{\partial t} + v_1\cdot\nabla\rho_0\big) &= \frac{\partial p_1}{\partial t}\end{aligned}$$

where the sound speed c is defined by

$$c^2(x) = \frac{\partial}{\partial \rho} f(\rho_0(x), S_0(x)).$$

From this we deduce that p_1 satisfies

$$\frac{\partial^2 p_1}{\partial t^2} = c^2(x)\rho_0(x)\nabla\cdot\big(\frac{1}{\rho_0(x)}\nabla p_1\big).$$

If we now assume that p_1 is time harmonic,

$$p_1(x,t) = \Re\{w(x)e^{-i\omega t}\},$$

then w satisfies

$$\rho_0(x)\nabla\cdot\frac{1}{\rho_0(x)}\nabla w + \frac{\omega^2}{c^2(x)}w = 0. \tag{2.9}$$

The above equation governs the propagation of time harmonic acoustic waves of small amplitude in a slowly varying inhomogeneous medium. Considering the wave motion is caused by a point source $\Phi(\cdot,y) = \frac{e^{ik|x-y|}}{4\pi|x-y|}$ located at $y \in \mathbb{R}^3$ inside a cavity being scattered by an inhomogeneous medium, and assuming the inhomogeneous medium is contained inside $D_1\backslash\overline{D}$, i.e., $c(x) = c_0 = \text{constant}, \rho_0(x) = \rho_{00} = \text{constant}$ outside $D_1\backslash\overline{D}$(see figure 1.1), we see the scattering problem under consideration is now modeled by

$$\nabla\cdot A(x)\nabla w + k^2 n(x) w = -\delta(x-y) \tag{2.10}$$

$$w = \Phi(\cdot,y) + u^s \tag{2.11}$$

$$\lim_{r\to\infty} r\big(\frac{\partial u^s}{\partial r} - iku^s\big) = 0 \tag{2.12}$$

where $k := \frac{\omega}{c(x)}$, $n(x) := \frac{c_0^2}{c^2(x)\rho_0(x)}$, $A(x) := \frac{1}{\rho_0(x)}$. In particular, continuity of w and $\nu \cdot A\nabla w$ is assumed across ∂D and ∂D_1. Wring in terms of $w \in H^1_{loc}(\mathbb{R}^3 \backslash \overline{D})$ and $u^s \in H^1(D)$ yields

$$\nabla \cdot A\nabla w + k^2 n w = 0 \qquad \text{in} \quad \mathbb{R}^d \setminus \overline{D} \tag{2.13}$$

$$\Delta u^s + k^2 u^s = 0 \qquad \text{in} \quad D \tag{2.14}$$

$$w - u^s = \Phi(\cdot, y) \qquad \text{on} \quad \partial D \tag{2.15}$$

$$\frac{\partial w}{\partial \nu_A} - \frac{\partial u^s}{\partial \nu} = \frac{\partial \Phi(\cdot, y)}{\partial \nu} \qquad \text{on} \quad \partial D \tag{2.16}$$

$$\lim_{r\to\infty} r\left(\frac{\partial w}{\partial r} - ikw\right) = 0 \tag{2.17}$$

Next we consider the wave propagation of electromagnetic wave in three dimensions with electric permittivity $\epsilon = \epsilon(x)$, magnetic permittivity $\mu = \mu(x)$, and electric conductivity $\sigma = \sigma(x)$. As well known the electromagnetic wave is described by the electric field $\mathcal{E}$ and the magnetic field $\mathcal{H}$ satisfying the Maxwell equations

$$\text{curl}\mathcal{E} + \mu\frac{\partial \mathcal{H}}{\partial t} = 0 \quad \text{and} \quad \text{curl}\mathcal{H} - \epsilon\frac{\partial \mathcal{E}}{\partial t} = \sigma\mathcal{E}.$$

For time harmonic electromagnetic waves of the form

$$\mathcal{E}(x,t) = \tilde{E}(x)e^{-i\omega t} \quad \text{and} \quad \mathcal{H}(x,t) = \tilde{H}(x)e^{-i\omega t}$$

with frequency $\omega > 0$, we deduce that the complex valued space dependent parts $\tilde{E}(x)$ and $\tilde{H}(x)$ satisfy

$$\text{curl}\tilde{E} - i\omega\mu(x)\tilde{H} = 0 \quad \text{and} \quad \text{curl}\tilde{H} + (i\omega\epsilon(x) - \sigma(x))\tilde{E} = 0.$$

Now let us suppose that the inhomogeneity occupies an infinitely long conducting cylindrical shell. Let $D_1 \backslash \overline{D}$ be the cross section of this cylindrical shell with ν being the unit outward normal to the boundary. We assume that the axis of the cylinder coincides with the z-axis. We further assume that the conductor is filled with a nonconducting homogeneous background, i.e., the electric permittivity $\epsilon_0 > 0$, the magnetic permittivity $\mu_0 > 0$, and the conductivity $\sigma_0 = 0$ inside D. Next we define

$$E^{int,ext} = \sqrt{\epsilon_0}\tilde{E}^{int,ext}, \quad H^{int,ext} = \sqrt{\mu_0}\tilde{H}^{int,ext}, \quad k^2 = \epsilon_0\mu_0\omega^2,$$

$$\mathcal{A}(x) = \frac{1}{\epsilon_0}\big(\epsilon(x) + i\frac{\sigma(x)}{\omega}\big), \quad \mathcal{N}(x) = \frac{1}{\mu_0}\mu(x)$$

where E^{int}, H^{int} and E^{ext}, H^{ext} denote the electric and magnetic fields inside and outside the homogeneous background D, respectively. For an orthotropic medium we have the matrices $\mathcal{A}$ and $\mathcal{N}$ are independent of the z-coordinate and are of the form

$$\mathcal{A} = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & a \end{pmatrix} \quad \text{and} \quad \mathcal{N} = \begin{pmatrix} n_{11} & n_{12} & 0 \\ n_{21} & n_{22} & 0 \\ 0 & 0 & n \end{pmatrix}.$$

In particular, the field E^{int} and H^{int} inside D satisfy

$$\text{curl}E^{int} - ikH^{int} = 0 \quad \text{and} \quad \text{curl}H^{int} + ikE^{int} = 0 \tag{2.18}$$

and the field E^{ext} and H^{ext} outside D satisfy

$$\text{curl}E^{ext} - ik\mathcal{N}H^{ext} = 0 \quad \text{and} \quad \text{curl}H^{ext} + ik\mathcal{A}E^{ext} = 0. \tag{2.19}$$

Across the boundary ∂D we have the continuity of the tangential component of both the electric and magnetic fields. Assuming that $\mathcal{A}$ is invertible, from (2.18) (2.19)

$$\text{curlcurl}H^{int} - k^2H^{int} = 0 \tag{2.20}$$

and

$$\text{curl}(\mathcal{A}^{-1}\text{curl}H^{ext}) - k^2\mathcal{N}H^{ext} = 0. \tag{2.21}$$

The electromagnetic wave is caused by incident fields E^i, H^i satisfying (2.18), i.e.

$$E^{ext} = E^s + E^i \quad \text{and} \quad H^{ext} = H^s + H^i$$

where E^s and H^s denote the scattered field and satisfy the Silver-Muller radiating condition

$$\lim_{r\to\infty}(H^s \times x - rE^s) = 0.$$

Now assume that the incident wave propagates perpendicular to the axis of the cylinder and is polarized perpendicular to the axis of the cylinder such that

$$H^i(x) = (0,0,u^i), \quad H^s(x) = (0,0,u^s), \quad H^{int}(x) = (0,0,w).$$

By elementary vector analysis, one can conclude (2.21) is equivalent to

$$\nabla \cdot A(x)\nabla w + k^2 n(x) w = 0 \quad \text{in} \quad \mathbb{R}^2\backslash\overline{D} \tag{2.22}$$

where

$$A := \frac{1}{\det(A)} \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}.$$

Analogously, (2.20) is equivalent to

$$\Delta u^s + k^2 u^s = 0 \quad \text{in} \quad D. \tag{2.23}$$

The transmission conditions

$$\nu \times (H^s + H^i) = \nu \times H^{int}, \quad \nu \times \mathrm{curl}(H^s + H^i) = \nu \times \mathcal{A}^{-1}\mathrm{curl} H^{int}$$

on the boundary of the conductor becomes

$$w - u^s = u^i, \quad \nu \cdot A\nabla w - \nu \cdot \nabla u^s = \nu \cdot \nabla u^i \quad \text{on} \quad \partial D.$$

The $\mathbb{R}^2$ analog of the Silver-Muller radiating condition is the Sommerfeld radiating condition

$$\lim_{r\to\infty} \sqrt{r}(\frac{\partial u^s}{\partial r} - iku^s) = 0.$$

If we consider the scattering of TM magnetic dipole located at $y \in \mathbb{R}^2$, i.e., $u^i = \Phi(\cdot, y) = \frac{e^{ik|x-y|}}{4\pi|x-y|}$, the scattering problem under consideration is modeled by

$$\nabla \cdot A\nabla w + k^2 n w = 0 \qquad \text{in} \quad \mathbb{R}^2\backslash\overline{D} \tag{2.24}$$

$$\Delta u^s + k^2 u^s = 0 \qquad \text{in} \quad D \tag{2.25}$$

$$w - u^s = \Phi(\cdot, y) \qquad \text{on} \quad \partial D \tag{2.26}$$

$$\frac{\partial w}{\partial \nu_A} - \frac{\partial u^s}{\partial \nu} = \frac{\partial \Phi(\cdot, y)}{\partial \nu} \qquad \text{on} \quad \partial D \tag{2.27}$$

$$\lim_{r\to\infty} \sqrt{r}(\frac{\partial w}{\partial r} - ikw) = 0 \tag{2.28}$$

In conclusion the scattering of acoustic wave in three dimension (2.13)-(2.17) and electromagnetic wave in two dimension (2.24)-(2.28) by the point source $\Phi(\cdot, y)$ inside a cavity are modeled by

$$\nabla \cdot A\nabla w + k^2 nw = 0 \qquad \text{in} \quad \mathbb{R}^d \setminus \overline{D} \tag{2.29}$$

$$\Delta u^s + k^2 u^s = 0 \qquad \text{in} \quad D \tag{2.30}$$

$$w - u^s = f \qquad \text{on} \quad \partial D \tag{2.31}$$

$$\frac{\partial w}{\partial \nu_A} - \frac{\partial u^s}{\partial \nu} = h \qquad \text{on} \quad \partial D \tag{2.32}$$

$$\lim_{r\to\infty} r^{\frac{d-1}{2}}(\frac{\partial w}{\partial r} - ikw) = 0 \tag{2.33}$$

where $f = \Phi(\cdot, y)$, $h = \frac{\partial \Phi(\cdot,y)}{\partial \nu}$, $d = 2, 3$ denotes the dimension and the point source located at $y \in \mathbb{R}^d$ is given by

$$\Phi(x,y) = \begin{cases} \frac{i}{4} H_0^{(1)}(k|x-y|) & \text{in } \mathbb{R}^2 \\ \\ \dfrac{e^{ik|x-y|}}{4\pi|x-y|} & \text{in } \mathbb{R}^3. \end{cases} \tag{2.34}$$

For later use, we make more precise mathematical setting of our time harmonic scattering problem. Let $D, D_1 \subset \mathbb{R}^d$, $d = 2, 3$, be simply connected bounded regions of $\mathbb{R}^d$ with Lipchitz boundary $\partial D, \partial D_1$ and denote by ν the outward unit normal to the boundary. We assume the medium inside D and outside D_1 is homogeneous with refractive index scaled to one and denote by k the corresponding wave number. The medium inside $D_1 \backslash \overline{D}$ is assumed to be inhomogeneous and possibly anisotropic. More specifically, the physical properties of the medium in $D_1 \backslash \overline{D}$ are described by the $d \times d$ symmetric matrix valued function with $C^2(D_1 \backslash \overline{D})$ entries(in fact, we can relax the assumption to piecewise Lipschitz continuous entries having only finitely many jumps) and the bounded function $n \in L^\infty(D_1 \backslash \overline{D})$ such that $\overline{\xi} \cdot \Re(A)\xi \geq \alpha \|\xi\|^2$, $\overline{\xi} \cdot \Im(A)\xi \leq 0$, for all $\xi \in \mathbb{C}$ and $\Re(n) \geq n_0 > 0$ in $\mathbb{R}^d \setminus \overline{D}$. Furthermore, we assume that $A \equiv I$ and $n \equiv 1$ in $\mathbb{R}^d \setminus D_1$ and D where $\text{supp}(A - I) = \text{supp}(n - 1) = D_1 \backslash D$ and B_R is a large ball containing D_1. Note that all the assumptions we make on the anisotropic medium hold thanks to the physical constitutions.

2.2 Wellposedness of the Forward Problem

In this section, we will study the uniqueness, solvability and stability of the forward problem (2.29)-(2.33). In particular, we use the variational methods for elliptic partial differential equations (c.f. [9]). Since the problem is formulated in an unbounded domain, we introduce the Dirichlet to Neumann map. Let B_R is a sufficiently large ball that contains D_1, then the (exterior) Dirichlet to Neumann map $T_k : H^{\frac{1}{2}}(\partial B_R) \to H^{-\frac{1}{2}}(\partial B_R)$ is defined by

$$T_k : g \to \frac{\partial u}{\partial \nu}|_{\partial B_R} \quad \text{for} \quad g \in H^{\frac{1}{2}}(\partial B_R) \tag{2.35}$$

where u is the radiating solution to the Helmholtz equation $\Delta u + k^2 u = 0$ outside B_R with boundary data $u = g$ on ∂B_R, and ν is the outward unit normal to ∂B_R. Now we state the following theorem (see e.g. [9]).

Theorem 2.2.1 *The Dirichlet to Neumann map T_k defined by (2.35) is a bounded linear operator from $H^{\frac{1}{2}}(\partial B_R)$ to $H^{-\frac{1}{2}}(\partial B_R)$. Furthermore, there exist a bounded operator $T_0 : H^{\frac{1}{2}}(\partial B_R) \to H^{-\frac{1}{2}}(\partial B_R)$ satisfying*

$$-\int_{\partial B_R} T_0 w \overline{w} \geq C||w||^2_{H^{\frac{1}{2}}(\partial B_R)} \tag{2.36}$$

for some constant $C > 0$ such that $T - T_0 : H^{\frac{1}{2}}(\partial B_R) \to H^{-\frac{1}{2}}(\partial B_R)$ is compact.

From the Dirichlet to Neumann map, equations (2.29)-(2.33) can be written as

$$\nabla \cdot A \nabla w + k^2 n w = 0 \qquad \text{in} \quad \mathbb{R}^d \setminus \overline{D} \tag{2.37}$$

$$\Delta u^s + k^2 u^s = 0 \qquad \text{in} \quad D \tag{2.38}$$

$$w - u^s = f \qquad \text{on} \quad \partial D \tag{2.39}$$

$$\frac{\partial w}{\partial \nu_A} - \frac{\partial u^s}{\partial \nu} = h \qquad \text{on} \quad \partial D \tag{2.40}$$

$$\frac{\partial w}{\partial \nu} = T_k w \qquad \text{on} \quad \partial B_R \tag{2.41}$$

The Dirichlet to Neumann map guarantees that one can extend $w \in H^1(B_R)$ in (2.37)-(2.41) to $w \in H^1_{loc}(\mathbb{R}^d \setminus \overline{D})$ in (2.29)-(2.33), therefore we have the following (see e.g. [9]).

Lemma 2.2.1 *The boundary value problems (2.29)-(2.33) and (2.37)-(2.41) are equivalent.*

From the classical Dirichlet boundary value problem, there exists a unique radiating solution $u_\ell \in H^1_{loc}(\mathbb{R}^d \setminus \overline{D})$ to

$$\Delta u_\ell + k^2 u_\ell = 0 \qquad \text{in} \quad \mathbb{R}^d \backslash \overline{D}$$

$$u_\ell = f \qquad \text{on} \quad \partial D$$

for any $f \in H^{\frac{1}{2}}(\partial D)$ and $||u_\ell||_{H^1(B_R\backslash\overline{D})} \leq C||f||_{H^{\frac{1}{2}}(\partial D)}$.

Now we will derive an equivalent variational formulation of the problem (2.37)-(2.41). Define

$$u := u^s|_D + (w - u_\ell)|_{B_R\backslash D}. \tag{2.42}$$

Multiplying (2.37) and (2.38) by $\overline{\varphi}$ and integrating by parts yields

$$\int_{\partial B_R} T_k w\overline{\varphi} ds - \int_{\partial D} \frac{\partial w}{\partial \nu_A}\overline{\varphi} ds - \int_{B_R\backslash\overline{D}} A\nabla w \cdot \nabla\overline{\varphi} dx + k^2 \int_{B_R\backslash\overline{D}} nw\overline{\varphi} dx = 0 \tag{2.43}$$

and

$$\int_{\partial D} \frac{\partial u^s}{\partial \nu}\overline{\varphi} ds - \int_D \nabla u^s \cdot \nabla\overline{\varphi} dx + k^2 \int_D u^s\overline{\varphi} dx = 0. \tag{2.44}$$

Then equations (2.43)-(2.44) and the definition of u yield

$$\begin{aligned} &\int_{\partial B_R} T_k u\overline{\varphi} ds - \int_{B_R} A\nabla u \cdot \nabla\overline{\varphi} dx + k^2 \int_{B_R} nu\overline{\varphi} dx \\ = \ &\int_{\partial D} h\overline{\varphi} ds + \int_{B_R\backslash\overline{D}} A\nabla u_\ell \cdot \nabla\overline{\varphi} dx - k^2 \int_{B_R\backslash\overline{D}} nu_\ell\overline{\varphi} dx - \int_{\partial B_R} T_k u_\ell\overline{\varphi} ds. \end{aligned} \tag{2.45}$$

Lemma 2.2.2 *Let $u \in H^1(B_R)$ be defined by (2.42). If $u^s \in H^1(D)$ and $w \in H^1(B_R\backslash\overline{D})$ satisfies (2.37)-(2.41), then u satisfies (2.45). Conversely if u satisfies (2.45), then $u^s := u|_D$ and $w := (u + u_\ell)|_{B_R\backslash D}$ satisfies (2.37)-(2.41).*

Proof. The above argument implies the first part. Conversely suppose $u \in H^1(B_R)$ satisfies (2.45). Let φ be a test function compactly supported in $B_R\backslash\overline{D}$ and D respectively, then a direct integration by parts yields (2.37) and (2.38). Choosing test function $\varphi \in H^1_{loc}(B_R)$ and matching the boundary conditions yields (2.39), (2.40) and (2.41). This proves the Lemma. $\square$

Now let

$$\begin{aligned} a_1(u,\varphi) &= \int_{B_R} A\nabla u \cdot \nabla\overline{\varphi}dx + \int_{B_R} \nabla u \nabla\overline{\varphi}dx - \int_{\partial B_R} T_0 u\overline{\varphi}ds \\ a_2(u,\varphi) &= -\int_{\partial B_R} (T_k - T_0)u\overline{\varphi}ds - \int_{B_R} (k^2 n + 1)u\overline{\varphi}dx \\ F(\varphi) &= \int_{\partial B_R} T_k u_\ell \overline{\varphi}ds - \int_{\partial D} h\overline{\varphi}ds - \int_{B_R\setminus\overline{D}} A\nabla u_\ell \cdot \nabla\overline{\varphi}dx + k^2 \int_{B_R\setminus\overline{D}} n u_\ell \overline{\varphi}dx. \end{aligned}$$

Then (2.45) is equivalent to

$$a_1(u,\varphi) + a_2(u,\varphi) = F(\varphi), \quad \forall\varphi \in H^1(B_R).$$

By Riesz theorem, one can define unique bounded linear operators $\mathscr{A}$ and $\mathscr{B}$ from $H^1(B_R)$ to $H^1(B_R)$ by

$$a_1(u,\varphi) = (\mathscr{A}u,\varphi) \quad \text{and} \quad a_2(u,\varphi) = (\mathscr{B}u,\varphi). \tag{2.46}$$

Let $f \in H^1(B_R)$ be the unique solution to $F(\varphi) = (f,\varphi)$. Then (2.45) is equivalent to

$$(\mathscr{A} + \mathscr{B})u = f. \tag{2.47}$$

Lemma 2.2.3 *The operators $\mathscr{A}$ and $\mathscr{B}$ defined by (2.46) satisfy the following: $\mathscr{A}$ is invertible with bounded inverse and $\mathscr{B}$ is compact.*

Proof. Since $\Re(A)$ is strictly positive definite, and $-T_0$ is non-negative from Theorem 2.2.1, then $a_1(\cdot,\cdot)$ is strictly coercive, by Lax-Milgram theorem one can conclude $\mathscr{A}$ is invertible with bounded inverse. Since $T - T_0$ is compact from Theorem 2.2.1 and $\int_{B_R}(k^2n+1)u\overline{\varphi}dx$ defines a compact perturbation due to compact embedding of $H^1(B_R)$ into $L^2(B_R)$, one can conclude $\mathscr{B}$ is compact. $\square$

Now we will show that there exists a solution to (2.45). In particular Fredholm alternative implies that it is sufficient to show the uniqueness of solution to (2.45).

Lemma 2.2.4 *There exists at most one pair of solution $u^s \in H^1(D)$ and $w \in H^1_{loc}(\mathbb{R}^d \setminus \overline{D})$ to (2.37)-(2.41).*

Proof. Let $u \in H^1(B_R)$ be defined by (2.42). Choosing $\varphi = u$ in (2.45) and taking the imaginary part yields

$$\Im(\int_{\partial D} \frac{\partial u}{\partial \nu}\overline{u}) = \int_{B_R} \Im A\nabla u \cdot \nabla \overline{u} dx - k^2 \int_{B_R} \Im(n) u\overline{u} dx.$$

Since $\Im(A) \leq 0, \Im(n) > 0$, one can conclude

$$\Im(\int_{\partial D} \frac{\partial u}{\partial \nu}\overline{u}) \leq 0.$$

Combining this with Rellich's Lemma yields $u \equiv 0$ outside D_1(c.f. [9]). Since $A \in (C^2(D_1 \setminus \overline{D}))^{3\times 3}$ and $n \in C^2(D_1 \setminus \overline{D})$, one can extend A, n to $\tilde{A}, \tilde{n}$ in B_R with continuously differentiable entries. From $u \equiv 0$ outside D_1, we have that

$$u = 0 \quad \text{and} \quad \frac{\partial u}{\partial \nu_A} = 0 \quad \text{on} \quad \partial D_1.$$

Then one can extend u to $\tilde{u} := u|_{D_1 \setminus \overline{D}} + 0|_{B_R \setminus \overline{D_1}}$ continuously, this gives $\tilde{u} \in H^1(B_R \setminus \overline{D})$ and satisfies

$$\nabla \cdot \tilde{A}\nabla \tilde{u} + k^2 \tilde{n}\tilde{u} = 0 \quad \text{in} \quad B_R \setminus \overline{D}.$$

From the classical regularity result ([36]), we have that $u \in H^2(B_R \setminus \overline{D})$ and is continuous across ∂D_1. Then one can conclude $\tilde{u} \equiv 0$ in $B_R \setminus \overline{D}$, in particular $u \equiv 0$ in $D_1 \setminus \overline{D}$. From the transmission boundary condition, we have that

$$u = 0 \quad \text{and} \quad \frac{\partial u}{\partial \nu_A} = 0 \quad \text{on} \quad \partial D.$$

Since u satisfies Helmholtz equation inside D, then $u \equiv 0$ inside D. Hence $u \equiv 0$ in B_R. This proves the Lemma. □

Theorem 2.2.2 *There exists a unique pair of solution $u^s \in H^1(D)$ and $w \in H^1_{loc}(\mathbb{R}^d \setminus \overline{D})$ to (2.29)-(2.33) for $f \in H^{\frac{1}{2}}(\partial D)$ and $h \in H^{-\frac{1}{2}}(\partial D)$ such that*

$$||u^s||_{H^1(D)} + ||w||_{H^1(B_R \setminus \overline{D})} \leq C(||f||_{H^{\frac{1}{2}}(\partial D)} + ||h||_{H^{-\frac{1}{2}}(\partial D)})$$

where $C > 0$ is some constant independent of f and h.

Proof. From Lemma 2.2.4, there exists a unique solution $u \in H^1(B_R)$ to (2.37)-(2.41). From (2.47), Lemma 2.2.3 and Riesz's lemma,

$$\begin{aligned} ||u||_{H^1(B_R)} &\leq C||F|| \\ &\leq C(||u_\ell||_{H^1(B_R\setminus\overline{D})} + ||h||_{H^{-\frac{1}{2}}(\partial D)}) \\ &\leq C(||f||_{H^{\frac{1}{2}}(\partial D)} + ||h||_{H^{-\frac{1}{2}}(\partial D)}) \end{aligned}$$

where we have used $||u_\ell||_{H^1(B_R\setminus\overline{D})} \leq C||f||_{H^{\frac{1}{2}}(\partial D)}$. Then the definition of u yields

$$||u^s||_{H^1(D)} + ||w||_{H^1(B_R\setminus\overline{D})} \leq C(||f||_{H^{\frac{1}{2}}(\partial D)} + ||h||_{H^{-\frac{1}{2}}(\partial D)})$$

where $C > 0$ is some constant independent of f and h. □

2.3 The Inverse Problem and Near Field Operator

The inverse problem we consider is to determine the boundary ∂D from a knowledge of internal measurements inside the cavity. The motivation for studying such a problem is to determine the shape of an underground reservoir by lowering receivers and transmitters into the reservoir through a bore hole drilled from the surface of the earth. Another application in non-destructive testing is to test the integrity of e.g. nuclear sectors. Mathematically, we assume that C is an open region in D such that $\overline{C} \subset D$. We place the point source $\Phi(\cdot, y)$ at every $y \in \partial C$ and measure the corresponding scattered field $u^s(x, y)$ for $x \in \partial C$. The *inverse problem* we consider is for fixed (but not necessarily known) A and n satisfying certain assumptions, determine the boundary of the cavity ∂D from a knowledge of $u^s(x, y)$ for all $x, y \in \partial C$.

It is known that there is one-to-one correspondence between the radiating solution to Helmholtz equation and its far field pattern. The near field has similar properties. If the near field vanishes, then the corresponding solution to the Helmholtz equation also vanishes. To make this rigorous, we begin with the following lemma.

Lemma 2.3.1 *Assume the open region C is such that k^2 is not a Dirichlet eigenvalue for $-\Delta$ in C. If $u^s(\cdot, y) = 0$ on ∂C, then $u^s(\cdot, y) = 0$ in D.*

Proof. Since k^2 is not a Dirichlet eigenvalue for $-\Delta$ in C and $u^s(\cdot,y)$ satisfies

$$\Delta u^s(\cdot,y) + k^2 u^s(\cdot,y) = 0 \quad \text{in} \quad C,$$

then we have $u^s(\cdot,y) = 0$ in C. Since $u^s(\cdot,y)$ satisfies

$$\Delta u^s(\cdot,y) + k^2 u^s(\cdot,y) = 0 \quad \text{in} \quad D,$$

then by unique continuation we have that $u^s(\cdot,y) = 0$ in D. $\square$

In our analysis of the inverse scattering problem, we shall always need the above assumption. Since the wave number k is known, one can choose C to satisfy the assumption and we assume this holds throughout the thesis.

Assumption 2.3.1 *The open region C is such that k^2 is not a Dirichlet eigenvalue for $-\Delta$ in C.*

Our analysis of the inverse scattering is based on an indicator function obtained by solving a linear integral equation of the first kind whose kernel is computed from internal measurements. Now let us introduce the near field operator.

Definition 2.3.1 *The near field operator N: $L^2(\partial C) \to L^2(\partial C)$ is defined by*

$$(Ng)(x) = \int_{\partial C} u^s(x,y)g(y)ds(y) \quad \textit{where} \quad g \in L^2(\partial C) \quad \textit{and} \quad x \in \partial C. \tag{2.48}$$

For later use, we establish a reciprocity relation of the scattered field $u^s(x,y)$.

Proposition 2.3.1 *The scattered field $u^s(x,y)$ respect to the point source $\Phi(x,y)$ satisfies the reciprocity relation*

$$u^s(x,y) = u^s(y,x) \quad \textit{where} \quad x,y \in \partial C. \tag{2.49}$$

Proof. From Green's formula we have that

$$u^s(x,y) = \int_{\partial D} \left\{ \frac{\partial u^s(\cdot,y)}{\partial \nu}\Phi(x,\cdot) - u^s(\cdot,y)\frac{\partial \Phi(x,\cdot)}{\partial \nu} \right\} ds, \qquad x \in C \tag{2.50}$$

$$u^s(y,x) = \int_{\partial D} \left\{ \frac{\partial u^s(\cdot,x)}{\partial \nu}\Phi(y,\cdot) - u^s(\cdot,x)\frac{\partial \Phi(y,\cdot)}{\partial \nu} \right\} ds, \qquad y \in C \tag{2.51}$$

$$0 = \int_{\partial D} \left\{ \frac{\partial u^s(\cdot,y)}{\partial \nu}u^s(\cdot,x) - u^s(\cdot,y)\frac{\partial u^s(\cdot,x)}{\partial \nu} \right\} ds. \tag{2.52}$$

Since $\Phi(\cdot,\cdot)$ satisfies the radiation condition, then we have again from Green's formula (applied in $\mathbb{R}^d\setminus\overline{D}$) that

$$0=\int\limits_{\partial D}\left\{\frac{\partial\Phi(x,\cdot)}{\partial\nu}\Phi(y,\cdot)-\Phi(x,\cdot)\frac{\partial\Phi(y,\cdot)}{\partial\nu}\right\}ds. \tag{2.53}$$

Since $\Phi(\cdot,\cdot)$ is symmetric, subtracting (2.51) from (2.50) and adding to the result the sum of (2.52) and (2.53) we obtain

$$u^s(y,x)-u^s(x,y)=\int\limits_{\partial D}\left\{\frac{\partial u(\cdot,y)}{\partial\nu}u(\cdot,x)-u(\cdot,y)\frac{\partial u(\cdot,x)}{\partial\nu}\right\}ds$$

where u is the total field. Now using the transmission conditions (2.39) (2.40), the fact that A is symmetric, the assumptions that $A-I$ and $n-1$ are zero in $\mathbb{R}^d\setminus B_R$ and the equation (2.37) we have that

$$\begin{aligned}
u^s(y,x)-u^s(x,y) &= \int\limits_{\partial D}\left\{\frac{\partial w(\cdot,y)}{\partial\nu_A}w(\cdot,x)-w(\cdot,y)\frac{\partial w(\cdot,x)}{\partial\nu_A}\right\}ds\\
&= -\int\limits_{B_R\setminus\overline{D}}\left\{A\nabla w(\cdot,y)\cdot\nabla w(\cdot,x)-A\nabla w(\cdot,x)\cdot\nabla w(\cdot,y)\right\}dv\\
&- \int\limits_{B_R\setminus\overline{D}}\left\{\nabla\cdot A\nabla w(\cdot,y)w(\cdot,x)-\nabla A\nabla w(\cdot,x)w(\cdot,y)\right\}dv\\
&+ \int\limits_{\partial B_R}\left\{\frac{\partial w(\cdot,y)}{\partial\nu}w(\cdot,x)-w(\cdot,y)\frac{\partial w(\cdot,x)}{\partial\nu}\right\}ds=0
\end{aligned}$$

because of the following: the first volume integral is zero due to the symmetry of A, the second volume integral is zero due the fact that $w(\cdot,x)$ and $w(\cdot,y)$ satisfy the same equation and the last integral is zero due to the fact that $w(\cdot,x)$ and $w(\cdot,y)$ are radiating solutions to the Helmholtz equation outside B_R. $\square$

2.4 Properties of Near Field Operator

In order to solve the near field equation, we first study the injectivity and the range of the near field operator. To this end, we define the single layer potential v_g by

$$v_g(x):=\int_{\partial C}\Phi(x,y)g(y)ds(y),\quad x\in\mathbb{R}^d\setminus\partial C. \tag{2.54}$$

Then by superposition Ng is the scattered field evaluated on ∂C due to v_g as incident field.

Theorem 2.4.1 *The near field operator $N : L^2(\partial C) \to L^2(\partial C)$ is injective with dense range if and only if there does not exist a non-zero $h \in L^2(\partial C)$ such that there exists $w \in H^1_{loc}(\mathbb{R}^d \backslash \overline{D})$ and $v := v_h$ solving the homogeneous problem*

$$\nabla \cdot A\nabla w + k^2 nw = 0 \qquad \text{in} \quad \mathbb{R}^d \setminus \overline{D} \tag{2.55}$$

$$\Delta v + k^2 v = 0 \qquad \text{in} \quad \mathbb{R}^d \setminus \overline{D} \tag{2.56}$$

$$w = v \qquad \text{on} \quad \partial D \tag{2.57}$$

$$\frac{\partial w}{\partial \nu_A} = \frac{\partial v}{\partial \nu} \qquad \text{on} \quad \partial D \tag{2.58}$$

$$\lim_{r\to\infty} r^{\frac{d-1}{2}}\left(\frac{\partial w}{\partial r} - ikw\right) = 0 \qquad \text{and} \qquad \lim_{r\to\infty} r^{\frac{d-1}{2}}\left(\frac{\partial v}{\partial r} - ikv\right) = 0 \tag{2.59}$$

Proof. The reciprocity relation of u^s implies that $N^*h = \overline{N\overline{h}}$ where N^* is the L^2-adjoint of N, i.e.,

$$(N^*g)(x) = \int_{\partial C} \overline{u^s(x,y)} g(y) ds(y) \quad \text{where} \quad g \in L^2(\partial C) \quad \text{and} \quad x \in \partial C. \tag{2.60}$$

Hence N is injective if and only if N^* is injective, Since $\text{Ker}(N^*)^\perp = \overline{\text{Range}(N)}$ to prove the theorem we must only prove that N is injective.

To this end, let a non-zero $h \in L^2(\partial C)$ be such that $(Nh)(x) = 0$ where $x \in \partial C$. Let $v_h(x) = \int_{\partial C} \phi(x,z) g(z) ds(z)$, and consider $(\tilde{w}, \tilde{v})$ the unique solution of (2.29)-(2.33) with $f := v_h$ and $h := \frac{\partial v_h}{\partial \nu}$. Since $u^s(\cdot, y)$ is the scattered field corresponding to $\Phi(\cdot, y)$, by superposition $\tilde{v}(x) = \int_{\partial C} u^s(x,y) h(y) ds(y)$. By assumption $(Nh)(x) = 0$ where $x \in \partial C$, then $\tilde{v}(x) = 0$ where $x \in \partial C$. From Lemma 2.3.1 $\tilde{v} = 0$ in D. This implies that $\tilde{w}$ and v_g satisfy the homogeneous exterior transmission problem.

Conversely, if (w, v_h) solves (2.55)-(2.59), then $(w, 0)$ solves (2.29)-(2.33) with $f := v_h$ and $h := \frac{\partial v_h}{\partial \nu}$. By superposition and uniqueness of (2.29)-(2.33), there is

$$\int_{\partial C} u^s(x,y) h(y) ds(y) = 0 \quad \text{in} \quad D.$$

In particular $Nh = 0$ on ∂C. This proves the theorem. $\square$

The above theorem implies the following.

Corollary 2.4.1 *Assume there does not exist a non-zero $h \in L^2(\partial C)$ such that there exists $w \in H^1_{loc}(\mathbb{R}^d \backslash \overline{D})$ and $v := v_h$ solves the homogeneous problem (2.55)-(2.59), then the operator $N : L^2(\partial C) \to L^2(\partial C)$ is injective with dense range.*

Remark 2.4.1 *The near field operator N is not normal since*

$$(NN^*g)(x) = \int_{\partial C}\int_{\partial C} u^s(x,z)\overline{u^s(z,y)}g(y)ds(y)ds(z) \quad \textit{where} \quad g \in L^2(\partial C) \quad \textit{and} \quad x \in \partial C$$

$$(N^*Ng)(x) = \int_{\partial C}\int_{\partial C} \overline{u^s(x,z)}u^s(z,y)g(z)ds(y)ds(x) \quad \textit{where} \quad g \in L^2(\partial C) \quad \textit{and} \quad x \in \partial C.$$

Chapter 3

THE EXTERIOR TRANSMISSION EIGENVALUE PROBLEM

In this chapter we will formulate and study the so-called exterior transmission problem which will play a fundamental role in our uniqueness proof and the justification of the qualitative methods. From Section 2.1 of Chapter 2, the homogeneous problem (2.55)-(2.59) arises from the analysis of the injectivity of the near field operator. The exterior transmission eigenvalue problem for anisotropic medium $A \neq I$ is finding $w \in H^1_{loc}(\mathbb{R}^d \setminus \overline{D})$ and $v \in H^1_{loc}(\mathbb{R}^d \setminus \overline{D})$ such that

$$\nabla \cdot A\nabla w + k^2 nw = 0 \qquad \text{in} \quad \mathbb{R}^d \setminus \overline{D} \tag{3.1}$$

$$\Delta v + k^2 v = 0 \qquad \text{in} \quad \mathbb{R}^d \setminus \overline{D} \tag{3.2}$$

$$w = v \qquad \text{on} \quad \partial D \tag{3.3}$$

$$\frac{\partial w}{\partial \nu_A} = \frac{\partial v}{\partial \nu} \qquad \text{on} \quad \partial D \tag{3.4}$$

$$\lim_{r\to\infty} r^{\frac{d-1}{2}}(\frac{\partial w}{\partial r} - ikw) = 0 \qquad \text{and} \qquad \lim_{r\to\infty} r^{\frac{d-1}{2}}(\frac{\partial v}{\partial r} - ikv) = 0. \tag{3.5}$$

Associated *exterior transmission problem* with the exterior transmission eigenvalue for anisotropic medium is: given $f \in H^{\frac{1}{2}}(\partial D)$, $h \in H^{-\frac{1}{2}}(\partial D)$, $\ell_1 \in L^2_{loc}(\mathbb{R}^d \setminus \overline{D})$ and $\ell_2 \in L^2_{loc}(\mathbb{R}^d \setminus \overline{D})$, finding $w \in H^1_{loc}(\mathbb{R}^d \setminus \overline{D})$ and $v \in H^1_{loc}(\mathbb{R}^d \setminus \overline{D})$ such that

$$\nabla \cdot A\nabla w + k^2 nw = \ell_1 \qquad \text{in} \quad \mathbb{R}^d \setminus \overline{D} \tag{3.6}$$

$$\Delta v + k^2 v = \ell_2 \qquad \text{in} \quad \mathbb{R}^d \setminus \overline{D} \tag{3.7}$$

$$w - v = f \qquad \text{on} \quad \partial D \tag{3.8}$$

$$\frac{\partial w}{\partial \nu_A} - \frac{\partial v}{\partial \nu} = h \qquad \text{on} \quad \partial D \tag{3.9}$$

$$\lim_{r\to\infty} r^{\frac{d-1}{2}}(\frac{\partial w}{\partial r} - ikw) = 0 \qquad \text{and} \qquad \lim_{r\to\infty} r^{\frac{d-1}{2}}(\frac{\partial v}{\partial r} - ikv) = 0 \tag{3.10}$$

where ℓ_1 and ℓ_2 vanish in $\mathbb{R}^d \setminus \overline{B_R}$ and R is the radius of the ball B_R outside of which $A = I$ and $n = 1$.

The exterior transmission eigenvalue problem for isotropic medium is finding $w \in L^2_{loc}(\mathbb{R}^d \setminus \overline{D})$ and $v \in L^2_{loc}(\mathbb{R}^d \setminus \overline{D})$ such that $w - v \in H^2_{loc}(\mathbb{R}^d \setminus \overline{D})$ and

$$\Delta w + k^2 n w = 0 \qquad \text{in} \quad \mathbb{R}^d \setminus \overline{D} \tag{3.11}$$

$$\Delta v + k^2 v = 0 \qquad \text{in} \quad \mathbb{R}^d \setminus \overline{D} \tag{3.12}$$

$$w = v \qquad \text{on} \quad \partial D \tag{3.13}$$

$$\frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} \qquad \text{on} \quad \partial D \tag{3.14}$$

$$\lim_{r\to\infty} r^{\frac{d-1}{2}}\left(\frac{\partial w}{\partial r} - ikw\right) = 0 \qquad \text{and} \qquad \lim_{r\to\infty} r^{\frac{d-1}{2}}\left(\frac{\partial v}{\partial r} - ikv\right) = 0 \tag{3.15}$$

Associated *exterior transmission problem* with the exterior transmission eigenvalue for isotropic medium is: given $f \in H^{-\frac{1}{2}}(\partial D)$, $h \in H^{-\frac{3}{2}}(\partial D)$, $\ell_1 \in L^2_{loc}(\mathbb{R}^d \setminus \overline{D})$ and $\ell_2 \in L^2_{loc}(\mathbb{R}^d \setminus \overline{D})$, finding $w \in L^2_{loc}(\mathbb{R}^d \setminus \overline{D})$ and $v \in L^2_{loc}(\mathbb{R}^d \setminus \overline{D})$ such that $w - v \in H^2_{loc}(\mathbb{R}^d \setminus \overline{D})$ and

$$\Delta w + k^2 n w = \ell_1 \qquad \text{in} \quad \mathbb{R}^d \setminus \overline{D} \tag{3.16}$$

$$\Delta v + k^2 v = \ell_2 \qquad \text{in} \quad \mathbb{R}^d \setminus \overline{D} \tag{3.17}$$

$$w - v = f \qquad \text{on} \quad \partial D \tag{3.18}$$

$$\frac{\partial w}{\partial \nu_A} - \frac{\partial v}{\partial \nu} = h \qquad \text{on} \quad \partial D \tag{3.19}$$

$$\lim_{r\to\infty} r^{\frac{d-1}{2}}\left(\frac{\partial w}{\partial r} - ikw\right) = 0 \qquad \text{and} \qquad \lim_{r\to\infty} r^{\frac{d-1}{2}}\left(\frac{\partial v}{\partial r} - ikv\right) = 0, \tag{3.20}$$

where ℓ_1 and ℓ_2 vanish in $\mathbb{R}^d \setminus \overline{B_R}$ and R is the radius of the ball B_R outside of which $A = I$ and $n = 1$.

Definition 3.0.1 *Values of $k \in \mathbb{R}$ for which the homogeneous problem (3.1)-(3.5)($A \neq I$) or (3.11)-(3.15) ($A = I$) has a nontrivial solution are called exterior transmission eigenvalues.*

As a physical motivation of the exterior transmission problem we ask the question if it is possible to send an outgoing incident field u^i from inside the cavity D that does not produce any scattered field in D and all the energy is transmitted to the exterior of D.

In practice, all the wave numbers are real valued. Mathematically, in order to prove the discreteness of transmission eigenvalues we need to consider complex wave numbers that lie in a neighborhood of the real axis. This motives us to consider a careful definition of the Sommerfeld radiation condition or the Dirichlet to Neumann map (2.35) for complex wave numbers k. To this end we make the following remarks.

Remark 3.0.2 *For real valued k, suppose $u \in H^1_{loc}(\mathbb{R}^d \backslash \overline{B_R})$ and $g \in H^{\frac{1}{2}}(\partial B_R)$ satisfy*

$$\Delta u + k^2 u = 0 \qquad in \qquad \mathbb{R}^d \backslash \overline{B_R} \tag{3.21}$$

$$u = g \qquad on \qquad \partial B_R \tag{3.22}$$

Assume u satisfies the Sommerfeld radiation condition for $k \in \mathbb{R}$

$$\lim_{r\to\infty} r^{\frac{d-1}{2}}\left(\frac{\partial u}{\partial r} - iku\right) = 0.$$

Then we have the boundary value problem (3.21)-(3.22) has a unique solution and we define

$$u = B_k g.$$

In the case that $k \in \mathbb{C}$, the boundary value problem (3.21)-(3.22) has a unique solution provided k is not a scattering pole. A complex number k is a scattering pole if and only if there is a nonzero $v \in H^1_{loc}(\mathbb{R}^d \backslash \overline{B_R})$ satisfying (3.21)-(3.22) and of the form

$$v = DL_k g_1 + SL_k g_2 \quad in \quad \mathbb{R}^d \backslash \overline{B_R}$$

for some $g_1 \in H^{\frac{1}{2}}(\partial B_R)$ and $g_2 \in H^{-\frac{1}{2}}(\partial B_R)$ where

$$\begin{aligned} DL_k\psi &:= \int_{\partial B_R} \frac{\partial \Phi(\cdot, y)}{\partial \nu}\psi(y)ds(y) \\ SL_k\psi &:= \int_{\partial B_R} \Phi(\cdot, y)\psi(y)ds(y). \end{aligned}$$

The operator B_k has a meromorphic continuation to the complex plane $\mathbb{C}$ and all the scattering poles lie in the lower half complex plane $\Im k < 0$. There exists a neighborhood of the real axis such that the operator B_k is well-defined and analytic [81]. We *remark that one can prove the same result for the acoustic scattering by inhomogeneous medium.*

Remark 3.0.3 *For real valued k, the Dirichlet to Neumann map (2.35) (for instance in two dimension, i.e. $d = 2$ [9]) is defined by*

$$T_k(w|_{\partial B_R}) := \sum_{-\infty}^{\infty} a_n \frac{kH_n^{(1)'}(kR)}{H_n^{(1)}(kR)} e^{in\theta}$$

for

$$w|_{\partial B_R} = \sum_{-\infty}^{\infty} a_n e^{in\theta}$$

where $w(r,\theta)$ is a radiating solution to the Helmholtz equation outside B_R and (r,θ) denotes the polar coordinates in $\mathbb{R}^2$, and $w(r,\theta)$ is defined by

$$w(r,\theta) = \sum_{-\infty}^{\infty} \alpha_n H_n^{(1)}(kr) e^{in\theta}, \quad r \geq R \quad \text{and} \quad 0 \leq \theta \leq 2\pi.$$

In the case that k is complex, we can still define the Dirichlet to Neumann operator as above for $k \in \mathbb{C}$ that are not zeros of Hankel functions $H_n^{(1)}(kR)$, $n = 0, \pm 1, \pm 2, \cdots$. Remark that all the zeros of $\{H_n^{(1)}(kR)\}$ (these zeros are scattering poles) form a discrete set in $\mathbb{C}$. Therefore for any fixed interval on $\mathbb{R}$, there exists a neighborhood of the real axis such that $\{H_n^{(1)}(kR)\}$ has no zeros in this neighborhood. Then we can define the Dirichlet to Neumann operator T_k in such a neighborhood and T_k is analytic in this neighborhood.

We will establish the Fredholm properties of the exterior transmission problem and show all the exterior transmission eigenvalues form a discrete set on the real line. In particular, we will use variational method for the anisotropic case and integral equation method for the isotropic case to prove the discreteness of exterior transmission eigenvalues.

Our results are based on the following lemma on analytic Fredholm theory [9].

Lemma 3.0.1 *Let D be a domain in $\mathbb{C}$ and let $A : D \to \mathcal{L}(X)$ be an operator valued analytic function such that $A(z)$ is compact for each $z \in D$. Suppose $I - A(z)$ is* *invertible at some point $z_0 \in D$. Then $I - A(z)$ is invertible except at most on a discrete set in D, and $(I - A(z))^{-1}$ is meromorphic on D.*

3.1 Exterior Transmission Eigenvalue Problem: Case $A \neq I$

In this section we will derive a variational formulation for the exterior transmission problem. To this end we first formulate the exterior transmission problem into a bounded domain problem using the Dirichlet to Neumann map. In fact, the exterior transmission eigenvalue problem is equivalent to

$$\nabla \cdot A\nabla w + k^2 n w = \ell_1 \quad \text{in} \quad \mathbb{R}^d \setminus \overline{D} \tag{3.23}$$

$$\Delta v + k^2 v = \ell_2 \quad \text{in} \quad \mathbb{R}^d \setminus \overline{D} \tag{3.24}$$

$$w - v = f \quad \text{on} \quad \partial D \tag{3.25}$$

$$\frac{\partial w}{\partial \nu_A} - \frac{\partial v}{\partial \nu} = h \quad \text{on} \quad \partial D \tag{3.26}$$

$$\frac{\partial w}{\partial \nu} = T_k w \quad \text{on} \quad \partial B_R \qquad \text{and} \qquad \frac{\partial v}{\partial \nu} = T_k v \quad \text{on} \quad \partial B_R \tag{3.27}$$

Now we take $v_l \in H^1_{loc}(\mathbb{R}^d \setminus \overline{D})$ to be the unique solution of the exterior Dirichlet problem

$$\Delta v_l + k^2 v_l = 0 \quad \text{in} \quad \mathbb{R}^d \setminus \overline{D}$$

$$v_l = f \quad \text{on} \quad \partial D$$

$$\lim_{r \to \infty} r^{\frac{d-1}{2}} \left(\frac{\partial v_l}{\partial r} - ikv_l\right) = 0$$

and set $v_0 = v + v_l$. Then (w, v_0) satisfies (3.23)-(3.27) with $(f, h) = (0, \tilde{h} := h - \frac{\partial v_l}{\partial \nu})$. Therefore it suffices to study (3.23)-(3.27) with $f = 0$. Next we define an appropriate Hilbert space to work on. Define the Hilbert space

$$H := \{(w, v) \in H^1(B_R \backslash \overline{D}) \times H^1(B_R \backslash \overline{D}), w - v = 0 \quad \text{on} \quad \partial D\}.$$

Taking a test function $(w', v') \in H$, multiplying both sides of (3.23) by w' and (3.24) by v', and integrating by parts yields

$$\int_{\partial B_R} T_k w \overline{w'} ds - \int_{\partial D} \frac{\partial w}{\partial \nu_A} \overline{w'} ds - \int_{B_R \backslash \overline{D}} A\nabla w \cdot \nabla \overline{w'} dx + \int_{B_R \backslash \overline{D}} n k^2 w \overline{w'} dx = \int_{B_R \backslash \overline{D}} \ell_1 \overline{w'} dx$$

and

$$\int\limits_{\partial B_R} T_k v\overline{v'}ds - \int\limits_{\partial D} \frac{\partial v}{\partial \nu}\overline{v'}ds - \int\limits_{B_R\setminus\overline{D}} \nabla v\cdot\nabla\overline{v'}dx + \int\limits_{B_R\setminus\overline{D}} k^2 v\overline{v'}dx = \int\limits_{B_R\setminus\overline{D}} \ell_2\overline{v'}dx.$$

Now taking the difference and and using the fact that $w' = v'$ on ∂D together with (3.25) (3.26) we have that

$$\begin{aligned}&\int\limits_{B_R\setminus\overline{D}} A\nabla w\cdot\nabla\overline{w'}dx - \int\limits_{B_R\setminus\overline{D}} \nabla v\cdot\nabla\overline{v'}dx + \int\limits_{B_R\setminus\overline{D}} (-nk^2 w\overline{w'} + k^2 v\overline{v'})dx - \int\limits_{\partial B_R} T_k w\overline{w'}ds \\ &+ \int\limits_{\partial B_R} T_k v\overline{v'}ds = -\int_{\partial D} h\overline{w'}ds - \int\limits_{B_R\setminus\overline{D}} \ell_1\overline{w'}dx + \int\limits_{B_R\setminus\overline{D}} \ell_1\overline{v'}dx.\end{aligned} \tag{3.28}$$

We define the sesquilinear form $a_k(\cdot,\cdot) : H\times H \to \mathbb{C}$ by

$$\begin{aligned}a_k((w,v),(w',v')) \;=\;& \int\limits_{B_R\setminus\overline{D}} A\nabla w\cdot\nabla\overline{w'}dx - \int\limits_{B_R\setminus\overline{D}} \nabla v\cdot\nabla\overline{v'}dx \\ &+ \int\limits_{B_R\setminus\overline{D}} (-nk^2 w\overline{w'} + k^2 v\overline{v'})dx - \int\limits_{\partial B_R} T_k w\overline{w'}ds + \int\limits_{\partial B_R} T_k v\overline{v'}ds\end{aligned}$$

and the conjugate linear functional $F(\cdot) : H \to \mathbb{C}$ by

$$F(w',v') := -\int_{\partial D} h\overline{w'}ds - \int\limits_{B_R\setminus\overline{D}} \ell_1\overline{w'}dx + \int\limits_{B_R\setminus\overline{D}} \ell_1\overline{v'}dx.$$

Conversely, assume that $(w,v)\in H$ satisfies $a_k((w,v),(w',v')) = F(w',v')$ for all $(w',v')\in H$. Taking $v'=0$, $w'\in C_0^\infty(B_R\setminus\overline{D})$, we have (3.23) and in a similar way we have (3.24). Taking $(w',v')\in H$ such that $w'=v'=0$ on ∂B_R, one can get (3.27). Finally, a choice of $(w',0)\in H$ implies (3.25) and in a similar way we obtain (3.26). Hence we have proven the following theorem.

Theorem 3.1.1 *The exterior transmission problem (3.23)-(3.27) is equivalent to the following problem: Find $(w,v)\in H$ such that for all $(w',v')\in H$*

$$a_k((w,v),(w',v')) = F(w',v'). \tag{3.29}$$

Note that by means of the Riesz representation theorem we can define the operator $\mathcal{A}_k : H \to H$ by

$$(\mathcal{A}_k(w,v),(w',v'))_H = a_k((w,v),(w',v')) \qquad \text{for all } ((w,v),(w',v')) \in H \times H.$$

We would like to show that $\mathcal{A}_{i\kappa} : H \to H$ for $\kappa > 0$ is invertible. To prove this we use the T-coercivity approach introduced in [8] and [22]. The idea behind the T-coercivity method is to consider an equivalent formulation of (3.29) where a_k is replaced by $a_k^{\mathcal{T}}$ defined by

$$a_k^{\mathcal{T}}((w,v),(w',v')) := a_k((w,v),\mathcal{T}(w',v')), \quad \forall((w,v),(w',v')) \in H \times H \tag{3.30}$$

with $\mathcal{T}$ being an *ad hoc* isomorphism of H. Indeed, $(w,v) \in H$ satisfies

$$a_k((w,v),(w',v')) = 0 \qquad \text{for all} \qquad (w',v') \in H$$

if and only if, it satisfies $a_k^{\mathcal{T}}((w,v),(w',v')) = 0$ for all $(w',v') \in H$. Assume that $\mathcal{T}$ and k are chosen so that $a_k^{\mathcal{T}}$ is coercive. Then using the Lax-Milgram theorem and the fact that $\mathcal{T}$ is an isomorphism of H, one deduces that $\mathcal{A}_k$ is an isomorphism on H.

In the following, in addition to the assumptions on A and n stated at the end of Section 2.1 in Chapter 2, we assume that there exists a neighborhood Ω of ∂D where both $\Im(A) = 0$ and $\Im(n) = 0$ in $B_R \backslash \overline{D} \cap \Omega$. Setting $\mathcal{N} := B_R \backslash \overline{D} \cap \Omega$, we denote by

$$\begin{aligned} A_* &:= \inf_{x\in\mathcal{N}} \inf_{|\xi|=1} \overline{\xi} \cdot A(x)\xi > 0, \quad & A^* &:= \sup_{x\in\mathcal{N}} \sup_{|\xi|=1} \overline{\xi} \cdot A(x)\xi < \infty, \\ n_* &:= \inf_{x\in\mathcal{N}} n(x) > 0, & n^* &:= \sup_{x\in\mathcal{N}} n(x) < \infty. \end{aligned} \tag{3.31}$$

for $\xi \in \mathbb{C}^d$. Then we can prove the following result.

Lemma 3.1.1 *Assume that either $A^* < 1$ and $n^* < 1$ or $A_* > 1$ and $n_* > 1$. Then there exists $\kappa > 0$ such that $\mathcal{A}_{i\kappa}$ is invertible.*

Proof. We first consider the case when $A^* < 1$ and $n^* < 1$. Take $\chi \in C^\infty(\overline{B_R \backslash \overline{D}})$ to be a cut off function equal to 1 in a neighborhood of ∂D with support in $\mathcal{N} := (B_R \backslash \overline{D}) \cap \Omega$ and let $\mathcal{T}(w, v) = (w - 2\chi v, -v)$. We then have that

$$\begin{aligned}
a_{i\kappa}^{\mathcal{T}}((w,v),(w,v)) &= (A\nabla w, \nabla w)_{B_R\backslash\overline{D}} + (\nabla v, \nabla v)_{B_R\backslash\overline{D}} - 2(A\nabla w, \nabla(\chi v))_{B_R\backslash\overline{D}} \\
&+ \kappa^2((nw, w)_{B_R\backslash\overline{D}} + (v, v)_{B_R\backslash\overline{D}} - 2(nw, \chi v)_{B_R\backslash\overline{D}}) \\
&- (T_{i\kappa} w, w)_{\partial B_R} - (T_{i\kappa} v, v)_{\partial B_R} + 2(T_{i\kappa} w, \chi v)_{\partial B_R}
\end{aligned}$$

where $(\cdot,\cdot)_X$ denotes the L^2-inner product in the generic space X. By Young's inequality we have

$$\begin{aligned}
2|(A\nabla w, \nabla \chi v)_{B_R\backslash\overline{D}}| &\leq 2|(\chi A\nabla w, \nabla v)_{\mathcal{N}}| + 2|(A\nabla w, \nabla(\chi) v)_{\mathcal{N}}| \\
&\leq \alpha(A\nabla w, \nabla w)_{\mathcal{N}} + \alpha^{-1}(A\nabla v, \nabla v)_{\mathcal{N}} \\
&+ \beta(A\nabla w, \nabla w)_{\mathcal{N}} + \beta^{-1}(A\nabla(\chi) v, \nabla(\chi) v)_{\mathcal{N}}
\end{aligned}$$

and

$$2|(nw, \chi v)_{B_R\backslash\overline{D}}| \leq 2|(nw, v)_{\mathcal{N}}| \leq \eta(nw, w)_{\mathcal{N}} + \eta^{-1}(nv, v)_{\mathcal{N}}$$

for some $\alpha > 0$, $\beta > 0$, and $\eta > 0$. Recall that A and n are real in $\mathcal{N}$. Furthermore, due to the exponential decay of w and v at ∞ we have that

$$-(T_{i\kappa} w, w)_{\partial B_R} = \int\limits_{\mathbb{R}^d\backslash\overline{B_R}} \left(|\nabla w|^2 + \kappa^2 |w|^2\right)\, dx$$

with a similar expression for $-(T_{i\kappa} v, v)_{\partial B_R}$. In a similar way we can have

$$(T_{i\kappa} w, \chi v)_{\partial B_R} = 0.$$

Using all the above estimates we finally obtain that

$$\begin{aligned}
|a_{i\kappa}^{\mathcal{T}}((w,v),(w,v))| &\geq \Re\left(a_{i\kappa}^{\mathcal{T}}((w,v),(w,v))\right) \\
&\geq \Re(A\nabla w, \nabla w)_{\{B_R\backslash\overline{D}\}\backslash\overline{\Omega}} + (\nabla v, \nabla v)_{\{B_R\backslash\overline{D}\}\backslash\overline{\Omega}} \\
&+ \kappa^2\left(\Re(nw, w)_{\{B_R\backslash\overline{D}\}\backslash\overline{\Omega}} + (v, v)_{\{B_R\backslash\overline{D}\}\backslash\overline{\Omega}}\right) \\
&+ (1-\alpha-\beta)(A\nabla w, \nabla w)_{\mathcal{N}} + ((I - \alpha^{-1}A)\nabla v, \nabla v)_{\mathcal{N}} \\
&+ \kappa^2(1-\eta)(nw, w)_{\mathcal{N}} + (\kappa^2(1-\eta^{-1}n) - sup|\nabla\chi|^2 A_+)v, v)_{\mathcal{N}}.
\end{aligned}$$

Taking $\alpha, \beta, \eta, \kappa$ such that $A^* < \alpha$, $n^* < \eta$, $\beta < 1 - \alpha$, and κ large enough yields that $a^{\mathcal{T}}_{i\kappa}$ is coercive.

The case when $A^* > 1$ and $n^* > 1$ can be proven the same way using $\mathcal{T}(w, v) = (w, -v + 2\chi w)$. $\square$

Remark 3.1.1 In Lemma 3.1.1 the assumption that A and n are real in a neighborhood $\mathcal{N}$ of ∂D can be relaxed. In particular, the proof of Lemma 3.1.1 goes through if we only assume that $-\Im(A) < \Re(A)$ snd $\Im(n) < \Re(n)$ in $\mathcal{N}$.

Theorem 3.1.2 *Assume that A and n satisfy the assumptions of Lemma 3.1.1. Then if k is not an exterior transmission eigenvalue, the exterior transmission problem (3.1)-(3.5) has a unique solution which depends continuously on the data f, h, ℓ_1 and ℓ_2, i.e.*

$$\begin{aligned} & ||w||_{H^1(B_R\backslash\overline{D})} + ||v||_{H^1(B_R\backslash\overline{D})} \\ \leq\ & C\left(||f||_{H^{\frac{1}{2}}(\partial D)} + ||h||_{H^{-\frac{1}{2}}(\partial D)} + ||\ell_1||_{L^2(B_R\backslash\overline{D})} + ||\ell_2||_{L^2(B_R\backslash\overline{D})}\right) \end{aligned}$$

where $C > 0$ is some constant independent of f, h, ℓ_1 and ℓ_2.

Proof. From Lemma 3.1.1, we can choose κ such that $\mathcal{A}_{i\kappa}$ is invertible. Since the embedding from H to $L^2(B_R\backslash\overline{D}) \times L^2(B_R\backslash\overline{D})$ is compact and $T_k - T_{i\kappa}$ is a compact operator from $H^{\frac{1}{2}}(\partial B_R)$ to $H^{-\frac{1}{2}}(\partial B_R)$ [9], we can conclude that $\mathcal{A}_k - \mathcal{A}_{i\kappa}$ is a compact, and hence the result follows from the Fredholm alternative. $\square$

We can now prove the following discreteness result for exterior transmission eigenvalues.

Theorem 3.1.3 *Assume that A and n satisfy the assumptions of Lemma 3.1.1. Then the set of exterior transmission eigenvalues is discrete on the real line.*

Proof. From Remark (3.0.3), we can define the Dirichlet to Neumann map in a neighborhood of the real axis such that T_k depends analytically on k. Then the operator $\mathcal{A}_k - \mathcal{A}_{i\kappa} : k \to \mathcal{L}(H)$ is analytic in such a neighborhood where we choose κ such that $\mathcal{A}_{i\kappa}$ is invertible. The theorem follows from Lemma 3.0.1. $\square$

Corollary 3.1.1 *Assume that A and n satisfy the assumptions of Lemma 3.1.1. Then the set of exterior transmission eigenvalues is discrete on the complex plane.*

Proof. From Remark (3.0.3), we can define the Dirichlet to Neumann map in a domain which avoids the scattering poles, such that T_k depends analytically on k. Then the operator $\mathcal{A}_k - \mathcal{A}_{i\kappa} : k \to \mathcal{L}(H)$ is analytic in such a domain. Then the exterior transmission eigenvalues form a discrete set in this domain. Since the scattering poles form a discrete set in the complex plane, this proves the corollary . $\square$

3.2 Exterior Transmission Eigenvalue Problem: Case $A = I$

The variational formulation does not work in this case due to $A = I$. In this section we will formulate the exterior transmission problem using integral equations. We will first consider the case where $n > 0$ is some constant, then we will proceed to the general case. Let us define the double and single layer potential by

$$(DL_k\psi)(x) := \int_{\partial D} \frac{\partial \phi_k(x,y)}{\partial \nu} \psi(y) ds_y \quad \text{where} \quad x \in \mathbb{R}^d \setminus \partial D \tag{3.32}$$

$$(SL_k\varphi)(x) := \int_{\partial D} \phi_k(x,y)\varphi(y) ds_y \quad \text{where} \quad x \in \mathbb{R}^d \setminus \partial D. \tag{3.33}$$

Since the double and single layer potential define pseudo-differential operators of order -1 and -2 respectively, this implies in particular that $DL_k : H^{-\frac{1}{2}}(\partial D) \to L^2_{loc}(\mathbb{R}^d \setminus \overline{D})$ and $SL_k : H^{-\frac{3}{2}}(\partial D) \to L^2_{loc}(\mathbb{R}^d \setminus \overline{D})$ are continuous. Since $DL_k\psi$ and $SL_k\varphi$ satisfy Helmholtz equation in the distributional sense in $\mathbb{R}^d \setminus \overline{D}$, by a denseness argument for $\psi \in H^{-\frac{1}{2}}(\partial D)$ and $\varphi \in H^{-\frac{3}{2}}(\partial D)$ one can show $DL_k : H^{-\frac{1}{2}}(\partial D) \to L^2_\Delta(\mathbb{R}^d \setminus \overline{D})$ and $SL_k : H^{-\frac{3}{2}}(\partial D) \to L^2_\Delta(\mathbb{R}^d \setminus \overline{D})$ are continuous where

$$L^2_\Delta(\mathbb{R}^d \setminus \overline{D}) := \{u \in L^2_{loc}(\mathbb{R}^d \setminus \overline{D}),\ \Delta u \in L^2_{loc}(\mathbb{R}^d \setminus \overline{D})\}.$$

We state the following lemmas(c.f. [33]).

