GAUSSIAN PROCESSES: KARHUNEN-LOÈVE EXPANSION, SMALL BALL ESTIMATES AND APPLICATIONS IN TIME SERIES MODELS

by

Shi Jin

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ABSTRACT

In this dissertation, we study the Karhunen-Loève (KL) expansion and the exact L^2 small ball probability for Gaussian processes. The exact L^2 small ball probability is connected to the Laplace transform of the Gaussian process via Sytaja Tauberian theorem. Using this technique, we solved the problem of finding the exact L^2 small ball estimates for the Slepian process $S(t)$ defined as $S(t) = W(t+a) - W(t), 0 \leq t \leq 1$ for $1/2 \leq a < 1$.

We also prove a conjecture raised by Tanaka on the first moment of the limiting distribution of the least squares estimator (LSE) of a unit root process. The limiting random variable is a ratio of quadratic functionals of the m-times integrated Brownian motion. Its expectation can be found by using Karhunen-Loéve expansion and a property of the orthonormal eigenfunctions of the covariance function of the m-times integrated Brownian motion.

<center>**Chapter 1**</center>

<center>**INTRODUCTION**</center>

1.1 Gaussian Processes

Gaussian processes have a long history in the studies of probability theory and statistics. They provide mathematical models of random phenomena ranging from movements of particles suspended in liquid, fluctuations of stock prices in financial markets and testing of goodness of fit in statistics.

A stochastic process X on a parametric set T is a family of random variables $X(t)$, $t \in T$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. A stochastic process is Gaussian if all of its finite-dimensional distributions are multivariate Gaussian distributions, that is, for any $t_1, \ldots, t_n \in T$ the distribution density of the random vector $(X(t_1), \ldots, X(t_n))$ is the n-dimensional normal distribution given by, see

$$f_X(x_1, \ldots, x_n) = \frac{1}{\sqrt{(2\pi)^n \det \Sigma}} \exp\left(-\frac{1}{2}(\boldsymbol{x} - \boldsymbol{\mu})'\Sigma^{-1}(\boldsymbol{x} - \boldsymbol{\mu}) \right)$$

where $\boldsymbol{x} = (x_1, \ldots, x_n)'$, $\boldsymbol{\mu}$ is the n-dimensional mean vector $E[\boldsymbol{x}]$ and Σ is the $n \times n$ positive definite covariance matrix $[\text{Cov}[X(t_i), X(t_j)], i = 1, 2, \ldots, n, \ j = 1, 2, \ldots, n$. Equivalently, $\{X(t), t \in T\}$ is a Gaussian process if every finite linear combination $\sum a_t X(t)$, $t \in T$ has a Gaussian distribution on $\mathbb{R}$ for a_t not all equal to zero. The mean vector and covariance matrix uniquely determine a Gaussian distribution; consequently, the mean function and covariance function of a Gaussian prLoèveocess completely determine all of the finite-dimensional distributions. Therefore, given a mean function and a positive definite covariance function, there exists a corresponding Gaussian process that is unique in distribution, see [24]. If a Gaussian process has zero mean, then it is called a centered Gaussian process, and its properties are entirely determined by its covariance function.

<center>1</center>

The most important one-parameter Gaussian processes are the Wiener process $\{W_t\}, t \geq 0$ (Brownian motion), the Ornstein-Uhlenbeck process $\{Y_t\}, t \in R$, and the Brownian bridge $\{B_t\}, t \in [0, 1]$. These are the centered Gaussian processes with covariance functions

$$E[W_s W_t] = s \wedge t,$$

$$E[Y_s Y_t] = \exp\{-|t - s|\},$$

$$E[B_s B_t] = s \wedge t - st.$$

In this dissertation, we investigate the Karhunen-Loève expansion for Gaussian processes and its applications in small ball probability estimate and statistical problems arising from time series models.

1.2 Karhunen-Loève Expansion

The Karhunen-Loève expansion is a representation of a stochastic process as an infinite linear combination of orthogonal functions according to a spectral decomposition of its correlation function. It is analogous to a Fourier series representation of a function on a bounded interval. The Karhunen-Loève (KL) expansion is well-known, and is widely used in many disciplines, including mechanics, signal analysis, imaging compression, biology, physics, statistics, and finance. It is also known as principal component analysis (PCA), proper orthogonal decomposition (POD) in the finite dimensional case. It has been proposed independently by different authors in the 1940's, see [27],[29], and [32].

The KL expansion provides an optimal representation of a process in the mean square sense. It also gives an important distribution identity for the L^2 norm of Gaussian processes, which is very useful in the study of exact L^2 small ball probability.

1.3 Summary

This dissertation consists of three parts, which are based on two papers: [17] and [26].

In Chapter 2, we review the KL expansion for some classical Gaussian processes. We also obtain the KL expansion for the Slepian process $W(t+a) - W(t), 0 \leq t \leq 1$ for $1/2 \leq a < 1$.

Chapter 3 is mainly about utilizing results from the KL expansion of the Gaussian processes to obtain their exact L^2 small ball estimate. In particular, we obtain the exact L^2 small ball estimate for the Slepian process $W(t+a) - W(t), 0 \leq t \leq 1$ for $1/2 \leq a < 1$.

We find a property regarding the eigenfunctions of the KL expansion for m-times integrated Brownian motion in Chapter 4. It is shown that this property can be applied to prove a conjecture raised by Tanaka on the limiting distribution of the least squares estimator of the unit root process, see [38], [40].

Chapter 2

KARHUNEN-LOÈVE EXPANSION OF GAUSSIAN PROCESSES

As mentioned in the Introduction, it is interesting to study the Karhunen-Loève (KL) expansion of Gaussian processes. We state the theory of KL expansion (see [23], [6], [11]) and show how to obtain the expansion for some classical Gaussian processes including Brownian motion, Brownian bridge, integrated Brownian motion, Ornstein-Uhlenbeck process, and etc. We also discuss the KL expansion of the Slepian process, i.e., $S(t) = W(t+a) - W(t), 0 \leq t \leq 1$ in three cases: $a \geq 1$, see [22] , $1/2 \leq a < 1$ and $0 < a < 1/2$. The result can be used to obtain the Laplace transform and exact L^2 small ball probability of the Slepian process which we mention in Chapter 3.

2.1   General Theory

Consider an L^2 process $\{X(t), a \leq t \leq b\}$ with zero mean and continuous covariance $K(s,t)$. It is desirable to find an orthogonal expansion of $X(t)$:

$$X(t) = \sum_{k=1}^{\infty} Z_k e_k(t), \quad a \leq t \leq b,$$

where the series converges in L^2. We want to have double orthogonality, that is,

1. The Z_k are orthogonal random variables with zero mean, i.e., $E(Z_j Z_k) = 0, j \neq k$.

2. The functions e_k are orthonormal, i.e.,

$$\int_a^b e_j(t) e_k(t) dt = \begin{cases} 0, & j \neq k \\ 1, & j = k. \end{cases}$$

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The covariance function $K(s,t)$ can be represented as

$$
\begin{aligned}
K(s,t) &= E[X(s)X(t)] \\
&= E\left[\sum_{j=1}^{\infty} Z_j e_j(s) \sum_{k=1}^{\infty} Z_k e_k(t)\right] \\
&= \sum_{k=1}^{\infty} \lambda_k e_k(s) e_k(t)
\end{aligned}
$$

where $\lambda_k = E(Z_k^2)$. Multiplying the covariance by $e_k(s)$ and applying term by term integration, we obtain

$$
\begin{aligned}
\int_a^b K(s,t) e_n(s) ds &= \int_a^b \sum_{k=1}^{\infty} \lambda_k e_k(s) e_k(t) e_n(s) ds \\
&= \sum_{k=1}^{\infty} \lambda_k e_k(t) \int_a^b e_k(s) e_n(s) ds \\
&= \lambda_n e_n(t),
\end{aligned}
$$

Therefore, if the above expansion exists, the functions $e_k(t)$ must be eigenfunctions of the integral operator associated with the covariance function $K(s,t)$, and the variances λ_k of the random variables Z_k must be the eigenvalues of the operator.

We recall some facts from Hilbert space theory. Suppose that A is an integral operator on $L^2[a,b]$ associated with a continuous, symmetric and nonnegative definite covariance kernel K defined by

$$
(Ax)(s) = \int_a^b K(s,t) x(t) dt, \quad a \le s \le b, x \in L^2[a,b].
$$

The eigenfunctions of A span $L^2[a,b]$. The operator A has at most countably many eigenvalues which are all real, with 0 as the only possible limit point. Furthermore, the nonnegative-definiteness of the covariance kernel K guarantees that the eigenvalues are all positive. That is, the eigenvalues of the operator A satisfy $\lambda_1 \ge \lambda_2 \ge \cdots > 0$. The eigenspace corresponding to each eigenvalue $\lambda_n > 0$ is finite dimensional. We take $\{e_k\}$ to be an orthonormal basis for the space. Then by the Mercer's theorem below, we have the convergence for the kernel $K(s,t)$, see [35].

Theorem 1 (Mercer's Theorem) *Let $K : D \times D \to \mathbb{R}$ be a continuous symmetric kernel, where $D = [a, b] \subset \mathbb{R}$. Suppose that the operator A generated by the kernel K is positive. If λ_k and e_k are the eigenvalues and eigenfunctions of A, then for all $s, t \in D$,*

$$K(s,t) = \sum_{k=1}^{\infty} \lambda_k e_k(s) e_k(t),$$

converges absolutely and uniformly on $D \times D$.

The following theorem gives the existence of Karhunen-Loève expansion of $X(t)$.

Theorem 2 *Let $\{X(t), a \leq t \leq b\}$ be an L^2 process with zero mean and continuous covariance K. Let $\{e_n, n = 1, 2, \ldots\}$ be an orthonormal basis for the space spanned by the eigenfunctions of the integral operator associated with K, with e_n taken as an eigenfunction corresponding to the eigenvalue λ_n. Then*

$$X(t) = \sum_{n=1}^{\infty} Z_n e_n(t), \quad a \leq t \leq b,$$

where $Z_n = \int_a^b X(t) e_n(t) dt$, and the Z_n are orthogonal random variables with $E(Z_n) = 0, E[Z_n^2] = \lambda_n$. The series converges in L^2 to $X(t)$, uniformly in t; in other words,

$$E\left[\left(X(t) - \sum_{k=1}^{n} Z_n e_n(t)\right)^2\right] \to 0 \quad as \quad n \to \infty$$

uniformly for $t \in [a, b]$

Proof. First we show Z_n are orthogonal.

$$\begin{aligned}
E(Z_j Z_k) &= E\left[\int_a^b X(t) e_j(t) dt \int_a^b X(t) e_k(t) dt\right] \\
&= \int_a^b e_j(s) \int_a^b E\left[X(s)X(t)\right] e_k(t) dt ds \\
&= \int_a^b e_j(s) \int_a^b K(s,t) e_k(t) dt ds \\
&= \lambda_k \int_a^b e_j(s) e_k(s) ds \\
&= \begin{cases} 0, & \text{if } j \neq k, \\ \lambda_k, & \text{if } j = k. \end{cases}
\end{aligned}$$

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Let $S_n(t) = \sum_{k=1}^{n} Z_k e_k(t)$. Then

$$E\left[(S_n(t) - X(t))^2\right] = E\left[S_n^2(t)\right] - 2E[X(t)S_n(t)] + E[X^2(t)]$$

$$= \sum_{k=1}^{n} \lambda_k e_k^2(t) - 2\sum_{k=1}^{n} E[X(t)Z_k]e_k(t) + K(t,t)$$

$$= \sum_{k=1}^{n} \lambda_k e_k^2(t) - 2\sum_{k=1}^{n} E[X(t)\int_a^b X(s)e_k(s)ds]e_k(t) + K(t,t)$$

$$= \sum_{k=1}^{n} \lambda_k e_k^2(t) - 2\sum_{k=1}^{n} \int_a^b E[X(t)X(s)]e_k(s)ds e_k(t) + K(t,t)$$

$$= \sum_{k=1}^{n} \lambda_k e_k^2(t) - 2\sum_{k=1}^{n} \int_a^b K(s,t)e_k(s)ds e_k(t) + K(t,t)$$

$$= \sum_{k=1}^{n} \lambda_k e_k^2(t) - 2\sum_{k=1}^{n} \lambda_k e_k^2(t) + K(t,t)$$

$$= -\sum_{k=1}^{n} \lambda_k e_k^2(t) + K(t,t)$$

Thus,

$$E[|S_n(t) - X(t)|^2] = K(t,t) - \sum_{k=1}^{n} \lambda_k |e_k(t)|^2 \to 0 \quad \text{as } n \to \infty,$$

uniformly for $t \in [a,b]$, by Mercer's theorem.

Corollary 1 *If $X(t)$ is a mean zero Gaussian process, then the random variables Z_k are independent and jointly Gaussian.*

Proof. Let $I_k = \sum_{m=1}^{n} X(t_m)e_k(t_m)(t_m - t_{m-1})$ be the Riemann sum of Z_k. Then the I_k's are jointly Gaussian and so are their limits Z_k. Since Z_k's are uncorrelated and Gaussian, they are independent.

Hence for a zero mean Gaussian process $Z(t), 0 \leq t \leq 1$, with continuous covariance function

$$K(s,t) = E[Z(s)Z(t)], \text{ for } 0 \leq s,t \leq 1$$

its Karhunen-Loève expansion is given by

$$X(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k} e_k(t)\xi_k$$

where $\{\lambda_k, e_k\}$ are the eigenvalues and eigenfunctions to the Fredholm integral equation

$$\lambda e(t) = \int_0^1 K(s,t)e(s)ds, \text{ for } 0 \leq t \leq 1,$$

ξ_k are i.i.d. $N(0,1)$ random variable, $\{e_k(t)\}$ are orthonormal in $L^2[0,1]$, and $K(s,t) = \sum_{k=1}^{\infty} \lambda_k e_k(s)e_k(t)$ where convergence is absolute and uniform on $[0,1]^2$.

In the following context, we focus on the Karhunen-Loève expansion for Gaussian processes. In fact, there are not many Gaussian processes that have eigenvalues and eigenfunctions associated with their covariance functions computed explicitly. The key is to solve the Fredholm integral equation of the second type with the covariance kernel. This is usually reduced to the problem of solving the corresponding differential equation with boundary conditions. We review some Gaussian processes that have known KL expansion and give the KL expansion of Slepian process for the case of $1/2 \leq a < 1$.

2.2 Examples

2.2.1 Brownian Motion

The best known Gaussian process is the Brownian motion, also called Wiener process, see [14], [33].

Definition 1 *A real-valued stochastic process $\{W(t) : t \geq 0\}$ is called a standard Brownian motion if the following hold:*

- *$W(0)=0$,*

- *the process has independent increments, i.e., for all times $0 \leq t_1 \leq t_2 \leq \ldots \leq t_n$ the increments $W(t_n) - W(t_{n-1}), W(t_{n-1}) - W(t_{n-2}), \ldots, W(t_2) - W(t_1)$, are independent random variables,*

- *for all $t \geq 0$ and $h > 0$, the increments $W(t+h) - W(t)$ are normally distributed with mean zero and variance h,*

- *the function $t \to W(t)$ is continuous almost surely.*

The covariance function of $W(t)$ is $K(s,t) = s \wedge t$. It is well-known that, see [2].

Theorem 3 *For the standard Brownian motion $W(t)$ with covariance function*

$$K(t, s) = Cov(W(t), W(s)) = s \wedge t.$$

The eigenfunctions of the covariance kernel are

$$e_k(t) = \sqrt{2} \sin((k - \frac{1}{2})\pi t)$$

and the corresponding eigenvalues are

$$\lambda_k = \frac{1}{(k - \frac{1}{2})^2 \pi^2}$$

Proof. We first compute the eigenvalues of $W(t)$ by substituting $K(s, t) = s \wedge t$ into

$$Tf(t) = \int_0^1 K(s, t)f(s)ds = \lambda f(t).$$

In order to handle the $s \wedge t$ term, we split the integration range and obtain

$$\int_0^t sf(s)ds + \int_t^1 tf(s)ds = \lambda f(t) \tag{2.1}$$

By differentiating both sides of of (2.1) with respect to t, we obtain

$$\int_t^1 f(s)ds = \lambda f'(t). \tag{2.2}$$

Differentiating again with repeat to t gives

$$-f(t) = \lambda f''(t). \tag{2.3}$$

Hence, the general solution has the form

$$f(t) = A\sin(\sqrt{\lambda^{-1}}t) + B\cos(\sqrt{\lambda^{-1}}t).$$

Setting $t = 0$ in (2.1) we obtain $f(0) = 0$, setting $t = 1$ in (2.2) gives $f'(1) = 0$ which implies $B = 0$ and $\cos(\sqrt{\lambda^{-1}}) = 0$. Thus, the eigenvalues of T are

$$\lambda_k = \frac{1}{(k - \frac{1}{2})^2 \pi^2}$$

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and the corresponding eigenfunctions are of the form

$$f_k(t) = A \sin\left((k - \frac{1}{2})\pi t\right).$$

Since

$$\int_0^1 \sin^2\left((k - \frac{1}{2})\pi t\right) dt = \frac{1}{2},$$

in order to normalize $f_k(t)$, we have $A = \sqrt{2}$.