Lemma 3.2.1 *The double layer potential $DL_k : H^{-\frac{1}{2}}(\partial D) \to L^2_\Delta(\mathbb{R}^d \setminus \overline{D})$ and the single layer potential $SL_k : H^{-\frac{3}{2}}(\partial D) \to L^2_\Delta(\mathbb{R}^d \setminus \overline{D})$ are continuous and give rise to bounded linear operator*

$$S_k : H^{-\frac{3}{2}}(\partial D) \to H^{-\frac{1}{2}}(\partial D), \quad K_k : H^{-\frac{1}{2}}(\partial D) \to H^{-\frac{1}{2}}(\partial D)$$

$$K'_k : H^{-\frac{3}{2}}(\partial D) \to H^{-\frac{3}{2}}(\partial D), \quad T_k : H^{-\frac{1}{2}}(\partial D) \to H^{-\frac{3}{2}}(\partial D)$$

where

$$\begin{aligned}
(S_k\psi)(x) : &= 2\int_{\partial D} \phi_k(x,y)\psi(y)ds_y \\
(K_k\psi)(x) : &= 2\int_{\partial D} \frac{\partial \phi_k(x,y)}{\partial \nu(y)}\psi(y)ds_y \\
(K'_k\psi)(x) : &= 2\int_{\partial D} \frac{\partial \phi_k(x,y)}{\partial \nu(x)}\psi(y)ds_y \\
(T_k\psi)(x) : &= 2\frac{\partial}{\partial \nu}\int_{\partial D} \frac{\partial \phi_k(x,y)}{\partial \nu(y)}\psi(y)ds_y
\end{aligned}$$

for all $x \in \partial D$. Furthermore the following jump properties hold

$$\begin{array}{ll}
(DL_k\psi)^{\pm} = \frac{1}{2}K_k\psi \pm \frac{1}{2}\psi, & (SL_k\varphi)^{\pm} = \frac{1}{2}S_k\varphi, \\
\frac{(\partial DL_k\psi)^{\pm}}{\partial \nu} = \frac{1}{2}T_k\psi, & \frac{(\partial SL_k\varphi)^{\pm}}{\partial \nu} = \frac{1}{2}K'_k\varphi \mp \frac{1}{2}\varphi.
\end{array} \tag{3.34}$$

Lemma 3.2.2 *The operators $DL_{k_1} - DL_k : H^{-\frac{1}{2}}(\partial D) \to H^2_{loc}(\mathbb{R}^d \backslash \overline{D})$ and $SL_{k_1} - SL_k : H^{-\frac{3}{2}}(\partial D) \to H^2_{loc}(\mathbb{R}^d \setminus \overline{D})$ are continuous for constants $k_1 \neq k$.*

Given Lipchitz boundaries Σ and Ω, let us define

$$S^{\Omega}_{k,\Sigma}(\psi)(x) := \int_{\Sigma} \frac{\partial \phi_k(x,y)}{\partial \nu}\psi(y)ds_y \quad \text{on} \quad \Omega$$

and similar for operators $K^{\Omega}_{k,\Sigma}$, $K'^{\Omega}_{k,\Sigma}$ and $T^{\Omega}_{k,\Sigma}$. Now we will derive an integral equation formulation for the exterior transmission problem. Let us first consider the case where $n > 0$ is some constant, and let $k_1 = \sqrt{nk^2}$. Suppose there exists $g \in H^{-\frac{1}{2}}(\partial D)$, $h \in H^{-\frac{3}{2}}(\partial D)$ and $g_1 \in H^{-\frac{1}{2}}(\partial D_1)$, $h_1 \in H^{-\frac{3}{2}}(\partial D_1)$ such that

$$w = v = g \quad \text{and} \quad \frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} = h \quad \text{on} \quad \partial D$$

and

$$w = g_1 \quad \text{and} \quad \frac{\partial w}{\partial \nu} = h_1 \quad \text{on} \quad \partial D_1.$$

By Green theorem, we have that

$$\begin{aligned}
w(x) &= \int_{\partial D} (g(y)\frac{\partial \phi_{k_1}(x,y)}{\partial \nu} - \phi_{k_1}(x,y)h(y))ds_y \\
&- \int_{\partial D_1} (g_1(y)\frac{\partial \phi_{k_1}(x,y)}{\partial \nu} - \phi_{k_1}(x,y)h_1(y))ds_y \\
&= DL_{k_1}g - SL_{k_1}h - DL_{k_1}g_1 + SL_{k_1}h_1 \quad \text{in} \quad D_1 \backslash \overline{D}
\end{aligned}$$

and

$$w(x) = DL_k g_1 - SL_k h_1 \quad \text{in} \quad \mathbb{R}^d \backslash \overline{D_1}.$$

Similarly we have that

$$v(x) = DL_k g - SL_k h \quad \text{in} \quad \mathbb{R}^d \setminus \overline{D}.$$

From (3.34) we have that

$$\begin{array}{ll} (DL_k g)^+ = \frac{1}{2}K_k g + \frac{1}{2}g, & (SL_k h)^+ = \frac{1}{2}S_k h, \\ \frac{(\partial DL_k g)^+}{\partial \nu} = \frac{1}{2}T_k g, & \frac{(\partial SL_k h)^+}{\partial \nu} = \frac{1}{2}K'_k h - \frac{1}{2}h. \end{array} \tag{3.35}$$

Let $x \to \partial D$, from jump relations of single and double layer potentials we have that

$$\begin{aligned} w &= \frac{1}{2}K^{\partial D}_{k_1,\partial D} g + \frac{1}{2}g - \frac{1}{2}S^{\partial D}_{k_1,\partial D} h - \frac{1}{2}K^{\partial D}_{k_1,\partial D_1} g_1 + \frac{1}{2}S^{\partial D}_{k_1,\partial D_1} h_1 \\ v &= \frac{1}{2}K^{\partial D}_{k,\partial D} g + \frac{1}{2}g - \frac{1}{2}S^{\partial D}_{k,\partial D} h \\ \frac{\partial w}{\partial \nu} &= \frac{1}{2}T^{\partial D}_{k_1,\partial D} g - \frac{1}{2}K'^{\partial D}_{k_1,\partial D} h + \frac{1}{2}h - \frac{1}{2}T^{\partial D}_{k_1,\partial D_1} g_1 + \frac{1}{2}K'^{\partial D}_{k_1,\partial D_1} h_1 \\ \frac{\partial v}{\partial \nu} &= \frac{1}{2}T^{\partial D}_{k,\partial D} g - \frac{1}{2}K'^{\partial D}_{k,\partial D} h + \frac{1}{2}h. \end{aligned} \tag{3.36}$$

Let $x \to \partial D_1$ we have that

$$\begin{aligned} w^- &= \frac{1}{2}K^{\partial D_1}_{k_1,\partial D} g - \frac{1}{2}S^{\partial D_1}_{k_1,\partial D} h - \frac{1}{2}K^{\partial D_1}_{k_1,\partial D_1} g_1 + \frac{1}{2}g_1 + \frac{1}{2}S^{\partial D_1}_{k_1,\partial D_1} h_1 \\ w^+ &= \frac{1}{2}K^{\partial D_1}_{k,\partial D_1} g_1 + \frac{1}{2}g_1 - \frac{1}{2}S^{\partial D_1}_{k,\partial D_1} h_1 \\ \frac{\partial w^-}{\partial \nu} &= \frac{1}{2}T^{\partial D_1}_{k_1,\partial D_1} g - \frac{1}{2}K'^{\partial D_1}_{k_1,\partial D} h - \frac{1}{2}T^{\partial D_1}_{k_1,\partial D_1} g_1 + \frac{1}{2}K'^{\partial D_1}_{k_1,\partial D_1} h_1 + \frac{1}{2}h_1 \\ \frac{\partial w^+}{\partial \nu} &= \frac{1}{2}T^{\partial D_1}_{k,\partial D_1} g_1 - \frac{1}{2}K'^{\partial D_1}_{k,\partial D_1} h_1 + \frac{1}{2}h_1. \end{aligned} \tag{3.37}$$

From (3.36) we have that

$$\begin{aligned} &\begin{pmatrix} -S^{\partial D}_{k_1,\partial D} + S^{\partial D}_{k,\partial D} & K^{\partial D}_{k_1,\partial D} - K^{\partial D}_{k,\partial D} \\ K'^{\partial D}_{k_1,\partial D} - K'^{\partial D}_{k,\partial D} & -T^{\partial D}_{k_1,\partial D} + T^{\partial D}_{k,\partial D} \end{pmatrix} \begin{pmatrix} h \\ g \end{pmatrix} \\ &\qquad + \begin{pmatrix} S^{\partial D}_{k_1,\partial D_1} & -K^{\partial D}_{k_1,\partial D_1} \\ -K'^{\partial D}_{k_1,\partial D_1} & T^{\partial D}_{k_1,\partial D_1} \end{pmatrix} \begin{pmatrix} h_1 \\ g_1 \end{pmatrix} = 0 \end{aligned} \tag{3.38}$$

From (3.37) we have that

$$\begin{pmatrix} -S^{\partial D_1}_{k_1,\partial D} & K^{\partial D_1}_{k_1,\partial D} \\ K'^{\partial D_1}_{k_1,\partial D} & -T^{\partial D_1}_{k_1,\partial D} \end{pmatrix} \begin{pmatrix} h \\ g \end{pmatrix}$$
$$+ \begin{pmatrix} S^{\partial D_1}_{k_1,\partial D_1} + S^{\partial D_1}_{k,\partial D_1} & -K^{\partial D_1}_{k_1,\partial D_1} - K^{\partial D_1}_{k,\partial D_1} \\ -K'^{\partial D_1}_{k_1,\partial D_1} - K'^{\partial D_1}_{k,\partial D_1} & T^{\partial D_1}_{k_1,\partial D_1} + T^{\partial D_1}_{k,\partial D_1} \end{pmatrix} \begin{pmatrix} h_1 \\ g_1 \end{pmatrix} = 0. \tag{3.39}$$

Now define

$$\begin{aligned} F^{\partial D}_{k,\partial D} &:= \begin{pmatrix} -S^{\partial D}_{k_1,\partial D} + S^{\partial D}_{k,\partial D} & K^{\partial D}_{k_1,\partial D} - K^{\partial D}_{k,\partial D} \\ K'^{\partial D}_{k_1,\partial D} - K'^{\partial D}_{k,\partial D} & -T^{\partial D}_{k_1,\partial D} + T^{\partial D}_{k,\partial D} \end{pmatrix} \\ F^{\partial D}_{k,\partial D_1} &:= \begin{pmatrix} S^{\partial D}_{k_1,\partial D_1} & -K^{\partial D}_{k_1,\partial D_1} \\ -K'^{\partial D}_{k_1,\partial D_1} & +T^{\partial D}_{k_1,\partial D_1} \end{pmatrix} \\ F^{\partial D_1}_{k,\partial D} &:= \begin{pmatrix} -S^{\partial D_1}_{k_1,\partial D} & K^{\partial D_1}_{k_1,\partial D} \\ K'^{\partial D_1}_{k_1,\partial D} & -T^{\partial D_1}_{k_1,\partial D} \end{pmatrix} \\ F^{\partial D_1}_{k,\partial D_1} &:= \begin{pmatrix} S^{\partial D_1}_{k_1,\partial D_1} + S^{\partial D_1}_{k,\partial D_1} & -K^{\partial D_1}_{k_1,\partial D_1} - K^{\partial D_1}_{k,\partial D_1} \\ -K'^{\partial D_1}_{k_1,\partial D_1} - K'^{\partial D_1}_{k,\partial D_1} & T^{\partial D_1}_{k_1,\partial D_1} + T^{\partial D_1}_{k,\partial D_1} \end{pmatrix}. \end{aligned}$$

Then (3.38) and (3.39) yields

$$\begin{aligned} F^{\partial D}_{k,\partial D} \begin{pmatrix} h \\ g \end{pmatrix} + F^{\partial D}_{k,\partial D_1} \begin{pmatrix} h_1 \\ g_1 \end{pmatrix} &= 0 \\ F^{\partial D_1}_{k,\partial D} \begin{pmatrix} h \\ g \end{pmatrix} + F^{\partial D_1}_{k,\partial D_1} \begin{pmatrix} h_1 \\ g_1 \end{pmatrix} &= 0. \end{aligned} \tag{3.40}$$

Lemma 3.2.3 *The operator $F^{\partial D_1}_{k,\partial D_1} : H^{-\frac{3}{2}}(\partial D_1) \times H^{-\frac{1}{2}}(\partial D_1) \to H^{\frac{3}{2}}(\partial D_1) \times H^{\frac{1}{2}}(\partial D_1)$ is invertible.*

Proof. Assume on the contrary there is $(h, g) \neq (0, 0)$ in $H^{-\frac{3}{2}}(\partial D_1) \times H^{-\frac{1}{2}}(\partial D_1)$ such that

$$F^{\partial D_1}_{k,\partial D_1} \begin{pmatrix} h \\ g \end{pmatrix} = 0. \tag{3.41}$$

Define

$$w = DL_k g - SL_k h \quad \text{and} \quad v = SL_{k_1} h - DL_{k_1} g \quad \text{in} \quad \mathbb{R}^d \backslash \partial D_1.$$

From jump properties of single and double layer potential, we have that

$$\begin{aligned} w^+ &= \frac{1}{2} K^{\partial D_1}_{k,\partial D_1} g + \frac{1}{2} g - \frac{1}{2} S^{\partial D_1}_{k,\partial D_1} h, & v^- &= -\frac{1}{2} K^{\partial D_1}_{k_1,\partial D_1} g + \frac{1}{2} g + \frac{1}{2} S^{\partial D_1}_{k_1,\partial D_1} h, \\ \frac{\partial w^+}{\partial \nu} &= \frac{1}{2} T^{\partial D_1}_{k,\partial D_1} g + \frac{1}{2} h - \frac{1}{2} K'^{\partial D_1}_{k,\partial D_1} h, & \frac{\partial v^-}{\partial \nu} &= -\frac{1}{2} T^{\partial D_1}_{k_1,\partial D_1} g + \frac{1}{2} h + \frac{1}{2} K'^{\partial D_1}_{k_1,\partial D_1} h. \end{aligned}$$

Then (3.41) yields

$$w^+ = v^- \quad \text{and} \quad \frac{\partial w}{\partial \nu}^+ = \frac{\partial v}{\partial \nu}^- .$$

Hence w and v satisfies

$$\Delta w + k^2 w = 0 \qquad \text{in} \quad \mathbb{R}^d \backslash \overline{D_1} \tag{3.42}$$

$$\Delta v + k_1^2 v = 0 \qquad \text{in} \quad D_1 \tag{3.43}$$

$$w - v = 0 \qquad \text{on} \quad \partial D_1 \tag{3.44}$$

$$\frac{\partial w}{\partial \nu} - \frac{\partial v}{\partial \nu} = 0 \qquad \text{on} \quad \partial D_1 \tag{3.45}$$

$$\lim_{r \to \infty} r^{\frac{d-1}{2}} \left(\frac{\partial w}{\partial r} - ikw \right) = 0 \tag{3.46}$$

the transmission problem has only trivial solution(c.f. [9]), i.e., $w = 0$ in $\mathbb{R}^d \backslash \overline{D_1}$ and $v = 0$ in D_1. This implies that $w^+ = v^- = 0$. Now from the jump properties of single and double layer potentials we have that

$$w^- = -g = v^+ \quad \text{and} \quad \frac{\partial w^-}{\partial \nu} = h = \frac{\partial v^+}{\partial \nu} \quad \text{on} \quad \partial D_1.$$

Then

$$\begin{aligned} \Delta v + k_1^2 v &= 0 \qquad && \text{in} \quad \mathbb{R}^d \backslash \overline{D_1} \\ \Delta w + k^2 w &= 0 && \text{in} \quad D_1 \\ v - w &= 0 && \text{on} \quad \partial D_1 \\ \frac{\partial v}{\partial \nu} - \frac{\partial w}{\partial \nu} &= 0 && \text{on} \quad \partial D_1 \\ \lim_{r \to \infty} r^{\frac{d-1}{2}} \left(\frac{\partial v}{\partial r} - ik_1 v \right) &= 0 \end{aligned}$$

therefore $v=0$ in $\mathbb{R}^d\backslash\overline{D_1}$ and $w=0$ in D_1, hence $g=-w^-=0$ and $h=-\frac{\partial w^-}{\partial\nu}^-=0$. This contradicts the assumption that $(h,g)\neq(0,0)$. This proves the theorem. $\square$

From the lemma we can define

$$F(k):=F^{\partial D}_{k,\partial D}-F^{\partial D}_{k,\partial D_1}(F^{\partial D_1}_{k,\partial D_1})^{-1}F^{\partial D_1}_{k,\partial D}.$$

Then (3.40) is equivalent to

$$F(k)\begin{pmatrix} h \\ g \end{pmatrix}=0. \tag{3.47}$$

We now proceed to the following theorem.

Theorem 3.2.1 *Suppose k is not an interior transmission eigenvalue in D. Then there exists a non-trivial solution $w\in L^2_{loc}(\mathbb{R}^d\setminus\overline{D})$ and $v\in L^2_{loc}(\mathbb{R}^d\setminus\overline{D})$ to (3.11)-(3.15) such that $w-v\in H^2_{loc}(\mathbb{R}^d\setminus\overline{D})$ if and only if there exists $(h,g)\neq(0,0)$ in $H^{-\frac{3}{2}}(\partial D)\times H^{-\frac{1}{2}}(\partial D)$ such that (3.47) holds.*

Proof. The above argument has proved that if there exists non-trivial solution $w\in L^2_{loc}(\mathbb{R}^d\setminus\overline{D})$, $v\in L^2_{loc}(\mathbb{R}^d\setminus\overline{D})$, $w-v\in H^2_{loc}(\mathbb{R}^d\setminus\overline{D})$ to (3.11)-(3.15), then there exists (g,h) such that (3.47) holds, since w,v are not trivial, we have by uniqueness that $g=w$ and $h=\frac{\partial w}{\partial\nu}$ on ∂D satisfies that $(h,g)\neq(0,0)$.

Conversely, assume there exists $(h,g)\neq(0,0)$ such that (3.47) holds. Let

$$\begin{pmatrix} h_1 \\ g_1 \end{pmatrix}:=-(F^{\partial D_1}_{k,\partial D_1})^{-1}F^{\partial D_1}_{k\partial D}\begin{pmatrix} h \\ g \end{pmatrix}\quad \text{on}\quad \partial D_1 \tag{3.48}$$

$$\begin{aligned} w(x) &= \begin{cases} DL_{k_1}g-SL_{k_1}h-DL_{k_1}g_1+SL_{k_1}h_1 & \text{in} \quad D_1\backslash\Gamma \\ \\ DL_kg_1-SL_kh_1 & \text{in} \quad \mathbb{R}^d\backslash\overline{D_1} \end{cases} \\ v(x) &= DL_kg-SL_kh \quad \text{in} \quad \mathbb{R}^d\backslash\Gamma. \end{aligned} \tag{3.49}$$

Then w, v satisfies equations (3.11) and (3.12). From equations (3.47) and (3.48), one can derive equation (3.40). From the jump properties of single and double layer potentials and equations (3.36) (3.37) (3.40) and (3.48) we have that

$$w^+ = v^+ \qquad \text{and} \qquad \frac{\partial w}{\partial \nu}^+ = \frac{\partial v}{\partial \nu}^+ \quad \text{on} \quad \partial D$$
$$w^+ = w^- \qquad \text{and} \qquad \frac{\partial w}{\partial \nu}^+ = \frac{\partial w}{\partial \nu}^- \quad \text{on} \quad \partial D_1.$$

Hence w and v satisfy equations (3.11)-(3.15). By regularity, we have $w \in L^2_{loc}(\mathbb{R}^d \setminus \overline{D})$, $v \in L^2_{loc}(\mathbb{R}^d \setminus \overline{D})$ and $w - v \in H^2_{loc}(\mathbb{R}^d \setminus \overline{D})$(c.f. [33]). It remains to show that (w, v) is not trivial. In fact, if $(w, v) = (0, 0)$ in $\mathbb{R}^d \setminus \overline{D}$, the jump properties of single and double layer potentials yield

$$w^- = -g = v^- \quad \text{and} \quad \frac{\partial w}{\partial \nu}^- = h = \frac{\partial v}{\partial \nu}^- \text{ on} \quad \partial D.$$

Then

$$\begin{aligned} \Delta w + k^2 n w = 0 & \qquad \text{in} \quad D \\ \Delta v + k^2 v = 0 & \qquad \text{in} \quad D \\ w = v & \qquad \text{on} \quad \partial D \\ \frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} & \qquad \text{on} \quad \partial D \end{aligned}$$

Since k is not an interior transmission eigenvalue, then $w = v = 0$ in D. Therefore $g = -v^- = 0$ and $h = -\frac{\partial v}{\partial \nu}^- = 0$. This contradicts the assumption that $(g, h) \neq (0, 0)$. This proves the theorem. $\square$

To proceed we state two properties of $F^{\partial D}_{k,\partial D}$(c.f. [33]).

Lemma 3.2.4 *The operator* $F^{\partial D}_{i|k|,\partial D} : H^{-\frac{3}{2}}(\partial D) \times H^{-\frac{1}{2}}(\partial D) \to H^{\frac{3}{2}}(\partial D) \times H^{\frac{1}{2}}(\partial D)$ *is coercive.*

Lemma 3.2.5 *The operator* $F^{\partial D}_{k,\partial D} : H^{-\frac{3}{2}}(\partial D) \times H^{-\frac{1}{2}}(\partial D) \to H^{\frac{3}{2}}(\partial D) \times H^{\frac{1}{2}}(\partial D)$ *is Fredholm of index zero and analytic on* $k \in \mathbb{C} \backslash \mathbb{R}^-$.

Then we have the following theorem.

Theorem 3.2.2 *Assume the constant $n \neq 1$ in $D_1 \backslash \overline{D}$, then $F(k) : H^{-\frac{3}{2}}(\partial D) \times H^{-\frac{1}{2}}(\partial D) \to H^{\frac{3}{2}}(\partial D) \times H^{\frac{1}{2}}(\partial D)$ is Fredholm of index zero and analytic in a neighborhood of the real axis.*

Proof. Since $F(k) = F^{\partial D}_{k,\partial D} - F^{\partial D}_{k,\partial D_1}(F^{\partial D_1}_{k,\partial D_1})^{-1}F^{\partial D_1}_{k,\partial D}$, from Lemma 3.2.4 and Lemma 3.2.5 it suffices to show $F^{\partial D}_{k,\partial D_1}(F^{\partial D_1}_{k,\partial D_1})^{-1}F^{\partial D_1}_{k,\partial D}$ is compact and analytic. Since ∂D and ∂D_1 are disjoint boundaries, by regularity $F^{\partial D}_{k,\partial D_1}$ and $F^{\partial D_1}_{k,\partial D}$ are compact, then the compactness follows immediately. The fact that $F^{\partial D}_{k,\partial D_1}(F^{\partial D_1}_{k,\partial D_1})^{-1}F^{\partial D_1}_{k,\partial D}$ is analytic in a neighborhood of the real axis is related to Remark 3.0.2. In fact, $(F^{\partial D_1}_{k,\partial D_1})^{-1}$ is analytic in $\{k : \Im k \geq 0\}$ since equations (3.42)-(3.46) are well-posed and we can prove that $F^{\partial D_1}_{k,\partial D_1}$ is invertible. Now from Remark 3.0.2, we can prove that equations (3.42)-(3.46) are well-posed in a neighborhood of the real axis. This can prove that $F^{\partial D_1}_{k,\partial D_1}$ is invertible and $(F^{\partial D_1}_{k,\partial D_1})^{-1}$ is analytic in a neighborhood of the real axis. This proves the theorem. $\square$

The following lemma shows injectivity of $F(i|k|)$ for general refractive index n.

Lemma 3.2.6 *Assume $\Re(n) - 1$ does not change sign and $\Im(n) \geq 0$ in $D_1 \backslash \overline{D}$, then there does not exist purely imaginary exterior transmission eigenvalues $i|k|$, i.e., $F(i|k|)$ is injective.*

Proof. Suppose ik with k real is an exterior transmission eigenvalue, then

$$\begin{aligned}
\Delta w - k^2 n w &= 0 && \text{in} \quad \mathbb{R}^d \setminus \overline{D} \\
\Delta v - k^2 v &= 0 && \text{in} \quad \mathbb{R}^d \setminus \overline{D} \\
w &= v && \text{on} \quad \partial D \\
\frac{\partial w}{\partial \nu} &= \frac{\partial v}{\partial \nu} && \text{on} \quad \partial D
\end{aligned}$$

$$\lim_{r\to\infty} r^{\frac{d-1}{2}}\left(\frac{\partial w}{\partial r} + kw\right) = 0 \qquad \text{and} \qquad \lim_{r\to\infty} r^{\frac{d-1}{2}}\left(\frac{\partial v}{\partial r} + kv\right) = 0$$

Let $u := w - v$, then

$$w = -\frac{1}{k^2}\frac{1}{1-n}(\Delta u - k^2 u) \quad \text{in} \quad D_1 \backslash \overline{D}$$

and

$$(\Delta - k^2 n)\frac{1}{1-n}(\Delta - k^2)u = 0 \quad \text{in} \quad D_1\backslash\overline{D}.$$

Multiplying the equation by $\overline{u}$ and integrating by parts yields

$$\begin{aligned} 0 &= \int_{\partial D_1} \frac{\partial}{\partial \nu}[\frac{1}{1-n}(\Delta u - k^2 u)]\overline{u} ds - \int_{\partial D_1} \frac{1}{1-n}(\Delta u - k^2 u)\frac{\partial \overline{u}}{\partial \nu} ds \\ &+ \int_{D_1\backslash\overline{D}} \frac{1}{1-n}(\Delta u - k^2 u)(\Delta \overline{u} - k^2 n \overline{u}) dx, \end{aligned}$$

i.e.

$$\begin{aligned} 0 &= -k^2 \int_{\partial D_1} \frac{\partial w^-}{\partial \nu}\overline{u} ds + k^2 \int_{\partial D_1} w^- \frac{\partial \overline{u}}{\partial \nu} ds \\ &+ \int_{D_1\backslash\overline{D}} \frac{1}{1-n}(\Delta u - k^2 u)(\Delta \overline{u} - k^2 n \overline{u}) dx. \end{aligned}$$

By regularity w and v are continuous across ∂D_1. If $\Re(n) - 1 > 0$ then

$$\begin{aligned} 0 &= -k^2 \int_{\partial D_1} \frac{\partial w^+}{\partial \nu}\overline{u} ds + k^2 \int_{\partial D_1} w^+ \frac{\partial \overline{u}}{\partial \nu} ds \\ &+ \int_{D_1\backslash\overline{D}} \frac{1}{1-n}(\Delta u - k^2 u)(\Delta \overline{u} - k^2 n \overline{u}) dx \\ &= -k^2 \int_{\partial B_R} \frac{\partial w}{\partial \nu}\overline{u} ds + k^2 \int_{\partial B_R} w \frac{\partial \overline{u}}{\partial \nu} ds \\ &+ \int_{D_1\backslash\overline{D}} \frac{1}{1-n}(\Delta u - k^2 u)(\Delta \overline{u} - k^2 n \overline{u}) dx \\ &= -k^2 \int_{\partial B_R} \frac{\partial w}{\partial \nu}\overline{u} ds + k^2 \int_{\partial B_R} w \frac{\partial \overline{u}}{\partial \nu} ds + k^2 \int_{\partial D_1} \frac{\partial u}{\partial \nu}\overline{u} ds \\ &+ \int_{D_1\backslash\overline{D}} \frac{1}{1-n}|\Delta u - k^2 u|^2 dx - k^2 \int_{D_1\backslash\overline{D}} |\nabla u|^2 dx - k^4 \int_{D_1\backslash\overline{D}} |u|^2 dx \\ &= -k^2 \int_{\partial B_R} \frac{\partial w}{\partial \nu}\overline{u} ds + k^2 \int_{\partial B_R} w \frac{\partial \overline{u}}{\partial \nu} ds + k^2 \int_{\partial B_R} \frac{\partial u}{\partial \nu}\overline{u} ds \\ &- k^2 \int_{B_R\backslash\overline{D_1}} |\nabla u|^2 dx - k^4 \int_{B_R\backslash\overline{D_1}} |u|^2 dx \\ &+ \int_{D_1\backslash\overline{D}} \frac{1}{1-n}|\Delta u - k^2 u|^2 dx - k^2 \int_{D_1\backslash\overline{D}} |\nabla u|^2 dx - k^4 \int_{D_1\backslash\overline{D}} |u|^2 dx \\ &= -k^2 \int_{\partial B_R} \frac{\partial w}{\partial \nu}\overline{u} ds + k^2 \int_{\partial B_R} w \frac{\partial \overline{u}}{\partial \nu} ds + k^2 \int_{\partial B_R} \frac{\partial u}{\partial \nu}\overline{u} ds \\ &+ \int_{D_1\backslash\overline{D}} \frac{1}{1-n}|\Delta u - k^2 u|^2 dx - k^2 \int_{B_R\backslash\overline{D}} |\nabla u|^2 dx - k^4 \int_{B_R\backslash\overline{D}} |u|^2 dx. \end{aligned}$$

Letting $R \to \infty$ yields

$$\int_{D_1 \backslash \overline{D}} \frac{1}{1-n} |\Delta u - k^2 u|^2 dx - k^2 \int_{\mathbb{R}^d \backslash \overline{D}} |\nabla u|^2 dx - k^4 \int_{\mathbb{R}^d \backslash \overline{D}} |u|^2 dx = 0. \tag{3.50}$$

Since $\Re(n) - 1 > 0$, then $u \equiv 0$ in $\mathbb{R}^d \backslash \overline{D}$. Hence $w = v = 0$ in $\mathbb{R}^d \backslash \overline{D}$.

Similarly, if $\Re(n) - 1 < 0$ and $\Im(n) = 0$, we can derive

$$\int_{D_1 \backslash \overline{D}} \frac{1}{n-1} |\Delta u - k^2 n u|^2 dx - k^2 \int_{\mathbb{R}^d \backslash \overline{D}} |\nabla u|^2 dx - k^4 \int_{\mathbb{R}^d \backslash \overline{D}} n|u|^2 dx = 0$$

from which we can conclude $w = 0$ and $v = 0$ in $\mathbb{R}^d \backslash \overline{D}$.

If $\Im(n) \neq 0$, taking the imaginary part of (3.50) yields

$$\Delta u - k^2 u = 0 \quad \text{in} \quad D_1 \backslash \overline{D}.$$

Hence $w = 0$ in $D_1 \backslash \overline{D}$, then we can conclude $w = 0$ and $v = 0$ in $\mathbb{R}^d \backslash \overline{D}$. This proves our theorem. $\square$

Now we apply Lemma 3.0.1 on the analytic Fredholm theory to obtain the following result.

Theorem 3.2.3 *Assume the constant $n \neq 1$, then the set of exterior transmission eigenvalues is discrete.*

Proof. Since $F(k)$ is Fredholm of index zero and is analytic in a neighborhood of the real axis, then we can apply the analytic Fredholm theory. From Lemma 3.2.6, there exists $i|\kappa|$ such that $F(i|\kappa|)$ is injective, then the analytic Fredholm theory yields the set of zeros of $F(k)$ is discrete. This proves our theorem.

Theorem 3.2.4 *Assume the constant $n \neq 1$, if k is not an exterior transmission eigenvalue, then the exterior transmission problem has a unique solution to (3.16)-(3.20) which depends continuously on the data ℓ_1, ℓ_2, f and h, i.e.*

$$\begin{aligned} & ||w||_{L^2(B_R \backslash \overline{D})} + ||v||_{L^2(B_R \backslash \overline{D})} \\ \leq \;& C \left(||f||_{H^{-\frac{3}{2}}(\partial D)} + ||h||_{H^{-\frac{1}{2}}(\partial D)} + ||\ell_1||_{L^2(B_R \backslash \overline{D})} + ||\ell_2||_{L^2(B_R \backslash \overline{D})} \right) \end{aligned}$$

where $C > 0$ is some constant independent of f and h.

Proof. Let

$$v = f_v \quad \text{and} \quad \frac{\partial v}{\partial \nu} = h_v \quad \text{on} \quad \partial D.$$

Then from Green's theorem we have that

$$w(x) \;=\; \begin{cases} DL_{k_1}(f_v + f) - SL_{k_1}(h_v + h) - DL_{k_1} f_1 + SL_{k_1} h_1 + V_{k_1, D_1\setminus\overline{D}} & \text{in} \quad D_1\setminus\overline{D} \\ \\ DL_k f_1 - SL_k h_1 + V_{k,\mathbb{R}^d\setminus\overline{D_1}} & \text{in} \quad \mathbb{R}^d\setminus\overline{D_1} \end{cases}$$

$$v(x) \;=\; DL_k f_v - SL_k h_v + V_{k,\mathbb{R}^d\setminus\overline{D}} \quad \text{in} \quad \mathbb{R}^d \setminus \overline{D}$$

where

$$w = f_1 \quad \text{and} \quad \frac{\partial w}{\partial \nu} = h_1 \quad \text{on} \quad \partial D_1,$$

$$V_{k_1, D_1\setminus\overline{D}}(x) := -\int_{D_1\setminus\overline{D}} \Phi_{k_1}(x,y)\ell_1(y)dy, \quad V_{k,\mathbb{R}^d\setminus\overline{D_1}}(x) := -\int_{\mathbb{R}^d\setminus\overline{D_1}} \Phi_k(x,y)\ell_1(y)dy$$

and

$$V_{k,\mathbb{R}^d\setminus\overline{D}}(x) := -\int_{\mathbb{R}^d\setminus\overline{D}} \Phi_k(x,y)\ell_2(y)dy.$$

Using similar arguments before Lemma 3.2.3 we have that

$$\begin{aligned} F(k)\begin{pmatrix} h_v \\ f_v \end{pmatrix} &= \begin{pmatrix} S^{\partial D}_{k_1,\partial D} & I - K^{\partial D}_{k_1,\partial D} \\ -I - K'^{\partial D}_{k_1,\partial D} & T^{\partial D}_{k_1,\partial D} \end{pmatrix}\begin{pmatrix} h \\ f \end{pmatrix} - F^{\partial D}_{k,\partial D_1} F^{-1,\partial D_1}_{k,\partial D_1} F^{\partial D_1}_{k,\partial D}\begin{pmatrix} h \\ f \end{pmatrix} \\ &+ \;2\begin{pmatrix} V_{k,\mathbb{R}^d\setminus\overline{D}}|_{\partial D} - V_{k_1,D_1\setminus\overline{D_1}}|_{\partial D} \\ -\frac{\partial}{\partial\nu}V_{k,\mathbb{R}^d\setminus\overline{D}}|_{\partial D} + \frac{\partial}{\partial\nu}V_{k_1,D_1\setminus\overline{D_1}}|_{\partial D} \end{pmatrix} \\ &- \;2F^{\partial D}_{k,\partial D_1}(F^{\partial D_1}_{k,\partial D_1})^{-1}\begin{pmatrix} V_{k,\mathbb{R}^d\setminus\overline{D_1}}|_{\partial D_1} - V_{k_1,D_1\setminus\overline{D_1}}|_{\partial D_1} \\ -\frac{\partial}{\partial\nu}V_{k,\mathbb{R}^d\setminus\overline{D_1}}|_{\partial D_1} + \frac{\partial}{\partial\nu}V_{k_1,D_1\setminus\overline{D_1}}|_{\partial D_1} \end{pmatrix}. \end{aligned}$$

Since $F(k)$ is Fredholm of index zero and k is not an exterior transmission eigenvalue, then $F(k)$ is invertible. From the properties of the volume potential we have for any ℓ_1, ℓ_2, f and h

$$\begin{aligned} &||f_v||_{H^{-\frac{1}{2}}(\partial D)} + ||h_v||_{H^{-\frac{3}{2}}(\partial D)} \\ \leq\;& C\left(||f||_{H^{-\frac{3}{2}}(\partial D)} + ||h||_{H^{-\frac{1}{2}}(\partial D)} + ||\ell_1||_{L^2(B_R\setminus\overline{D})} + ||\ell_2||_{L^2(B_R\setminus\overline{D})}\right) \end{aligned}$$

where $C > 0$ is some constant independent of f and h. Note that

$$||w||_{L^2(B_R\backslash\overline{D})} \leq C\left(||f_v + f||_{H^{-\frac{1}{2}}(\partial D)} + ||h_v + h||_{H^{-\frac{3}{2}}(\partial D)} + ||\ell_1||_{L^2(B_R\backslash\overline{D})}\right)$$

and

$$||v||_{L^2(B_R\backslash\overline{D})} \leq C(||f_v||_{H^{-\frac{1}{2}}(\partial D)} + ||h_v||_{H^{-\frac{3}{2}}(\partial D)} + ||\ell_2||_{L^2(B_R\backslash\overline{D})}),$$

then we can prove the theorem. □

In general, Theorem 3.2.3 also holds for non constant $n(x)$ under certain assumptions. In fact if $n(x)$ is constant only in a neighborhood $\mathcal{N} \in \mathbb{R}^d \setminus \overline{D}$ of the boundary ∂D, first suppose $k_1 = \sqrt{k^2 n_1}$ in $D_1\backslash\overline{D \cup \mathcal{N}}$, $k_2 = \sqrt{k^2 n_2}$ in $\mathcal{N}$ are constants. We denote the interface between $\mathcal{N}$ and $D_1\backslash\overline{D \cup \mathcal{N}}$ by $\partial\mathcal{N}$. Let

$$w = g \quad \text{and} \quad \frac{\partial w}{\partial \nu} = h \quad \text{on} \quad \partial D,$$

$$w = g_2 \quad \text{and} \quad \frac{\partial w}{\partial \nu} = h_2 \quad \text{on} \quad \partial \mathcal{N}$$

and

$$w = g_1 \quad \text{and} \quad \frac{\partial w}{\partial \nu} = h_1 \quad \text{on} \quad \partial D_1.$$

We can then define w in $\mathbb{R}^d\backslash\overline{D_1}$, $D_1\backslash\overline{D \cup \mathcal{N}}$, $\mathcal{N}$ respectively using single and double layer potentials. Matching the boundary conditions on ∂D, ∂D_1 and $\partial\mathcal{N}$ we can derive

$$\begin{aligned} 0 \;=\; & \begin{pmatrix} -S^{\partial D}_{k_2,\partial D} + S^{\partial D}_{k,\partial D} & K^{\partial D}_{k_2,\partial D} - K^{\partial D}_{k,\partial D} \\ K'^{\partial D}_{k_2,\partial D} - K'^{\partial D}_{k,\partial D} & -T^{\partial D}_{k_2,\partial D} + T^{\partial D}_{k,\partial D} \end{pmatrix} \begin{pmatrix} h \\ g \end{pmatrix} \\ & + \begin{pmatrix} S^{\partial D}_{k_2,\partial\mathcal{N}} & -K^{\partial D}_{k_2,\partial\mathcal{N}} \\ -K'^{\partial D}_{k_2,\partial\mathcal{N}} & +T^{\partial D}_{k_2,\partial\mathcal{N}} \end{pmatrix} \begin{pmatrix} h_2 \\ g_2 \end{pmatrix} \end{aligned} \tag{3.51}$$

$$\begin{aligned} 0 \;=\; & \begin{pmatrix} -S^{\partial D_1}_{k_1,\partial\mathcal{N}} & K^{\partial D_1}_{k_1,\partial\mathcal{N}} \\ K'^{\partial D_1}_{k_1,\partial\mathcal{N}} & -T^{\partial D_1}_{k_1,\partial\mathcal{N}} \end{pmatrix} \begin{pmatrix} h_2 \\ g_2 \end{pmatrix} \\ & + \begin{pmatrix} S^{\partial D_1}_{k_1,\partial D_1} + S^{\partial D_1}_{k,\partial D_1} & -K^{\partial D_1}_{k_1,\partial D_1} - K^{\partial D_1}_{k,\partial D_1} \\ -K'^{\partial D_1}_{k_1,\partial D_1} - K'^{\partial D_1}_{k,\partial D_1} & T^{\partial D_1}_{k_1,\partial D_1} + T^{\partial D_1}_{k,\partial D_1} \end{pmatrix} \begin{pmatrix} h_1 \\ g_1 \end{pmatrix} \end{aligned} \tag{3.52}$$

and

$$
\begin{aligned}
0 &= \begin{pmatrix} -S^{\partial\mathcal{N}}_{k_2,\partial D} & K^{\partial\mathcal{N}}_{k_2,\partial D} \\ K'^{\partial\mathcal{N}}_{k_2,\partial D} & -T^{\partial\mathcal{N}}_{k_2,\partial D} \end{pmatrix} \begin{pmatrix} h \\ g \end{pmatrix} \\
&+ \begin{pmatrix} S^{\partial\mathcal{N}}_{k_2,\partial\mathcal{N}} + S^{\partial\mathcal{N}}_{k_1,\partial\mathcal{N}} & -K^{\partial\mathcal{N}}_{k_2,\partial\mathcal{N}} - K^{\partial\mathcal{N}}_{k_1,\partial\mathcal{N}} \\ -K'^{\partial\mathcal{N}}_{k_2,\partial\mathcal{N}} - K'^{\partial\mathcal{N}}_{k_1,\partial\mathcal{N}} & T^{\partial\mathcal{N}}_{k_2,\partial\mathcal{N}} + T^{\partial\mathcal{N}}_{k_1,\partial\mathcal{N}} \end{pmatrix} \begin{pmatrix} h_2 \\ g_2 \end{pmatrix} \\
&+ \begin{pmatrix} -S^{\partial\mathcal{N}}_{k_1,\partial D_1} & K^{\partial\mathcal{N}}_{k_1,\partial D_1} \\ K'^{\partial\mathcal{N}}_{k_1,\partial D_1} & -T^{\partial\mathcal{N}}_{k_1,\partial D_1} \end{pmatrix} \begin{pmatrix} h_1 \\ g_1 \end{pmatrix}. \qquad (3.53)
\end{aligned}
$$

If we define

$$
\begin{aligned}
F^{\partial D}_{02,\partial D} : &= \begin{pmatrix} -S^{\partial D}_{k_2,\partial D} + S^{\partial D}_{k,\partial D} & K^{\partial D}_{k_2,\partial D} - K^{\partial D}_{k,\partial D} \\ K'^{\partial D}_{k_2,\partial D} - K'^{\partial D}_{k,\partial D} & -T^{\partial D}_{k_2,\partial D} + T^{\partial D}_{k,\partial D} \end{pmatrix} \\
F^{\partial D}_{2,\partial\mathcal{N}} : &= \begin{pmatrix} S^{\partial D}_{k_2,\partial\mathcal{N}} & -K^{\partial D}_{k_2,\partial\mathcal{N}} \\ -K'^{\partial D}_{k_2,\partial\mathcal{N}} & T^{\partial D}_{k_2,\partial\mathcal{N}} \end{pmatrix} \\
F^{\partial D_1}_{1,\partial\mathcal{N}} : &= \begin{pmatrix} -S^{\partial D_1}_{k_1,\partial\mathcal{N}} & K^{\partial D_1}_{k_1,\partial\mathcal{N}} \\ K'^{\partial D_1}_{k_1,\partial\mathcal{N}} & -T^{\partial D_1}_{k_1,\partial\mathcal{N}} \end{pmatrix} \\
F^{\partial D_1}_{01,\partial D_1} : &= \begin{pmatrix} S^{\partial D_1}_{k_1,\partial D_1} + S^{\partial D_1}_{k,\partial D_1} & -K^{\partial D_1}_{k_1,\partial D_1} - K^{\partial D_1}_{k,\partial D_1} \\ -K'^{\partial D_1}_{k_1,\partial D_1} - K'^{\partial D_1}_{k,\partial D_1} & T^{\partial D_1}_{k_1,\partial D_1} + T^{\partial D_1}_{k,\partial D_1} \end{pmatrix} \\
F^{\partial\mathcal{N}}_{2,\partial D} : &= \begin{pmatrix} -S^{\partial\mathcal{N}}_{k_2,\partial D} & K^{\partial\mathcal{N}}_{k_2,\partial D} \\ K'^{\partial\mathcal{N}}_{k_2,\partial D} & -T^{\partial\mathcal{N}}_{k_2,\partial D} \end{pmatrix} \\
F^{\partial\mathcal{N}}_{21,\partial\mathcal{N}} : &= \begin{pmatrix} S^{\partial\mathcal{N}}_{k_2,\partial\mathcal{N}} + S^{\partial\mathcal{N}}_{k_1,\partial\mathcal{N}} & -K^{\partial\mathcal{N}}_{k_2,\partial\mathcal{N}} - K^{\partial\mathcal{N}}_{k_1,\partial\mathcal{N}} \\ -K'^{\partial\mathcal{N}}_{k_2,\partial\mathcal{N}} - K'^{\partial\mathcal{N}}_{k_1,\partial\mathcal{N}} & T^{\partial\mathcal{N}}_{k_2,\partial\mathcal{N}} + T^{\partial\mathcal{N}}_{k_1,\partial\mathcal{N}} \end{pmatrix} \\
F^{\partial\mathcal{N}}_{1,\partial D} : &= \begin{pmatrix} -S^{\partial\mathcal{N}}_{k_1,\partial D_1} & K^{\partial\mathcal{N}}_{k_1,\partial D_1} \\ K'^{\partial\mathcal{N}}_{k_1,\partial D_1} & -T^{\partial\mathcal{N}}_{k_1,\partial D_1} \end{pmatrix}.
\end{aligned}
$$

Similarly we can show $F^{\partial D_1}_{01,\partial D_1}, F^{\partial\mathcal{N}}_{21,\partial\mathcal{N}}$ are invertible, and $F^{\partial\mathcal{N}}_{21,\partial\mathcal{N}} - F^{\partial\mathcal{N}}_{1,\partial D} F^{-1,\partial D_1}_{01,\partial D_1} F^{\partial D_1}_{1,\partial\mathcal{N}}$ which corresponds to the transsmision problem with $k_{12} := k_1$ in $D_1 \backslash \overline{D \cup \mathcal{N}}$ and $k_{12} :=$

k_2 in $D \cup \mathcal{N}$

$$\begin{aligned}
\Delta w + k^2 w = 0 &\qquad \text{in} \quad \mathbb{R}^d \backslash \overline{D_1} \\
\Delta v + k_{12}^2 v = 0 &\qquad \text{in} \quad D_1 \\
w - v = 0 &\qquad \text{on} \quad \partial D_1 \\
\frac{\partial w}{\partial \nu} - \frac{\partial v}{\partial \nu} = 0 &\qquad \text{on} \quad \partial D_1 \\
\lim_{r \to \infty} r^{\frac{d-1}{2}} (\frac{\partial w}{\partial r} - ikw) = 0 &
\end{aligned}$$

is invertible. Now define

$$F(k) := F_{02,\partial D}^{\partial D} + F_{2,\partial \mathcal{N}}^{\partial D} \left(F_{21,\partial \mathcal{N}}^{\partial \mathcal{N}} - F_{1,\partial D}^{\partial \mathcal{N}} F_{01,\partial D_1}^{-1,\partial D_1} F_{1,\partial \mathcal{N}}^{\partial D_1} \right)^{-1} F_{2,\partial D}^{\partial \mathcal{N}}.$$

Then equations (3.51) (3.52) and (3.53) are equivalent to

$$F(k) \begin{pmatrix} h \\ g \end{pmatrix} = 0.$$

We can prove Theorem 3.2.4 and Theorem 3.2.1 hold. Now suppose k_1 is not constant, let us replace $\Phi_{k_1}(\cdot, y)$ with the fundamental solution $\mathbb{G}(\cdot, y) \in H^1_{loc}(\mathbb{R}^d \backslash \{y\})$ satisfying

$$\Delta \mathbb{G}(x, y) + k^2 n(x) \mathbb{G}(x, y) = -\delta(x - y) \quad \text{in} \quad \mathbb{R}^d$$

and the Sommerfeld radiation condition, where n in D is defined to be equal with the constant in $\mathcal{N}$. Then $\mathbb{G}(\cdot, y) - \Phi_{k_1}(\cdot, y)$ satisfies the Helmholtz equation in $\mathcal{N}$, therefore is an analytic function in $\mathcal{N}$. Replacing $\Phi_{k_1}(\cdot, y)$ by $\mathbb{G}(\cdot, y)$, one can conclude the mapping properteis 3.34, Theorem 3.2.1 holds. The properties (3.2.4) of $Z(k)$ holds provided $n \in L^\infty(\mathbb{R}^d \setminus \overline{D})$, $|\Re(n) - 1|$ is bounded away from zero and $\Im(n) \geq 0$.

Theorem 3.2.5 *Suppose $\Re(e^{i\theta})(n-1) > m_*$ for some $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ in some neighborhood $\mathcal{N}$ of ∂D or that $n-1$ is real in all of $D_1 \backslash \overline{D}$ and satiesfies $n - 1 < -m_*$ in some neighborhood $\mathcal{N}$ of ∂D, then pure imaginary $i|k|$ with large enough modulus cannot be exterior transmission eigenvalues, i.e., $F(i|k|)$ is injective for k large enough.*

Proof. We apply the idea of Sylvester's result on interior transmission eigenvalue problems. Since the wave number lies on the positive imaginary axis, then the solution decays exponetially at infinity. This allows us to use Sylvester's proof by only changing $\phi(x) \in C_0^\infty(D)$ to $\phi(x) \in C_0^\infty(\mathbb{R}^d \setminus \overline{D})$ with $\phi(x) = 1$ in $\mathbb{R}^d \setminus \overline{D_1}$ and a neighborhood $\mathcal{N}$ of ∂D in Proposition 2.1 in [79]. $\square$

Then from Theorem 3.2.5 and Lemma 3.0.1 we have the following theorem.

Theorem 3.2.6 *Assume $n \in L^\infty(\mathbb{R}^d \setminus \overline{D})$, $|\Re(n) - 1|$ is bounded away from zero and $\Im(n) \geq 0$, furthermore assume n is constant in a neighborhood of $\mathcal{N} \in \mathbb{R}^d \setminus \overline{D}$ of the boundary ∂D , then the set of exterior transmission eigenvalue is discrete on the real line and the exterior transmission problem (3.16)-(3.20) is well-posed.*

Corollary 3.2.1 *Assume $n \in L^\infty(\mathbb{R}^d \setminus \overline{D})$, $|\Re(n) - 1|$ is bounded away from zero and $\Im(n) \geq 0$, furthermore assume n is constant in a neighborhood of $\mathcal{N} \in \mathbb{R}^d \setminus \overline{D}$ of the boundary ∂D , then the set of exterior transmission eigenvalue is discrete in the complex plane and the exterior transmission problem (3.16)-(3.20) is well-posed.*

Proof. In Theorem 3.2.6 we have used $F(k)$ is analytic in a neighborhood of the real axis. Now from Remark 3.0.2, we can prove that $F(k)$ is analytic in a domain which avoids the scattering poles. Then the exterior transmission eigenvalues form a discrete set in this domain. Since the scattering poles form a discrete set in the complex plane, this proves the corollary . $\square$

Chapter 4

SOLUTION OF THE INVERSE SCATTERING PROBLEM

In this chapter, we use qualitative methods to solve the inverse scattering problem, in particular linear sampling method and factorization method. The use of sampling methods has played an important role in inverse scattering theory for the past fifteen years and for a survey of recent results in this area we refer the reader to [9]. These methods are concerned with the inverse scattering problem for an inhomogeneous medium and seek to determine the support and bounds on the constitutive parameters of the scattering object by solving a linear integral equation of the first kind called the far field equation. A central role in this approach is an investigation of a class of non-selfadjoint eigenvalue problems called interior transmission eigenvalue problems. On the other hand, in the case of scattering by an impenetrable obstacle with Dirichlet, Neumann or impedance boundary conditions, there has been a recent interest in the inverse scattering problem with measured data inside a cavity [44, 45, 65, 72–74, 84]. In this class of problems the object is to determine the shape of the cavity from the use of sources and measurements along a curve or surface inside the cavity. A possible motivation for studying such a problem is to determine the shape of an underground reservoir by lowering receivers and transmitters into the reservoir through a bore hole drilled from the surface of the earth. In this chapter we will combine the above two directions of research and consider the inverse scattering problem for a cavity that is bounded by a penetrable inhomogeneous medium of compact support and seek to determine the shape of the cavity from internal measurements. Of particular interest in this investigation is the central role played by an unusual non-selfajoint eigenvalue problem called the *exterior transmission eigenvalue problem.* Before deriving the qualitative method, in the next section we first prove the data uniquely determine the support of the cavity.

4.1 Uniqueness of the Inverse Problem

In this section we prove that the boundary of the cavity is uniquely determined from a knowledge of the scattered field $u^s(x, y)$ for all $x, y \in \partial C$ where ∂C is the measurement manifold introduced in Section 2.3 Chapter 2 . It is not necessary to know the physical properties of the inhomogeneous medium as long as they satisfy appropriate a priori assumptions. The proof of uniqueness for the inverse problem of penetrable cavity is more complicated than for the case of scattering by an impenetrable cavity considered in [73]. The idea of the uniqueness proof for the inverse medium scattering problem originates from [46, 47]. Here we make use of the exterior transmission problem inspired by the idea in [39]. Since we are using some regularity results, in this section we assume more regularity of the boundary ∂D and material properties A and n than in previous sections.

Let ∂C be the smooth $d-1$ manifold of measurement satisfying Assumption 2.3.1 and let us define the admissible set of cavities

$$\mathbb{S} := \{D \subset \mathbb{R}^d : \partial D \text{ is of class } C^1,\ C \subset D\}.$$

Furthermore, we assume that the media outside the cavity has the material properties (A, n) which belong to

$$\mathcal{A} := \left\{ \begin{array}{c} A, n \in C^1(\Omega_{\partial D} \setminus \overline{D}) \cap L^\infty(\mathbb{R}^d \setminus \overline{D}),\ \Omega_{\partial D} \text{ is a neighborhood of } \partial D \\ \text{and } A, n \text{ satisfy the assumptions in Section 2.1 and in Lemma 3.1.1.} \end{array} \right\}.$$

We begin with a simple lemma.

Lemma 4.1.1 *Assume that $A, n \in \mathcal{A}$. Let $\{v_n, w_n\} \in H^1(\mathbb{R}^d \setminus \overline{D}) \times H^1(\mathbb{R}^d \setminus \overline{D})$, $n \in \mathbb{N}$, be a sequence of solutions to the exterior transmission problem (3.6)-(3.10) with boundary data $f_n \in H^{\frac{1}{2}}(\partial D)$, $h_n \in H^{-\frac{1}{2}}(\partial D)$. If the sequences $\{f_n\}$ and $\{h_n\}$*

converge in $H^{\frac{1}{2}}(\partial D)$ *and* $H^{-\frac{1}{2}}(\partial D)$ *respectively, and if the sequences* $\{v_n\}$ *and* $\{w_n\}$ *are bounded in* $H^1(B_R \setminus \overline{D})$*, then there exists a subsequence* $\{(v_{n_k}, w_{n_k})\}$ *which converges in* $H^1(B_R \setminus \overline{D})$*.*

Proof. Let $\{v_n, w_n\}$ be as in the statement of the lemma. Due to the compact imbedding of $H^1(B_R \setminus \overline{D})$ into $L^2(B_R \setminus \overline{D})$ we can select L^2-convergent subsequences $\{v_{n_k}\}$ and $\{w_{n_k}\}$. Hence, $\{v_{n_k}\}$ and $\{w_{n_k}\}$ satisfy

$$\nabla \cdot A\nabla w_{n_k} - \kappa^2 n w_{n_k} = -(\kappa^2 + k^2) w_{n_k} \qquad \text{in} \quad B_R \backslash \overline{D} \tag{4.1}$$

$$\Delta v_{n_k} - \kappa^2 v_{n_k} = -(\kappa^2 + k^2) v_{n_k} \qquad \text{in} \quad B_R \backslash \overline{D} \tag{4.2}$$

$$w_{n_k} - v_{n_k} = f_{n_k} \qquad \text{on} \quad \partial D \tag{4.3}$$

$$\frac{\partial w_{n_k}}{\partial \nu_A} - \frac{\partial v_{n_k}}{\partial \nu} = h_{n_k} \qquad \text{on} \quad \partial D \tag{4.4}$$

$$\frac{\partial w_{n_k}}{\partial \nu} - T_{i\kappa} w_{n_k} = (T_k - T_{i\kappa}) w_{n_k} \qquad \text{on} \quad \partial B_R \tag{4.5}$$

$$\frac{\partial v_{n_k}}{\partial \nu} - T_{i\kappa} v_{n_k} = (T_k - T_{i\kappa}) v_{n_k} \qquad \text{on} \quad \partial B_R \tag{4.6}$$

for $\kappa > 0$ chosen as in Lemma 3.1.1. Note that the left hand side of (4.1)-(4.6) in the variational setting is equivalent to the bounded invertible map $\mathcal{A}_{i\kappa}$. Thus v_{n_k} and w_{n_k} are bounded by the right hand side with respect to the appropriate norm. Now, due to compactly embedding of H^1 into L^2, there is a subsequence of the right hand sides of (4.1) and (4.2) that converge in L^2. Since $T_k - T_{i\kappa}$ is a compact operator there is a subsequence of the right hand side of (4.5) and (4.6) that converge in $H^{-\frac{1}{2}}(\partial B_R)$. Hence the result follows from the boundeness of $\mathcal{A}_{ik}$. $\square$

Note that Lemma 4.1.1 allows us to prove the uniqueness result without assuming that k is not an exterior transmission eigenvalue.

Theorem 4.1.1 *Assume that* $D_1, D_2 \in \mathbb{S}$ *are two penetrable cavities having material properties* $A_1, n_1 \in \mathcal{A}$ *and* $A_2, n_2 \in \mathcal{A}$ *in the exterior of* D_1 *and* D_2*, respectively, such that the corresponding scattered fields coincide on* ∂C *for all point sources located on* ∂C *and any fixed wave number* k*. Then* $D_1 = D_2$*.*

Proof. We denote by G the connected component of $D_1 \cap D_2$ which contains C. Let $u_j^s(\cdot, z)$ be the solution of (2.29)-(2.33) corresponding to $D_j,\ A_j, n_j,\ j = 1, 2$. We have that $u_1^s(x, z) = u_2^s(x, z)$ for $x, z \in \partial C$. Following the argument in [73], the latter implies that $u_1^s(x, z) = u_2^s(x, z)$ for $x, z \in \overline{G}$. Next, assume that $\bar{D}_1$ is not included in $\bar{D}_2$. We can find a point $z \in \partial D_1$ and $\epsilon > 0$ with the following properties, where $\Omega_\delta(z)$ denotes the ball of radius δ centered at z:

1. $\Omega_{8\epsilon}(z) \cap \bar{D}_2 = \emptyset$.
2. The intersection $\bar{D}_1 \cap \Omega_{8\epsilon}(z)$ is contained in the connected component of $\bar{D}_1$ to which z belongs.
3. There are points from this connected component of $\bar{D}_1$ to which z belongs which are not contained in $\bar{D}_1 \cap \bar{\Omega}_{8\epsilon}(z)$.
4. The points $z_n := z + \dfrac{\epsilon}{n}\nu(z)$ lie in G for all $n \in \mathbb{N}$, where $\nu(z)$ is the innerward unit normal to ∂D_1 at z.

Due to the singular behavior of $\Phi(\cdot,\, z_n)$ at the point z_n, it can be shown that

$$\|\Phi(\cdot,\, z_n)\|_{H^1(B_R \setminus \overline{D_1})} \to \infty \quad \text{as} \quad n \to \infty$$

where B_R is a large ball of radius R containing D_1 and D_2. We now define

$$v_n(x) := \frac{1}{\|\Phi(\cdot,\, z_n)\|_{H^1(B_R \setminus \overline{D_1})}} \Phi(x,\, z_n), \qquad x \in \mathbb{R}^d \setminus \overline{G}$$

and let $w_{1,n}, u_{1,n}^s$ and $w_{2,n}, u_{2,n}^s$ be the solutions of the scattering problem (2.29)-(2.33) with boundary data $f := v_n$ and $h := \partial v_n/\partial\nu$ corresponding to D_1 and D_2, respectively. Note that for each n, v_n is a radiating solution of the Helmholtz equation outside D_1 and D_2. Our aim is to prove that if $\bar{D}_1 \not\subset \bar{D}_2$ then the equality $u_1(\cdot, z) = u_2(\cdot, z)$ for $z \in G$ allows the selection of a subsequence $\{v_{n_k}\}$ from $\{v_n\}$ that converges to zero with respect to $H^1(B_R \setminus \overline{D_1})$. This certainly contradicts the definition of $\{v^n\}$ as a sequence of functions with $H^1(B_R \setminus \overline{D_1})$-norm equal to one. Note that as mentioned above we have $u_{1,n}^s = u_{2,n}^s$ in G.

We begin by noting that, since the functions $\Phi(\cdot,\, z_n)$ together with their derivatives are uniformly bounded in every compact subset of $\mathbb{R}^d \setminus \Omega_{2\epsilon}(z)$ and

$$\|\Phi(\cdot,\, z_n)\|_{H^1(B_R\setminus\overline{D_1})} \to \infty$$

as $n \to \infty$. Then $\|v_n\|_{H^1(B_R\setminus\overline{D_2})} \to 0$ as $n \to \infty$. Hence, $\|u^s_{2,n}\|_{H^1(D_2)} \to 0$ as $n \to \infty$ from the well-posedness of the forward scattering problem. Since $u^s_{1,n} = u^s_{2,n}$ in G then $\|u_{1,n}\|_{H^1(G)} \to 0$ as $n \to \infty$ as well. Now, with the help of a cutoff function $\chi \in C_0^\infty(\Omega_{8\epsilon}(z))$ satisfying $\chi(x) = 1$ in $\Omega_{7\epsilon}(z)$, we see that $\|u_{1,n}\|_{H^1(G)} \to 0$ implies that

$$(\chi u_{1,n}) \to 0 \quad \text{and} \quad \frac{\partial(\chi u_{1,n})}{\partial\nu} \to 0 \quad \text{as} \quad n \to \infty \tag{4.7}$$

with respect to the $H^{\frac{1}{2}}(\partial D_1)$-norm and $H^{-\frac{1}{2}}(\partial D_1)$-norm, respectively. Indeed, for the first convergence we simply apply the trace theorem while for the convergence of $\partial(\chi u_{1,n})/\partial\nu$, we first deduce the convergence of $\Delta(\chi u_{1,n})$ in $L^2(D_1)$, which follows from $\Delta(\chi u_{1,n}) = \chi\Delta u_{1,n} + 2\nabla\chi\cdot\nabla u_{1,n} + u_{1,n}\Delta\chi$, and then apply Theorem 5.5 in [9]. Note here that we need conditions 2 and 4 on z to ensure $\Omega_{8\epsilon}(z) \cap D_1^e = \Omega_{8\epsilon}(z) \cap G$.

We next note that in the exterior of $\Omega_{2\epsilon}(z)$ the $H^2(\Omega_R \setminus \Omega_{2\epsilon}(z))$-norms of v_n remain uniformly bounded. Then using the interior elliptic regularity and localization techniques as in Theorem 8.8 in [36] we can conclude that $u^s_{1,n}$ is uniformly bounded with respect to the $H^2((\Omega_{\partial D}\cap D_1)\setminus\Omega_{4\epsilon}(z))$-norm, where $\Omega_{\partial D}$ is an open neighborhood of ∂D. Therefore, using the compact imbedding of H^2 into H^1, we can select a $H^1(\Omega_{\partial D}\cap D_1)$ convergent subsequence $\{(1-\chi)u^s_{1,n_k}\}$ from $\{(1-\chi)u^s_{1,n}\}$. Hence, $\{(1-\chi)u^s_{1,n_k}\}$ is a convergent sequence in $H^{\frac{1}{2}}(\partial D_1)$ and similarly to the above reasoning we also have that $\{\partial((1-\chi)u^s_{1,n_k})/\partial\nu\}$ converges in $H^{-\frac{1}{2}}(\partial D_1)$. This, together with (4.7), implies that the sequences

$$\{u^s_{1,n_k}\} \qquad \text{and} \qquad \left\{\frac{\partial u^s_{1,n_k}}{\partial\nu}\right\}$$

converge in $H^{\frac{1}{2}}(\partial D_1)$ and $H^{-\frac{1}{2}}(\partial D_1)$, respectively.

Finally, the functions v_{n_k} and w_{1,n_k} are solutions to the exterior transmission problem (3.6)-(3.10) for the domain $\mathbb{R}^d \setminus \overline{D}_1$ with boundary data $f = u^s_{1,n_k}$ and $h =$

$\partial u^s_{1,n_k}/\partial\nu$. Since, the $H^1(B_r \setminus \overline{D_1})$-norms of v_{n_k} and w_{1,n_k} remain uniformly bounded, from Lemma 4.1.1 we can select a subsequence of $\{v_{n_k}\}$, denoted again by $\{v_{n_k}\}$, which converges in $H^1(B_r \setminus \overline{D_1})$ to some v. As H^1-limit of weak solutions to the Helmholtz equation, v is a distributional solution to the Helmholtz equation. We also have that $v|_{B_R\setminus(D_1\cup\Omega_{2\epsilon}(z))} = 0$ because the functions v_{n_k} converge uniformly to zero in the exterior of $\Omega_{2\epsilon}(z)$. Hence, v must be zero in all of $B_R \setminus \overline{D_1}$ (here we make use of condition 3). This contradicts the fact that $\|v_{n_k}\|_{H^1(B_R\setminus\overline{D_1})} = 1$. Hence the assumption $\bar{D}_1 \not\subset \bar{D}_2$ is false.

Since we can derive the analogous contradiction for the assumption $\bar{D}_2 \not\subset \bar{D}_1$, we have proved that $D_1 = D_2$. $\square$

Remark 4.1.1 The assumptions of Theorem 3.1.2 required for A and n can be replaced by any other assumptions that guaranty the well-posedness of the exterior transmission problem. Also the assumption that ∂D is smooth can be relaxed as long as it guaranties $H^{1+\epsilon}$-regularity near the boundary of the solution of the corresponding transmission problem (e.g. piecewise smooth [34]).

4.2 The Linear Sampling Method

Our analysis of the inverse scattering is based on an indicator function obtained by solving a linear integral equation of the first kind abtained from internal measurements. To this end, we define the near field equation

$$(Ng_z)(x) = \Phi(x,z) \quad \text{where} \quad x \in \partial C \tag{4.8}$$

for the unknown $g_z \in L^2(\partial C)$ where $z \in \mathbb{R}^d$ is a sampling point. To derive our linear sampling method, we need to define various operators and analyze them in appropriate function spaces. To this end, we define $\overline{\mathcal{U}}$ to be the closure of the set

$$\mathcal{U} := \left\{ \int_C \phi(\cdot,z)g(y)ds(y),\ g \in L^2(\partial C) \right\} \text{ with respect to } H^1_{loc}(\mathbb{R}^d\setminus\overline{D})$$

and

$$\mathcal{W} = \left\{ u \in H^1_{loc}(\mathbb{R}^d\setminus\overline{D}) :\ \Delta u + k^2 u = 0 \text{ in } \mathbb{R}^d\setminus\overline{D},\ \lim_{r\to\infty} r^{\frac{d-1}{2}}\left(\frac{\partial v}{\partial r} - ikv\right) = 0 \right\}.$$

It is clearly seen that $\mathcal{U} \subset \mathcal{W}$, in fact we have

Lemma 4.2.1 $\overline{\mathcal{U}} = \mathcal{W}$.

Proof. By the well-posedness of the problem

$$\Delta u + k^2 u = 0 \qquad \text{in} \quad \mathbb{R}^d \backslash \overline{D} \tag{4.9}$$

$$u = g \qquad \text{on} \quad \partial D \tag{4.10}$$

$$\lim_{r\to\infty} r^{\frac{d-1}{2}} (\frac{\partial v}{\partial r} - ikv) = 0 \tag{4.11}$$

we have $||u||_{H^1_{loc}(\mathbb{R}^d \backslash \overline{D})} \leq c||g||_{H^{\frac{1}{2}}(\partial D)}$ where c is some constant. Then $\mathcal{U}$ is dense in $\mathcal{W}$ if we can show that $\{\int_C \phi(\cdot, z)g(z)ds(z)|_{\partial D}, g \in L^2(\partial C)\}$ is dense in $H^{\frac{1}{2}}(\partial D)$. In fact, let $f \in H^{-\frac{1}{2}}(\partial D)$ be such that for any $g \in L^2(\partial C)$

$$\int_{\partial D} \int_C \phi(x, y)g(y)ds(y)f(x)ds(x) = 0.$$

Then

$$\int_C \int_{\partial D} \phi(x, y)f(x)ds(x)g(y)ds(y) = 0.$$

Since g is arbitrary, we have that

$$\int_{\partial D} \phi(x, y)f(x)ds(x) = 0, \quad \forall y \in \partial C.$$

Then the single layer potential $v_f(x) = \int_{\partial D} \phi(x, y)f(y)ds(y)$ satisfies $v_f|_{\partial C} = 0$ and the Helmholtz equation in C. From assumption 2.3.1 and Lemma 2.3.1 we have $v_f = 0$ in D. From the jump conditions across ∂D we have that

$$v_f^- = v_f^+ \quad \text{on} \quad \partial D$$

and

$$f = \frac{\partial v_f^-}{\partial \nu} - \frac{\partial v_f^+}{\partial \nu} \quad \text{on} \quad \partial D$$

where $+$ and $-$ denote approaching the boundary from outside and inside ∂D, respectively. We now have that $v_f^+ = 0$. Since v_f is a radiating solution to the Helmholtz equation, from uniqueness of the exterior Dirichlet problem we have that $v_f = 0$ in

$\mathbb{R}^d\backslash\overline{D}$. Then we have $f = \frac{\partial v_f^-}{\partial \nu} - \frac{\partial v_f^+}{\partial \nu} = 0$. Thus the set $\{v_g,\, g \in L^2(\partial C)\}$ where v_g defined by (2.54), is dense in $H^{\frac{1}{2}}(\partial D)$. Finally, note that since $\mathcal{W}$ is closed in $H^1_{loc}(\mathbb{R}^d\backslash\overline{D})$ and $\{v_g, g \in L^2(\partial C)\}$ is dense in $\mathcal{W}$ we have that $\overline{\mathcal{U}} = \mathcal{W}$. $\square$

Having defined U and W, we need to define an appropriate space for the traces of functions in W. To this end, we define $\mathcal{U}(\partial D) := \{(u|_{\partial D}, \frac{\partial u}{\partial \nu}|_{\partial D}), u \in \overline{\mathcal{U}}\}$.

Lemma 4.2.2 *$\mathcal{U}(\partial D)$ is closed in $H^{\frac{1}{2}}(\partial D) \times H^{-\frac{1}{2}}(\partial D)$ and hence is a Hilbert space.*

Proof. Let $(f, h) \in H^{\frac{1}{2}}(\partial D) \times H^{-\frac{1}{2}}(\partial D)$. If $(f, h) \in \overline{U}(\partial D)$ then there exists a sequence $\{u_n\}_{n=1}^{\infty}$ in $\mathcal{U}$ such that

$$\left(u_n|_{\partial D}, \frac{\partial u_n}{\partial \nu}|_{\partial D}\right) \to (f, h) \quad \text{as} \quad n \to \infty.$$

Clearly, $(u_n|_{\partial D}, \frac{\partial u_n}{\partial \nu}|_{\partial D})$ is bounded in $H^{\frac{1}{2}}(\partial D) \times H^{-\frac{1}{2}}(\partial D)$ and u_n satisfies

$$\begin{aligned} \Delta u_n + k^2 u_n = 0 \qquad & \text{in} \quad \mathbb{R}^d\backslash\overline{D} \\ u_n = u_n|_{\partial D} \qquad & \text{on} \quad \partial D \\ \lim_{r\to\infty} r^{\frac{d-1}{2}}\left(\frac{\partial u_n}{\partial r} - iku_n\right) = 0. & \end{aligned}$$

From the well-posedness of the exterior Dirichlet problem we have that $||u_n||_{H^1_{loc}(\mathbb{R}^d\backslash\overline{D})}$ is bounded by $||u_n||_{H^{\frac{1}{2}}(\partial D)}$ and therefore $\{u_n\}$ is bounded in $H^1_{loc}(\mathbb{R}^d \setminus \overline{D})$. Then there exists $u \in \overline{\mathcal{U}}$ such that u_n converges to u weakly. Since the trace operator $H^1_{loc}(\mathbb{R}^d\backslash\overline{D}) \to H^{\frac{1}{2}}(\partial D)$ and $\{u \in H^1_{loc}(\mathbb{R}^d \setminus \overline{D}) : \Delta u \in L^2_{loc}(\mathbb{R}^d \setminus \overline{D})\} \to H^{-\frac{1}{2}}(\partial D)$ are bounded [9], we obtain that

$$(u_n|_{\partial D}, \frac{\partial u_n}{\partial \nu}|_{\partial D}) \text{ converges to } (u|_{\partial D}, \frac{\partial u}{\partial \nu}|_{\partial D}) \text{ weakly.}$$

This shows that $f = u|_{\partial D}$ and $h = \frac{\partial u}{\partial \nu}|_{\partial D}$. Then one can conclude that $\mathcal{U}(\partial D)$ is closed in $H^{\frac{1}{2}}(\partial D) \times H^{-\frac{1}{2}}(\partial D)$. $\square$

Definition 4.2.1 *We define the operator $B : \mathcal{U}(\partial D) \to L^2(\partial C)$ which maps $\left(v|_{\partial D}, \frac{\partial v}{\partial \nu}|_{\partial D}\right)$ where $v \in \overline{\mathcal{U}}$, to $u^s_v|_C$ where (u^s_v, w_v) is the unique solution of (2.29)-(2.33) with $f := v|_{\partial D}$ and $g := \frac{\partial v}{\partial \nu}|_{\partial D}$.*

Theorem 4.2.1 *Assume that there does not exsit non-trivial $w \in H^1_{loc}(\mathbb{R}^d \setminus \overline{D})$ and $v \in H^1_{loc}(\mathbb{R}^d \setminus \overline{D})$ solves the homogeneous problem*

$$\begin{aligned}
\nabla \cdot A\nabla w + k^2 nw &= 0 && \textit{in} \quad \mathbb{R}^d \setminus \overline{D} \\
\Delta v + k^2 v &= 0 && \textit{in} \quad \mathbb{R}^d \setminus \overline{D} \\
w &= v && \textit{on} \quad \partial D \\
\frac{\partial w}{\partial \nu_A} &= \frac{\partial v}{\partial \nu} && \textit{on} \quad \partial D
\end{aligned}$$
$$\lim_{r\to\infty} r^{\frac{d-1}{2}}(\frac{\partial w}{\partial r} - ikw) = 0 \qquad \textit{and} \qquad \lim_{r\to\infty} r^{\frac{d-1}{2}}(\frac{\partial v}{\partial r} - ikv) = 0.$$

Then $B : \mathcal{U}(\partial D) \to L^2(C)$ is compact, injective and has dense range in $L^2(C)$.

Proof. The solution $u^s_v \in H^1(D)$ depends continuously on $\big(v|_{\partial D}, \frac{\partial v}{\partial \nu}|_{\partial D}\big)$. Since $u^s_v|_{\partial C} \in H^{\frac{1}{2}}(\partial C)$ and the imbedding $H^{\frac{1}{2}}(\partial C) \to L^2(\partial C)$ is compact, we have B is compact.

Next if $B(v|_{\partial D}, \frac{\partial v}{\partial \nu}|_{\partial D}) = 0$, then we have that $u^s_v|_{\partial C} = 0$. In addition we have $\Delta u^s_v + k^2 u^s_v = 0$ in C. Then from assumption 2.3.1 and Lemma 2.3.1, we have that $u^s_v = 0$ in D. Then w_v and v satisfy

$$\begin{aligned}
\nabla \cdot A\nabla w_v + k^2 nw_v &= 0 && \text{in} \quad \mathbb{R}^d \backslash \overline{D} \\
\Delta v + k^2 v &= 0 && \text{in} \quad \mathbb{R}^d \backslash \overline{D} \\
w_v &= v && \text{on} \quad \partial D \\
\frac{\partial w_v}{\partial \nu_A} &= \frac{\partial v}{\partial \nu} && \text{on} \quad \partial D
\end{aligned}$$
$$\lim_{r\to+\infty} r^{\frac{d-1}{2}}(\frac{\partial w_v}{\partial r} - ikw_v) \qquad \text{and} \qquad \lim_{r\to+\infty} r^{\frac{d-1}{2}}(\frac{\partial v}{\partial r} - ikv) = 0.$$

By assumption, we have $v = 0$ in $\mathbb{R}^d \backslash \overline{D}$ and thus $(v|_{\partial D}, \frac{\partial v}{\partial \nu}|_{\partial D}) = 0$. Hence B is injective.

Finally, since Range$(N) \subset$ Range(B), from Corollary 2.4.1 we can conclude that the range of B is dense in $L^2(\partial C)$. $\square$

To factorize the near field operator we define the bounded linear operator $S : L^2(\partial C) \to \mathcal{U}(\partial D)$ by

$$(Sg)(x) = \left(v_g|_{\partial D}, \frac{\partial v_g}{\partial \nu}|_{\partial D} \right) \quad \text{where } v_g \text{ is defined by (2.54).}$$

We can prove the following denseness result for the operator S.

Theorem 4.2.2 *The bounded linear operator $S : L^2(\partial C) \to \mathcal{U}(\partial D)$ is injective with dense range.*

Proof. If g is such that $Sg = 0$ then $v_g(x) = \int_{\partial C} \phi(x,y)g(y)ds(y)$ satisfies

$$\begin{aligned} \Delta v_g + k^2 v_g = 0 \qquad & \text{in} \quad \mathbb{R}^d \setminus \overline{D} \\ v_g = 0 \qquad & \text{on} \quad \partial D \\ \lim_{r \to +\infty} r^{\frac{d-1}{2}} \left(\frac{\partial v_g}{\partial r} - ikv_g \right) = 0. & \end{aligned}$$

Then $v_g = 0$ in $\mathbb{R}^d \setminus \overline{D}$. Since $\Delta v_g + k^2 v_g = 0$ in $\mathbb{R}^d \setminus \partial C$, by the unique continuation principle we have that $v_g = 0$ outside C. In particular the single layer boundary integral operator

$$g \to \int_{\partial C} \phi(x,y)g(y)ds(y) \quad \text{where} \quad g \in L^2(C) \quad \text{and} \quad x \in \partial C$$

is invertible as long as k^2 is not Dirichlet eigenvalue for $-\Delta$ in C [67]. Hence $g = 0$.

Now since $\{v_g : g \in L^2(\partial C)\}$ is dense in $\overline{\mathcal{U}}$ by definition, we have that S has dense range in $\mathcal{U}(\partial D)$. □

As the last ingredient to the main theorem is the follow which charactorizes the boundary.

Theorem 4.2.3 *Assume A and n satisfies the assumptions of Theorem 3.1.2 and k is not an exterior transmission eigenvalue. Then for $z \in \mathbb{R}^d \setminus \overline{C}$, $\Phi(\cdot, z)$ is in the range of B if and only if $z \in \mathbb{R}^d \setminus \overline{D}$.*

Proof. If $z \in \mathbb{R}^d \setminus \overline{D}$ and k is not an exterior transmission eigenvalue, then from Theorem 3.1.2, we have that the exterior transmission problem

$$\nabla \cdot A\nabla w_z + k^2 n w_z = 0 \qquad \text{in} \quad \mathbb{R}^d \setminus \overline{D} \tag{4.12}$$

$$\Delta v_z + k^2 v_z = 0 \qquad \text{in} \quad \mathbb{R}^d \setminus \overline{D} \tag{4.13}$$

$$w_z - v_z = \Phi(\cdot, z) \qquad \text{on} \quad \partial D \tag{4.14}$$

$$\frac{\partial w_z}{\partial \nu_A} - \frac{\partial v_z}{\partial \nu} = \frac{\partial \Phi(\cdot, z)}{\partial \nu} \qquad \text{on} \quad \partial D \tag{4.15}$$

$$\lim_{r\to\infty} r^{\frac{d-1}{2}}\left(\frac{\partial w}{\partial r} - ikw\right) = 0 \qquad \text{and} \qquad \lim_{r\to\infty} r^{\frac{d-1}{2}}\left(\frac{\partial v}{\partial r} - ikv\right) = 0 \tag{4.16}$$

has a unique solution $(w_z, v_z) \in H^1_{loc}(\mathbb{R}^d \setminus \overline{D}) \times H^1_{loc}(\mathbb{R}^d \setminus \overline{D})$. Then $(w_z, \Phi(\cdot, z))$ satisfies (2.29)-(2.33) with $(f, h) = (v_z, \frac{\partial v_z}{\partial \nu})|_{\partial D}$. Since $v_z \in \overline{\mathcal{U}}$, we have that $B(v_z, \frac{\partial v_z}{\partial \nu}) = \Phi(\cdot, z)|_C$. Then $\Phi(x, z)$ for $x \in \partial C$ is in the range of B.

Now assume that, for $z \in D$, $\Phi(\cdot, z)$ is in the range of B. Then there exists $v \in \overline{\mathcal{U}}$ such that

$$B(v|_{\partial D}, \frac{\partial v}{\partial \nu}|_{\partial D}) = \Phi(x, z), \quad x \in \partial C.$$

Let w_v, u_v^s be the solution to (2.29)-(2.33) with $(f, h) = (v|_{\partial D}, \frac{\partial v}{\partial \nu}|_{\partial D})$. By definition of B, $u_v^s = \Phi(\cdot, z)$ in D but this is not possible since $\Phi(\cdot, z) \notin H^1(D)$. $\Box$

Now we are ready to prove the main theorem of this section which provides the basis for the linear sampling method.

Theorem 4.2.4 *Assume that k is not an exterior transmission eigenvalue. Let u^s be the scattered field corresponding to the scattering problem (2.29)-(2.33) and N is the near field operator. Then the following holds:*

1. For $z \in \mathbb{R}^d \setminus \overline{D}$ and a given $\epsilon > 0$ there exists a function $g_z^\epsilon \in L^2(\partial C)$ such that

$$\|Ng_z^\epsilon - \Phi(\cdot, z)\|_{L^2(\partial C)} < \epsilon$$

and as $\epsilon \to 0$, the potential $v_{g_z^\epsilon}$ given by (2.54) with kernel g_z^ϵ converges to the solution v_z in the $H^1(B_R \setminus \overline{D})$-norm where (w_z, v_z) is the solution of (4.12)-(4.16).

2. For $z \in D \setminus \overline{C}$ and a given $\epsilon > 0$, every $g_z^\epsilon \in L^2(\partial C)$ that satisfies

$$\|Ng_z^\epsilon - \Phi(\cdot, z)\|_{L^2(\partial C)} < \epsilon$$

is such that

$$\lim_{\epsilon \to 0} \|v_{g_z^\epsilon}\|_{H^1(B_R \setminus \overline{D})} = \infty.$$

Proof. 1. Let $z \in \mathbb{R}^d \setminus \overline{D}$, then from Theorem 4.2.3, $\Phi(\cdot, z)$ is in the range of B and

$$B(v_z|_{\partial D}, \frac{\partial v_z}{\partial \nu}|_{\partial D}) = \Phi(\cdot, z)$$

where (w_z, v_z) is the solution of (4.12)-(4.16). Now for $\epsilon > 0$, since S has dense range in $\mathcal{U}(\partial D)$ by Theorem 4.2.2, then there exists $g_z^\epsilon \in L^2(\partial C)$ satisfying

$$\left\| Sg_z^\epsilon - \left(v_z|_{\partial D}, \frac{\partial v_z}{\partial \nu}|_{\partial D} \right) \right\|_{H^{\frac{1}{2}}(\partial D) \times H^{-\frac{1}{2}}(\partial D)} < \frac{\epsilon}{||B||} \tag{4.17}$$

which yields

$$\left\| BSg_z^\epsilon - B(v_z|_{\partial D}, \frac{\partial v_z}{\partial \nu}|_{\partial D}) \right\|_{L^2(\partial C)} < \epsilon.$$

The latter can be written as

$$\|Ng_z^\epsilon - \Phi(\cdot, z)\|_{L^2(\partial C)} < \epsilon.$$

Furthermore we have that

$$\lim_{\epsilon \to 0} \left\| Sg_z^\epsilon - \left(v_z|_{\partial D}, \frac{\partial v_z}{\partial \nu}|_{\partial D} \right) \right\|_{H^{\frac{1}{2}}(\partial D) \times H^{-\frac{1}{2}}(\partial D)} = 0$$

and hence

$$\lim_{\epsilon \to 0} ||v_{g_z^\epsilon} - v_z||_{H^1_{loc}(B_R \setminus \overline{D})} = 0.$$

For a fixed $\epsilon > 0$, we observe that $u^s := \Phi(\cdot, z)$ and $w := w_z$ satisfy the scattering problem (2.29)-(2.33) with data $f := v_z|_{\partial D}$ and $h := \frac{\partial v_z}{\partial \nu}|_{\partial D}$. From the well-posedness of (2.29)-(2.33) and the fact that $\|\Phi(\cdot, z)\|_{H^1(D)}$ goes to infinity as $z \to \partial D$, we obtain that

$$\lim_{z \to \partial D} \left\| \left(v_z|_{\partial D}, \frac{\partial v_z}{\partial \nu}|_{\partial D} \right) \right\|_{H^{\frac{1}{2}}(\partial D) \times H^{-\frac{1}{2}}(\partial D)} = \infty$$

and hence

$$\lim_{z\to\partial D} ||Sg_z^\epsilon||_{H^{\frac{1}{2}}(\partial D)\times H^{-\frac{1}{2}}(\partial D)} = \infty.$$

Since $||Sg_z^\epsilon||_{H^{\frac{1}{2}}(\partial D)\times H^{-\frac{1}{2}}(\partial D)}$ is bounded bellow by $||v_{g_z^\epsilon}||_{H^1(B_R\setminus\overline{D})}$, we can conclude that

$$\lim_{z\to\partial D} ||v_{g_z^\epsilon}||_{H^1(B_R\setminus\overline{D})} = \infty \quad \text{and} \quad \lim_{z\to\partial D} ||g_z^\epsilon||_{L^2(\partial C)} = \infty.$$

2. In order to prove the second statement, for $z \in D\setminus\overline{C}$ assume to the contrary that there exists a sequence

$$\{\epsilon_n\} \to 0$$

and corresponding functions v_{g_n} with kernels $g_n := g_z^{\epsilon_n}$ satisfying

$$\|Ng_n - \Phi(\cdot,z)\|_{L^2(\partial C)} < \epsilon_n$$

i.e.

$$Ng_n \to \Phi(\cdot,z) \quad \text{in} \quad L^2(\partial C) \quad \text{as} \quad n\to\infty$$

such that $\|v_n\|_{H^1(B_R\setminus\overline{D})}$ remains bounded. Then without loss of generality we may assume weak convergence v_n to some $v \in H^1(B_R\setminus\overline{D})$. Let us define

$$\tau : v \to \left(v|_{\partial D}, \frac{\partial v}{\partial \nu}|_{\partial D} \right)$$

which is obviously a bounded operator from $H^1(B_R\setminus\overline{D})$ to $H^{\frac{1}{2}}(\partial D)\times H^{-\frac{1}{2}}(\partial D)$. Since $B\tau$ is also bounded, we can conclude the weak convergence

$$(B\tau)v_{g_n} \rightharpoonup (B\tau)v \quad \text{in} \quad L^2(\partial C) \quad \text{as} \quad n\to\infty.$$

But $(B\tau)v_{g_n} = Ng_n$ converges strongly to $\Phi(\cdot,z)|_{\partial C}$ as $n\to\infty$, which means $\Phi(\cdot,z) = B(\tau v)$. This contradicts Theorem 4.2.3. $\square$

This theorem can be used to reconstruct the boundary ∂D. Roughly speaking if g_z^ϵ is the approximate solution of $Ng_z^\epsilon = \Phi(\cdot,z)$, then $\|v_{g_z^\epsilon}\|_{H^1(B_R\setminus\overline{D})}$ is large z in D and small for z outside D for a fixed ϵ. Unfortunately, $\|v_{g_z^\epsilon}\|_{H^1(B_R\setminus\overline{D})}$ cannot be used as an indicator function for D since it depends on D. Instead in practice we use the indicator function $\frac{1}{\|g_z^\epsilon\|_{L^2(\partial C)}}$. Since the near field equation (4.8) is ill-posed, it is necessary to use regularization techniques, e.g. Tikhonov regularization.

Now we provide some preliminary numerical results to show the viability of the linear sampling method to determine the support of a cavity surrounded by an anisotropic inhomogeneous media. For a given anisotropic medium and point sources on the given manifold ∂C, we can compute the near field data using a finite element method with perfectly matched layer. Having the simulated data $u^s(x, y)$ for $x, y \in \partial C$, i.e. the near field matrix $A(i, j) := u^s(x_i, y_j)$, then we can consider a discretized near field equation, and then apply the criterion described in Theorem 4.2.4 to reconstruct the interior of the cavity D. In particular, adding white Gaussian noise N_δ to the near field matrix yields perturbed near field matrix with $A^\delta(i, j) := A(i, j)(1 + N_\delta(i, j))$ and $\delta = ||A - A^\delta||$, we compute the regularized equation

$$((A^\delta)^* A^\delta + \alpha_z I)g_z^\alpha = (A^\delta)^* \Phi(\cdot, z)$$

where the regularized parameter α_z is determined by the Morozov principle

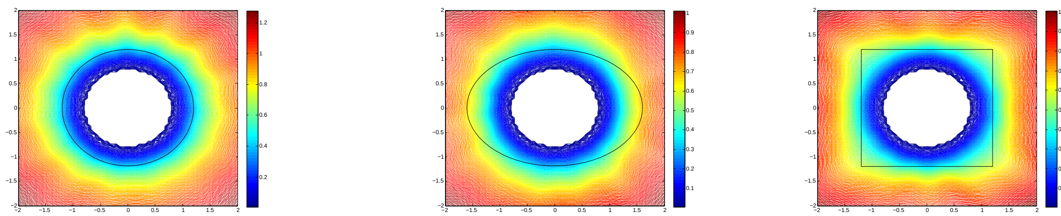
$$||A^\delta g_z^\alpha - \Phi_z||^2 = \delta^2 ||g_z^\alpha||$$

and Φ_z is the discretized representation of $\Phi(\cdot, z)$. To visualize the cavity we plot the contour lines of

$$W(z) := \frac{1}{||g_z^\alpha||_{L^2(\partial C)}}$$

for z varying in a region containing D. The cavity is the region where $W(z)$ takes values close to zero. For more details in the implementation of the linear sampling method see [13].

Now we present the reconstruction of a circle, an ellipse and a square in the two dimensional case. The exact geometry and the reconstructions are shown in the figures below. In all the examples presented here the region D_1 is the disk of radius 2, C is the disk of radius 0.8 (30 incident point sources and 30 corresponding measurements equally distributed on ∂C), the anisotropic medium has the constitutive parameters $A = [1.2 \quad 0; 0 \quad 1.5]$, $n = 0.8$, and the wave number is $k = 5$. Reconstructions are given for 0.1% white noise added and 1% white noise added. The sampling point z moves in a grid covering the square $[-2, 2]^2$.



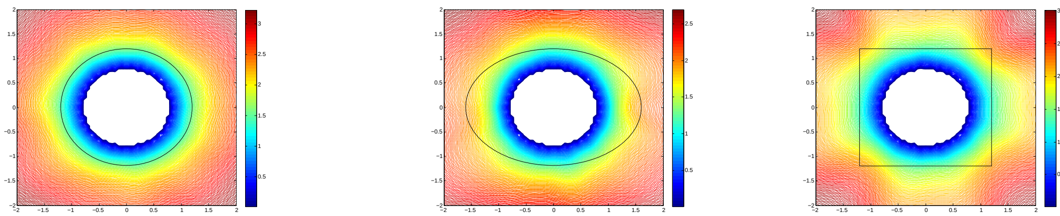
(a) **(b)** **(c)**

Figure 4.1: Panels (a), (b) and (c) show the reconstruction of a circle with radius 1.2, of an ellipse with x-axis 3.2 and y-axis 2.4 and of a square with length 2.4, respectively, with 0.1% noise data. The wavelength is $2\pi/5$ and ∂C is a circle of radius 0.8. Here $A = [1.2 \quad 0; 0 \quad 1.5]$, $n = 0.8$ and the true geometry of the cavity is indicated by the solid line.

4.3 The Factorization Method

As explained in section 4.2, the question whether the regularized solution of (4.8) captures the approximate solution g_z^ϵ provided by Theorem 4.2.4 is not justified by the linear sampling method. This is a typical theorical gap for this method. To get a more rigorous mathematical charactorization of the support of the cavity, we derive the factorization method. To begin with we define various operators, appropriate functional spaces and prove their properties. To this end, let us define the bounded linear operator H: $L^2(\partial C) \to H^1(D_1 \backslash \overline{D})$ by

$$(Hg)(x) := \int_{\partial C} \overline{\Phi(x, y)} g(y) ds(y), \quad x \in D_1 \backslash \overline{D} \tag{4.18}$$



(a) **(b)** **(c)**

Figure 4.2: Panels (a), (b) and (c) show the reconstruction of a circle with radius 1.2, of an ellipse with x-axis 3.2 and y-axis 2.4 and of a square with length 2.4, respectively, with 1% noise data. The wavelength is $2\pi/5$ and ∂C is a circle of radius 0.8. Here $A = [1.2 \quad 0; 0 \quad 1.5]$, $n = 0.8$ and the true geometry of the cavity is indicated by the solid line.

and the bounded linear operator $G : H^1(D_1\backslash\overline{D}) \to L^2(\partial C)$ which map w_0 to the trace of radiating solution w^* on ∂C, where $w^* \in H^1_{loc}(R^d)$ is the radiating solution

$$\nabla \cdot A\nabla w^* + k^2 n w^* = \nabla(I - A)\nabla w_0 + k^2(1-n)w_0 \quad \text{in} \quad \mathbb{R}^d. \tag{4.19}$$

Lemma 4.3.1 *The adjoint operator $H^* : H^1(D_1\backslash\overline{D}) \to L^2(\partial C)$ is given by*

$$(H^* v_0)(x) = \int_{\partial C} \frac{\partial \Phi(x,y)}{\partial \nu_y} v(y) ds(y) - \frac{1}{2} v(x) \quad \textit{for} \quad x \in \partial C \tag{4.20}$$

where $v \in H^1(B_R\backslash\overline{C})$ is uniquely determined by the variational formula

$$-\int_{B_R\backslash\overline{C}} \nabla v \cdot \nabla\overline{\psi}\, dx + k^2 \int_{B_R\backslash\overline{C}} v\,\overline{\psi}\, dx + \int_{\partial B_R} T_k v\,\overline{\psi}\, dx = \Big(v_0, \psi|_{D_1\backslash\overline{D}}\Big)_{H^1(D_1\backslash\overline{D})} \tag{4.21}$$

$\forall \psi \in H^1(B_R\backslash\overline{C})$.

Proof. First we remark that based on Lax-Milgram lemma and the properties of the Dirichlet to Neumann operator T_k (see e.g [9]), it is easy to see that (4.21) has a unique solution $v \in H^1(B_R\backslash\overline{C})$. Now, let $u = \int_{\partial C} \Phi(x,y)\overline{g(y)}ds(y)$ in $B_R\backslash\overline{C}$. Then $u \in H^1(B_R\backslash\overline{C})$ satisfies

$$\begin{aligned}
&\Delta u + k^2 u = 0 && \text{in } \ B_R\backslash\overline{C} \\
&\frac{\partial u^+}{\partial \nu} = \frac{\partial(\overline{Hg})^+}{\partial \nu} && \text{on } \ \partial C \\
&\frac{\partial u}{\partial \nu} = T_k u && \text{on } \ \partial B_R
\end{aligned}$$

and $\overline{u} = Hg$ in $D_1\backslash\overline{D}$. From (4.21) and the above equation for u, we obtain that

$$\begin{aligned}
(H^* v_0, g)_{L^2(\partial C)} &= (v_0, Hg)_{H^1(D_1\backslash\overline{D})} = (v_0, \overline{u})_{H^1(D_1\backslash\overline{D})} \\
&= -\int_{B_R\backslash\overline{C}} \nabla v \cdot \nabla u \, dx + k^2 \int_{B_R\backslash\overline{C}} vu \, dx + \int_{\partial B_R} T_k v u \, ds \\
&= -\int_{B_R\backslash\overline{C}} \nabla v \cdot \nabla u \, dx + k^2 \int_{B_R\backslash\overline{C}} vu \, dx + \int_{\partial B_R} T_k u v \, ds \\
&= \int_{\partial C} \frac{\partial u^+}{\partial \nu} v \, ds = \int_{\partial C} \left[\int_{\partial C} \frac{\partial \Phi(x,y)}{\partial \nu_x} \overline{g(y)} ds(y) - \frac{1}{2}\overline{g(x)} \right] v(x) ds(x) \\
&= \left(\int_{\partial C} \frac{\partial \Phi(x,\cdot)}{\partial \nu_x} v(x) ds(x) - \frac{1}{2} v, g \right)_{L^2(\partial C)}.
\end{aligned}$$

Therefore, we have that

$$(H^* v_0)(x) = \int_{\partial C} \frac{\partial \Phi(x,y)}{\partial \nu_y} v(y) ds(y) - \frac{1}{2} v(x) \quad \text{for} \quad x \in \partial C$$

which ends the proof. □

Now let us define an operator in $H^1(D_1 \setminus \overline{D})$. To this end, for a given $w_0 \in H^1(D_1\backslash\overline{D})$, let us consider the second kind integral equation

$$\int_{\partial C} \frac{\partial \Phi(x,y)}{\partial \nu_y} \varphi(y) ds(y) - \frac{1}{2}\varphi(x) = w^*(x) \quad \text{for} \quad x \in \partial C \tag{4.22}$$

where w^* is the radiating solution to (4.19) with this w_0. Since k^2 is not Dirichlet eigenvalue for $-\Delta$ in C, and C is smooth, the above second kind integral equation has a unique solution $\varphi \in H^{\frac{1}{2}}(\partial C)$ (see e.g. [58, 67]). Then we define $v \in H^1(B_R\backslash\overline{C})$ by the double layer potential

$$v(x) = \int_{\partial C} \frac{\partial \Phi(x,y)}{\partial \nu_y} \varphi(y) ds(y) - w^*(x) \quad \text{for} \quad x \in B_R\backslash\overline{C}. \tag{4.23}$$

(Note that the jump relation of the double layer potential implies that $\varphi := v|_{\partial C}$.) Having defined $v \in H^1(B_R \backslash \overline{C})$ we can now define the unique $v_0 \in H^1(D_1 \backslash \overline{D})$ by means of Riesz representation theorem as

$$\begin{aligned}(v_0, \psi)_{H^1(D_1\backslash\overline{D})} &= -\int_{D_1\backslash\overline{D}} \nabla v \cdot \nabla\overline{\psi}\, dx + k^2 \int_{D_1\backslash\overline{D}} v\overline{\psi}\, dx \\ &+ \int_{\partial D_1} \frac{\partial v^+}{\partial \nu}\overline{\psi}\, ds - \int_{\partial D} \frac{\partial v^-}{\partial \nu}\overline{\psi}\, ds. \end{aligned} \tag{4.24}$$

Hereon the subscripts " + " and " − " indicate that we approach the boundary from outside and inside the enclosed region, respectively. Also hereon the integrals over $d-1$ dimensional manifolds are defined in the sense of duality between $H^{1/2}$ and $H^{-1/2}$.

Definition 4.3.1 *The bounded linear operator $S : H^1(D_1\backslash\overline{D}) \to H^1(D_1\backslash\overline{D})$ is defined by*

$$S : w_0 \mapsto v_0$$

where v_0 is given by (4.24) corresponding to v defined by (4.23) with w^* satisfying (4.19) for the given w_0.

Before we start to factorize the near field operator, let us derive an explicit formula for $(Sw_0, u_0)_{H^1(D_1\backslash\overline{D})}$. To this end, we recall the double layer potential

$$\mathcal{D}(\varphi)(\cdot) = \int_{\partial C} \frac{\partial \Phi(\cdot, y)}{\partial \nu_y}\varphi(y)ds(y) \qquad \text{in} \quad B_R\backslash\partial C.$$

For a given w_0, let w^*, v and v_0 be as stated in Definition 4.3.1. Then

$$
\begin{aligned}
(v_0, u_0)_{H^1(D_1\setminus\overline{D})} &= -\int\limits_{D_1\setminus\overline{D}} \nabla v\cdot\nabla\overline{u_0}\,dx + k^2\int\limits_{D_1\setminus\overline{D}} v\overline{u_0}\,dx + \int\limits_{\partial D_1}\frac{\partial v^+}{\partial\nu}\overline{u_0}\,ds - \int\limits_{\partial D}\frac{\partial v^-}{\partial\nu}\overline{u_0}\,ds\\
&= \int\limits_{D_1\setminus\overline{D}} \nabla w^*\cdot\nabla\overline{u_0}\,dx - k^2\int\limits_{D_1\setminus\overline{D}} w^*\overline{u_0}\,dx - \int\limits_{\partial D_1}\frac{\partial (w^*)^+}{\partial\nu}\overline{u_0}\,ds\\
&+ \int\limits_{\partial D}\frac{\partial (w^*)^-}{\partial\nu}\overline{u_0}\,dx - \int\limits_{D_1\setminus\overline{D}} \nabla\mathcal{D}(v)\cdot\nabla\overline{u_0}\,dx + k^2\int\limits_{D_1\setminus\overline{D}}\mathcal{D}(v)\overline{u_0}\,dx\\
&+ \int\limits_{\partial D_1}\frac{\partial\mathcal{D}(v)^+}{\partial\nu}\overline{u_0}\,ds - \int\limits_{\partial D}\frac{\partial\mathcal{D}(v)^-}{\partial\nu}\overline{u_0}\,ds\\
&= -\int\limits_{\partial D_1}\frac{\partial (w^*)^+}{\partial\nu}\overline{u_0}\,ds + \int\limits_{D_1\setminus\overline{D}}\nabla w^*\nabla\overline{u_0}\,dx - k^2\int\limits_{D_1\setminus\overline{D}} w^*\overline{u_0}\,dx + \int\limits_{\partial D}\frac{\partial (w^*)^-}{\partial\nu}\overline{u_0}\,ds
\end{aligned}
$$

which gives that

$$
\begin{aligned}
(Sw_0, u_0)_{H^1(D_1\setminus\overline{D})} &= -\int_{\partial D_1}\frac{\partial (w^*)^+}{\partial\nu}\overline{u_0}\,dx + \int_{\partial D}\frac{\partial (w^*)^-}{\partial\nu}\overline{u_0}\,dx\\
&+ \int_{D_1\setminus\overline{D}}\nabla w^*\cdot\nabla\overline{u_0}\,dx - k^2\int_{D_1\setminus\overline{D}} w^*\overline{u_0}\,dx. \qquad (4.25)
\end{aligned}
$$

Now we are ready to construct the main factorization of our data operator.

Theorem 4.3.1 *The data operator N: $L^2(\partial C)\to L^2(\partial C)$ can be factorized as $N = H^*SH$ where $H : L^2(\partial C)\to H^1(D_1\setminus\overline{D})$ is defined by (4.18), $S : H^1(D_1\setminus\overline{D})\to H^1(D_1\setminus\overline{D})$ is defined by Definition 4.3.1, and $H^* : H^1(D_1\setminus\overline{D})\to L^2(\partial C)$ is given by Lemma 4.3.1.*

Proof. Given $g\in L^2(\partial C)$ and let $w_0 = Hg$ we have that $Ng = w^*|_{\partial C}$. From (4.23), we have that v satisfies Helmholtz equation in $B_R\setminus\overline{D_1}$ and $D\setminus\overline{C}$ and satisfies radiation condition, whence from (4.24) for any $\psi\in H^1(B_R\setminus\overline{C})$

$$
\begin{aligned}
\left(v_0, \psi|_{D_1\setminus\overline{D}}\right)_{H^1(D_1\setminus\overline{D})} &= -\int\limits_{D_1\setminus\overline{D}}\nabla v\cdot\nabla\overline{\psi}\,dx + k^2\int\limits_{D_1\setminus\overline{D}} v\overline{\psi}\,dx + \int\limits_{\partial D_1}\frac{\partial v^+}{\partial\nu}\overline{\psi}\,ds - \int\limits_{\partial D}\frac{\partial v^-}{\partial\nu}\overline{\psi}\,ds\\
&= -\int\limits_{B_R\setminus\overline{C}}\nabla v\cdot\nabla\overline{\psi}\,dx + k^2\int\limits_{B_R\setminus\overline{C}} v\overline{\psi}\,dx + \int\limits_{\partial B_R} T_k v\overline{\psi}\,ds - \int\limits_{\partial C}\frac{\partial v^+}{\partial\nu}\overline{\psi}\,ds.
\end{aligned}
$$

Next we show $\dfrac{\partial v^+}{\partial \nu} = 0$ on ∂C. From (4.22) and jump properties of double layer potential, we have that

$$\left[\int_{\partial C} \frac{\partial \Phi(x,y)}{\partial \nu_y} v(y) ds(y)\right]^- = w^*(x) \quad \text{for} \quad x \in \partial C.$$

Next, since both w^* and the double layer potential satisfy Helmholtz equation in C, the fact that they have the same Dirichlet boundary data on ∂C implies

$$\int_{\partial C} \frac{\partial \Phi(x,y)}{\partial \nu_y} v(y) ds(y) = w^*(x) \quad \text{for} \quad x \in C,$$

by making use that k^2 is not Dirichlet eigenvalue for $-\Delta$ in C. (Note that w^* is an H^1-solution of the Helmholtz equation in D, and therefore its normal derivative is continuous across ∂C.) Therefore

$$\frac{\partial}{\partial \nu_x}\left[\int_{\partial C} \frac{\partial \Phi(x,y)}{\partial \nu_y} v(y) ds(y)\right]^- = \frac{\partial w^*(x)}{\partial \nu_x} \quad \text{for} \quad x \in \partial C.$$

From the expression

$$v(x) = \int_{\partial C} \frac{\partial \Phi(x,y)}{\partial \nu_y} v(y) ds(y) - w^*(x) \quad \text{for} \quad x \in B_R \backslash \overline{C}$$

and the fact that the normal derivative of the double layer potential is continuous, we obtain that

$$\frac{\partial v^+}{\partial \nu} = 0 \quad \text{on} \quad \partial C$$

which now implies that

$$\left(v_0, \psi|_{D_1 \backslash \overline{D}}\right)_{H^1(D_1 \backslash \overline{D})} = -\int_{B_R \backslash \overline{C}} \nabla v \cdot \nabla \overline{\psi}\, dx + k^2 \int_{B_R \backslash \overline{C}} v \overline{\psi}\, dx + \int_{\partial B_R} T_k v \overline{\psi}\, ds. \quad (4.26)$$

Therefore from the definition of H^*, we have that

$$H^* v_0(\cdot) = \int_{\partial C} \frac{\partial \Phi(\cdot,y)}{\partial \nu_y} v(y) ds(y) - \frac{1}{2} v(\cdot) \quad \text{on} \quad \partial C.$$

Finally (4.23) and the jump properties of double layer potential yield

$$\int_{\partial C} \frac{\partial \Phi(x,y)}{\partial \nu_y} v(y) ds(y) - \frac{1}{2} v(x) = w^*(x) \quad \text{for} \quad x \in \partial C$$

which means tha $H^*v_0 = w^*|_C$. Thus $H^*SHg = H^*v_0 = w^*|_C = Ng$ and this holds for any $g \in L^2(\partial C)$, therefore we can conclude that $N = H^*SH$. $\square$

The above factorization of the data operator will enable us to characterize the cavity D in terms of the range of an operator know from the (measured) data operator. To do so we recall Theorem 2.1 in [61] which provides the theoretical basis of the factorization method that we use for our problem. For sake of reader's convenience we state this theorem below and for the proof we refer the reader to [61]. For a generic bounded linear operator K between two Banach spaces, we define its real and imaginary parts by $\Re(K) = \frac{K+K^*}{2}$ and $\Im(A) = \frac{K-K^*}{2i}$ where K^* is the adjoint of K.

Theorem 4.3.2 *Let $X \subset U \subset X^*$ be a Gelfand triple with Hilbert space U and reflexive Banach space X such that the embeddings are dense. Furthermore, let V be a second Hilbert space and $F : V \to V$, $H : V \to X$ and $T : X \to X^*$ be linear bounded operators with $F = H^*TH$. Assume*

(a) H is compact and injective.

(b) $\Re(T) = T_0 + T_1$ with some positive definite selfadjoint operator T_0 and some compact operator $T_1 : X \to X^*$.

(c) $(\Im(T\phi), \phi) \geq 0$ for all $\phi \in X$.

Furthermore, assume that one of the following two conditions is satisfied.

(d) T is injective.

(e) $\Im(T)$ is positive on the (finite dimensional) null space of $\Re(T)$, i.e $(\Im(T\phi), \phi) > 0$ for all $\phi \neq 0$ such that $\Re(T\phi) = 0$.

Then the operator $F_\# := |\Re(F)| + \Im(F)$ is positive definite and the range of $H^* : X^* \to V$ and the range of $F_\#^{1/2} : V \to V$ coincide.

We will apply this theorem to our near field operator $N = H^*SH$ and the rest of the paper is to make sure that the operator H, S and H^* fulfill the assumptions of the above theorem. To this end we make the following assumption on the wave number k.

Assumption 4.3.1 *The wave number $k > 0$ is such that there does not exist a nonzero $w_0 \in H^1(D_1\backslash\overline{D})$ satisfying*

$$\int_{D_1\backslash\overline{D}}(I-A)\nabla w_0\cdot\nabla\overline{\psi}\,dx - k^2\int_{D_1\backslash\overline{D}}(1-n)w_0\overline{\psi}\,dx = 0, \qquad \forall\,\psi\in H^1(D_1\backslash\overline{D}).$$

Theorem 4.3.3 *The operators H, S, H^* have the following properties.*

1. *H is compact and injective.*
2. *The imaginary part $\Im(S)$ of S is non-negative.*
3. *S is injective on $H^1(D_1\backslash\overline{D})$ provided that $k > 0$ satisfies Assumption 4.3.1.*

Proof. *(i)* Since the embedding of $H^2(D_1\backslash\overline{D})$ to $H^1(D_1\backslash\overline{D})$ is compact, and from the regularity of single layer potential aways from ∂C we obviously have that H is compact. Furthermore if $Hg = 0$ then since Hg solves the Helmholtz equation up to ∂C we have that $Hg|_{\partial C} = 0$. Now since k^2 is not a Dirichlet eigenvalue for $-\Delta$ and by the continuity of single layer potential we have that $Hg = 0$ in C. Now the jump relation gives that $g = 0$ and hence H is injective.

(ii) From (4.25) we have that,

$$\begin{aligned}(Sw_0,u_0)_{H^1(D_1\backslash\overline{D})} &= -\int\limits_{\partial D_1}\frac{\partial(w^*)^+}{\partial\nu}\overline{u_0}\,ds + \int\limits_{\partial D}\frac{\partial(w^*)^-}{\partial\nu}\overline{u_0}\,dx \\ &\quad + \int\limits_{D_1\backslash\overline{D}}\nabla w^*\cdot\nabla\overline{u_0}\,dx - k^2\int\limits_{D_1\backslash\overline{D}}w^*\overline{u_0}\,dx. \end{aligned} \tag{4.27}$$

Multiplying both sides (4.19) by $\overline{u_0}$ and integrating by parts yield

$$\int\limits_{\partial D_1}\frac{\partial(w^*)^-}{\partial\nu_A}\overline{u_0}\,ds - \int\limits_{\partial D}\frac{\partial(w^*)^+}{\partial\nu_A}\overline{u_0}\,ds - \int\limits_{D_1\backslash\overline{D}}A\nabla w^*\cdot\nabla\overline{u_0}\,dx + k^2\int\limits_{D_1\backslash\overline{D}}nw^*\overline{u_0}\,dx =$$

$$\int\limits_{\partial D_1}\frac{\partial w_0^-}{\partial\nu_{(I-A)}}\overline{u_0}\,ds - \int\limits_{\partial D}\frac{\partial w_0^+}{\partial\nu_{(I-A)}}\overline{u_0}\,ds - \int\limits_{D_1\backslash\overline{D}}(I-A)\nabla w_0\cdot\nabla\overline{u_0}\,dx + k^2\int\limits_{D_1\backslash\overline{D}}(1-n)w_0\overline{u_0}\,dx.$$

Now using the boundary conditions in the above

$$\frac{\partial(w_0)^+}{\partial\nu_{A-I}} = \frac{\partial(w^*)^-}{\partial\nu} - \frac{\partial(w^*)^+}{\partial\nu_A} \qquad \text{on} \qquad \partial D \tag{4.28}$$

and

$$\frac{\partial(w_0)^-}{\partial\nu_{I-A}} = -\frac{\partial(w^*)^+}{\partial\nu} + \frac{\partial(w^*)^-}{\partial\nu_A} \qquad \text{on} \qquad \partial D_1, \tag{4.29}$$

we have that

$$-\int\limits_{\partial D_1} \frac{\partial(w^*)^+}{\partial\nu}\overline{u_0}\,ds + \int\limits_{\partial D} \frac{\partial(w^*)^-}{\partial\nu}\overline{u_0}\,ds = -\int\limits_{D_1\setminus\overline{D}} A\nabla w^*\cdot\nabla\overline{u_0}\,dx + k^2\int\limits_{D_1\setminus\overline{D}} nw^*\overline{u_0}\,dx$$
$$+\int\limits_{D_1\setminus\overline{D}} (I-A)\nabla w_0\cdot\nabla\overline{u_0}\,dx - k^2\int\limits_{D_1\setminus\overline{D}} (1-n)w_0\overline{u_0}\,dx.$$

Let u^* be the solution of (4.19) corresponding to u_0 in the righthand side. Plugging the above expression in (4.27) we can now get

$$(Sw_0,u_0)_{H^1(D_1\setminus\overline{D})} = -\int_{D_1\setminus\overline{D}} A\nabla w^*\cdot\nabla\overline{u_0}\,dx + k^2\int\limits_{D_1\setminus\overline{D}} nw^*\overline{u_0}\,dx + \tag{4.30}$$
$$\int\limits_{D_1\setminus\overline{D}} (I-A)\nabla w_0\cdot\nabla\overline{u_0}\,dx - k^2\int\limits_{D_1\setminus\overline{D}} (1-n)w_0\overline{u_0}\,dx + \int_{D_1\setminus\overline{D}} \nabla w^*\cdot\nabla\overline{u_0}\,dx - k^2\int\limits_{D_1\setminus\overline{D}} w^*\overline{u_0}\,dx.$$

The latter can be rewritten as

$$(Sw_0,u_0)_{H^1(D_1\setminus\overline{D})} = -\int\limits_{D_1\setminus\overline{D}} (A-I)\nabla(w^*+w_0)\cdot\nabla\overline{(u^*+u_0)}\,dx$$
$$+k^2\int\limits_{D_1\setminus\overline{D}} (n-1)(w^*+w_0)\overline{(u^*+u_0)}\,dx + \int\limits_{D_1\setminus\overline{D}} (A-I)\nabla(w^*+w_0)\cdot\nabla\overline{u}^*\,dx$$
$$-k^2\int\limits_{D_1\setminus\overline{D}} (n-1)(w^*+w_0)\overline{u}^*\,dx. \tag{4.31}$$

Next noting that

$$-\nabla\cdot(A-I)\nabla(w^*+w_0) - k^2(n-1)(w^*+w_0) = \Delta w^* + k^2 w^* \qquad \text{in} \qquad D_1\setminus\overline{D},$$

multiplying both sides by $\overline{u^*}$ and integrating by parts we obtain

$$-\int_{D_1\setminus\overline{D}} \nabla w^*\cdot\overline{\nabla u^*}\,dx + k^2\int_{D_1\setminus\overline{D}} w^*\overline{u^*}\,dx + \int_{\partial D_1}\frac{\partial(w^*)^-}{\partial\nu}\overline{u^*}\,dx - \int_{\partial D}\frac{\partial(w^*)^+}{\partial\nu}\overline{u^*}\,dx$$
$$= -\int_{\partial D_1}\frac{\partial(w^*+w_0)^-}{\partial\nu_{(A-I)}}\overline{u^*}\,ds + \int_{\partial D}\frac{\partial(w^*+w_0)^+}{\partial\nu_{(A-I)}}\overline{u^*}\,ds$$
$$+\int_{D_1\setminus\overline{D}}(A-I)\nabla(w^*+w_0)\cdot\nabla\overline{u}^*\,dx - k^2\int_{D_1\setminus\overline{D}}(n-1)(w^*+w_0)\overline{u}^*\,dx. \tag{4.32}$$

Next using the boundary conditions (4.28) and (4.29) along with (6.43) in (4.31) yield

$$\begin{aligned}(Sw_0,u_0)_{H^1(D_1\backslash\overline{D})} = &-\int_{D_1\backslash\overline{D}}(A-I)\nabla(w^*+w_0)\cdot\nabla\overline{(u^*+u_0)}\,dx \\ &+k^2\int_{D_1\backslash\overline{D}}(n-1)(w^*+w_0)\overline{(u^*+u_0)}\,dx-\int_{D_1\backslash\overline{D}}\nabla w^*\cdot\nabla\overline{u^*}\,dx \\ &+k^2\int_{D_1\backslash\overline{D}}w^*\overline{u^*}\,dx+\int_{\partial D_1}\frac{\partial(w^*)^+}{\partial\nu}\overline{u^*}\,ds-\int_{\partial D}\frac{\partial(w^*)^-}{\partial\nu}\overline{u^*}\,ds.\end{aligned}$$

Since w^* satisfies Helmholtz equation in D and outside D_1, we can rewrite the above expression as

$$\begin{aligned}(Sw_0,u_0)_{H^1(D_1\backslash\overline{D})} &= -\int_{D_1\backslash\overline{D}}(A-I)\nabla(w^*+w_0)\cdot\nabla\overline{(u^*+u_0)}\,dx \\ &+k^2\int_{D_1\backslash\overline{D}}(n-1)(w^*+w_0)\overline{(u^*+u_0)}\,dx-\int_{B_R\backslash\overline{D_1}}\nabla w^*\cdot\nabla\overline{u^*}\,dx \\ &-\int_{D_1\backslash\overline{D}}\nabla w^*\cdot\nabla\overline{u^*}\,dx+k^2\int_{D_1\backslash\overline{D}}w^*\overline{u^*}\,dx+\int_{\partial B_R}T_kw^*\overline{u^*}\,dx \\ &+k^2\int_{B_R\backslash\overline{D_1}}w^*\overline{u^*}\,dx-\int_D\nabla w^*\cdot\nabla\overline{u^*}\,dx+k^2\int_D w^*\overline{u^*}\,dx\end{aligned}$$

which can be finally transformed to

$$\begin{aligned}(Sw_0,u_0)_{H^1(D_1\backslash\overline{D})} &= -\int_{D_1\backslash\overline{D}}(A-I)\nabla(w^*+w_0)\cdot\nabla\overline{(u^*+u_0)}\,dx \\ &+k^2\int_{D_1\backslash\overline{D}}(n-1)(w^*+w_0)\overline{(u^*+u_0)}\,dx \\ &-\int_{B_R}\nabla w^*\cdot\nabla\overline{u^*}\,dx+k^2\int_{B_R}w^*\overline{u^*}\,dx+\int_{\partial B_R}T_kw^*\overline{u^*}\,ds.\end{aligned}\tag{4.33}$$

Now taking the imaginary part of S, we can see that

$$\begin{aligned}&(\Im(S)w_0,w_0)_{H^1(D_1\backslash\overline{D})} \\ = \;&\Im\Big(\int_{\partial B_R}T_kw^*\overline{w^*}\,ds-\int_{D_1\backslash\overline{D}}(A-I)\nabla(w^*+w_0)\cdot\nabla\overline{(w^*+w_0)}\,dx \\ + \;&k^2\int_{D_1\backslash\overline{D}}(n-1)(w^*+w_0)\overline{(w^*+w_0)}\,dx\Big) \\ \geq \;&k\int_{S^2}|w^*_\infty|^2\,ds-\int_{D_1\backslash\overline{D}}\Im(A)\nabla(w^*+w_0)\cdot\nabla\overline{(w^*+w_0)}\,dx \\ + \;&k^2\int_{D_1\backslash\overline{D}}\Im(n)|w^*+w_0|^2\,dx \\ \geq \;&0\end{aligned}$$

because $\Im(A) \leq 0$ and $\Im(n) \geq 0$, where the far field pattern w^*_∞ of the radiating solution w^* is defined from the asymptotic expansion

$$w^*(x) = \frac{e^{ikr}}{r^{\frac{d-1}{2}}} w^*_\infty(\hat{x}) + O\left(r^{-\frac{d+1}{2}}\right), \qquad r = |x|,\ \hat{x} = x/|x|.$$

(iii) To prove the third part we assume that $Sw_0 = 0$. Then for any $\psi \in H^1(D_1 \backslash \overline{D})$ from (4.25) we have that

$$\int_{\partial D} \frac{\partial (w^*)^-}{\partial \nu} \overline{\psi}\, ds - \int_{\partial D_1} \frac{\partial (w^*)^+}{\partial \nu} \overline{\psi}\, ds + \int_{D_1 \backslash \overline{D}} \nabla w^* \cdot \nabla \overline{\psi}\, dx - k^2 \int_{D_1 \backslash \overline{D}} w^* \overline{\psi}\, dx = 0$$

which means that w^* satisfies

$$\Delta w^* + k^2 w^* = 0 \quad \text{in} \quad D_1 \backslash \overline{D}$$

and the transmission conditions

$$\frac{\partial (w^*)^+}{\partial \nu} = \frac{\partial (w^*)^-}{\partial \nu} \quad \text{on} \quad \partial D_1 \qquad \text{and} \qquad \frac{\partial (w^*)^+}{\partial \nu} = \frac{\partial (w^*)^-}{\partial \nu} \quad \text{on} \quad \partial D.$$

Therefore from (4.19), we can conclude that $w^* \in H^1_{loc}(\mathbb{R}^d)$ is a radiating solution to the Helmholtz equation in $\mathbb{R}^d$, and hence $w^* = 0$. Now multiplying both sides of (4.19) by $\overline{\psi}$ and integrating by parts, we obtain that w_0 satisfies

$$\int_{D_1 \backslash \overline{D}} (I - A) \nabla w_0 \cdot \nabla \overline{\psi} - k^2 \int_{D_1 \backslash \overline{D}} (1 - n) w_0 \overline{\psi} = 0, \quad \forall \quad \psi \in H^1(D_1 \backslash \overline{D}).$$

Then we have that $w_0 = 0$ providing that $k > 0$ satisfies Assumption 4.3.1. This implies that S is injective. □

Theorem 4.3.4 *The operator S satisfies in addition the following property:*

1. *If $\Re(A) > I$ then $-\Re(S)$ is the sum of a compact operator and a self-adjoint positive definite operator.*
2. *If $I - \Re(A) - \alpha |\Im(A)| > 0$ and $\Re(A) - \frac{1}{\alpha} |\Im(A)| \geq 0$ for some $\alpha > 0$, then $\Re(S)$ is the sum of a compact operator and a self-adjoint positive definite operator.*

Proof. *(i)* From (4.33) the real part of the operator S is given by

$$\begin{aligned}
(\Re(S) w_0, u_0)_{H^1(D_1 \backslash \overline{D})} = &- \int_{D_1 \backslash \overline{D}} (\Re(A) - I) \nabla (w^* + w_0) \cdot \nabla \overline{(u^* + u_0)} \\
&+ k^2 \int_{D_1 \backslash \overline{D}} (\Re(n) - 1)(w^* + w_0) \overline{(u^* + u_0)}\, dx \\
&- \int_{B_R} \nabla w^* \cdot \nabla \overline{u^*}\, dx + k^2 \int_{B_R} w^* \overline{u^*}\, dx + \int_{\partial B_R} \Re(T_k) w^* \overline{u^*}\, dx.
\end{aligned}$$

In the case when $\Re(A) > I$ we define the operator $K : H^1(D_1\backslash\overline{D}) \to H^1(D_1\backslash\overline{D})$ by

$$
\begin{aligned}
(Kw_0, u_0)_{H^1(D_1\backslash\overline{D})} = -\int_{D_1\backslash\overline{D}} (\Re(A) - I)\nabla(w^* + w_0)\cdot\nabla\overline{(u^* + u_0)}\,dx \\
-\int_{D_1\backslash\overline{D}} w_0\overline{u_0}\,dx - \int_{B_R}\nabla w^*\cdot\nabla\overline{u^*}\,dx + \int_{\partial B_R}\Re(T_k)w^*\overline{u^*}\,dx \quad (4.34)
\end{aligned}
$$

which is obviously self-adjoint. Using the known fact that the real part of the Dirichlet to Neumann operator $\Re(T_k)$ is nonpositive (see e.g. [69] in $\mathbb{R}^3$) and applying Young's inequality yield

$$
\begin{aligned}
(-Kw_0, w_0)_{H^1(D_1\backslash\overline{D})} &\geq (1-\alpha)\left((\Re(A) - I)\nabla w_0, \nabla w_0)_{L^2(D_1\backslash\overline{D})} + (w_0, w_0)_{L^2(D_1\backslash\overline{D})}\right. \\
&+ \left(1 - \frac{1}{\alpha}\right)\left((\Re(A) - I)\nabla w^*, \nabla w^*)_{L^2(D_1\backslash\overline{D})} + (\nabla w^*, \nabla w^*)_{L^2(B_R)} \geq c||w_0||^2_{H^1(D_1\backslash\overline{D})}\right.
\end{aligned}
$$

where $0 < \alpha < 1$ is such that $(1 - \frac{1}{\alpha})\sup_{D_1\backslash\overline{D}}(\Re(A) - I) + 1 > 0$, and c is some positive constant depending on A. Now, the fact that $\Re(S) - K$ is compact thanks to the compactly imbedding of $H^1(D_1\backslash\overline{D})$ into $L^2(D_1\backslash\overline{D})$, proves the first claim.

(ii) Next, we consider the case when $\Re(A) < I$. To prove the claim we need to derive a new expression for $(Sw_0, u_0)_{H^1(D_1\backslash\overline{D})}$. To this end from the expression (4.30) we have

$$
\begin{aligned}
& (Sw_0, u_0)_{H^1(D_1\backslash\overline{D})} \\
= & -\int_{D_1\backslash\overline{D}} A\nabla w^*\cdot\nabla\overline{u_0}\,dx + k^2\int_{D_1\backslash\overline{D}} nw^*\overline{u_0}\,dx + \int_{D_1\backslash\overline{D}}(I - A)\nabla w_0\cdot\nabla\overline{u_0}\,dx \\
- & k^2\int_{D_1\backslash\overline{D}}(1-n)w_0\overline{u_0}\,dx + \int_{D_1\backslash\overline{D}}\nabla w^*\cdot\nabla\overline{u_0} - k^2\int_{D_1\backslash\overline{D}} w^*\overline{u_0}\,dx \\
= & \int_{D_1\backslash\overline{D}}(I - A)\nabla w^*\cdot\nabla\overline{u_0}\,dx - k^2\int_{D_1\backslash\overline{D}}(1-n)w^*\overline{u_0}\,dx \\
+ & \int_{D_1\backslash\overline{D}}(I - A)\nabla w_0\cdot\nabla\overline{u_0}\,dx - k^2\int_{D_1\backslash\overline{D}}(1-n)w_0\overline{u_0}\,dx. \quad (4.35)
\end{aligned}
$$

For given $u_0 \in H^1(D_1\backslash\overline{D})$ let u^* be the radiating solution of (4.19). Multiplying both sides of

$$
\nabla\cdot A\nabla u^* + k^2 nu^* = \nabla(I - A)\cdot\nabla u_0 + k^2(1-n)u_0 \quad \text{in} \quad \mathbb{R}^d
$$

by $\overline{w^*}$ and integrating by parts, we obtain

$$\begin{aligned}
&\int_{\partial D_1} \frac{\partial (u^*)^-}{\partial \nu_A}\overline{w^*}\,dx - \int_{\partial D} \frac{\partial (u^*)^+}{\partial \nu_A}\overline{w^*}\,dx - \int_{D_1\setminus\overline{D}} A\nabla u^*\cdot\nabla\overline{w^*}\,dx + k^2\int_{D_1\setminus\overline{D}} nu^*\overline{w^*}\,dx \\
= &\int_{\partial D_1} \frac{\partial (u_0)^-}{\partial \nu_{I-A}}\overline{w^*}\,dx - \int_{\partial D} \frac{\partial (u_0)^+}{\partial \nu_{I-A}}\overline{w^*}\,dx - \int_{D_1\setminus\overline{D}} (I-A)\nabla u_0\cdot\nabla\overline{w^*}\,dx \\
+ \; & k^2\int_{D_1\setminus\overline{D}}(1-n)u_0\overline{w^*}\,dx.
\end{aligned}$$

Therefore, from the transmission conditions (4.28) and (4.29) for u^* and u_0 the above expression can be written as

$$\begin{aligned}
&\int_{D_1\setminus\overline{D}}(I-A)\nabla u_0\cdot\nabla\overline{w^*}\,dx - k^2\int_{D_1\setminus\overline{D}}(1-n)u_0\overline{w^*}\,dx \\
= &-\int_{\partial D_1}\frac{\partial (u^*)^+}{\partial \nu}\overline{w^*}\,ds + \int_{\partial D}\frac{\partial (u^*)^-}{\partial \nu}\overline{w^*}\,ds + \int_{D_1\setminus\overline{D}} A\nabla u^*\cdot\nabla\overline{w^*}\,dx - k^2\int_{D_1\setminus\overline{D}} nu^*\overline{w^*}\,dx \\
= &-\int_{\partial B_R} T_k u^*\overline{w^*}\,ds + \int_{B_R}\nabla u^*\cdot\nabla\overline{w^*}\,ds - k^2\int_{B_R} u^*\overline{w^*}\,dx + \int_{D_1\setminus\overline{D}}(A-I)\nabla u^*\cdot\nabla\overline{w^*}\,dx \\
- \; & k^2\int_{D_1\setminus\overline{D}}(n-1)u^*\overline{w^*}\,dx
\end{aligned}$$

where $T_k : H^{1/2}(\partial B_R) \to H^{-1/2}(\partial B_R)$ is the exterior Dirichlet to Neumann operator defined by (2.35). Conjugating the above expression we obtain

$$\begin{aligned}
&\int_{D_1\setminus\overline{D}}(I-\overline{A})\nabla w^*\cdot\nabla\overline{u_0}\,dx - k^2\int_{D_1\setminus\overline{D}}(1-\overline{n})w^*\overline{u_0}\,dx \\
= &-\int_{\partial B_R}\overline{T_k u^*}w^*\,ds + \int_{B_R}\nabla w^*\cdot\nabla\overline{u^*}\,dx - k^2\int_{B_R} w^*\overline{u^*}\,dx \\
+ &\int_{D_1\setminus\overline{D}}(\overline{A}-I)\nabla w^*\cdot\nabla\overline{u^*}\,dx - k^2\int_{D_1\setminus\overline{D}}(\overline{n}-1)w^*\overline{u^*}\,dx
\end{aligned}\tag{4.36}$$

and substituting (4.36) in (4.35) yields

$$\begin{aligned}
&(Sw_0, u_0)_{H^1(D_1\setminus\overline{D})} \\
= &\int_{D_1\setminus\overline{D}}(I-A)\nabla w_0\cdot\nabla\overline{u_0}\,dx - k^2\int_{D_1\setminus\overline{D}}(1-n)w_0\overline{u_0}\,dx - \int_{\partial B_R}\overline{T_k u^*}w^*\,ds \\
+ &\int_{B_R}\nabla w^*\cdot\nabla\overline{u^*}\,dx - k^2\int_{B_R} w^*\overline{u^*}\,dx + \int_{D_1\setminus\overline{D}}(\overline{A}-I)\nabla w^*\cdot\nabla\overline{u^*}\,dx \\
- \; & k^2\int_{D_1\setminus\overline{D}}(\overline{n}-1)w^*\overline{u^*}\,dx + \int_{D_1\setminus\overline{D}}(\overline{A}-A)\nabla w^*\cdot\nabla\overline{u_0}\,dx - k^2\int_{D_1\setminus\overline{D}}(\overline{n}-n)w^*\overline{u_0}\,dx.
\end{aligned}$$

Hence, taking the real part of S, i.e. computing $(S+S^*)/2$

$$
\begin{aligned}
&(\Re(S)w_0, u_0)_{H^1(D_1\backslash\overline{D})} \\
= & \int_{D_1\backslash\overline{D}} (I-\Re(A))\nabla w_0\cdot\nabla\overline{u_0}\,dx - k^2\int_{D_1\backslash\overline{D}}(1-\Re(n))w_0\overline{u_0}\,dx \\
+ & \ i\int_{D_1\backslash\overline{D}}\left(-\Im(A)\nabla w^*\cdot\nabla\overline{u_0}+\Im(A)\nabla\overline{u^*}\cdot\nabla w_0\right)\,dx \\
- & \ ik^2\int_{D_1\backslash\overline{D}}\left(-\Im(n)w^*\overline{u_0}+\Im(n)\overline{u^*}w_0\right)\,dx \\
+ & \int_{B_R}\Re(A)\nabla w^*\cdot\nabla\overline{u^*}\,dx - k^2\int_{B_R}\Re(n)w^*\overline{u^*}\,dx - \int_{\partial B_R}\overline{\Re(T_k)u^*}w^*\,ds.
\end{aligned}
$$

Now let us define K by

$$
\begin{aligned}
&(Kw_0, u_0)_{H^1(D_1\backslash\overline{D})} \\
= & \int_{D_1\backslash\overline{D}} (I-\Re(A))\nabla w_0\cdot\nabla\overline{u_0}\,dx + \int_{D_1\backslash\overline{D}} w_0\overline{u_0}\,dx + \int_{B_R}\Re(A)\nabla w^*\cdot\nabla\overline{u^*}\,dx \\
+ & \ i\int_{D_1\backslash\overline{D}}\left(-\Im(A)\nabla w^*\cdot\nabla\overline{u_0}+\Im(A)\nabla\overline{u^*}\cdot\nabla w_0\right)\,dx - \int_{\partial B_R}\overline{\Re(T_k)u^*}w^*\,ds
\end{aligned}
$$

which obviously is a self-adjoint. Again, using that the real part of the Dirichlet to Neumann operator $\Re(T_k)$ is nonpositive and applying Young's inequality yield

$$
\begin{aligned}
(Kw_0, w_0)_{H^1(D_1\backslash\overline{D})} \ \geq & \ ((I-\Re(A)-\alpha|\Im(A)|)\nabla w_0, \nabla w_0)_{L^2(D_1\backslash\overline{D})} \\
+ & \left((\Re(A)-\frac{1}{\alpha}|\Im(A)|)\nabla w^*, \nabla w^*\right)_{L^2(D_1\backslash\overline{D})} + (w_0, w_0)_{L^2(D_1\backslash\overline{D})} \\
\geq & \ c||w_0||^2_{H^1(D_1\backslash\overline{D})}
\end{aligned}
$$

where α is such that $I-\Re(A)-\alpha|\Im(A)|>0$, $\Re(A)-\frac{1}{\alpha}|\Im(A)|\geq 0$, and c is some constant depending on A, n only.

Finally the difference $\Re(S)-K$ is compact due to the compactly imbedding of $H^1(D_1\backslash\overline{D})$ into $L^2(D_1\backslash\overline{D})$. □

Remark 4.3.1 Injectivity of the operator $S: H^1(D_1\backslash\overline{D})\to H^1(D_1\backslash\overline{D})$ holds true if Assumption 4.3.1 is satisfied. Based on the analytic Fredholm theory it is easy to show that such $k>0$ form at most a discrete set with $+\infty$ as the only possible accumulation point. It is easy to see that if $\Im(A) \leq 0$ and $\Im(n) > 0$ in $D_1 \setminus \overline{D}$, or $\Im(A) < 0$ and $n-1$ does not change sign in $D_1 \setminus \overline{D}$ (more generally it suffices that $\int_{D_1 \setminus \overline{D}}(n-1)\,dx \neq 0$), then Assumption 4.3.1 holds for all real $k > 0$. In addition, the latter is also the case when A and n are real valued and the contrasts $(A - I)$ and $n - 1$ have the opposite signs.

Using the factorization in Theorem 4.3.1 along with Theorem 4.3.3 and Theorem 4.3.4, and applying Theorem 4.3.2 to the data operator N we can conclude the following range characterization result.

Corollary 4.3.1 *Under the assumptions of Theorem 4.3.3 and Theorem 4.3.4, the range of the operator* $N_{\#}^{1/2} : L^2(\partial C) \to L^2(\partial C)$ *and the range of the operator* $H^* : H^1(D_1 \setminus \overline{D}) \to L^2(\partial C)$ *coincide, where* $N_{\#} := |\Re(N)| + \Im(N)$.

The last step of our approach is to characterize the range of H^* in term of the support of the cavity D. At this point we introduce the so-called exterior transmission eigenvalue problem which in the current settings is a slight modification of the problem considered in [12] due to the fact that the incident field is the complex conjugate of the point source. This problem reads as: find $w \in H^1_{loc}(\mathbb{R}^d \setminus \overline{D})$, $v \in H^1_{loc}(\mathbb{R}^d \setminus \overline{D})$ such that

$$\nabla \cdot A\nabla w + k^2 n w = 0 \qquad \text{in} \quad \mathbb{R}^d \setminus \overline{D} \tag{4.37}$$

$$\Delta v + k^2 v = 0 \qquad \text{in} \quad \mathbb{R}^d \setminus \overline{D} \tag{4.38}$$

$$w - v = f \qquad \text{on} \quad \partial D \tag{4.39}$$

$$\frac{\partial w}{\partial \nu_A} - \frac{\partial v}{\partial \nu} = h \qquad \text{on} \quad \partial D \tag{4.40}$$

$$\lim_{r \to +\infty} r^{\frac{d-1}{2}} \left(\frac{\partial (w - v)}{\partial r} - ik(w - v) \right) = 0 \tag{4.41}$$

$$\lim_{r \to +\infty} r^{\frac{d-1}{2}} \left(\frac{\partial v}{\partial r} + ikv \right) = 0 \tag{4.42}$$

for $f \in H^{1/2}(\partial D)$ and $h \in H^{-1/2}(\partial D)$. Values of $k > 0$ for which the homogeneous exterior transmission problem (i.e (4.37)-(4.42) with $f = 0$ and $h = 0$) has non-trivial solution are called *exterior transmission eigenvalues*. Using the same technique as in

[12], it can be proven that the problem (4.37)-(4.42) satisfies the Fredholm alternative and the exterior transmission eigenvalues form at most a discrete set with $+\infty$ as the only possible accumulation point. Hence one can prove that provided that $k > 0$ is not an exterior transmission eigenvalue the problem (4.37)-(4.42) has a unique solution $w \in H^1_{loc}(\mathbb{R}^d \backslash \overline{D})$, $v \in H^1_{loc}(\mathbb{R}^d \backslash \overline{D})$ that depends continuously on f and h.

Assumption 4.3.2 *The wave number $k > 0$ is not an exterior transmission eigenvalue corresponding to (4.37)-(4.42).*

We can now prove the following theorem that relates the range of H^* with the support of the cavity D.

Theorem 4.3.5 *Suppose that Assumption 4.3.2 holds. Then for $z \in \mathbb{R}^d \backslash \overline{C}$ we have that $\Phi(\cdot, z)$ is in the range of H^* if and only if $z \in \mathbb{R}^d \backslash \overline{D}$.*

Proof. Let $z \in \mathbb{R}^d \backslash \overline{D}$ and since k is not an exterior transmission eigenvalue we can construct the unique solution $w_z \in H^1_{loc}(\mathbb{R}^d \backslash \overline{D})$, $v_z \in H^1_{loc}(\mathbb{R}^d \backslash \overline{D})$ of (4.37)-(4.42) with $f := \Phi(\cdot, z)$ and $h := \dfrac{\partial \Phi(\cdot, z)}{\partial \nu}$. Setting $u_z = w_z - v_z$, we have that from (4.41) u_z is an outgoing radiating solution of

$$\nabla \cdot A \nabla u_z + k^2 n u_z = \nabla (I - A) \cdot \nabla v_z + k^2 (1 - n) v_z \qquad \text{in} \quad \mathbb{R}^d \backslash \overline{D}$$

satisfying $u_z := \Phi(\cdot, z)$ and $\dfrac{\partial u_z}{\partial \nu} = \dfrac{\partial \Phi(\cdot, z)}{\partial \nu}$ on ∂D from (4.39) and (4.40). Define $u := u_z$ in $\mathbb{R}^d \backslash \overline{D}$ and $u := \Phi(\cdot, z)$ in D. The continuity of the Cauchy data guaranties that $u \in H^1_{loc}(\mathbb{R}^d)$ and in addition u is an outgoing radiating solution of

$$\nabla \cdot A \nabla u + k^2 n u = \nabla \cdot (I - A) \nabla v_z + k^2 (1 - n) v_z \qquad \text{in} \quad \mathbb{R}^d$$

which from the definition of operator $G : H^1(D_1 \backslash \overline{D}) \to L^2(\partial C)$ means that $\Phi(\cdot, z)|_{\partial C} = G v_z$. Note that $v_z \in H^1(D_1 \backslash \overline{D})$ satisfies the Helmholtz equation and the incoming radiation condition and therefore it is in the closure of the range of H. Finally since $G = H^* S$, we now have that $\Phi(\cdot, z)$ is in the range of H^*.

Next assume that $z \in \overline{D}\backslash\overline{C}$ and to the contrary that $\Phi(\cdot, z)|_{\partial C}$ is in the range of H^*. Let $v_0 \in H^1(D_1\backslash\overline{D})$ be such that $H^*v_0 = \Phi(\cdot, z)$. Then there is $v \in H^1(\mathbb{R}^d\backslash\overline{C})$ uniquely determined by (4.21) such that

$$(H^*v_0)(x) = \int_{\partial C} \frac{\partial\Phi(\cdot, y)}{\partial\nu_y} v(y)ds(y) - \frac{1}{2}v(x) \quad \text{for} \quad x \in \partial C.$$

From the jump property of the double layer potential we have that

$$\left[\int_{\partial C} \frac{\partial\Phi(\cdot, y)}{\partial\nu_y} v(y)ds(y)\right]^- = \Phi(\cdot, z) \quad \text{on} \quad \partial C$$

approaching ∂C from inside. From (4.21), we can also see that v satisfies the Helmholtz equation in $D\backslash\overline{C}$ and $\frac{\partial v^+}{\partial\nu}|_{\partial C} = 0$ where $+$ indicates that ∂C is approached from outside C. Now define

$$w(\cdot) = \begin{cases} \displaystyle\int_{\partial C} \frac{\partial\Phi(\cdot, y)}{\partial\nu_y} v(y)ds(y) & \text{in} \quad C \\ \\ \displaystyle\int_{\partial C} \frac{\partial\Phi(\cdot, y)}{\partial\nu_y} v(y)ds(y) - v(\cdot) & \text{in} \quad D\backslash\overline{C} \end{cases}$$

then $w \in H^1(D)$, satisfies Helmholtz equation in D, and $w^- = \Phi(\cdot, z)$ on ∂C. Hence the Assumption (2.3.1) guaranties that $w = \Phi(\cdot, z)$ in C since both satisfy the same Dirichlet boundary value problem for the Helmholtz equation in C. Now, if $z \in D$, by analytic continuation we have that $w = \Phi(\cdot, z)$ in $D\backslash z$, but since $\Phi(x, z)$ has singularity at $x = z$ whereas w is analytic, we arrive at a contradiction. Furthermore, if $z \in \partial D$, then equality of w and $\Phi(\cdot, z)$ up to the boundary ∂D requires that $\Phi(\cdot, z) \in H^{\frac{1}{2}}(\partial D)$, in the sense of the trace, which is not true, whence we again arrive at a contradiction. Therefore we can conclude that for $z \in \overline{D}\backslash\overline{C}$, $\Phi(\cdot, z)$ is not in the range of H^*. $\square$

Theorem 4.3.5 can be modified to remove Assumption 4.3.2.