Theorem 4 *There is a sequence $\{\eta_i\}$ of independent Gaussian random variables with mean zero and variance 1 such that*

$$W(t) = \sqrt{2} \sum_{k=1}^{\infty} \frac{\sin((k - \frac{1}{2})\pi t)}{(k - \frac{1}{2})\pi} \eta_k.$$

2.2.2 Demeaned Brownian Motion

Let $\widetilde{W}(t)$ be the demeaned Brownian motion defined as $\widetilde{W}(t) = W(t) - \int_0^1 W(u)du$. It has many applications in econometrics.

Theorem 5 *For the demeaned Brownian motion $\{\widetilde{W}(t), 0 \leq t \leq 1\}$, its covariance function is*

$$K(s, t) = t \wedge s - \left(s - \frac{s^2}{2}\right) - \left(t - \frac{t^2}{2}\right) + \frac{1}{3}.$$

The eigenvalues of the covariance kernel are

$$\lambda_k = \frac{1}{k^2 \pi^2}$$

and the orthonormal eigenfunctions

$$e_k(t) = \sqrt{2} \cos(k\pi t).$$

Therefore, the demeaned Brownian motion $\widetilde{W}(t)$ has the following Karhunen-Loève expansion

$$\widetilde{W}(t) = \sqrt{2} \sum_{k=1}^{\infty} \frac{e_k(t)}{k\pi} \xi_k$$

where $\{\xi_k\}$ are i.i.d. $N(0,1)$.

Proof. Consider the following integral equation.

$$\lambda f(t) = \int_0^1 \left[t \wedge s - \left(s - \frac{s^2}{2} \right) - \left(t - \frac{t^2}{2} \right) + \frac{1}{3} \right] f(s) ds$$

$$= \int_0^t s f(s) ds + \int_t^1 t f(s) ds - (t - \frac{t^2}{2}) \int_0^1 f(s) ds - \int_0^1 (s - \frac{s^2}{2} - \frac{1}{3}) f(s) ds$$

We differentiate the equation with respect to t to get

$$\lambda f'(t) = t f(t) + \int_t^1 f(s) ds - t f(t) - (1 - t) \int_0^1 f(s) ds$$

$$= \int_t^1 f(s) ds - (1 - t) \int_0^1 f(s) ds.$$

One more differentiation gives

$$\lambda f''(t) = -f(t) + \int_0^1 f(s) ds$$

Let $\int_0^1 f(s) ds = C$. Then the equation becomes

$$\lambda f''(t) + f(t) = C$$

We use the boundary conditions $f'(0) = f'(1) = 0$ to find the general solution. It is easy to see that the differential equation has the complementary solution,

$$f(t) = C_1 \cos(\sqrt{\lambda^{-1}} t) + C_2 \sin(\sqrt{\lambda^{-1}} t).$$

We compute the derivative of the solution and use the boundary conditions.

$$f'(t) = -C_1 \sqrt{\lambda^{-1}} \sin(\sqrt{\lambda^{-1}} t) + C_2 \sqrt{\lambda^{-1}} \cos(\sqrt{\lambda^{-1}} t)$$

$$0 = f'(0) = C_2 \sqrt{\lambda^{-1}} \Rightarrow C_2 = 0$$

$$0 = f'(1) = -C_1 \sqrt{\lambda^{-1}} \sin(\sqrt{\lambda^{-1}}) \Rightarrow \sin(\sqrt{\lambda^{-1}}) = 0 \Rightarrow \lambda = \frac{1}{k^2 \pi^2}.$$

Thus, we have the eigenvalues and eigenfunctions,

$$\lambda_k = \frac{1}{k^2 \pi^2}$$

and

$$f_k(t) = C_1 \cos(k\pi t).$$

When we orthonormalize the eigenfunctions, we get

$$e_k(t) = \sqrt{2} \cos(k\pi t).$$

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2.2.3 Brownian Bridge

A Brownian bridge $\{B(t), 0 \leq t \leq 1\}$ is defined as the Brownian motion $W(t)$ conditioned on $W(1) = 0$. It can be represented as $B(t) = W(t) - tW(1)$ or $B(t) = (1-t)W(\frac{t}{1-t})$ for $t \in [0,1]$. A Brownian bridge is the result of Donsker's theorem in the area of empirical processes. It is also used in the Kolmogorov-Smirnov test in the area of statistical inference. Its KL expansion can also be computed by solving the integral equation of its covariance kernel.

Theorem 6 *For the Brownian bridge $B(t) = W(t) - tW(1)$, with covariance function*

$$K(t,s) = t \wedge s - ts$$

can be represented as the series

$$B(t) = \sum_{k=1}^{\infty} \frac{\sqrt{2}\sin(k\pi t)}{k\pi}\xi_k.$$

Proof.

$$Tf(t) = \int_0^1 (t \wedge s - ts)f(s)ds = \lambda f(t) \tag{2.4}$$

That is

$$\int_0^t (s - ts)f(s)ds + \int_t^1 (t - ts)f(s)ds = \lambda f(t) \tag{2.5}$$

Differentiating once with respect t yields

$$-\int_0^1 sf(s)ds + \int_t^1 f(s)ds = \lambda f'(t) \tag{2.6}$$

Differentiating once more with respect t yields

$$-f(t) = \lambda f''(t) \tag{2.7}$$

So the general solution has the form

$$f(t) = A\sin(\sqrt{\lambda^{-1}}t) + B\cos(\sqrt{\lambda^{-1}}t).$$

Setting $t = 0$, (2.5) gives $f(0) = 0$. Setting $t = 1$, (2.5) gives $f(1) = 0$. Thus, $B = 0$ and $\sin(\sqrt{\lambda^{-1}}) = 0$. The eigenvalues are

$$\lambda_k = \frac{1}{k^2\pi^2}.$$

The corresponding normalized eigenfunctions are

$$e_k(t) = \frac{f(t)}{||f(t)||_2} = \sqrt{2}\sin(k\pi t).$$

2.2.4 Integrated Brownian Motion

The m-times integrated Brownian motion is defined recursively as

$$X_m(t) = \int_0^t X_{m-1}(s)ds, \quad t \geq 0, m \geq 1$$

for all positive integer m and $X_0(t) = W(t)$ where $W(t)$ is the standard Brownian motion. Using integration by parts, we also have the representation

$$X_m(t) = \frac{1}{m!}\int_0^t (t-s)^m dW(s), \quad m \geq 0.$$

When $m = 1$, the one-time integrated Brownian motion $X_1(t)$ is $\int_0^t W(s)ds$, see [16]. Then,

$$E[X_1(t)] = 0,$$
$$\text{Var}(X_1(t)) = 2\int_0^t \int_0^u E[W(u)W(v)]dvdu$$
$$= 2\int_0^t \int_0^u vdvdu = \frac{1}{3}t^3,$$
$$K(s,t) = \text{Cov}(X_1(s), X_1(t))$$
$$= \frac{1}{2}st(s \wedge t) - \frac{1}{6}(s \wedge t)^3.$$

Consider the eigenvalue problem

$$\int_0^1 K(s,t)\phi(s)ds = \lambda\phi(t). \tag{2.8}$$

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Plug in $K(s,t)$ and rewrite (2.8), we get

$$\int_0^t \left(\frac{1}{2}s^2 t - \frac{1}{6}s^3\right)\phi(t)ds + \int_t^1 \left(\frac{1}{2}st^2 - \frac{1}{6}t^3\right)\phi(s)ds = \lambda\phi(t). \qquad (2.9)$$

Differentiating with respect to t successively gives

$$\int_0^t \frac{1}{2}s^2\phi(s)ds + \int_t^1 (st - \frac{1}{2}t^2)\phi(s)ds = \lambda\phi'(s), \qquad (2.10)$$

$$\int_t^1 (s - t)\phi(s)ds = \lambda\phi''(t), \qquad (2.11)$$

$$-\int_t^1 \phi(s)ds = \lambda\phi^{(3)}(t), \qquad (2.12)$$

$$\phi(t) = \lambda\phi^{(4)}(t). \qquad (2.13)$$

The general solution to (2.13) is of the form

$$\phi(t) = A\cos(t/\lambda^{1/4}) + B\sin(t/\lambda^{1/4}) + C\cosh(t/\lambda^{1/4}) + D\sinh(t/\lambda^{1/4}), \qquad (2.14)$$

where $A, B, C,$ and D are constants. The boundary conditions that determine the constants are obtained from (2.9)-(2.12):

$$\phi(0) = 0, \quad \phi'(0) = 0, \quad \phi''(1) = 0, \quad \phi^{(3)}(1) = 0.$$

From $\phi(0) = 0$, we have $C = -A$. $\phi'(0) = 0$ gives $D = -B$. The remaining two boundary conditions lead to

$$A(\cos(1/\lambda^{1/4}) + \cosh(1/\lambda^{1/4})) + B(\sin(1/\lambda^{1/4}) + \sinh(1/\lambda^{1/4})) = 0,$$
$$A(-\sin(1/\lambda^{1/4}) + sinh(1/\lambda^{1/4})) + B(\cos(1/\lambda^{1/4}) + \cosh(1/\lambda^{1/4})) = 0.$$

In order to have nontrivial solution for A and B, we must have

$$\det\begin{pmatrix} \cos(1/\lambda^{1/4}) + \cosh(1/\lambda^{1/4}) & \sin(1/\lambda^{1/4}) + \sinh(1/\lambda^{1/4})) \\ -\sin(1/\lambda^{1/4}) + \sinh(1/\lambda^{1/4})) & \cos(1/\lambda^{1/4}) + \cosh(1/\lambda^{1/4})) \end{pmatrix}$$

$$= (\cos(1/\lambda^{1/4}) + \cosh(1/\lambda^{1/4}))^2 - (\sinh(1/\lambda^{1/4})^2 - \sin(1/\lambda^{1/4})^2)$$

$$= 2\cos(1/\lambda^{1/4})\cosh(1/\lambda^{1/4}) + 2 = 0.$$

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Theorem 7 *([9], [16]) For the one-time integrated Brownian motion, that is, $X_1(t) = \int_0^t W(s)ds$, the eigenvalues λ satisfy*

$$\cosh\left(\frac{1}{\lambda^{1/4}}\right)\cos\left(\frac{1}{\lambda^{1/4}}\right) = -1,$$

and $\lambda_k \sim (k\pi)^{-4}$. The corresponding eigenfunction is given by (2.14), with

$$A = B, \quad C = -D, \quad A/B = -(\sin(1/\lambda^{1/4})+\sinh(1/\lambda^{1/4}))/(\cos(1/\lambda^{1/4})+\cosh(1/\lambda^{1/4})),$$

while B is chosen so the function has norm 1.

As m increases, the complexity of the characteristic equation for eigenvalues also increases. For example, for $m = 2$ the eigenvalues of $X_2(t)$ are given by, see [21],

$$4 + 4\cos\left(\frac{1}{\lambda^{1/6}}\right) + \cos^2\left(\frac{1}{\lambda^{1/6}}\right) + 8\cos\left(\frac{1}{2\lambda^{1/6}}\right)\cosh\left(\frac{\sqrt{3}}{2\lambda^{1/6}}\right)$$

$$+ \cos\left(\frac{1}{\lambda^{1/6}}\right)\cosh\left(\frac{\sqrt{3}}{\lambda^{1/6}}\right) = 0.$$

For the general m-times integrated Brownian motion $X_m(t)$, the covariance kernel is, see [18], [21],

$$K(s,t) = \frac{1}{(m!)^2}\int_0^{s\wedge t}(s-u)^m(t-u)^m du.$$

Successive differentiations of the integral equation

$$\int_0^1 K(s,t)f(s)ds = \lambda f(t)$$

give the Sturm-Liouville equation:

$$\lambda f^{(2m+2)}(t) = (-1)^{m+1}f(t) = (i)^{2m+2}f(t) \tag{2.15}$$

with boundary conditions

$$f^{(k)}(0) = f^{(m+1+k)}(1) = 0$$

for $k = 0, 1, \ldots, m$. The eigenfunctions thus are the nontrivial functions of the form

$$f(t) = \sum_{j=0}^{2m+1} c_j e^{\alpha_j t}$$

with $\alpha_j = \lambda^{-1/(2d+2)} i \omega_j$ and $\omega_j = \exp(\frac{j\pi}{d+1} i)$ satisfying the boundary conditions. The eigenvalues λ's are determined by setting the determinant of the following matrix

$$
M_W = \begin{pmatrix}
1 & 1 & \cdots & 1 \\
\omega_0 & \omega_1 & \cdots & \omega_{2d+1} \\
\vdots & \vdots & \ddots & \vdots \\
\omega_0^d & \omega_1^d & \cdots & \omega_{2d+1}^d \\
\omega_0^{d+1} e^{\alpha_0} & \omega_1^{d+1} e^{\alpha_1} & \cdots & \omega_{2d+1}^{d+1} e^{\alpha_{2d+1}} \\
\vdots & \vdots & \ddots & \vdots \\
\omega_0^{2d+1} e^{\alpha_0} & \omega_1^{2d+1} e^{\alpha_1} & \cdots & \omega_{2d+1}^{2d+1} e^{\alpha_{2d+1}}
\end{pmatrix}
$$

to be zero.

The authors in [21] simplified the determinant of M_W and computed the asymptotics of the eigenvalues λ_k.

Theorem 8 *The eigenvalues λ_k of $K(s,t)$ are*

$$
\lambda_k = \left(\frac{1}{(k - \frac{1}{2})\pi} \right)^{2m+2} + O\left(\frac{1}{k^{2m+3}} \exp\left(-k\pi \sin\left(\frac{\pi}{m+1} \right) \right) \right).
$$

In fact, if we modify the boundary conditions for the Sturm-Liouville problem above to

$$
f(t_0) = f'(t_1) = \cdots = f^{(m)}(t_m) = f^{(m+1)}(t_{m+1}) = \cdots = f^{(2m)}(t_{2m}) = f^{(2m+1)}(t_{2m+1}) = 0.
$$

where $t_j \in \{0,1\}$ for all j, $\sum_j t_j = m + 1$ and $t_{2m+1-j} = 1 - t_j$, then we have a class of m-times integrated Brownian motions depending on the choice of t_j. For a particular choice of $\{t_0, t_1, \ldots, t_{2m+1}\}$, ew call the associated centered Gaussian process a generalized integrated Brownian motion and denote it by $X_{\{t_0,\ldots,t_m\}}(t)$. If $t_0 = t_2 = \ldots = t_{2m} = 0$, then the process is called an Euler-integrated Brownian motion since the covariance kernel is just the difference of two Euler polynomials. The covariance operator of Euler integrated Brownian motion has the eigenvalues exactly equal to

$$
b_n = ((n - 1/2)\pi)^{-2m-2}
$$

2.2.5 Integrated Brownian Bridge

The m-times integrated Brownian bridge is defined similar to the m-times integrated Brownian motion. Let $B(t)$ be a standard Brownian bridge for $0 \leq t \leq 1$. For integer $m \geq 0$, the m-times integrated Brownian bridge on $[0, 1]$ is the Gaussian process

$$Y_m(t) = \int_0^t \int_0^{s_m} \cdots \int_0^{s_2} B(s_1) ds_1 ds_2 \cdots ds_m \quad 0 \leq t \leq 1.$$

The eigenfunctions are given by the same Sturm-Liouville equation as integrated Brownian motion, see [19]:

$$\lambda f^{(2m+2)}(t) = (-1)^{m+1} f(t) = (i)^{2m+2} f(t) \tag{2.16}$$

with boundary conditions

$$f(0) = f'(0) = \cdots = f^{(m-1)}(0) = f^{(m)}(0)$$
$$= f^{(m)}(1) = f^{(m+2)}(1) = f^{(m+3)}(1) = \cdots = f^{(2m+1)}(1) = 0.$$

The eigenvalues are then determined by setting the determinant

$$\det(M_B(\lambda^{-1/(2m+2)}))$$

equals to zero, where

$$M_B(z) = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \omega_0 & \omega_1 & \cdots & \omega_{2m+1} \\ \cdots & \cdots & \cdots & \cdots \\ \omega_0^m & \omega_1^m & \cdots & \omega_{2m+1}^m \\ \omega_0^m e^{i\omega_0 z} & \omega_1^m e^{i\omega_1 z} & \cdots & \omega_{2m+1}^m e^{i\omega_{2m+1} z} \\ \omega_0^{m+2} e^{i\omega_0 z} & \omega_1^{m+2} e^{i\omega_1 z} & \cdots & \omega_{2m+1}^{m+2} e^{i\omega_{2m+1} z} \\ \omega_0^{m+3} e^{i\omega_0 z} & \omega_1^{m+3} e^{i\omega_1 z} & \cdots & \omega_{2m+1}^{m+3} e^{i\omega_{2m+1} z} \\ \cdots & \cdots & \cdots & \cdots \\ \omega_0^{2m+1} e^{i\omega_0 z} & \omega_1^{2m+1} e^{i\omega_1 z} & \cdots & \omega_{2m+1}^{2m+1} e^{i\omega_{2m+1} z} \end{pmatrix}.$$

2.2.6 Ornstein-Uhlenbeck Process

The Ornstein-Uhlenbeck (OU) process is defined by the stochastic differential equation, see [13]

$$dX(t) = \theta(\mu - X(t))dt + \sigma dW(t), \quad \sigma > 0, \theta \geq 0$$

Solving the above SDE, we get

$$X(t) = X_0 e^{-\theta t} + \mu(1 - e^{-\theta t}) + \int_0^t \sigma e^{\theta(s-t)} dW(s).$$

The Ornstein-Uhlenbeck process is mean-reverting. Applications of this property includes interest rate modeling (Vasicek short rate model), see [41], [36], pair trading, see [15] and etc. It can also be considered as the continuous-time analogue of the discrete-time AR(1) process. The Ornstein-Uhlenbeck process can be interpreted as a scaling limit of a discrete process, in the same way that Brownian motion is a scaling limit of random walks.

We consider two cases depending on how the initial condition $X_0 := X(0)$ is specified. First, let $X(0)$ to be normal random variable $N(m_0, \sigma_0^2)$ independent of $W(t)$. Then $E[X(t)] = m_0 e^{-\theta t} + \mu(1 - e^{-\theta t})$. Consider the centered process $Y(t) = X_0 - m_0 e^{-\theta t} - \mu(1 - e^{-\theta t})$, then

$\text{Cov}(Y(s), Y(t))$

$$= E\left[\left((X_0 - m_0)e^{-\theta t} + \int_0^t \sigma e^{\theta(u-t)} dW(u)\right)\left((X_0 - m_0)e^{-\theta s} + \int_0^s \sigma e^{\theta(v-s)} dW(v)\right)\right]$$

$$= E\left[\int_0^t \sigma e^{\theta(u-t)} dW(u) \int_0^s \sigma e^{\theta(v-s)} dW(v)\right] + E\left[(X_0 - m_0)^2 e^{-\theta(s+t)}\right]$$

$$= \sigma^2 e^{-\theta(s+t)} E\left[\int_0^t e^{\theta u} dW(u) \int_0^s e^{\theta v} dW(v)\right] + \sigma_0^2 e^{-\theta(s+t)}$$

$$= \frac{\sigma^2}{2\theta} e^{-\theta(s+t)} \left(e^{2\theta(s \wedge t)} - 1\right) + \sigma_0^2 e^{-\theta(s+t)}$$

$$= \frac{\sigma^2}{2\theta} \left(e^{-\theta|s-t|} - e^{-\theta(s+t)}\right) + \sigma_0^2 e^{-\theta(s+t)}$$

As $t \to \infty$, $\text{Var}(Y(t)) \to \frac{\sigma^2}{2\theta}$ the long run variance. If the initial variance σ_0^2 is equal to the long run variance $\frac{\sigma^2}{2\theta}$, then $Y(t)$ is stationary and the covariance function is

$$K_Y(s, t) = \text{Cov}(Y(s), Y(t)) = \frac{\sigma^2}{2\theta} e^{-\theta|s-t|}.$$

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Let

$$\lambda f(t) = \int_0^T K_Y(s,t)f(s)ds \tag{2.17}$$

$$= \int_0^t \frac{\sigma^2}{2\theta}e^{-\theta(t-s)}f(s)ds + \int_t^T \frac{\sigma^2}{2\theta}e^{-\theta(s-t)}f(s)ds + \int_0^T \left(\sigma_0^2 - \frac{\sigma^2}{2\theta}\right)e^{-\theta(s+t)}f(s)ds. \tag{2.18}$$

Differentiating with respect to t yields,

$$\lambda f'(t) = -\frac{\sigma^2}{2}\int_0^t e^{-\theta(t-s)}f(s)ds + \frac{\sigma^2}{2}\int_t^T e^{-\theta(s-t)}f(s)ds - \left(\theta\sigma_0^2 - \frac{\sigma^2}{2}\right)\int_0^T e^{-\theta(s+t)}f(s)ds.$$

Differentiating with respect to t again,

$$\lambda f''(t) = \frac{\sigma^2\theta}{2}\left(\int_0^t e^{-\theta(t-s)}f(s)ds + \int_t^T e^{-\theta(s-t)}f(s)ds\right)$$
$$+\theta^2\left(\sigma_0^2 - \frac{\sigma^2}{2\theta}\right)\int_0^T e^{-\theta(s+t)}f(s)ds - \sigma^2 f(t)$$

Hence, we obtain

$$\lambda f''(t) + (\sigma^2 - \lambda\theta^2)f(t) = 0,$$

and two boundary conditions

$$\begin{cases} \sigma_0^2 f'(0) &= (\sigma^2 - \theta\sigma_0^2)f(0), \\ f'(T) &= -\theta f(T). \end{cases}$$

The solution of the ODE on $[0,T]$ has the form $f(t) = A\cos(\omega t) + B\sin(\omega t)$ where

$$\omega = \sqrt{\frac{\sigma^2 - \lambda\theta^2}{\lambda}}$$

From the boundary conditions we have

$$\omega\sigma^2\cos(\omega T) + (-\omega^2\sigma_0^2 + \theta\sigma^2 - \theta^2\sigma_0^2)\sin(\omega T) = 0.$$

Theorem 9 *The centered OU process $Y(t)$ has eigenvalues $\lambda_n = \frac{\sigma^2}{\omega_n^2 + \theta^2}$ and eigenfunctions $e_n(t) = K_n(\omega_n\sigma_0^2\cos(\omega_n t) + (\sigma^2 - \theta\sigma_0^2)\sin(\omega_n t))$ where K_n is the normalization constant.*

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If the initial condition $X_0 = 0$, then $\sigma_0 = 0$, we have

$$e_n(t) = \frac{1}{\sqrt{\frac{T}{2} - \frac{\sin(2\omega_n T)}{4\omega_n}}} \sin(\omega_n t)$$

In the stationary case when $\sigma_0^2 = \frac{\sigma^2}{2\theta}$,

$$e_n(t) = C_n \left(\omega_n \cos(\omega_n t) + \theta \sin(\omega_n t)\right)$$

where

$$C_n = \left(\frac{\theta}{2}\left(1 - \cos(2\omega_n T)\right) + \frac{\omega_n^2}{2}\left(T + \frac{\sin(2\omega_n T)}{2\omega_n}\right) + \frac{\theta^2}{2}\left(T - \frac{\sin(2\omega_n T)}{2\omega_n}\right)\right)^{-1/2}.$$

2.2.7 Anderson-Darling process

Consider the center Gaussian process $\{Z_1(t), 0 < t < 1\}$ with almost surely continuous sample paths and covariance function

$$K_1(s,t) = \frac{s \wedge t - st}{\sqrt{s(1-s)t(1-t)}} \quad \text{for } 0 < s, t < 1. \tag{2.19}$$

The square integral of this process $Z_1(t)$ arises as the limit of the statistic

$$\mathcal{A}_{1,n}^2 = \int_0^1 \frac{U_n^2(t)}{t(1-t)} dt,$$

where

$$U_n(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \{1_{\{X_i \leq t\}} - t\}, \quad 0 \leq t \leq 1$$

is the uniform empirical process. Here $X_i = F(Y_i), 1 \leq i \leq n$, F is a specified continuous distribution function and $Y_1, \ldots, Y_n$ are independent, identically distributed (i.i.d.) random variables, see [4], [5], [34].

The covariance function K_1 in (2.19) can be decomposed into

$$K_1(s,t) = \sum_{k=1}^\infty \lambda_k f_k(s) f_k(t), \quad \text{for } 0 < s, t < 1.$$

where the eigenvalues are

$$\lambda_k = \frac{1}{k(k+1)}$$

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and the eigenfunctions are

$$f_k(t) = 2\sqrt{\frac{2k+1}{k(k+1)}}\sqrt{t(1-t)}P_k'(2t-1)$$

with

$$P_k(x) = \frac{1}{2^k k!}\frac{d^k}{dx^k}(x^2-1)^k$$

denoting the kth Legendre polynomial.

Therefore, the KL expansion of the Anderson-Darling process $Z_1(t)$ is

$$Z_1(t) = \sum_{k=1}^{\infty}\sqrt{\lambda_k}f_k(t)\xi_k,$$

where $\{\xi_k, k \geq 1\}$ are i.i.d. $N(0,1)$ random variables.

Consequently, the asymptotic distribution of $\mathcal{A}_{1,n}^2$ is the distribution of the random variable

$$\mathcal{A}_1^2 = \int_0^1 Z_1(t) = \sum_{k=1}^{\infty}\lambda_k\xi_k^2.$$

And the characteristic function can be obtained as

$$E[e^{iu\mathcal{A}_1^2}] = \prod_{k=1}^{\infty}\left(1 - \frac{2iu}{k(k+1)}\right)^{-1/2}.$$

For large n, the distribution of $\mathcal{A}_1^2$ provides a good approximation to the distribution of the test statistic $\mathcal{A}_{1,n}^2$.

Furthermore, [34] generalized the covariance function K_1 to

$$K_\mu(s,t) = \left(\frac{s \wedge t - st}{\sqrt{s(1-s)t(1-t)}}\right)^\mu = K_1(s,t)^\mu \quad \text{for } 0 < s,t < 1. \tag{2.20}$$

for $\mu > 0$. The corresponding centered Gaussian process $Z_\mu(t)$ admits the following decomposition:

$$Z_\mu(t) = \sum_{k=1}^{\infty}\sqrt{\lambda_{\mu,k}}f_{\mu,k}(t)\xi_k,$$

where

$$\lambda_{\mu,k} = \frac{\mu}{(\mu+k-1)(\mu+k)},$$

$$f_{\mu,k}(t) = \sqrt{\frac{(2\mu+2k-1)\Gamma(2\mu+k)}{(k-1)!}}P_{\mu+k-1}^{-\mu}(2t-1).$$

and P_v^μ is the Legendre function of the first kind which is the solution to the differential equation

$$(1 - x^2)y'' - 2xy' - \frac{\mu^2}{1 - x^2}y = -v(v + 1)y,$$

where μ, v are two arbitrary complex numbers, $-1 \le x \le 1$.

The result is then used in [34] to test independence of multi-dimensional samples.

2.2.8 Mean-centered Brownian Bridge

The mean-centered Brownian bridges are defined by

$$y_K(t) = B(t) - 6Kt(1 - t)\int_0^1 B(u)du \quad \text{for } 0 \le t \le 1,$$

where $B(t) : 0 \le t \le 1$ is a Brownian bridge and $K \in \mathbb{R}$ is a constant. The covariance function can be computed as

$$K(s, t) = s \wedge t - st - 3K(2 - K)s(1 - s)t(1 - t).$$

[10] gives the KL expansion for $K = 1$.

Theorem 10 *The process $\{y_1(t) = B(t) - 6t(1 - t)\int_0^1 B(u)du : 0 \le t \le 1\}$ has a KL expansion given by*

$$y_1(t) = \sum_{k=1}^\infty \frac{1}{2k\pi}Z_k\sqrt{2}\sin(2k\pi t) + \sum_{k=1}^\infty \frac{1}{2z_{3/2,k}}Z_k^*\sqrt{2}(\cos(z_{3/2,k}(2t - 1)) - \cos(z_{3/2,k}))$$

where $\{Z_k : k \ge 0\}$ and $\{Z_k^* : k \ge 1\}$ denote two independent sequences of i.i.d. $N(0, 1)$ random variables and $z_{v,k}$ is the positive zeros of the Bessel function of the first order $J_v(\cdot)$.

2.2.9 Detrended Brownian Motion and Bridge

Consider the optimization problem, see [3],

$$\min_{a,b\in\mathbb{R}} \int_0^1 (X(t) - a - bt)^2 dt.$$

The optimal constant a, b must satisfy

$$\frac{\partial}{\partial a}\int_0^1 (X(t) - a - bt)^2 dt = 0, \quad \frac{\partial}{\partial b}\int_0^1 (X(t) - a - bt)^2 dt = 0.$$

It follows that

$$a = 4 \int_0^1 X(s)ds - 6 \int_0^1 sX(s)ds, \quad b = 12 \int_0^1 sX(s)ds - 6 \int_0^1 X(s)ds.$$

Thus, we define the detrended Gaussian process to be the orthogonal component of the projection,

$$\widehat{X}(t) = X(t) - a - bt = X(t) + (6t - 4) \int_0^1 X(s)ds + (6 - 12t) \int_0^1 sX(s)ds,$$

Let $X(t) = W(t)$, then we have the detrended Brownian motion

$$\widehat{W}(t) = W(t) + (6t - 4) \int_0^1 W(s)ds + (6 - 12t) \int_0^1 sW(s)ds.$$

Similarly, we have the detrended Brownian bridge

$$\widehat{B}(t) = B(t) + (6t - 4) \int_0^1 B(s)ds + (6 - 12t) \int_0^1 sB(s)ds.$$

The covariance function can be computed as

$$K_{\widehat{W}}(s,t) = K_{\widehat{B}}(s,t) = s \wedge t - \frac{11}{10}(t+s) + 2(t^2+s^2) - (t^3+s^3) - 3st^2 - 3ts^2 + 2st^3 + 2ts^3 + \frac{6}{5}st + \frac{2}{15}$$

Thus, $\widehat{W}(t)$ and $\widehat{B}(t)$ are the same process on $C[0,1]$.

Solving the eigenvalue problem

$$\int_0^1 K_{\widehat{W}}(s,t)f(s)ds = \lambda f(t)$$

by differentiation and using boundary conditions, we obtain the characteristic equation for the eigenvalues

$$2(1 - \cos(\lambda^{-1/2})) - \lambda^{-1/2}\sin(\lambda^{-1/2}) = 0$$

which can be rewritten as

$$2^{-1}\pi\lambda^{-1}J_{1/2}(2^{-1}\lambda^{-1/2})J_{3/2}(2^{-1}\lambda^{-1/2}) = 0, \tag{2.21}$$

where $J_{1/2}(x)$ and $J_{3/2}(x)$ are the Bessel function of the first kind. Specifically,

$$J_{1/2}(x) = (2/(\pi x))^{1/2}\sin(x),$$

$$J_{3/2}(x) = (2/(\pi x))^{1/2}(\sin(x)/x - \cos(x)).$$

The solutions of (2.21) are

$$\lambda_{2k-1} = (2k\pi)^{-2}, k = 1, 2, \ldots, \tag{2.22}$$

$$\lambda_{2k} = (2z_{3/2,k})^{-2}, k = 1, 2, \ldots, \tag{2.23}$$

where $z_{3/2,k}$ are the ordered positive zeros of Bessel function and $\lambda_1 > \lambda_2 > \cdots > 0$.