Theorem 4.3.6 *For $z \in D_1\backslash\overline{C}$ we have that $\Phi(\cdot, z)$ is in the range of H^* if and only if $z \in D_1\backslash\overline{D}$.*

Proof. We only need to prove the statement for $z \in D_1\backslash\overline{D}$ since the complimentary case holds under no restriction on the wave and is proven in the second part of Theorem

4.3.5. To this end, for $z \in D_1 \backslash \overline{D}$, we need to show that there exists $v_0 \in H^1(D_1 \backslash \overline{D})$ such that $H^* v_0 = \Phi(\cdot, z)$. Fix $\epsilon > 0$ small enough and consider $w^* := \Phi(\cdot, z)\chi_\epsilon$, where χ_ϵ is a cut-off function such that $\chi_\epsilon = 0$ in $B(z, \epsilon)$ and $\chi_\epsilon = 1$ outside $B(z, 2\epsilon)$ where $B(z, \epsilon)$ is a ball centered at z with radius ϵ, and $B(z, 2\epsilon) \subset D_1 \backslash \overline{D}$. Obviously, $w^* \in H^1_{loc}(\mathbb{R}^d)$. Let now $v \in H^1(B_R \backslash \overline{C})$ be defined by (4.23) and $v_0 \in H^1(D_1 \backslash \overline{D})$ be defined by (4.24). We need to show that v_0, v satisfy (4.21). Indeed, by constructions, w^* satisfies Helmholtz equation in $D \backslash \overline{C}$ and $\mathbb{R}^d \backslash \overline{D_1}$ and so does v. Therefore

$$\int_{\partial D_1} \frac{\partial v^+}{\partial \nu} \overline{\psi} \, ds = \int_{\partial B_R} T_k v \overline{\psi} \, ds - \int_{B_R \backslash \overline{D_1}} \nabla v \cdot \nabla \overline{\psi} \, dx + k^2 \int_{B_R \backslash \overline{D_1}} v \overline{\psi} \, dx$$

and

$$\int_{\partial D} \frac{\partial v^-}{\partial \nu} \overline{\psi} \, ds = \int_{\partial C} \frac{\partial v^+}{\partial \nu} \overline{\psi} \, ds + \int_{D \backslash \overline{C}} \nabla v \cdot \nabla \overline{\psi} \, dx - k^2 \int_{D \backslash \overline{C}} v \overline{\psi} \, dx.$$

Plugging both the above equations in (4.24), we have that for any $\psi \in H^1(B_R \backslash \overline{C})$

$$\left(v_0, \psi|_{D_1 \backslash \overline{D}} \right)_{H^1(D_1 \backslash \overline{D})} = -\int\limits_{B_R \backslash \overline{C}} \nabla v \cdot \nabla \overline{\psi} \, dx + k^2 \int\limits_{B_R \backslash \overline{C}} v \overline{\psi} \, dx + \int\limits_{\partial B_R} T_k v \overline{\psi} \, dx - \int\limits_{\partial C} \frac{\partial v^+}{\partial \nu} \overline{\psi} \, ds.$$

From the definition of v and using jump properties of double layer potential we have that

$$\left[\int_{\partial C} \frac{\partial \Phi(x, y)}{\partial \nu_y} v(y) ds(y) \right]^- = w^*(x) \quad \text{for} \quad x \in \partial C$$

where " $-$ " indicates approaching ∂C from inside C. Then

$$\frac{\partial}{\partial \nu_x} \left[\int_{\partial C} \frac{\partial \Phi(\cdot, y)}{\partial \nu_y} v(y) ds(y) \right]^- = \frac{\partial w^*}{\partial \nu} \quad \text{on} \quad \partial C$$

and another application of the jump properties of double layer potential implies

$$\frac{\partial}{\partial \nu_x} \left[\int_{\partial C} \frac{\partial \Phi(\cdot, y)}{\partial \nu_y} v(y) ds(y) \right]^+ = \frac{\partial w^*}{\partial \nu} \quad \text{on} \quad \partial C$$

whence by construction of v we have that $\dfrac{\partial v^+}{\partial \nu} = 0$ on ∂C, where " $+$ " indicates approaching ∂C from outside C. Therefore (4.21) holds for v and v_0, hence by definition of H^* (4.20) holds true. From the construction of v and jump properties of the double layer potential we have that

$$\int_{\partial C} \frac{\partial \Phi(\cdot, y)}{\partial \nu_y} v(y) ds(y) - \frac{1}{2} v(x) = w^*(x) \quad \text{for} \quad x \in \partial C$$

and therefore $H^* v_0 = w^*$. Now since $w^* = \Phi(\cdot, z)$ in D we finally obtain $H^* v_0 = \Phi(\cdot, z)$ on ∂C. $\square$

Now we are ready to state the main theorem of the paper. Let us recall the compact data operator $N : L^2(\partial C) \to L^2(\partial C)$ given by (2.48) and define $\Re(N) = \frac{N+N^*}{2}$, $\Im(N) = \frac{N-N^*}{2i}$ and $N_\# := |\Re(N)| + \Im(N)$ which is also compact. In addition $N_\#$ is also selfadjoint. We denote by $(\phi_j, \lambda_j)_{j\in\mathbb{N}}$ an orthonormal eigen-system for $N_\#$. Then we have the following result.

Theorem 4.3.7 *Suppose that all Assumption 2.3.1, Assumption 4.3.1 and Assumption 4.3.2 are valid for the wave number* $k > 0$, *and either* $\Re(A) > I$, *or* $I - \Re(A) - \alpha|\Im(A)| > 0$ *and* $\Re(A) - \frac{1}{\alpha}|\Im(A)| \geq 0$ *for some* $\alpha > 0$. *Then for* $z \in \mathbb{R}^d \backslash \overline{C}$

$$z \in \mathbb{R}^d \backslash \overline{D} \quad \textit{if and only if} \quad \sum_j \frac{|\,(\Phi_z, \phi_j)\,|^2}{\lambda_j} < \infty$$

where $\Phi_z := \Phi(\cdot, z)|_{\partial C}$, *with* $\Phi(\cdot, z)$ *being the fundamental solution of the Helmholtz equation given by (2.34).*

Proof. The result follows from Corollary 4.3.1 and Theorem 4.3.5 along with an application of the Picard's theorem [9] and [24]. $\square$

Using now Theorem 4.3.6 instead of Theorem 4.3.5 we can drop Assumption 4.3.2. Note it is more difficult to handle the existence of exterior transmission eigenvalues than checking whether the wave number $k > 0$ satisfies Assumption 4.3.1.

Theorem 4.3.8 *Suppose that both Assumption 2.3.1 and Assumption 4.3.1 are valid for the wave number* $k > 0$, *and either* $\Re(A) > I$, *or* $I - \Re(A) - \alpha|\Im(A)| > 0$ *and* $\Re(A) - \frac{1}{\alpha}|\Im(A)| \geq 0$ *for some* $\alpha > 0$. *Then for* $z \in D_1 \backslash \overline{C}$

$$z \in D_1 \backslash \overline{D} \quad \textit{if and only if} \quad \sum_j \frac{|\,(\Phi_z, \phi_j)\,|^2}{|\lambda_j|} < \infty$$

where $\Phi_z := \Phi(\cdot, z)|_{\partial C}$, *with* $\Phi(\cdot, z)$ *being the fundamental solution of the Helmholtz equation given by (2.34).*

From practical point of view in order to determine the support of D from interior sources and measurements it suffices to sample only within the region D_1.

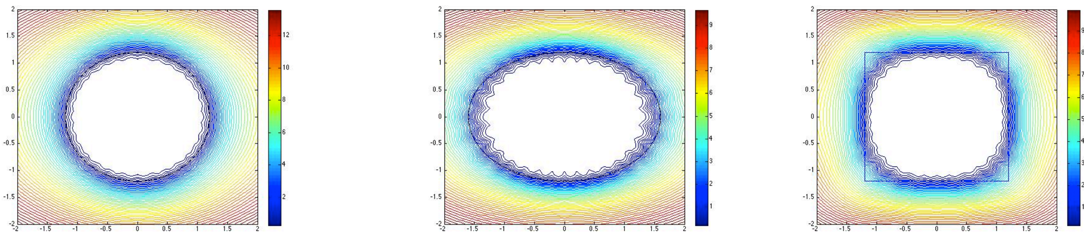
Now we provide some preliminary numerical results to show the viability of the factorization method to determine the support of a cavity surrounded by anisotropic inhomogeneous media. For a given anisotropic medium and artificial point sources on the given manifold ∂C, we can compute the near field data using a finite element method combined with PML on the artificial boundary. Having the simulated data $u^s(x,y), x, y \in \partial C$, we compute a discretized version of the near field operator and of $N_{\#}$, and then apply the criterion described in Theorem 4.3.7 to reconstruct the interior of the cavity D. In particular, we compute the eigensystem $(\phi_j, \lambda_j)_{j=1..M}$ of the symmetric matrix that approximate $N_{\#}$ and then use the discrete version of the Picard's criteria. To visualize the cavity we plot the contour lines of

$$W(z) := \left[\sum_{j=1}^{M} \frac{| < \Phi_z, \phi_j > |^2}{|\lambda_j|} \right]^{-1}$$

for z varying in a region large enough to contain the D. The cavity is the region where $W(z)$ takes values close to zero. For more details in the implementation of the factorization method see [55].

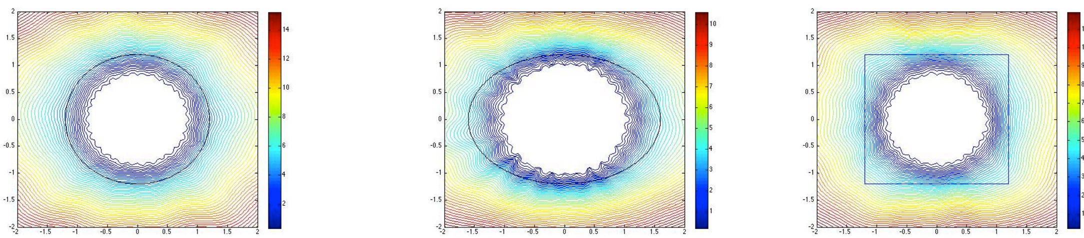
Now we present the reconstruction of a circle, an ellipse and a square in the two dimensional case. The exact geometry and the reconstructions are shown in the figures below. In all the examples presented here the region D_1 is the disk of radius 2. In the examples presented in Figure 4.3 and Figure 4.4, C is the disk of radius 0.8 (30 incident point sources and 30 corresponding measurements equally distributed on ∂C), the anisotropic medium has the constitutive parameters $A = [1.2 \quad 0; 0 \quad 1.5]$, $n = 0.8$, and the wave number is $k = 5$. Reconstructions are given for noise free data and 1% white noise added. The sampling point z moves in a grid covering the square $[-2, 2]^2$.

In order to study the sensitivity of reconstructions on the size of the measurement manifold ∂C, we show reconstructions for the configuration of the examples in Figure 4.3 where now ∂C is the circle of radius 0.4. The results presented in Figure 4.5 confirm that the reconstructions become worse as C gets smaller although the number



(a) **(b)** **(c)**

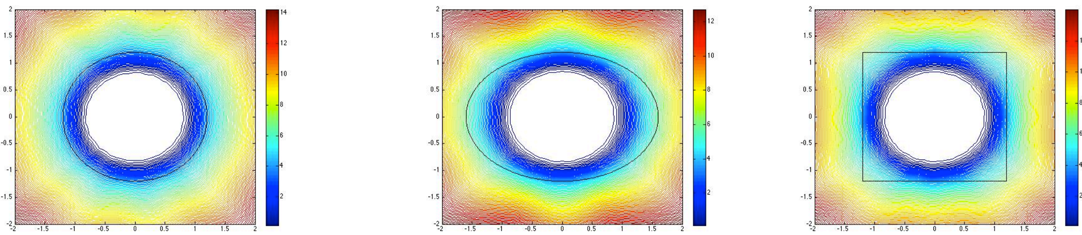
Figure 4.3: Panels (a), (b) and (c) show the reconstruction of a circle with radius 1.2, of an ellipse with x-axis 3.2 and y-axis 2.4 and of a square with length 2.4, respectively, with noise free data. The wavelength is $2\pi/5$ and ∂C is a circle of radius 0.8. Here $A = [1.2 \quad 0; 0 \quad 1.5]$, $n = 0.8$ and the true geometry of the cavity is indicated by the solid line.



(a) **(b)** **(c)**

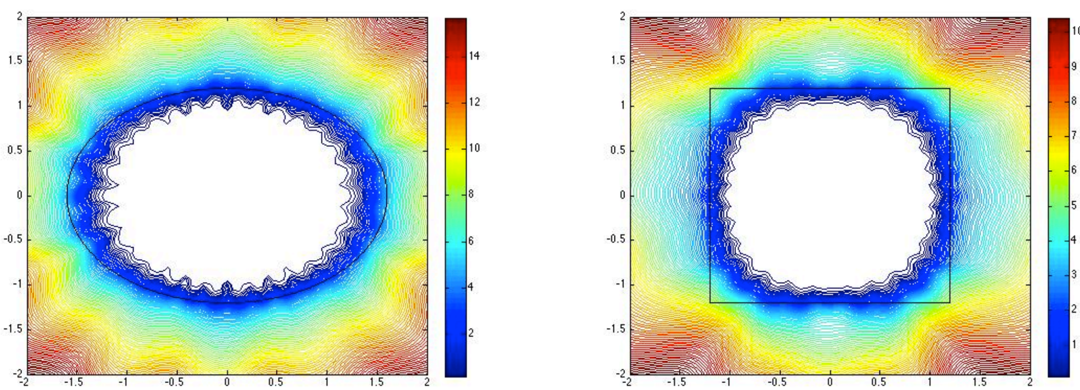
Figure 4.4: Panels (a), (b) and (c) show the reconstruction of a circle with radius 1.2, of an ellipse with x-axis 3.2 and y-axis 2.4 and of a square with length 2.4, respectively, with 1% noise. The wavelength is $2\pi/5$ and ∂C is a circle of radius 0.8. Here $A = [1.2 \quad 0; 0 \quad 1.5]$, $n = 0.8$ and the true geometry is indicated by the solid line. The sampling points z are in $[-2, 2]^2$.

of sources and receivers remains the same. We also consider the anisotropic media with matrix A satisfying (loosely speaking) $A - I < 0$, namely $A = [0.6, 0; 0, 0.8]$ for the ellipse and square and the reconstructions are presented in Figure 4.6. Finally as explained in Theorem 4.3.8 it is possible to avoid the (real) exterior transmission eigenvalues (which in particular cases are proven to exists c.f. [26]) if the sampling point z remains only inside D_1, i.e. in the inhomogeneous layer and the cavity. The examples presented in Figure 4.7 with sampling region $D_1 \setminus \overline{C}$ for the ellipse and the



(a) **(b)** **(c)**

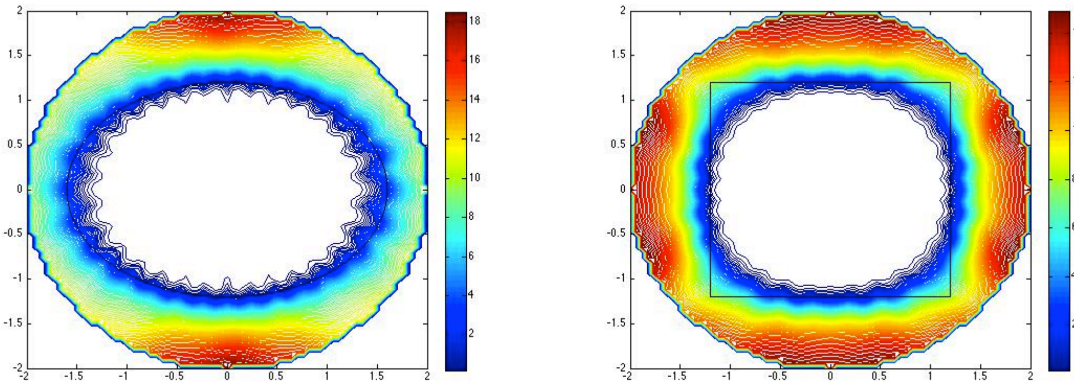
Figure 4.5: Panels (a), (b) and (c) show the reconstruction of a circle with radius 1.2, of an ellipse with x-axis 3.2 and y-axis 2.4 and of a square with length 2.4, respectively, with noise free data. The wavelength is $2\pi/5$ and ∂C is a circle of radius 0.4. Here $A = [1.2 \quad 0; 0 \quad 1.5]$, $n = 0.8$ and the true geometry is indicated by the solid line. The sampling points z are in $[-2, 2]^2$.



(a) **(b)**

Figure 4.6: Panels (a) and (b) show the reconstruction of an ellipse with x-axis 3.2 and y-axis 2.4 and of a square with length 2.4, respectively, with noise free data. The wavelength is $2\pi/5$ and ∂C is a circle of radius 0.8. Here $A = [0.6, 0; 0, 0.8]$, $n = 0.8$ and the true geometry of the cavity is indicated by the solid line. The sampling points z are in $[-2, 2]^2$.

square confirm that this confinement of sampling region does not affect the quality of reconstructions.



(a) **(b)**

Figure 4.7: Panels (a) and (b) show the reconstruction of an ellipse with x-axis 3.2 and y-axis 2.4 and of a square with length 2.4, respectively, with noise free data. The wavelength is $2\pi/5$ and ∂C is a circle of radius 0.8. Here $A = [1.2 \quad 0; 0 \quad 1.5]$, $n = 0.8$ and the true geometry of the cavity is indicated by the solid line. The sampling points z are in $D_1 \setminus \overline{C}$.

Remark 4.3.2 (non-physical incident sources) *Our justification of the factorization method works for incident waves being complex conjugate of point sources, which are non-physical. However, it is well known that these non-physical sources can be approximated arbitrarily close by linear combination of physical point sources (these fact is also discussed in [51]).*

Remark 4.3.3 *It is interesting that for inverse scattering in bounded domain the factorization method can be justified for physical incident waves. Our analysis can be carried through for the problem when D_1 is contained in a large ball B_R with homogeneous medium in $B_R \setminus \overline{D_1}$ and zero Dirichlet or Neumann conditions on ∂B_R. In particular if the cavity is embedded in a perfect conductor or sound-soft object, we could use physical source $\Phi_0(\cdot, y)$ instead of the artificial source $\overline{\Phi(\cdot, y)}$ where*

$$\begin{aligned} \Delta\Phi_0(\cdot, y) + k^2\Phi_0(\cdot, y) &= -\delta(, \cdot, y) \qquad (4.43) \\ \Phi_0(\cdot, y) &= 0 \quad on \quad \partial B_R \end{aligned}$$

then we exclude the Dirichlet to Neumann mapping in the analysis, everything esle for the artificial source works exactly in the same way.

Chapter 5

THE TRANSMISSION EIGENVALUE PROBLEM FOR SPHERICALLY STRATIFIED MEDIUM

Of particular interest in the investigation of transmission eigenvalue problem is the inverse spectral problem for transmission eigenvalues which was originally studied by McLaughlin and Polyakov [66] and more recently by Aktosun, Gintides and Papanicolaou [2], Aktosun and Papanicolaou [3], Colton and Leung [25], Wei and Xu [83] and many others. This interior transmission eigenvalue problem is characterized by its formulation as two elliptic equations defined in a bounded domain which have the same Cauchy data on the boundary. In Section 5.1 we study the inverse spectral problem for the *exterior* transmission eigenvalues and in Section 5.2 we study the distribution of the *interior* transmission eigenvalues.

5.1 The Exterior Transmission Eigenvalue Problem

More recently a complementary class of transmission eigenvalue problems has appeared in inverse scattering theory which is characterized by the problem of finding a nontrivial solution of two elliptic equations in an *unbounded* domain that have the same Cauchy data on the boundary and both of which satisfy the Sommerfeld radiation condition at infinity refered as the *exterior* transmission eigenvalue problem. More specifically, we are concerned here with the inverse spectral problem for a special case of such problem in which the index of refraction is spherically stratified and the resulting spectral problem can be reduced to a spectral problem for a coupled set of ordinary differential equations.

As in the case for the interior transmission eigenvalue problem studied in [25], our approach for the exterior problem is based on the use of transformation operators and special results in the theory of entire functions of a complex variable. However in the case of the exterior problem special difficulties arise due to the fact that the fundamental determinant of the exterior problem is no longer an even entire function that is real on the real axis. In addition, it is no long possible to choose special values of the spectral parameter in order to simplify the fundamental determinant. As a consequence we now need to use two sets of spectral data in order to uniquely determine the spherically stratified index of refraction $n(r)$. We first show that for constant $n(r)$ all eigenvalues are real. (We also later give an example to show that when $n(r)$ is allowed to be piecewise constant the results are drastically different.) In contrast to this simple result, for non constant $n(r)$ we show that there exist cases in which there are an infinite number of complex eigenvalues and at most a finite number of real eigenvalues. Having examined the existence and distribution of transmission eigenvalues we then turn our attention to the inverse spectral problem and give conditions under which two sets of spectral data uniquely determine $n(r)$. This result is based on Hadamard's factorization theorem together with a theorem of Rundell and Sacks which show that a coefficient in a certain class of hyperbolic equations is uniquely determined by an appropriate set of overdetermined initial data.

5.1.1 Existence of Exterior Transmission Eigenvalues

In this section we are concerned with the existence of exterior transmission eigenvalue for a special case of such problems in which the index of refraction is spherically stratified and the resulting spectral problem can be reduced to a spectral problem for a coupled set of ordinary differential equations.

Here we consider the exterior transmission eigenvalue problem for isotropic spherically stratified medium with strong solutions in $\mathbb{R}^3$, i.e., finding functions $u, v \in$

$C^2(\mathbb{R}^3\backslash\overline{B}) \cap C^2(\mathbb{R}^3\backslash B)$ such that

$$\Delta u + k^2 n(r) u = 0 \qquad \text{in} \quad \mathbb{R}^3\backslash\overline{B} \tag{5.1}$$

$$\Delta v + k^2 v = 0 \qquad \text{in} \quad \mathbb{R}^3\backslash\overline{B} \tag{5.2}$$

$$u = v \qquad \text{on} \quad \partial B \tag{5.3}$$

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} \qquad \text{on} \quad \partial B \tag{5.4}$$

$$\lim_{r\to+\infty} r\left(\frac{\partial u}{\partial r} - iku\right) = 0 \tag{5.5}$$

$$\lim_{r\to+\infty} r\left(\frac{\partial v}{\partial r} - ikv\right) = 0 \tag{5.6}$$

where $r := |x|$, $x \in \mathbb{R}^3$, $B := \{x : |x| < a\}$, $n \in C[a,b]$, $n(r) = 1$ for $r > b > a$ and the radiation conditions in (5.5) and (5.6) are assumed to hold uniformly with respect to the angular variable. Values of k such that there exists a nontrivial solution to (5.1)-(5.6) are called *exterior transmission eigenvalues* with corresponding eigenfunctions u and v. We are interested in the special case when the eigenfunctions are spherically stratified and set $u(r) = a_0\frac{y(r)}{r}$, $v(r) = b_0\frac{y_0(r)}{r}$. In this case (5.1)-(5.6) become

$$y'' + k^2 n(r) y = 0 \qquad \text{in} \quad [a,\infty) \tag{5.7}$$

$$y_0'' + k^2 y_0 = 0 \qquad \text{in} \quad [a,\infty) \tag{5.8}$$

$$a_0 y(a) = b_0 y_0(a) \tag{5.9}$$

$$a_0 y'(a) = b_0 y_0'(a) \tag{5.10}$$

and y and y_0 are normalized such that

$$y(r) = y_0(r) = e^{ikr}, \quad r > b. \tag{5.11}$$

Then k is a transmission eigenvalue if and only if

$$D_1(k) := \det\begin{vmatrix} y(a) & e^{ika} \\ y'(a) & ike^{ika} \end{vmatrix} = 0. \tag{5.12}$$

From now on we make the stronger assumption on $n(r)$ that $n(r) \in C^3[a,b]$. In addition we always assume that $n(r)$ is not identically equal to 1.

We begin with the simple case when $n(r)$ is a constant, in particular $n(r) = n_0^2$ for $a \le r \le b$. Then

$$y(r) = c_1 e^{ikn_0 r} + c_2 e^{-ikn_0 r}, \quad a \le r \le b$$

and requiring $y(r)$ to be continuously differentiable across $r = b$ gives

$$c_1 = \frac{n_0 + 1}{2n_0} e^{ikb(1-n_0)}, \; c_2 = \frac{n_0 - 1}{2n_0} e^{ikb(1+n_0)}.$$

We now have that

$$D_1(k) = \frac{k e^{ik(a+b)}(1 - n_0^2)}{n_0} \sin\left(kn_0(b-a)\right)$$

and hence $D_1(k) = 0$ if and only if

$$k = \frac{m\pi}{n_0(b-a)}$$

for m an integer. In particular all eigenvalues are real. In addition, n_0 is uniquely determined by the first transmission eigenvalue.

We now turn our attention to the case when $n(r)$ is no longer a constant. In this case we will make use of transformation operators (c.f. [24, 64]) to represent $y(r)$ in terms of solutions to $y_0'' + k^2 y_0 = 0$. In particular, let

$$\xi := \int_a^r \sqrt{\eta(t)}dt \quad \text{and} \quad p(\xi) := \frac{\eta''(r)}{4\eta(r)^2} - \frac{5}{16}\frac{\eta'(r)^2}{\eta(r)^3}.$$

Then $w_1(\xi) := \eta(r)^{1/4} y(r)$ is a solution to

$$w_1'' + [k^2 - p(\xi)]w_1 = 0, \qquad 0 \le \xi \le \gamma \tag{5.13}$$

$$w_1(0) = 1, \quad w_1'(0) = 0 \tag{5.14}$$

and can be represented in the form

$$w_1(\xi) = \cos(k\xi) + \int_0^\xi K_1(\xi, t)\cos(kt)dt \tag{5.15}$$

where

$$\frac{\partial^2 K_1}{\partial \xi^2} - \frac{\partial^2 K_1}{\partial t^2} - p(\xi)K_1 = 0, \qquad 0 < t < \xi < \gamma \tag{5.16}$$

$$\frac{\partial K_1}{\partial t}(\xi, 0) = 0, \qquad 0 \le \xi \le \gamma \tag{5.17}$$

$$K_1(\xi, \xi) = \frac{1}{2}\int_0^\xi p(s)ds, \qquad 0 \le \xi \le \gamma. \tag{5.18}$$

If $w_2 \in C^2[0,\gamma]$ satisfies

$$w_2'' + [k^2 - p(\xi)]w_2 = 0, \qquad 0 \le \xi \le \gamma \tag{5.19}$$

$$w_2(0) = 0, \quad w_2'(0) = 1 \tag{5.20}$$

then $w_2(\xi)$ can be represented in the form

$$w_2(\xi) = \frac{\sin(k\xi)}{k} + \int_0^\xi K_2(\xi,t)\frac{\sin(kt)}{k}dt \tag{5.21}$$

where

$$\frac{\partial^2 K_2}{\partial \xi^2} - \frac{\partial^2 K_2}{\partial t^2} - p(\xi)K_2 = 0, \qquad 0 < t < \xi < \gamma \tag{5.22}$$

$$K_2(\xi,0) = 0, \qquad 0 \le \xi \le \gamma \tag{5.23}$$

$$K_2(\xi,\xi) = \frac{1}{2}\int_0^\xi p(s)ds, \qquad 0 \le \xi \le \gamma. \tag{5.24}$$

In particular, the general solution of $w'' + (k^2 - p(\xi))w = 0$ can be represented in the form $w(\xi) = w(0)w_1(\xi) + w'(0)w_2(\xi)$ with w_1 and w_2 represented in the form (5.15) and (5.21) respectively.

We now look for a solution of $y'' + k^2 n(r)y = 0$ such that $y(r)$ is continuously differentiable across $r = b$ and $y(r) = e^{ikr}$ for $r > b$. If we translate to the origin and use the Liouville transformation

$$\begin{aligned} \xi(r) &:= \int_0^r [\eta(\rho)]^{\frac{1}{2}} d\rho \\ w(\xi) &:= [\eta(r)]^{\frac{1}{4}} y(b-r)e^{-ikb} \end{aligned} \tag{5.25}$$

where $\eta(r) := n(b-r)$, we can represent $y(r)$ in terms of $\cos(k\xi)$ and $\frac{1}{k}\sin(k\xi)$ by using the above transformation operators where in this case $\gamma := \xi(d)$ and $d := b - a$. In fact, from (5.7)-(5.11) we have that

$$\begin{aligned} y'' + k^2 ny &= 0, \qquad a \le r \le b \\ y(b) &= e^{ikb} \\ y'(b) &= ike^{ikb}. \end{aligned}$$

From above and (5.25),

$$w'' + [k^2 - q(\xi)]w = 0, \quad 0 \le \xi \le \lambda$$

where

$$p(\xi) := \frac{\eta''(r)}{4\eta(r)^2} - \frac{5}{16}\frac{\eta'(r)^2}{\eta(r)^3}.$$

Representing w in terms of w_1, w_2 yields

$$w(\xi) = w(0)w_1 + w'(0)w_2.$$

From (5.25) we have that

$$\begin{aligned} w(0) &= [\eta(0)]^{-\frac{1}{4}} w(0), \\ w'(0) &= -ik[\eta(0)]^{-\frac{1}{4}} + \frac{1}{4}[\eta(0)]^{-\frac{5}{4}}\eta'(0). \end{aligned}$$

Evaluating $w(\xi)$ and $w'(\xi)$ at $\xi = \lambda$ yields

$$\begin{aligned} w(\lambda) &= [\eta(0)]^{-\frac{1}{4}}[\cos(k\lambda) + \int_0^\lambda K_1(\lambda, t)\cos(kt)dt] \\ &+ [-ik[\eta(0)]^{-\frac{1}{4}} + \frac{1}{4}[\eta(0)]^{-\frac{5}{4}}\eta'(0)][\frac{\sin(k\lambda)}{k} \\ &+ \int_0^\lambda K_2(\lambda, t)\frac{\sin(kt)}{k}dt], \end{aligned} \tag{5.26}$$

similiarly

$$\begin{aligned} w'(\lambda) &= [\eta(0)]^{-\frac{1}{4}}[-k\sin(k\lambda) + K_1(\lambda, \lambda)\cos(k\lambda) + \int_0^\lambda K_{1,\xi}(\lambda, t)\cos(kt)dt] \\ &+ [-ik[\eta(0)]^{-\frac{1}{4}} + \frac{1}{4}[\eta(0)]^{-\frac{5}{4}}\eta'(0)][\cos(k\lambda) + K_2(\lambda, \lambda)\frac{\sin(k\lambda)}{k} \\ &+ \int_0^\lambda K_{2,\xi}(\lambda, t)\frac{\sin(kt)}{k}dt]. \end{aligned} \tag{5.27}$$

From (5.25) we have that

$$\begin{aligned} y(a) &= e^{ikb}[\eta(d)]^{-\frac{1}{4}} w(\lambda) \\ y'(a) &= -e^{ikb}[-\frac{1}{4}[\eta(d)]^{-\frac{1}{4}}\eta'(d)w(\lambda) + [\eta(d)]^{\frac{1}{4}} w'(\lambda)]. \end{aligned}$$

Pluging into (5.12) yields

$$D_1(k) = e^{ik(a+b)}\{ik[\eta(d)]^{-\frac{1}{4}}w(\lambda) - \frac{1}{4}[\eta(d)]^{-\frac{1}{4}}w(\lambda)]\eta'(d)w(\lambda) + [\eta(d)]^{\frac{1}{4}}w'(\lambda)\}.$$

From (5.26) and (5.27)

$$\begin{aligned}
e^{-ik(a+b)}D_1(k) &= [\eta(d)]^{-\frac{1}{4}}[\eta(0)]^{\frac{1}{4}}[\cos(k\lambda) + \int_0^\lambda K_1(\lambda,t)\cos(kt)dt] \\
&+ ik[\eta(d)]^{-\frac{1}{4}}[-ik[\eta(0)]^{-\frac{1}{4}} + \frac{1}{4}[\eta(0)]^{-\frac{5}{4}}\eta'(0)] \cdot \\
&\quad [\frac{\sin(k\lambda)}{k} + \int_0^\lambda K_2(\lambda,t)\frac{\sin(kt)}{k}dt] \\
&- \frac{1}{4}[\eta(d)]^{-\frac{1}{4}}\eta'(d)[\eta(0)]^{-\frac{1}{4}}[\cos(k\lambda) + \int_0^\lambda K_1(\lambda,t)\cos(kt)dt] \\
&- \frac{1}{4}[\eta(d)]^{-\frac{1}{4}}\eta'(d)[-ik[\eta(0)]^{-\frac{1}{4}} + \frac{1}{4}[\eta(0)]^{-\frac{5}{4}}\eta'(0)] \cdot \\
&\quad [\frac{\sin(k\lambda)}{k} + \int_0^\lambda K_2(\lambda,t)\frac{\sin(kt)}{k}dt] \\
&+ [\eta(d)]^{\frac{1}{4}}[\eta(0)]^{\frac{1}{4}}[-k\sin(k\lambda) + K_1(\lambda,\lambda)\cos(k\lambda) + \int_0^\lambda K_{1,\xi}(\lambda,t)\cos(kt)dt] \\
&+ [\eta(d)]^{\frac{1}{4}}[-ik[\eta(0)]^{-\frac{1}{4}} + \frac{1}{4}[\eta(0)]^{-\frac{5}{4}}\eta'(0)] \cdot \\
&\quad [\cos(k\lambda) + k_2(\lambda,\lambda)\frac{\sin(k\lambda)}{k} + \int_0^\lambda K_{2,\xi}(\lambda,t)\frac{\sin(kt)}{k}dt].
\end{aligned}$$

Integrating by parts yields

$$\begin{aligned}
\int_0^\lambda K_1(\lambda,t)\cos(kt)dt &= \frac{\sin(k\lambda)}{k}k_1(\lambda,\lambda) - \frac{1}{k}\int_0^\lambda K_{1,t}(\lambda,t)\sin(kt)dt \\
\int_0^\lambda K_2(\lambda,t)\sin(kt)dt &= -\frac{\cos(k\lambda)}{k}k_2(\lambda,\lambda) + \frac{1}{k}\int_0^\lambda K_{2,t}(\lambda,t)\cos(kt)dt \\
\int_0^\lambda K_{1,\xi}(\lambda,t)\cos(kt)dt &= \frac{\sin(kt)}{k}k_{1,\xi}(\lambda,t)|_0^\lambda - \frac{1}{k}\int_0^\lambda K_{1,\xi t}(\lambda,t)\sin(kt)dt = O(\frac{e^{\gamma|\Im(k)|}}{k}) \\
\int_0^\lambda K_{2,\xi}(\lambda,t)\sin(kt)dt &= -\frac{\cos(kt)}{k}k_{2,\xi}(\lambda,t)|_0^\lambda + \frac{1}{k}\int_0^\lambda K_{2,\xi t}(\lambda,t)\cos(kt)dt = O(\frac{e^{\gamma|\Im(k)|}}{k}).
\end{aligned}$$

From above equations,

$$\begin{aligned}
e^{-ik(a+b)}D_1(k) &= ik\{[\eta(d)]^{-\frac{1}{4}}[\eta(0)]^{\frac{1}{4}}\cos(k\lambda) - i[\eta(d)]^{-\frac{1}{4}}[\eta(0)]^{-\frac{1}{4}}\sin(k\lambda) \\
&+ i[\eta(d)]^{\frac{1}{4}}[\eta(0)]^{\frac{1}{4}}\sin(k\lambda) - [\eta(d)]^{\frac{1}{4}}[\eta(0)]^{-\frac{1}{4}}\cos(k\lambda) \\
&+ O(\frac{e^{\gamma|\Im(k)|}}{k})\}.
\end{aligned}$$

From $\eta(r) := n(b-r)$, we have the following asymptotic expansion for $D_1(k)$:

$$D_1(k) = ike^{ik(a+b)}\{\frac{c_1+c_2}{2}e^{ik\gamma} + \frac{c_1-c_2}{2}e^{-ik\gamma} + O(\frac{e^{\gamma|\Im(k)|}}{k})\} \tag{5.28}$$

where

$$c_1 : = (n(a))^{-\frac{1}{4}}(n(b))^{\frac{1}{4}} - (n(a))^{\frac{1}{4}}(n(b))^{-\frac{1}{4}} \tag{5.29}$$

$$c_2 : = (n(a))^{\frac{1}{4}}(n(b))^{\frac{1}{4}} - (n(a))^{-\frac{1}{4}}(n(b))^{-\frac{1}{4}}. \tag{5.30}$$

We define $E(k)$ as

$$E(k) := -ie^{-ik(a+b)}D_1(k).$$

Then $E(k)$ and $D_1(k)$ have the same roots and hence it suffices to only consider the function $E(k)$. It is easily seen that $E(k)$ is an entire function of k of exponential type γ. We are now in a position to prove the following theorem.

Theorem 5.1.1 *1. Assume that $n(a) \neq 1, n(b) \neq 1$ and that either $n(a) = n(b)$ or $n(a)n(b) = 1$. Then there exist infinitely many real transmission eigenvalues.*

2. Assume that $n(a) \neq 1, n(b) \neq 1, n(a) \neq n(b)$ and $n(a)n(b) \neq 1$. Then there exist infinitely many complex transmission eigenvalues which all lie in a strip in the complex plane parallel to the real axis and at most a finite number of real transmission eigenvalues.

Proof. 1. If $n(a) = n(b)$ or $n(a)n(b) = 1$, then either $c_1 = 0$ or $c_2 = 0$ (they cannot be zero at the same time since $n(a)$ and $n(b)$ are not 1). Without loss of generality we assume that $c_1 = 0$ and $c_2 \neq 0$. Then

$$\frac{E(k)}{k} = c_2\frac{e^{ik\gamma} - e^{-ik\gamma}}{2} + O(\frac{e^{\gamma|\Im(k)|}}{k})$$

i.e.

$$\frac{E(k)}{ki} = c_2 \sin(k\gamma) + O(\frac{e^{\gamma|\Im(k)|}}{k}). \tag{5.31}$$

This implies that there exists an infinite number of real zeros of $E(k)$, i.e. an infinite number of real transmission eigenvalues. In particular γ can be determined by the limiting spacing between two consecutive real eigenvalues.

2. If $n(a) \neq n(b)$ and $n(a)n(b) \neq 1$, then $c_1 \neq 0, c_2 \neq 0$ and since $n(a) \neq 1$ and $n(b) \neq 1$, $c_1 + c_2 \neq 0$ and $c_1 - c_2 \neq 0$. Then from (5.28)-(5.30) it follows that there exists at most a finite number of real transmission eigenvalues (i.e. real values of k such that $E(k) = 0$). By Hadamard's factorization theorem and (5.28)-(5.30) again, it follows that there must exist an infinite number of complex eigenvalues. To show that all the complex eigenvalues lie in a strip in the complex plane parallel to the real axis, let

$$T(k) := \frac{c_1 + c_2}{2} e^{ik\gamma} + \frac{c_1 - c_2}{2} e^{-ik\gamma}.$$

Then

$$e^{-\gamma|\Im(k)|}(\frac{E(k)}{k} - T(k)) \to 0$$

as $|\Im(k)| \to \infty$. If $E(k_j) = 0$ and $|\Im(k_j)| \to \infty$, then $e^{-\gamma|\Im(k_j)|}T(k_j) \to 0$ and this is a contradiction since

$$e^{-\lambda|\Im(k_j)|}T(k_j) \to \min\{\frac{|c_1 + c_2|}{2}, \frac{|c_1 - c_2|}{2}\} > 0.$$

Hence all the complex transmission eigenvalues must lie in a strip in the complex plane. □

In the case when $n(a) = n(b) = 1$ we have that $c_1 - c_2 = 0$ and $c_1 + c_2 = 0$. However in this case direct computation shows that

$$D_1(k) = e^{ik(a+b)}\{\frac{d_1 + d_2}{2} e^{ik\gamma} + \frac{d_1 - d_2}{2} e^{-ik\gamma} + O(\frac{e^{\gamma|\Im(k)|}}{k})\}$$

where

$$d_1 := -\frac{n'(b) - n'(a)}{4}, \; d_2 := -\frac{n'(a) + n'(b)}{4}.$$

Then using the same arguments as above, we have the following theorem:

Theorem 5.1.2 *Assume that $n(a) = n(b) = 1$. Then if either $n'(a) = n'(b) \neq 0$ or $n'(a) = -n'(b) \neq 0$ there exist infinitely many real transmission eigenvalues. On the other hand, if $n'(a) \neq n'(b)$ and $n'(a) \neq -n'(b)$, then there exist an infinite number of complex transmission eigenvalues which all lie in a strip in the complex plane parallel to the real axis and at most a finite number of real transmission eigenvalues.*

5.1.2 The Inverse Spectral Exterior Transmission Eigenvalue Problem

We now turn our attention to the inverse spectral problem and give conditions under which two sets of spectral data uniquely determine $n(r)$. This result is based on Hadamard's factorization theorem together with a theorem of Rundell and Sacks which show that a coefficient in a certain class of hyperbolic equations is uniquely determined by an appropriate set of overdetermined initial data. We consider the reduced exterior transmission eigenvalue problem (5.7)-(5.10) where $y(r)$ is continuously differentiable across $r = b$ and $y(r), y_0(r)$ are normalized such that $y(r) = e^{ikr}$ for $r > b$ and $y_0(r) = e^{-ikr}$ for $r > b$. In particular

$$y'' + k^2 n(r) y = 0 \qquad \text{in} \quad [a, \infty) \tag{5.32}$$

$$y_0'' + k^2 y_0 = 0 \qquad \text{in} \quad [a, \infty) \tag{5.33}$$

$$a_0 y(a) = b_0 y_0(a) \tag{5.34}$$

$$a_0 y'(a) = b_0 y_0'(a) \tag{5.35}$$

$$y(r) = e^{ikr}, \quad y_0(r) = e^{-ikr} \qquad \text{in} \quad [b, \infty] \tag{5.36}$$

The problem corresponds to the incident field being the "nonphysical" source $\frac{e^{-ikr}}{r}$ which radiates inwards instead of outwards (such sources can be approximated arbitrarily closely by a finite number of "physical" sources c.f. [51]). Our aim in this section is to show that under appropriate assumptions a knowledge of the spectrum for both (5.7)-(5.10) and (5.32)-(5.36) is sufficient to determine $n(r)$. An investigation of the location of the eigenvalues of (5.32)-(5.36) would be of interest but will not be done here.

We define $D_2(k)$ to be the determinant

$$D_2(k) := \det \begin{vmatrix} y(a) & e^{-ika} \\ y'(a) & -ike^{-ika} \end{vmatrix} \tag{5.37}$$

and note that k is a transmission eigenvalue for (5.32)-(5.36) if and only if $D_2(k) = 0$. Under the assumption that $n(a) = n(b), n(a) \neq 1$ and $n'(a) = n'(b) = 0$ and representing $y(r)$ in terms of transformation operators we have that

$$\begin{aligned} e^{-ik(a+b)}D_1(k) &= k\left(n^{-\frac{1}{2}}(a) - n^{\frac{1}{2}}(a)\right)\sin(k\gamma) \\ &+ \frac{\cos(k\gamma)}{2}\left(n^{-\frac{1}{2}}(a) - n^{\frac{1}{2}}(a)\right)\int_0^\gamma p(s)ds \\ &- i\int_0^\gamma \sin(kt)f_1(t)dt + \int_0^\gamma \cos(kt)g_1(t)dt \end{aligned} \tag{5.38}$$

and

$$\begin{aligned} e^{ik(a-b)}D_2(k) &= -2ik\cos(k\gamma) - k\left(n^{-\frac{1}{2}}(a) - n^{\frac{1}{2}}(a)\right)\sin(k\gamma) \\ &- \frac{i\sin(k\gamma)}{2}\int_0^\gamma p(s)ds + \frac{\cos(k\gamma)}{2}\left(n^{-\frac{1}{2}}(a) - n^{\frac{1}{2}}(a)\right)\int_0^\gamma p(s)ds \\ &+ i\int_0^\gamma \sin(kt)f_2(t)dt + \int_0^\gamma \cos(kt)g_2(t)dt \end{aligned} \tag{5.39}$$

where

$$\begin{aligned} f_1(t) &:= \frac{\partial}{\partial t}K_1(\gamma,t) + \frac{\partial}{\partial \xi}K_2(\gamma,t) \\ g_1(t) &:= n^{-\frac{1}{2}}(a)\frac{\partial}{\partial t}K_2(\gamma,t) + n^{\frac{1}{2}}(a)\frac{\partial}{\partial \xi}K_1(\gamma,t) \\ f_2(t) &:= \frac{\partial}{\partial t}K_1(\gamma,t) - \frac{\partial}{\partial \xi}K_2(\gamma,t) \\ g_2(t) &:= -n^{-\frac{1}{2}}(a)\frac{\partial}{\partial t}K_2(\gamma,t) + n^{\frac{1}{2}}(a)\frac{\partial}{\partial \xi}K_1(\gamma,t). \end{aligned} \tag{5.40}$$

We are now in a position to prove the main result of this section.

Theorem 5.1.3 *Assume that $n(a)$ is known, $n(a) = n(b)$, $n(a) \neq 1$, $n'(a) = n'(b) = 0$ and $n(r) \in C^3[a,b]$. Then $n(r)$ is uniquely determined from a knowledge of the transmission eigenvalues (including multiplicities) for (5.7)-(5.10) and (5.32)-(5.36).*

Proof. From (5.31) we have that γ is uniquely determined. Since $D_1(k)$ is an entire function of exponential type we have from Hadamard's factorization theorem that

$$e^{-ik(a+b)}D_1(k) = ck^m e^{\alpha k}\Pi_{k=1}^{\infty}(1 - \frac{k}{k_n})e^{\frac{k}{k_n}} \tag{5.41}$$

for constants c, α and some integer m where $\{k_n\}_{n=1}^{\infty}$ are the transmission eigenvalues for (5.7)-(5.10). Since the transmission eigenvalues are assumed to be known, we know

$$G(k) := \Pi_{k=1}^{\infty}(1 - \frac{k}{k_n})e^{\frac{k}{k_n}} \quad . \tag{5.42}$$

Hence from (5.31) and (5.41) we can determine α by taking logarithms and letting k tend to infinity. Setting $T_1(k) := e^{-ik(a+b)}D_1(k)$ we have from (5.38) that

$$\begin{aligned} T_1(k) &= ck^m e^{\alpha k} G(k) \qquad (5.43)\\ &= k\left(n^{-\frac{1}{2}}(a) - n^{\frac{1}{2}}(a)\right)\sin(k\gamma) \\ &+ \frac{\cos(k\gamma)}{2}\left(n^{-\frac{1}{2}}(a) - n^{\frac{1}{2}}(a)\right)\int_0^{\gamma} p(s)ds + O(\frac{e^{\gamma|\Im(k)|}}{k}) \end{aligned}$$

and hence for ℓ an integer we can compute

$$\frac{T_1^2(\frac{4\ell+\frac{1}{2}}{\gamma}\pi)}{T_1(\frac{2\ell+\frac{1}{2}}{\gamma}\pi)T_1(\frac{6\ell+\frac{1}{2}}{\gamma}\pi)} = [\frac{(4\ell+\frac{1}{2})^2}{(2\ell+\frac{1}{2})(6\ell+\frac{1}{2})}]^m \frac{G^2(\frac{4\ell+\frac{1}{2}}{\gamma}\pi)}{G(\frac{2\ell+\frac{1}{2}}{\gamma}\pi)G(\frac{6\ell+\frac{1}{2}}{\gamma}\pi)} \tag{5.44}$$

as $\ell \to \infty$ noting that by (5.43) the denominator in (5.44) is nonzero for ℓ sufficiently large. From (5.43) we see that the left hand side (5.44) tends to $\frac{4}{3}$ and hence the integer m can be determined from (5.44). Finally, the constant c in (5.41) can be uniquely determined from (5.31) and (5.41) if $n(a)$ is known, i.e. in this case $D_1(k)$ is uniquely determined from the set $\{k_n\}_{n=1}^{\infty}$ of the transmission eigenvalues for (5.7)-(5.10). In exactly the same way we can determine $D_2(k)$ from a knowledge of the transmission eigenvalues for (5.32)-(5.36).

We can now conclude from (5.38) and (5.39) that

$$-i\int_0^{\gamma} \sin(kt)f_1(t)dt + \int_0^{\gamma}\cos(kt)g_1(t)dt \tag{5.45}$$

$$i\int_0^{\gamma} \sin(kt)f_2(t)dt + \int_0^{\gamma}\cos(kt)g_2(t)dt \tag{5.46}$$

are both uniquely determined (noting that these terms are $O(\frac{1}{k})$). Now note that $f_1(t), f_2(t), g_1(t), g_2(t)$ are all real valued. Hence, setting $k = \frac{\ell\pi}{\gamma}$ in (5.45) and (5.46) and noting that $\{\sin(\frac{\ell\pi t}{\gamma})\}_{\ell=1}^{\infty}$ and $\{\cos(\frac{\ell\pi t}{\gamma})\}_{\ell=0}^{\infty}$ are basis for $L^2[0,\gamma]$, we see that

$f_1(t), f_2(t), g_1(t), g_2(t)$ are uniquely determined. From (5.40) we now have that $\frac{\partial}{\partial \xi}K_1(\gamma, t)$ and $\frac{\partial}{\partial \xi}K_2(\gamma, t)$ are uniquely determined. Hence, following Rundell and Sacks [75], we can conclude that $p \in C^1[0,\gamma]$ is uniquely determined and from this it is easily seen that $n(r)$ is uniquely determined (c.f [25] or section 9.4 of [9]). $\square$

5.2 Distribution of Interior Transmission Eigenvalues

We consider the case of a spherically stratified medium with (normalized) support $\{x : |x| \leq 1\ \}$ and spherically symmetric eigenfunctions, i.e. the eigenvalue problem

$$w'' + \frac{2}{r}w' + k^2 n(r)w = 0, \qquad v'' + \frac{2}{r}v' + k^2 v = 0, \quad 0 < r < 1$$
$$w(1) = v(1), \qquad w'(1) = v'(1)$$

where $n(r) > 0$ and both $w(0)$ and $v(0)$ must be finite. Setting $y(r) = rw(r),\ y_0(r) = rv(r)$, then

$$y'' + k^2 n(r)y = 0, \qquad 0 < r < 1 \tag{5.47}$$

$$y_0'' + k^2 y_0 = 0, \qquad 0 < r < 1 \tag{5.48}$$

$$y(0) = y_0(0) = 0, \quad y(1) = y_0(1), \qquad y'(1) = y_0'(1). \tag{5.49}$$

The eigenvalue problem (6.42)-(5.49) is called the *interior transmission eigenvalue problem* for a spherically stratified medium and values of k for which a nontrivial solution of (6.42)-(5.49) exist are called *interior transmission eigenvalues*. As shown in [29], and subsequently in many papers and books (c.f. [9, 24, 66]), the eigenvalues are the zeros of the entire function

$$d(k) := y(1)\cos(k) - y'(1)\sin(k)/k. \tag{5.50}$$

The function $d(k)$ is entire as a function of k and goes to zero in the order of $O(1/k)$ as k goes to infinity along the real line [9].

Let $\delta := \int_0^1 \sqrt{n(t)}\,dt$. It was shown in [9] (see also [24]) that an infinite number of *real* transmission eigenvalues exist under the assumptions that $n(1) \neq 1$ and $\delta \neq 1$.

There was the question whether complex eigenvalues could exist. It was shown that the function $d(k)$ has the asymptotic expansion

$$d(k) = \frac{1}{k}(B\sin(k\delta)\cos(k) - C\cos(k\delta)\sin(k)) + O(1/k^2)$$

for k going to infinity along the real axis where

$$B := \frac{1}{(n(0)n(1))^{1/4}}, \quad C := \left(\frac{n(1)}{n(0)}\right)^{1/4} \quad \text{and} \quad \delta = \int_0^1 \sqrt{n(t)}dt. \tag{5.51}$$

Using this expansion plus the assumptions that both $\delta \neq 1$ and $n(1) \neq 1$, it was shown in [25] that infinitely many complex transmission eigenvalues in fact exist and they lie in a strip parallel to the real axis. Lastly, a recent article [80] of Sylvester has a detailed study on the distribution of transmission eigenvalues when $n(r)$ is a constant.

Our main goal here is to investigate the cases when one of the parameters δ or $n(1)$ is 1 with the extra assumption that the refractive index $n \in C^2[0,1]$. In the case when $\delta = 1$ we show that it is possible to have all the eigenvalues being real. If $n(1) = 1$, then in general an infinite number of complex eigenvalues are present. However, in contrast to the case when $n(1) \neq 1$, these eigenvalues no longer lie in a strip parallel to the real axis. We will also provide an example with all the transmission eigenvalues being complex when both parameters δ and $n(1)$ are 1. Finally we will consider the case when the medium is absorbing and show that under appropriate assumption there are an infinite number of eigenvalues that accumulate near the real axis.

We will always assume that $n(r)$ is not identically equal to one.

5.2.1 Non-absorbing Medium

We first recall a classical result due to Levinson .

Definition 5.2.1 *The entire function $f(z)$ is of order ρ if*

$$\overline{\lim_{r\to\infty}} \frac{\log\log M(r)}{\log r} = \rho.$$

Here $M(r)$ denotes the maximum modulus of $f(z)$ on $|z| = r$.

Definition 5.2.2 *The entire function $f(z)$ of positive order ρ is of type τ if*

$$\overline{\lim_{r\to\infty}} \frac{\log M(r)}{r^{\rho}} = \tau.$$

One of the important theorems involving entire functions of exponential type is the Paley-Wiener Theorem [57].

Theorem 5.2.1 *The entire function $f(z)$ is of exponential type $\leq \tau$ and belongs to L^2 on the real axis if and only if*

$$f(z) = \int_{-\tau}^{\tau} \phi(t)\, e^{izt}\, dt \tag{5.52}$$

for some $\phi(t) \in L^2(-\tau, \tau)$.

$f(z)$ is of type τ if $\phi(t)$ does not vanish almost everywhere in a neighborhood of τ (or of $-\tau$) .

We say that an entire function belongs to the Paley Wiener class if it has the representation given in (5.52).

Corollary 5.2.1 *Suppose $f(z)$ and $g(z)$ are in the Paley Wiener class of types τ and σ respectively. If $\sigma < \tau$, then the sum $f(z) + g(z)$ is of type τ.*

To employ the theorem in the next section, we note that a sine transform $\int_0^{\tau} \psi(t)\sin(zt)dt$ can be expressed as $\int_{-\tau}^{\tau} \phi(t)e^{izt}dt$ for some complex valued $\phi(t)$ on $[-\tau, \tau]$ if $\psi(t)$ is extended onto the interval $[-\tau, 0]$ appropriately.

Let $n_+(r)$ denote the number of zeros of an entire function $f(z)$ in the right half plane with $|z| \leq r$. One can also define a corresponding function $n_-(r)$ for the zeros in the left half plane. Our tool of counting the density of the complex zeros of the entire function $d(k)$ is the following extension of a theorem due to Cartwright [57].

Theorem 5.2.2 *Let the entire function $f(z)$ of exponential type be such that*

$$(a) \quad \int_{-\infty}^{\infty} \frac{\log^+ |f(x)|}{1 + x^2} dx < \infty \tag{5.53}$$

and suppose that

$$(b) \quad \overline{\lim_{y\to\pm\infty}} \frac{\log|f(iy)|}{|y|} = \tau. \tag{5.54}$$

Then

$$\lim_{r\to\infty} \frac{n_+(r)}{r} = \frac{\tau}{\pi}.$$

This limit τ/π will be called the *density* of all the zeros of $f(z)$ on the right half plane. To apply these two theorems to count the number of complex zeros of a given entire function, we first establish the following results.

Corollary 5.2.2 *Let $\tau > 0$ be fixed. Suppose a real valued entire function $f(z)$ has the form*

$$f(z) := \sin(\tau z + \alpha) + \int_{-\tau}^{\tau} \phi(t)e^{izt}\, dt$$

with α being a real constant and $\phi(t)$ a possibly complex valued function continuous on $[-\tau, \tau]$. Then the zeros of $f(z)$ have density τ/π on the right half plane.

Proof. The function $|f(x)|$ is bounded on the real axis for x real. Condition (a) in the previous theorem holds trivially. Along the positive imaginary y axis, i.e. $z = iy$ with $y > 0$,

$$f(iy) = e^{\tau y}\left(\frac{e^{i\alpha}e^{-2\tau y} - e^{-i\alpha}}{2i} + \int_{-\tau}^{\tau} \phi(t)e^{y(-\tau-t)}\, dt\right),$$

$$\log|f(iy)| = \tau y + \log|(\frac{e^{i\alpha}e^{-2\tau y} - e^{-i\alpha}}{2i} + \int_{-\tau}^{\tau} \phi(t)e^{y(-\tau-t)}dt)|.$$

Inside the logarithm on the right, the limit is $(ie^{-i\alpha})/2$ as y goes up to infinity, so we get

$$\lim_{y\to\infty} \frac{\log|f(iy)|}{y} = \tau.$$

The proof of the limit along the negative y axis runs similarly. Thus we have verified condition (b) in Theorem 2.5. □

Corollary 5.2.3 *Let $f(z)$ be a real entire function in the Paley Wiener class of type at most τ. Suppose $x^2 f(x) = \sin(\tau x) + O(1/x)$ as x goes to infinity on the real axis. Then $f(z)$ is of type τ.*

Proof. The density of the positive zeros of $f(z)$ is τ/π. So the type of $f(z)$ must be at least τ, so it is equal to τ. $\square$

In the next result, we are setting up conditions to prove the finiteness of the number of complex roots. The assumptions are not the best possible. The number τ below is assumed to be a positive number. The following theorem is a consequence of the Phragmén-Lindelöf maximum principle([7], Theorem 6.2.6).

Theorem 5.2.3 *Let $g(z)$ be a real entire function of exponential type. Suppose*

1. $|g(x)| \leq M \ \ \forall x \in \ (-\infty, \infty)$, *and*
2. $$\lim_{y \to \pm\infty} \frac{\log|g(iy)|}{|y|} \leq \tau.$$

Then $|g(x+iy)| \leq M \cosh(\tau y)$.

For later use, we note that functions of the form $\sin(z)\int_0^\delta \phi(t)\sin(zt)dt$ and $\cos(z)\int_0^\delta \phi(t)\sin(zt)dt$ satisfy the assumptions in the theorem with $\tau = 1+\delta$ when $\phi(t)$ is continuous on $[0,\delta]$.

Corollary 5.2.4 *Let $h(z)$ be a real entire function in the Paley Wiener class of type at most τ and $h(x) = O(1/x)$ when x is large. Then $f(z) := \sin(\tau z) + h(z)$ is of type τ and has at most a finite number of complex zeros.*

Proof. To prove the first part of the corollary, we note that $f(z)$ is of type at most τ based on the property of $h(z)$. On the real axis, $h(x)$ goes to 0 as x goes to infinity. Hence the density of the real zeros on the positive real axis is τ/π. So the density of all the zeros on the right half plane is at least τ/π. Using Levinson's Theorem, we see that the type of $f(z)$ is τ.

Let $\tau = 1$. The general case follows from dilation. From the theorem above, there exists a real number M such that

$$|h(x+iy)| \leq M \,\frac{\cosh(y)}{|z|}.$$

We set up a symmetric rectangle $\mathcal{R}$ with vertices at $\pm(n+1/2)\pi \pm iY$, with n being an integer and Y a large positive real number. Our aim is to show that $|f(z) - \sin(z)| = |h(z)| < |\sin(z)|$ for z on the boundary of $\mathcal{R}$. An application of Rouché's theorem shows that $f(z)$ and $\sin(z)$ have the same number of zeros inside $\mathcal{R}$.

For $z = x + iy$, $|\sin(z)|^2 = \sin^2(x) + \sinh^2(y)$ whose value is $1 + \sinh^2(y)$ on a vertical side of $\mathcal{R}$. Since

$$M\cosh(y) \ < |z|\sqrt{1+\sinh^2(y)}$$

when $Re(z)$ is large, $|h(z)| < |\sin(z)|$ there. On a horizontal segment of $\mathcal{R}$ $|\sin(z)| \geq |\sinh(y)|$. So

$$|h(x+iy)| \leq M\,\frac{\cosh(y)}{|z|} < |\sinh(y)| \leq |\sin(z)|$$

when $Im(z) = Y$ is large enough. Altogether, $|f(z) - \sin(z)| < |\sin(z)|$ on the four edges of the rectangle.

When $Re(z) = x$ is large, $|h(x)|$ is small. So all the zeros of $f(z)$ are real and are close to that of $\sin(z)$. $\square$

As noted earlier, when both parameters $\delta \neq 1$ and $n(1) \neq 1$, the entire function $d(k)$ has infinitely many real and complex zeros. The main theme of this paper is to show that this situation is drastically different when one of these parameters is 1. If both are 1, then it is possible to have all zeros complex.

Our method to locate the zeros of the function $d(k)$ as a function of the parameters $n(1)$ and $\delta := \int_0^1 \sqrt{n(t)}dt$ hinges on the Levitan-Gelfand formulation of the Sturm-Liouville problem. We assume that $n \in C^2[0,1]$. Using the Liouville transformation

$$\xi := \int_0^r \sqrt{n(t)}\,dt \tag{5.55}$$

and setting

$$z(\xi) := n(r)^{1/4}y(r), \quad r \ = \ r(\xi), \tag{5.56}$$

we can rewrite

$$y'' + k^2 n(r) y = 0, \quad y(0) = 0, \quad y'(0) = 1$$

as

$$z'' + (k^2 - p(\xi))z = 0 \tag{5.57}$$

$$z(0) = 0, \quad z'(0) = (n(0))^{-1/4}$$

where

$$p(\xi) = \frac{n''(r)}{4n^2(r)} - \frac{5}{16}\frac{(n'(r))^2}{n^3(r)}. \tag{5.58}$$

From [9], we can represent $z(\xi)$ in the form

$$z(\xi) = \frac{1}{n(0)^{1/4}}\left[\frac{\sin(k\xi)}{k} + \int_0^\xi K(\xi,t)\frac{\sin(kt)}{k}\,dt\right]. \tag{5.59}$$

Then

$$z'(\xi) = \frac{1}{n(0)^{1/4}}\left[\cos(k\xi) + K(\xi,\xi)\frac{\sin(k\xi)}{k} + \int_0^\xi K_\xi(\xi,t)\frac{\sin(kt)}{k}\,dt\right]. \tag{5.60}$$

Here $K(\xi,t)$ is the unique solution of

$$\begin{aligned} K_{\xi\xi} - K_{tt} - p(\xi)K &= 0 \\ K(\xi,0) &= 0 \\ K(\xi,\xi) &= \frac{1}{2}\int_0^\xi p(s)ds. \end{aligned}$$

This partial differential equation for $K(\xi,t)$ is defined on the triangular region $\Delta_o := 0 \le t \le \xi \le \delta = \xi(1)$. It is shown in [9] that $K(\xi,t)$ can be constructed in a straight forward manner by the method of successive approximations. It is a C^2 function on the closure of Δ_o if $p(\xi)$ is assumed to be continuous on $[0,\delta]$. Set $\alpha := n(0)^{1/4}$. From (5.59) and (5.60), we have

$$\begin{aligned} z(\delta) &= \frac{1}{\alpha k}\left[\sin(k\delta) + \int_0^\delta K(\delta,t)\sin(kt)dt\right] \\ z'(\delta) &= \frac{1}{\alpha k}\left[k\cos(k\delta) + K(\delta,\delta)\sin(k\delta) + \int_0^\delta K_\xi(\delta,t)\sin(kt)\,dt\right]. \end{aligned}$$

We note that each of these two entire functions is of type δ as a function of k. Since $z(\xi) = n(r)^{1/4} y(r)$ we have that

$$\begin{aligned} y(1) &= \frac{z(a)}{n(1)^{1/4}} \\ y'(1) &= n(1)^{1/4} z'(a) - \frac{n'(1)}{4n(1)^{5/4}} y(1). \end{aligned}$$

The entire function $d(k) = y(1)\cos(k) - y'(1)\sin(k)/k$ first defined in (5.50) is of type at most $\delta + 1$ and can be rewritten as

$$d(k) = \left[\frac{\cos(k)}{n(1)^{1/4}} + \frac{n'(1)}{4n(1)^{5/4}}\frac{\sin(k)}{k}\right] z(\delta) - n(1)^{1/4}\frac{\sin(k)}{k} z'(\delta). \tag{5.61}$$

Before expanding $d(k)$ out, let us perform one integration by parts on $z(\delta)$ to transform it into

$$z(\delta) = \frac{1}{\alpha k}\left[\sin(\delta k) - K(\delta,\delta)\frac{\cos(\delta k)}{k} + \int_0^\delta K_t(\delta,t)\frac{\cos(kt)}{k}\,dt\right]. \tag{5.62}$$

In terms of the kernel function $K(\xi, t)$, we have

$$\begin{aligned} d(k) &= \left(\frac{\cos(k)}{\alpha\, k\, n(1)^{1/4}} + \frac{n'(1)}{4\alpha n(1)^{5/4}}\frac{\sin(k)}{k^2}\right) \\ &\times \left(\sin(k\delta) - K(\delta,\delta)\frac{\cos(k\delta)}{k} + \int_0^\delta K_t(\delta,t)\frac{\cos(kt)}{k}\,dt\right) \\ &- \frac{n(1)^{1/4}\sin(k)}{\alpha k}\left[k\cos(k\delta) + K(\delta,\delta)\sin(k\delta) + \int_0^\delta K_\xi(\delta,t)\sin(kt)\,dt\right]. \end{aligned}$$

We multiply both sides above by $\alpha n(1)^{1/4}\, k$ to arrive at

$$\begin{aligned} \alpha n(1)^{1/4} k\, d(k) &= \left(\cos(k) + \frac{n'(1)\sin(k)}{4n(1)k}\right) \\ &\times \left(\sin(k\delta) - K(\delta,\delta)\frac{\cos(k\delta)}{k} + \int_0^\delta K_t(\delta,t)\frac{\cos(kt)}{k} dt\right) \\ &- \sqrt{n(1)}\frac{\sin(k)}{k}\left[k\cos(k\delta) + K(\delta,\delta)\sin(k\delta) + \int_0^\delta K_\xi(\delta,t)\sin(kt) dt\right]. \end{aligned}$$

Let $D(k) := \alpha\, n(1)^{1/4}\, k\, d(k)$. After expanding the right hand side and collecting terms of similar order of decay as k goes to infinity along the real line, we have the following formulation

$$D(k) := \alpha\, n(1)^{1/4}\, k\, d(k) = \cos(k)\sin(k\delta) - \sqrt{n(1)}\sin(k)\cos(k\delta) + H(k) \tag{5.63}$$

where

$$\begin{aligned} H(k) &:= \left(\frac{n'(1)}{4n(1)} - \sqrt{n(1)}K(\delta,\delta)\right)\frac{\sin(k)\sin(k\delta)}{k} - K(\delta,\delta)\frac{\cos(k)\cos(k\delta)}{k} \\ &- \frac{n'(1)}{4n(1)}K(\delta,\delta)\frac{\sin(k)\cos(k\delta)}{k^2} + \frac{\cos(k)}{k}\int_0^\delta K_t(\delta,t)\cos(kt)dt \\ &- \sqrt{n(1)}\frac{\sin(k)}{k}\int_0^\delta K_\xi(\delta,t)\sin(kt)dt + \frac{n'(1)}{4n(1)}\frac{\sin(k)}{k^2}\int_0^\delta K_t(\delta,t)\cos(kt)dt. \end{aligned}$$

The function $kH(k)$ is bounded on the real line and is of exponential type $\leq \delta + 1$. The first two terms on the right hand side of (5.63) can be written as

$$T(k) := \frac{1-\sqrt{n(1)}}{2}\sin((\delta+1)k) + \frac{1+\sqrt{n(1)}}{2}\sin((\delta-1)k) \tag{5.64}$$

while all the other terms are $O(1/k)$ for k large. When both $n(1) \neq 1$ and $\delta \neq 1$, we see that density of the zeros of $d(k)$ is $(\delta+1)/\pi$. In general, there are many situations that infinitely many of them are complex. Interesting patterns of the location of the zeros can be generated by picking an $n(r)$ with $\delta = \int_0^1 \sqrt{n(t)}dt$ close to 1 as the example below shows. However the exact conditions to determine the existence of complex eigenvalues are still lacking.

Since the refractive index $n(r)$ is defined to be one for $r \geq 1$, a natural assumption is to let $n(1) = 1$ and $n'(1) = 0$. We intend to show that an infinite number of complex eigenvalues are present under the additional assumptions $n''(1) \neq 0$ and $\delta \neq 1$.

Theorem 5.2.4 *Suppose the refractive index $n \in C^2[0,1]$ with $n(1) = 1$, $n'(1) = 0$ and $\delta \neq 1$. Then under the extra assumption that $n''(1) \neq 0$ the entire function $d(k)$ has infinitely many complex zeros and infinitely many real zeros.*

Proof. With the given parameters $n(1) = 1$ and $n'(1) = 0$, we have that

$$\begin{aligned} D(k) &= \sin((\delta-1)k) - K(\delta,\delta)\frac{\cos((\delta-1)k)}{k} \\ &+ \frac{\cos(k)}{k}\int_0^\delta K_t(\delta,t)\cos(kt)\,dt - \frac{\sin(k)}{k}\int_0^\delta K_\xi(\delta,t)\sin(kt)\,dt. \end{aligned}$$

If we perform an integration by parts on the last two integrals, we see that

$$\begin{aligned} D(k) &= -K(\delta,\delta)\frac{\cos((\delta-1)k)}{k} + K_t(\delta,\delta)\frac{\cos(k)\sin(k\delta)}{k^2} + K_\xi(\delta,\delta)\frac{\sin(k)\cos(k\delta)}{k^2} \\ &+ \sin((\delta-1)k) - \frac{\cos(k)}{k^2}\int_0^\delta K_{tt}(\delta,t)\sin(kt)\,dt - \frac{\sin(k)}{k^2}\int_0^\delta K_{\xi t}(\delta,t)\cos(kt)\,dt. \end{aligned}$$

Note that we simplified one of the integrated terms using the fact that $K_\xi(\delta,0)=0$ since $K(\xi,0)\equiv 0$.

Using a trigonometric identity, the terms of order $O(1/k^2)$ can be written as

$$\frac{K_t(\delta,\delta)}{2k^2}\left(\sin((\delta+1)k)+\sin((\delta-1)k)\right) + \frac{K_\xi(\delta,\delta)}{2k^2}\left(\sin((\delta+1)k)-\sin((\delta-1)k)\right).$$

According to Corollary 5.2.3, the sum of this expression with the remainder term which is of order $O(1/k^3)$ is an entire function of type $(\delta+1)$ if the coefficient of $\sin((\delta+1)k)$ (which equals to $(K_t(\delta,\delta)+K_\xi(\delta,\delta))/2$) is nonzero. Since

$$K(\xi,\xi) = \frac{1}{2}\int_0^\xi p(s)ds$$

for $0\le\xi\le\delta$, the term $K_t(\delta,\delta)+K_\xi(\delta,\delta)$ is equal to $\frac{p(\delta)}{2}$. From (3.4), we see that $p(\delta)=n''(1)/4$ since $n(1)=1$ and $n'(1)=0$.

In summary, under the given assumptions, the asymptotic expansion of $D(k)$ has the form

$$\begin{aligned} D(k) &= \sin((\delta-1)k) - \frac{K(\delta,\delta)}{k}\cos((\delta-1)k) \\ &+ \frac{K_t-K_\xi}{2k^2}\sin((\delta-1)k) + \frac{K_t+K_\xi}{2k^2}\sin((\delta+1)k) + O(1/k^3) \end{aligned}$$

with $K(\delta,\delta) = (\int_0^\delta p(s)ds)/2$ and $(K_t+K_\xi)/2 = n''(1)/8$.

If $\delta\neq 1$, we see from Corollaries 5.2.1 and 5.2.3 that $D(k)$ is of type $\delta+1$. Since the leading term $\sin((\delta-1)k)$ generates an infinite set of positive real zeros with density equal to $|1-\delta|/\pi$ while the density of all the zeros on right half plane equal to $(\delta+1)/\pi$ we have both infinitely many real and complex zeros. □

It was proved in [25] that the zeros of $D(k)$ lie in a strip parallel to the real axis if $n(1)\neq 1$. We now show that if $n(1)=1$, the imaginary parts of the zeros cannot stay bounded as their real parts move to the right.

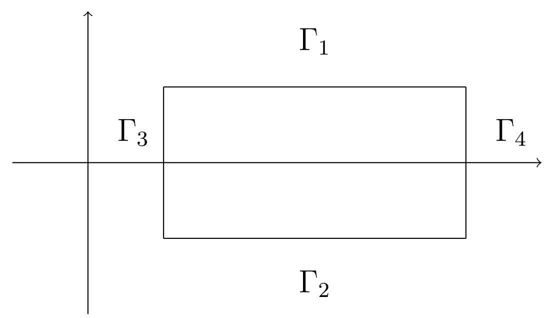


Figure 5.1: An example of the strip

Theorem 5.2.5 *Suppose the refractive index $n \in C^2[0,1]$ with $n(1) = 1$ and $\delta \neq 1$. If either $n'(1)$ or $n''(1)$ is non-zero, the zeros of $D(k)$ do not lie inside a fixed strip parallel to the real axis.*

Proof. Recalling from (5.63)-(5.64) with $n(1) = 1$, $D(k) := \alpha\, k\, d(k) = \sin((\delta-1)k) + H(k)$, where $H(k)$ can be written as

$$-\frac{n'(1)\cos((\delta+1)k)}{8k} - (K(\delta,\delta) - n'(1)/2)\,\frac{\cos((\delta-1)k)}{k} + O(1/k^2).$$

The real entire function $H(k)$ is in the Paley Wiener class. We express $H(k) = h(k)/k$ with $h(k)$ being an entire function bounded on the real axis. According to Theorem 5.2.3, there is a constant M such that $|h(k)| < M\cosh(\tau y)$ for $k = x + iy$.

Assume on the contrary that the zeros of $D(k)$ lie in a fixed strip parallel to the real axis. Now consider a rectangular region lying in the strip as in Figure 5.1 with Γ_3 and Γ_4 intersecting the real axis at $\frac{(2m+1)\pi}{2|\delta-1|}$ for an integer m. On the two vertical boundaries, $|\sin((\delta-1)k)|^2 = \sinh^2((\delta-1)y) + 1$. This value is at least 1. So $|D(k) - \sin((\delta-1)k)| = |h(k)/k| < M\cosh(\tau y)/|k| < |\sin((\delta-1)k)|$ for y bounded and $|k|$ large. The inequality also holds on two fixed horizontal boundaries for $|k|$ large. Thus we have proved that $|D(k) - \sin((\delta-1)k)| < |\sin((\delta-1)k)|$ on all four sides of the rectangle when $Re(k)$ is large. By Rouché theorem $D(k)$ has the same density of zeros as $\sin((\delta-1)k)$ inside a rectangle with fixed height. When Γ_4 moves out to infinity, the density of zeros inside this infinite strip is $|\delta-1|/\pi$. However the density of all the complex zeros is $(\delta+1)/\pi$. This shows that the zeros of $D(k)$ cannot lie inside a fixed horizontal strip. $\square$

Remark 5.2.1 *We did not consider the case $n(1)=1$ and $n'(1)\neq 0$ in the theorem above. However in this case it is quite easy to deduce from (5.63) that*

$$D(k)=\sin((\delta-1)k)+\frac{1}{k}\left(\frac{n'(1)}{4}\sin(k)\sin(\delta k)-K(\delta,\delta)\cos((\delta-1)k)\right)+O(1/k^2).$$

When $n'(1)\neq 0$, the density of all the zeros on the right half plane is still $(\delta+1)/\pi$ and the density of the real zeros is $|\delta-1|/\pi$.

An investigation of eigenvalues in the case $\delta=1$ gives a number of surprising results. In particular we will show that in this case it is possible to have all real eigenvalues or all complex eigenvalues.

Theorem 5.2.6 *Let the refractive index $n\in C^2[0,1]$. Suppose $\delta=1$ and $n(1)\neq 1$. Then there are at most finitely many complex transmission eigenvalues. However if both $\delta=1$ and $n(1)=1$, then it is possible to have only finitely many real eigenvalues.*

Proof. The theoretical aspect is pretty straight forward. If $\delta=1$, then from (5.63)-(5.64) we have that $D(k)$ has the form

$$D(k)=\frac{1-\sqrt{n(1)}}{2}\sin(2k)+G(k) \tag{5.65}$$

where

$$\begin{aligned} G(k) &:= \left(\frac{n'(1)}{4n(1)}-\sqrt{n(1)}K(1,1)\right)\frac{\sin^2(k)}{k}-K(1,1)\frac{\cos^2(k)}{k}+O(1/k^2)\\ &= \left(\frac{n'(1)}{4n(1)}-\sqrt{n(1)}K(1,1)+K(1,1)\right)\frac{\sin^2(k)}{k}-\frac{K(1,1)}{k}+O(1/k^2). \end{aligned}$$

The term $\sin(2k)$ dominates the sum for $D(k)$ when k is large along the real axis and we see that the density of the real zeros is $2/\pi$. The function $D(k)$ vanishes at the origin, so the term $G(k)$ has a zero at the origin.

If we multiply the entire equation by k, we see that $kG(k)$ is an entire function of type two and there are at most finitely many complex roots as shown by Corollary

5.2.3 and Corollary 5.2.3 (We will show examples below with one where all the roots are real and another with a few complex roots at the beginning and then all real roots afterwards).

If both $\delta = 1$ and $n(1) = 1$, the expression in (5.65) gives

$$D(k) = (\frac{n'(1)\sin^2(k)}{4} - K(1,1))\frac{1}{k} + O(1/k^2).$$

So if $|n'(1)| < 4|K(1,1)|$, then $D(k)$ will be either strictly positive or strictly negative for large k. Hence there are at most finitely many real zeros.

Finally if in addition $n'(1) = 0$ then

$$D(k) = G(k) = -\frac{K(1,1)}{k} + O(1/k^2).$$

Again, there will be only a finite number of real zeros if $K(1,1) \neq 0$. Surprisingly, a simple constraint like $n'(0) < 0$ will show that $K(1,1) > 0$ since

$$\begin{aligned}
2K(1,1) = \int_0^1 p(\xi)d\xi &= \int_0^1 \left(\frac{n''(x)}{4n(x)^2} - \frac{5}{16}\frac{(n'(x))^2}{n(x)^3}\right)\xi'(x)\,dx \\
&= \int_0^1 \frac{n''(x)}{4n(x)^{3/2}} - \frac{5}{16}\frac{n'(x)^2}{n(x)^{5/2}}\,dx \\
&= -\frac{n'(0)}{4n(0)^{3/2}} + \int_0^1 \frac{3}{8}\frac{n'(x)^2}{n(x)^{5/2}} - \frac{5}{16}\frac{n'(x)^2}{n(x)^{5/2}}\,dx \\
&= -\frac{n'(0)}{4n(0)^{3/2}} + \frac{1}{16}\int_0^1 \frac{n'(x)^2}{n(x)^{5/2}}\,dx.
\end{aligned}$$

□

5.2.2 Absorbing Medium

In this last section of this chapter we turn our attention to the case when the medium is absorbing, i.e. the index of refraction is complex valued. In this case, we cannot in general expect that real transmission eigenvalues exist (Theorem 8.12 of [24]). However we will show that under appropriate assumption there exist an infinite number of transmission eigenvalues that lie arbitrary close to the real axis.

For the case of absorbing media, the interior transmission eigenvalue problem becomes (c.f. [11])

$$w'' + \tfrac{2}{r}w' + k^2(\epsilon_1(r) + i\tfrac{\gamma_1(r)}{k})w = 0, \quad 0 < r < 1 \tag{5.66}$$

$$v'' + \tfrac{2}{r}v' + k^2(\epsilon_0 + i\tfrac{\gamma_0}{k})v = 0, \quad 0 < r < 1 \tag{5.67}$$

$$w(1) = v(1), \quad w'(1) = v'(1); \tag{5.68}$$

where $\epsilon_1(r)$ and $\gamma_1(r)$ are continuous for $0 \le r < 1$, $\epsilon_1(1) = \epsilon_0$ and ϵ_0 and γ_0 are positive constants. We look for a solution of (5.66) - (5.68) in the form

$$v(r) = c_1 j_0(kn_o r) \tag{5.69}$$

$$w(r) = c_2\tfrac{y(r)}{r} \tag{5.70}$$

where $n_o = \sqrt{\epsilon_0 + i\frac{\gamma_0}{k}}$ (where the branch cut is chosen such that n_o has a positive real part), j_0 is a spherical Bessel function of order zero, $y(r)$ is a solution of

$$y'' + k^2(\epsilon_1(r) + i\frac{\gamma_1(r)}{k})y = 0 \tag{5.71}$$

$$y(0) = 0, \quad y'(0) = 1 \tag{5.72}$$

for $0 < r < 1$ and c_1 and c_2 are constants. Then there exist constants c_1 and c_2 not both zero, such that (5.69) will be a nontrivial solution of (5.66) - (5.68) provided that

$$d(k) := \ Det\begin{pmatrix} y(1) & -\frac{\sin(kn_o)}{k} \\ y'(1) & -n_o\cos(kn_o) \end{pmatrix} = 0.$$

Theorem 5.2.7 *If* $\frac{\gamma_0}{\sqrt{\epsilon_0}} = \int_0^1 \frac{\gamma_1(\rho)}{\sqrt{\epsilon_1(\rho)}}d\rho$ *and* $\sqrt{\epsilon_0} \neq \int_0^1 \sqrt{\epsilon_1(\rho)}d\rho$. *Then there exist an infinite number of transmission eigenvalues that are arbitrarily near the real axis.*

Proof. Assume the contrary. Then we can choose a semi-infinite strip parallel to the real axis such that there are no (complex) eigenvalues in the srip. (see Figure 5.1). Since

$$\frac{\gamma_0}{\sqrt{\epsilon_0}} = \int_0^1 \frac{\gamma_1(\rho)}{\sqrt{\epsilon_1(\rho)}}d\rho,$$

it is seen from [11] that

$$y(r) = \frac{1}{ik[\epsilon_1(0)\epsilon_1(r)]^{\frac{1}{4}}} \sinh\left[ik\int_0^r \sqrt{\epsilon(\rho)}d\rho - \frac{1}{2}\int_0^r \frac{\gamma_1(\rho)}{\sqrt{\epsilon_1(\rho)}}d\rho\right] + O(\frac{1}{k^2})$$

holds uniformly for k in the strip. Then

$$d(k) = \frac{1}{ik[\epsilon_1(0)\epsilon_0]^{\frac{1}{4}}} \sin\left((\sqrt{\epsilon_0} - \int_0^1 \sqrt{\epsilon_1(\rho)}d\rho)k\right) + O(\frac{1}{k^2}). \tag{5.73}$$

Let $\tau := \sqrt{\epsilon_0} - \int_0^1 \sqrt{\epsilon_1(\rho)}d\rho$ and $S(k) := \frac{1}{ik[\epsilon_1(0)\epsilon_0]^{\frac{1}{4}}} \sin(\tau k)$. Consider the finite strip shown in Figure 5.1 where Γ_3 and Γ_4 intersect with the real axis at $\frac{(2m+1)\pi}{2\tau}$ for an integer m.

We will show that $|d(k) - S(k)| < |S(k)|$ for large enough $|k|$ on the boundary of the rectangular strip. Indeed, let $k = \alpha + i\beta$. Then

$$|S(k)| = \frac{1}{|k|[\epsilon_1(0)\epsilon_0]^{\frac{1}{4}}} e^{-\beta\tau}\sqrt{(1 - e^{2\beta\tau})^2 + 4e^{2\beta\tau}\sin^2(\alpha\tau)}.$$

Then on the boundaries Γ_3 and Γ_4 we have

$$|S(k)| \geq \frac{1}{|k|[\epsilon_1(0)\epsilon_0]^{\frac{1}{4}}} > |d(k) - S(k)|$$

and on the boundaries Γ_1 and Γ_2

$$|S(k)| > |d(k) - S(k)|$$

for the real part of k sufficiently large. Now we can apply Rouché's theorem to see that $d(k)$ and $S(k)$ have the same number of zeros in the strip shown in Figure 5.1. By letting the right hand side of the strip move to infinity, we see that $d(k)$ and $S(k)$ have the same number of zeros in this semi-infinite strip. Note that since $\frac{m\pi}{\tau}$ are the zeros of $S(k)$, then $S(k)$ has infinitely many zeros in the strip. Hence $d(k)$ has infinitely many zeros in the strip which is a contradiction to our assumption. $\square$

Chapter 6

BOUNDARY INTEGRAL EQUATIONS FOR THE INTERIOR TRANSMISSION PROBLEM FOR MAXWELL'S EQUATIONS

The transmission eigenvalue problem is genuinely related to the scattering problem for an inhomogeneous media. In this chapter the underlying scattering problem is the scattering of electromagnetic waves by a (possibly anisotropic) non-magnetic material of bounded support D situated in homogenous background, which in terms of the electric field reads:

$$\operatorname{curl}\operatorname{curl}\mathbf{E}^s - k^2\mathbf{E}^s = 0 \qquad \text{in} \quad \mathbb{R}^3 \setminus \overline{D} \tag{6.1}$$

$$\operatorname{curl}\operatorname{curl}\mathbf{E} - k^2 N\mathbf{E} = 0 \qquad \text{in} \quad D \tag{6.2}$$

$$\nu \times \mathbf{E} = \nu \times \mathbf{E}^s + \nu \times \mathbf{E}^i \qquad \text{on} \quad \partial D \tag{6.3}$$

$$\nu \times \operatorname{curl}\mathbf{E} = \nu \times \operatorname{curl}\mathbf{E}^s + \nu \times \operatorname{curl}\mathbf{E}^i \qquad \text{on} \quad \partial D \tag{6.4}$$

$$\lim_{r\to\infty} (\operatorname{curl}\mathbf{E}^s \times x - ikr\mathbf{E}^s) = 0 \tag{6.5}$$

where $\mathbf{E}^i$ is the incident electric field, $\mathbf{E}^s$ is the scattered electric field and $N(x) = \dfrac{\epsilon(x)}{\epsilon_0} + i\dfrac{\sigma(x)}{\omega\epsilon_0}$ is the matrix index of refraction, $k = \omega\sqrt{\epsilon_0\mu_0}$ is the wave number corresponding to the background and the frequency ω and the Silver-Müller radiation condition is satisfied uniformly with respect to $\hat{x} = x/r$, $r = |x|$. The difference $N - I$, in the following, is refereed to as the contrast in the media. In scattering theory, transmission eigenvalues can be seen as the extension of the notion of resonant frequencies for impenetrable objects to the case of penetrable media. The transmission eigenvalue problem is related to non-scattering incident fields. Indeed, if $\mathbf{E}^i$ is such that $\mathbf{E}^s = 0$ then $\mathbf{E}|_D$ and $\mathbf{E}_0 = \mathbf{E}^i|_D$ satisfy the following homogenous problem

$$\operatorname{curl}\operatorname{curl}\mathbf{E} - k^2 N\mathbf{E} = 0 \qquad \text{in} \quad D \tag{6.6}$$

$$\operatorname{curl}\operatorname{curl}\mathbf{E}_0 - k^2\mathbf{E}_0 = 0 \qquad \text{in} \quad D \tag{6.7}$$

$$\nu \times \mathbf{E} = \nu \times \mathbf{E}_0 \qquad \text{on} \quad \partial D \tag{6.8}$$

$$\nu \times \operatorname{curl}\mathbf{E} = \nu \times \operatorname{curl}\mathbf{E}_0 \qquad \text{on} \quad \partial D \tag{6.9}$$

which is referred to as the transmission eigenvalue problem. Conversely, if (6.6)-(6.9) has a nontrivial solution $\mathbf{E}$ and $\mathbf{E}_0$ and $\mathbf{E}_0$ can be extended outside D as a solution to $\operatorname{curl}\operatorname{curl}\mathbf{E}_0 - k^2\mathbf{E}_0 = 0$, then if this extended $\mathbf{E}_0$ is considered as the incident field the corresponding scattered field is $\mathbf{E}^s = 0$.

The transmission eigenvalue problem is a nonlinear and non-selfadjoint eigenvalue problem that is not covered by the standard theory of eigenvalue problems for elliptic equations. For a long time research on the transmission eigenvalue problem mainly focussed on showing that transmission eigenvalues form at most a discrete set and we refer the reader to the survey paper [18] for the state of the art on this question up to 2010. From a practical point of view the question of discreteness was important to answer, since sampling methods for reconstructing the support of an inhomogeneous medium [9, 24] fail if the interrogating frequency corresponds to a transmission eigenvalue. On the other hand, due to the non-selfadjointness of the transmission eigenvalue problem, the existence of transmission eigenvalues for non-spherically stratified media remained open for more than 20 years until Sylvester and Päivärinta [70] showed the existence of at least one transmission eigenvalue provided that the contrast in the medium is large enough for the scalar. A full answer on the existence of transmission eigenvalues was given by Cakoni, Gintides and Haddar [15] where the existence of an infinite set of transmission eigenvalue was proven only under the assumption that the contrast in the medium does not change sign and is bounded away from zero (see also [14, 20, 32, 52] for Maxwell's equation). Since the appearance of these papers there has been an explosion of interest in the transmission eigenvalue problem and the papers in the Special Issue of Inverse Problems on Transmission Eigenvalues, Volume 29, Number 10, October 2013, are representative of the myriad directions that this research has taken.

The discreteness and existence of transmission eigenvalues is very well understood under the assumption that the contrast does not change sign in all of D. Recently, for the scalar Helmholtz type equation, several papers have appeared addressing both the question of discreteness and existence of transmission eigenvalue assuming that the contrast is of one sign only in a neighborhood of the inhomogeneity's boundary ∂D, [8,28,33,59,60,76,79]. The picture is not the same for the transmission eigenvalue problem for the Maxwell's equation. The only result in this direction is the proof of discreteness of transmission eigenvalues in [22] for magnetic materials, i.e. when there is contrast in both the electric prematurity and magnetic permeability. The T-coercivity approach used in [22] does not apply to our problem (6.6)-(6.9), which mathematically has a different structure form the case of magnetic materials and this paper is dedicated to study the discreteness of transmission eigenvalues for the considered problem under weaker assumption of $N - I$. Before specifying our assumptions and approach let us rigorously formulate our transmission eigenvalue problem.

Formulation of the Problem: Let $D \in \mathbb{R}^3$ be a bounded open and connected region with C^2-smooth boundary $\partial D := \Gamma$ (we call it Γ for notational convenience as will be seen later) and let ν denote the outward unit normal vector on Γ. In general we consider a 3×3 matrix-valued function N with $L^\infty(D)$ entries such that $\overline{\xi} \cdot \mathrm{Re}(N)\xi \geq \alpha > 0$ and $\overline{\xi} \cdot \mathrm{Im}(N)\xi \geq 0$ in D for every $\xi \in \mathbb{C}^3$, $|\xi| = 1$. The transmission eigenvalue problem can be formulated as finding $\mathbf{E}, \mathbf{E}_0 \in \mathbf{L}^2(D)$, $\mathbf{E} - \mathbf{E}_0 \in \mathbf{H}_0(\mathrm{curl}^2, D)$ that satisfy

$$\mathrm{curl}\,\mathrm{curl}\,\mathbf{E} - k^2 N\mathbf{E} = 0 \qquad \text{in} \quad D \tag{6.10}$$

$$\mathrm{curl}\,\mathrm{curl}\,\mathbf{E}_0 - k^2\mathbf{E}_0 = 0 \qquad \text{in} \quad D \tag{6.11}$$

$$\nu \times \mathbf{E} = \nu \times \mathbf{E}_0 \qquad \text{on} \quad \Gamma \tag{6.12}$$

$$\nu \times \mathrm{curl}\,\mathbf{E} = \nu \times \mathrm{curl}\,\mathbf{E}_0 \qquad \text{on} \quad \Gamma \tag{6.13}$$

where

$$\mathbf{L}^2(D) := \left\{\mathbf{u} : \mathbf{u}_j \in L^2(D), j = 1, 2, 3\right\},$$

$$\mathbf{H}(\mathrm{curl}^{\,2}, D) := \left\{\mathbf{u} : \mathbf{u} \in \mathbf{L}^2(D), \mathrm{curl}\,\mathbf{u} \in \mathbf{L}^2(D) \text{ and } \mathrm{curl}\,\mathrm{curl}\,\mathbf{u} \in \mathbf{L}^2(D)\right\},$$

$$\mathbf{H}_0(\mathrm{curl}^{\,2}, D) := \left\{\mathbf{u} : \mathbf{u} \in \mathbf{H}(\mathrm{curl}^{\,2}, D), \gamma_t\mathbf{u} = 0 \text{ and } \gamma_t\mathrm{curl}\,\mathbf{u} = 0 \text{ on } \Gamma\right\}.$$

Definition 6.0.3 *Values of $k \in \mathbb{C}$ for which the (6.10)-(6.13) has a nontrivial solution $\mathbf{E}, \mathbf{E}_0 \in \mathbf{L}^2(D)$, $\mathbf{E} - \mathbf{E}_0 \in \mathbf{H}_0(\mathrm{curl}^2, D)$ are called transmission eigenvalues.*

It is well-known [20,37] that, if $\Re(N-I)$ has one sign in D the transmission eigenvalues form at most a discrete set with $+\infty$ as the only possible accumulation point, and if in addition $\Im(N) = 0$, there exists an infinite set of real transmission eigenvalues. Our main concern is to understand the structure of the transmission eigenvalue problem in the case when $\Re(N - I)$ changes sign inside D. More specifically in this case we show that the transmission eigenvalues form at most a discrete set using an equivalent integral equation formulation of the transmission eigenvalue problem following the boundary integral equations approach developed in [33]. The assumption on the real part of the contract $N - I$ that we need in our analysis will become more precise later in the paper, but roughly speaking in our approach we allow for $\Re(N - I)$ to change sign in a compact subset of D. To this end, in the next section we consider the simplest case when the electric permittivity is constant, i.e. $N = nI$ with positive $n \neq 1$, for which we develop and analyze an equivalent system of integral equations formulation of the corresponding transmission eigenvalue problem. This system of integral equations will then be a building block to study the more general case of the electric permittivity N. We note that the extension to Maxwell's equations of the approach in [33] is not a trivial task due to the more peculiar mapping properties of the electromagnetic boundary integral operators.

6.1 Boundary Integral Equations for Constant Electric Permittivity

Let $n > 0$ be a constant such that $n \neq 1$ and consider the problem of finding $\mathbf{E}, \mathbf{E}_0 \in \mathbf{L}^2(D)$, $\mathbf{E} - \mathbf{E}_0 \in \mathbf{H}_0(\mathrm{curl}^2, D)$ that satisfy

$$\mathrm{curl}\,\mathrm{curl}\,\mathbf{E} - k^2 n\mathbf{E} = 0 \qquad \text{in} \qquad D \tag{6.14}$$

$$\mathrm{curl}\,\mathrm{curl}\,\mathbf{E}_0 - k^2\mathbf{E}_0 = 0 \qquad \text{in} \qquad D \tag{6.15}$$

$$\nu \times \mathbf{E} = \nu \times \mathbf{E}_0 \qquad \text{on} \qquad \Gamma \tag{6.16}$$

$$\nu \times (\mathrm{curl}\,\mathbf{E}) = \nu \times (\mathrm{curl}\,\mathbf{E}_0) \qquad \text{on} \qquad \Gamma \tag{6.17}$$

In the following we denote by $k_1 := k\sqrt{n}$. Before formulating the transmission eigenvalue problem as an equivalent system of boundary integral equations, we recall several integral operators and study their mapping properties. To this end, let us define the Hilbert spaces of tangential fields defined on Γ:

$$\mathbf{H}^{s_1,s_2}(\mathrm{div},\Gamma) := \{\mathbf{u} \in \mathbf{H}_t^{s_1}(\Gamma), \mathrm{div}_\Gamma \mathbf{u} \in H^{s_2}(\Gamma)\},$$

$$\mathbf{H}^{s_1,s_2}(\mathrm{curl},\Gamma) := \{\mathbf{u} \in \mathbf{H}_t^{s_1}(\Gamma), \mathrm{curl}_\Gamma \mathbf{u} \in \mathbf{H}^{s_2}(\Gamma)\}$$

endowed with the respective natural norms, where curl_Γ and div_Γ are the surface curl and divergence operator, respectively, and for later use ∇_Γ denotes the tangential gradient operator. (Note that the boldface indicate vector spaces of vector fields, whereas non-bold face indicate vector spaces of scalar fields.) If $\gamma_\Gamma \mathbf{u} = \nu \times (\mathbf{u} \times \nu)$ denotes the tangential trace of a vector field $\mathbf{u}$ on the boundary Γ, we define the boundary integral operators:

$$\mathbf{T}_k(\mathbf{u}) := \frac{1}{k}\gamma_\Gamma \left(k^2 \int_\Gamma \Phi_k(\cdot,\mathbf{y})\mathbf{u}(\mathbf{y})\,ds_y + \nabla_\Gamma \int_\Gamma \Phi_k(\cdot,\mathbf{y})\mathrm{div}_\Gamma \mathbf{u}(\mathbf{y})\,ds_y \right) \tag{6.18}$$

and

$$\mathbf{K}_k(\mathbf{u}) := \gamma_\Gamma \left(\mathrm{curl} \int_\Gamma \Phi_k(\cdot,y)\mathbf{u}(\mathbf{y})\,ds_y \right) \tag{6.19}$$

where

$$\Phi_k(x,y) = \frac{1}{4\pi}\frac{e^{ik|x-y|}}{|x-y|}$$

is the fundament solution of the Helmholtz equation $\Delta u + k^2 u = 0$. Referring to [33] and [67] for the mapping properties of the single layer potential

$$S_k(\varphi) := \int_\Gamma \Phi_k(\cdot, \mathbf{y})\varphi(\mathbf{y}) ds_y \tag{6.20}$$

with scalar densities φ, we have that the boundary integral operator

$$\mathbf{S}_k(\mathbf{u}) = \int_\Gamma \Phi_k(\cdot, \mathbf{y})\mathbf{u}(\mathbf{y})\, ds \tag{6.21}$$

acting on vector fields $\mathbf{u}$, is bounded from $\mathbf{H}^{-\frac{1}{2}+s}(\Gamma)$ to $\mathbf{H}^{\frac{1}{2}+s}(\Gamma)$ for $-1 \le s \le 1$, hence

$$\mathbf{T}_k : \mathbf{H}^{-\frac{1}{2},-\frac{3}{2}}(\mathrm{div}, \Gamma) \to \mathbf{H}^{-\frac{1}{2},-\frac{3}{2}}(\mathrm{curl}, \Gamma)$$

$$\mathbf{K}_k : \mathbf{H}^{-\frac{3}{2},-\frac{1}{2}}(\mathrm{div}, \Gamma) \to \mathbf{H}^{-\frac{3}{2},-\frac{1}{2}}(\mathrm{curl}, \Gamma)$$

are bounded linear operators. Now from Stratton-Chu formula [24] we have that

$$\begin{aligned} \mathbf{E}_0(\mathbf{x}) &= \mathrm{curl} \int_\Gamma (\mathbf{E}_0 \times \nu)(\mathbf{y})\Phi_k(\mathbf{x}, \mathbf{y}) ds_y + \int_\Gamma (\mathrm{curl}\, \mathbf{E}_0 \times \nu)(\mathbf{y})\Phi_k(\mathbf{x}, \mathbf{y}) ds_y \\ &+ \frac{1}{k^2}\nabla \int_\Gamma \mathrm{div}_\Gamma (\mathrm{curl}\, \mathbf{E}_0 \times \nu)(\mathbf{y})\Phi_k(\mathbf{x}, \mathbf{y}) ds_y \qquad \text{for} \quad \mathbf{x} \in D \end{aligned}$$

with similar expression for $\mathbf{E}$ where k is replaced by $k_1 := k\sqrt{n}$, then we have the integral expression for $\mathbf{E} - \mathbf{E}_0$. Note by taking the difference $\mathbf{E} - \mathbf{E}_0$ we have the corresponding kernel $\Phi_{k_1}(x, y) - \Phi_k(x, y)$ is a smooth function of x, y, then approaching the boundary Γ and noting $\mathbf{E} \times \nu = \mathbf{E}_0 \times \nu$ and $\mathrm{curl}\, \mathbf{E} \times \nu = \mathrm{curl}\, \mathbf{E}_0 \times \nu$ we have

$$\begin{aligned} \gamma_\Gamma(\mathbf{E} - \mathbf{E}_0) &= (\mathbf{K}_k - \mathbf{K}_{k_1})(\mathbf{E}_0 \times \nu) + \frac{1}{k}(\mathbf{T}_k - \mathbf{T}_{k_1})(\mathrm{curl}\, \mathbf{E}_0 \times \nu) \\ \gamma_\Gamma \mathrm{curl}\, (\mathbf{E} - \mathbf{E}_0) &= (\mathbf{K}_k - \mathbf{K}_{k_1})(\mathrm{curl}\, \mathbf{E}_0 \times \nu) + k(\mathbf{T}_k - \mathbf{T}_{k_1})(\mathbf{E}_0 \times \nu). \end{aligned}$$

From the boundary conditions (6.16) and (6.17) we have $\gamma_\Gamma(\mathbf{E} - \mathbf{E}_0) = 0$ and $\gamma_\Gamma \mathrm{curl}\, (\mathbf{E} - \mathbf{E}_0) = 0$, i.e.

$$\mathbf{K}_k(\mathbf{E}_0 \times \nu) + \frac{1}{k}\mathbf{T}_k(\mathrm{curl}\, \mathbf{E}_0 \times \nu) - \mathbf{K}_{k_1}(\mathbf{E} \times \nu) + \frac{1}{k_1}\mathbf{T}_{k_1}(\mathrm{curl}\, \mathbf{E} \times \nu) = 0 \tag{6.22}$$

$$\mathbf{K}_k(\mathrm{curl}\, \mathbf{E}_0 \times \nu) + k\mathbf{T}_k(\mathbf{E}_0 \times \nu) - \mathbf{K}_{k_1}(\mathrm{curl}\, \mathbf{E} \times \nu) + k_1\mathbf{T}_{k_1}(\mathbf{E} \times \nu) = 0. \tag{6.23}$$

Introducing $\mathbf{M} = \mathbf{E} \times \nu = \mathbf{E}_0 \times \nu$ and $\mathbf{J} = \operatorname{curl} \mathbf{E} \times \nu = \operatorname{curl} \mathbf{E}_0 \times \nu$, we arrive at the following homogeneous system of boundary integral equations

$$\begin{pmatrix} k_1 \mathbf{T}_{k_1} - k\mathbf{T}_k & \mathbf{K}_{k_1} - \mathbf{K}_k \\ \mathbf{K}_{k_1} - \mathbf{K}_k & \frac{1}{k_1}\mathbf{T}_{k_1} - \frac{1}{k}\mathbf{T}_k \end{pmatrix} \begin{pmatrix} \mathbf{M} \\ \mathbf{J} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{6.24}$$

for the unknowns $\mathbf{M}$ and $\mathbf{J}$. Let us define by

$$\mathbf{L}(k) =: \begin{pmatrix} k_1 \mathbf{T}_{k_1} - k\mathbf{T}_k & \mathbf{K}_{k_1} - \mathbf{K}_k \\ \mathbf{K}_{k_1} - \mathbf{K}_k & \frac{1}{k_1}\mathbf{T}_{k_1} - \frac{1}{k}\mathbf{T}_k \end{pmatrix} = \begin{pmatrix} k\sqrt{n}\mathbf{T}_{k\sqrt{n}} - k\mathbf{T}_k & \mathbf{K}_{k\sqrt{n}} - \mathbf{K}_k \\ \mathbf{K}_{k\sqrt{n}} - \mathbf{K}_k & \frac{1}{k\sqrt{n}}\mathbf{T}_{k\sqrt{n}} - \frac{1}{k}\mathbf{T}_k \end{pmatrix} \tag{6.25}$$

Note that while the operator $\mathbf{K}_{k_1} - \mathbf{K}_k$ is smoothing pseudo-differential operator of order -2 (see e.g. [33] and [42]), the operators in the main diagonal have a mixed structure. Indeed, from the expressions

$$\begin{aligned} k_1 \mathbf{T}_{k_1} - k\mathbf{T}_k &= (k_1^2 \mathbf{S}_{k_1} - k^2 \mathbf{S}_k) + \nabla_\Gamma \circ (S_{k_1} - S_k) \circ \operatorname{div}_\Gamma \\ \frac{1}{k_1}\mathbf{T}_{k_1} - \frac{1}{k}\mathbf{T}_k &= (\mathbf{S}_{k_1} - \mathbf{S}_k) + \nabla_\Gamma \circ \left(\frac{1}{k_1^2} S_{k_1} - \frac{1}{k^2} S_k \right) \circ \operatorname{div}_\Gamma \end{aligned} \tag{6.26}$$

where S and $\mathbf{S}$ are defined by (6.20) and (6.21) respectively, we can see that these operators have different behavior component-wise. Hence a more delicate analysis is called for to find the correct functional spaces for $\mathbf{M}, \mathbf{J}$ and their dual spaces in order to analyze the mapping properties of the operator $\mathbf{L}(k)$.

Lemma 6.1.1 *The dual space of* $\mathbf{H}^{-\frac{3}{2},-\frac{1}{2}}(\operatorname{div},\Gamma)$ *is* $\mathbf{H}^{-\frac{1}{2},\frac{1}{2}}(\operatorname{curl},\Gamma)$*. For* $\mathbf{u}^t \in \mathbf{H}^{-\frac{1}{2},\frac{1}{2}}(\operatorname{curl},\Gamma)$ *and* $\mathbf{u} \in \mathbf{H}^{-\frac{3}{2},-\frac{1}{2}}(\operatorname{div},\Gamma)$*,* $\langle \mathbf{u}^t, \mathbf{u} \rangle$ *is a understood by duality with respect to* $\mathbf{L}^2(\Gamma)$ *as a pivot space.*

Proof. For any tangential fields $\mathbf{u} \in \mathbf{H}^{-\frac{3}{2},-\frac{1}{2}}(\operatorname{div},\Gamma)$ and $\mathbf{u}^t \in \mathbf{H}^{-\frac{1}{2},\frac{1}{2}}(\operatorname{curl},\Gamma)$, we consider the corresponding Helmholtz orthogonal decomposition

$$\mathbf{u} = \overrightarrow{\operatorname{curl}}_\Gamma q + \nabla_\Gamma p, \quad \mathbf{u}^t = \overrightarrow{\operatorname{curl}}_\Gamma q^t + \nabla_\Gamma p^t.$$

Since $\operatorname{div}_\Gamma \mathbf{u} = \operatorname{div}_\Gamma \nabla_\Gamma p = \Delta_\Gamma p \in H^{-\frac{1}{2}}(\Gamma)$ we have by eigensystem expansion (e.g. [69]) that $\nabla_\Gamma p \in \mathbf{H}^{\frac{1}{2}}(\Gamma)$. Similarly, from the fact that $\operatorname{curl}_\Gamma \mathbf{u}^t \in \mathbf{H}^{\frac{1}{2}}(\Gamma)$ we obtain that $\overrightarrow{\operatorname{curl}}_\Gamma q^t \in \mathbf{H}^{\frac{3}{2}}(\Gamma)$. Now

$$\begin{aligned} \left\langle \mathbf{u}^t, \mathbf{u} \right\rangle &= \left\langle \overrightarrow{\operatorname{curl}}_\Gamma q^t + \nabla_\Gamma p^t, \overrightarrow{\operatorname{curl}}_\Gamma q + \nabla_\Gamma p \right\rangle \\ &= \left\langle \overrightarrow{\operatorname{curl}}_\Gamma q^t, \overrightarrow{\operatorname{curl}}_\Gamma q \right\rangle + \left\langle \nabla_\Gamma p, \nabla_\Gamma p^t \right\rangle. \end{aligned}$$

Hence the right hand side is well defined in the sense of duality of $\mathbf{H}^{\frac{3}{2}}(\Gamma)$-$\mathbf{H}^{-\frac{3}{2}}(\Gamma)$ and $\mathbf{H}^{\frac{1}{2}}(\Gamma)$-$\mathbf{H}^{-\frac{1}{2}}(\Gamma)$, hence $\mathbf{H}^{-\frac{1}{2},\frac{1}{2}}(\mathrm{curl},\Gamma)$ is in the dual space of $\mathbf{H}^{-\frac{3}{2},-\frac{1}{2}}(\mathrm{div},\Gamma)$.

Furthermore, if $\mathbf{u}^t = \overrightarrow{\mathrm{curl}}_\Gamma q^t + \nabla_\Gamma p^t$ is in the dual space of $\mathbf{H}^{-\frac{3}{2},-\frac{1}{2}}(\mathrm{div},\Gamma)$, then $\langle \mathbf{u}^t, \cdot\rangle$ is continuous and linear on $\mathbf{H}^{-\frac{3}{2},-\frac{1}{2}}(\mathrm{div},\Gamma)$. Then for $\mathbf{u} = \overrightarrow{\mathrm{curl}}_\Gamma q$

$$\left\langle \mathbf{u}^t, \mathbf{u}\right\rangle = \left\langle \overrightarrow{\mathrm{curl}}_\Gamma q^t, \overrightarrow{\mathrm{curl}}_\Gamma q\right\rangle.$$

Notice $\overrightarrow{\mathrm{curl}}_\Gamma q$ is only in $\mathbf{H}^{-\frac{3}{2}}(\Gamma)$, therefore by eigensystem analysis $\overrightarrow{\mathrm{curl}}_\Gamma q^t \in \mathbf{H}^{\frac{3}{2}}(\Gamma)$ and $\mathrm{curl}_\Gamma \overrightarrow{\mathrm{curl}}_\Gamma q^t \in H^{\frac{1}{2}}(\Gamma)$, i.e. $\mathrm{curl}_\Gamma \mathbf{u}^t \in H^{\frac{1}{2}}(\Gamma)$. Now for $\mathbf{u} = \nabla_\Gamma p$ where $\nabla_\Gamma p \in \mathbf{H}^{\frac{1}{2}}(\Gamma)$

$$\left\langle \mathbf{u}^t, \mathbf{u}\right\rangle = \left\langle \nabla_\Gamma p^t, \nabla_\Gamma p\right\rangle.$$

Then $\nabla_\Gamma p^t \in \mathbf{H}^{-\frac{1}{2}}(\Gamma)$. Therefore $\mathbf{u}^t \in \mathbf{H}^{-\frac{1}{2},\frac{1}{2}}(\mathrm{curl},\Gamma)$. Now we have proved the lemma. $\square$

In the following the spaces $\mathbf{H}^{-\frac{3}{2},-\frac{1}{2}}(\mathrm{div},\Gamma)$ and $\mathbf{H}^{-\frac{1}{2},\frac{1}{2}}(\mathrm{curl}\,,\Gamma)$ are considered dual to each other in the duality defined in Lemma 6.1.1. In the next lemma we establish some mapping properties of the operator $\mathbf{L}(k)$ given by (6.25).

Lemma 6.1.2 *For a fixed k, the linear operator*

$$\mathbf{L}(k) : \mathbf{H}_t^{-\frac{1}{2}}(\Gamma) \times \mathbf{H}^{-\frac{3}{2},-\frac{1}{2}}(\mathrm{div},\Gamma) \to \mathbf{H}_t^{\frac{1}{2}}(\Gamma) \times \mathbf{H}^{-\frac{1}{2},\frac{1}{2}}(\mathrm{curl},\Gamma)$$

is bounded. Moreover, the family of operators $\mathbf{L}(k)$ depends analytically on $k \in \mathbb{C}\backslash\mathbb{R}_-$.

Proof. Let $\mathbf{E}, \mathbf{E}_0 \in \mathbf{L}^2(D)$, $\mathbf{E} - \mathbf{E}_0 \in \mathbf{H}_0(\mathrm{curl}^2, D)$ be a solution to the transmission eigenvalue problem (6.14)-(6.17). Hence

$$\mathbf{M} = \mathbf{E} \times \nu \in \mathbf{H}_t^{-\frac{1}{2}}(\Gamma), \quad \mathbf{J} = \mathrm{curl}\,\mathbf{E} \times \nu \in \mathbf{H}_t^{-\frac{3}{2}}(\Gamma).$$

Noting that $\mathrm{div}_\Gamma(\mathrm{curl}\,\mathbf{E} \times \nu) = \mathrm{curl}_\Gamma \mathrm{curl}\,\mathbf{E} = \mathrm{curl}^2\mathbf{E} \cdot \nu|_\Gamma$, we have that $\mathrm{div}_\Gamma \mathbf{J} \in \mathbf{H}_t^{-\frac{1}{2}}(\Gamma)$ and therefore $(\mathbf{M}, \mathbf{J}) \in \mathbf{H}_t^{-\frac{1}{2}}(\Gamma) \times \mathbf{H}^{-\frac{3}{2},-\frac{1}{2}}(\mathrm{div},\Gamma)$. It is known from [33] that

$\mathbf{S}_k$, $\mathbf{S}_{k_1} - \mathbf{S}_k$, $\mathbf{K}_{k_1} - \mathbf{K}_k$ are smoothing operators of order -1, -3 and -2 respectively. Then using (6.26) we have that the following operators are bounded

$$\begin{aligned} k_1\mathbf{T}_{k_1} - k\mathbf{T}_k \quad &: \quad \mathbf{H}_t^{-\frac{1}{2}}(\Gamma) \to \mathbf{H}_t^{\frac{1}{2}}(\Gamma) \\ \mathbf{K}_{k_1} - \mathbf{K}_k \quad &: \quad \mathbf{H}_t^{-\frac{3}{2}}(\Gamma) \to \mathbf{H}_t^{\frac{1}{2}}(\Gamma) \\ \tfrac{1}{k_1}\mathbf{T}_{k_1} - \tfrac{1}{k}\mathbf{T}_k \quad &: \quad \mathbf{H}^{-\frac{3}{2},-\frac{1}{2}}(\mathrm{div},\Gamma) \to \mathbf{H}_t^{-\frac{1}{2}}(\Gamma). \end{aligned}$$

Moreover

$$\begin{aligned} & \mathrm{curl}_\Gamma\left((\mathbf{K}_{k_1} - \mathbf{K}_k)\mathbf{M} + (\frac{1}{k_1}\mathbf{T}_{k_1} - \frac{1}{k}\mathbf{T}_k)\mathbf{J}\right) \\ = \quad & \mathrm{curl}_\Gamma(\mathbf{K}_{k_1} - \mathbf{K}_k)\mathbf{M} + \mathrm{curl}_\Gamma(\mathbf{S}_{k_1} - \mathbf{S}_k)\mathbf{J} \in \mathbf{H}_t^{\frac{1}{2}}(\Gamma), \end{aligned}$$

and hence

$$\begin{aligned} (k_1\mathbf{T}_{k_1} - k\mathbf{T}_k)\mathbf{M} + (\mathbf{K}_{k_1} - \mathbf{K}_k)\mathbf{J} \quad &\in \quad \mathbf{H}_t^{\frac{1}{2}}(\Gamma), \\ (\mathbf{K}_{k_1} - \mathbf{K}_k)\mathbf{M} + \left(\frac{1}{k_1}\mathbf{T}_{k_1} - \frac{1}{k}\mathbf{T}_k\right)\mathbf{J} \quad &\in \quad \mathbf{H}^{-\frac{1}{2},\frac{1}{2}}(\mathrm{curl},\Gamma), \end{aligned}$$

Hence $\mathbf{L}(k)$ is bounded. Note every component of $\mathbf{L}(k)$ is analytic on $\mathbb{C}\backslash\mathbb{R}_-$, then $\mathbf{L}(k)$ is analytic on $\mathbb{C}\backslash\mathbb{R}_-$ (recall that $k_1 = k\sqrt{n}$). $\square$

We need the following lemma to show the equivalence between the transmission eigenvalue problem and the system of integral equations (6.24).

Lemma 6.1.3 *Let Ω be any bounded open region in $\mathbb{R}^3$ and denote $\mathbf{V}(\mathrm{curl}^2,\Omega) := \{\mathbf{u} : \mathbf{u} \in \mathbf{L}^2(\Omega), \mathrm{curl}^2\mathbf{u} \in \mathbf{L}^2(\Omega)\}$. For $\varphi \in \mathbf{H}_t^{-\frac{1}{2}}(\Gamma)$, $\psi \in \mathbf{H}^{-\frac{3}{2},-\frac{1}{2}}(\mathrm{div},\Gamma)$, we define*

$$\tilde{\mathbf{M}}_1(\varphi)(\mathbf{x}) := \mathrm{curl}\int_\Gamma \Phi_k(\mathbf{x},\mathbf{y})\varphi(\mathbf{y})ds_y, \quad x \in \mathbb{R}^3\backslash\Gamma,$$

and

$$\tilde{\mathbf{M}}_2(\psi)(\mathbf{y}) := \int_\Gamma \Phi_k(\mathbf{x},\mathbf{y})\psi(\mathbf{y})ds_y, \quad x \in \mathbb{R}^3\backslash\Gamma.$$

Then $\tilde{\mathbf{M}}_1$ is continuous from $\mathbf{H}_t^{-\frac{1}{2}}(\Gamma)$ to $\mathbf{V}(\mathrm{curl}^2, D^\pm)$ and $\tilde{\mathbf{M}}_2$ is continuous from $\mathbf{H}^{-\frac{3}{2},-\frac{1}{2}}(\mathrm{div},\Gamma)$ to $\mathbf{V}(\mathrm{curl}^2, D^\pm)$ where $D^- = D$ and $D^+ = B_R\backslash\overline{D}$ with a sufficient *large ball B_R containing the closure of D. Furthermore the following jump relations hold*

$$[\gamma_t \tilde{\mathbf{M}}_1(\varphi)] = \varphi \quad in \quad \mathbf{H}_t^{-\frac{1}{2}}(\Gamma), \tag{6.27}$$

$$[\gamma_t \mathrm{curl} \tilde{\mathbf{M}}_1(\varphi)] = 0 \quad in \quad \mathbf{H}_t^{-\frac{3}{2}}(\Gamma), \tag{6.28}$$

$$[\gamma_t \mathrm{curl} \tilde{\mathbf{M}}_2(\psi)] = \psi \quad in \quad \mathbf{H}_t^{-\frac{3}{2}}(\Gamma), \tag{6.29}$$

$$[\mathrm{div}_\Gamma \gamma_t \mathrm{curl} \tilde{\mathbf{M}}_2(\psi)] = \mathrm{div}_\Gamma \psi \quad in \quad H^{-\frac{1}{2}}(\Gamma). \tag{6.30}$$

Proof. Let us denote by $<\ \cdot,\cdot\ >$ the $\mathbf{H}_t^{\frac{1}{2}}(\Gamma)$-$\mathbf{H}_t^{-\frac{1}{2}}(\Gamma)$ or $H^{\frac{1}{2}}(\Gamma)$-$H^{-\frac{1}{2}}(\Gamma)$ duality product. Since $\varphi \in \mathbf{H}_t^{-\frac{1}{2}}(\Gamma)$, then from the classical results for single layer potentials

$$\|\tilde{\mathbf{M}}_1(\varphi)\|_{\mathbf{L}^2(D^\pm)} \leq c \left\| \int_\Gamma \Phi_k(\mathbf{x},\mathbf{y})\varphi(\mathbf{y}) ds_y \right\|_{\mathbf{H}^1(D^\pm)} \leq c\|\varphi\|_{\mathbf{H}_t^{-\frac{1}{2}}(\Gamma)}$$

and since $\mathrm{curl}\,^2 \tilde{\mathbf{M}}_1(\varphi) - k^2 \tilde{\mathbf{M}}_1(\varphi) = 0$ in $D^\pm$, then

$$\|\mathrm{curl}\,^2 \tilde{\mathbf{M}}_1(\varphi)\|_{\mathbf{L}^2(D^\pm)} = |k^2| \|\tilde{\mathbf{M}}_1(\varphi)\|_{\mathbf{L}^2(D^\pm)} \leq c\|\varphi\|_{\mathbf{H}_t^{-\frac{1}{2}}(\Gamma)}$$

where c is some constant depending on k. For $\psi \in \mathbf{H}^{-\frac{3}{2},-\frac{1}{2}}(\mathrm{div},\Gamma)$, we have from [33]

$$\|\tilde{\mathbf{M}}_2(\psi)\|_{\mathbf{L}^2(D^\pm)} \leq c\|\psi\|_{\mathbf{H}_t^{-\frac{3}{2}}(\Gamma)}.$$

Notice that

$$\mathrm{curl}\,^2 \tilde{\mathbf{M}}_2(\psi)(\mathbf{x}) = k^2 \int_\Gamma \Phi_k(\mathbf{x},\mathbf{y})\psi(\mathbf{y}) ds_y + \nabla \int_\Gamma \mathrm{div}\,_\Gamma \psi(\mathbf{y}) \Phi_k(\cdot,\mathbf{y}) ds_y$$

and $\mathrm{div}\,_\Gamma \psi \in H^{-\frac{1}{2}}(\Gamma)$, hence we have from [33]

$$\|\mathrm{curl}\,^2 \tilde{\mathbf{M}}_2(\psi)\|_{\mathbf{L}^2(D^\pm)} \leq c \left(\|\psi\|_{\mathbf{H}_t^{-\frac{3}{2}}(\Gamma)} + \|\mathrm{div}\,_\Gamma \psi\|_{H^{-\frac{1}{2}}(\Gamma)} \right).$$

This proves the continuity property of $\tilde{\mathbf{M}}_1$ and $\tilde{\mathbf{M}}_2$. To prove the jump relations, we will use a density argument. Let

$$\mathbf{u}^\pm = \mathrm{curl} \int_\Gamma \Phi_k(\mathbf{x},\mathbf{y})\varphi(\mathbf{y}) ds_y \quad \text{in} \quad D^\pm.$$

We define the tangential component $\gamma_t \mathbf{u}^\pm$ by duality. For $\alpha \in \mathbf{H}_t^{\frac{1}{2}}(\Gamma)$, $\|\alpha\|_{\mathbf{H}_t^{\frac{1}{2}}(\Gamma)} = 1$, there exists $\mathbf{w}^\pm \in \mathbf{H}^2(D^\pm)$ and $\mathbf{w}^+$ compactly supported in B_R such that $\gamma_t \mathrm{curl}\, \mathbf{w} = \alpha, \gamma_t \mathbf{w} = 0$ and $\|\mathbf{w}\|_{\mathbf{H}^2(D^\pm)} \leq c\|\alpha\|_{\mathbf{H}_t^{\frac{1}{2}}(\Gamma)}$ (see [37]). Moreover,

$$< \alpha, \gamma_t \mathbf{u}^\pm >= \pm \int_{D^\pm} (\mathbf{u}^\pm \cdot \mathrm{curl}\,^2 \mathbf{w}^\pm - \mathbf{w}^\pm \cdot \mathrm{curl}\,^2 \mathbf{u}^\pm) d\mathbf{x}.$$

Then

$$\begin{aligned} | < \alpha, \gamma_t \mathbf{u}^\pm > | &\leq (\|\mathbf{u}\|_{\mathbf{L}^2(D^\pm)} + \|\mathrm{curl}\,^2 \mathbf{u}\|_{\mathbf{L}^2(D^\pm)}) \|\mathbf{w}\|_{\mathbf{H}^2(D^\pm)} \\ &\leq c_1 (\|\mathbf{u}\|_{\mathbf{L}^2(D^\pm)} + \|\mathrm{curl}\,^2 \mathbf{u}\|_{\mathbf{L}^2(D^\pm)}) \\ &\leq c_2 \|\varphi\|_{\mathbf{H}_t^{-\frac{1}{2}}(\Gamma)} \end{aligned}$$

where c_1 and c_2 are independent from u, therefore $\|\gamma_t \mathbf{u}^\pm\|_{\mathbf{H}_t^{-\frac{1}{2}}(\Gamma)} \leq c_2 \|\varphi\|_{\mathbf{H}_t^{-\frac{1}{2}}(\Gamma)}$. Choosing $\varphi_n \in \mathbf{H}^{-\frac{1}{2},-\frac{1}{2}}(\mathrm{div}, \Gamma)$ such that $\varphi_n \to \varphi$ in $\mathbf{H}_t^{-\frac{1}{2}}(\Gamma)$ yields

$$\|\gamma_t \mathbf{u}^\pm - \gamma_t \mathbf{u}_n^\pm\|_{\mathbf{H}_t^{-\frac{1}{2}}(\Gamma)} \leq c\|\varphi - \varphi_n\|_{\mathbf{H}_t^{-\frac{1}{2}}(\Gamma)} \to 0.$$

Since $[\gamma_t \mathbf{u}_n] = \varphi_{\mathbf{n}}$ for $\varphi_n \in \mathbf{H}^{-\frac{1}{2},-\frac{1}{2}}(\mathrm{div}, \Gamma)$(see [69]), letting $n \to \infty$ yields $[\gamma_t \mathbf{u}] = \varphi$ in $\mathbf{H}_t^{-\frac{1}{2}}(\Gamma)$, hence (6.27) holds. In a similar argument we can prove (6.28) (6.29). From (6.29) we have

$$[\gamma_t \mathrm{curl}\, \tilde{\mathbf{M}}_2(\psi)] = \psi \quad \text{in} \quad \mathbf{H}_t^{-\frac{3}{2}}(\Gamma).$$

Then

$$[\mathrm{div}_\Gamma \gamma_t \mathrm{curl}\, \tilde{\mathbf{M}}_2(\psi)] = \mathrm{div}_\Gamma \psi$$

in the distributional sense. Notice $\mathrm{div}_\Gamma \psi$ and $\left(\mathrm{div}_\Gamma \gamma_t \mathrm{curl}\, \tilde{\mathbf{M}}_2(\psi)\right)^\pm$ are in $H^{-\frac{1}{2}}(\Gamma)$, then (6.30) holds. □

Now we are ready to prove the equivalence between the transmission eigenvalue problem and the system of integral equations (6.24). Our proof follow the lines of the proof of Theorem 2.2 in [33].

Theorem 6.1.1 *The following statements are equivalent:*

(1) There exists non trivial $\mathbf{E}$, $\mathbf{E}_0 \in \mathbf{L}^2(D)$, $\mathbf{E} - \mathbf{E}_0 \in \mathbf{H}(\mathrm{curl}^2, D)$ such that (6.14)-(6.17) holds.

(2) There exists non trivial $(\mathbf{M},\mathbf{J}) \in \mathbf{H}_t^{-\frac{1}{2}}(\Gamma) \times \mathbf{H}^{-\frac{3}{2},-\frac{1}{2}}(\mathrm{div},\Gamma)$ *such that (6.24) holds and either* $\mathbf{E}_0^\infty(\mathbf{M},\mathbf{J}) = 0$ *or* $\mathbf{E}^\infty(\mathbf{M},\mathbf{J}) = 0$ *where*

$$\begin{aligned} \mathbf{E}_0^\infty(\mathbf{M},\mathbf{J})(\hat{x}) &= \hat{x} \times \left(\frac{1}{4\pi}\mathrm{curl} \int_\Gamma \mathbf{M}(y) e^{-ik\hat{x}\cdot y} ds_y \right. \\ &\left. + \frac{1}{4\pi k^2} \nabla \int_\Gamma \mathrm{div}_\Gamma \mathbf{J}(y) e^{-ik\hat{x}\cdot y} ds_y + \int_\Gamma \mathbf{J}(y) e^{-ik\hat{x}\cdot y} ds_y \right) \times \hat{x} \end{aligned} \tag{6.31}$$

with the same expression for $\mathbf{E}^\infty(\mathbf{M},\mathbf{J})$ *where* k *is replaced by* k_1*.*

Proof. Assume (1) holds, then from the argument above (6.24) we have that $\mathbf{M}$ and $\mathbf{J}$ satisfy (6.24) and hence it suffices to show $\mathbf{E}_0^\infty(\mathbf{M},\mathbf{J}) = 0$ and $\mathbf{E}^\infty(\mathbf{M},\mathbf{J}) = 0$. To this end, recall that $\mathbf{E}_0$ has the following representation

$$\begin{aligned} \mathbf{E}_0(x) &= \mathrm{curl} \int_\Gamma \mathbf{M}(y)\Phi_k(x,y) ds_y + \int_\Gamma \mathbf{J}(y)\Phi_k(\cdot,y) ds_y \\ &+ \frac{1}{k^2}\nabla \int_\Gamma \mathrm{div}_\Gamma \mathbf{J}(y)\Phi_k(\cdot,y) ds_y \end{aligned} \tag{6.32}$$

where $\mathbf{E}_0 \times \nu = \mathbf{E} \times \nu = \mathbf{M}$ and $\mathrm{curl}\,\mathbf{E}_0 \times \nu = \mathrm{curl}\,\mathbf{E} \times \nu = \mathbf{J}$. Then, from the jump relations (6.27)-(6.30) of the vector potentials applied to (6.32) and (6.24) (see also [33]), we obtain that $(\mathbf{E}_0 \times \nu)^+ = 0$, $(\mathrm{curl}\,\mathbf{E}_0 \times \nu)^+ = 0$ ($+$ denotes the traces from outside of D) and hence the far field pattern $\mathbf{E}_0^\infty(\mathbf{M},\mathbf{J})$ varnishes. The asymptotic expression of the fundamental solution $\Phi(\cdot,\cdot)$ in [24] yields (6.31). Similarly we can prove that $\mathbf{E}^\infty(\mathbf{M},\mathbf{J}) = 0$.

Next assume that (2) holds and define

$$\begin{aligned} \mathbf{E}_0(x) &= \mathrm{curl} \int_\Gamma \mathbf{M}(y)\Phi_k(x,y) ds_y + \int_\Gamma \mathbf{J}(y)\Phi_k(\cdot,y) ds_y \\ &+ \frac{1}{k^2}\nabla \int_\Gamma \mathrm{div}_\Gamma \mathbf{J}(y)\Phi_k(\cdot,y) ds_y \qquad x \in \mathbb{R}^3 \setminus \Gamma \end{aligned}$$

with same expression for $\mathbf{E}$ where k is replaced by k_1. Again from the jump relations of vector potentials and (6.24) we have

$$\begin{aligned} &\mathrm{curl}\,\mathrm{curl}\,\mathbf{E} - k^2 n\mathbf{E} = 0, \quad \mathrm{curl}\,\mathrm{curl}\,\mathbf{E}_0 - k^2\mathbf{E}_0 = 0 && \text{in} \quad D \\ &\mathbf{E} \times \nu = \mathbf{E}_0 \times \nu, \quad \mathrm{curl}\,\mathbf{E} \times \nu = \mathrm{curl}\,\mathbf{E}_0 \times \nu && \text{on} \quad \Gamma \end{aligned}$$

(note that E and E_0 are in $L^2(D)$. Hence it suffices to show $\mathbf{E}_0$ and $\mathbf{E}$ are non trivial. Assume to the contrary that $\mathbf{E}_0 = \mathbf{E} = 0$, and without loss of generality $\mathbf{E}^\infty(\mathbf{M}, \mathbf{J}) = 0$, then by Rellich's Lemma (see e.g. [24]) $\mathbf{E} = 0$ in $\mathbb{R}^3 \backslash \overline{D}$. Hence jump relations imply $\mathbf{M} = 0$ and $\mathbf{J} = 0$ which is a contradiction to the assumptions in (2). This proves the theorem. $\square$

The above discussion allows us to conclude that in order to prove the discreteness of transmission eigenvalues we need to show that the kernel of the operator $\mathbf{L}(k)$ is non-trivial for at most discrete set of wave numbers k. In the following, we will show the operator $\mathbf{L}(k)$ is Fredholm of index zero and use the analytic Fredholm theory to obtain our main theorem. To this end we first show that for purely complex wave number $k := i\kappa$, $\kappa > 0$, $\mathbf{L}(k)$ restricted to

$$\mathbf{H}_0^{-\frac{3}{2},-\frac{1}{2}}(\mathrm{div}, \Gamma) := \{\mathbf{u} \in \mathbf{H}^{-\frac{3}{2},-\frac{1}{2}}(\mathrm{div}, \Gamma),\ \mathrm{div}_\Gamma \mathbf{u} = 0\}$$

satisfies the coercive property. In the following lemma we use the shorthand notation $\mathbf{H}_0(\Gamma) := \mathbf{H}_t^{-\frac{1}{2}}(\Gamma) \times \mathbf{H}_0^{-\frac{3}{2},-\frac{1}{2}}(\mathrm{div}, \Gamma)$ and its dual space $\mathbf{H}^*(\Gamma) := \mathbf{H}_t^{\frac{1}{2}}(\Gamma) \times \left(\mathbf{H}_0^{-\frac{3}{2},-\frac{1}{2}}(\mathrm{div}, \Gamma)\right)'$ where the dual $\left(\mathbf{H}_0^{-\frac{3}{2},-\frac{1}{2}}(\mathrm{div}, \Gamma)\right)'$ of the subspace $\mathbf{H}_0^{-\frac{3}{2},-\frac{1}{2}}(\mathrm{div}, \Gamma) \subset \mathbf{H}^{-\frac{3}{2},-\frac{1}{2}}(\mathrm{div}, \Gamma)$ is understood in the sense of the duality defined by Lemma 6.1.1.

Lemma 6.1.4 *Let $\kappa > 0$. The operator $\mathbf{L}(i\kappa) : \mathbf{H}_0(\Gamma) \to \mathbf{H}^*(\Gamma)$ is strictly coercive, i.e.*

$$\left| \left\langle \mathbf{L}(i\kappa) \begin{pmatrix} \mathbf{M} \\ \mathbf{J} \end{pmatrix}, \begin{pmatrix} \mathbf{M} \\ \mathbf{J} \end{pmatrix} \right\rangle \right| \geq c \left(\|\mathbf{M}\|_{\mathbf{H}_t^{-\frac{1}{2}}(\Gamma)} + \|\mathbf{J}\|_{\mathbf{H}^{-\frac{3}{2},-\frac{1}{2}}(\mathrm{div},\Gamma)} \right),$$

where c is a constant depending only on κ.

Proof. We consider the following problem: for given $(\mathbf{M}, \mathbf{J}) \in \mathbf{H}_t^{-\frac{1}{2}}(\Gamma) \times \mathbf{H}_0^{-\frac{3}{2},-\frac{1}{2}}(\mathrm{div}, \Gamma)$ find $\mathbf{U} \in \mathbf{L}^2(\mathbb{R}^3)$, $\mathrm{curl}\, \mathbf{U} \in \mathbf{L}^2(\mathbb{R}^3)$, $\mathrm{curl}^2 \mathbf{U} \in \mathbf{L}^2(\mathbb{R}^3)$ such that

$$(\mathrm{curl}^2 + n\kappa^2)(\mathrm{curl}^2 + \kappa^2)\mathbf{U} = 0 \quad \text{in} \quad \mathbb{R}^3 \backslash \Gamma \tag{6.33}$$

$$[\nu \times \mathrm{curl}^2 \mathbf{U}] = (n\kappa^2 - \kappa^2)\mathbf{M} \quad \text{on} \quad \Gamma \tag{6.34}$$

$$[\nu \times \mathrm{curl}^3 \mathbf{U}] = (n\kappa^2 - \kappa^2)\mathbf{J} \quad \text{on} \quad \Gamma \tag{6.35}$$

where $[\cdot]$ denotes the jump across Γ. Multiplying (6.33) by a test function $\mathbf{W}$ and integrating by parts yield

$$\int_{\mathbb{R}^3\setminus\Gamma}(\operatorname{curl}^2+n\kappa^2)\mathbf{U}\cdot(\operatorname{curl}^2+\kappa^2)\overline{\mathbf{W}}dx$$
$$=(n\kappa^2-\kappa^2)\left(\int_\Gamma \gamma_\Gamma \operatorname{curl}\overline{\mathbf{W}}\cdot\mathbf{M}ds+\int_\Gamma\gamma_\Gamma\overline{\mathbf{W}}\cdot\mathbf{J}ds\right). \quad (6.36)$$

First we show that the right hand side is well defined. Note that $\operatorname{div}(\operatorname{curl}\mathbf{W})=0$, hence from [69] $\operatorname{curl}\mathbf{W}\in\mathbf{H}^1(\mathbb{R}^3)$ and and thus $\gamma_\Gamma\operatorname{curl}\mathbf{W}\in\mathbf{H}_t^{\frac{1}{2}}(\Gamma)$, which implies $\int_\Gamma\gamma_\Gamma\operatorname{curl}\overline{\mathbf{W}}\cdot\mathbf{M}ds$ is defined in $\mathbf{H}_t^{\frac{1}{2}}(\Gamma)$, $\mathbf{H}_t^{-\frac{1}{2}}(\Gamma)$ duality. Since $\gamma_\Gamma\mathbf{W}\in\mathbf{H}_t^{-\frac{1}{2}}(\Gamma)$ and $\operatorname{curl}_\Gamma\mathbf{W}=\gamma_\Gamma\operatorname{curl}\mathbf{W}\in\mathbf{H}_t^{\frac{1}{2}}(\Gamma)$ then from Lemma 6.1.1 $\int_\Gamma\gamma_\Gamma\overline{\mathbf{W}}\cdot\mathbf{J}ds$ is well defined. Now let

$$\mathbf{V}:=\{\mathbf{U}\in\mathbf{L}^2(\mathbb{R}^3),\operatorname{curl}\mathbf{U}\in\mathbf{L}^2(\mathbb{R}^3),\operatorname{curl}^2\mathbf{U}\in\mathbf{L}^2(\mathbb{R}^3)\}$$

equipped with the norm

$$\|\mathbf{U}\|^2_{\mathbf{V}}=\int_{\mathbb{R}^3}(|\operatorname{curl}^2\mathbf{U}|^2+|\operatorname{curl}\mathbf{U}|^2+|\mathbf{U}|^2)dx.$$

Next taking $\mathbf{W}=\mathbf{U}$ in the continuous sesquilinear form in the left-hand side of (6.36), and after integrating by parts (note that $\mathbf{U}$ and $\operatorname{curl}\mathbf{U}$ are continuous across Γ, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^3\setminus\Gamma}(\operatorname{curl}^2+n\kappa^2)\mathbf{U}\cdot(\operatorname{curl}^2+\kappa^2)\overline{\mathbf{U}}dx\\ =\quad&\int_{\mathbb{R}^3}(|\operatorname{curl}^2\mathbf{U}|^2+(n\kappa^2+\kappa^2)|\operatorname{curl}\mathbf{U}|^2+n\kappa^2\kappa^2|\mathbf{U}|^2)dx\geq c\|\mathbf{U}\|_{\mathbf{V}}\end{aligned}$$

where c is a constant depending on κ. The Lax-Milgram lemma guaranties the existence of a unique solution to (6.36). Up to here we did not need that $\operatorname{div}_\Gamma\mathbf{J}=0$. Next we define

$$\begin{aligned}\mathbf{U}&=\operatorname{curl}\int_\Gamma\mathbf{M}(y)(\Phi_{\sqrt{n}\kappa}(\cdot,y)-\Phi_\kappa(\cdot,y))ds+\int_\Gamma\mathbf{J}(y)(\Phi_{\sqrt{n}\kappa}(\cdot,y)-\Phi_\kappa(\cdot,y))ds\\ &+\frac{1}{(i\sqrt{n}\kappa)^2}\nabla\int_\Gamma\operatorname{div}_\Gamma\mathbf{J}(y)\Phi_{\sqrt{n}\kappa}(\cdot,y)ds-\frac{1}{(i\kappa)^2}\nabla\int_\Gamma\operatorname{div}_\Gamma\mathbf{J}(y)\Phi_\kappa(\cdot,y)ds.\end{aligned}$$

Then $\mathbf{U}\in\mathbf{L}^2(\mathbb{R}^3)$, $\operatorname{curl}\mathbf{U}\in\mathbf{L}^2(\mathbb{R}^3)$, $\operatorname{curl}^2\mathbf{U}\in\mathbf{L}^2(\mathbb{R}^3)$ and $\mathbf{U}$ satisfies (6.33)-(6.35), hence $\mathbf{U}$ defined above is the unique solution to (6.36). Now for a given $\gamma_\Gamma\operatorname{curl}\mathbf{W}\in$

$\mathbf{H}^{\frac{1}{2}}(\Gamma)$, let us construct a lifting function $\tilde{\mathbf{W}} \in \mathbf{H}^2(\mathbb{R}^3)$ [37] such that $\gamma_\Gamma \mathrm{curl}\, \tilde{\mathbf{W}} = \gamma_\Gamma \mathrm{curl}\, \mathbf{W}$, $\gamma_\Gamma \tilde{\mathbf{W}} = 0$ and $\|\tilde{\mathbf{W}}\|_{\mathbf{H}^2(\mathbb{R}^3)} \leq c\|\gamma_\Gamma \mathrm{curl}\, \tilde{\mathbf{W}}\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}$ for some constant c. Then

$$
\begin{aligned}
\left|\int_\Gamma \gamma_\Gamma \mathrm{curl}\, \mathbf{W} \cdot \mathbf{M} ds\right| &= \left|\int_\Gamma \gamma_\Gamma \mathrm{curl}\, \tilde{\mathbf{W}} \cdot \mathbf{M} ds\right| \\
&= \frac{1}{|n\kappa^2 - \kappa^2|}\left|\int_{\mathbb{R}^3\setminus\Gamma} (\mathrm{curl}^{\,2} + n\kappa^2)\mathbf{U} \cdot (\mathrm{curl}^{\,2} + \kappa^2)\tilde{\mathbf{W}} dx\right| \\
&\leq \|\mathbf{U}\|_{\mathbf{V}} \|\tilde{\mathbf{W}}\|_{\mathbf{V}} \\
&\leq c\|\mathbf{U}\|_{\mathbf{V}} \|\gamma_\Gamma \mathrm{curl}\, \tilde{\mathbf{W}}\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}.
\end{aligned}
$$

Hence $\|\mathbf{M}\|_{\mathbf{H}_t^{-\frac{1}{2}}(\Gamma)} \leq c\|\mathbf{U}\|_{\mathbf{V}}$. Similarly for given $\gamma_\Gamma \mathbf{W} \in \mathbf{H}^{\frac{3}{2}}(\Gamma)$ we construct the lifting $\tilde{\mathbf{W}}_2 \in \mathbf{H}^2(\mathbb{R}^3)$ [37] such that $\gamma_\Gamma \tilde{\mathbf{W}}_2 = \gamma_\Gamma \mathbf{W}$, $\gamma_\Gamma \mathrm{curl}\, \tilde{\mathbf{W}}_2 = 0$ and $\|\tilde{\mathbf{W}}_2\|_{\mathbf{H}^2(\mathbb{R}^3)} \leq c\|\gamma_T \tilde{\mathbf{W}}_2\|_{\mathbf{H}^{\frac{3}{2}}(\Gamma)}$ for some constant c. We recall that $\mathrm{div}_\Gamma \mathbf{J} = 0$ hence from the Helmoltz decomposition $\mathbf{J} = \overrightarrow{\mathrm{curl}}_\Gamma q \in \mathbf{H}^{-\frac{3}{2}}(\Gamma)$. Thus we have

$$
\begin{aligned}
\left|\int_\Gamma \gamma_\Gamma \mathbf{W} \cdot \mathbf{J} ds\right| &= \left|\int_\Gamma \gamma_\Gamma \tilde{\mathbf{W}}_2 \cdot \mathbf{J} ds\right| \\
&= \frac{1}{|n\kappa^2 - \kappa^2|}\left|\int_{\mathbb{R}^3\setminus\Gamma} (\mathrm{curl}^{\,2} + n\kappa^2)\mathbf{U} \cdot (\mathrm{curl}^{\,2} + \kappa^2)\tilde{\mathbf{W}}_2 dx\right| \\
&\leq c\|\mathbf{U}\|_{\mathbf{V}} \|\tilde{\mathbf{W}}_2\|_{\mathbf{V}} \\
&\leq c\|\mathbf{U}\|_{\mathbf{V}} \|\gamma_T \mathbf{W}\|_{\mathbf{H}^{\frac{3}{2}}(\Gamma)}.
\end{aligned}
$$

Since $\mathbf{J} = \overrightarrow{\mathrm{curl}}_\Gamma q \in \mathbf{H}^{-\frac{3}{2}}(\Gamma)$, then by duality $\|\mathbf{J}\|_{\mathbf{H}_0^{-\frac{3}{2},-\frac{1}{2}}(\mathrm{div},\Gamma)} \leq c\|\mathbf{U}\|_{\mathbf{V}}$. Finally

$$
\begin{aligned}
&\left|\left\langle \mathbf{L}(i\kappa)\begin{pmatrix}\mathbf{M}\\ \mathbf{J}\end{pmatrix}, \begin{pmatrix}\mathbf{M}\\ \mathbf{J}\end{pmatrix}\right\rangle\right| \\
&\quad = \left\|\int_\Gamma \gamma_\Gamma \mathrm{curl}\, \mathbf{W} \cdot \overline{\mathbf{M}} ds + \int_\Gamma \gamma_\Gamma \mathbf{W} \cdot \overline{\mathbf{J}} ds\right\| \\
&\quad \geq c\|\mathbf{U}\|_{\mathbf{V}} \geq c\left(\|\mathbf{M}\|_{\mathbf{H}_t^{-\frac{1}{2}}(\Gamma)} + \|\mathbf{J}\|_{\mathbf{H}^{-\frac{3}{2},-\frac{1}{2}}(\mathrm{div},\Gamma)}\right).
\end{aligned}
$$

where c is a constant depending on κ. This proves our lemma. □

Next we proceed with the following lemma.