Theorem 11 *The spectrum of the KL expansion for the detrended Brownian motion* $\widehat{W}(t), t \in [0,1]$ *and detrended Brownian bridge is given by (2.22), (2.23), and we have the distribution identities*

$$\int_0^1 \widehat{W}(t)^2 dt = \int_0^1 \widehat{B}(t)^2 dt = \sum_{k \geq 1} \frac{\eta_k^2}{4\pi^2 k^2} + \sum_{k \geq 1} \frac{\eta_k^{*2}}{4z_{3/2,k}^2}.$$

2.2.10 Slepian Process

Consider the one-dimensional Slepian process which is the increment of a Wiener process during a fixed time interval $a > 0$, i.e.,

$$S(t) = W(t + a) - W(t), \quad 0 \leq t \leq 1,$$

with $W(t)$ is the standard Brownian motion. Let us consider the Karhunen-Loève expansion in one-dimensional setting. That is, we need to solve the eigenvalue problem

$$\lambda f(t) = \int_0^1 K(s,t) f(s) ds, \quad 0 \leq t \leq 1, \quad a > 0,$$

where the covariance function $K(s,t)$ can be computed as follows,

$$\begin{aligned}
K(s,t) &= \mathbb{E}[S(s)S(t)] \\
&= \mathbb{E}[(W(s+a) - W(s))(W(t+a) - W(t))] \\
&= \mathbb{E}[W(s+a)W(t+a) + W(s)W(t) - W(s+a)W(t) - W(s)W(t+a)] \\
&= (s+a) \wedge (t+a) + s \wedge t - (s+a) \wedge t - s \wedge (t+a) \\
&= a + 2s \wedge t - (s+a) \wedge t - s \wedge (t+a) \\
&= 2s \wedge t - s \wedge (t-a) - s \wedge (t+a) \\
&= \begin{cases} a - |t - s| & \text{if } |t - s| \leq a, \\ 0 & \text{otherwise,} \end{cases}
\end{aligned}$$

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for $s, t \in [0, 1]$. Figure 2.1 illustrates the covariance function for $a = 1$ and $a = 1/2$. We discuss the KL decomposition of $S(t)$ in three cases: $a \geq 1$, $1/2 \leq a < 1$ and $0 < a < 1/2$.
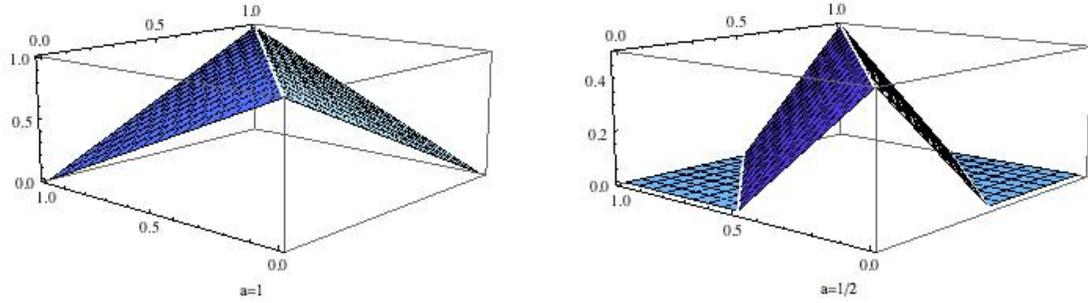
Figure 2.1: Covariance function of the Slepian process for $a = 1$ and $a = 1/2$

2.2.10.1 Case $a \geq 1$

The $a \geq 1$ case was studied in [22]. When $a \geq 1$, $|s - t| < a$. So the covariance function of $S(t)$ is

$$K(s, t) = a - |s - t|$$

for $s, t \in [0, 1]$. Plugging $K(s, t)$ into the integral equation

$$\lambda f(t) = \int_0^1 K(s, t) f(s) ds, \quad 0 \leq t \leq 1,$$

we have

$$\lambda f(t) = \int_0^t (a - t + s) f(s) ds + \int_t^1 (a + t - s) f(s) ds, \quad 0 \leq t \leq 1. \tag{2.24}$$

Differentiating (2.24) with respect t yields

$$\lambda f'(t) = - \int_0^t f(s) ds + \int_t^1 f(s) ds, \tag{2.25}$$

Differentiating (2.25) gives $\lambda f''(t) = -2 f(t)$. Hence, the general solution is

$$f(t) = c_1 \sin \sqrt{2\lambda^{-1}} t + c_2 \cos \sqrt{2\lambda^{-1}} t. \tag{2.26}$$

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We set $t = 0$ and $t = 1$ in (2.24) and (2.25) to obtain the boundary conditions

$$f'(0) + f'(1) = 0 \quad \text{and} \quad f(0) + f(1) = (2a - 1)f'(0). \tag{2.27}$$

Substituting into (2.26) into (2.27) yields

$$(1 + \cos \sqrt{2\lambda^{-1}})c_1 - \sin \sqrt{2\lambda^{-1}}c_2 = 0,$$
$$(\sin \sqrt{2\lambda^{-1}} - (2a - 1)\sqrt{2\lambda^{-1}})c_1 + (1 + \cos \sqrt{2\lambda^{-1}})c_2 = 0,$$

Then the equation for the eigenvalues is obtained by setting the determinant of the above two equations to be zero. That is,

$$2 + 2\cos \sqrt{2\lambda^{-1}} - (2a - 1)\sqrt{(2\lambda)^{-1}} \sin \sqrt{2\lambda^{-1}} = 0,$$

which simplifies to

$$\cos \sqrt{(2\lambda)^{-1}}(\cos \sqrt{(2\lambda)^{-1}} - (2a - 1)\sqrt{(2\lambda)^{-1}} \sin \sqrt{(2\lambda)^{-1}}) = 0. \tag{2.28}$$

Therefore, the eigenvalues are implicitly determined by (2.28).

2.2.10.2   Case $1/2 \leq a < 1$

In this case the covariance function is

$$K(s, t) = 2s \wedge t - s \wedge (t - a) - s \wedge (t + a),$$

for $s, t \in [0, 1]$. We consider $1/2 < a < 1$ and $a = 1/2$ separately.

(i) $1/2 < a < 1$.

Noticing that the differential equations obtained by successively differentiate the integral equation

$$\lambda f(t) = \int_0^1 K(s, t)f(s)ds, \quad 0 \leq t \leq 1,$$

are different for $t \in [0, 1 - a]$, $t \in (1 - a, a)$, and $t \in [a, 1]$. We let

$$f(t) = \begin{cases} f_1(t), & t \in [0, 1 - a], \\ f_2(t), & t \in (1 - a, a), \\ f_3(t), & t \in [1 - a, 1], \end{cases}$$

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for $t \in [0, 1]$ and obtain the differential equations for t in the above three subintervals.

1) For $t \in [1 - a, a]$,

$$t - a < 0 < t < 1 < t + a.$$

Then,

$$\lambda f(t) = \int_0^1 K(s, t) f(s) ds, \quad 1 - a \leq t \leq a$$

is equivalent to

$$\lambda f_2(t) = \int_0^{1-a} (a - t + s) f_1(s) ds + \int_{1-a}^t (a - t + s) f_2(s) ds$$
$$+ \int_t^a (a + t - s) f_2(s) ds + \int_a^1 (a + t - s) f_3(s) ds. \qquad (2.29)$$

Differentiating with respect to t, we get

$$\lambda f_2'(t) = - \int_0^{1-a} f_1(s) ds - \int_{1-a}^t f_2(s) ds + \int_t^a f_2(s) ds + \int_a^1 f_3(s) ds.$$

Differentiating twice to obtain

$$\lambda f_2''(t) = -2 f_2(t).$$

Hence, the solution is

$$f_2(t) = d_1 \cos(\sqrt{2\lambda^{-1}}t) + d_2 \sin(\sqrt{2\lambda^{-1}}t).$$

2) For $t \in [0, 1 - a]$,

$$t - a < 0 < t < t + a < 1.$$

Again we solve

$$\lambda f(t) = \int_0^1 K(s, t) f(s) ds, \quad 0 \leq t \leq 1 - a,$$

which is,

$$\lambda f_1(t) = \int_0^t (a - t + s) f_1(s) ds + \int_t^{1-a} (a + t - s) f_1(s) ds$$
$$+ \int_{1-a}^a (a + t - s) f_2(s) ds + \int_a^{t+a} (a + t - s) f_3(s) ds. \qquad (2.30)$$

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Differentiating with respect to t, we get

$$\lambda f_1'(t) = -\int_0^t f_1(s)ds + \int_t^{1-a} f_1(s)ds + \int_{1-a}^a f_2(s)ds + \int_a^{t+a} f_3(s)ds.$$

Differentiating twice to obtain

$$\lambda f_1''(t) + 2f_1(t) = f_3(t+a).$$

3) For $t \in [a, 1]$,

$$0 < t - a < t < 1 < t + a.$$

We have

$$\lambda f(t) = \int_0^1 K(s,t)f(s)ds, \quad a \le t \le 1$$

which is,

$$\lambda f_3(t) = \int_{t-a}^{1-a} (a - t + s)f_1(s)ds + \int_{1-a}^a (a - t + s)f_2(s)ds$$
$$+ \int_a^t (a - t + s)f_3(s)ds + \int_t^1 (a + t - s)f_3(s)ds. \tag{2.31}$$

Differentiating with respect to t, we get

$$\lambda f_3'(t) = -\int_{t-a}^{1-a} f_1(s)ds - \int_{1-a}^a f_2(s)ds - \int_a^t f_3(s)ds + \int_t^1 f_3(s)ds.$$

Differentiating twice to obtain

$$\lambda f_3''(t) + 2f_3(t) = f_1(t-a).$$

From the discussion above, we have

$$\begin{cases} \lambda f_1''(t) + 2f_1(t) = f_3(t+a), & t \in [0, 1-a], \\ \lambda f_3''(t) + 2f_3(t) = f_1(t-a), & t \in [a, 1]. \end{cases}$$

Noticing that $t + a \in [a, 1]$ for $t \in [0, 1-a]$ and $t - a \in [0, 1-a]$ for $t \in [a, 1]$, we solve the two differential equations together gives

$$\begin{cases} \lambda^2 f_1^{(4)}(t) + 4\lambda f_1''(t) + 3f_1(t) = 0, & t \in [0, 1-a], \\ \lambda^2 f_3^{(4)}(t) + 4\lambda f_3''(t) + 3f_3(t) = 0, & t \in [a, 1]. \end{cases}$$

and the solutions are

$$f_1(t) = c_1 \cos(\sqrt{\lambda^{-1}}t) + c_2 \sin(\sqrt{\lambda^{-1}}t) + c_3 \cos(\sqrt{3\lambda^{-1}}t) + c_4 \sin(\sqrt{3\lambda^{-1}}t) \quad \text{for } t \in [0, 1-a],$$

and

$$f_3(t) = c_5 \cos(\sqrt{\lambda^{-1}}t) + c_6 \sin(\sqrt{\lambda^{-1}}t) + c_7 \cos(\sqrt{3\lambda^{-1}}t) + c_8 \sin(\sqrt{3\lambda^{-1}}t) \quad \text{for } t \in [a, 1].$$

Therefore, for $t \in [0, 1]$, we have

$$f(t) = \begin{cases} c_1 \cos(\sqrt{\lambda^{-1}}t) + c_2 \sin(\sqrt{\lambda^{-1}}t) + c_3 \cos(\sqrt{3\lambda^{-1}}t) + c_4 \sin(\sqrt{3\lambda^{-1}}t), & t \in [0, 1-a], \\ d_1 \cos(\sqrt{2\lambda^{-1}}t) + d_2 \sin(\sqrt{2\lambda^{-1}}t), & t \in (1-a, a), \\ c_5 \cos(\sqrt{\lambda^{-1}}t) + c_6 \sin(\sqrt{\lambda^{-1}}t) + c_7 \cos(\sqrt{3\lambda^{-1}}t) + c_8 \sin(\sqrt{3\lambda^{-1}}t), & t \in [a, 1]. \end{cases}$$

$$(2.32)$$

as the eigenfunction of the integral equation

$$\lambda f(t) = \int_0^1 K(s, t) f(s) ds, \quad 0 \le t \le 1.$$

For convenience, we write

$$f(t) = \begin{cases} c_1 \cos(\sqrt{\lambda^{-1}}t) + c_2 \sin(\sqrt{\lambda^{-1}}t) + c_3 \cos(\sqrt{3\lambda^{-1}}t) + c_4 \sin(\sqrt{3\lambda^{-1}}t), & t \in [0, 1-a], \\ D_1 \cos(\sqrt{2\lambda^{-1}}(t-(1-a))) + D_2 \sin(\sqrt{2\lambda^{-1}}(t-(1-a))), & t \in (1-a, a), \\ C_5 \cos(\sqrt{\lambda^{-1}}(t-a)) + C_6 \sin(\sqrt{\lambda^{-1}}(t-a)) \\ \quad + C_7 \cos(\sqrt{3\lambda^{-1}}(t-a)) + C_8 \sin(\sqrt{3\lambda^{-1}}(t-a)), & t \in [a, 1]. \end{cases}$$

$$(2.33)$$

Now the equation $\lambda f_1''(t) + 2f_1(t) = f_3(t+a)$, $t \in [0, 1-a]$ gives $C_5 = c_1$, $C_6 = c_2$, $C_7 = -c_3$, and $C_8 = -c_4$. That is,

$$f(t) = \begin{cases} c_1 \cos(\sqrt{\lambda^{-1}}t) + c_2 \sin(\sqrt{\lambda^{-1}}t) + c_3 \cos(\sqrt{3\lambda^{-1}}t) + c_4 \sin(\sqrt{3\lambda^{-1}}t), & t \in [0, 1-a], \\ D_1 \cos(\sqrt{2\lambda^{-1}}(t-(1-a))) + D_2 \sin(\sqrt{2\lambda^{-1}}(t-(1-a))), & t \in (1-a, a), \\ c_1 \cos(\sqrt{\lambda^{-1}}(t-a)) + c_2 \sin(\sqrt{\lambda^{-1}}(t-a)) \\ \quad - c_3 \cos(\sqrt{3\lambda^{-1}}(t-a)) - c_4 \sin(\sqrt{3\lambda^{-1}}(t-a)), & t \in [a, 1]. \end{cases}$$

$$(2.34)$$

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Furthermore, by using $f_2(1-a) = f_1(1-a)$, and $f_2'(1-a) = f_1'(1-a)$, we obtain

$$D_1 = c_1\cos(\sqrt{\lambda^{-1}}(1-a)) + c_2\sin(\sqrt{\lambda^{-1}}(1-a)) + c_3\cos(\sqrt{3\lambda^{-1}}(1-a)) + c_4\sin(\sqrt{3\lambda^{-1}}(1-a)),$$

$$(2.35)$$

$$\sqrt{2}D_2 = -c_1\sin(\sqrt{\lambda^{-1}}(1-a)) + c_2\cos(\sqrt{\lambda^{-1}}(1-a))$$
$$-\sqrt{3}c_3\sin(\sqrt{3\lambda^{-1}}(1-a)) + \sqrt{3}c_4\cos(\sqrt{3\lambda^{-1}}(1-a)).$$

$$(2.36)$$

Using $f_2(a) = f_3(a)$ and $f_2'(a) = f_3'(a)$, we get

$$c_1 - c_3 = \cos(\sqrt{2/\lambda}(2a-1))D_1 + \sin(\sqrt{2/\lambda}(2a-1))D_2, \qquad (2.37)$$

$$c_2 - \sqrt{3}c_4 = -\sqrt{2}\sin(\sqrt{2/\lambda}(2a-1))D_1 + \sqrt{2}\cos(\sqrt{2/\lambda}(2a-1))D_2. \qquad (2.38)$$

Computing

$$\lambda f_2'(1-a) = -\int_0^{1-a} f_1(s)ds + \int_{1-a}^{a} f_2(s)ds + \int_a^1 f_3(s)ds,$$

we obtain

$$-\sin(\sqrt{3/\lambda}(1-a))c_3 + [\cos(\sqrt{3/\lambda}(1-a)) - 1]c_4 \qquad (2.39)$$

$$+\frac{\sqrt{6}}{4}\sin(\sqrt{2/\lambda}(2a-1))D_1 - \frac{\sqrt{6}}{4}[\cos(\sqrt{2/\lambda}(2a-1)) + 1]D_2 = 0. \qquad (2.40)$$

Finally, we compute $\lambda f_3(a)$ using (2.31). That is,

$$\lambda f_3(a) = \int_0^{1-a} sf_1(s)ds + \int_{1-a}^{a} sf_2(s)ds + \int_a^1 (2a-s)f_3(s)ds.$$