Lemma 6.1.5 *Let* $\gamma(k) := \frac{k_1^2 - k^2}{|k_1|^2 - |k|^2}$ *and* $k_1 = k\sqrt{n}$ *for* $k \in \mathbb{C} \setminus \mathbb{R}_-$. *Then* $\mathbf{L}(k) + \gamma(k)\mathbf{L}(i|k|) : \mathbf{H}_0(\Gamma) \to \mathbf{H}^*(\Gamma)$ *is compact.*

Proof. From [33], Theorem 3.8 the operator

$$(\mathbf{S}_{k_1} - \mathbf{S}_k) + \gamma(k)(\mathbf{S}_{i|k_1|} - \mathbf{S}_{i|k|}) : \mathbf{H}^{-\frac{3}{2}}(\Gamma) \to \mathbf{H}^{\frac{3}{2}}(\Gamma)$$

is compact. Then from (6.26) we have

$$\nabla_\Gamma \circ (S_{k_1} - S_k) \circ \operatorname{div}_\Gamma + \gamma(k)\nabla_\Gamma \circ (S_{i|k_1|} - S_{i|k|}) \circ \operatorname{div}_\Gamma : \mathbf{H}^{-\frac{1}{2}}(\Gamma) \to \mathbf{H}^{\frac{1}{2}}(\Gamma)$$

$$(\mathbf{K}_{k_1} - \mathbf{K}_k) + \gamma(k)(\mathbf{K}_{i|k_1|} - \mathbf{K}_{i|k|}) : \mathbf{H}^{-\frac{3}{2}}(\Gamma) \to \mathbf{H}^{\frac{1}{2}}(\Gamma)$$

$$(\mathbf{K}_{k_1} - \mathbf{K}_k) + \gamma(k)(\mathbf{K}_{i|k_1|} - \mathbf{K}_{i|k|}) : \mathbf{H}^{-\frac{1}{2}}(\Gamma) \to \mathbf{H}^{\frac{3}{2}}(\Gamma)$$

$$\left(\frac{1}{k_1}\mathbf{K}_{k_1} - \frac{1}{k}\mathbf{K}_k\right) + \gamma(k)\left(\frac{1}{i|k_1|}\mathbf{K}_{i|k_1|} - \frac{1}{i|k|}\mathbf{K}_{i|k|}\right) : \mathbf{H}^{-\frac{3}{2}}(\Gamma) \to \mathbf{H}^{\frac{3}{2}}(\Gamma)$$

are compact. It remains to show that

$$({k_1}^2\mathbf{S}_{k_1} - k^2\mathbf{S}_k) + \gamma(k)((i|k_1|)^2\mathbf{S}_{i|k_1|} - (i|k|)^2\mathbf{S}_{i|k|}) : \mathbf{H}^{-\frac{1}{2}}(\Gamma) \to \mathbf{H}^{\frac{1}{2}}(\Gamma)$$

is compact. Since

$$\begin{aligned} & ({k_1}^2\mathbf{S}_{k_1} - k^2\mathbf{S}_k) + \gamma(k)((i|k_1|)^2\mathbf{S}_{i|k_1|} - (i|k|)^2\mathbf{S}_{i|k|}) \\ = \quad & ({k_1}^2(\mathbf{S}_{k_1} - \mathbf{S}_0) - k^2(\mathbf{S}_k - \mathbf{S}_0)) + \gamma(k)((i|k_1|)^2(\mathbf{S}_{i|k_1|} - \mathbf{S}_0) - (i|k|)^2(\mathbf{S}_{i|k|} - \mathbf{S}_0)) \end{aligned}$$

and $\mathbf{S}_k - \mathbf{S}_0$ is compact, then the compactness follows. Hence the proof of the lemma is completed. □

In order to handle the non divergence free part of $\mathbf{J}$, we will split $\mathbf{J} := \mathbf{Q} + \mathbf{P}$ where $\mathbf{Q} \in \mathbf{H}_0^{-\frac{3}{2},-\frac{1}{2}}(\operatorname{div}, \Gamma)$, $\mathbf{P} = \nabla_\Gamma p \in \mathbf{H}_t^{\frac{1}{2}}(\Gamma)$ and rewrite the equation (6.24) for the unknowns $(\mathbf{M}, \mathbf{Q}, \mathbf{P})$. To this end let us define

$$\mathbf{H}_1(\Gamma) := \left\{\mathbf{P} \in \mathbf{H}_t^{\frac{1}{2}}(\Gamma), \operatorname{curl}_\Gamma \mathbf{P} = 0\right\}$$

and introduce the operator

$$\tilde{\mathbf{L}}(k) = \begin{pmatrix} k_1\mathbf{T}_{k_1} - k\mathbf{T}_k & \mathbf{K}_{k_1} - \mathbf{K}_k & \mathbf{K}_{k_1} - \mathbf{K}_k \\ \mathbf{K}_{k_1} - \mathbf{K}_k & \mathbf{S}_{k_1} - \mathbf{S}_k & \mathbf{S}_{k_1} - \mathbf{S}_k \\ \mathbf{K}_{k_1} - \mathbf{K}_k & \mathbf{S}_{k_1} - \mathbf{S}_k & (\mathbf{S}_{k_1} - \mathbf{S}_k) + \nabla_\Gamma \circ (\frac{1}{k_1^2}\mathbf{S}_{k_1} - \frac{1}{k^2}\mathbf{S}_k) \circ \operatorname{div}_\Gamma \end{pmatrix}. \tag{6.37}$$

From from Lemma 6.1.1 and Lemma 6.1.2 $\tilde{\mathbf{L}}(k) : \mathbf{H}_0(\Gamma) \times \mathbf{H}_1(\Gamma) \to \mathbf{H}^*(\Gamma) \times \mathbf{H}^{-\frac{1}{2}}(\Gamma)$ is bounded and furthermore the family of operators $\tilde{\mathbf{L}}(k)$ depends analytically on $k \in \mathbb{C}\backslash\mathbb{R}_-$, where recall $\mathbf{H}_0(\Gamma) := \mathbf{H}_t^{-\frac{1}{2}}(\Gamma) \times \mathbf{H}_0^{-\frac{3}{2},-\frac{1}{2}}(\mathrm{div},\Gamma)$ with its dual $\mathbf{H}^*(\Gamma)$. We first notice that (6.24) is equivalent to the following:

$$\left\langle \mathbf{L}(k) \begin{pmatrix} \mathbf{M} \\ \mathbf{J} \end{pmatrix}, \begin{pmatrix} \tilde{\mathbf{M}} \\ \tilde{\mathbf{J}} \end{pmatrix} \right\rangle = 0$$

for any $(\tilde{\mathbf{M}}, \tilde{\mathbf{J}}) \in \mathbf{H}_t^{\frac{1}{2}}(\Gamma) \times \mathbf{H}^{-\frac{1}{2},\frac{1}{2}}(\mathrm{curl},\Gamma)$ which equivalently can be written as

$$\left\langle \tilde{\mathbf{L}}(k) \begin{pmatrix} \mathbf{M} \\ \mathbf{Q} \\ \mathbf{P} \end{pmatrix}, \begin{pmatrix} \tilde{\mathbf{M}} \\ \tilde{\mathbf{Q}} \\ \tilde{\mathbf{P}} \end{pmatrix} \right\rangle = 0$$

for any $(\tilde{\mathbf{M}}, \tilde{\mathbf{Q}}, \tilde{\mathbf{P}}) \in \mathbf{H}^* \times \mathbf{H}_t^{-\frac{1}{2}}(\Gamma)$. Now we are ready to prove the following lemma.

Lemma 6.1.6 *The operator $\tilde{\mathbf{L}}(k) : \mathbf{H}_0(\Gamma) \times \mathbf{H}_1(\Gamma) \to \mathbf{H}^*(\Gamma) \times \mathbf{H}^{-\frac{1}{2}}(\Gamma)$ is a Fredholm with index zero, i.e. it can be written as a sum of an invertible operator and a compact operator.*

Proof. We rewrite the operator $\tilde{\mathbf{L}}(k)$ as follows

$$\begin{aligned}
\tilde{\mathbf{L}}(k) = &- \begin{pmatrix} \gamma(k)(i|k_1|\mathbf{T}_{i|k_1|} - i|k|\mathbf{T}_{i|k|}) & \gamma(k)(\mathbf{K}_{i|k_1|} - \mathbf{K}_{i|k|}) & 0 \\ \gamma(k)(\mathbf{K}_{i|k_1|} - \mathbf{K}_{i|k|}) & \gamma(k)(\mathbf{S}_{i|k_1|} - \mathbf{S}_{i|k|}) & 0 \\ 0 & 0 & \nabla_\Gamma \circ (-\frac{1}{k_1^2} + \frac{1}{k^2})\mathbf{S}_0 \circ \mathrm{div}_\Gamma \end{pmatrix} \\
&+ \begin{pmatrix} \gamma(k)\left(i|k_1|\mathbf{T}_{i|k_1|} - i|k|\mathbf{T}_{i|k|}\right) & \gamma(k)\left(\mathbf{K}_{i|k_1|} - \mathbf{K}_{i|k|}\right) & 0 \\ \gamma(k)\left(\mathbf{K}_{i|k_1|} - \mathbf{K}_{i|k|}\right) & \gamma(k)\left(\mathbf{S}_{i|k_1|} - \mathbf{S}_{i|k|}\right) & 0 \\ 0 & 0 & \nabla_\Gamma \circ (-\frac{1}{k_1^2} + \frac{1}{k^2})\mathbf{S}_0 \circ \mathrm{div}_\Gamma \end{pmatrix} \\
&+ \begin{pmatrix} k_1\mathbf{T}_{k_1} - k\mathbf{T}_k & \mathbf{K}_{k_1} - \mathbf{K}_k & \mathbf{K}_{k_1} - \mathbf{K}_k \\ \mathbf{K}_{k_1} - \mathbf{K}_k & \mathbf{S}_{k_1} - \mathbf{S}_k & \mathbf{S}_{k_1} - \mathbf{S}_k \\ \mathbf{K}_{k_1} - \mathbf{K}_k & \mathbf{S}_{k_1} - \mathbf{S}_k & (\mathbf{S}_{k_1} - \mathbf{S}_k) + \nabla_\Gamma \circ (\frac{1}{k_1^2}\mathbf{S}_{k_1} - \frac{1}{k^2}\mathbf{S}_k) \circ \mathrm{div}_\Gamma \end{pmatrix} \\
=&: \tilde{\mathbf{L}}_1(k) + \tilde{\mathbf{L}}_2(k)
\end{aligned} \tag{6.38}$$

where $\tilde{\mathbf{L}}_1(k)$ is the first operator and $\tilde{\mathbf{L}}_2(k)$ is the sum of the last two operators. Then from Lemma 6.1.5 and $\mathbf{S}_{k_1} - \mathbf{S}_k$,$\mathbf{K}_{k_1} - \mathbf{K}_k$ are smoothing operators of order 3,2 respectively, we have $\tilde{\mathbf{L}}_2(k)$ are compact. From Lemma 6.1.4 and $\mathbf{S}_0$ is invertible, whence we have $\tilde{\mathbf{L}}_1(k)$ is invertible. This proves our lemma. $\square$

6.2 The Case When $N - I$ Changes Sign

In this section we will discuss the Fredholm properties of $\mathbf{L}(k)$ when N is not a constant any longer. Our approach to handle the more general case follows exactly the lines of the discussion in Section 4 of [33], and here for sake of reader's convenience we sketch the main steps of the analysis. To begin with, we assume that $D = \overline{D}_1 \cup \overline{D}_2$ such that $D_1 \subset D$ and $D_2 := D \setminus \overline{D}_1$ and consider the simple case when $N = n_2 I$ in D_2 and $N = n_1 I$ in D_1 where $n_1 > 0$, $n_2 > 0$ are two positive constants such that $(n_1 - 1)(n_2 - 1) < 0$. Let $\Gamma = \partial D$, $\Sigma = \partial D_1$ which are assumed to be C^2 smooth surfaces and ν denotes the unit normal vector to either Γ or Σ outward to D and D_1 respectively (see Figure 6.1). Let us recall the notations $k_1 = k\sqrt{n_1}$ and $k_2 = k\sqrt{n_2}$. For convenience, we denote $\mathbf{K}_k^{\Sigma,\Gamma}$ and $\mathbf{T}_k^{\Sigma,\Gamma}$ be the potential $\mathbf{K}_k$ and $\mathbf{T}_k$

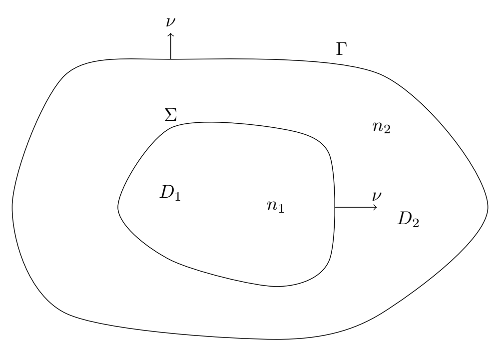


Figure 6.1: Configuration of the geometry for two constants

given by (6.18) and (6.19) for densities defined on Σ and evaluated on Γ. The solution of the transmission eigenvalue problem (6.10)-(6.13) by means of the Stratton-Chu formula can be represented as

$$\begin{aligned}\mathbf{E}_0(x) &= \operatorname{curl}\int_\Gamma(\mathbf{E}_0\times\nu)(\mathbf{y})\Phi_k(x,y)ds_y+\int_\Gamma(\operatorname{curl}\mathbf{E}_0\times\nu)(\mathbf{y})\Phi_k(\cdot,\mathbf{y})ds_y\\ &+ \frac{1}{k^2}\nabla\int_\Gamma \operatorname{div}_T(\operatorname{curl}\mathbf{E}_0\times\nu)(\mathbf{y})\Phi_k(\cdot,\mathbf{y})ds_y \quad \text{in} \quad D \end{aligned}\tag{6.39}$$

$$\begin{aligned}\mathbf{E}(x) &= \operatorname{curl}\int_\Sigma(\mathbf{E}\times\nu)(\mathbf{y})\Phi_{k_1}(x,y)ds_y+\int_\Sigma(\operatorname{curl}\mathbf{E}\times\nu)(\mathbf{y})\Phi_{k_1}(\cdot,\mathbf{y})ds_y\\ &+ \frac{1}{k_1^2}\nabla\int_\Sigma \operatorname{div}_T(\operatorname{curl}\mathbf{E}\times\nu)(\mathbf{y})\Phi_{k_1}(\cdot,\mathbf{y})ds_y \quad \text{in} \quad D_1 \end{aligned}\tag{6.40}$$

$$\begin{aligned}\mathbf{E}(x) &= \operatorname{curl}\int_\Gamma(\mathbf{E}\times\nu)(\mathbf{y})\Phi_{k_2}(x,y)ds_y+\int_\Gamma(\operatorname{curl}\mathbf{E}\times\nu)(\mathbf{y})\Phi_{k_2}(\cdot,\mathbf{y})ds_y\\ &+ \frac{1}{k_2^2}\nabla\int_\Gamma \operatorname{div}_T(\operatorname{curl}\mathbf{E}\times\nu)(\mathbf{y})\Phi_{k_2}(\cdot,\mathbf{y})ds_y\\ &- \operatorname{curl}\int_\Sigma(\mathbf{E}\times\nu)(\mathbf{y})\Phi_{k_2}(x,y)ds_y-\int_\Sigma(\operatorname{curl}\mathbf{E}\times\nu)(\mathbf{y})\Phi_{k_2}(\cdot,\mathbf{y})ds_y\\ &- \frac{1}{k_2^2}\nabla\int_\Sigma \operatorname{div}_T(\operatorname{curl}\mathbf{E}\times\nu)(\mathbf{y})\Phi_{k_2}(\cdot,\mathbf{y})ds_y \quad \text{in} \quad D_2 \end{aligned}\tag{6.41}$$

Let $\mathbf{E}\times\nu=\mathbf{E}_0\times\nu=\mathbf{M}$, $\operatorname{curl}\mathbf{E}\times\nu=\operatorname{curl}\mathbf{E}_0\times\nu=\mathbf{J}$ on Γ and $\mathbf{E}\times\nu=\mathbf{M}'$, $\operatorname{curl}\mathbf{E}\times\nu=\mathbf{J}'$ on Σ. From the jump relations of the boundary integral operators across Γ and Σ, we have that

$$\begin{pmatrix} k_2\mathbf{T}^\Gamma_{k_2}-k\mathbf{T}^\Gamma_k & \mathbf{K}^\Gamma_{k_2}-\mathbf{K}^\Gamma_k\\ \mathbf{K}^\Gamma_{k_2}-\mathbf{K}^\Gamma_k & \frac{1}{k_2}\mathbf{T}^\Gamma_{k_2}-\frac{1}{k}\mathbf{T}^\Gamma_k\end{pmatrix}\begin{pmatrix}\mathbf{M}\\ \mathbf{J}\end{pmatrix}=\begin{pmatrix} k_2\mathbf{T}^{\Sigma,\Gamma}_{k_2} & \mathbf{K}^{\Sigma,\Gamma}_{k_2}\\ \mathbf{K}^{\Sigma,\Gamma}_{k_2} & \frac{1}{k_2}\mathbf{T}^{\Sigma,\Gamma}_{k_2}\end{pmatrix}\begin{pmatrix}\mathbf{M}'\\ \mathbf{J}'\end{pmatrix}\tag{6.42}$$

$$\begin{pmatrix} k_2\mathbf{T}^\Sigma_{k_2}+k_1\mathbf{T}^\Sigma_{k_1} & \mathbf{K}^\Sigma_{k_2}+\mathbf{K}^\Sigma_{k_1}\\ \mathbf{K}^\Sigma_{k_2}+\mathbf{K}^\Sigma_{k_1} & \frac{1}{k_2}\mathbf{T}^\Sigma_{k_2}+\frac{1}{k_1}\mathbf{T}^\Sigma_{k_1}\end{pmatrix}\begin{pmatrix}\mathbf{M}'\\ \mathbf{J}'\end{pmatrix}=\begin{pmatrix} k_2\mathbf{T}^{\Gamma,\Sigma}_{k_2} & \mathbf{K}^{\Gamma,\Sigma}_{k_2}\\ \mathbf{K}^{\Gamma,\Sigma}_{k_2} & \frac{1}{k_2}\mathbf{T}^{\Gamma,\Sigma}_{k_2}\end{pmatrix}\begin{pmatrix}\mathbf{M}\\ \mathbf{J}\end{pmatrix}\tag{6.43}$$

Let us denote by $\mathbf{L}_{20}(k)$, $\mathbf{L}^{\Sigma,\Gamma}(k)$, $\mathbf{L}_{21}(k)$, $\mathbf{L}^{\Gamma,\Sigma}(k)$ the matrix-valued operators in the above two equations in the order from the left to the right from the top to the bottom, respectively. By the regularity of the solution of the Maxwell's equations inside D_2 (see e.g. [54]), we have $(\mathbf{M}', \mathbf{J}') \in \mathbf{H}_t^{-\frac{1}{2}}(\Sigma, \mathrm{div}\,) \times \mathbf{H}_t^{-\frac{1}{2}}(\Sigma, \mathrm{div}\,)$. Then the equation

$$\mathbf{L}_{21}(k)\begin{pmatrix}\mathbf{M}'\\ \mathbf{J}'\end{pmatrix} = \begin{pmatrix}\mathbf{g}\\ \mathbf{h}\end{pmatrix}$$

where $(\mathbf{g}, \mathbf{h}) \in \mathbf{H}_\mathbf{t}^{-\frac{1}{2}}(\boldsymbol{\Sigma}, \mathrm{div}\,) \times \mathbf{H}_\mathbf{t}^{-\frac{1}{2}}(\boldsymbol{\Sigma}, \mathrm{div}\,)$ corresponds to the transmission problem which is to find $(\mathbf{E}_2, \mathbf{E}_1) \in \mathbf{H}_{loc}(\mathrm{curl}\,, \mathbb{R}^3\backslash\overline{D_1}) \times \mathbf{H}(\mathrm{curl}\,, D_1)$ and $\mathbf{E}_2$ such that

$$\begin{aligned}
\mathrm{curl}\,\mathrm{curl}\,\mathbf{E}_2 - k_2^2\mathbf{E}_2 = 0 &\quad \text{in} \quad \mathbb{R}^3\backslash\overline{D_1}\\
\mathrm{curl}\,\mathrm{curl}\,\mathbf{E}_1 - k_1^2\mathbf{E}_1 = 0 &\quad \text{in} \quad D_1\\
\nu \times \mathbf{E}_2 - \nu \times \mathbf{E}_1 = \mathbf{g} &\quad \text{on} \quad \Sigma\\
\nu \times (\mathrm{curl}\,\mathbf{E}_2) - \nu \times (\mathrm{curl}\,\mathbf{E}_1) = \mathbf{h} &\quad \text{on} \quad \Sigma
\end{aligned}$$

and $\mathbf{E}_2$ satisfies the Silver-Mueller radiation condition. By well-posedeness of the transmission problem we have $\mathbf{L}_{21}(k)$ is invertible (for real valued k). Hence pugging in (6.42) $\mathbf{M}'$ and $\mathbf{J}'$ from (6.43) we obtain the following equation for $\mathbf{M}$ and $\mathbf{J}$

$$\mathbf{L}(k)\begin{pmatrix}\mathbf{M}\\ \mathbf{J}\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \tag{6.44}$$

where $\mathbf{L}(k) := \mathbf{L}_{20}(k) - \mathbf{L}^{\Sigma,\Gamma}(k)\mathbf{L}_{21}(k)^{-1}\mathbf{L}^{\Gamma,\Sigma}(k)$. Then in a similar way to Theorem 6.1.1, we can prove the following theorem.

Theorem 6.2.1 *The following statements are equivalent:*

(1) There exist non trivial $\mathbf{E} \in L^2(D)$ *and* $\mathbf{E}_0 \in L^2(D)$ *such that* $\mathbf{E}-\mathbf{E}_0 \in \mathbf{H}(\mathrm{curl}^2, D)$ *and (6.14)-(6.17) holds.*

(2) There exists non trivial $(\mathbf{M}, \mathbf{J}) \in \mathbf{H}_t^{-\frac{1}{2}}(\Gamma) \times \mathbf{H}^{-\frac{3}{2},-\frac{1}{2}}(\mathrm{div}, \Gamma)$ *such that (6.44) holds and* $\mathbf{E}_0^\infty(\mathbf{M}, \mathbf{J}) = 0$ *where*

$$\begin{aligned}
\mathbf{E}_0^\infty(\mathbf{M}, \mathbf{J})(\hat{x}) \;&=\; \hat{x} \times \left(\frac{1}{4\pi}\mathrm{curl}\int_\Gamma \mathbf{M}(y)e^{-ik\hat{x}\cdot y}ds_y\right.\\
&+\; \left.\frac{1}{4\pi k^2}\nabla\int_\Gamma \mathrm{div}_\Gamma\mathbf{J}(y)e^{-ik\hat{x}\cdot y}ds_y + \int_\Gamma \mathbf{J}(y)e^{-ik\hat{x}\cdot y}ds_y\right) \times \hat{x}
\end{aligned}$$

Now we note Σ and Γ are two disjoint curves, we have $\mathbf{L}^{\Sigma,\Gamma}(k)$, $\mathbf{L}^{\Gamma,\Sigma}(k)$ are compact. By writing $\mathbf{L}(k)$ as a 3×3 matrix operator $\tilde{\mathbf{L}}(k)$ similar to (6.38), we can prove the following lemma.

Lemma 6.2.1 *The operator $\tilde{\mathbf{L}}(k) : \mathbf{H}_0(\Gamma) \times \mathbf{H}_1(\Gamma) \to \mathbf{H}^*(\Gamma) \times \mathbf{H}^{-\frac{1}{2}}(\Gamma)$ is a Fredholm with index zero, i.e. it can be written as a sum of an invertible operator and a compact operator. Furthermore the family of the operators $\tilde{\mathbf{L}}(k)$ depends analytically on k in a neighborhood of the real axis.*

Proof. From Lemma 6.1.6, it is sufficient to show $\tilde{\mathbf{L}}(k)$ is analytic in a neighborhood of the real axis. Since

$$\mathbf{L}(k) = \mathbf{L}_{20}(k) - \mathbf{L}^{\Sigma,\Gamma}(k)\mathbf{L}_{21}(k)^{-1}\mathbf{L}^{\Gamma,\Sigma}(k),$$

it is sufficient to show $\mathbf{L}_{21}(k)$ is invertible in a neighborhood of the real axis. Note that $\mathbf{L}_{21}(k)$ is invertible in the closed upper half complex plane $\Im k \geq 0$, then we have that $\mathbf{L}_{21}(k)$ is analytic in a neighborhood of the closed upper half complex plane $\Im k \geq 0$. Now there exists a neighborhood of the real axis that $\tilde{\mathbf{L}}(k)$ is analytic. This proves the Lemma. $\square$

This approach can be readily generalized to the case when the medium consists of finitely many homogeneous layers.

In a more general case where $N = n(x)I$ in D_1, where $n \in L^\infty(D_1)$ such that $n(x) \geq \alpha > 0$ but still constant in D_2, we can prove the same result as in Lemma 6.2.1 by replacing the fundamental solution $\Phi_{k_1}(\cdot, y)$ with the free space fundamental $\mathbb{G}(\cdot, y)$ of

$$\Delta\mathbb{G}(\cdot, y) + k^2 n(x)\mathbb{G}(\cdot, y) = -\delta_y \qquad \text{in } \mathbb{R}^3$$

in the distributional sense together with the Sommerfeld radiation condition, where $n(x)$ is extended by its constant value in D_2 to the whole space $\mathbb{R}^3$. Because $\Phi_{k_2}(\cdot, y) - \mathbb{G}(\cdot, y)$ solves the Helmholtz equation with wave number k_2 in the neighborhood of Γ the mapping properties of the integral operators do not change. We refer the reader to Section 4.2 of [33] for more details.

In fact the above idea can be applied even in a more general case, provided that N is positive constant not equal to one in a neighborhood of Γ. More precisely, consider a neighborhood $\mathcal{O}$ of Γ in D (above denoted by D_2) with C^2 smooth boundary (e.g. one can take $\mathcal{O}$ the region in D bounded by Γ and $\Sigma := \{x - \epsilon\nu(x),\ x \in \Gamma\}$ for some $\epsilon > 0$ where ν is the outward unit normal vector to Γ). Assume that $N = nI$ in $\mathcal{O}$, where $n \neq 1$ is a positive constant, whereas in $D \setminus \overline{\mathcal{O}}$ N satisfies the assumptions at the beginning of the paper, i.e. N is a 3×3 matrix-valued function with $L^\infty(D)$ entries such that $\overline{\xi} \cdot \mathrm{Re}(N)\xi \geq \alpha > 0$ and $\overline{\xi} \cdot \mathrm{Im}(N)\xi \geq 0$ for every $\xi \in \mathbb{C}^3$. Then, similar result as in Theorem 6.2.1 and Lemma 6.2.1 holds true in this case. Indeed, without going into details, we can express $\mathbf{E}_0$ by (6.39) and $\mathbf{E}$ by (6.41) in $\mathcal{O}$ and in $D \setminus \overline{\mathcal{O}}$ we can leave it in the form of partial differential equation with Cauchy data connected to $\mathbf{E}$ in $\mathcal{O}$. Hence it is possible to obtain an equation of the form (6.44) where the operator $\mathbf{L}(k)$ is written as

$$\mathbf{L}(k) = \mathbf{L}_n(k) - \mathbf{L}^{\Sigma,\Gamma}(k)\mathbf{A}^{-1}(k)\mathbf{L}^{\Gamma,\Sigma}(k) \tag{6.45}$$

where $\mathbf{L}_n(k)$ is the boundary integral operator corresponding to the transmission eigenvalue problem with contrast $n - 1$, the compact operators $\mathbf{L}^{\Sigma,\Gamma}(k)$ and $\mathbf{L}^{\Gamma,\Sigma}(k)$ are defined right below (6.42) and (6.43) and $\mathbf{A}(k)$ is the invertible solution operator corresponding to the well-posed transmission problem

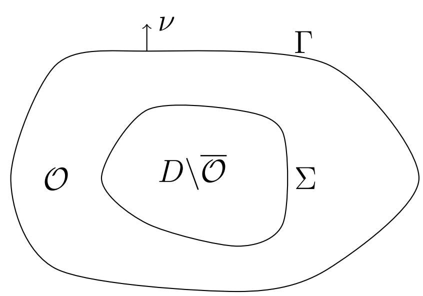


Figure 6.2: Example of the geometry of the problem

$$\operatorname{curl}\operatorname{curl}\mathbf{E}_2 - k^2 n_2 \mathbf{E}_2 = 0 \qquad \text{in} \qquad \mathbb{R}^3 \backslash \{\overline{D\backslash\overline{\mathcal{O}}}\} \tag{6.46}$$

$$\operatorname{curl}\operatorname{curl}\mathbf{E}_1 - k^2 N \mathbf{E}_1 = 0 \qquad \text{in} \qquad D\backslash\overline{\mathcal{O}} \tag{6.47}$$

$$\nu \times \mathbf{E}_2 - \nu \times \mathbf{E}_1 = \mathbf{g} \qquad \text{on} \qquad \Sigma \tag{6.48}$$

$$\nu \times (\operatorname{curl}\mathbf{E}_2) - \nu \times (\operatorname{curl}\mathbf{E}_1) = \mathbf{h} \qquad \text{on} \qquad \Sigma \tag{6.49}$$

and $\mathbf{E}_2$ satisfies the Silver-M$\ddot{u}$ller radiation condition. Hence the above analysis can apply to prove Theorem 6.2.1 and Lemma 6.2.1.

For later use in the following we formally state the assumptions on N (here $\mathcal{O}$ is a neighborhood of Γ as explained above).

Assumption 6.2.1 *N is a 3×3 symmetric matrix-valued function with $L^\infty(D)$ entries such that $\overline{\xi}\cdot Re(N)\xi \geq \alpha > 0$ and $\overline{\xi}\cdot Im(N)\xi \geq 0$ for every $\xi \in \mathbb{C}^3$, $|\xi| = 1$ and $N = nI$ in $\mathcal{O}$ where $n \neq 1$ is a positive constant.*

6.3 The Existence of Non Transmission Eigenvalue Wave Numbers

In this section we assume that N satisfies Assumption 6.2.1 and consider pure imaginary wave numbers k and, for convenience, let $\lambda := -k^2$ be a real positive number and start by proving a priori estimate following the idea of [79] for the scalar case.

Lemma 6.3.1 *Assume that N satisfies 6.2.1 and $\chi(x) \in \mathbb{C}_0^\infty(D)$ is real valued cutoff function with $0 \leq \chi \leq 1$ and $\chi \equiv 1$ in $D\backslash\overline{\mathcal{O}}$. If $\mathbf{v} \in \mathbf{L}^2(D)$ and*

$$(\operatorname{curl}\operatorname{curl} + \lambda)\mathbf{v} = 0 \quad \textit{in} \quad D$$

then there exists a constant $K(\chi)$ such that for sufficiently large λ

$$\|\chi\mathbf{v}\|^2 \leq K\frac{\|(1-\chi)\mathbf{v}\|^2}{\lambda}. \tag{6.50}$$

Here $\|\cdot\|$ donotes the $\mathbf{L}^2$ norm.

Proof. Since $\chi \in \mathbb{C}_0^\infty(D)$ we have

$$\begin{aligned}
0 &= \int_D (\operatorname{curl}\operatorname{curl} + \lambda)\mathbf{v}\cdot(\chi^2\overline{\mathbf{v}})dx = \int_D \operatorname{curl}\operatorname{curl}\mathbf{v}\cdot(\chi^2\overline{\mathbf{v}})dx + \lambda\int_D \mathbf{v}\cdot(\chi^2\overline{\mathbf{v}})dx \\
&= \int_D \operatorname{curl}\mathbf{v}\cdot\operatorname{curl}(\chi^2\overline{\mathbf{v}})dx + \lambda\int_D \mathbf{v}\cdot(\chi^2\overline{\mathbf{v}})dx \\
&= \int_D \operatorname{curl}\mathbf{v}\cdot(\chi\operatorname{curl}(\chi\overline{\mathbf{v}}))dx + \int_D \operatorname{curl}\mathbf{v}\cdot(\nabla\chi\times(\chi\overline{\mathbf{v}}))dx + \lambda\int_D \mathbf{v}\cdot(\chi^2\overline{\mathbf{v}})dx \\
&= \int_D \operatorname{curl}(\chi\mathbf{v})\cdot\operatorname{curl}(\chi\overline{\mathbf{v}})dx - \int_D \operatorname{curl}(\chi\overline{\mathbf{v}})\cdot(\nabla\chi\times\mathbf{v})dx \\
&+ \int_D \operatorname{curl}\mathbf{v}\cdot(\nabla\chi\times(\chi\overline{\mathbf{v}}))dx + \lambda\int_D \mathbf{v}\cdot(\chi^2\overline{\mathbf{v}})dx \\
&= \int_D |\operatorname{curl}(\chi\mathbf{v})|^2dx - \int_D (\chi\operatorname{curl}\overline{\mathbf{v}} + \nabla\chi\times\overline{\mathbf{v}})\cdot(\nabla\chi\times\mathbf{v})dx \\
&+ \int_D \operatorname{curl}\mathbf{v}\cdot(\nabla\chi\times(\chi\overline{\mathbf{v}}))dx + \lambda\int_D \mathbf{v}\cdot(\chi^2\overline{\mathbf{v}})dx \\
&= \int_D |\operatorname{curl}(\chi\mathbf{v})|^2dx - \int_D |(\nabla\chi\times\mathbf{v})|^2dx + \lambda\int_D |\chi\mathbf{v}|^2dx \\
&+ \int_D \left((\chi\operatorname{curl}\mathbf{v})\cdot(\nabla\chi\times\overline{\mathbf{v}}) - (\chi\operatorname{curl}\overline{\mathbf{v}})\cdot(\nabla\chi\times\mathbf{v})\right)dx.
\end{aligned}$$

Taking the real part yields

$$\int_D |\operatorname{curl}(\chi\mathbf{v})|^2dx + \lambda\int_D |\chi\mathbf{v}|^2dx = \int_D |(\nabla\chi\times\mathbf{v})|^2dx$$

and then

$$\lambda\|\chi\mathbf{v}\|^2 \le K(\chi)\|\mathbf{v}\|^2 \le K(\chi)\left(\|\chi\mathbf{v}\|^2 + \|(1-\chi)\mathbf{v}\|^2\right)$$

which yields (6.50) for sufficiently large λ.

□

Now we are ready to prove the following theorem.

Theorem 6.3.1 *Under the assumption 6.2.1, there exists a sufficiently large real $\lambda > 0$ where $\lambda = -k^2$ such that (6.10)-(6.13) has only trivial solutions.*

Proof. Assume first $n-1<0$ in $\mathcal{O}$, let $\mathbf{u}=\mathbf{E}-\mathbf{E}_0\in\mathbf{H}_0(\mathrm{curl}^2,D)$, $\mathbf{v}=\lambda\mathbf{E}_0\in\mathbf{L}^2(D)$, then

$$\mathrm{curl}\,\mathrm{curl}\,\mathbf{u}+\lambda N\mathbf{u}=-(N-I)\mathbf{v}\qquad \text{in}\qquad D \tag{6.51}$$

$$\mathrm{curl}\,\mathrm{curl}\,\mathbf{v}+\lambda\mathbf{v}=0\qquad \text{in}\qquad D \tag{6.52}$$

$$\nu\times\mathbf{u}=\nu\times(\mathrm{curl}\,\mathbf{u})=0\qquad \text{on}\qquad \Gamma \tag{6.53}$$

For any $\varphi\in\mathbf{C}_0^\infty(D)$, interpreting (6.52) in the distributional sense yields

$$\int_D\mathbf{v}(\mathrm{curl}\,\mathrm{curl}\,\varphi+\lambda\varphi)=0.$$

Then the denseness of $\mathbf{C}_0^\infty(D)$ in $\mathbf{H}_0(\mathrm{curl}^2,D)$ (see [37]) yields

$$\int_D\overline{\mathbf{v}}\cdot\mathrm{curl}^2\mathbf{u}+\lambda\int_D\overline{\mathbf{v}}\cdot\mathbf{u}=0 \tag{6.54}$$

Multiplying (6.51) by $\overline{\mathbf{v}}$ yields

$$\int_D\overline{\mathbf{v}}\cdot\mathrm{curl}^2\mathbf{u}dx+\lambda\int_D N\mathbf{u}\cdot\overline{\mathbf{v}}dx+\int_D(N-I)\mathbf{v}\cdot\overline{\mathbf{v}}dx=0.$$

Combining above with (6.54) yields

$$\lambda\int_D(N-I)\overline{\mathbf{u}}\cdot\mathbf{v}dx+\int_D(N-I)\mathbf{v}\cdot\overline{\mathbf{v}}dx=0. \tag{6.55}$$

Multiplying (6.51) by $\overline{\mathbf{u}}$ and integrating by parts yields

$$\int_D|\mathrm{curl}\,\mathbf{u}|^2dx+\lambda\int_D N\mathbf{u}\cdot\overline{\mathbf{u}}dx+\int_D(N-I)\mathbf{v}\cdot\overline{\mathbf{u}}dx=0.$$

Note N is symmetric, then $(N-I)\overline{\mathbf{u}}\cdot\mathbf{v}=(N-I)\mathbf{v}\cdot\overline{\mathbf{u}}$ and hence

$$\int_D|\mathrm{curl}\,\mathbf{u}|^2dx+\lambda\int_D N\mathbf{u}\cdot\overline{\mathbf{u}}dx+\int_D(N-I)\overline{\mathbf{u}}\cdot\mathbf{v}dx=0. \tag{6.56}$$

By regularity [69] $\mathbf{v}$ is sufficiently smooth in D away from the boundary, then by unique continuation we can see $\int_{\mathcal{O}}(n-1)(1-\chi^2)|\mathbf{v}|^2dx \neq 0$. Then combining (6.55) with (6.56) yields

$$\int_D |\mathrm{curl}\,\mathbf{u}|^2 dx + \lambda \int_D N\mathbf{u}\cdot\overline{\mathbf{u}}\,dx = \frac{1}{\lambda}\int_D (N-I)\mathbf{v}\cdot\overline{\mathbf{v}}\,dx \tag{6.57}$$

$$\begin{aligned}
&= \frac{1}{\lambda}\left(\int_D (N-I)\chi^2\mathbf{v}\cdot\overline{\mathbf{v}}\,dx + \int_D (N-I)(1-\chi^2)\mathbf{v}\cdot\overline{\mathbf{v}}\,dx\right)\\
&= \frac{1}{\lambda}\int_D (N-I)(1-\chi^2)\mathbf{v}\cdot\overline{\mathbf{v}}\,dx\left(1+\frac{\int_D (N-I)\chi^2\mathbf{v}\cdot\overline{\mathbf{v}}\,dx}{\int_D (N-I)(1-\chi^2)\mathbf{v}\cdot\overline{\mathbf{v}}\,dx}\right)\\
&= \frac{1}{\lambda}(n-1)\int_{\mathcal{O}}(1-\chi^2)|\mathbf{v}|^2\,dx\left(1+\frac{\int_D (N-I)\chi^2\mathbf{v}\cdot\overline{\mathbf{v}}\,dx}{(n-1)\int_{\mathcal{O}}(1-\chi^2)|\mathbf{v}|^2\,dx}\right).
\end{aligned} \tag{6.58}$$

From Lemma 6.3.1 we have for sufficiently large λ

$$\frac{\left|\int_D (N-I)\chi^2\mathbf{v}\cdot\overline{\mathbf{v}}\,dx\right|}{(1-n)\int_{\mathcal{O}}(1-\chi^2)|\mathbf{v}|^2dx} < \frac{K(N_{max}+1)}{\lambda} < 1$$

where N_{max} is supreme in D of 2-norm of N, which implies

$$\Re\left(1+\frac{\int_D (N-I)\chi^2\mathbf{v}\cdot\overline{\mathbf{v}}dx}{(n-1)\int_{\mathcal{O}}(1-\chi^2)|\mathbf{v}|^2dx}\right) > 0.$$

Then, since $n-1<0$, the real part of (6.58) is non positive for sufficiently large λ but the real part of (6.57) is non negative hence the only possibility is $\mathbf{u}=0, \mathbf{v}=0$, i.e. $\mathbf{E}=\mathbf{E}_0=0$.

Let us next consider $n-1>0$ in $\mathcal{O}$, and let $\mathbf{u}=\mathbf{E}-\mathbf{E}_0$, $\mathbf{v}=\lambda\mathbf{E}$, then

$$\mathrm{curl}\,\mathrm{curl}\,\mathbf{u} + \lambda\mathbf{u} = -(N-I)\mathbf{v} \quad \text{in} \quad D \tag{6.59}$$

$$\mathrm{curl}\,\mathrm{curl}\,\mathbf{v} + \lambda N\mathbf{v} = 0 \quad \text{in} \quad D \tag{6.60}$$

$$\nu\times\mathbf{u} = \nu\times(\mathrm{curl}\,\mathbf{u}) = 0 \quad \text{on} \quad \Gamma \tag{6.61}$$

Using same argument as for (6.54)

$$\int_D \mathrm{curl}^2\overline{\mathbf{u}}\cdot\mathbf{v}dx + \lambda\int_D N\mathbf{v}\cdot\overline{\mathbf{u}}dx = 0. \tag{6.62}$$

Multiplying (6.59) by $\overline{\mathbf{v}}$ yields

$$\int_D \overline{\mathbf{v}}\cdot\mathrm{curl}^2\mathbf{u}dx + \lambda\int_D \overline{\mathbf{v}}\cdot\mathbf{u}dx + \int_D (N-I)\mathbf{v}\cdot\overline{\mathbf{v}}dx = 0.$$

Combining the conjugate of above with (6.62) yields

$$\lambda \int_D (N-I)\overline{\mathbf{u}}\cdot\mathbf{v}dx = \int_D \overline{N-I}\,\overline{\mathbf{v}}\cdot\mathbf{v}dx. \tag{6.63}$$

Multiplying (6.59) by $\overline{\mathbf{u}}$ and integrating by parts yields

$$\int_D |\mathrm{curl}\,\mathbf{u}|^2 dx + \lambda \int_D |\mathbf{u}|^2 dx + \int_D (N-I)\mathbf{v}\cdot\overline{\mathbf{u}}dx = 0.$$

Note that N is symmetric, then $(N-I)\overline{\mathbf{u}}\cdot\mathbf{v} = (N-I)\mathbf{v}\cdot\overline{\mathbf{u}}$ and hence

$$\int_D |\mathrm{curl}\,\mathbf{u}|^2 dx + \lambda \int_D |\mathbf{u}|^2 dx + \int_D (N-I)\overline{\mathbf{u}}\cdot\mathbf{v}dx = 0. \tag{6.64}$$

Then combining (6.63) with (6.64) yields

$$\int_D |\mathrm{curl}\,\mathbf{u}|^2\,dx + \lambda \int_D |\mathbf{u}|^2\,dx = -\frac{1}{\lambda}\int_D \overline{N-I}\,\overline{\mathbf{v}}\cdot\mathbf{v}\,dx \tag{6.65}$$

$$\begin{aligned}
&= -\frac{1}{\lambda}\left(\int_D \chi^2\overline{N-I}\,\overline{\mathbf{v}}\cdot\mathbf{v}\,dx + \int_D (1-\chi^2)\overline{N-I}\,\overline{\mathbf{v}}\cdot\mathbf{v}\,dx\right)\\
&= -\frac{1}{\lambda}\int_D (1-\chi^2)\overline{N-I}\,\overline{\mathbf{v}}\cdot\mathbf{v}\,dx\left(1+\frac{\int_D \chi^2\overline{N-I}\,\overline{\mathbf{v}}\cdot\mathbf{v}dx}{\int_D (1-\chi^2)\overline{N-I}\,\overline{\mathbf{v}}\cdot\mathbf{v}dx}\right)\\
&= -\frac{1}{\lambda}\int_{\mathcal{O}} (n-1)(1-\chi^2)|\mathbf{v}|^2 dx\left(1+\frac{\int_D \chi^2\overline{N-I}\,\overline{\mathbf{v}}\cdot\mathbf{v}\,dx}{(n-1)\int_{\mathcal{O}}(1-\chi^2)|\mathbf{v}|^2\,dx}\right).
\end{aligned} \tag{6.66}$$

From Lemma 6.3.1 we have for sufficiently large λ

$$\frac{\left|\int_D \chi^2\overline{N-I}\,\overline{\mathbf{v}}\cdot\mathbf{v}\,dx\right|}{(n-1)\int_{\mathcal{O}}(1-\chi^2)|\mathbf{v}|^2\,dx} < \frac{K(N_{max}+1)}{\lambda} < 1.$$

Then

$$\Re\left(1+\frac{\int_D \chi^2\overline{N-I}\,\overline{\mathbf{v}}\cdot\mathbf{v}\,dx}{\int_{\mathcal{O}}(n-1)(1-\chi^2)|\mathbf{v}|^2\,dx}\right) > 0.$$

Therefore, since $n-1>0$, the real part of (6.66) is non positive for sufficiently large λ but the real part of (6.65) is non negative hence the only possibility is $\mathbf{u}=0, \mathbf{v}=0$, i.e. $\mathbf{E}=\mathbf{E}_0=0$. □

6.4 Discreteness of Transmission Eigenvalues

Recall that in Section 6.2, we have proved that $\tilde{\mathbf{L}}(k)$ is a Fredholm operator, hence to show discreteness we will use the analytic Fredholm theory [24]. To this end we must to show that there exists k such that $\tilde{\mathbf{L}}(k)$ is injective.

Lemma 6.4.1 *Assume that N satisfies 6.2.1. There exists a purely imaginary k with sufficiently large $|k| > 0$ such that $\tilde{\mathbf{L}}(k)$ is injective.*

Proof. Let us extend N to $\mathbb{R}^3 \backslash \overline{D}$ by $N = nI$ where n is the constant $N|_{\mathcal{O}}$. Assume there exists $\begin{pmatrix} \mathbf{M} \\ \mathbf{J} \end{pmatrix}$ such that $\mathbf{L}(k) \begin{pmatrix} \mathbf{M} \\ \mathbf{J} \end{pmatrix} = 0$, then we show that if k is purely imaginary with large modulus, then $\begin{pmatrix} \mathbf{M} \\ \mathbf{J} \end{pmatrix} = 0$. Recalling (6.45), we define

$$\begin{pmatrix} \mathbf{M}' \\ \mathbf{J}' \end{pmatrix} = \mathcal{A}^{-1}(k)\mathbf{L}^{\Gamma,\Sigma}(k) \begin{pmatrix} \mathbf{M} \\ \mathbf{J} \end{pmatrix}$$

and

$$\begin{aligned} \mathbf{E}_0(x) &= \operatorname{curl} \int_\Gamma \mathbf{M}(\mathbf{y})\Phi_k(x, y)ds_y + \int_\Gamma \mathbf{J}(\mathbf{y})\Phi_k(\cdot, \mathbf{y})ds_y \\ &+ \frac{1}{k^2}\nabla \int_\Gamma \operatorname{div}_T \mathbf{J}(\mathbf{y})\Phi_k(\cdot, \mathbf{y})ds_y \quad \text{in} \quad \mathbb{R}^3 \backslash \Gamma. \end{aligned}$$

From the definition of $\begin{pmatrix} \mathbf{M}' \\ \mathbf{J}' \end{pmatrix}$ there exists $\mathbf{E} \in \mathbf{L}(D_1)$, $D_1 := D \setminus \overline{\mathcal{O}}$, such that

$$\begin{aligned} \operatorname{curl}\operatorname{curl}\mathbf{E} - k^2 N\mathbf{E} = 0 &\quad \text{in} \quad D_1 \\ [\mathbf{E} \times \nu]^+ = \mathbf{M}' &\quad \text{on} \quad \Sigma \\ [\operatorname{curl}\mathbf{E} \times \nu]^+ = \mathbf{J}' &\quad \text{on} \quad \Sigma \end{aligned}$$

Also we define

$$\begin{aligned} \mathbf{E}(x) &= \operatorname{curl} \int_\Gamma \mathbf{M}(\mathbf{y})\Phi_{k_2}(x, y)ds_y + \int_\Gamma \mathbf{J}(\mathbf{y})\Phi_{k_2}(\cdot, \mathbf{y})ds_y \\ &+ \frac{1}{k_2^2}\nabla \int_\Gamma \operatorname{div}_T \mathbf{J}(\mathbf{y})\Phi_{k_2}(\cdot, \mathbf{y})ds_y \\ &- \operatorname{curl} \int_\Sigma \mathbf{M}'(\mathbf{y})\Phi_{k_2}(x, y)ds_y - \int_\Sigma \mathbf{J}'(\mathbf{y})\Phi_{k_2}(\cdot, \mathbf{y})ds_y \\ &- \frac{1}{k_2^2}\nabla \int_\Sigma \operatorname{div}_T \mathbf{J}'(\mathbf{y})\Phi_{k_2}(\cdot, \mathbf{y})ds_y \quad \text{in} \quad \mathbb{R}^3 \backslash (\overline{D}_1 \cup \Gamma) \end{aligned}$$

Jump relations across Γ applied to $\mathbf{E}, \mathbf{E}_0$ along with the equation (6.44) yield

$$\operatorname{curl}\operatorname{curl}\mathbf{E} - k^2 N\mathbf{E} = 0 \qquad \text{in} \qquad \mathbb{R}^3 \backslash \Gamma \tag{6.67}$$

$$\operatorname{curl}\operatorname{curl}\mathbf{E}_0 - k^2 \mathbf{E}_0 = 0 \qquad \text{in} \qquad \mathbb{R}^3 \backslash \Gamma \tag{6.68}$$

$$(\nu \times \mathbf{E})^{\pm} = (\nu \times \mathbf{E}_0)^{\pm} \qquad \text{on} \qquad \Gamma \tag{6.69}$$

$$(\nu \times \operatorname{curl}\mathbf{E})^{\pm} = (\nu \times \operatorname{curl}\mathbf{E}_0)^{\pm} \qquad \text{on} \qquad \Gamma \tag{6.70}$$

From Theorem 6.3.1 if k is purely imaginary with large enough modulus then (6.67)-(6.70) in D only has trivial solutions. Since $N = nI$ where n is a constant in $\mathbb{R}^3 \backslash \overline{D}$, then the variational formulation of (6.67)-(6.70) in $\mathbb{R}^3 \backslash \overline{D}$ is (6.36) where the right hand is 0 and $\mathbb{R}^3 \backslash \Gamma$ is replaced by $\mathbb{R}^3 \backslash \overline{D}$, then $\mathbf{U} = 0$ and hence $\mathbf{E} = 0$, $\mathbf{E}_0 = 0$ in $\mathbb{R}^3 \backslash \Gamma$. The jump relations (6.27)-(6.30) yield $\mathbf{M} = 0$ and $\mathbf{J} = 0$ and this proves the lemma. $\square$

Now we have the main theorem.

Theorem 6.4.1 *Assume that N satisfies Assumption 6.2.1, then the transmission eigenvalues form a discrete set in a neighborhood of the real axis.*

Proof. Combining Lemma 6.2.1 and Lemma 6.4.1, we can prove that the set of the transmission eigenvalues is discrete in a neighborhood of the real axis. $\square$

Chapter 7

THE SPECTRAL ANALYSIS OF THE INTERIOR TRANSMISSION PROBLEM FOR MAXWELL'S EQUATIONS

In this Chapter we study the spectral analysis of the interior transmission problem (6.6)-(6.9) for Maxwell's equations. The existence of transmission eigenvalues for Maxwell's equations for which the electric permittivity changes sign is an open problem. It is our concern to study the existence of transmission eigenvalues in the complex plane under the assumption that the electric permittivity is constant near the boundary. Although the index of refraction may be a complex valued function, our analysis does not cover the case with absorption where the imaginary part of n is proportional to $1/k$. For the case with absorption, some non-linear eigenvalue techniques would be more relevant [21, 40, 78]. We also remark that, similar to the scalar case in [76], our analysis does not yield information on the existence of real transmission eigenvalues.

In Section 7.1 we give an appropriate formulation of the transmission eigenvalue problem and relate transmission eigenvalues to the eigenvalues of an unbounded linear operator $\mathbf{B}_\lambda$.

This motivates us to derive desired regularity results in Section 7.2 that are needed to show the invertibility of $\mathbf{B}_\lambda$ and prove the main theorem. The derivation of these results mainly uses the semi-classical pseudo-differential calculus introduced in [76] for the scalar case with appropriate adaptations to Maxwell's system. The assumption that the electric permittivity is constant near the boundary considerably eases the technicality of this section and allows us to use results from the scalar problem that are summarized in Section 7.5. The main technical difficulty related to non constant electric permittivity is that the divergence free condition is different for $\mathbf{E}$

and $\mathbf{E}_0$ near the boundary. One therefore cannot impose a "simple" control of the divergence of the difference which is needed to establish regularity results.

Using the regularity results obtained in Section 7.2, we show that $\mathbf{B}_\lambda$ has a bounded inverse for certain λ in Section 7.3.

Section 7.4 is dedicated to proving the main results on transmission eigenvalues following the approach in [76] which is based on Agmon's theory for the spectrum of non self-adjoint PDE [1]. We prove for instance that the inverse $\mathbf{B}_\lambda^{-1}$ composed with a projection operator is a Hilbert-Schmidt operator with desired growth properties for its resolvent. This allows us to prove that the set of transmission eigenvalues is discrete, infinite and without finite accumulation points. Moreover, a notion of generalized eigenfunctions is introduced and a denseness result is obtained in an appropriate solution space. Throughout this chapter we denote $m := N - 1$ and shall make the

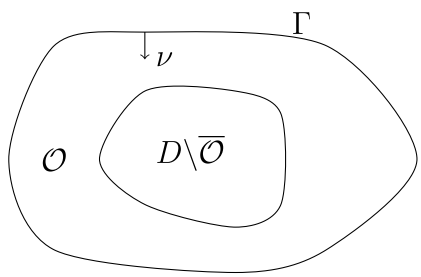


Figure 7.1: Example of the geometry of the problem

following assumption on the index of refraction N.

Assumption 7.0.1 *We assume that the complex valued function $N \in C^\infty(\overline{D})$ and that $\Re(N) > 0$ in D. Moreover we assume the existence of a neighborhood $\mathcal{O}$ of Γ such that N is constant in $\mathcal{O}$ and that this constant is different from 1 (which means that m is constant and different from zero in $\mathcal{O}$).*

7.1 Formulation of the Transmission Eigenvalue Problem

In the following $D \subset \mathbb{R}^3$ denotes a bounded open and connected region with C^∞-smooth boundary $\partial D := \Gamma$ and ν denotes the inward unit normal vector on Γ

(see Figure 7.1 for an example of the geometry). We set $\mathbf{L}^2(D) := L^2(D)^3$, $\mathbf{H}^m(D) := H^m(D)^3$ and define

$$\mathbf{H}(\mathrm{curl}^2, D) := \left\{\mathbf{u} \in \mathbf{L}^2(D); \mathrm{curl}\,\mathbf{u} \in \mathbf{L}^2(D) \text{ and } \mathrm{curl}\,\mathrm{curl}\,\mathbf{u} \in \mathbf{L}^2(D)\right\}$$

$$\mathbf{L}(\mathrm{curl}^2, D) := \left\{\mathbf{u} \in \mathbf{L}^2(D); \mathrm{curl}\,\mathrm{curl}\,\mathbf{u} \in \mathbf{L}^2(D)\right\}$$

endowed with the graph norm and define

$$\mathbf{H}_0(\mathrm{curl}^2, D) := \left\{\mathbf{u} \in \mathbf{H}(\mathrm{curl}^2, D); \gamma_t \mathbf{u} = 0 \text{ and } \gamma_t \mathrm{curl}\,\mathbf{u} = 0 \text{ on } \Gamma\right\}$$

where $\gamma_t \mathbf{u} := \nu \times \mathbf{u}|_\Gamma$.

Definition 7.1.1 *Values of $k \in \mathbb{C}$ for which (6.6)-(6.9) has a nontrivial solution $\mathbf{E}, \mathbf{E}_0 \in \mathbf{L}(curl^2, D)$ and $\mathbf{E} - \mathbf{E}_0 \in \mathbf{H}_0(curl^2, D)$ are called transmission eigenvalues.*

Following the approach in [76,79] for the scalar case, we rewrite the transmission eigenvalue problem in an equivalent form in terms of $\mathbf{u} := \mathbf{E} - \mathbf{E}_0 \in \mathbf{H}_0(\mathrm{curl}^2, D)$ and $\mathbf{v} := k^2 \mathbf{E}_0 \in \mathbf{L}(\mathrm{curl}^2, D)$

$$\mathrm{curl}\,\mathrm{curl}\,\mathbf{u} - k^2(1+m)\mathbf{u} - m\mathbf{v} = 0 \qquad \text{in} \quad D \tag{7.1}$$

$$\mathrm{curl}\,\mathrm{curl}\,\mathbf{v} - k^2\mathbf{v} = 0 \qquad \text{in} \quad D \tag{7.2}$$

Definition 7.1.2 *Normalized non-trivial solutions $\mathbf{u} \in \mathbf{H}_0(curl^2, D)$ and $\mathbf{v} \in \mathbf{L}(curl^2, D)$ to equations (7.1)-(7.2) are called transmission eigenvectors corresponding to k.*

To study the PDEs (7.1)-(7.2) and formulate the transmission eigenvalue problem, we first investigate the function spaces that transmission eigenvectors $\mathbf{u}$ and $\mathbf{v}$ belong to. This is the motivation of the next lemma.

Lemma 7.1.1 *Assume that assumption 7.0.1 holds and $\mathbf{u} \in \mathbf{H}_0(curl^2, D)$ and $\mathbf{v} \in \mathbf{L}(curl^2, D)$ are transmission eigenvectors corresponding to k. Then $div\,\mathbf{u} \in H^1(D)$ and $div\,\mathbf{v} \in H^1(D)$.*

Proof. Taking the divergence of (7.2) implies $\operatorname{div}\mathbf{v} = 0$ and therefore $\operatorname{div}\mathbf{v} \in H^1(D)$. Taking the divergence of equation (7.1) yields

$$(1+m)\operatorname{div}\mathbf{u} + \nabla m \cdot \mathbf{u} = -k^{-2}(\nabla m \cdot \mathbf{v} + m\operatorname{div}\mathbf{v}). \tag{7.3}$$

Since ∇m has compact support in D and $\mathbf{v}$ satisfies a vectorial Helmholtz equation in D, then standard regularity results give $\nabla m \cdot \mathbf{v} \in H^1(D)$. Since $\operatorname{div}\mathbf{v} \in H^1(D)$ and $\mathbf{u} \in \mathbf{L}^2(D)$, we deduce from (7.3) that $\operatorname{div}\mathbf{u} \in L^2(D)$. Since $\operatorname{curl}\mathbf{u} \in \mathbf{L}^2(D)$ and $\gamma_t\mathbf{u} = 0$, $\mathbf{u} \in \mathbf{H}^1(D)$ (c.f. [5]). Hence, using again (7.3), $\operatorname{div}\mathbf{u} \in H^1(D)$ and we have proved the lemma. $\square$

We now define the following spaces:

$$\mathbf{U}(D) := \left\{\mathbf{u} \in \mathbf{H}_0(\operatorname{curl}^2, D);\ \operatorname{div}\mathbf{u} \in H^1(D)\right\}$$

and

$$\mathbf{V}(D) := \left\{\mathbf{v} \in \mathbf{L}^2(D); \operatorname{curl}\operatorname{curl}\mathbf{v} \in \mathbf{L}^2(D) \text{ and } \operatorname{div}\mathbf{v} \in H^1(D)\right\}.$$

Having studied the function spaces that transmission eigenvectors belong to, we are ready to introduce an operator which plays an important role in our analysis. We introduce the operator $\mathbf{B}_\lambda$ defined on $\mathbf{U}(D) \times \mathbf{V}(D)$ by

$$\mathbf{B}_\lambda(\mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{g})$$

where

$$\operatorname{curl}\operatorname{curl}\mathbf{u} - \lambda(1+m)\mathbf{u} - m\mathbf{v} = (1+m)\mathbf{f} \qquad \text{in} \quad D \tag{7.4}$$

$$\operatorname{curl}\operatorname{curl}\mathbf{v} - \lambda\mathbf{v} = \mathbf{g} \qquad \text{in} \quad D \tag{7.5}$$

and $\lambda \in \mathbb{C}$ is a fixed parameter (we will choose λ later). We can now relate the transmission eigenvalue with the eigenvalues of $\mathbf{B}_\lambda$. In fact, one observes that k is a transmission eigenvalue if and only if $k^2 - \lambda$ is an eigenvalue of $\mathbf{B}_\lambda$ (this also explains the motivation to define the operator $\mathbf{B}_\lambda$).

To study the invertibility of the operator $\mathbf{B}_\lambda$, we first investigate the range of $\mathbf{B}_\lambda$.

Lemma 7.1.2 *Assume* $\mathbf{B}_\lambda(\mathbf{u},\mathbf{v}) = (\mathbf{f},\mathbf{g})$ *and* $(\mathbf{u},\mathbf{v}) \in \mathbf{U}(D) \times \mathbf{V}(D)$. *Then* $\mathbf{f} \in \mathbf{L}^2(D)$, *div* $((1+m)\,\mathbf{f}) \in H^1(D)$, $\mathbf{g} \in \mathbf{L}^2(D)$ *and div* $\mathbf{g} \in H^1(D)$.

Proof. Noting that $\mathbf{v} \in \mathbf{V}$ and $\text{curl}^2 = \nabla\text{div} - \Delta$, we have that

$$\Delta\mathbf{v} = \nabla\text{div}\,\mathbf{v} - \text{curl}^2\mathbf{v} \in \mathbf{L}^2(D).$$

Since ∇m has compact support in D, standard elliptic regularity results yield $\nabla m \cdot \mathbf{v} \in H^2(D)$. Since

$$\text{div}\ (m\mathbf{v}) = \nabla m \cdot \mathbf{v} + m\text{div}\,v,$$

we have that

$$\text{div}\ (m\mathbf{v}) \in H^1(D).$$

Since $\mathbf{u} \in \mathbf{U}$, $\mathbf{u} \in \mathbf{H}^2(D)$(c.f. [5]). Therefore

$$\text{div}\ ((1+m)\,\mathbf{f}) = -\lambda\text{div}\ ((1+m)\,\mathbf{u}) - \text{div}\ (m\mathbf{v}) \in H^1(D).$$

$\text{div}\,\mathbf{g} \in H^1(D)$ follows directly from $\text{div}\,\mathbf{v} \in H^1(D)$. This proves our lemma. $\square$

We now define the following spaces:

$$\mathbf{F}(D) := \left\{\mathbf{f} \in \mathbf{L}^2(D); \text{div}\ ((1+m)\,\mathbf{f}) \in H^1(D)\right\}$$

and

$$\mathbf{G}(D) := \left\{\mathbf{g} \in \mathbf{L}^2(D); \text{div}\,\mathbf{g} \in H^1(D)\right\}.$$

7.2 Regularity Results for Transmission Eigenvectors

As is seen from Section 7.1, the analysis of transmission eigenvalues will be obtained from the analysis of the spectrum of the operator $\mathbf{B}_\lambda$ or more precisely of its inverse $\mathbf{R}_\lambda$. To show the existence of $\mathbf{R}_\lambda$ for well chosen λ, we need certain regularity results and this is the purpose of this section. Moreover, the regularity results in this section (in particular Theorem 7.2.2) is important to apply the spectral theory of Hilbert-Schmidt operator in section 7.4. The reader may proceed to read section 7.3

and section 7.4 by assuming Theorem 7.2.1 and 7.2.2 and come back to the technical details in this section after that.

In this section we will derive a detailed study of equations (7.4)-(7.5). Roughly speaking we will show that, for appropriate λ the solutions $\mathbf{u}$ and $\mathbf{v}$ are bounded by $\mathbf{f}$ and $\mathbf{g}$ in appropriate norms. The idea is based on applying the semiclassical pseudo-differential calculus used in [76] for the scalar problem. The analysis for Maxwell's equations requires non trivial adaptations since the normal component of the trace of $\mathbf{u}$ does not necessarily vanish, the $\operatorname{curl}\operatorname{curl}$ operator is not strongly elliptic and the compact embedding for Maxwell's equations are more complicated. Restricting ourselves to the case m is constant near the boundary simplifies the analysis since one can first derive a semiclassical estimate for the normal component of the trace of $\mathbf{v}$. This allows us to then derive estimates for $\mathbf{u}$ and $\mathbf{v}$. In order to write the equation for the normal trace of $\mathbf{v}$ and apply the analysis in [76] we first need to rewrite (7.4)-(7.5) as a problem in $\mathbb{R}^3$.

To begin with, we introduce a tubular neighborhood D_ϵ of Γ, where

$$D_\epsilon = \{x : x = y + s\nu(y),\, y \in \Gamma,\, 0 \leq s < \epsilon\}\,.$$

We define

$$\Gamma_s = \{x : x = y + s\nu(y),\, y \in \Gamma\}\,.$$

The boundary Γ corresponds to Γ_s with $s = 0$.