This gives us

$$
\sqrt{\lambda}\left[a\sin(\sqrt{\tfrac{1}{\lambda}}(1-a))c_1 + \left(-a\cos(\sqrt{\tfrac{1}{\lambda}}(1-a))+a\right)c_2\right.
$$

$$
-\frac{3a-2}{\sqrt{3}}\sin(\sqrt{\tfrac{3}{\lambda}}(1-a))c_3 + \left(\frac{3a-2}{\sqrt{3}}\cos(\sqrt{\tfrac{3}{\lambda}}(1-a))-\frac{a}{\sqrt{3}}\right)c_4
$$

$$
\left.+\frac{a}{\sqrt{2}}\sin(\sqrt{\tfrac{2}{\lambda}}(2a-1))D_1 + \left(-\frac{a}{\sqrt{2}}\cos(\sqrt{\tfrac{2}{\lambda}}(2a-1))+\frac{1-a}{\sqrt{2}}\right)D_2\right]
$$

$$
+\lambda\left[-c_1 + \left(\frac{1}{3}+\frac{2}{3}\cos(\sqrt{\tfrac{3}{\lambda}}(1-a))\right)c_3 + \frac{2}{3}\sin(\sqrt{\tfrac{3}{\lambda}}(1-a))c_4\right.
$$

$$
\left.+\left(\frac{1}{2}\cos(\sqrt{\tfrac{2}{\lambda}}(2a-1))-\frac{1}{2}\right)D_1 + \left(\frac{1}{2}\sin(\sqrt{\tfrac{2}{\lambda}}(2a-1))\right)D_2\right]=0. \qquad (2.41)
$$

Putting equations (2.35), (2.36), (2.37), (2.38), (2.40), and (2.41) together, we must have the determinant of the coefficient matrix to be singular for c_1, c_2, c_3, c_4, D_1 and D_2 to have a nontrivial solution. After some matrix simplification, we have the equation that determines the eigenvalues as

$$
\sqrt{\lambda}M(\lambda)+\lambda N(\lambda)=0, \qquad (2.42)
$$

or equivalently, $\frac{1}{\sqrt{\lambda}}M(\lambda)+N(\lambda)=0$, where

$$
M(\lambda)
$$

$$
=\frac{4a-1}{\sqrt{6}}\begin{vmatrix}
\cos\sqrt{\tfrac{1}{\lambda}}(1-a) & \sin\sqrt{\tfrac{1}{\lambda}}(1-a) & \cos\sqrt{\tfrac{3}{\lambda}}(1-a) & \sin\sqrt{\tfrac{3}{\lambda}}(1-a) & -1 & 0 \\
-\sin\sqrt{\tfrac{1}{\lambda}}(1-a) & \cos\sqrt{\tfrac{1}{\lambda}}(1-a) & 0 & 0 & 0 & -\tfrac{\sqrt{2}}{4} \\
1 & 0 & -1 & 0 & -\cos\sqrt{\tfrac{2}{\lambda}}(2a-1) & -\sin\sqrt{\tfrac{2}{\lambda}}(2a-1) \\
0 & 0 & 0 & -\tfrac{4\sqrt{3}}{3} & \sqrt{2}\sin\sqrt{\tfrac{2}{\lambda}}(2a-1) & -\sqrt{2}\cos\sqrt{\tfrac{2}{\lambda}}(2a-1) \\
0 & 0 & -\sin\sqrt{\tfrac{3}{\lambda}}(1-a) & \cos\sqrt{\tfrac{3}{\lambda}}(1-a) & 0 & -\tfrac{\sqrt{6}}{4} \\
0 & \sqrt{3} & 0 & 1 & 0 & 0
\end{vmatrix}
$$

and

$N(\lambda)$

$$
=
\begin{vmatrix}
\cos\sqrt{\frac{1}{\lambda}}(1-a) & \sin\sqrt{\frac{1}{\lambda}}(1-a) & \cos\sqrt{\frac{3}{\lambda}}(1-a) & \sin\sqrt{\frac{3}{\lambda}}(1-a) & -1 & 0 \\
-\sin\sqrt{\frac{1}{\lambda}}(1-a) & \cos\sqrt{\frac{1}{\lambda}}(1-a) & -\sqrt{3}\sin\sqrt{\frac{3}{\lambda}}(1-a) & \sqrt{3}\cos\sqrt{\frac{3}{\lambda}}(1-a) & 0 & -\sqrt{2} \\
1 & 0 & -1 & 0 & -\cos\sqrt{\frac{2}{\lambda}}(2a-1) & -\sin\sqrt{\frac{2}{\lambda}}(2a-1) \\
0 & 1 & 0 & -\sqrt{3} & \sqrt{2}\sin\sqrt{\frac{2}{\lambda}}(2a-1) & -\sqrt{2}\cos\sqrt{\frac{2}{\lambda}}(2a-1) \\
0 & 0 & -\sin\sqrt{\frac{3}{\lambda}}(1-a) & \cos\sqrt{\frac{3}{\lambda}}(1-a)-1 & \frac{\sqrt{6}}{4}\sin\sqrt{\frac{2}{\lambda}}(2a-1) & -\frac{\sqrt{6}}{4}\cos\sqrt{\frac{2}{\lambda}}(2a-1)-\frac{\sqrt{6}}{4} \\
-1 & 0 & \frac{1}{3}+\frac{2}{3}\cos\sqrt{\frac{3}{\lambda}}(1-a) & \frac{2}{3}\sin\sqrt{\frac{3}{\lambda}}(1-a) & \frac{1}{2}\cos\sqrt{\frac{2}{\lambda}}(2a-1)-\frac{1}{2} & \frac{1}{2}\sin\sqrt{\frac{2}{\lambda}}(2a-1)
\end{vmatrix}.
$$

It is difficult to directly solve for λ's from (2.42). However, the equation (2.42) can give us the small ball estimate of the process and we will show this in Chapter 3.

(i) $a = 1/2$.

Let

$$
f(t) =
\begin{cases}
f_1(t), & t \in [0, 1/2], \\
f_3(t), & t \in (1/2, 1],
\end{cases}
$$

for $t \in [0, 1]$. By expanding the integral equation and applying successive differentiations similar to the case $1/2 < a < 1$, we have

$$
\lambda f_1''(t) + 2f_1(t) = f_3(t + 1/2) \qquad t \in [0, 1/2]
$$
$$
\lambda f_3''(t) + 2f_3(t) = f_1(t - 1/2) \qquad t \in (1/2, 1].
$$

Let $\phi_1(t) := f_1(t)$ and $\phi_2(t) := f_3(t + 1/2)$ for $0 \le t \le 1/2$. On the interval $[0, 1/2]$, the system of differential equations above becomes

$$
\lambda \phi_1''(t) + 2\phi_1(t) = \phi_2(t),
$$
$$
\lambda \phi_2''(t) + 2\phi_2(t) = \phi_1(t).
$$

From the original integral equation setting, we have the following boundary conditions at both end points of $[0, 1/2]$:

$$\phi_1(1/2) = \phi_2(0),$$

$$\phi_1'(1/2) = \phi_2'(0),$$

$$\phi_1(0) = \int_0^{1/2} (1/2 - s)\phi_1(s)\, ds,$$

$$\phi_2(1/2) = \int_0^{1/2} s\, \phi_2(s)\, ds.$$

The boundary conditions give the equation for determining the eigenvalue λ as

$$\lambda^{-1}\left(\sqrt{3}\cos\frac{\sqrt{\lambda^{-1}}}{2}\sin\frac{\sqrt{3\lambda^{-1}}}{2} + 3\sin\frac{\sqrt{\lambda^{-1}}}{2}\cos\frac{\sqrt{3\lambda^{-1}}}{2} - \sqrt{3}\sin\frac{\sqrt{3\lambda^{-1}}}{2} + 3\sin\frac{\sqrt{\lambda^{-1}}}{2}\right)$$

$$+\sqrt{\lambda^{-1}}\left(2\sqrt{3}\sin\frac{\sqrt{\lambda^{-1}}}{2}\sin\frac{\sqrt{3\lambda^{-1}}}{2} - 5\cos\frac{\sqrt{\lambda^{-1}}}{2}\cos\frac{\sqrt{3\lambda^{-1}}}{2} - 4\cos\frac{\sqrt{\lambda^{-1}}}{2} - 4\cos\frac{\sqrt{3\lambda^{-1}}}{2} - 5\right) = 0.$$

2.2.10.3   Case $0 < a < 1/2$

The method for the case $1/2 \le a < 1$ does not work for the case $0 < a < 1/2$ and the characteristic equation that determines the eigenvalues is still unknown. We will study this in our future research.

Chapter 3

SMALL BALL PROBABILITY OF GAUSSIAN PROCESSES

3.1 Introduction

For a given continuous stochastic process $X(t), t \in [0, 1]$, the small deviation probability concerns the asymptotic behavior of $P(||X|| \leq \epsilon)$ as $\epsilon \to 0^+$, where $|| \cdot ||$ is a norm on the space $C[0, 1]$. We use

$$||f||_p = \begin{cases} (\int_0^1 |f(t)|^p dt)^{1/p} & \text{for} \quad 1 \leq p < \infty, \\ \sup_{0 \leq t \leq 1} |f(t)| & \text{for} \quad p = \infty \end{cases}$$

to denote the L_p-norm on $C[0, 1]$, $1 \leq p \leq \infty$. The survey paper [31] covers the development of theory on small ball probability for L_p-norm of the stochastic process X. In this chapter, we discuss the exact small ball probability in the most explored case of L^2-norm. By Karhunen-Loève theorem, we have the distributional identity $||X||_2^2 = \sum_{n=1}^{\infty} \lambda_n \xi_n^2$, where λ_n are the eigenvalues of the associated covariance kernel and ξ_n are i.i.d. standard normal random variables. The small ball problem $P(||X||_2^2 \leq \epsilon) = P(\sum_{n=1}^{\infty} \lambda_n \xi_n^2 \leq \epsilon)$ was solved in [37] in an implicit way if the eigenvalues are known. However for most of the Gaussian processes, the eigenvalues of their covariance functions are still unknown. When the Karhunen-Loève expansion for a given Gaussian process can be found in some reasonable form, the L^2 small ball probabilities can be treated using the the comparison theorems in [30] and [20]. If we can obtain approximation of the eigenvalues, the comparison theorems build connection between the original process and the process with the approximated eigenvalues and the problem can be solved. This is the case for Brownian motion, fractional Brownian motion and Brownian sheets, etc.

We review the method and examples in [19] and compute the small ball rate for Slepian process in this chapter.

3.2 General Theory

Suppose that a centered Gaussian process $\{X_t, a \leq t \leq b\}$ has KL expansion $X_t = \sum_{i=1}^{\infty} \sqrt{\lambda_i} e_i(t) \xi_i$. Then by

$$||X||_2^2 = \int_a^b X_t^2 dt \stackrel{d}{=} \sum_{n=1}^{\infty} \lambda_n \xi_n^2,$$

finding the exact L^2 small ball probability $P(||X_t||_2 \leq \epsilon)$ is equivalent to finding the asymptotic behavior of

$$P\left(\sum_{n=1}^{\infty} \lambda_n \xi_n^2 \leq \epsilon\right).$$

This problem has been solved in [37] by the following theorem when the eigenvalues λ_n are known.

Theorem 12 *If $\lambda_n > 0$ and $\sum_{n=1}^{\infty} \lambda_n < +\infty$, then as $\epsilon \to 0$*

$$\mathbb{P}\left(\sum_{n=1}^{\infty} \lambda_n \xi_n^2 \leq \epsilon^2\right) \sim \left(4\pi \sum_{n=1}^{\infty} (\frac{\lambda_n \gamma_\lambda}{1 + 2\lambda_n \gamma_\lambda})^2\right)^{-1/2} \cdot \exp\left(\epsilon^2 \gamma_\lambda - \frac{1}{2} \sum_{n=1}^{\infty} \log(1 + 2\lambda_n \gamma_\lambda)\right)$$

where $\gamma_\lambda = \gamma_\lambda(\epsilon)$ is uniquely determined, for $\epsilon > 0$ small enough, by the equation

$$\epsilon^2 = \sum_{n=1}^{\infty} \frac{\lambda_n}{1 + 2\lambda_n \gamma_\lambda}.$$

Although the above theorem is still difficult to apply due to the series and implicit relation between ϵ and γ_λ, the problem is considered solved theoretically. If the eigenvalues of X_t can not be found explicitly, since the Laplace transform of the process is

$$L(t) = E(\exp(-t||X||_2^2)) = \left(\prod_{n=1}^{\infty}(1 + 2t\lambda_n)\right)^{-1/2}$$

one can obtain the small ball probability by the function $L(t)$.

Gao, Hanning, Lee and Torcaso gives a method to compute the Laplace transform for many Gaussian processes via Hadamard factorization of an entire function

$f(z)$ in [19]. Let $M(r) = \max |f(z)|$ for $|z| = r$. The order $\lambda \geq 0$ of an entire function $f(z)$ is the smallest number that $M(r) \leq e^{r^{\lambda+\epsilon}}$ for any $\epsilon > 0$ and sufficiently large r, formally defined as

$$\lambda = \limsup_{r \to \infty} \frac{\log \log M(r)}{\log r}.$$

Theorem 13 (Hadamard's Factorization Theorem) *Let $f(z)$ be an entire function and $\{z_k\}$ be its zeros with 0 excluded and all zeros are repeated according to their multiplicity. Suppose the order of $f(z)$ is λ, then*

$$f(z) = z^m e^{H(z)} \prod_{k=1}^{\infty} \left(1 - \frac{z}{z_k}\right) e^{P_d(z/z_k)},$$

where the integer $m \geq 0$ is the multiplicity of 0, $d \geq 0$ is an integer such that $d \leq \lambda < d+1$, $H(z)$ is a polynomial of degree $\leq d$, and $P_d(z) = z + z^2/2 + \cdots + z^d/d$.

If f is an entire function of order $\lambda < 1$, and $f(0) \neq 0$. Then $d = 0$, $H(z)$ is a constant, and

$$\frac{f(z)}{f(0)} = \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right).$$

It is well-known that

$$L(s) = E(\exp(-s||X||_2^2)) = \left(\prod_{n=1}^{\infty} (1 + 2s\lambda_n)\right)^{-1/2}$$

If the zeros of f are $z_n = 1/\lambda_n$ and f is entire with order $\lambda < 1$ and $f(0) \neq 0$, then

$$L(s) = E(\exp(-s||X||_2^2)) = \left(\prod_{n=1}^{\infty} (1 + 2s\lambda_n)\right)^{-1/2} = \left(\frac{f(-2t)}{f(0)}\right)^{-1/2}.$$

That is,

Theorem 14 *Let X be a Gaussian process whose covariance operator has nonzero eigenvalues λ_n, repeated according to their multiplicity. Suppose there is an entire function $f(z)$ of order $\lambda < 1$, such that, $z_n = 1/\lambda_n$, $n \geq 1$, are the only zeros, counting multiplicities, of $f(z)$. Then the Laplace transform of $||X||_2^2 = \sum_{n=1}^{\infty} \lambda_n \xi_n^2$ can be expressed as*

$$E(\exp(-t||X||_2^2)) = \left(\frac{f(-2t)}{f(0)}\right)^{-1/2}.$$

Thus, Theorem 1 can be rewritten as

Theorem 15 ([37] Sytaja Tauberian Theorem)

$$\mathbb{P}\left(\sum_{n=1}^{\infty} \lambda_n \xi_n^2 \le \epsilon^2\right) \sim \left(-2\pi t^2 h''(t)\right)^{-1/2} \cdot \exp\left(t\epsilon^2 - h(t)\right)$$

where $h(t) = -\log L(t)$ where $L(t)$ is the Laplace transform and $t = t(\epsilon)$ satisfies

$$\frac{t\epsilon^2 - th'(t)}{\sqrt{-t^2 h''(t)}} \to 0.$$

When the eigenvalues can not be found explicitly, but can be approximated, the comparison theorem by [30] provides a way to obtain the exact small ball rate for the original process when the small ball rate of the process with the asymptotic eigenvalues is known.

Theorem 16 *If $\sum_{n=1}^{\infty} |1 - a_n/b_n| < \infty$, then as $\epsilon \to 0$*

$$P\left(\sum_{n=1}^{\infty} a_n \xi_n^2 \le \epsilon^2\right) \sim \left(\prod_{n=1}^{\infty} b_n/a_n\right)^{1/2} P\left(\sum_{n=1}^{\infty} b_n \xi_n^2 \le \epsilon^2\right),$$

where a_n, b_n are positive and $\sum_{n=1}^{\infty} a_n < \infty$, $\sum_{n=1}^{\infty} b_n < \infty$. Furthermore, if $a_n \ge b_n$ for n large, then $P(\sum_{n=1}^{\infty} a_n \xi_n^2 \le \epsilon^2)$ and $P(\sum_{n=1}^{\infty} b_n \xi_n^2 \le \epsilon^2)$ have the same order of magnitude as $\epsilon \to 0$ if and only if $\sum_{n=1}^{\infty} |1 - a_n/b_n| < \infty$.

Gao, Hannig, Lee, Torcaso [19] refined the comparison theorem and gives the optimal condition as follows

Theorem 17 *If $\prod_{n=1}^{\infty}(a_n/b_n) < \infty$, then as $\epsilon \to 0$*

$$P\left(\sum_{n=1}^{\infty} a_n \xi_n^2 \le \epsilon^2\right) \sim \left(\prod_{n=1}^{\infty} b_n/a_n\right)^{1/2} \cdot P\left(\sum_{n=1}^{\infty} b_n \xi_n^2 \le \epsilon^2\right)$$

In the Chapter 2, we reviewed the Karhunen-Loève expansion for several Gaussian processes. In the following context, we will review the exact L^2 small ball probability for some of the Gaussian processed mentioned in Chapter 2 and obtain the exact L^2 small ball rate for the Slepian process when the parameter a satisfies $1/2 \le a \le 1$.

3.3 Examples

3.3.1 Brownian Motion

The exact L^2 small ball probability for Brownian motion was given by [7]. That is,

$$P(||W||_2 \leq \epsilon) \sim \frac{4}{\sqrt{\pi}} \epsilon \exp\left(-\frac{1}{8\epsilon^2}\right)$$

for $\epsilon \to 0$.

This result is the most fundamental result for the exact L^2 small ball of Gaussian processes. Since the eigenvalues of the Brownian motion is $\lambda_n = ((n - 1/2)^2 \pi^2)^{-1}$, we can apply the comparison theorems to processes that have asymptotic eigenvalues to obtain their exact L^2 small ball probability. For example, [25] studied

$$P\left(\sum_{n=1}^{\infty} a_n \xi_n^2 \leq \epsilon^2\right)$$

where ξ_n are i.i.d. standard normal random variables, and

$$a_n = \left(2\left\lfloor \frac{n+1}{2}\right\rfloor - 1\right)^{-2}$$

We take $b_n = (n - 1/2)^{-2}$, then

$$P\left(\sum_{n=1}^{\infty} b_n \xi_n^2 \leq \epsilon^2\right) = P(||X||_2^2 \leq \epsilon^2/\pi^2) \sim \frac{4}{\pi\sqrt{\pi}} \epsilon \exp\left(-\frac{\pi^2}{8\epsilon^2}\right)$$

It is easy to see that

$$\left(\prod_{n=1}^{\infty} \frac{a_n}{b_n}\right)^{1/2} = \prod_{k=1}^{\infty}\left(1 - \frac{1}{4(2k-1)^2}\right) = \frac{1}{\sqrt{2}}.$$

By comparison theorem,

$$P\left(\sum_{n=1}^{\infty} a_n \xi_n^2 \leq \epsilon^2\right) \sim 4\sqrt{2}\pi^{-3/2} \epsilon \exp\left(-\frac{\pi^2}{8\epsilon^2}\right)$$

3.3.2 Brownian Bridge

Let $B(t), 0 \leq t \leq 1$ be the Brownian bridge. The exact small ball probability of $B(t)$ was first given by [4]

Proposition 1 *Let $\{B(t) : 0 \le t \le 1\}$ be a Brownian bridge. Then as $\epsilon \to 0$,*

$$P\left(\int_0^1 B^2(t)dt < \epsilon\right) = P\left(\sum_{n=1}^\infty \frac{1}{\pi^2 n^2}\xi_n^2 < \epsilon\right) \sim \frac{4}{\sqrt{2\pi}} \cdot \exp\left(-\frac{1}{8\epsilon}\right).$$

3.3.3   Transformed Brownian Bridge

Proposition 2 *For $\alpha > 0$ and $\beta = 1 - \alpha^{-1} < 1$,*

$$P\left(\int_0^1 B^2(t^\alpha)dt \le \epsilon^2\right)\left(= P\left(\int_0^1 \frac{1}{t^\beta}B^2(t)dt \le \alpha\epsilon^2\right)\right) \sim c_\alpha \epsilon^{-\frac{\alpha-1}{2(\alpha+1)}} \exp\left(-\frac{\alpha}{2(\alpha+1)^2}\cdot\frac{1}{\epsilon^2}\right)$$

as $\epsilon \to 0$, where c_α is a positive constant.

Proposition 3 *For $\alpha > 0$ and $\beta = 1 - \alpha^{-1} < 1$,*

$$P\left(\int_0^1 B^2(t^\alpha)dt \le \epsilon^2\right)\left(= P\left(\int_0^1 \frac{1}{t^\beta}B^2(t)dt \le \alpha\epsilon^2\right)\right) \sim c_\alpha \epsilon^{1/2-v} \exp\left(-\frac{v}{2(\alpha+1)}\cdot\epsilon^{-2}\right)$$

as $\epsilon \to 0$, where

$$c_\alpha = 2\pi^{-1/4}\left(\frac{v}{\alpha+1}\right)^{\frac{v}{2}-\frac{1}{4}}(\Gamma(v+1))^{-1/2} \quad and \quad v = \frac{\alpha}{\alpha+1}$$

Lemma 1 *The Laplace transform of the squared L^2 norm of $B(t^\alpha)$ is*

$$E(\exp\{-t\|B(t^\alpha)\|_2^2\}) = \left(\frac{c\sqrt{2t}}{2}\right)^{v/2}\left(\Gamma(1+v)I_v\left(c\sqrt{2t}\right)\right)^{-1/2},$$

where I_v is the modified Bessel function of fractional order

$$v = \frac{\alpha}{\alpha+1} \quad and \quad c = \frac{2\sqrt{\alpha}}{\alpha+1}.$$

Proof. For $\alpha > 0$, $\{B(t^\alpha) : 0 \le t \le 1\}$ is a Gaussian process with mean zero and covariance function $K(s,t) = s^\alpha \wedge t^\alpha - s^\alpha t^\alpha$. We need to find the eigenvalues λ_n of the equation $\lambda f(t) = \int_0^1 K(s,t)f(s)ds$, which is

$$\lambda f(t) = (1-t^\alpha)\int_0^1 s^\alpha f(s)ds + t^\alpha\int_t^1(1-s^\alpha)f(s)ds$$

with boundary condition $f(0) = 0$ and $f(1) = 0$. By differentiation the equation becomes

$$\lambda t f''(t) - \lambda(\alpha-1)f'(t) + \alpha t^\alpha f(t) = 0$$

The general solution is, see [28]

$$f(t) = c_1 t^{\alpha/2} J_{\alpha/(\alpha+1)}(2(\alpha+1)^{-1}\sqrt{\alpha/\lambda}t^{(\alpha+1)/2}) + c_2 t^{\alpha/2} J_{-\alpha/(\alpha+1)}(2(\alpha+1)^{-1}\sqrt{\alpha/\lambda}t^{(\alpha+1)/2})$$

$$= c_1 t^{\alpha/2} J_v(c\lambda^{-1/2}t^{(\alpha+1)/2}) + c_2 t^{\alpha/2} J_{-v}(c\lambda^{-1/2}t^{(\alpha+1)/2})$$

where $J_v(x)$ is the Bessel function. Using the boundary condition $f(0) = 0$ and $f(1) = 0$, we obtain the eigenvalues as the solutions of

$$J_v\left(c\lambda^{-1/2}\right) = 0.$$

We take $f(t) = J_v(ct^{1/2})/t^{v/2}$, then $f(t)$ is entire of order $1/2$, see [1] formula 9.1.10, 9.1.62 and 9.2.1. Since $\lim_{t\to 0} f(t) = 1/(2^v\Gamma(1+v))$ and $I_v(x) = e^{-\frac{1}{2}v\pi i}J_v(ix)$ for $x > 0$ see formula 9.6.3, it can be obtained that

$$L(t) = \left(\frac{f(-2t)}{f(0)}\right)^{-1/2} = \left(\frac{c\sqrt{2t}}{2}\right)^{v/2}\left(\Gamma(1+v)I_v(c\sqrt{2t})\right)^{-1/2}.$$

Apply Sytaja Tauberian theorem to the Laplace theorem of $B(t^\alpha)$, the small ball rate can be directly obtained as in the above theorem.

On the other hand, the limit comparison theorem can also be used to find the small ball rate. Using the asymptotic formula for zeros of the Bessel function, see [42], we have

$$\frac{2}{\alpha+1}\sqrt{\frac{\alpha}{\lambda_n}} = \left(n + \frac{\alpha-1}{4(\alpha+1)}\right)\pi + O\left(\frac{1}{n}\right)$$

which shows that

$$\sum_{n=1}^{\infty}\left|\frac{4\alpha}{(\alpha+1)^2\pi^2}\left(n + \frac{\alpha-1}{4(\alpha+1)}\right)^{-2}\cdot\frac{1}{\lambda_n} - 1\right| < \infty$$

Thus, by Theorem 15, we obtain

$$P\left(\int_0^1 B^2(t^\alpha)dt \le \epsilon^2\right) = P\left(\sum_{n=1}^{\infty}\lambda_n\xi_n^2 \le \epsilon^2\right)$$

$$\sim D_\alpha P\left(\sum_{n=1}^{\infty}\frac{4\alpha}{(\alpha+1)^2\pi^2}\cdot\left(n + \frac{\alpha-1}{4(\alpha+1)}\right)^{-2}\xi_n^2 \le \epsilon^2\right)$$

$$\sim C_\alpha\epsilon^{-(\alpha-1)/(2(\alpha+1))}\exp\left(-\frac{\alpha}{2(\alpha+1)^2}\cdot\frac{1}{\epsilon^2}\right)$$

3.3.4 Integrated Brownian Motion

The m-times integrated Brownian motion is defined recursively by

$$X_m(t) = \int_0^t X_{m-1}(s)ds, \quad t \geq 0, m \geq 1$$

for all positive integer m and $X_0(t) = W(t)$ where $W(t)$ is the standard Brownian motion. Using integration by parts, we also have the representation

$$X_m(t) = \frac{1}{m!} \int_0^t (t-s)^m dW(s), \quad m \geq 0.$$

The covariance kernel is

$$K(s,t) = \frac{1}{(m!)^2} \int_0^{s \wedge t} (s-u)^m (t-u)^m du.$$

By successively differentiating

$$\int_0^1 K(s,t)f(s)ds = \lambda f(t)$$

$(2m+2)$ times, we obtain the following Sturm-Liouville problem:

$$\lambda f^{(2m+2)}(t) = (-1)^{m+1} f(t), \quad 0 < t < 1$$

$$f(0) = f'(0) = \cdots = f^{(m)}(0) = f^{(m+1)}(1) = \cdots = f^{(2m)}(1) = f^{(2m+1)}(1) = 0$$

The eigenfunctions are the nontrivial functions of the form

$$f(t) = \sum_{j=0}^{2m+1} c_j e^{i\alpha_j t}$$

with $\alpha_j = \lambda^{-1/(2m+2)i\omega_j}$ and $\omega_j = \exp(\frac{j\pi}{m+1}i)$. Using boundary conditions, we obtain that

$$M_W(\lambda^{-1/(2m+2)})C = 0$$

where

$$M_W(t) = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \omega_0 & \omega_1 & \cdots & \omega_{2m+1} \\ \cdots & \cdots & \cdots & \cdots \\ \omega_0^m & \omega_1^m & \cdots & \omega_{2m+1}^m \\ \omega_0^{m+1}e^{i\omega_0 z} & \omega_1^{m+1}e^{i\omega_1 z} & \cdots & \omega_{2m+1}^{m+1}e^{i\omega_{2m+1} z} \\ \omega_0^{m+2}e^{i\omega_0 z} & \omega_1^{m+2}e^{i\omega_1 z} & \cdots & \omega_{2m+1}^{m+2}e^{i\omega_{2m+1} z} \\ \cdots & \cdots & \cdots & \cdots \\ \omega_0^{2m+1}e^{i\omega_0 z} & \omega_1^{2m+1}e^{i\omega_1 z} & \cdots & \omega_{2m+1}^{2m+1}e^{i\omega_{2m+1} z} \end{pmatrix}$$

and

$$C = [c_0, c_1, \ldots, c_{2m+1}]'.$$

The characteristic determinant for the eigenvalues is $\det(M_W(\lambda^{-1/(2m+2)})) = 0$. We get

$$g(t) = \det(M_W(t^{1/(2m+2)}))$$

is of order $1/(2m+2)$ and $g(0) = (-i)^m(2m+2)^{m+1}$. We use the following notation

$$v_j = e^{i\frac{2j+1}{2m+2}\pi}, \text{ and } \beta_j = (2t)^{1/(2m+2)}iv_j.$$

Lemma 2 *The Laplace transform of the squared L^2 norm of m-times integrated Brownian motion is*

$$E(\exp\{-t||X_m||_2^2\}) = (2m+2)^{(m+1)/2}|\det N_W(t)|^{-1/2}$$

where

$$N_W(t) = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \omega_0 & \omega_1 & \cdots & \omega_{2m+1} \\ \cdots & \cdots & \cdots & \cdots \\ \omega_0^m & \omega_1^m & \cdots & \omega_{2m+1}^m \\ \omega_0^{m+1}e^{\beta_0} & \omega_1^{m+1}e^{\beta_1} & \cdots & \omega_{2m+1}^{m+1}e^{\beta_{2m+1}} \\ \cdots & \cdots & \cdots & \cdots \\ \omega_0^{2m+1}e^{\beta_0} & \omega_1^{2m+1}e^{\beta_1} & \cdots & \omega_{2m+1}^{2m+1}e^{\beta_{2m+1}} \end{pmatrix}$$

For the case of $m = 1, 2$, the above formula simplify to:

$$E(\exp\{-t||X_1||_2^2\}) = 2\left(2 + \cos(2^{3/4}t^{1/4}) + \cosh(2^{3/4}t^{1/4})\right)^{-1/2}$$

$$E(\exp\{-t||X_2||_2^2\}) = 6\left(9 + 16\cos(3^{1/2}2^{-5/6}t^{1/6})\cosh(2^{-5/6}t^{1/6})\right.$$

$$\left. + 8\cosh(2^{1/6}t^{1/6}) + 2\cos(2^{1/6}3^{1/2}t^{1/6})\cosh(2^{1/6}t^{1/6}) + \cosh(2^{7/6}t^{1/6})\right)^{-1/2}$$

Again by Sytaja Tauberin theorem,

Theorem 18

$$P(||X_m||_2 \leq \epsilon) \sim C_m^W \epsilon^{\frac{1}{2m+2}} \exp\{-D_m\epsilon^{-\frac{2}{2m+1}}\}$$

where

$$D_m = \frac{2m+1}{2}\left((2m+2)\sin\frac{\pi}{2m+2}\right)^{-\frac{2m+2}{2m+1}}$$

$$C_m^W = \frac{(2m+2)^{(m+1)/2}}{|\det U|}\left(\frac{2m+2}{(2m+1)\pi}\right)\left[(2m+2)\sin\frac{\pi}{2m+2}\right]^{(m+1)/(2m+1)}$$

and

$$U = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \omega_0 & \omega_1 & \cdots & \omega_m \\ \omega_0^2 & \omega_1^2 & \cdots & \omega_m^2 \\ \cdots & \cdots & \cdots & \cdots \\ \omega_0^m & \omega_1^m & \cdots & \omega_m^m \end{pmatrix}$$

Remark: For the usual 1- and 2-times integrated Brownian mitons, respectively, as $\epsilon \to 0$,

$$P(||X_1|| \leq \epsilon) \sim \frac{8\sqrt{2}}{\sqrt{3\pi}}\epsilon^{1/3}\exp\{-\frac{3}{8}\epsilon^{-2/3}\}$$

$$P(||X_2|| \leq \epsilon) \sim \frac{36(3^{1/10})}{\sqrt{5\pi}}\epsilon^{1/5}\exp\{-\frac{5}{6(3^{1/5})}\epsilon^{-2/5}\}$$

The authors in [21] used comparison theorem and obtained the small ball rate for general m-times integrated Brownian motion, and in [18] they used complex analysis method and obtained further result.