To deal with the boundary conditions on Γ, we follow the idea in [76] and extend the transmission eigenvectors by 0 outside D. To begin with, let us introduce

$$\underline{\mathbf{u}} = \begin{cases} \mathbf{u}(x) & \text{in} \quad D \\ 0 & \text{in} \quad \mathbb{R}^3 \backslash \overline{D}. \end{cases}$$

Lemma 7.2.1 *Assume* $(\mathbf{f}, \mathbf{g}) = \mathbf{B}_\lambda(\mathbf{u}, \mathbf{v})$ *as defined by equations (7.4) and (7.5). Then* $\underline{\mathbf{u}}$ *and* $\underline{\mathbf{v}}$ *satisfy the following*

$$-\Delta\underline{\mathbf{u}} - \lambda(1+m)\underline{\mathbf{u}} - m\underline{\mathbf{v}} = (1+m)\underline{\mathbf{f}} - \nabla\, \underline{div\,\mathbf{u}} - \nabla_\Gamma(\mathbf{u}_N\cdot\nu)\otimes\delta_{s=0} - \mathbf{u}_N \otimes D_s\delta_{s=0} \quad (7.6)$$

$$-\Delta\underline{\mathbf{v}} - \lambda\underline{\mathbf{v}} = \underline{\mathbf{g}} + \lambda^{-1}\nabla\underline{div\,\mathbf{g}} - (2H\mathbf{v}_T + \frac{\partial \mathbf{v}_T}{\partial \nu} - \nu\, div_\Gamma \mathbf{v}_T) \otimes \delta_{s=0} - \mathbf{v} \otimes D_s\delta_{s=0} \quad (7.7)$$

where $\gamma\mathbf{u} := \mathbf{u}|_\Gamma$, $\mathbf{u}_T := \gamma_T\mathbf{u} := \nu \times (\mathbf{u} \times \nu)|_\Gamma$ *and* $\mathbf{u}_N := \gamma_N\mathbf{u} = \nu(\mathbf{u}\cdot\nu)|_\Gamma$. *Here* $\delta_{s=0}$ *is the delta distribution on* Γ *and* D_s *is the normal derivative.*

Proof. From Δ in geodesic coordinates (c.f. [76] and [69]), we have that

$$\Delta\underline{\mathbf{u}} = \underline{\Delta\mathbf{u}} + (2H\mathbf{u}_T + \frac{\partial \mathbf{u}_T}{\partial \nu} + 2H\mathbf{u}_N + \frac{\partial \mathbf{u}_N}{\partial \nu}) \otimes \delta_{s=0} + (\mathbf{u}_T + \mathbf{u}_N) \otimes D_s\delta_{s=0}$$

where H is a smooth function on Γ (see Appendix). From $\text{curl}^2 = \nabla\text{div} - \Delta$ we are able to rewrite the equations (7.4)-(7.5) as follows

$$\begin{aligned} -\Delta\underline{\mathbf{u}} - \lambda(1+m)\underline{\mathbf{u}} - m\underline{\mathbf{v}} &= (1+m)\underline{\mathbf{f}} - \underline{\nabla\text{div}\,\mathbf{u}} - (\mathbf{u}_T + \mathbf{u}_N) \otimes D_s\delta_{s=0} \\ &- (2H\mathbf{u}_T + \frac{\partial \mathbf{u}_T}{\partial \nu} + 2H\mathbf{u}_N + \frac{\partial \mathbf{u}_N}{\partial \nu}) \otimes \delta_{s=0} \end{aligned} \quad (7.8)$$

and

$$\begin{aligned} -\Delta\underline{\mathbf{v}} - \lambda\underline{\mathbf{v}} &= \underline{\mathbf{g}} - \underline{\nabla\text{div}\,\mathbf{v}} - (2H\mathbf{v}_T + \frac{\partial \mathbf{v}_T}{\partial \nu} + 2H\mathbf{v}_N + \frac{\partial \mathbf{v}_N}{\partial \nu}) \otimes \delta_{s=0} \\ &- (\mathbf{v}_T + \mathbf{v}_N) \otimes D_s\delta_{s=0}. \end{aligned} \quad (7.9)$$

We now use the fact that (c.f. [69])

$$\begin{aligned} \nabla\underline{\text{div}\,\mathbf{u}} &= \underline{\nabla\text{div}\,\mathbf{u}} + (\nu\text{div}\,\mathbf{u}) \otimes \delta_{s=0} \\ \nu\text{div}\,\mathbf{u} &= \nu\text{div}_\Gamma\mathbf{u}_T + 2H\mathbf{u}_N + \frac{\partial \mathbf{u}_N}{\partial \nu} \quad \text{on} \quad \Gamma \end{aligned}$$

with the same equations hold for $\mathbf{v}$. Using above two equations to simplify equations (7.8)-(7.9) we get

$$\begin{aligned} -\Delta\underline{\mathbf{u}} - \lambda(1+m)\underline{\mathbf{u}} - m\underline{\mathbf{v}} &= (1+m)\underline{\mathbf{f}} - \nabla\underline{\text{div}\,\mathbf{u}} - (\mathbf{u}_T + \mathbf{u}_N) \otimes D_s\delta_{s=0} \\ &- (2H\mathbf{u}_T + \frac{\partial \mathbf{u}_T}{\partial \nu} - \nu\text{div}_\Gamma\mathbf{u}_T) \otimes \delta_{s=0} \end{aligned}$$

and

$$\begin{aligned} -\Delta\underline{\mathbf{v}} - \lambda\underline{\mathbf{v}} &= \underline{\mathbf{g}} - \nabla\underline{\text{div}\,\mathbf{v}} - (2H\mathbf{v}_T + \frac{\partial \mathbf{v}_T}{\partial \nu} - \nu\text{div}_\Gamma\mathbf{v}_T) \otimes \delta_{s=0} \\ &- (\mathbf{v}_T + \mathbf{v}_N) \otimes D_s\delta_{s=0}. \end{aligned}$$

We now use (c.f. [69])

$$\nu \times \operatorname{curl} \mathbf{u} = \nabla_\Gamma(\mathbf{u}_N \cdot \nu) + \nu \times (R(\mathbf{u} \times \nu)) - 2H\mathbf{u}_T - \frac{\partial \mathbf{u}_T}{\partial \nu} \quad \text{on} \quad \Gamma.$$

Then $\mathbf{u}_T = 0$ and $(\operatorname{curl} \mathbf{u})_T = 0$ yields

$$\frac{\partial \mathbf{u}_T}{\partial \nu} = \nabla_\Gamma(\mathbf{u}_N \cdot \nu) \quad \text{on} \quad \Gamma$$

and therefore we get (7.6). From equation (7.5)

$$-\lambda \operatorname{div} \mathbf{v} = \operatorname{div} \mathbf{g}.$$

This yields equation (7.7). □

The following lemma is important in our analysis as it allows us in subsection 7.2.1 to derive an estimate only involving $\mathbf{v}_N$.

Lemma 7.2.2 *Assume* $(\mathbf{f}, \mathbf{g}) = \mathbf{B}_\lambda(\mathbf{u}, \mathbf{v})$ *as defined by equations (7.4) and (7.5). Then*

$$\lambda \mathbf{u}_N = -\frac{m}{(1+m)}\mathbf{v}_N - \mathbf{f}_N.$$

In particular for $\lambda = h^{-2}\mu$ *where* $h > 0$ *and* $\mu \neq 0 \in \mathbb{C}$, *we have*

$$\mathbf{u}_N = -h^2 \frac{m}{\mu(1+m)}\mathbf{v}_N - h^2 \frac{1}{\mu}\mathbf{f}_N. \tag{7.10}$$

Proof. Equation (7.4) yields

$$\lambda(1+m)\mathbf{u}_N = -m\mathbf{v}_N - (1+m)\mathbf{f}_N + \operatorname{curl}\operatorname{curl} \mathbf{u} \cdot \nu.$$

Since $\operatorname{curl} \mathbf{u} \times \nu = 0$, then $\operatorname{curl}\operatorname{curl} \mathbf{u} \cdot \nu = -\operatorname{div}_\Gamma(\operatorname{curl} \mathbf{u} \times \nu) = 0$. Then we can prove the lemma. □

7.2.1 A First Regularity Result

We prove in this subsection a first explicit continuity result for $(\mathbf{u}, \mathbf{v}) \in \mathbf{U}(D) \times \mathbf{V}(D)$ satisfying

$$\mathbf{B}_\lambda(\mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{g})$$

for certain large values of λ. We refer to Section 7.5 for notations related to pseudo-differential calculus and some key results from [76]. Readers may need to read Section 7.5 first to be able to understand the proof.

Throughout this section, we let $h := \frac{1}{|\lambda|^{\frac{1}{2}}}$ and $\mu := h^2\lambda$. Multiplying equations (7.6) and (7.7) by h^2 yields

$$\begin{aligned} -h^2\Delta\underline{\mathbf{u}} - \mu(1+m)\underline{\mathbf{u}} - h^2 m\underline{\mathbf{v}} &= h^2(1+m)\underline{\mathbf{f}} + \frac{h}{i}\nabla_h \underline{\mathrm{div}\,\mathbf{u}} \\ &+ \frac{h}{i}\nabla_\Gamma^h(\mathbf{u}_N\cdot\nu)\otimes\delta_{s=0} + \frac{h}{i}\mathbf{u}_N\otimes D_s^h\delta_{s=0} \end{aligned} \tag{7.11}$$

and

$$\begin{aligned} -h^2\Delta\underline{\mathbf{v}} - \mu\underline{\mathbf{v}} &= h^2\underline{\mathbf{g}} - \frac{h^3}{i\mu}\nabla_h\underline{\mathrm{div}\,\mathbf{g}} \\ &+ \frac{h}{i}(2\frac{h}{i}H\mathbf{v}_T + \frac{\partial_h\mathbf{v}_T}{\partial\nu} - \nu\mathrm{div}_\Gamma^h\mathbf{v}_T)\otimes\delta_{s=0} + \frac{h}{i}\mathbf{v}\otimes D_s^h\delta_{s=0} \end{aligned} \tag{7.12}$$

(see Appendix for notations of $D_{x_j}^h$, ∇_h, $\frac{\partial_h}{\partial\nu}$). We define $\mathbf{J}(\mathbf{v}_T)$ by

$$\mathbf{J}(\mathbf{v}_T) := 2\frac{h}{i}H\mathbf{v}_T + \frac{\partial_h\mathbf{v}_T}{\partial\nu} - \nu\mathrm{div}_\Gamma^h\mathbf{v}_T.$$

Based on these two equations, we will derive the desired regularity results.

Before digging into the technical estimates, we first explain the ideas and what we are doing in each Lemma and Theorem. The general idea is to get first an estimate for $\mathbf{v}_N$ and $\mathbf{u}_N$. This will allow us to derive estimates for $\mathbf{v}$ and $\mathbf{J}(\mathbf{v}_T)$ and consequently estimates for $\mathbf{v}$ and $\mathbf{u}$.

More specifically, it will be seen in Theorem 7.2.1 that the estimates of $\mathbf{u}$ and $\mathbf{v}$ stems from the estimates of $\mathbf{v}_N$ in $\mathbf{H}_{sc}^{-\frac{1}{2}}(\Gamma)$ and of $\mathbf{J}(\mathbf{v}_T)$ in $\mathbf{H}_{sc}^{-\frac{3}{2}}(\Gamma)$ evidenced from (7.27) and (7.28). To get an estimate for $\mathbf{v}_N$ in $\mathbf{H}_{sc}^{-\frac{1}{2}}(\Gamma)$ we will need to get an estimate for $\mathbf{g}_5$ in $\mathbf{H}_{sc}^{\frac{3}{2}}(\Gamma)$ as is seen from (7.25). The estimate for $\mathbf{g}_5$ is obtained by establishing an equation for $\mathbf{u}_N$ that allows us to control the $\mathbf{H}_{sc}^{\frac{3}{2}}(\Gamma)$ norm of this boundary term. This is the first main additional technical difference between the scalar problem treated in [76] and the present one. For the scalar case this step in not needed since the solution has vanishing traces on the boundary.

Therefore, Lemma 7.2.3, Lemma 7.2.4, Lemma 7.2.5 and Lemma 7.2.6 serve to derive the desired estimate for $\mathbf{u}_N$ in $\mathbf{H}^{\frac{3}{2}}_{sc}(\Gamma)$. In Lemma 7.2.6, we derive an estimate for $\mathbf{u}_N$ that only involves $\mathbf{v}$, $\mathbf{f}$ and $\mathbf{g}$. This will serve to obtain an estimate for $\mathbf{v}$ in Theorem 7.2.1. The estimate of $\mathbf{u}_N$ in $\mathbf{H}^{\frac{3}{2}}_{sc}(\Gamma)$ stems from estimate of $\mathbf{v}_N$ in $\mathbf{H}^{-\frac{1}{2}}_{sc}(\Gamma)$. This is the motivation of Lemma 7.2.5: an a priori estimate on $\mathbf{v}_N$ independent of $\mathbf{u}$. To fullfill this, we derive an a priori estimate for $\mathbf{v}_N$ (involving $\mathbf{u}$) in Lemma 7.2.4 and an a priori estimate on $\mathbf{u}$ involving $\mathbf{v}_N$ in Lemma 7.2.3 (such that we can eliminate $\mathbf{u}$ in Lemma 7.2.5).

Now we begin with the following lemma.

Lemma 7.2.3 *Assume that assumption 7.0.1 holds. Assume in addition that* $|\xi|^2-\mu \neq 0$, $|\xi|^2-(1+m)\mu \neq 0$ *for any* ξ *and* $x \in \overline{D}$. *Then for sufficiently small* h

$$\begin{aligned}\|\mathbf{u}\|_{\mathbf{L}^2(D)} &\lesssim h^2\|\mathbf{v}\|_{\mathbf{L}^2(D)} + h^2\|\mathbf{f}\|_{\mathbf{L}^2(D)} + h^5\|\mathbf{g}\|_{\mathbf{L}^2(D)} + h^5\|div\,\mathbf{g}\|_{L^2(D)} \\ &+ h^2\|div\,\mathbf{f}\|_{L^2(D)} + h^{\frac{5}{2}}|\mathbf{v}_N|_{\mathbf{H}^{-\frac{1}{2}}_{sc}(\Gamma)}. \end{aligned} \tag{7.13}$$

Proof. From Section 7.5, Q is a parametrix of $-h^2\Delta - \mu(1+m)$, then applying Q to equation (7.11)

$$\begin{aligned}\underline{\mathbf{u}} &= hK_{-M}\underline{\mathbf{u}} + h^2Q(m\underline{\mathbf{v}}) + h^2Q((1+m)\underline{\mathbf{f}}) + \frac{h}{i}Q(\nabla_h\underline{\operatorname{div}\mathbf{u}}) \\ &+ Q(\frac{h}{i}\nabla^h_\Gamma(\mathbf{u}_N\cdot\nu)\otimes\delta_{s=0}) + Q(\frac{h}{i}\mathbf{u}_N\otimes D^h_s\delta_{s=0}) \end{aligned} \tag{7.14}$$

where K_{-M} denotes a semiclassical pseudo-differential operator of order $-M$ with M positive and sufficiently large. From equation (7.14), estimate (7.62) and Lemma 7.5.1

$$\begin{aligned}\|\mathbf{u}\|_{\mathbf{L}^2(D)} &\lesssim h^2\|\mathbf{v}\|_{\mathbf{L}^2(D)} + h^2\|\mathbf{f}\|_{L^2(D)} + h\|\operatorname{div}\mathbf{u}\|_{\mathbf{L}^2(D)} + h^{\frac{1}{2}}|\mathbf{u}_N|_{\mathbf{H}^{-\frac{1}{2}}_{sc}(\Gamma)} \\ &+ h^{\frac{1}{2}}|\nabla^h_\Gamma(\mathbf{u}_N\cdot\nu)|_{\mathbf{H}^{-\frac{3}{2}}_{sc}(\Gamma)}. \end{aligned} \tag{7.15}$$

Then a direct calculation (see the Calculation subsection 7.2.2) yields the lemma. □

Lemma 7.2.4 *Assume that assumption 7.0.1 holds. Assume in addition that* $|\xi|^2 - \mu \neq 0$, $|\xi|^2 - (1+m)\mu \neq 0$ *for any* ξ *and* $x \in \overline{D}$ *and* $R_0(x,\xi') - \frac{1+m}{2+m}\mu \neq 0$ *for any* ξ' *and* $x \in \Gamma$. *Then for sufficiently small* h

$$
\begin{aligned}
|\mathbf{v}_N|_{\mathbf{H}_{sc}^{-\frac{1}{2}}(\Gamma)} &\lesssim h^{\frac{3}{2}}\|\mathbf{g}\|_{\mathbf{L}^2(D)} + h^{\frac{5}{2}}\|div\,\mathbf{g}\|_{\mathbf{L}^2(D)} + h^{-\frac{1}{2}}\|\mathbf{f}\|_{\mathbf{L}^2(D)} + h^{-\frac{1}{2}}\|div\,\mathbf{f}\|_{L^2(D)} \\
&+ h^{\frac{1}{2}}\|\mathbf{v}\|_{\mathbf{L}^2(D)} + h^{-\frac{3}{2}}\|\mathbf{u}\|_{\mathbf{L}^2(D)} + h^{-\frac{3}{2}}\|div\,\mathbf{u}\|_{L^2(D)} \\
&+ h|\mathbf{J}(\mathbf{v}_T)|_{\mathbf{H}_{sc}^{-\frac{3}{2}}(\Gamma)} + h|\gamma\mathbf{v}|_{\mathbf{H}_{sc}^{-\frac{1}{2}}(\Gamma)}. \qquad (7.16)
\end{aligned}
$$

Proof. The idea is to derive an equation for $\mathbf{v}_N$, which we will do in Steps 1, 2, and 3. In Step 4, we then derive an a priori estimate for $\mathbf{v}_N$.

Step 1: Relating $\mathbf{v}_N$ to $\mathrm{div}_\Gamma^h \mathbf{v}_\Gamma$.

From Section 7.5, $\tilde{Q}$ is a parametrix of $-h^2\Delta - \mu(1+m)$. Then applying $\tilde{Q}$ to equation (7.12) we have that

$$
\begin{aligned}
\underline{\mathbf{v}} &= hK_{-M}\underline{\mathbf{v}} + h^2\tilde{Q}\underline{\mathbf{g}} - h^3\tilde{Q}(\frac{1}{i\mu}\nabla_h \underline{\mathrm{div}\,\mathbf{g}}) \\
&+ \tilde{Q}[\frac{h}{i}\mathbf{J}(\mathbf{v}_T)\otimes\delta_{s=0}] + \tilde{Q}[\frac{h}{i}\mathbf{v}\otimes D_s^h\delta_{s=0}]. \qquad (7.17)
\end{aligned}
$$

Taking the traces on the boundary Γ and a direct calculation (see the Calculation subsection 7.2.2) yields

$$
\begin{aligned}
-\nu\mathrm{div}_\Gamma^h\mathbf{v}_T + \mathrm{op}(\rho_2)\mathbf{v}_N &= \mathrm{op}(r_1)\left(h\gamma_N K_{-M}\underline{\mathbf{v}} + h^2\gamma_N\tilde{Q}\underline{\mathbf{g}} - h^3\gamma_N\tilde{Q}(\frac{1}{i\mu}\nabla_h\underline{\mathrm{div}\,\mathbf{g}})\right) \\
&+ h\mathrm{op}(r_{-1})\mathbf{J}(\mathbf{v}_T) + h\mathrm{op}(r_0)\mathbf{v} \\
&+ h\mathrm{op}(r_{-1})(-\nu\mathrm{div}_\Gamma^h\mathbf{v}_T) + h\mathrm{op}(r_0)\mathbf{v}_N \\
&:= \mathbf{g}_1 \qquad (7.18)
\end{aligned}
$$

where we denote the right hand side as $\mathbf{g}_1$.

Step 2. Relating $\mathbf{u}_N$ to $\mathbf{v}_N$.

Using a similar argument as in Step 1 (see the Calculation subsection 7.2.2) yields

$$\begin{aligned}
\mathbf{u}_N &= h\gamma_N K_{-M}\underline{\mathbf{u}} + h^3\gamma_N Q(mK_{-M}\underline{\mathbf{v}}) + h^4\gamma_N Qm\tilde{Q}\underline{\mathbf{g}} - h^5\gamma_N Qm\tilde{Q}(\frac{1}{i\mu}\nabla_h\underline{\operatorname{div}\mathbf{g}}) \\
&+ h^2\gamma_N Q((1+m)\underline{\mathbf{f}}) + \frac{h}{i}\gamma_N Q(\nabla_h\underline{\operatorname{div}\mathbf{u}}) \\
&+ h^2\mathrm{op}\left(\frac{m(\rho_2-\rho_1+\lambda_2-\lambda_1)}{(\lambda_1-\lambda_2)(\lambda_1-\rho_2)(\rho_1-\lambda_2)(\rho_1-\rho_2)}\right)(-\nu\operatorname{div}_\Gamma^h\mathbf{v}_T) \\
&+ h^2\mathrm{op}\left(\frac{m(\rho_2\lambda_2-\rho_1\lambda_1)}{(\lambda_1-\lambda_2)(\lambda_1-\rho_2)(\rho_1-\lambda_2)(\rho_1-\rho_2)}\right)\mathbf{v}_N + \mathrm{op}(\frac{\lambda_1}{\lambda_1-\lambda_2})\mathbf{u}_N \\
&+ h^3\mathrm{op}(r_{-4})\mathbf{J}(\mathbf{v}_T) + h^3\mathrm{op}(r_{-3})\mathbf{v} + h\mathrm{op}(r_{-1})\mathbf{u}_N + h\mathrm{op}(r_{-2})\nabla_\Gamma^h(\mathbf{u}_N\cdot\nu) \\
&:= h^2\mathrm{op}\left(\frac{m(\rho_2-\rho_1+\lambda_2-\lambda_1)}{(\lambda_1-\lambda_2)(\lambda_1-\rho_2)(\rho_1-\lambda_2)(\rho_1-\rho_2)}\right)(-\nu\operatorname{div}_\Gamma^h\mathbf{v}_T) \\
&+ h^2\mathrm{op}\left(\frac{m(\rho_2\lambda_2-\rho_1\lambda_1)}{(\lambda_1-\lambda_2)(\lambda_1-\rho_2)(\rho_1-\lambda_2)(\rho_1-\rho_2)}\right)\mathbf{v}_N + \mathrm{op}(\frac{\lambda_1}{\lambda_1-\lambda_2})\mathbf{u}_N + \mathbf{g}_2
\end{aligned} \tag{7.19}$$

Step 3. Derive an equation for $\mathbf{v}_N$.

From equation (7.10) $\mathbf{u}_N = -h^2\frac{m}{\mu(1+m)}\mathbf{v}_N - h^2\frac{1}{\mu}\mathbf{f}_N$. Then, combining this with equations (7.18) and (7.19) yields

$$\begin{aligned}
& -h^2\frac{m}{\mu(1+m)}\mathbf{v}_N - h^2\frac{1}{\mu}\mathbf{f}_N \\
= \quad & h^2\mathrm{op}(\frac{m(\rho_2\lambda_2-\rho_1\lambda_1)}{(\lambda_1-\lambda_2)(\lambda_1-\rho_2)(\rho_1-\lambda_2)(\rho_1-\rho_2)})\mathbf{v}_N + \mathrm{op}(\frac{\lambda_1}{\lambda_1-\lambda_2})(-h^2\frac{m}{\mu(1+m)}\mathbf{v}_N - h^2\frac{1}{\mu}\mathbf{f}_N) \\
+ \quad & h^2\mathrm{op}(\frac{m(\rho_2-\rho_1+\lambda_2-\lambda_1)}{(\lambda_1-\lambda_2)(\lambda_1-\rho_2)(\rho_1-\lambda_2)(\rho_1-\rho_2)})(-\mathrm{op}(\rho_2)\mathbf{v}_N + \mathbf{g}_1) + \mathbf{g}_2.
\end{aligned}$$

Hence

$$\begin{aligned}
& h^2\mathrm{op}\left(-\frac{m}{\mu(1+m)} - \frac{m(\rho_2\lambda_2-\rho_1\lambda_1) - m\rho_2(\rho_2-\rho_1+\lambda_2-\lambda_1)}{(\lambda_1-\lambda_2)(\lambda_1-\rho_2)(\rho_1-\lambda_2)(\rho_1-\rho_2)} + \frac{m}{\mu(1+m)}\frac{\lambda_1}{\lambda_1-\lambda_2}\right)\mathbf{v}_N \\
= \quad & h^2\mathrm{op}(r_0)\mathbf{f}_N + \mathbf{g}_2 + h^2\mathrm{op}(r_{-3})\mathbf{g}_1 := \mathbf{g}_3.
\end{aligned}$$

Step 4. Getting an a priori estimate for $\mathbf{v}_N$.

From equations (7.59) and (7.60) we have $\lambda_1 = -\lambda_2$, $\rho_1 = -\rho_2$, $-\lambda_2^2 = R - \mu(1+m)$ and $-\rho_2^2 = R - \mu$. Then a direct calculation yields

$$\begin{aligned}
& -\frac{m}{\mu(1+m)} - \frac{m(\rho_2\lambda_2-\rho_1\lambda_1) - m\rho_2(\rho_2-\rho_1+\lambda_2-\lambda_1)}{(\lambda_1-\lambda_2)(\lambda_1-\rho_2)(\rho_1-\lambda_2)(\rho_1-\rho_2)} + \frac{m}{\mu(1+m)}\frac{\lambda_1}{\lambda_1-\lambda_2} \\
= \quad & \frac{1}{2(1+m)\mu}\frac{\lambda_2-(1+m)\rho_2}{\lambda_2}.
\end{aligned}$$

Then

$$\text{op}(\lambda_2^2 - (1+m)^2\rho_2^2)\mathbf{v}_N = h^{-2}\text{op}(2(1+m)\mu\lambda_2(\lambda_2 + (1+m)\rho_2))\mathbf{g}_3 + h\text{op}(r_1)\mathbf{v}_N,$$

which implies that

$$\text{op}\left(m((m+2)R - (1+m)\mu)\right)\mathbf{v}_N = h^{-2}\text{op}(r_2)\mathbf{g}_3 + h\text{op}(r_1)\mathbf{v}_N.$$

Let $R_0(x,\xi')$ be the principle symbol of $R(x,\xi')$. Then

$$\text{op}\left(m((m+2)R_0 - (1+m)\mu)\right)\mathbf{v}_N = h^{-2}\text{op}(r_2)\mathbf{g}_3 + h\text{op}(r_1)\mathbf{v}_N.$$

Note that

$$(m+2)R_0 - (1+m)\mu \neq 0 \tag{7.20}$$

for any ξ' and $x \in \Gamma$. Then there exists a parametrix of $(m+2)R_0 - (1+m)\mu$ and consequently

$$\begin{aligned} & |\mathbf{v}_N|_{\mathbf{H}_{sc}^{-\frac{1}{2}}(\Gamma)} \\ \lesssim\ & h^{-2}|\mathbf{g}_3|_{\mathbf{H}_{sc}^{-\frac{1}{2}}(\Gamma)} + h|\mathbf{v}_N|_{\mathbf{H}_{sc}^{-\frac{3}{2}}(\Gamma)} \\ \lesssim\ & |\mathbf{f}_N|_{\mathbf{H}_{sc}^{-\frac{1}{2}}(\Gamma)} + h^{-2}|\mathbf{g}_2|_{\mathbf{H}_{sc}^{-\frac{1}{2}}(\Gamma)} + |\mathbf{g}_1|_{\mathbf{H}_{sc}^{-\frac{1}{2}}(\Gamma)} + h|\mathbf{v}_N|_{\mathbf{H}_{sc}^{-\frac{3}{2}}(\Gamma)} \end{aligned}$$

A direct calculation (see the Calculation subsection 7.2.2) yields the lemma. □

Now Lemma 7.2.3 and Lemma 7.2.4 now yield the following.

Lemma 7.2.5 *Assume that assumption 7.0.1 holds. Assume in addition that* $|\xi|^2 - \mu \neq 0$, $|\xi|^2 - (1+m)\mu \neq 0$ *for any* ξ *and* $x \in \overline{D}$, *and* $R_0(x,\xi') - \frac{1+m}{2+m}\mu \neq 0$ *for any* ξ' *and* $x \in \Gamma$. *Then for sufficiently small* h

$$\begin{aligned} |\mathbf{v}_N|_{\mathbf{H}_{sc}^{-\frac{1}{2}}(\Gamma)} \lesssim\ & h^{\frac{3}{2}}\|\mathbf{g}\|_{\mathbf{L}^2(D)} + h^{\frac{5}{2}}\|div\,\mathbf{g}\|_{L^2(D)} + h^{-\frac{1}{2}}\|\mathbf{f}\|_{\mathbf{L}^2(D)} \\ +\ & h^{\frac{1}{2}}\|\mathbf{v}\|_{\mathbf{L}^2(D)} + h^{-\frac{1}{2}}\|div\,\mathbf{f}\|_{L^2(D)} + h|\mathbf{J}(\mathbf{v}_T)|_{\mathbf{H}_{sc}^{-\frac{3}{2}}(\Gamma)} + h|\gamma\mathbf{v}|_{\mathbf{H}_{sc}^{-\frac{1}{2}}(\Gamma)} \end{aligned} \tag{7.21}$$

and

$$\begin{aligned} \|\mathbf{u}\|_{\mathbf{L}^2(D)} \lesssim\ & h^2\|\mathbf{v}\|_{\mathbf{L}^2(D)} + h^2\|\mathbf{f}\|_{\mathbf{L}^2(D)} + h^4\|\mathbf{g}\|_{\mathbf{L}^2(D)} + h^5\|div\,\mathbf{g}\|_{L^2(D)} \\ +\ & h^2\|div\,\mathbf{f}\|_{L^2(D)} + h^{\frac{7}{2}}|\mathbf{J}(\mathbf{v}_T)|_{\mathbf{H}_{sc}^{-\frac{3}{2}}(\Gamma)} + h^{\frac{7}{2}}|\gamma\mathbf{v}|_{\mathbf{H}_{sc}^{-\frac{1}{2}}(\Gamma)}. \end{aligned} \tag{7.22}$$

Proof. The assumptions in Lemma 7.2.3 and Lemma 7.2.4 are satisfied. Therefore we substitue estimates (7.13) and (7.32) into estimate (7.16) to get

$$\begin{aligned}|\mathbf{v}_N|_{\mathbf{H}_{sc}^{-\frac{1}{2}}(\Gamma)} &\lesssim h^{\frac{3}{2}}\|\mathbf{g}\|_{\mathbf{L}^2(D)} + h^{\frac{5}{2}}\|\operatorname{div}\mathbf{g}\|_{L^2(D)} + h^{-\frac{1}{2}}\|\mathbf{f}\|_{\mathbf{L}^2(D)} + h^{\frac{1}{2}}\|\mathbf{v}\|_{\mathbf{L}^2(D)} \\ &+ h^{-\frac{1}{2}}\|\operatorname{div}\mathbf{f}\|_{L^2(D)} + h|\mathbf{v}_N|_{\mathbf{H}_{sc}^{-\frac{1}{2}}(\Gamma)} + h|\mathbf{J}(\mathbf{v}_T)|_{\mathbf{H}_{sc}^{-\frac{3}{2}}(\Gamma)} + h|\gamma\mathbf{v}|_{\mathbf{H}_{sc}^{-\frac{1}{2}}(\Gamma)}.\end{aligned}$$

Since $\mathbf{v}_N \in \mathbf{H}_{sc}^{-\frac{1}{2}}(\Gamma)$, for h small enough we get estimate (7.21). Inequality (7.13) then yields estimate (7.22). This proves the lemma. □

Lemma 7.2.6 *Assume that assumption 7.0.1 holds. Assume in addition that* $|\xi|^2-\mu \neq 0$, $|\xi|^2-(1+m)\mu \neq 0$ *for any* ξ *and* $x \in \overline{D}$, *and* $R_0(x,\xi') - \frac{1+m}{2+m}\mu \neq 0$ *for any* ξ' *and* $x \in \Gamma$. *Then for sufficiently small* h

$$\begin{aligned}|\mathbf{u}_N|_{\mathbf{H}_{sc}^{\frac{3}{2}}(\Gamma)} &\lesssim h^{\frac{7}{2}}\|\mathbf{g}\|_{\mathbf{L}^2(D)} + h^{\frac{9}{2}}\|div\,\mathbf{g}\|_{\overline{\mathbf{H}}_{sc}^{1}(D)} + h^{\frac{3}{2}}\|\mathbf{f}\|_{\mathbf{L}^2(D)} + h^{\frac{5}{2}}\|\mathbf{v}\|_{\mathbf{L}^2(D)} \\ &+ h^{\frac{3}{2}}\|div\,((1+m)\mathbf{f})\|_{\overline{\mathbf{H}}_{sc}^{1}(D)} + h^{3}|\mathbf{J}(\mathbf{v}_T)|_{\mathbf{H}_{sc}^{-\frac{3}{2}}(\Gamma)} + h^{3}|\gamma\mathbf{v}|_{\mathbf{H}_{sc}^{-\frac{1}{2}}(\Gamma)}\end{aligned} \tag{7.23}$$

Proof. From equation (7.19) we have

$$\begin{aligned}\operatorname{op}(\frac{\lambda_2}{\lambda_2-\lambda_1})\mathbf{u}_N &= h^2\operatorname{op}(\frac{m(\rho_2-\rho_1+\lambda_2-\lambda_1)}{(\lambda_1-\lambda_2)(\lambda_1-\rho_2)(\rho_1-\lambda_2)(\rho_1-\rho_2)})(-\nu\operatorname{div}_\Gamma^h\mathbf{v}_T) \\ &+ h^2\operatorname{op}(\frac{m(\rho_2\lambda_2-\rho_1\lambda_1)}{(\lambda_1-\lambda_2)(\lambda_1-\rho_2)(\rho_1-\lambda_2)(\rho_1-\rho_2)})\mathbf{v}_N + \mathbf{g}_2.\end{aligned}$$

Applying $\lambda_2-\lambda_1$ to both sides and combining this with equation (7.18) yields

$$\begin{aligned}\operatorname{op}(\lambda_2)\mathbf{u}_N &= h^2\operatorname{op}(r_{-2})(-\operatorname{op}(\rho_2)\mathbf{v}_N + \mathbf{g}_1) \\ &+ h^2\operatorname{op}(r_{-1})\mathbf{v}_N + \operatorname{op}(r_1)\mathbf{g}_2 + h\operatorname{op}(r_0)\mathbf{u}_N.\end{aligned}$$

Since $\lambda_2 \neq 0$, for small enough h we have that

$$|\mathbf{u}_N|_{\mathbf{H}_{sc}^{\frac{3}{2}}(\Gamma)} \lesssim h^2|\mathbf{v}_N|_{\mathbf{H}_{sc}^{-\frac{1}{2}}(\Gamma)} + h^2|\mathbf{g}_1|_{\mathbf{H}_{sc}^{-\frac{1}{2}}(\Gamma)} + |\mathbf{g}_2|_{\mathbf{H}_{sc}^{\frac{3}{2}}(\Gamma)}.$$

Then a direct calculation (see the Calculation subsection 7.2.2) yields the lemma. □

Now we are ready to prove the main theorem.

Theorem 7.2.1 *Assume that assumption 7.0.1 holds. Assume in addition that* $|\xi|^2 - \mu \neq 0$, $|\xi|^2 - (1+m)\mu \neq 0$ *for any* ξ *and* $x \in \overline{D}$ *and* $R_0(x,\xi') - \frac{1+m}{2+m}\mu \neq 0$ *for any* ξ' *and* $x \in \Gamma$. *Then for sufficiently small* h

$$\|\mathbf{v}\|_{\mathbf{L}^2(D)} \lesssim h^2\|\mathbf{g}\|_{\mathbf{L}^2(D)} + h^3\|div\,\mathbf{g}\|_{\overline{\mathbf{H}}^1_{sc}(D)} + \|\mathbf{f}\|_{\mathbf{L}^2(D)} + \|div\,((1+m)\mathbf{f})\,\|_{\overline{\mathbf{H}}^1_{sc}(D)},$$
$$\|\mathbf{u}\|_{\overline{\mathbf{H}}^2_{sc}(D)} \lesssim h^2\|\mathbf{f}\|_{\mathbf{L}^2(D)} + h^4\|\mathbf{g}\|_{\mathbf{L}^2(D)} + h^5\|div\,\mathbf{g}\|_{\overline{\mathbf{H}}^1_{sc}(D)} + h^2\|div\,((1+m)\mathbf{f})\,\|_{\overline{\mathbf{H}}^1_{sc}(D)}.$$

Proof. From (7.35) we have that

$$\begin{aligned}
\mathbf{v} &= h\gamma K_{-M}\underline{\mathbf{v}} + h^2\gamma\tilde{Q}\underline{\mathbf{g}} - h^3\tilde{Q}(\frac{1}{i\mu}\nabla_h\underline{\mathrm{div}\,\mathbf{g}}) \\
&+ \mathrm{op}(\frac{1}{\rho_1-\rho_2})\mathbf{J}(\mathbf{v}_T) + \mathrm{op}(\frac{\rho_1}{\rho_1-\rho_2})\mathbf{v} + h\mathrm{op}(r_{-2})\mathbf{J}(\mathbf{v}_T) + h\mathrm{op}(r_{-1})\mathbf{v}.
\end{aligned}$$

Then

$$\begin{aligned}
\mathbf{J}(\mathbf{v}_T) + \mathrm{op}(\rho_2)\mathbf{v} &= \mathrm{op}(r_1)(h\gamma_N K_{-M}\underline{\mathbf{v}} + h^2\gamma_N\tilde{Q}\underline{\mathbf{g}} - h^3\gamma_N\tilde{Q}(\frac{1}{i\mu}\nabla_h\underline{\mathrm{div}\,\mathbf{g}})) \\
&+ h\mathrm{op}(r_{-1})\mathbf{J}(\mathbf{v}_T) + h\mathrm{op}(r_0)\mathbf{v} := \mathbf{g}_4. \qquad (7.24)
\end{aligned}$$

From (7.36) we have that

$$\begin{aligned}
\mathbf{u}_N &= h\gamma K_{-M}\underline{\mathbf{u}} + h^3\gamma Q(mK_{-M}\underline{\mathbf{v}}) + h^4\gamma Qm\tilde{Q}\underline{\mathbf{g}} - h^5\gamma Qm\tilde{Q}(\frac{1}{i\mu}\nabla_h\underline{\mathrm{div}\,\mathbf{g}}) \\
&+ h^2\gamma Q((1+m)\underline{\mathbf{f}}) + \frac{h}{i}\gamma Q(\nabla_h\underline{\mathrm{div}\,\mathbf{u}}) \\
&+ h^2\mathrm{op}(\frac{m(\rho_2-\rho_1+\lambda_2-\lambda_1)}{(\lambda_1-\lambda_2)(\lambda_1-\rho_2)(\rho_1-\lambda_2)(\rho_1-\rho_2)})\mathbf{J}(\mathbf{v}_T) \\
&+ h^2\mathrm{op}(\frac{m(\rho_2\lambda_2-\rho_1\lambda_1)}{(\lambda_1-\lambda_2)(\lambda_1-\rho_2)(\rho_1-\lambda_2)(\rho_1-\rho_2)})\mathbf{v} \\
&+ \mathrm{op}(\frac{1}{\lambda_1-\lambda_2})\nabla^h_\Gamma(\mathbf{u}_N\cdot\nu) + \mathrm{op}(\frac{\lambda_1}{\lambda_1-\lambda_2})\mathbf{u}_N \\
&+ h^3\mathrm{op}(r_{-4})\mathbf{J}(\mathbf{v}_T) + h^3\mathrm{op}(r_{-3})\mathbf{v} + h\mathrm{op}(r_{-2})\nabla^h_\Gamma(\mathbf{u}_N\cdot\nu) + h\mathrm{op}(r_{-1})\mathbf{u}_N.
\end{aligned}$$

Combining the above with equation (7.24) yields

$$
\begin{aligned}
& \mathrm{op}\Big(-\frac{m\rho_2(\rho_2-\rho_1+\lambda_2-\lambda_1)}{(\lambda_1-\lambda_2)(\lambda_1-\rho_2)(\rho_1-\lambda_2)(\rho_1-\rho_2)}+\frac{m(\rho_2\lambda_2-\rho_1\lambda_1)}{(\lambda_1-\lambda_2)(\lambda_1-\rho_2)(\rho_1-\lambda_2)(\rho_1-\rho_2)}\Big)\mathbf{v} \\
= \quad & -h^{-2}\left(h\gamma K_{-M}\underline{\mathbf{u}}+h^3\gamma Q(mK_{-M}\underline{\mathbf{v}})+h^4\gamma Qm\tilde{Q}\underline{\mathbf{g}}-h^5\gamma Qm\tilde{Q}(\frac{1}{i\mu}\nabla_h\underline{\mathrm{div}\,\mathbf{g}})\right) \\
- \quad & h^{-2}\left(h^2\gamma Q((1+m)\underline{\mathbf{f}})+\frac{h}{i}\gamma Q(\nabla_h\underline{\mathrm{div}\,\mathbf{u}})\right) \\
+ \quad & h^{-2}\mathrm{op}(r_{-1})\nabla_\Gamma^h(\mathbf{u}_N\cdot\nu)+h^{-2}\mathrm{op}(r_0)\mathbf{u}_N+\mathrm{op}(r_{-3})\mathbf{g}_4+h\mathrm{op}(r_{-4})\mathbf{J}(\mathbf{v}_T)+h\mathrm{op}(r_{-3})\mathbf{v} \\
:= \quad & \mathbf{g}_5.
\end{aligned}
$$

As in [76], the symbol

$$
-\frac{m\rho_2(\rho_2-\rho_1+\lambda_2-\lambda_1)}{(\lambda_1-\lambda_2)(\lambda_1-\rho_2)(\rho_1-\lambda_2)(\rho_1-\rho_2)}+\frac{m(\rho_2\lambda_2-\rho_1\lambda_1)}{(\lambda_1-\lambda_2)(\lambda_1-\rho_2)(\rho_1-\lambda_2)(\rho_1-\rho_2)}
$$

is not zero and we can apply its parametrix to the above equation. Then

$$
|\mathbf{v}|_{\mathbf{H}_{sc}^{-\frac{1}{2}}(\Gamma)} \lesssim |\mathbf{g}_5|_{\mathbf{H}_{sc}^{\frac{3}{2}}(\Gamma)}. \tag{7.25}
$$

Estimates (7.24) and (7.43) yields

$$
\begin{aligned}
|\mathbf{J}(\mathbf{v}_T)|_{\mathbf{H}_{sc}^{-\frac{3}{2}}(\Gamma)} & \lesssim |\mathbf{v}|_{\mathbf{H}_{sc}^{-\frac{1}{2}}(\Gamma)}+|\mathbf{g}_4|_{\mathbf{H}_{sc}^{-\frac{3}{2}}(\Gamma)} \\
& \lesssim h^{\frac{1}{2}}\|\mathbf{v}\|_{\mathbf{L}^2(D)}+h^{\frac{3}{2}}\|\mathbf{g}\|_{\mathbf{L}^2(D)}+h^{\frac{5}{2}}\|\mathrm{div}\,\mathbf{g}\|_{\overline{\mathbf{H}}_{sc}^1(D)} \\
& + h|\mathbf{J}(\mathbf{v}_T)|_{\mathbf{H}_{sc}^{-\frac{3}{2}}(\Gamma)}+|\mathbf{v}|_{\mathbf{H}_{sc}^{-\frac{1}{2}}(\Gamma)}.
\end{aligned} \tag{7.26}
$$

A direct calculation (see the Calculation subsection 7.2.2) yields for small enough h

$$
\begin{aligned}
& |\mathbf{v}|_{\mathbf{H}_{sc}^{-\frac{1}{2}}(\Gamma)}+|\mathbf{J}(\mathbf{v}_T)|_{\mathbf{H}_{sc}^{-\frac{3}{2}}(\Gamma)} \\
\lesssim \quad & h^{\frac{1}{2}}\|\mathbf{v}\|_{\mathbf{L}^2(D)}+h^{\frac{3}{2}}\|\mathbf{g}\|_{\mathbf{L}^2(D)}+h^{\frac{5}{2}}\|\mathrm{div}\,\mathbf{g}\|_{\overline{\mathbf{H}}_{sc}^1(D)} \\
+ \quad & h^{-\frac{1}{2}}\|\mathbf{f}\|_{\mathbf{L}^2(D)}+h^{-\frac{1}{2}}\|\mathrm{div}\,((1+m)\mathbf{f})\|_{\overline{\mathbf{H}}_{sc}^1(D)}.
\end{aligned} \tag{7.27}
$$

Notice that $\mathbf{v}$ satisfies equation (7.17). Then applying estimates (7.62) and (7.27) gives

$$
\|\mathbf{v}\|_{\mathbf{L}^2(D)} \lesssim h^2\|\mathbf{g}\|_{\mathbf{L}^2(D)}+h^3\|\mathrm{div}\,\mathbf{g}\|_{\overline{\mathbf{H}}_{sc}^1(D)}+\|\mathbf{f}\|_{\mathbf{L}^2(D)}+\|\mathrm{div}\,((1+m)\mathbf{f})\|_{H^1(D)}. \tag{7.28}
$$

From equation (7.14) we have that

$$
\|\mathbf{u}\|_{\overline{\mathbf{H}}_{sc}^2(D)} \lesssim h^2\|\mathbf{v}\|_{\mathbf{L}^2(D)}+h^2\|\mathbf{f}\|_{\mathbf{L}^2(D)}+h\|\mathrm{div}\,\mathbf{u}\|_{\overline{\mathbf{H}}_{sc}^1(D)}+h^{\frac{1}{2}}|\mathbf{u}_N|_{\mathbf{H}_{sc}^{\frac{3}{2}}(\Gamma)}.
$$

From estimates (7.40) (7.23) (7.27) and (7.28) we have

$$\|\mathbf{u}\|_{\overline{\mathbf{H}}^2_{sc}(D)} \lesssim h^2\|\mathbf{f}\|_{\mathbf{L}^2(D)} + h^4\|\mathbf{g}\|_{\mathbf{L}^2(D)} + h^5\|\mathrm{div}\,\mathbf{g}\|_{\overline{\mathbf{H}}^1_{sc}(D)} + h^2\|\mathrm{div}\,\left((1+m)\mathbf{f}\right)\|_{\overline{\mathbf{H}}^1_{sc}(D)}.$$

This completes the proof. □

7.2.2 Calculation

In this subsection, we will show the necessary calculations for subsection 7.2.1.

1. Calculation for Lemma 7.2.3

Taking the divergence of equation (7.4) and noticing that $\lambda = \mu h^{-2}$ yields

$$-\mu((1+m)\mathrm{div}\,\mathbf{u} + \nabla m\cdot\mathbf{u}) - h^2(\nabla m\cdot\mathbf{v} + m\mathrm{div}\,\mathbf{v}) = h^2\mathrm{div}\,\left((1+m)\,\mathbf{f}\right). \quad (7.29)$$

Since ∇m has compact support in D and $|\xi|^2 - \mu \neq 0$, estimate (7.63) yields

$$\|\nabla m\cdot\mathbf{v}\|_{L^2(D)} \lesssim h\|\mathbf{v}\|_{\mathbf{L}^2(D)} + h^2\|\mathbf{g}\|_{\mathbf{L}^2(D)} + h^3\|\mathrm{div}\,\mathbf{g}\|_{L^2(D)}.$$

Therefore

$$\begin{aligned}\|\mathrm{div}\,\mathbf{u}\|_{L^2(D)} &\lesssim \|\mathbf{u}\|_{\mathbf{L}^2(D)} + h^3\|\mathbf{v}\|_{\mathbf{L}^2(D)} + h^2\|\mathrm{div}\,\mathbf{v}\|_{\mathbf{L}^2(D)} + h^4\|\mathbf{g}\|_{\mathbf{L}^2(D)} + h^5\|\mathrm{div}\,\mathbf{g}\|_{L^2(D)} \\ &+ h^2\|\mathrm{div}\,(1+m)\mathbf{f}\|_{L^2(D)}. \end{aligned} \quad (7.30)$$

Since $\mathrm{div}\,(-\lambda\mathrm{div}\,\mathbf{v}) = \mathrm{div}\,\mathbf{g}$, we have that

$$\mu\mathrm{div}\,\mathbf{v} = -h^2\mathrm{div}\,\mathbf{g}$$

and therefore

$$\|\mathbf{v}\|_{\overline{\mathbf{H}}^s_{sc}(D)} \lesssim h^2\|\mathbf{g}\|_{\overline{\mathbf{H}}^s_{sc}(D)}. \quad (7.31)$$

Substituting (7.31) (with s=0) into (7.30) yields

$$\begin{aligned}\|\mathrm{div}\,\mathbf{u}\|_{L^2(D)} &\lesssim \|\mathbf{u}\|_{\mathbf{L}^2(D)} + h^3\|\mathbf{v}\|_{\mathbf{L}^2(D)} + h^2\|\mathrm{div}\,\mathbf{v}\|_{\mathbf{L}^2(D)} + h^4\|\mathbf{g}\|_{\mathbf{L}^2(D)} + h^4\|\mathrm{div}\,\mathbf{g}\|_{L^2(D)} \\ &+ h^2\|\mathrm{div}\,(1+m)\mathbf{f}\|_{L^2(D)}. \end{aligned} \quad (7.32)$$

Notice that since $|\mathbf{f}_N|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma)} \lesssim \|\mathbf{f}\|_{\mathbf{L}^2(D)} + \|\operatorname{div} \mathbf{f}\|_{L^2(D)}$, then

$$|\mathbf{f}_N|_{\mathbf{H}_{sc}^{-\frac{1}{2}}(\Gamma)} \lesssim h^{-\frac{1}{2}} \left(\|\mathbf{f}\|_{\mathbf{L}^2(D)} + \|\operatorname{div} \mathbf{f}\|_{L^2(D)}\right). \tag{7.33}$$

From equation (7.10) and estimate (7.33) we have that

$$\begin{aligned} |\mathbf{u}_N|_{\mathbf{H}_{sc}^{-\frac{1}{2}}(\Gamma)} &\lesssim h^2|\mathbf{v}_N|_{\mathbf{H}_{sc}^{-\frac{1}{2}}(\Gamma)} + h^2|\mathbf{f}_N|_{\mathbf{H}_{sc}^{-\frac{1}{2}}(\Gamma)} \\ &\lesssim h^2|\mathbf{v}_N|_{\mathbf{H}_{sc}^{-\frac{1}{2}}(\Gamma)} + h^{\frac{3}{2}} \left(\|\mathbf{f}\|_{\mathbf{L}^2(D)} + \|\operatorname{div} \mathbf{f}\|_{L^2(D)}\right). \end{aligned} \tag{7.34}$$

Plugging estimates (7.32) and (7.34) into (7.15) yields for h small enough

$$\begin{aligned} \|\mathbf{u}\|_{\mathbf{L}^2(D)} &\lesssim h^2\|\mathbf{v}\|_{\mathbf{L}^2(D)} + h^2\|\mathbf{f}\|_{\mathbf{L}^2(D)} + h^5\|\mathbf{g}\|_{\mathbf{L}^2(D)} + h^5\|\operatorname{div} \mathbf{g}\|_{L^2(D)} \\ &+ h^2\|\operatorname{div} \mathbf{f}\|_{L^2(D)} + h^{\frac{5}{2}}|\mathbf{v}_N|_{\mathbf{H}_{sc}^{-\frac{1}{2}}(\Gamma)}. \end{aligned}$$

2. Calculation for Lemma 7.2.4

Calculation for Step 1

Taking the traces on the boundary Γ and using equations (7.64)-(7.65) we have

$$\begin{aligned} \gamma\mathbf{v} &= h\gamma K_{-M}\underline{\mathbf{v}} + h^2\gamma\tilde{Q}\underline{\mathbf{g}} - h^3\gamma\tilde{Q}(\frac{1}{i\mu}\nabla_h\underline{\operatorname{div} \mathbf{g}}) \\ &+ \operatorname{op}(\frac{1}{\rho_1-\rho_2})\mathbf{J}(\mathbf{v}_T) + \operatorname{op}(\frac{\rho_1}{\rho_1-\rho_2})\mathbf{v} + h\operatorname{op}(r_{-2})\mathbf{J}(\mathbf{v}_T) + h\operatorname{op}(r_{-1})\mathbf{v} \end{aligned} \tag{7.35}$$

where γ is the trace operator on Γ. Taking the normal component yields

$$\begin{aligned} \mathbf{v}_N &= h\gamma_N K_{-M}\underline{\mathbf{v}} + h^2\gamma_N\tilde{Q}\underline{\mathbf{g}} - h^3\gamma_N\tilde{Q}(\frac{1}{i\mu}\nabla_h\underline{\operatorname{div} \mathbf{g}}) \\ &+ \operatorname{op}(\frac{1}{\rho_1-\rho_2})(-\nu\operatorname{div}_\Gamma^h\mathbf{v}_T) + \operatorname{op}(\frac{\rho_1}{\rho_1-\rho_2})\mathbf{v}_N \\ &+ [\gamma_N, \operatorname{op}(\frac{1}{\rho_1-\rho_2})]\mathbf{J}(\mathbf{v}_T) + [\gamma_N, \operatorname{op}(\frac{\rho_1}{\rho_1-\rho_2})]\mathbf{v} + h\operatorname{op}(r_{-2})\mathbf{J}(\mathbf{v}_T) + h\operatorname{op}(r_{-1})\mathbf{v}. \end{aligned}$$

Since $[\gamma_N, \operatorname{op}(\frac{1}{\rho_1-\rho_2})]$ and $[\gamma_N, \operatorname{op}(\frac{\rho_1}{\rho_1-\rho_2})]$ are pseudo-differential operators in $h\operatorname{op}_h S^{-2}$ and $h\operatorname{op}_h S^{-1}$ respectively, then

$$\begin{aligned} \mathbf{v}_N &= h\gamma_N K_{-M}\underline{\mathbf{v}} + h^2\gamma_N\tilde{Q}\underline{\mathbf{g}} - h^3\gamma_N\tilde{Q}(\frac{1}{i\mu}\nabla_h\underline{\operatorname{div} \mathbf{g}}) \\ &+ \operatorname{op}(\frac{1}{\rho_1-\rho_2})(-\nu\operatorname{div}_\Gamma^h\mathbf{v}_T) + \operatorname{op}(\frac{\rho_1}{\rho_1-\rho_2})\mathbf{v}_N + h\operatorname{op}(r_{-2})\mathbf{J}(\mathbf{v}_T) + h\operatorname{op}(r_{-1})\mathbf{v}. \end{aligned}$$

Applying $\mathrm{op}(\rho_2 - \rho_1)$ to both sides yields equation (7.18).

Calculation for Step 2

Substituting equation (7.17) into equation (7.14) yields

$$\begin{aligned}
\mathbf{u} &= hK_{-M}\underline{\mathbf{u}} + h^3 Q(mK_{-M}\underline{\mathbf{v}}) + h^4 Qm\tilde{Q}\underline{\mathbf{g}} - h^5 Qm\tilde{Q}(\frac{1}{i\mu}\nabla_h \underline{\mathrm{div}\,\mathbf{g}}) \\
&+ h^2 Q((1+m)\underline{\mathbf{f}}) + \frac{h}{i}Q(\nabla_h \underline{\mathrm{div}\,\mathbf{u}}) \\
&+ h^2 Qm\tilde{Q}[\frac{h}{i}\mathbf{J}(\mathbf{v}_T)\otimes\delta_{s=0} + \frac{h}{i}\mathbf{v}\otimes D_s^h\delta_{s=0}] \\
&+ Q(\frac{h}{i}\nabla_\Gamma^h(\mathbf{u}_N\cdot\nu)\otimes\delta_{s=0}) + Q(\frac{h}{i}\mathbf{u}_N\otimes D_s^h\delta_{s=0}).
\end{aligned}$$

Taking the traces on Γ and using equations (7.66) (7.67) yields

$$\begin{aligned}
\mathbf{u}|_\Gamma &= h\gamma K_{-M}\underline{\mathbf{u}} + h^3\gamma Q(mK_{-M}\underline{\mathbf{v}}) + h^4\gamma Qm\tilde{Q}\underline{\mathbf{g}} - h^5\gamma Qm\tilde{Q}(\frac{1}{i\mu}\nabla_h\underline{\mathrm{div}\,\mathbf{g}}) \\
&+ h^2\gamma Q((1+m)\underline{\mathbf{f}}) + \frac{h}{i}\gamma Q(\nabla_h\underline{\mathrm{div}\,\mathbf{u}}) \\
&+ h^2\mathrm{op}\left(\frac{m(\rho_2-\rho_1+\lambda_2-\lambda_1)}{(\lambda_1-\lambda_2)(\lambda_1-\rho_2)(\rho_1-\lambda_2)(\rho_1-\rho_2)}\right)\mathbf{J}(\mathbf{v}_T) \\
&+ h^2\mathrm{op}\left(\frac{m(\rho_2\lambda_2-\rho_1\lambda_1)}{(\lambda_1-\lambda_2)(\lambda_1-\rho_2)(\rho_1-\lambda_2)(\rho_1-\rho_2)}\right)\mathbf{v} \\
&+ \mathrm{op}(\frac{1}{\lambda_1-\lambda_2})\nabla_\Gamma^h(\mathbf{u}_N\cdot\nu) + \mathrm{op}(\frac{\lambda_1}{\lambda_1-\lambda_2})\mathbf{u}_N \\
&+ h^3\mathrm{op}(r_{-4})\mathbf{J}(\mathbf{v}_T) + h^3\mathrm{op}(r_{-3})\mathbf{v} + h\mathrm{op}(r_{-2})\nabla_\Gamma^h(\mathbf{u}_N\cdot\nu) + h\mathrm{op}(r_{-1})\mathbf{u}_N.
\end{aligned} \tag{7.36}$$

Taking the normal component and noticing that $\nu\cdot\nabla_\Gamma^h(\mathbf{u}_N\cdot\nu) = 0$ yields equation (7.19).

Calculation for Step 4

Applying estimates (7.61) and (7.68) gives

$$\begin{aligned}
|\mathbf{g}_2|_{\mathbf{H}_{sc}^{-\frac{1}{2}}(\Gamma)} &\lesssim h^{\frac{1}{2}}\|\mathbf{u}\|_{\mathbf{L}^2(D)} + h^{\frac{5}{2}}\|\mathbf{v}\|_{\mathbf{L}^2(D)} + h^{\frac{7}{2}}\|\mathbf{g}\|_{\mathbf{L}^2(D)} + h^{\frac{9}{2}}\|\mathrm{div}\,\mathbf{g}\|_{\mathbf{L}^2(D)} + h^{\frac{3}{2}}\|\mathbf{f}\|_{\mathbf{L}^2(D)} \\
&+ h^{\frac{1}{2}}\|\mathrm{div}\,\mathbf{u}\|_{\mathbf{L}^2(D)} + h^3|\mathbf{J}(\mathbf{v}_T)|_{\mathbf{H}_{sc}^{-\frac{3}{2}}(\Gamma)} + h^3|\mathbf{v}|_{\mathbf{H}_{sc}^{-\frac{1}{2}}(\Gamma)} + h|\mathbf{u}_N|_{\mathbf{H}_{sc}^{-\frac{1}{2}}(\Gamma)}.
\end{aligned}$$

From equation (7.10) $\mathbf{u}_N = -h^2 \frac{m}{\mu(1+m)} \mathbf{v}_N - h^2 \frac{1}{\mu} \mathbf{f}_N$, and therefore

$$\begin{aligned} |\mathbf{g}_2|_{\mathbf{H}_{sc}^{-\frac{1}{2}}(\Gamma)} &\lesssim h^{\frac{1}{2}} \|\mathbf{u}\|_{\mathbf{L}^2(D)} + h^{\frac{5}{2}} \|\mathbf{v}\|_{\mathbf{L}^2(D)} + h^{\frac{7}{2}} \|\mathbf{g}\|_{\mathbf{L}^2(D)} + h^{\frac{9}{2}} \|\operatorname{div} \mathbf{g}\|_{\mathbf{L}^2(D)} + h^{\frac{3}{2}} \|\mathbf{f}\|_{\mathbf{L}^2(D)} \\ &+ h^{\frac{1}{2}} \|\operatorname{div} \mathbf{u}\|_{\mathbf{L}^2(D)} + h^3 |\mathbf{J}(\mathbf{v}_T)|_{\mathbf{H}_{sc}^{-\frac{3}{2}}(\Gamma)} + h^3 |\mathbf{v}|_{\mathbf{H}_{sc}^{-\frac{1}{2}}(\Gamma)} \\ &+ h^3 |\mathbf{v}_N|_{\mathbf{H}_{sc}^{-\frac{1}{2}}(\Gamma)} + h^3 |\mathbf{f}_N|_{\mathbf{H}_{sc}^{-\frac{1}{2}}(\Gamma)}. \end{aligned} \tag{7.37}$$

Applying estimates (7.61) and (7.68) yield

$$\begin{aligned} |\mathbf{g}_1|_{\mathbf{H}_{sc}^{-\frac{1}{2}}(\Gamma)} &\lesssim h^{\frac{1}{2}} \|\mathbf{v}\|_{\mathbf{L}^2(D)} + h^{\frac{3}{2}} \|\mathbf{g}\|_{\mathbf{L}^2(D)} + h^{\frac{5}{2}} \|\operatorname{div} \mathbf{g}\|_{\mathbf{L}^2(D)} \\ &+ h |\mathbf{J}(\mathbf{v}_T)|_{\mathbf{H}_{sc}^{-\frac{3}{2}}(\Gamma)} + h |\mathbf{v}|_{\mathbf{H}_{sc}^{-\frac{1}{2}}(\Gamma)} + h |\mathbf{v}_N|_{\mathbf{H}_{sc}^{-\frac{1}{2}}(\Gamma)} + h |\nu \operatorname{div}_\Gamma^h \mathbf{v}_T|_{\mathbf{H}_{sc}^{-\frac{3}{2}}(\Gamma)}. \end{aligned}$$

Since $\mathbf{v}_N$ and $-\nu \operatorname{div}_\Gamma^h \mathbf{v}_T$ are the normal components of $\mathbf{v}$ and $\mathbf{J}(\mathbf{v}_T)$ respectively, then

$$\begin{aligned} |\mathbf{g}_1|_{\mathbf{H}_{sc}^{-\frac{1}{2}}(\Gamma)} &\lesssim h^{\frac{1}{2}} \|\mathbf{v}\|_{\mathbf{L}^2(D)} + h^{\frac{3}{2}} \|\mathbf{g}\|_{\mathbf{L}^2(D)} + h^{\frac{5}{2}} \|\operatorname{div} \mathbf{g}\|_{\mathbf{L}^2(D)} \\ &+ h |\mathbf{J}(\mathbf{v}_T)|_{\mathbf{H}_{sc}^{-\frac{3}{2}}(\Gamma)} + h |\mathbf{v}|_{\mathbf{H}_{sc}^{-\frac{1}{2}}(\Gamma)}. \end{aligned} \tag{7.38}$$

Then estimates (7.33) (7.37) and (7.38) yield for small enough that

$$\begin{aligned} |\mathbf{v}_N|_{\mathbf{H}_{sc}^{-\frac{1}{2}}(\Gamma)} &\lesssim h^{\frac{3}{2}} \|\mathbf{g}\|_{\mathbf{L}^2(D)} + h^{\frac{5}{2}} \|\operatorname{div} \mathbf{g}\|_{\mathbf{L}^2(D)} + h^{-\frac{1}{2}} \|\mathbf{f}\|_{\mathbf{L}^2(D)} + h^{-\frac{1}{2}} \|\operatorname{div} \mathbf{f}\|_{L^2(D)} \\ &+ h^{\frac{1}{2}} \|\mathbf{v}\|_{\mathbf{L}^2(D)} + h^{-\frac{3}{2}} \|\mathbf{u}\|_{\mathbf{L}^2(D)} + h^{-\frac{3}{2}} \|\operatorname{div} \mathbf{u}\|_{L^2(D)} \\ &+ h |\mathbf{J}(\mathbf{v}_T)|_{\mathbf{H}_{sc}^{-\frac{3}{2}}(\Gamma)} + h |\gamma \mathbf{v}|_{\mathbf{H}_{sc}^{-\frac{1}{2}}(\Gamma)}. \end{aligned}$$

3. Calculation for Lemma 7.2.6

From inequalities (7.61) and (7.68) one get

$$\begin{aligned} |\mathbf{g}_2|_{\mathbf{H}_{sc}^{\frac{3}{2}}(\Gamma)} &\lesssim h^{\frac{1}{2}} \|\mathbf{u}\|_{\mathbf{L}^2(D)} + h^{\frac{5}{2}} \|\mathbf{v}\|_{\mathbf{L}^2(D)} + h^{\frac{7}{2}} \|\mathbf{g}\|_{\mathbf{L}^2(D)} + h^{\frac{9}{2}} \|\operatorname{div} \mathbf{g}\|_{\mathbf{L}^2(D)} + h^{\frac{3}{2}} \|\mathbf{f}\|_{\mathbf{L}^2(D)} \\ &+ h \| |\gamma_N Q \nabla_h \underline{\operatorname{div} \mathbf{u}}|_{\mathbf{H}_{sc}^{\frac{3}{2}}(\Gamma)} + h^3 |\mathbf{J}(\mathbf{v}_T)|_{\mathbf{H}_{sc}^{-\frac{3}{2}}(\Gamma)} + h^3 |\mathbf{v}|_{\mathbf{H}_{sc}^{-\frac{1}{2}}(\Gamma)} + h |\mathbf{u}_N|_{\mathbf{H}_{sc}^{\frac{1}{2}}(\Gamma)} \end{aligned} \tag{7.39}$$

This motivates us to derive an estimate for $|\gamma_N Q \nabla_h \underline{\operatorname{div} \mathbf{u}}|_{\mathbf{H}_{sc}^{\frac{3}{2}}(\Gamma)}$. Since ∇m has compact support in D, then estimate (7.63) yields

$$\|\nabla m \cdot \mathbf{v}\|_{\overline{\mathbf{H}}_{sc}^{1}(D)} \lesssim h \|\mathbf{v}\|_{\mathbf{L}^2(D)} + h^2 \|\mathbf{g}\|_{\mathbf{L}^2(D)} + h^3 \|\operatorname{div} \mathbf{g}\|_{L^2(D)}$$

and

$$\|\nabla m\cdot\mathbf{u}\|_{\overline{\mathbf{H}}^1_{sc}(D)}\lesssim h\|\mathbf{u}\|_{\mathbf{L}^2(D)}+h^2\|\mathbf{v}\|_{\mathbf{L}^2(D)}+h^2\|\mathbf{f}\|_{\mathbf{L}^2(D)}+h\|\operatorname{div}\mathbf{u}\|_{L^2(D)}.$$

From equation (7.29) and estimate (7.31) (with s=1) we have that for small h

$$\begin{aligned}\|\operatorname{div}\mathbf{u}\|_{\overline{\mathbf{H}}^1_{sc}(D)} &\lesssim \|\nabla m\cdot\mathbf{u}\|_{\overline{\mathbf{H}}^1_{sc}(D)}+h^2\|\nabla m\cdot\mathbf{v}\|_{\overline{\mathbf{H}}^1_{sc}(D)}+h^2\|\operatorname{div}\mathbf{v}\|_{\overline{\mathbf{H}}^1_{sc}(D)}\\ &+ h^2\|\operatorname{div}\left((1+m)\mathbf{f}\right)\|_{\overline{\mathbf{H}}^1_{sc}(D)}\\ &\lesssim h^2\|\mathbf{v}\|_{\mathbf{L}^2(D)}+h^4\|\mathbf{g}\|_{\mathbf{L}^2(D)}+h^4\|\operatorname{div}\mathbf{g}\|_{\overline{\mathbf{H}}^1_{sc}(D)}+h\|\mathbf{u}\|_{\mathbf{L}^2(D)}\\ &+ h^2\|\operatorname{div}\left((1+m)\mathbf{f}\right)\|_{\overline{\mathbf{H}}^1_{sc}(D)}+h^2\|\mathbf{f}\|_{\mathbf{L}^2(D)}. \end{aligned}\tag{7.40}$$

From Lemma 7.5.1 and estimate (7.68) we have

$$\begin{aligned}|\gamma_N Q\nabla_h\underline{\operatorname{div}\mathbf{u}}|_{\mathbf{H}^{\frac{3}{2}}_{sc}(\Gamma)} &\lesssim h^{-\frac{1}{2}}\|Q\nabla_h\underline{\operatorname{div}\mathbf{u}}\|_{\overline{\mathbf{H}}^2_{sc}(D)}\\ &\lesssim h^{-\frac{1}{2}}\|\operatorname{div}\mathbf{u}\|_{\overline{\mathbf{H}}^1_{sc}(D)}.\end{aligned}$$

Combined with (7.40), this inequality gives

$$\begin{aligned}|\gamma_N Q\nabla_h\underline{\operatorname{div}\mathbf{u}}|_{\mathbf{H}^{\frac{3}{2}}_{sc}(\Gamma)} &\lesssim h^{\frac{3}{2}}\|\mathbf{v}\|_{\mathbf{L}^2(D)}+h^{\frac{7}{2}}\|\mathbf{g}\|_{\mathbf{L}^2(D)}+h^{\frac{7}{2}}\|\operatorname{div}\mathbf{g}\|_{\overline{\mathbf{H}}^1_{sc}(D)}+h^{\frac{1}{2}}\|\mathbf{u}\|_{\mathbf{L}^2(D)}\\ &+ h^{\frac{3}{2}}\|\operatorname{div}\left((1+m)\mathbf{f}\right)\|_{\overline{\mathbf{H}}^1_{sc}(D)}+h^{\frac{3}{2}}\|\mathbf{f}\|_{\mathbf{L}^2(D)}.\end{aligned}\tag{7.41}$$

Substituting estimates (7.22) and (7.41) into (7.39) yields

$$\begin{aligned}|\mathbf{g}_2|_{\mathbf{H}^{\frac{3}{2}}_{sc}(\Gamma)} &\lesssim h^{\frac{1}{2}}\|\mathbf{u}\|_{\mathbf{L}^2(D)}+h^{\frac{5}{2}}\|\mathbf{v}\|_{\mathbf{L}^2(D)}+h^{\frac{7}{2}}\|\mathbf{g}\|_{\mathbf{L}^2(D)}+h^{\frac{9}{2}}\|\operatorname{div}\mathbf{g}\|_{\overline{\mathbf{H}}^1_{sc}(D)}+h^{\frac{3}{2}}\|\mathbf{f}\|_{\mathbf{L}^2(D)}\\ &+ h^{\frac{5}{2}}\|\operatorname{div}\left((1+m)\mathbf{f}\right)\|_{\overline{\mathbf{H}}^1_{sc}(D)}+h^3|\mathbf{J}(\mathbf{v}_T)|_{\mathbf{H}^{-\frac{3}{2}}_{sc}(\Gamma)}+h^3|\mathbf{v}|_{\mathbf{H}^{-\frac{1}{2}}_{sc}(\Gamma)}+h|\mathbf{u}_N|_{\mathbf{H}^{\frac{1}{2}}_{sc}(\Gamma)}\\ &\lesssim h^{\frac{7}{2}}\|\mathbf{g}\|_{\mathbf{L}^2(D)}+h^{\frac{9}{2}}\|\operatorname{div}\mathbf{g}\|_{\overline{\mathbf{H}}^1_{sc}(D)}+h^{\frac{3}{2}}\|\mathbf{f}\|_{\mathbf{L}^2(D)}+h^{\frac{5}{2}}\|\operatorname{div}\left((1+m)\mathbf{f}\right)\|_{\overline{\mathbf{H}}^1_{sc}(D)}\\ &+ h^{\frac{5}{2}}\|\mathbf{v}\|_{\mathbf{L}^2(D)}+h^3|\mathbf{J}(\mathbf{v}_T)|_{\mathbf{H}^{-\frac{3}{2}}_{sc}(\Gamma)}+h^3|\mathbf{v}|_{\mathbf{H}^{-\frac{1}{2}}_{sc}(\Gamma)}+h|\mathbf{u}_N|_{\mathbf{H}^{\frac{1}{2}}_{sc}(\Gamma)}.\end{aligned}\tag{7.42}$$

Combining estimates (7.38) (7.21) and (7.42) implies that

$$\begin{aligned}|\mathbf{u}_N|_{\mathbf{H}^{\frac{3}{2}}_{sc}(\Gamma)} &\lesssim h^2|\mathbf{v}_N|_{\mathbf{H}^{-\frac{1}{2}}_{sc}(\Gamma)}+h^2|\mathbf{g}_1|_{\mathbf{H}^{-\frac{1}{2}}_{sc}(\Gamma)}+|\mathbf{g}_2|_{\mathbf{H}^{\frac{3}{2}}_{sc}(\Gamma)}\\ &\lesssim h^{\frac{7}{2}}\|\mathbf{g}\|_{\mathbf{L}^2(D)}+h^{\frac{9}{2}}\|\operatorname{div}\mathbf{g}\|_{\overline{\mathbf{H}}^1_{sc}(D)}+h^{\frac{3}{2}}\|\mathbf{f}\|_{\mathbf{L}^2(D)}+h^{\frac{5}{2}}\|\mathbf{v}\|_{\mathbf{L}^2(D)}\\ &+ h^{\frac{3}{2}}\|\operatorname{div}\left((1+m)\mathbf{f}\right)\|_{\overline{\mathbf{H}}^1_{sc}(D)}+h^3|\mathbf{J}(\mathbf{v}_T)|_{\mathbf{H}^{-\frac{3}{2}}_{sc}(\Gamma)}+h^3|\gamma\mathbf{v}|_{\mathbf{H}^{-\frac{1}{2}}_{sc}(\Gamma)}.\end{aligned}$$

4. Calculation for Theorem 7.2.1

Applying estimates (7.61) and (7.68) gives

$$\begin{aligned}|\mathbf{g}_5|_{\mathbf{H}^{\frac{3}{2}}_{sc}(\Gamma)} &\lesssim h^{-\frac{3}{2}}\|\mathbf{u}\|_{\mathbf{L}^2(D)} + h^{\frac{1}{2}}\|\mathbf{v}\|_{\mathbf{L}^2(D)} + h^{\frac{3}{2}}\|\mathbf{g}\|_{\mathbf{L}^2(D)} + h^{\frac{5}{2}}\|\mathrm{div}\,\mathbf{g}\|_{\mathbf{L}^2(D)} + h^{-\frac{1}{2}}\|\mathbf{f}\|_{\mathbf{L}^2(D)}\\ &+ h^{-1}\|\gamma Q\nabla_h \underline{\mathrm{div}\,\mathbf{u}}\|_{\mathbf{H}^{\frac{3}{2}}_{sc}(\Gamma)} + h^{-2}|\mathbf{u}_N|_{\mathbf{H}^{\frac{3}{2}}_{sc}(\Gamma)}\\ &+ h|\mathbf{J}(\mathbf{v}_T)|_{\mathbf{H}^{-\frac{3}{2}}_{sc}(\Gamma)} + h|\mathbf{v}|_{\mathbf{H}^{-\frac{1}{2}}_{sc}(\Gamma)} + |\mathbf{g}_4|_{\mathbf{H}^{-\frac{3}{2}}_{sc}(\Gamma)}.\end{aligned}$$

Applying estimates (7.61) and (7.68) gives

$$\begin{aligned}|\mathbf{g}_4|_{\mathbf{H}^{-\frac{3}{2}}_{sc}(\Gamma)} &\lesssim h^{\frac{1}{2}}\|\mathbf{v}\|_{\mathbf{L}^2(D)} + h^{\frac{3}{2}}\|\mathbf{g}\|_{\mathbf{L}^2(D)} + h^{\frac{5}{2}}\|\mathrm{div}\,\mathbf{g}\|_{\mathbf{L}^2(D)}\\ &+ h|\mathbf{J}(\mathbf{v}_T)|_{\mathbf{H}^{-\frac{3}{2}}_{sc}(\Gamma)} + h|\mathbf{v}|_{\mathbf{H}^{-\frac{1}{2}}_{sc}(\Gamma)}. \qquad (7.43)\end{aligned}$$

Combining estimates (7.22) (7.23) (7.41) and (7.43) yields

$$\begin{aligned}|\mathbf{v}|_{\mathbf{H}^{-\frac{1}{2}}_{sc}(\Gamma)} &\lesssim |\mathbf{g}_5|_{\mathbf{H}^{\frac{3}{2}}_{sc}(\Gamma)} \qquad (7.44)\\ &\lesssim h^{\frac{1}{2}}\|\mathbf{v}\|_{\mathbf{L}^2(D)} + h^{\frac{3}{2}}\|\mathbf{g}\|_{\mathbf{L}^2(D)} + h^{\frac{5}{2}}\|\mathrm{div}\,\mathbf{g}\|_{\overline{\mathbf{H}}^1_{sc}(D)} + h^{-\frac{1}{2}}\|\mathbf{f}\|_{\mathbf{L}^2(D)}\\ &+ h^{-\frac{1}{2}}\|\mathrm{div}\,((1+m)\mathbf{f})\|_{\overline{\mathbf{H}}^1_{sc}(D)} + h|\mathbf{J}(\mathbf{v}_T)|_{\mathbf{H}^{-\frac{3}{2}}_{sc}(\Gamma)} + h|\mathbf{v}|_{\mathbf{H}^{-\frac{1}{2}}_{sc}(\Gamma)}. \qquad (7.45)\end{aligned}$$

Combining estimates (7.26) and (7.45) yields

$$\begin{aligned}&|\mathbf{v}|_{\mathbf{H}^{-\frac{1}{2}}_{sc}(\Gamma)} + |\mathbf{J}(\mathbf{v}_T)|_{\mathbf{H}^{-\frac{3}{2}}_{sc}(\Gamma)}\\ \lesssim\ & h^{\frac{1}{2}}\|\mathbf{v}\|_{\mathbf{L}^2(D)} + h^{\frac{3}{2}}\|\mathbf{g}\|_{\mathbf{L}^2(D)} + h^{\frac{5}{2}}\|\mathrm{div}\,\mathbf{g}\|_{\overline{\mathbf{H}}^1_{sc}(D)} + h^{-\frac{1}{2}}\|\mathbf{f}\|_{\mathbf{L}^2(D)}\\ +\ & h^{-\frac{1}{2}}\|\mathrm{div}\,((1+m)\mathbf{f})\|_{\overline{\mathbf{H}}^1_{sc}(D)} + h|\mathbf{J}(\mathbf{v}_T)|_{\mathbf{H}^{-\frac{3}{2}}_{sc}(\Gamma)} + h|\mathbf{v}|_{\mathbf{H}^{-\frac{1}{2}}_{sc}(\Gamma)}.\end{aligned}$$

Then for small enough h we have estimate (7.27).

7.2.3 A Second Regularity Result

In this section we study the regularity under the restriction that $\mathrm{div}\,((1+m)\,\mathbf{f}) = 0$ and $\mathrm{div}\,\mathbf{g} = 0$. The reason to consider this case is to obtain a regularizing effect of the operator R_z. In particular, from equation (7.29), we see that $\mathrm{div}\,\mathbf{u}$ has the same regularity as $\mathrm{div}\,((1+m)\mathbf{f})$ (with a similar situation for $\mathbf{v}$) and therefore the regularizing effect does not hold in general. On the other hand, if the right hand side of equation

(7.29) vanishes, then the regularity of $\operatorname{div}\mathbf{u}$ is controlled by $\mathbf{u}$ and $\nabla m\cdot\mathbf{v}$. This allows us to obtain the desired regularity of $\mathbf{u}$.

Theorem 7.2.2 *Assume that the hypothesis of Theorem 7.2.1 hold. If* $\mathbf{f}\in\overline{\mathbf{H}}^2_{sc}(D)$, *div* $((1+m)\,\mathbf{f})=0$ *and* $\operatorname{div}\mathbf{g}=0$, *then for sufficiently small* $h:=\frac{1}{|\lambda|^{\frac{1}{2}}}$

$$\|\mathbf{v}\|_{\overline{\mathbf{H}}^2_{sc}(D)}\lesssim h^2\|\mathbf{g}\|_{\mathbf{L}^2(D)}+\|\mathbf{f}\|_{\overline{\mathbf{H}}^2_{sc}(D)},$$
$$\|\mathbf{u}\|_{\overline{\mathbf{H}}^4_{sc}(D)}\lesssim h^4\|\mathbf{g}\|_{\mathbf{L}^2(D)}+h^2\|\mathbf{f}\|_{\overline{\mathbf{H}}^2_{sc}(D)}.$$

Moreover if $\mathbf{f}\in\overline{\mathbf{H}}^4_{sc}(D)$ *and* $\mathbf{g}\in\overline{\mathbf{H}}^2_{sc}(D)$, *then for sufficiently small* $h:=\frac{1}{|\lambda|^{\frac{1}{2}}}$

$$\|\mathbf{v}\|_{\overline{\mathbf{H}}^4_{sc}(D)}\lesssim h^2\|\mathbf{g}\|_{\overline{\mathbf{H}}^2_{sc}(D)}+\|\mathbf{f}\|_{\overline{\mathbf{H}}^4_{sc}(D)},$$
$$\|\mathbf{u}\|_{\overline{\mathbf{H}}^6_{sc}(D)}\lesssim h^4\|\mathbf{g}\|_{\overline{\mathbf{H}}^2_{sc}(D)}+h^2\|\mathbf{f}\|_{\overline{\mathbf{H}}^4_{sc}(D)}.$$

Proof. We use similar arguments as in Section 7.2.1 and we shall only highlight here the differences. We first prove that $\mathbf{v}\in\overline{\mathbf{H}}^1_{sc}(D)$ and $\mathbf{u}\in\overline{\mathbf{H}}^3_{sc}(D)$ if $\mathbf{v}\in\mathbf{L}^2(D)$ and $\mathbf{u}\in\overline{\mathbf{H}}^2_{sc}(D)$ for $\mathbf{g}\in\mathbf{L}^2(D)$ and $\mathbf{f}\in\overline{\mathbf{H}}^2_{sc}(D)$, then we can prove $\mathbf{v}\in\overline{\mathbf{H}}^2_{sc}(D)$ and $\mathbf{u}\in\overline{\mathbf{H}}^4_{sc}(D)$.

1. (Similarly to Lemma 7.2.3) An a priori estimate for $\mathbf{u}$.

   Since $\operatorname{div}\,((1+m)\,\mathbf{f})=0$ and $\operatorname{div}\mathbf{g}=0$, Theorem 7.2.1 yields

$$\|\mathbf{u}\|_{\overline{\mathbf{H}}^2_{sc}(D)}\lesssim h^2\|\mathbf{f}\|_{\mathbf{L}^2(D)}+h^4\|\mathbf{g}\|_{\mathbf{L}^2(D)}. \tag{7.46}$$

2. (Similarly to Lemma 7.2.4 and Lemma 7.2.5). An a priori estimate for $\mathbf{v}_N$.

   The argument can also be divided into four steps. Steps 1, 2 and 3 follow exactly the same way as in Section 7.2.1. We shall only indicate the changes in step 4.

Step 4. From Step 4 of Lemma 7.2.4 we have that

$$\operatorname{op}\left(m((m+2)R-(1+m)\mu)\right)\mathbf{v}_N=h^{-2}\operatorname{op}(r_2)\mathbf{g}_3+h\operatorname{op}(r_1)\mathbf{v}_N.$$

Then

$$|\mathbf{v}_N|_{\mathbf{H}^{\frac{1}{2}}_{sc}(\Gamma)}\lesssim h^{-2}|\mathbf{g}_3|_{\mathbf{H}^{\frac{1}{2}}_{sc}(\Gamma)}+h|\mathbf{v}_N|_{\mathbf{H}^{-\frac{1}{2}}_{sc}(\Gamma)}$$

and for h small enough

$$|\mathbf{v}_N|_{\mathbf{H}^{\frac{1}{2}}_{sc}(\Gamma)} \lesssim h^{-2}|\mathbf{g}_3|_{\mathbf{H}^{\frac{1}{2}}_{sc}(\Gamma)}.$$

Following the arguments in the proof of Lemma 7.2.4, the only difference is to replace estimate (7.33) by

$$|\mathbf{f}_N|_{\mathbf{H}^{\frac{1}{2}}_{sc}(\Gamma)} \lesssim h^{-\frac{1}{2}}\|\mathbf{f}\|_{\overline{\mathbf{H}}^1_{sc}(D)}$$

Notice from (7.27) and Theorem 7.2.1 that

$$|\mathbf{v}|_{\mathbf{H}^{-\frac{1}{2}}_{sc}(\Gamma)} \lesssim h^{\frac{3}{2}}\|\mathbf{g}\|_{\mathbf{L}^2(D)} + h^{-\frac{1}{2}}\|\mathbf{f}\|_{L^2(D)}.$$

This gives the following estimate (corresponding to estimate (7.16) in Section 7.2.1)

$$|\mathbf{v}_N|_{\mathbf{H}^{\frac{1}{2}}_{sc}(\Gamma)} \lesssim h^{\frac{3}{2}}\|\mathbf{g}\|_{\mathbf{L}^2(D)} + h^{-\frac{1}{2}}\|\mathbf{f}\|_{\overline{\mathbf{H}}^1_{sc}(D)} + h^{\frac{1}{2}}\|\mathbf{v}\|_{\mathbf{L}^2(D)} + h^{-\frac{3}{2}}\|\mathbf{u}\|_{\overline{\mathbf{H}}^1_{sc}(D)}.$$

Then Theorem 7.2.1 yields

$$|\mathbf{v}_N|_{\mathbf{H}^{\frac{1}{2}}_{sc}(\Gamma)} \lesssim h^{\frac{3}{2}}\|\mathbf{g}\|_{\mathbf{L}^2(D)} + h^{-\frac{1}{2}}\|\mathbf{f}\|_{\overline{\mathbf{H}}^1_{sc}(D)}. \tag{7.47}$$

3. (Similarly to Lemma 7.2.6) A priori estimate for $\mathbf{u}_N$.

   From Lemma 7.2.6 of Section 7.2.1

$$\begin{aligned} \mathrm{op}(\lambda_2)\mathbf{u}_N &= h^2\mathrm{op}(r_{-2})(\mathrm{op}(\rho_2)\mathbf{v}_N + \mathbf{g}_1) \\ &+ h^2\mathrm{op}(r_{-1})\mathbf{v}_N + \mathrm{op}(r_1)\mathbf{g}_2 + h\mathrm{op}(r_0)\mathbf{u}_N. \end{aligned}$$

Then for small enough h

$$|\mathbf{u}_N|_{\mathbf{H}^{\frac{5}{2}}_{sc}(\Gamma)} \lesssim h^2|\mathbf{v}_N|_{\mathbf{H}^{\frac{1}{2}}_{sc}(\Gamma)} + h^2|\mathbf{g}_1|_{\mathbf{H}^{-\frac{1}{2}}_{sc}(\Gamma)} + |\mathbf{g}_2|_{\mathbf{H}^{\frac{5}{2}}_{sc}(\Gamma)}.$$

As in estimate (7.39), we need to estimate $|\gamma_N Q\nabla_h \underline{\operatorname{div}\mathbf{u}}|_{\mathbf{H}^{\frac{5}{2}}_{sc}(\Gamma)}$. The argument here is different, since $\|\operatorname{div}\mathbf{u}\|_{\overline{\mathbf{H}}^2_{sc}}$ can only be bounded by $\|\mathbf{v}\|_{\overline{\mathbf{H}}^1_{sc}}$ from equation (7.29). But $\mathbf{v}$ is only in $\mathbf{L}^2(D)$. However, from Lemma 7.5.2,

$$|\gamma_N Q\nabla_h \underline{\operatorname{div}\mathbf{u}}|_{\mathbf{H}^{\frac{5}{2}}_{sc}(\Gamma)} \lesssim h^{\frac{1}{2}}\|\operatorname{div}\mathbf{u}\|_{\overline{H}^1_{sc}(D)}.$$

Using estimate (7.47) and Theorem 7.2.1, direct calculations yield

$$|\mathbf{u}_N|_{\mathbf{H}^{\frac{5}{2}}_{sc}(\Gamma)} \lesssim h^{\frac{7}{2}}\|\mathbf{g}\|_{\mathbf{L}^2(D)} + h^{\frac{3}{2}}\|\mathbf{f}\|_{\overline{\mathbf{H}}^1_{sc}(D)} + h^{\frac{5}{2}}\|\mathbf{v}\|_{\mathbf{L}^2(D)}.$$

From Theorem 7.2.1 and estimate (7.46) we now have that

$$|\mathbf{u}_N|_{\mathbf{H}^{\frac{5}{2}}_{sc}(\Gamma)} \lesssim h^{\frac{7}{2}}\|\mathbf{g}\|_{\mathbf{L}^2(D)} + h^{\frac{3}{2}}\|\mathbf{f}\|_{\overline{\mathbf{H}}^1_{sc}(D)}. \tag{7.48}$$

4. New a priori estimates for $\mathbf{v}$ and $\mathbf{u}$.

As in Section 7.2.1, we have the following equation for $\mathbf{v}$:

$$\begin{aligned}
& \mathrm{op}\Big(-\frac{m\rho_2(\rho_2-\rho_1+\lambda_2-\lambda_1)}{(\lambda_1-\lambda_2)(\lambda_1-\rho_2)(\rho_1-\lambda_2)(\rho_1-\rho_2)} + \frac{m(\rho_2\lambda_2-\rho_1\lambda_1)}{(\lambda_1-\lambda_2)(\lambda_1-\rho_2)(\rho_1-\lambda_2)(\rho_1-\rho_2)}\Big)\mathbf{v} \\
= \; & -h^{-2}\left(h\gamma K_{-M}\underline{\mathbf{u}} + h^3\gamma Q(mK_{-M}\underline{\mathbf{v}}) + h^4\gamma Q m\tilde{Q}\underline{\mathbf{g}} + h^2\gamma Q((1+m)\underline{\mathbf{f}}) + \frac{h}{i}\gamma Q(\nabla_h \underline{\mathrm{div}\,\mathbf{u}})\right) \\
+ \; & h^{-2}\mathrm{op}(r_{-1})\nabla_\Gamma(\mathbf{u}_N\cdot\nu) + h^{-2}\mathrm{op}(r_0)\mathbf{u}_N + \mathrm{op}(r_{-3})\mathbf{g}_4 + h\mathrm{op}(r_{-4})\mathbf{J}(\mathbf{v}_T) + h\mathrm{op}(r_{-3})\mathbf{v} \\
:= \; & \mathbf{g}_5.
\end{aligned}$$

Then, using estimate (7.48), we obtain

$$|\mathbf{v}|_{\mathbf{H}^{\frac{1}{2}}_{sc}(\Gamma)} + |\mathbf{J}(\mathbf{v}_T)|_{\mathbf{H}^{-\frac{1}{2}}_{sc}(\Gamma)} \lesssim h^{\frac{3}{2}}\|\mathbf{g}\|_{\mathbf{L}^2(D)} + h^{-\frac{1}{2}}\|\mathbf{f}\|_{\overline{\mathbf{H}}^1_{sc}(D)} + h^{\frac{1}{2}}\|\mathbf{v}\|_{\mathbf{L}^2(D)}.$$

Therefore

$$\|\mathbf{v}\|_{\overline{\mathbf{H}}^1_{sc}(D)} \lesssim h^2\|\mathbf{g}\|_{\mathbf{L}^2(D)} + \|\mathbf{f}\|_{\overline{\mathbf{H}}^1_{sc}(D)}. \tag{7.49}$$

Then

$$\|\mathbf{u}\|_{\overline{\mathbf{H}}^3_{sc}(D)} \lesssim h^2\|\mathbf{v}\|_{\overline{\mathbf{H}}^1_{sc}(D)} + h^2\|\mathbf{f}\|_{\overline{\mathbf{H}}^1_{sc}(D)} + h\|\mathrm{div}\,\mathbf{u}\|_{\overline{H}^2_{sc}(D)} + h^{\frac{1}{2}}|\mathbf{u}_N|_{\mathbf{H}^{\frac{5}{2}}_{sc}(\Gamma)}.$$

Since ∇m has compact support in D and div $((1+m)\mathbf{f}) = 0$, then from equation (7.29) we have that

$$\|\mathrm{div}\,\mathbf{u}\|_{\overline{H}^2_{sc}(D)} \lesssim \|\mathbf{u}\|_{\overline{\mathbf{H}}^2_{sc}(D)} + h^3\|\mathbf{v}\|_{\overline{\mathbf{H}}^1_{sc}(D)} + h^4\|\mathbf{g}\|_{\mathbf{L}^2(D)}.$$

Combining this inequality with (7.48) and (7.49) yields for small enough h that

$$\|\mathbf{u}\|_{\overline{\mathbf{H}}^3_{sc}(D)} \lesssim h^2\|\mathbf{f}\|_{\overline{\mathbf{H}}^1_{sc}(D)} + h^4\|\mathbf{g}\|_{\mathbf{L}^2(D)}.$$

We finally arrive at the following estimates

$$\|\mathbf{v}\|_{\overline{\mathbf{H}}^1_{sc}(D)} \lesssim h^2\|\mathbf{g}\|_{\mathbf{L}^2(D)} + \|\mathbf{f}\|_{\overline{\mathbf{H}}^1_{sc}(D)},$$
$$\|\mathbf{u}\|_{\overline{\mathbf{H}}^3_{sc}(D)} \lesssim h^4\|\mathbf{g}\|_{\mathbf{L}^2(D)} + h^2\|\mathbf{f}\|_{\overline{\mathbf{H}}^1_{sc}(D)}.$$

5. We use a bootstrap argument to prove the results of the theorem by repeating the above arguments line by line. □

7.3 The Inverse of $\mathbf{B}_z$

In this section we will show that $\mathbf{B}_z$ has a bounded inverse for some z with sufficiently large $|z|$. We begin with the following. For a complex number $z = |z|e^{i\theta}$, $\theta \in [0, 2\pi[$ we define $\arg z := \theta$. Now we define

$$C(m) := \{\arg \frac{1}{1+m(x)};\ x \in \overline{D}\}.$$

Before we prove the main results in this section, we first make a connection between the set $C(m)$ and the assumptions made in Theorem 7.2.1.

Lemma 7.3.1 *If there exists θ such that $\theta \notin C(m) \cup \{0\} \cup \{\arg\left(\frac{N(x)+1}{N(x)}\right);\ x \in \Gamma\}$, then $\mu = e^{i\theta}$ satisfies the assumptions in Theorem 7.2.1, i.e. $|\xi|^2 - \mu \neq 0$, $|\xi|^2 - N(x)\mu \neq 0$ for any ξ and $x \in \overline{D}$ and $R_0(x,\xi') - \frac{N(x)}{1+N(x)}\mu \neq 0$ for any ξ' and $x \in \Gamma$.*

Proof. Assume on the contrary that there exists $\xi \in \mathbb{R}^d$ such that

$$\frac{1}{N(x)}|\xi|^2 - \mu = 0 \quad \text{or} \quad |\xi|^2 - \mu = 0 \quad \text{for some} \quad x \in \overline{D}$$

or

$$R_0(x,\xi') - \frac{N(x)}{1+N(x)}\mu = 0 \quad \text{for some} \quad x \in \Gamma.$$

This implies $\theta = \arg \mu \in C(m) \cup \{0\} \cup \{\arg\left(\frac{N(x)+1}{N(x)}\right);\ x \in \Gamma\}$. This contradicts the assumption. Hence we have proved the lemma. □

Now we are ready to prove the following.

Theorem 7.3.1 *Assume that assumption 7.0.1 holds and that* $C(m)\cup\{0\}\cup\{\arg\left(\frac{N(x)+1}{N(x)}\right);\ x\in\Gamma\}\neq[0,2\pi[$. *Then there exists z with sufficiently large $|z|>0$ such that $\mathbf{B}_z$ has a bounded inverse* $\mathbf{R}_z:\mathbf{F}(D)\times\mathbf{G}(D)\to\mathbf{U}(D)\times\mathbf{V}(D)$.

Proof. Since $C(m)\cup\{0\}\cup\{\arg\left(\frac{N(x)+1}{N(x)}\right);\ x\in\Gamma\}\neq[0,2\pi[$, then from Lemma 7.3.1 there exists $\mu=e^{i\theta}$ satisfying the assumption of Theorem 7.2.1. Let $h>0$ and define $z:=\mu h^{-2}$. Let $\mathbf{B}_z(\mathbf{u},\mathbf{v})=(\mathbf{f},\mathbf{g})$ where $(\mathbf{u},\mathbf{v})\in\mathbf{U}(D)\times\mathbf{V}(D)$. From Theorem 7.2.1, for a sufficiently small h, we have that

$$\|\mathbf{v}\|_{\mathbf{L}^2(D)}\lesssim|z|^{-1}\|\mathbf{g}\|_{\mathbf{L}^2(D)}+|z|^{-\frac{3}{2}}\|\operatorname{div}\mathbf{g}\|_{\overline{\mathbf{H}}^1_{sc}(D)}+\|\mathbf{f}\|_{\mathbf{L}^2(D)}+\|\operatorname{div}\left((1+m)\mathbf{f}\right)\|_{\overline{\mathbf{H}}^1_{sc}(D)}\tag{7.50}$$

and

$$\begin{aligned}&\|\mathbf{u}\|_{\mathbf{L}^2(D)}+|z|^{-\frac{1}{2}}\|\mathbf{u}\|_{\mathbf{H}^1(D)}+|z|^{-1}\|\mathbf{u}\|_{\mathbf{H}^2(D)}\\ \lesssim\ &|z|^{-1}\left(\|\mathbf{f}\|_{\mathbf{L}^2(D)}+\|\operatorname{div}\left((1+m)\mathbf{f}\right)\|_{\overline{\mathbf{H}}^1_{sc}(D)}\right)+|z|^{-2}\|\mathbf{g}\|_{\mathbf{L}^2(D)}+|z|^{-\frac{5}{2}}\|\operatorname{div}\mathbf{g}\|_{\overline{\mathbf{H}}^1_{sc}(D)}\end{aligned}\tag{7.51}$$

From (7.31) (with s=1), we have that

$$\|\operatorname{div}\mathbf{v}\|_{H^1(D)}\lesssim|z|^{-1}\|\operatorname{div}\mathbf{g}\|_{H^1(D)}.\tag{7.52}$$

Therefore $\mathbf{B}_z$ is injective and has closed range in $\mathbf{F}(D)\times\mathbf{G}(D)$ (the latter follows from a Cauchy sequence argument).

Now we prove that $\mathbf{B}_z$ has dense range. The argument will be divided into three steps.

Step 1: First we show that for any $(\mathbf{p}^d,\mathbf{q}^d)\in\mathbf{F}(D)\times\mathbf{G}(D)$ with $\operatorname{div}\left((1+m)\,\mathbf{p}^d\right)=0$ and $\operatorname{div}\mathbf{q}^d=0$, there exists $(\mathbf{u}_{1,\ell},\mathbf{v}_{1,\ell})\in\mathbf{U}(D)\times\mathbf{V}(D)$ such that

$$\mathbf{B}_z(\mathbf{u}_{1,\ell},\mathbf{v}_{1,\ell})\to(\mathbf{p}^d,\mathbf{q}^d)\quad\text{in}\quad\mathbf{F}(D)\times\mathbf{G}(D).$$

Indeed assume that $(\mathbf{p}^d,\mathbf{q}^d)\in\mathbf{F}(D)\times\mathbf{G}(D)$ with $\operatorname{div}\left((1+m)\,\mathbf{p}^d\right)=0$ and $\operatorname{div}\mathbf{q}^d=0$ and that

$$\left\langle\mathbf{B}_z(\mathbf{u},\mathbf{v}),(\mathbf{p}^d,\mathbf{q}^d)\right\rangle=0,\quad\forall(\mathbf{u},\mathbf{v})\in\mathbf{U}(D)\times\mathbf{V}(D)$$

where $\langle\cdot,\cdot\rangle$ denotes the natural $\mathbf{F}(D)\times\mathbf{G}(D)$ inner product. It is sufficient to show that $\mathbf{p}^d = 0$ and $\mathbf{q}^d = 0$ to conclude the proof in this step. As $(\mathbf{p}^d, \mathbf{q}^d)$ satisfies $\operatorname{div}\big((1+m)\,\mathbf{p}^d\big) = 0$ and $\operatorname{div}\mathbf{q}^d = 0$, then the inner product reduces to the $\mathbf{L}^2$ inner product. Letting $(\mathbf{u},\mathbf{v}) \in \mathbf{C}_0^\infty(D)\times\mathbf{C}_0^\infty(D)$, one gets, with $\tilde{\mathbf{p}} := \overline{\mathbf{p}^d}/(1+m)$,

$$\begin{aligned}
\operatorname{curl}\operatorname{curl}\overline{\mathbf{q}^d} - z\overline{\mathbf{q}^d} - m\tilde{\mathbf{p}} &= 0 \qquad \text{in} \quad D\\
\operatorname{curl}\operatorname{curl}\tilde{\mathbf{p}} - z(1+m)\tilde{\mathbf{p}} &= 0 \qquad \text{in} \quad D
\end{aligned}$$

in the distributional sense. We observe that $\operatorname{curl}\operatorname{curl}\overline{\mathbf{q}^d} \in \mathbf{L}^2(D)$ and therefore the tangential traces $\nu\times\operatorname{curl}\overline{\mathbf{q}^d}$ and $\nu\times\overline{\mathbf{q}^d}$ are well defined in $\mathbf{H}^{-3/2}(\Gamma)$ and $\mathbf{H}^{-1/2}(\Gamma)$ respectively. Since $\mathbf{u}\times\nu = 0$ and $\operatorname{curl}\mathbf{u}\times\nu = 0$ on Γ, then for all $\mathbf{v}\in\mathbf{C}^\infty(\overline{D})$ we have that

$$\int_\Gamma (\nu\times\operatorname{curl}\overline{\mathbf{q}^d})\cdot\mathbf{v}\,ds - \int_\Gamma(\nu\times\operatorname{curl}\mathbf{v})\cdot\overline{\mathbf{q}^d}\,ds = 0$$

where the integrals are understood as duality products. Hence

$$\nu\times\overline{\mathbf{q}^d} = 0 \quad \text{and} \quad \nu\times\operatorname{curl}\overline{\mathbf{q}^d} = 0 \quad \text{on} \quad \Gamma$$

(see for instance [41, Lemma 3.1]). Now Let $\mathbf{p}_1 = z\overline{\mathbf{q}^d} - \tilde{\mathbf{p}}$. Then one gets

$$\operatorname{curl}\operatorname{curl}\overline{\mathbf{q}^d} - z(1+m)\overline{\mathbf{q}^d} + m\mathbf{p}_1 = 0 \qquad \text{in} \quad D \tag{7.53}$$

$$\operatorname{curl}\operatorname{curl}\mathbf{p}_1 - z\mathbf{p}_1 = 0 \qquad \text{in} \quad D. \tag{7.54}$$

Now we want to apply Theorem 7.2.1 (one can check that we can relax the condition $\operatorname{curl}\mathbf{q}^d\in\mathbf{L}^2(D)$ from the proof of Theorem 7.2.1) to $(\overline{\mathbf{q}^d},\mathbf{p}_1)$. Since $z = \mu h^{-2}$ and μ satisfies the assumption in Theorem 7.2.1, we obtain $\overline{\mathbf{q}^d} = 0$, $\mathbf{p}_1 = 0$ which implies $\mathbf{p}^d = 0$, $\mathbf{q}^d = 0$. This proves the first part.

Step 2: We show that for any given $(\mathbf{p}^c,\mathbf{q}^c)\in\mathbf{F}(D)\times\mathbf{G}(D)$ with $\operatorname{curl}\mathbf{p}^c = 0$, $\mathbf{p}^c\times\nu|_\Gamma = 0$ and $\operatorname{curl}\mathbf{q}^c = 0$, $\mathbf{q}^c\times\nu|_\Gamma = 0$, there exists $(\mathbf{u}_{2,\ell},\mathbf{v}_{2,\ell})$ such that

$$\mathbf{B}_z(\mathbf{u}_{2,\ell},\mathbf{v}_{2,\ell}) \to (\mathbf{p}^c,\mathbf{q}^c) \quad \text{in} \quad \mathbf{F}(D)\times\mathbf{G}(D).$$

Assume

$$\langle\mathbf{B}_z(\mathbf{u},\mathbf{v}),(\mathbf{p}^c,\mathbf{q}^c)\rangle = 0, \quad \forall(\mathbf{u},\mathbf{v})\in\mathbf{U}(D)\times\mathbf{V}(D).$$

It is sufficient to show $\mathbf{p}^c = 0$ and $\mathbf{q}^c = 0$ to conclude the proof in this step. Indeed from $\operatorname{curl}\mathbf{p}^c = 0$, div $((1+m)\mathbf{p}^c) \in H^1(D)$ and $\mathbf{p}^c \times \nu|_\Gamma = 0$, one gets $\mathbf{p}^c \in \mathbf{H}^2(D)$ (see [5]), then $\operatorname{curl}\mathbf{p}^c = 0$ implies $\mathbf{p}^c \in \mathbf{U}(D)$. We obviously have $\mathbf{q}^c \in \mathbf{V}(D)$. Then, letting $\mathbf{u} = \mathbf{p}^c$ and $\mathbf{v} = 0$, one gets

$$\|\mathbf{p}^c\|_{\mathbf{F}(D)} = 0.$$

This implies $\mathbf{p}^c = 0$. Second, let $\mathbf{v} = \mathbf{q}^c$ which implies

$$\|\mathbf{q}^c\|_{\mathbf{G}(D)} = 0$$

and therefore $\mathbf{q}^c = 0$.

Step 3: Now we are ready to prove that $\mathbf{B}_z$ has dense range in $\mathbf{F}(D) \times \mathbf{G}(D)$. Indeed let $(\mathbf{p}, \mathbf{q}) \in \mathbf{F}(D) \times \mathbf{G}(D)$. By the Helmholtz decomposition (see for instance [54]), there exist unique $\mathbf{p}^d \in \mathbf{L}^2(D)$, $\mathbf{p}^c \in \mathbf{L}^2(D)$ and $\mathbf{q}^d \in \mathbf{L}^2(D)$, $\mathbf{q}^c \in \mathbf{L}^2(D)$ such that

$$\mathbf{p} = \mathbf{p}^d + \mathbf{p}^c, \quad \mathbf{q} = \mathbf{q}^d + \mathbf{q}^c \tag{7.55}$$

where

$$\operatorname{div}\left((1+m)\,\mathbf{p}^d\right) = 0, \quad \operatorname{curl}\mathbf{p}^c = 0, \quad \mathbf{p}^c \times \nu|_\Gamma = 0.$$

$$\operatorname{div}\mathbf{q}^d = 0, \quad \operatorname{curl}\mathbf{q}^c = 0, \quad \mathbf{q}^c \times \nu|_\Gamma = 0.$$

The existence of $(\mathbf{p}^d, \mathbf{p}^c)$ is guaranteed by the strict positiveness of $\Re(1+m)$. As shown above, there exists $(\mathbf{u}_{1,\ell}, \mathbf{v}_{1,\ell}) \in \mathbf{U}(D) \times \mathbf{V}(D)$ and $(\mathbf{u}_{2,\ell}, \mathbf{v}_{2,\ell}) \in \mathbf{U}(D) \times \mathbf{V}(D)$ such that

$$\mathbf{B}_z(\mathbf{u}_{1,\ell}, \mathbf{v}_{1,\ell}) \to (\mathbf{p}^d, \mathbf{q}^d) \quad \text{in} \quad \mathbf{F}(D) \times \mathbf{G}(D)$$

and

$$\mathbf{B}_z(\mathbf{u}_{2,\ell}, \mathbf{v}_{2,\ell}) \to (\mathbf{p}^c, \mathbf{q}^c) \quad \text{in} \quad \mathbf{F}(D) \times \mathbf{G}(D).$$

Now let $\mathbf{u}_\ell = \mathbf{u}_{1,\ell} + \mathbf{u}_{2,\ell}$ and $\mathbf{v}_\ell = \mathbf{v}_{1,\ell} + \mathbf{v}_{2,\ell}$. Then

$$\mathbf{B}_z(\mathbf{u}_\ell, \mathbf{v}_\ell) \to (\mathbf{p}^d + \mathbf{p}^c, \mathbf{q}^d + \mathbf{q}^c) = (\mathbf{p}, \mathbf{q})$$

in $\mathbf{F}(D)\times\mathbf{G}(D)$. Now we have proved that $\mathbf{B}_z$ has dense range in $\mathbf{F}(D)\times\mathbf{G}(D)$. Since $\mathbf{B}_z$ is injective and has closed dense range in $\mathbf{F}(D)\times\mathbf{G}(D)$, $\mathbf{R}_z := \mathbf{B}_z^{-1}$ is well-defined. $\square$

7.4 Main Results on Transmission Eigenvalues

We shall state and prove here the main results of our paper on the existence of transmission eigenvalues and the completeness of associated eigenvectors. The results of this section rely heavily on the regularity results obtained in section 7.2.

Let us first introduce the Helmholtz decomposition. The motivation for introducing the Helmholtz decomposition is to get the desired compact imbedding (which will be proved to be a Hilbert-Schmidt operator) for Maxwell's equations. For any $\mathbf{u}\in\mathbf{L}^2(D)$ there exists a unique $\mathbf{u}^d\in\mathbf{L}^2(D)$ and $\mathbf{u}^c\in\mathbf{L}^2(D)$ such that

$$\mathbf{u} = \mathbf{u}^d + \mathbf{u}^c \tag{7.56}$$

and

$$\operatorname{div}\,\big((1+m)\,\mathbf{u}^d\big) = 0, \quad \operatorname{curl}\mathbf{u}^c = 0, \quad \mathbf{u}^c\times\nu|_\Gamma = 0.$$

This is guaranteed by the strict positiveness of $\Re(1+m)$ (see for instance [54]). We now define $\mathbf{P}^d$ as the projection operator in $\mathbf{L}^2(D)\times\mathbf{L}^2(D)$ defined by

$$\mathbf{P}^d(\mathbf{u},\mathbf{v}) = (\mathbf{u}^d,\mathbf{v})$$

where $\mathbf{u}^d$ is defined by (7.56).

For z chosen as in Theorem 7.3.1, we now consider the operator

$$\mathbf{S}_z := \mathbf{P}^d\mathbf{R}_z : \mathbf{H}(D)\to\mathbf{H}(D)$$

with

$$\mathbf{H}(D) := \{\mathbf{u}\in\mathbf{U}(D);\operatorname{div}\,((1+m)\,\mathbf{u}) = 0\}\times\{\mathbf{v}\in\mathbf{L}^2(D);\operatorname{div}\mathbf{v}=0\}.$$

Since $\mathbf{H}(D)$ is a subspace of $\mathbf{H}^2(D)\times\mathbf{G}$, we also get from Theorem 7.2.2 that $\mathbf{S}_z^2$ continuously map $\mathbf{H}(D)$ into $\mathbf{H}^6(D)\times\mathbf{H}^4(D)$. Observing that the $\mathbf{H}^2(D)\times\mathbf{L}^2(D)$

norm is an equivalent norm in $\mathbf{H}(D)$, we have from [76, Lemma 4.1] (see also [1]) that $\mathbf{S}_z^2 : \mathbf{H}(D) \to \mathbf{H}(D)$ is a Hilbert-Schmidt operator.

We shall now apply Agmon's theory on the spectrum of Hilbert-Schmidt operators in [1] to get the desired main results. More specifically we shall apply the result of the following lemma that is a direct consequence of Proposition 4.2 and the proof of Theorem 5 in [76].

Lemma 7.4.1 *Let H be a Hilbert space and S be a bounded linear operator from H to H. If λ^{-1} is in the resolvent of S, define*

$$(S)_\lambda = S(I - \lambda S)^{-1}.$$

Assume $S^p : H \to H$ is a Hilbert-Schmidt operator for some $p \geq 2$. For the operator S, assume there exists N rays with bounded growth where the angle between any two adjacent rays is less that $\frac{\pi}{2p}$: more precisely assume there exist $0 \leq \theta_1 < \theta_2 < \cdots < \theta_N < 2\pi$ such that $\theta_k - \theta_{k-1} < \frac{\pi}{2p}$ for $k = 2, \cdots, N$ and $2\pi - \theta_N + \theta_1 < \frac{\pi}{2p}$ satisfying the condition that there exists $r_0 > 0$, $c > 0$ such that $sup_{r \geq r_0} \|(S)_{re^{i\theta_k}}\| \leq c$ for $k = 1, \cdots, N$. Then the space spanned by the nonzero generalized eigenfunctions of S is dense in the closure of the range of S^p.

We shall first apply this lemma to the operator $\mathbf{S}_z$, then deduce the spectral decomposition of the operator $\mathbf{B}_z$ and the main result on transmission eigenvalues. In order to prove the existence of rays with bounded growth we need the following two lemmas on $(\mathbf{R}_z)_\lambda$ which will be used in the proof of Theorem 7.4.1.

Lemma 7.4.2 *Let $z \in \mathbb{C}$ such $\mathbf{R}_z = \mathbf{B}_z^{-1}$ is well defined as in Theorem 7.3.1. Then one has the following identities:*

$$\mathbf{P}^d \mathbf{R}_z \mathbf{P}^d \mathbf{B}_z = \mathbf{I}, \quad \textit{and} \quad \mathbf{P}^d \mathbf{B}_z \mathbf{P}^d \mathbf{R}_z = \mathbf{I}$$

where $\mathbf{I}$ is the identity operator on $\mathbf{H}(D)$.

Proof. On one hand, for any $(\mathbf{f}^d, \mathbf{g}) \in \mathbf{H}(D)$, let $(\mathbf{u}, \mathbf{v}) = \mathbf{R}_z(\mathbf{f}^d, \mathbf{g})$, then

$$\begin{aligned} \operatorname{curl}\operatorname{curl}\mathbf{u} - z(1+m)\mathbf{u} - m\mathbf{v} &= (1+m)\mathbf{f}^d && \text{in} \quad D \\ \operatorname{curl}\operatorname{curl}\mathbf{v} - z\mathbf{v} &= \mathbf{g} && \text{in} \quad D \end{aligned}$$

Let $(\mathbf{u}^d, \mathbf{v}) = \mathbf{P}^d(\mathbf{u}, \mathbf{v})$, then

$$\begin{aligned} \operatorname{curl}\operatorname{curl}\mathbf{u}^d - z(1+m)\mathbf{u}^d - m\mathbf{v} &= (1+m)\mathbf{f}^d + z(1+m)\mathbf{u}^c && \text{in} \quad D \\ \operatorname{curl}\operatorname{curl}\mathbf{v} - z\mathbf{v} &= \mathbf{g} && \text{in} \quad D \end{aligned}$$

This implies that

$$\mathbf{P}^d\mathbf{B}_z\mathbf{P}^d\mathbf{R}_z(\mathbf{f}^d, \mathbf{g}) = \mathbf{P}^d\mathbf{B}_z\mathbf{P}^d(\mathbf{u}, \mathbf{v}) = \mathbf{P}^d\mathbf{B}_z(\mathbf{u}^d, \mathbf{v}) = \mathbf{P}^d(\mathbf{f}^d + z\mathbf{u}^c, \mathbf{g}) = (\mathbf{f}^d, \mathbf{g}).$$

On the other hand, for any $(\mathbf{u}^d, \mathbf{v}) \in \mathbf{H}(D)$, let $(\mathbf{f}, \mathbf{g}) = \mathbf{B}_z(\mathbf{u}^d, \mathbf{v})$, then

$$\begin{aligned} \operatorname{curl}\operatorname{curl}\mathbf{u}^d - z(1+m)\mathbf{u}^d - m\mathbf{v} &= (1+m)\mathbf{f} && \text{in} \quad D \\ \operatorname{curl}\operatorname{curl}\mathbf{v} - z\mathbf{v} &= \mathbf{g} && \text{in} \quad D \end{aligned}$$

This implies that

$$\begin{aligned} \operatorname{curl}\operatorname{curl}(\mathbf{u}^d + \tfrac{1}{z}\mathbf{f}^c) - z(1+m)(\mathbf{u}^d + \tfrac{1}{z}\mathbf{f}^c) - m\mathbf{v} &= (1+m)\mathbf{f}^d && \text{in} \quad D \\ \operatorname{curl}\operatorname{curl}\mathbf{v} - z\mathbf{v} &= \mathbf{g} && \text{in} \quad D \end{aligned}$$

Therefore

$$\mathbf{P}^d\mathbf{R}_z\mathbf{P}^d\mathbf{B}_z(\mathbf{u}^d, \mathbf{v}) = \mathbf{P}^d\mathbf{R}_z(\mathbf{f}^d, \mathbf{g}) = \mathbf{P}^d(\mathbf{u}^d + \frac{1}{z}\mathbf{f}^c, \mathbf{v}) = (\mathbf{u}^d, \mathbf{v}).$$

Hence we have proved the lemma. $\square$

We now have the following expression for $(\mathbf{S}_z)_\lambda$.

Lemma 7.4.3 *Let $\lambda \in \mathbb{C}$ and assume that $\mathbf{R}_{z+\lambda} = \mathbf{B}_{z+\lambda}^{-1}$ is well defined. Then $(\mathbf{S}_z)_\lambda = \mathbf{P}^d\mathbf{R}_{z+\lambda}$.*

Proof. By definition, $(\mathbf{S}_z)_\lambda = \mathbf{P}^d\mathbf{R}_z(\mathbf{I} - \lambda \mathbf{P}^d\mathbf{R}_z)^{-1}$. From Lemma 7.4.2 and the fact that $\mathbf{P}^d\mathbf{I} = \mathbf{I}$ where $\mathbf{I}$ is the identity operator on $\mathbf{H}(D)$, we have that

$$\begin{aligned}
(\mathbf{S}_z)_\lambda &= \mathbf{P}^d\mathbf{R}_z(\mathbf{I} - \lambda \mathbf{P}^d\mathbf{R}_z)^{-1} \\
&= \mathbf{P}^d\mathbf{R}_z(\mathbf{P}^d\mathbf{B}_z\mathbf{P}^d\mathbf{R}_z - \lambda \mathbf{P}^d\mathbf{R}_z)^{-1} \\
&= \mathbf{P}^d\mathbf{R}_z((\mathbf{P}^d\mathbf{B}_z - \lambda\mathbf{I})\mathbf{P}^d\mathbf{R}_z)^{-1} \\
&= \mathbf{P}^d\mathbf{R}_z(\mathbf{P}^d(\mathbf{B}_z - \lambda\mathbf{I})\mathbf{P}^d\mathbf{R}_z)^{-1} \\
&= \mathbf{P}^d\mathbf{R}_z(\mathbf{P}^d\mathbf{B}_{z+\lambda}\mathbf{P}^d\mathbf{R}_z)^{-1} \\
&= \mathbf{P}^d\mathbf{R}_{z+\lambda}.
\end{aligned}$$

where for the last equality we used that $\mathbf{P}^d\mathbf{R}_{z+\lambda}\mathbf{P}^d\mathbf{B}_{z+\lambda} = \mathbf{I}$. $\square$

We are now in position to prove the following result on the spectral decomposition of $\mathbf{S}_z$.

Theorem 7.4.1 *Assume that Assumption 1 holds and assume that $C(m)$ is contained in an interval of length $< \frac{\pi}{4}$. Then there are infinitely many eigenvalues of $\mathbf{S}_z$ and the associated generalized eigenfunctions are dense in $\{\mathbf{u} \in \mathbf{U}(D); div\,((1+m)\,\mathbf{u}) = 0\} \times \{\mathbf{v} \in \mathbf{V}(D); div\,\mathbf{v} = 0\}$.*

Proof. We prove the theorem in two steps.

Step 1. We shall apply Lemma 7.4.1 with $S = \mathbf{S}_z$, $H = \mathbf{H}(D)$ and $p = 2$.

Since $C(m)$ is contained in an interval of length $< \frac{\pi}{4}$, then we can choose $0 \le \theta_1 < \theta_2 < \cdots < \theta_N < 2\pi$ such that (recall that since n is a constant on Γ, then $\{\arg\left(\frac{N(x)+1}{N(x)}\right);\ x \in \Gamma\}$ is a fixed angle)

$$\theta_k - \theta_{k-1} < \frac{\pi}{4}$$

for $k = 2, \cdots, N$ and $2\pi - \theta_N + \theta_1 < \frac{\pi}{4}$ satisfying

$$\theta_j \notin C(m) \cup \{0\} \cup \{\arg\left(\frac{N(x)+1}{N(x)}\right);\ x \in \Gamma\}.$$

From Lemma 7.3.1 and Theorem 7.3.1, $\mathbf{R}_{re^{i\theta_k}}$ is well-defined as the bounded inverse of $\mathbf{B}_{re^{i\theta_k}}$. Moreover $\mathbf{R}_{re^{i\theta_k}}$ is uniformly bounded with respect to r because of the estimates (7.50), (7.51) and (7.52). Now for sufficiently large $r > 0$, the angle of $z + re^{i\theta_k}$ is sufficiently close to $re^{i\theta_k}$. Therefore $\mathbf{R}_{z+re^{i\theta_k}}$ is also uniformly bounded with respect to r. Hence there exist r_0 such that

$$sup_{r \geq r_0} \|\mathbf{R}_{z+re^{i\theta_k}}\| \leq c.$$

From Lemma 7.4.3 we have that

$$S_{re^{i\theta_k}} = (\mathbf{S}_z)_{re^{i\theta_k}} = \mathbf{P}^d \mathbf{R}_{z+re^{i\theta_k}}.$$

Therefore

$$sup_{r \geq r_0} \|S_{re^{i\theta_k}}\| \leq c.$$

Now we have found directions θ_j as required in Lemma 7.4.1 for which the bounded growth conditions are satisfied.

Step 2. It only remains to prove that the closure of the range of $\mathbf{S}_z^2$ is dense in $\{\mathbf{u} \in \mathbf{U} : \mathrm{div}\,((1+m)\,\mathbf{u}) = 0\} \times \{\mathbf{v} \in \mathbf{V} : \mathrm{div}\,\mathbf{v} = 0\}$. By a denseness argument, it is sufficient to show that the closure of the range of $\mathbf{S}_z$ is $\{\mathbf{u} \in \mathbf{U}(D) : \mathrm{div}\,((1+m)\,\mathbf{u}) = 0\} \times \{\mathbf{v} \in \mathbf{V}(D) : \mathrm{div}\,\mathbf{v} = 0\}$. Indeed for $(\mathbf{u}, \mathbf{v}) \in \{\mathbf{u} \in \mathbf{U}(D) : \mathrm{div}\,((1+m)\,\mathbf{u}) = 0\} \times \{\mathbf{v} \in \mathbf{V}(D) : \mathrm{div}\,\mathbf{v} = 0\}$, we define $p \in H_0^1(D)$ such that

$$-z\mathrm{div}\,[(1+m)\nabla p] = \nabla m \cdot \mathbf{v}$$

Since $\mathrm{curl}\,\nabla p = 0$, $\mathrm{div}\,\nabla p \in L^2(D)$ and $\nu \times \nabla p = 0$ then $\nabla p \in \mathbf{H}^1(D)$ (see for instance [?]), the same argument yields again $\nabla p \in \mathbf{H}^2(D)$ since $\mathrm{div}\,[(1+m)\nabla p] \in H^1(D)$ (this come from the fact that ∇m has compact support in D and $\mathbf{v}$ is regular on that support by elliptic regularity).

Let $\mathbf{u}^* = \mathbf{u} + \nabla p$. Then we have $(\mathbf{u}^*, \mathbf{v}) \in \mathbf{U}(D) \times \mathbf{V}(D)$ and $\mathbf{P}^d(\mathbf{u}^*, \mathbf{v}) = (\mathbf{u}, \mathbf{v})$. Moreover by a direct calculation we have that

$$\mathrm{div}\,(-z(1+m)\mathbf{u}^* - m\mathbf{v}) = 0.$$

Now define $(\mathbf{f}, \mathbf{g}) = \mathbf{B}_z(\mathbf{u}^*, \mathbf{v})$. Then

$$(\mathbf{f}, \mathbf{g}) \in \{\mathbf{f} \in \mathbf{F}(D); \mathrm{div}\,((1+m)\,\mathbf{f}) = 0\} \times \{\mathbf{g} \in \mathbf{G}(D); \mathrm{div}\,\mathbf{g} = 0\}.$$

Let $(\mathbf{f}_\ell, \mathbf{g}_\ell) \in \mathbf{F}(D) \times \mathbf{G}(D)$ be a Cauchy sequence such that

$$(\mathbf{f}_\ell, \mathbf{g}_\ell) \to (\mathbf{f}, \mathbf{g})$$

in the space $\mathbf{F}(D) \times \mathbf{G}(D)$. Since $\mathbf{R}_z$ is bounded, we have that

$$\mathbf{R}_z(\mathbf{f}_\ell, \mathbf{g}_\ell) \to \mathbf{R}_z(\mathbf{f}, \mathbf{g}) = (\mathbf{u}^*, \mathbf{v}) \quad \text{in} \quad \mathbf{U}(D) \times \mathbf{V}(D).$$

Therefore

$$\mathbf{S}_z(\mathbf{f}_\ell, \mathbf{g}_\ell) = \mathbf{P}^d \mathbf{R}_z(\mathbf{f}_\ell, \mathbf{g}_\ell) \to \mathbf{P}^d(\mathbf{u}^*, \mathbf{v}) = (\mathbf{u}, \mathbf{v})$$

in $\{\mathbf{u} \in \mathbf{U}(D); \operatorname{div}\, ((1+m)\,\mathbf{u}) = 0\} \times \{\mathbf{v} \in \mathbf{V}(D); \operatorname{div} \mathbf{v} = 0\}$. This proves the theorem.

□

Now we relate the transmission eigenvalues to the operator $\mathbf{B}_z$.

Theorem 7.4.2 *The number k and $(\mathbf{u}, \mathbf{v}) \in \mathbf{U}(D) \times \{\mathbf{v} \in \mathbf{V}(D) : \mathit{div}\, \mathbf{v} = 0\}$ are a transmission eigenvalue and a non trivial solution of (7.1)-(7.2) respectively if and only if $\mu^{-1} = k^2 - z$ and $\mathbf{P}^d(\mathbf{u}, \mathbf{v})$ are respectively an eigenvalue and an eigenvector of $\mathbf{S}_z$.*

Proof. First we show that for each eigenvalue μ^{-1} of $\mathbf{S}_z$ we can find a transmission eigenvalue k and and non trivial solution of (7.1)-(7.2). Indeed, suppose $(\mathbf{u}^d, \mathbf{v}) \in \mathbf{H}(D)$ is such that

$$\mathbf{P}^d \mathbf{B}_z^{-1}(\mathbf{u}^d, \mathbf{v}) = \mu^{-1}(\mathbf{u}^d, \mathbf{v}). \tag{7.57}$$

Since $\mathbf{B}_z^{-1}$ is well-defined, $(\tilde{\mathbf{u}}, \tilde{\mathbf{v}}) := \mu \mathbf{B}_z^{-1}(\mathbf{u}^d, \mathbf{v})$ satisfies

$$\begin{aligned} \operatorname{curl}\operatorname{curl} \tilde{\mathbf{u}} - z(1+m)\tilde{\mathbf{u}} - m\mathbf{v} &= \mu(1+m)\mathbf{u}^d \qquad &&\text{in} \quad D \\ \operatorname{curl}\operatorname{curl} \tilde{\mathbf{v}} - z\tilde{\mathbf{v}} &= \mu\mathbf{v} &&\text{in} \quad D. \end{aligned}$$

Define $\tilde{\mathbf{u}}^d$ such that $(\tilde{\mathbf{u}}^d, \tilde{\mathbf{v}}) = \mathbf{P}^d(\tilde{\mathbf{u}}, \tilde{\mathbf{v}})$. Then, equation (7.57) yields

$$\tilde{\mathbf{u}}^d = \mathbf{u}^d, \qquad \tilde{\mathbf{v}} = \mathbf{v}.$$

Now set

$$\mathbf{u} = \tilde{\mathbf{u}}^d + \frac{z}{\mu + z}\tilde{\mathbf{u}}^c = \mathbf{u}^d + \frac{z}{\mu + z}\tilde{\mathbf{u}}^c,$$

where $\tilde{\mathbf{u}}^c = \tilde{\mathbf{u}} - \tilde{\mathbf{u}}^d$. Then a direct calculation yields

$$\begin{aligned} \operatorname{curl}\operatorname{curl}\mathbf{u} - (z+\mu)(1+m)\mathbf{u} - m\mathbf{v} = 0 \qquad & \text{in} \quad D \\ \operatorname{curl}\operatorname{curl}\mathbf{v} - (z+\mu)\mathbf{v} = 0 \qquad & \text{in} \quad D. \end{aligned}$$

The definition of $\mathbf{u}$ and (7.56) ensures that $\gamma_t \mathbf{u} = 0$ and $\gamma_t \operatorname{curl}\mathbf{u} = 0$ on Γ and that $(\mathbf{u}, \mathbf{v})$ are non trivial solutions of (7.1)-(7.2) with $k := \sqrt{z+\mu}$ (with appropriate branch).

The converse is easily seen by reversing the above arguments and defining $(\mathbf{u}^d, \mathbf{v}) = \mathbf{P}^d(\mathbf{u}, \mathbf{v})$. This completes the proof. □

Note that since $\mathbf{S}_z^2$ is a Hilbert-Schmidt operator then the reciprocal of the eigenvalues form a discrete set without finite accumulation points. We therefore can summarize the results on transmission eigenvalues in the following main theorem.

Theorem 7.4.3 *Assume that the assumptions of Theorem 7.4.1 hold. Then there exist infinitely many transmission eigenvalues in the complex plane and they form a discrete set $\mathcal{T}$ without finite accumulation points. Moreover, if z is such in Theorem 7.4.1, then the set $\{\mu = (k^2 - z)^{-1},\ k \in \mathcal{T}\}$ form the set of eigenvalues of the operator $\mathbf{S}_z$ and the associated eigenvectors are dense in $\{\mathbf{u} \in \mathbf{U}(D);\, div\,((1+m)\,\mathbf{u}) = 0\} \times \{\mathbf{v} \in \mathbf{V}(D);\, div\,\mathbf{v} = 0\}$.*

7.5 A Semiclassical Pseudo-differential Calculus

In this section, we will state some results from semiclassical pseudo-differential calculus that will be used in the thesis. We introduce a small parameter h. We define $D^h_{x_j} = \frac{h}{i}\frac{\partial}{\partial x_j}$. Similar notations hold for $\nabla_h, \frac{\partial_h}{\partial \nu}$. For an open bounded manifold D in $\mathbb{R}^3$ we introduce the semiclassical Sobolev spaces $\overline{\mathbf{H}}^s_{sc}(D)$ equipped with the norm $\|\cdot\|_{\overline{\mathbf{H}}^s_{sc}(D)}$, where $\|\mathbf{u}\|_{\overline{\mathbf{H}}^s_{sc}(D)} := \inf\{\|\tilde{\mathbf{u}}\|_{\mathbf{H}^s_{sc}(\mathbb{R}^3)}, \tilde{\mathbf{u}}|_D = \mathbf{u}\}$ and $\|\mathbf{u}\|^2_{\mathbf{H}^s_{sc}(\mathbb{R}^3)} := \int_{\mathbb{R}^3}(1+h^2|\xi|^2)^s|\hat{\mathbf{u}}(\xi)|^2 d\xi$. For a two dimensional manifold Γ, we denote the semiclassical norm as $|\cdot|_{\mathbf{H}^s_{sc}(\Gamma)}$. We denote the commutator of two semiclassical pseudo-differential operators as $[\cdot,\cdot]$. We refer to [4] and [85] for details. By $a \lesssim b$ we mean that $a \leq Cb$ for some independent constant C.

Definition 7.5.1 *Let $a(x,\xi)$ be in $C^\infty(\mathbb{R}^{2d})$, we say a is a symbol of order m, denoted as $a \in S^m$, if*

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq C_{\alpha\beta}\langle\xi\rangle^{m-|\beta|}$$

for all α and β where $\langle\xi\rangle := (1+|\xi|^2)^{\frac{1}{2}}$. For $a \in S^m$ we define the semiclassical operator $Op_h(a)$ by

$$Op_h(a)u = \frac{1}{(2\pi)^d}\int e^{ix\xi}a(x,h\xi)\hat{u}(\xi)d\xi$$

and we define the class of such operators as Op_hS^m.

In particular we need the following results from [76]. Let $x = (x', x_n)$ and $\xi = (\xi', \xi_n)$ where (x,ξ) is the local coordinate in the cotangent bundle $T^*(\Gamma \times (0,\epsilon))$ and (x',ξ') is the local coordinate in the cotangent bundle $T^*\Gamma$.

For the case that $-h^2\Delta - \mu$ is elliptic with the symbol $|\xi|^2 - \mu \neq 0$ for any ξ and $x \in \overline{D}$, we have in the tubular neighborhood of Γ the semiclassical symbol of

$$-h^2\Delta - \mu$$

is

$$\xi_n^2 + 2hH(x)\frac{1}{i}\xi_n + R(x,\xi') - \mu$$

where $H(x)$ is a smooth function depending on x. We denote by

$$R_0(x,\xi') \tag{7.58}$$

the principle semiclassical symbol of $R(x,\xi')$. Moreover we can have

$$\xi_n^2 + R(x,\xi') - \mu = (\xi_n - \rho_1(x,\xi'))(\xi_n - \rho_2(x,\xi')) \tag{7.59}$$

where ρ_1 and ρ_2 are symbols of order 1 with $\Im(\rho_1) > 0$ and $\Im(\rho_2) < 0$.

For the case that $-h^2\Delta-\mu(1+m)$ is elliptic with the symbol $|\xi|^2-\mu(1+m)\neq 0$ for any ξ and $x\in\overline{D}$ we have similarly

$$\xi_n^2+R(x,\xi')-\mu(1+m)=(\xi_n-\lambda_1(x,\xi'))(\xi_n-\lambda_2(x,\xi')) \tag{7.60}$$

where λ_1 and λ_2 are symbols of order 1 with $\Im(\lambda_1)>0$ and $\Im(\lambda_2)<0$.

Also we will use frequently that if the symbol $|\xi|^2-\mu(1+m)\neq 0$ for all ξ and $x\in\overline{D}$, then the parametrix Q of $-h^2\Delta-\mu(1+m)$ exists where

$$Q\left(-h^2\Delta-\mu(1+m)\right)=I$$

modulo a smoothing operator. The following holds

$$\|(Q\underline{\mathbf{f}})|_D\|_{\overline{\mathbf{H}}_{sc}^{s+2}(D)}\lesssim\|\mathbf{f}\|_{\overline{\mathbf{H}}_{sc}^{s}(D)} \tag{7.61}$$

for any $\mathbf{f}\in\overline{\mathbf{H}}_{sc}^{s}(D)$ with $s\geq 0$. The same holds true for the parametrix $\tilde{Q}$ of $-h^2\Delta-\mu$. Also we have

$$\|\left(Q(\psi\otimes(D_s^h)^k\delta_{s=0})\right)|_D\|_{\overline{\mathbf{H}}_{sc}^{s-k+\frac{3}{2}}(D)}\lesssim h^{-\frac{1}{2}}\|\phi\|_{\mathbf{H}_{sc}^{s}(\Gamma)} \tag{7.62}$$

where $s-k+\frac{3}{2}\geq 0$.

Moreover if $-h^2\Delta\mathbf{v}-\mu\mathbf{v}=h^2\mathbf{g}$ in D and $|\xi|^2-\mu\neq 0$ then

$$\|\mathbf{v}\|_{\overline{\mathbf{H}}_{sc}^{s+1}(D\setminus\overline{N})}\lesssim h\|\mathbf{v}\|_{\overline{\mathbf{H}}_{sc}^{s}(D)}+h^2\|\mathbf{g}\|_{\overline{\mathbf{H}}_{sc}^{max\{s-1,0\}}(D)} \tag{7.63}$$

for $s\geq 0$ when the right hand side makes sense.

Next we introduce $\mathrm{op}(r_M)$ as the semiclassical pseudo-differential operator of order M on Γ. We have that

$$\tilde{Q}[\frac{h}{i}\psi\otimes D_s^h\delta_{s=0}]=\mathrm{op}(\frac{\rho_1}{\rho_1-\rho_2})\psi+h\mathrm{op}(r_{-1})\psi \tag{7.64}$$

$$\tilde{Q}[\frac{h}{i}\psi\otimes\delta_{s=0}]=\mathrm{op}(\frac{1}{\rho_1-\rho_2})\psi+h\mathrm{op}(r_{-2})\psi \tag{7.65}$$

$$Qm\tilde{Q}[\frac{h}{i}\psi\otimes\delta_{s=0}]=\mathrm{op}\left(\frac{m(\rho_2-\rho_1+\lambda_2-\lambda_1)}{(\lambda_1-\lambda_2)(\lambda_1-\rho_2)(\rho_1-\lambda_2)(\rho_1-\rho_2)}\right)\psi+h\mathrm{op}(r_{-4})\psi \tag{7.66}$$

$$Qm\tilde{Q}[\frac{h}{i}\psi \otimes D_s^h \delta_{s=0}] = \mathrm{op}\left(\frac{m(\rho_2\lambda_2 - \rho_1\lambda_1)}{(\lambda_1-\lambda_2)(\lambda_1-\rho_2)(\rho_1-\lambda_2)(\rho_1-\rho_2)}\right)\psi + h\mathrm{op}(r_{-3})\psi \tag{7.67}$$

where ψ is a distribution on the boundary.

In the framework of semiclassical norms, the trace formula reads

$$|\gamma u|_{H_{sc}^{s-\frac{1}{2}}(\Gamma)} \lesssim h^{-\frac{1}{2}} \|u\|_{\overline{H}_{sc}^{s}(D)} \tag{7.68}$$

for $s > \frac{1}{2}$.

Moreover we need the following two lemmas.

Lemma 7.5.1 *Assume $u \in H^s(D)$. Then for $s \geq 0$*

$$\|Q\nabla_h \underline{u}\|_{\overline{\mathbf{H}}_{sc}^{s+1}(D)} \lesssim \|u\|_{\overline{H}_{sc}^{s}(D)}.$$

Proof. If $s = 0$, then this is a consequence of the mapping properties of semiclassical pseudo-differential operators on $L^2(\mathbb{R}^d)$. Now assume $s \geq 1$. From classical jump relations (c.f. [69])

$$\nabla_h \underline{u} \quad = \quad \underline{\nabla_h u} + (\nu \frac{h}{i} u) \otimes \delta_{s=0}.$$

Then

$$\|Q\nabla_h \underline{u}\|_{\overline{\mathbf{H}}_{sc}^{s+1}(D)} \lesssim \|Q\underline{\nabla_h u}\|_{\overline{H}_{sc}^{s+1}(D)} + \|Q(\frac{h}{i}\nu u \otimes \delta_{s=0})\|_{\overline{H}_{sc}^{s+1}(D)}.$$

From the estimates (7.62) and (7.68) we have that

$$\|Q(\frac{h}{i}\nu u \otimes \delta_{s=0})\|_{\overline{H}_{sc}^{s+1}(D)} \lesssim \|u\|_{\overline{H}_{sc}^{s}(D)}.$$

Noting that $s \geq 1$, we can proceed to have

$$\|Q\underline{\nabla_h u}\|_{\overline{H}_{sc}^{s+1}(D)} \lesssim \|\nabla_h u\|_{\overline{H}_{sc}^{s-1}(D)} \lesssim \|u\|_{\overline{H}_{sc}^{s}(D)}.$$

This completes our proof. ☐

Lemma 7.5.2 *Assume $f \in H^1(D)$ and $f = 0$ in the neighborhood N of the boundary Γ. Then for $f \in \overline{H}_{sc}^{s}(D)$ and small enough h*

$$|\gamma Q\nabla_h f|_{\mathbf{H}^{s+\frac{3}{2}}(\Gamma)} \lesssim h^{\frac{1}{2}} \|f\|_{\overline{H}_{sc}^{s}(D)}. \tag{7.69}$$

Proof. Note that if $f = 0$ in N, then $\underline{f} \in H^s_{sc}(\mathbb{R}^3)$ and $\underline{f} \in H^1(\mathbb{R}^3)$. Let $\mathbf{u} \in \mathbf{H}^s_{sc}(\mathbb{R}^3)$ satisfy

$$-h^2\Delta\mathbf{u} - \mu(1+m)\mathbf{u} = \nabla_h \underline{f}.$$

Then $\mathbf{u} = Q\nabla_h \underline{f} + hK_{-M}\mathbf{u}$ for sufficiently large $M > 0$. Let $\chi \in C_0^\infty(\mathbb{R}^3)$ be supported in $N_\epsilon = \{x : x = y + s\nu(y),\ y \in \Gamma,\ -\epsilon \leq s < \epsilon\}$ with sufficiently small $\epsilon > 0$ such that $\chi\nabla_h f = 0$, and $\chi = 1$ on Γ. Then we have

$$\|\chi Q\nabla_h \underline{f}\|_{\mathbf{H}^{s+2}_{sc}(\mathbb{R}^3)} \leq \|\chi\mathbf{u}\|_{\mathbf{H}^{s+2}_{sc}(\mathbb{R}^3)} + h\|\mathbf{u}\|_{\mathbf{H}^{s+1}_{sc}(\mathbb{R}^3)}. \tag{7.70}$$

Since $\chi\nabla_h f = 0$ then

$$-h^2\Delta(\chi\mathbf{u}) - \mu(1+m)\chi\mathbf{u} = \chi\nabla_h \underline{f} - hK_1\mathbf{u} = -hK_1\mathbf{u}$$

where K_1 is a differential operator of order 1. Therefore

$$\|\chi\mathbf{u}\|_{\mathbf{H}^{s+2}_{sc}(\mathbb{R}^3)} \lesssim h\|\mathbf{u}\|_{\mathbf{H}^{s+1}_{sc}(\mathbb{R}^3)}.$$

Then estimate (7.70) yields

$$\|\chi Q\nabla_h f\|_{\mathbf{H}^{s+2}_{sc}(\mathbb{R}^3)} \lesssim h\|\mathbf{u}\|_{\mathbf{H}^{s+1}_{sc}(\mathbb{R}^3)}.$$

Recall that $\mathbf{u} = Q\nabla_h \underline{f} + hK_{-M}\mathbf{u}$. Then for h small enough

$$\|\mathbf{u}\|_{\mathbf{H}^{s+1}_{sc}(\mathbb{R}^3)} \lesssim \|f\|_{\overline{H}^s_{sc}(D)},$$

and therefore

$$\|\chi Q\nabla_h f\|_{\mathbf{H}^{s+2}_{sc}(\mathbb{R}^3)} \lesssim h\|f\|_{\overline{H}^s_{sc}(D)}.$$

From the inequality (7.68) we have that

$$|\gamma Q\nabla_h f|_{\mathbf{H}^{s+\frac{3}{2}}(\Gamma)} = |\gamma(\chi Q\nabla_h f)|_{\mathbf{H}^{s+\frac{3}{2}}(\Gamma)} \lesssim h^{\frac{1}{2}}\|f\|_{\overline{H}^s_{sc}(D)}.$$

This completes the proof. $\square$

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