3.3.5 Integrated Brownian Bridge

Lemma 3 *The Laplace transform of the squared L^2-norm of m-times integrated Brownian bridge is*

$$E(\exp\{-t\|Y_m\|_2^2\}) = (2t)^{1/(4m+4)}(2m+2)^{(m+1)/2}|\det N_B(t)|^{-1/2}$$

where

$$N_B(t) = \begin{pmatrix}
1 & 1 & \cdots & 1 \\
\omega_0 & \omega_1 & \cdots & \omega_{2m+1} \\
\cdots & \cdots & \cdots & \cdots \\
\omega_0^m & \omega_1^m & \cdots & \omega_{2m+1}^m \\
\omega_0^m e^{\beta_0} & \omega_1^m e^{\beta_1} & \cdots & \omega_{2m+1}^m e^{\beta_{2m+1}} \\
\omega_0^{m+2} e^{\beta_0} & \omega_1^{m+2} e^{\beta_1} & \cdots & \omega_{2m+1}^{m+2} e^{\beta_{2m+1}} \\
\omega_0^{m+3} e^{\beta_0} & \omega_1^{m+3} e^{\beta_1} & \cdots & \omega_{2m+1}^{m+3} e^{\beta_{2m+1}} \\
\cdots & \cdots & \cdots & \cdots \\
\omega_0^{2m+1} e^{\beta_0} & \omega_1^{2m+1} e^{\beta_1} & \cdots & \omega_{2m+1}^{2m+1} e^{\beta_{2m+1}}
\end{pmatrix}$$

The m-times integrated Brownian bridge $Y_m(t), 0 \le t \le 1$ is defined recursively by

$$Y_m(t) = \int_0^1 Y_{m-1}(s)ds, \quad t \ge 0, m \ge 1$$

for all positive integer m and $Y_0(t) = B(t)$ where $B(t)$ is the standard Brownian bridge. or equivalently,

$$Y_m(t) = \int_0^t \int_0^{s_m} \int_0^{s_{m-1}} \cdots \int_0^{s_2} B(s_1)ds_1 ds_2 \cdots ds_m.$$

The covariance kernel is

$$K(s,t) = \frac{1}{(m!)^2} \int_0^{s \wedge t} (s-u)^m (t-u)^m du.$$

By successively differentiating

$$\int_0^1 K(s,t)f(s)ds = \lambda f(t)$$

$(2m + 2)$ times, we obtain the following Sturm-Liouville problem:

$$\lambda f^{(2m+2)}(t) = (-1)^{m+1} f(t), \quad 0 < t < 1$$

$$f(0) = f'(0) = \cdots = f^{(m)}(0) = f^{(m)}(1) = f^{(m+2)}(1) = f^{(m+3)}(1) = \cdots = f^{(2m)}(1) = f^{(2m+1)}(1) = 0.$$

The eigenfunctions are the nontrivial functions of the form

$$f(t) = \sum_{j=0}^{2m+1} c_j e^{i\alpha_j t}$$

with $\alpha_j = \lambda^{-1/(2m+2)i\omega_j}$ and $\omega_j = \exp(\frac{j\pi}{m+1}i)$. Using boundary conditions, we obtain that

$$M_B(\lambda^{-1/(2m+2)})C = 0$$

where

$$M_B(z) = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \omega_0 & \omega_1 & \cdots & \omega_{2m+1} \\ \cdots & \cdots & \cdots & \cdots \\ \omega_0^m & \omega_1^m & \cdots & \omega_{2m+1}^m \\ \omega_0^m e^{i\omega_0 z} & \omega_1^m e^{i\omega_1 z} & \cdots & \omega_{2m+1}^m e^{i\omega_{2m+1} z} \\ \omega_0^{m+2} e^{i\omega_0 z} & \omega_1^{m+2} e^{i\omega_1 z} & \cdots & \omega_{2m+1}^{m+2} e^{i\omega_{2m+1} z} \\ \omega_0^{m+3} e^{i\omega_0 z} & \omega_1^{m+3} e^{i\omega_1 z} & \cdots & \omega_{2m+1}^{m+3} e^{i\omega_{2m+1} z} \\ \cdots & \cdots & \cdots & \cdots \\ \omega_0^{2m+1} e^{i\omega_0 z} & \omega_1^{2m+1} e^{i\omega_1 z} & \cdots & \omega_{2m+1}^{2m+1} e^{i\omega_{2m+1} z} \end{pmatrix}$$

and

$$C = [c_0, c_1, \ldots, c_{2m+1}]'.$$

The characteristic determinant for the eigenvalues is $\det(M_B(\lambda^{-1/(2m+2)})) = 0$. Then we get

$$g(t) = \det(M_B(t^{1/(2m+2)}))$$

is of order $1/(2m + 2)$.

For the case $m = 1$, the Lemma 3 simplifies to

$$E(\exp\{-t||Y_1||_2^2\}) = \left(\frac{2^{7/4}t^{1/4}}{\sin(2^{3/4}t^{1/4}) + \sinh(2^{3/4}t^{1/4})}\right)^{1/2}$$

Theorem 19

$$P(||Y_m||_2 \leq \epsilon) \sim C_m^B \exp\{-D_m\epsilon^{-\frac{2}{2m+1}}\}$$

where

$$D_m = \frac{2m+1}{2}\left((2m+2)\sin\frac{\pi}{2m+2}\right)^{-\frac{2m+2}{2m+1}}$$

$$C_m^B = \left(\frac{(2m+2)^{m+3}\sin\frac{\pi}{2m+2}}{(2m+1)\pi|\det U \det V|}\right)^{1/2}$$

and

$$U = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \omega_0 & \omega_1 & \cdots & \omega_m \\ \omega_0^2 & \omega_1^2 & \cdots & \omega_m^2 \\ \cdots & \cdots & \cdots & \cdots \\ \omega_0^m & \omega_1^m & \cdots & \omega_m^m \end{pmatrix}, \quad V = \begin{pmatrix} \omega_{m+1}^m & \omega_{m+2}^m & \cdots & \omega_{2m+1}^m \\ \omega_{m+1}^{m+2} & \omega_{m+2}^{m+2} & \cdots & \omega_{2m+1}^{m+2} \\ \omega_{m+1}^{m+3} & \omega_{m+2}^{m+3} & \cdots & \omega_{2m+1}^{m+3} \\ \cdots & \cdots & \cdots & \cdots \\ \omega_{m+1}^{2m+1} & \omega_{m+2}^{2m+1} & \cdots & \omega_{2m+1}^{2m+1} \end{pmatrix}$$

3.3.6   Ornstein-Uhlenbeck Process

Consider the stationary Ornstein-Uhlenbeck process $X(t)$ on the interval $[0,1]$, that is, the centered Gaussian process determined by the covariance kernel $K(s,t) = e^{-\alpha|t-s|}/(2\alpha)$. We have the Laplace transform of $X(t)$ as

Theorem 20 *For the stationary Ornstein-Uhlenbeck process X with parameter $\alpha > 0$,*

$$E(\exp(-\sigma||X||_2^2)) = e^{\alpha/2}\left(\frac{\sigma+\alpha^2}{\alpha\sqrt{\alpha^2+2\sigma}}\sinh(\sqrt{\alpha^2+2\sigma}) + \cosh(\sqrt{\alpha^2+2\sigma})\right)^{-1/2}.$$

Now consider the Ornstein-Uhlenbeck process X_0 starting at 0, that it, the centered Gaussian process with the covariance kernel $K(s,t) = (e^{-\alpha|t-s|} - e^{-\alpha(t+s)})/(2\alpha)$.

Theorem 21 *For the Ornstein-Uhlenbeck process X_0 starting at 0 with parameter $\alpha \in \mathbb{R}$,*

$$E(\exp(-\sigma||X_0||_2^2)) = e^{\alpha/2}\left(\frac{\alpha}{\sqrt{\alpha^2+2\sigma}}\sinh(\sqrt{\alpha^2+2\sigma}) + \cosh(\sqrt{\alpha^2+2\sigma})\right)^{-1/2}.$$

Corollary 2 *Let X be the stationary Ornstein-Uhlenbeck process with parameter $\alpha > 0$, then*

$$P(||X||_2 \leq \epsilon) \sim 8\sqrt{\frac{\alpha}{\pi}} e^{\alpha/2} \epsilon^2 \exp(-\frac{1}{8}\epsilon^{-2}), \quad \text{as } \epsilon \to 0.$$

Let X_0 be the Ornstein-Uhlenbeck process starting at 0 with parameter $\alpha \in \mathbb{R}$, then

$$P(||X_0||_2 \leq \epsilon) \sim \frac{4}{\sqrt{\pi}} e^{\alpha/2} \epsilon^2 \exp(-\frac{1}{8}\epsilon^{-2}), \quad \text{as } \epsilon \to 0.$$

We omit the proofs.

3.3.7 Slepian Process

In this section, we review the small ball probability for Slepian process for $a \geq 1$ and obtain the small ball probability for Slepian process for $1/2 \leq a < 1$ based on the KL expansion from Chapter 2.

3.3.7.1 Case $a \geq 1$

From [22], the Laplace transform for Slepian process $S(t) \overset{d}{=} W(t+a) - W(t)$ for $a \geq 1$ is

$$L(\lambda) = \mathbb{E}e^{-\lambda \int_0^1 S^2(t)dt} = \left[\cosh\sqrt{\lambda}\left(\cosh\sqrt{\lambda} + (2a-1)\sqrt{\lambda}\sinh\sqrt{\lambda}\right)\right]^{-1/2}.$$

Let $h(t) = -\log L(t)$. We compute the asymptotic behavior of $h(t)$, $th'(t)$ and $t^2 h''(t)$ as $t \to \infty$.

$$h(t) = t^{1/2} + \frac{1}{4}\log t + \frac{1}{2}\log\left(\frac{1}{4}(2a-1)\right) + o(1),$$

$$th'(t) = \frac{1}{2}t^{1/2} + \frac{1}{4} + o(1),$$

$$t^2 h''(t) = -\frac{1}{4}t^{1/2} - \frac{1}{4} + o(1).$$

Choose $t^{1/2} = \frac{1}{2}\epsilon^{-2}$, then

$$\frac{t\epsilon^2 - th'(t)}{\sqrt{-t^2 h''(t)}} \to 0$$

as $t \to \infty$. Therefore,

$$\mathbb{P}(||S||_2^2 \leq \epsilon^2) \sim \frac{4\sqrt{2}}{\sqrt{\pi(2a-1)}} \epsilon^2 \exp\left(-\frac{1}{4}\epsilon^{-2}\right).$$

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In particular if $a = 1$,

$$\mathbb{P}(||S||_2^2 \le \epsilon^2) \sim \frac{4\sqrt{2}}{\sqrt{\pi}} \epsilon^2 \exp\left(-\frac{1}{4}\epsilon^{-2}\right),$$

which is the same as in [22].

3.3.7.2 Case $1/2 \le a < 1$

For $1/2 \le a < 1$, we have the following two theorems for $1/2 < a < 1$ and $a = 1/2$ respectively.

Theorem 22 *For $1/2 < a < 1$,*

$$\mathbb{P}(\|S\|_2 < \varepsilon) \sim \frac{8\varepsilon^2}{\sqrt{\beta\pi}\alpha^{3/2}} e^{-\frac{1}{16}\alpha^2\varepsilon^{-2}},$$

where

$$\alpha = \sqrt{6} + \sqrt{2} - 2 + 4a - \sqrt{6}a - \sqrt{2}a,$$
$$\beta = \frac{6\sqrt{3} + 6\sqrt{2} + 2\sqrt{6} + 3}{288}(4a - 1).$$

Proof. If we let $g(t) = t^{1/2}M(1/t) + N(1/t)$, then by Theorem 2 of [19], we have

$$L(s) := \mathbb{E} \exp\left(-s \int_0^1 |X(t)|^2 dt\right) = \left(\frac{g(-2s)}{g(0)}\right)^{-1/2}.$$

Denote $x = \lambda^{-1/2}(1 - a)$ and $b = \frac{\sqrt{2}(2a-1)}{1-a}$, then direct computation gives

$$M(\lambda) = \frac{4a - 1}{2\sqrt{6}}\left[4\sqrt{3}\sin x[\cos(\sqrt{3}x)\cos(\sqrt{2}(2a - 1)x) + \sqrt{6}\sin(\sqrt{2}(2a - 1)x)[\cos x \cos(\sqrt{3}x) + 1]\right.$$
$$\left. +4\sin(\sqrt{3}x)[\cos x \cos[\sqrt{2}(2a - 1)x] - 1] - 6\sqrt{2}\sin x \sin\sqrt{3}x \sin[\sqrt{2}(2a - 1)x]\right],$$

and

$$N(\lambda) = \frac{-13\sqrt{2}}{6} - 2\sqrt{2}\cos(x) + \frac{2\sqrt{2}}{3}\cos(\sqrt{3}x) + \frac{\sqrt{2}}{2}\cos(x)\cos(\sqrt{3}x) + \frac{\sqrt{2}}{2}\cos(bx)$$
$$+ \frac{2\sqrt{2}}{3}\cos(x)\cos(bx) - 2\sqrt{2}\cos(\sqrt{3}x)\cos(bx) - \frac{13\sqrt{2}}{6}\cos(x)\cos(\sqrt{3}x)\cos(bx)$$
$$+ \frac{\sqrt{6}}{3}\sin(x)\sin(\sqrt{3}x) + \frac{\sqrt{6}}{3}\cos(bx)\sin(x)\sin(\sqrt{3}x) + \sin(x)\sin(bx)$$
$$- \cos(\sqrt{3}x)\sin(x)\sin(bx) - \frac{5\sqrt{3}}{3}\sin(\sqrt{3}x)\sin(bx) - \frac{5\sqrt{3}}{3}\cos(x)\sin(\sqrt{3}x)\sin(bx)$$

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Thus,

$$
\begin{aligned}
L(t) = \Big[& 13/36 + \cosh(\sqrt{2}\sqrt{t}(a-1))/3 - \cosh(2\sqrt{t}(2a-1))/12 - \cosh(\sqrt{6}\sqrt{t}(a-1))/9 \\
& - (\cosh(2\sqrt{t}(2a-1))\cosh(\sqrt{2}\sqrt{t}(a-1)))/9 + (\cosh(2\sqrt{t}(2a-1))\cosh(\sqrt{6}\sqrt{t}(a-1)))/3 \\
& - (\cosh(\sqrt{2}\sqrt{t}(a-1))\cosh(\sqrt{6}\sqrt{t}(a-1)))/12 - (\sqrt{t}\sinh(2\sqrt{t}(2a-1)))/12 \\
& + (\sqrt{2}\sinh(2\sqrt{t}(2a-1))\sinh(\sqrt{2}\sqrt{t}(a-1)))/12 \\
& - (5\sqrt{6}\sinh(2\sqrt{t}(2a-1))\sinh(\sqrt{6}\sqrt{t}(a-1)))/36 \\
& + (\sqrt{3}\sinh(\sqrt{2}\sqrt{t}(a-1))\sinh(\sqrt{6}\sqrt{t}(a-1)))/18 + (\sqrt{2}\sqrt{t}\sinh(\sqrt{2}\sqrt{t}(a-1)))/6 \\
& - (\sqrt{6}\sqrt{t}\sinh(\sqrt{6}\sqrt{t}(a-1)))/18 + (a\sqrt{t}\sinh(2\sqrt{t}(2a-1)))/3 \\
& + (13\cosh(2\sqrt{t}(2a-1))\cosh(\sqrt{2}\sqrt{t}(a-1))\cosh(\sqrt{6}\sqrt{t}(a-1)))/36 \\
& - (\sqrt{2}\cosh(\sqrt{6}\sqrt{t}(a-1))\sinh(2\sqrt{t}(2a-1))\sinh(\sqrt{2}\sqrt{t}(a-1)))/12 \\
& + (\sqrt{3}\cosh(2\sqrt{t}(2a-1))\sinh(\sqrt{2}\sqrt{t}(a-1))\sinh(\sqrt{6}\sqrt{t}(a-1)))/18 \\
& - (5\sqrt{6}\cosh(\sqrt{2}\sqrt{t}(a-1))\sinh(2\sqrt{t}(2a-1))\sinh(\sqrt{6}\sqrt{t}(a-1)))/36 \\
& - (\sqrt{t}\cosh(\sqrt{2}\sqrt{t}(a-1))\cosh(\sqrt{6}\sqrt{t}(a-1))\sinh(2\sqrt{t}(2a-1)))/12 \\
& - (2\sqrt{2}a\sqrt{t}\sinh(\sqrt{2}\sqrt{t}(a-1)))/3 + (2\sqrt{6}a\sqrt{t}\sinh(\sqrt{6}\sqrt{t}(a-1)))/9 \\
& + (\sqrt{2}\sqrt{t}\cosh(2\sqrt{t}(2a-1))\cosh(\sqrt{6}\sqrt{t}(a-1))\sinh(\sqrt{2}\sqrt{t}(a-1)))/6 \\
& + (\sqrt{6}\sqrt{t}\cosh(2\sqrt{t}(2a-1))\cosh(\sqrt{2}\sqrt{t}(a-1))\sinh(\sqrt{6}\sqrt{t}(a-1)))/18 \\
& + (a\sqrt{t}\cosh(\sqrt{2}\sqrt{t}(a-1))\cosh(\sqrt{6}\sqrt{t}(a-1))\sinh(2\sqrt{t}(2a-1)))/3 \\
& - (\sqrt{3}\sqrt{t}\sinh(2\sqrt{t}(2a-1))\sinh(\sqrt{2}\sqrt{t}(a-1))\sinh(\sqrt{6}\sqrt{t}(a-1)))/6 \\
& - (2\sqrt{2}a\sqrt{t}\cosh(2\sqrt{t}(2a-1))\cosh(\sqrt{6}\sqrt{t}(a-1))\sinh(\sqrt{2}\sqrt{t}(a-1)))/3 \\
& - (2\sqrt{6}a\sqrt{t}\cosh(2\sqrt{t}(2a-1))\cosh(\sqrt{2}\sqrt{t}(a-1))\sinh(\sqrt{6}\sqrt{t}(a-1)))/9 \\
& + (2\sqrt{3}a\sqrt{t}\sinh(2\sqrt{t}(2a-1))\sinh(\sqrt{2}\sqrt{t}(a-1))\sinh(\sqrt{6}\sqrt{t}(a-1)))/3 \Big]^{-1/2} \\
= \Big[& \beta\sqrt{t}e^{\alpha\sqrt{t}}(1 + O(1/\sqrt{t})) \Big]^{-1/2}
\end{aligned}
$$

where

$$\alpha = \sqrt{6} + \sqrt{2} - 2 + 4a - \sqrt{6}a - \sqrt{2}a,$$

$$\beta = \frac{6\sqrt{3} + 6\sqrt{2} + 2\sqrt{6} + 3}{288}(4a - 1),$$

which is valid for $1/2 < a < 1$. Then

$$\log L(s) = \log \mathbb{E} \exp\left(-s \int_0^1 |X(t)|^2 dt\right) = -\frac{1}{2}\alpha\sqrt{s} - \frac{1}{4}\log s - \frac{1}{2}\log \beta + O(s^{-1/2})$$

as $s \to \infty$. Let $h(s) = -\log L(s)$. We have

$$h(s) = \frac{1}{2}\alpha s^{1/2} + \frac{1}{4}\log s + \frac{1}{2}\log \beta + O(1/\sqrt{s}),$$

$$sh'(s) = \frac{1}{4}\alpha s^{1/2} + O(1),$$

$$s^2 h''(s) = -\frac{1}{8}\alpha s^{1/2} + O(1),$$

as $s \to \infty$. In particular, if we choose $\sqrt{s} = \frac{1}{4}\alpha\varepsilon^{-2}$, by using the Sytaja Tauberian theorem, we have

$$\mathbb{P}(\|S\|_2 < \varepsilon) \sim (-2\pi s^2 h''(s))^{-1/2} \exp(s\varepsilon^2 - h(s)) \sim \frac{8\varepsilon^2}{\sqrt{\beta\pi}\alpha^{3/2}}e^{-\frac{1}{16}\alpha^2\varepsilon^{-2}}.$$

Theorem 23 *For $a = 1/2$,*

$$\mathbb{P}(\|S\|_2 < \varepsilon) \sim \frac{8\varepsilon^2}{\sqrt{\beta\pi}\alpha^{3/2}}e^{-\frac{1}{16}\alpha^2\varepsilon^{-2}}, \quad as \ \epsilon \to \infty,$$

where

$$\alpha = \frac{\sqrt{6} + \sqrt{2}}{2}, \quad \beta = \frac{\sqrt{6} + 3\sqrt{2}}{72}.$$

Proof. For $a = 1/2$, the eigenvalues of S is determined by the equation

$$\lambda^{-1}\left(\sqrt{3}\cos\frac{\sqrt{\lambda^{-1}}}{2}\sin\frac{\sqrt{3\lambda^{-1}}}{2} + 3\sin\frac{\sqrt{\lambda^{-1}}}{2}\cos\frac{\sqrt{3\lambda^{-1}}}{2} - \sqrt{3}\sin\frac{\sqrt{3\lambda^{-1}}}{2} + 3\sin\frac{\sqrt{\lambda^{-1}}}{2}\right)$$

$$+\sqrt{\lambda^{-1}}\left(2\sqrt{3}\sin\frac{\sqrt{\lambda^{-1}}}{2}\sin\frac{\sqrt{3\lambda^{-1}}}{2} - 5\cos\frac{\sqrt{\lambda^{-1}}}{2}\cos\frac{\sqrt{3\lambda^{-1}}}{2} - 4\cos\frac{\sqrt{\lambda^{-1}}}{2} - 4\cos\frac{\sqrt{3\lambda^{-1}}}{2} - 5\right) = 0.$$

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Define

$$
\begin{aligned}
g(t) =\ & t\left(\sqrt{3}\cos\frac{\sqrt{t}}{2}\sin\frac{\sqrt{3t}}{2}+3\sin\frac{\sqrt{t}}{2}\cos\frac{\sqrt{3t}}{2}-\sqrt{3}\sin\frac{\sqrt{3t}}{2}+3\sin\frac{\sqrt{t}}{2}\right)\\
&+\sqrt{t}\left(2\sqrt{3}\sin\frac{\sqrt{t}}{2}\sin\frac{\sqrt{3t}}{2}-5\cos\frac{\sqrt{t}}{2}\cos\frac{\sqrt{3t}}{2}-4\cos\frac{\sqrt{t}}{2}-4\cos\frac{\sqrt{3t}}{2}-5\right).
\end{aligned}
$$

Then,

$$
\begin{aligned}
L(t) =\ & \left(\frac{g(-2t)}{g(0)}\right)^{-1/2}\\
=\ & \frac{5}{18}+\frac{2}{9}\cosh\left(\sqrt{\frac{3t}{2}}\right)+\frac{2}{9}\cosh\left(\sqrt{\frac{t}{2}}\right)+\frac{5}{18}\cosh\left(\sqrt{\frac{3t}{2}}\right)\cosh\left(\sqrt{\frac{t}{2}}\right)\\
& +\frac{\sqrt{3}}{9}\sinh\left(\sqrt{\frac{3t}{2}}\right)\sinh\left(\sqrt{\frac{t}{2}}\right)+\sqrt{t}\left[\frac{\sqrt{2}}{6}\sinh\left(\sqrt{\frac{t}{2}}\right)-\frac{\sqrt{6}}{18}\sinh\left(\sqrt{\frac{3t}{2}}\right)\right.\\
& \left.+\frac{\sqrt{6}}{18}\cosh\left(\sqrt{\frac{t}{2}}\right)\sinh\left(\sqrt{\frac{3t}{2}}\right)+\frac{\sqrt{2}}{6}\cosh\left(\sqrt{\frac{3t}{2}}\right)\sinh\left(\sqrt{\frac{t}{2}}\right)\right]\\
=\ & \left[\beta\sqrt{t}\,e^{\alpha\sqrt{t}}(1+O(1/\sqrt{t}))\right]^{-1/2},
\end{aligned}
$$

where

$$
\alpha=\frac{\sqrt{6}+\sqrt{2}}{2},\quad \beta=\frac{\sqrt{6}+3\sqrt{2}}{72}.
$$

The theorem is proved by exactly the same argument as in Theorem 22.

<div align="center">

Chapter 4

**APPLICATION OF KARHUNEN-LOÈVE EXPANSION IN TIME
SERIES MODELS**

</div>

In this Chapter, we show the weak convergence of discrete processes to continuous Gaussian processes by functional central limit theorem and continuous mapping theorem. Then, we prove a conjecture on the expectation of the least squares estimate of the integrated process by [38].

4.1 Convergence of Discrete Processes

Theorem 24 (Functional Central Limit Theorem) *Suppose that $u_t \sim iid(0, \sigma^2)$, and the stochastic process X_n is defined by*

$$X_n(t) = \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} u_k + \frac{1}{\sqrt{n}} \left(nt - [nt]\right) u_{[nt]+1}, \quad \frac{k-1}{n} \leq t \leq \frac{k}{n},$$

Then

$$\frac{X_n(t)}{\sigma} \xrightarrow{d} W(t).$$

Theorem 25 *Suppose that $u_t \sim iid(0, \sigma^2)$, $0 \leq t \leq 1$ and the stochastic process X_n is defined by*

$$\bar{X}_n(t) = \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} (u_k - \bar{u}) + \frac{1}{\sqrt{n}} \left(nt - [nt]\right) \left(u_{[nt]+1} - \bar{u}\right) = X_n(t) - tX_n(1),$$

where $\bar{u} = \sum_{j=1}^n u_j / n$. Then

$$\frac{\bar{X}_n(t)}{\sigma} \xrightarrow{d} B(t),$$

where $B(t) = W(t) - tW(1)$ is the one-dimensional Brownian bridge on $[0, 1]$.

Theorem 26 *Suppose that $u_t \sim iid(0, \sigma^2)$, and the stochastic process X_n is defined by*

$$\tilde{X}_n(t) = \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} u_k - \frac{1}{n} \sum_{k=1}^{n} \left(\frac{1}{\sqrt{n}} \sum_{i=1}^{k} u_i \right) + \frac{1}{\sqrt{n}} (nt - [nt]) \, u_{[nt]+1},$$

Then

$$\frac{\tilde{X}_n(t)}{\sigma} \xrightarrow{d} Y(t),$$

where $Y(t) = W(t) - \int_0^1 W(t) dt$ is the demeaned Brownian motion on $[0, 1]$.

Theorem 27 (Continuous Mapping Theorem) *Let $h(x)$ be a continuous function. If $X_n \xrightarrow{d} X$, then $h(X_n) \xrightarrow{d} h(X)$.*

Now we look at some examples.

Example 1 Consider the discrete process y_j modeled by

$$y_j = \rho y_{j-1} + \epsilon_j, \quad y_0 = 0, \quad (j = 1, \ldots, T),$$

where the true value of ρ is 1 and $\{\epsilon_j\} \sim$ i.i.d. $(0, \sigma^2)$. We find the weak convergence of the statistic $T(\hat{\rho} - 1)$ where

$$\hat{\rho} = \frac{\sum_{j=2}^{T} y_{j-1} y_j}{\sum_{j=2}^{T} y_{j-1}^2}.$$

It can be directly computed that

$$T(\hat{\rho} - 1) = \frac{1}{T} \sum_{j=2}^{T} y_{j-1}(y_j - y_{j-1}) \left/ \left[\frac{1}{T^2} \sum_{j=2}^{T} y_{j-1}^2 \right] \right.$$

$$= U_T / V_T$$

where

$$U_T = \frac{1}{T} \sum_{j=2}^{T} y_{j-1}(y_j - y_{j-1})$$

$$= \frac{1}{2} X_T^2(1) - \frac{1}{2T} \sum_{j=1}^{T} \epsilon_j^2,$$

$$V_T = \frac{1}{T^2} \sum_{j=2}^{T} y_{j-1}^2$$

$$= \frac{1}{T} \sum_{j=1}^{T} X_T^2 \left(\frac{j}{T} \right) - \frac{1}{T^2} y_T^2.$$

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Define a continuous function $h(x) = (h_1(x), h_2(x))$ for $x \in C$, where

$$h_1(x) = \frac{1}{2}x^2(1), \quad h_2(x) = \int_0^1 x^2(t)dt.$$

Then

$$U_T = h_1(X_T) - \frac{1}{2T}\sum_{j=1}^T \epsilon_j^2,$$

$$V_T = h_2(X_T) + R_T - \frac{1}{T^2}y_T^2,$$

where

$$R_T = \frac{1}{T}\sum_{j=1}^T X_T^2\left(\frac{j}{T}\right) - \int_0^1 X_T^2(t)dt$$

$$= \sum_{j=1}^T \int_{(j-1)/T}^{j/T} \left[X_T^2\left(\frac{j}{T}\right) - X_T^2(t)\right]dt$$

Now we can deduce that

$$\left(\frac{U_T}{\sigma^2}, \frac{V_T}{\sigma^2}\right) \xrightarrow{d} \left(h_1(W) - \frac{1}{2}, h_2(W)\right)$$

The theorem of continuous mapping yields

$$T(\hat\rho - 1) \xrightarrow{d} \frac{h_1(W) - \frac{1}{2}}{h_2(W)}$$

$$= \frac{\frac{1}{2}(W^2(1) - 1)}{\int_0^1 W^2(t)dt}$$

$$= \frac{\int_0^1 W(t)dW(t)}{\int_0^1 W^2(t)dt}.$$

Example 2 Let us construct the $I(2)$ process $\{y_j^{(2)}\}$ generated by

$$(1 - L)^2 y_j^{(2)} = \epsilon_j, \quad y_{-1}^{(2)} = y_0^{(2)} = 0, \quad (j = 1, \ldots, n),$$

where $\{\epsilon_j\}$ is i.i.d.$(0, \sigma^2)$ with $\sigma^2 > 0$. Obviously,

$$y_j^{(2)} = y_{j-1}^{(2)} + y_j^{(1)} = y_1^{(1)} + \cdots + y_j^{(1)},$$

where $\{y_j^{(1)}\}$ is the $I(1)$ process following

$$y_j^{(1)} = y_{j-1}^{(1)} + \epsilon_j, \quad y_0^{(1)} = 0.$$

Let

$$Y_n^{(1)}(t) = \frac{1}{\sqrt{n}} y_{[nt]}^{(1)} + (nt - [nt]) \frac{1}{\sqrt{n}} \epsilon_{[nt]+1}$$

$$= \frac{1}{\sqrt{n}} \sum_{i=1}^{j} \epsilon_i + n \left(t - \frac{j}{n} \right) \frac{1}{\sqrt{n}} \epsilon_j, \quad \frac{j-1}{n} \le t \le \frac{j}{n}$$

and

$$Y_n^{(2)}(t) = \frac{1}{n\sqrt{n}} y_{[nt]}^{(2)} + (nt - [nt]) \frac{1}{n\sqrt{n}} y_{[nt]+1}^{(1)}$$

$$= \frac{1}{n} \sum_{i=1}^{j} Y_n^{(1)} \left(\frac{i}{n} \right) + n \left(t - \frac{j}{n} \right) \frac{1}{n\sqrt{n}} y_j^{(1)}, \quad \frac{j-1}{n} \le t \le \frac{j}{n}$$

It follows from Donsker's theorem that

$$\frac{Y_n^{(1)}}{\sigma} \to W.$$

Also it can be shown that

$$\frac{Y_n^{(2)}}{\sigma} \to F_1.$$

Now we consider weak convergence to the general n-fold integrated Brownian motion X_n. Construct the $I(d)$ process $\{y_j^{(d)}\}$ generated by

$$(1 - L)^d y_j^{(d)} = \epsilon_j, \quad (j = 1, \ldots, n),$$

with $y_{-(d-1)}^{(d)} = y_{-(d-2)}^{(d)} = \cdots = y_0^{(d)} = 0$ and $\{\epsilon_j\}$ being i.i.d.$(0, \sigma^2)$. We have

$$y_j^{(d)} = y_{j-1}^{(d)} + y_j^{(d-1)} = y_1^{(d-1)} + \cdots + y_j^{(d-1)}, \quad y_j^{(0)} = \epsilon_j$$

and put, for $d \ge 2$,

$$Y_n^{(d)}(t) = \frac{1}{n^{d-1/2}} y_{[nt]}^{(d)} + (nt - [nt]) \frac{1}{n^{d-1/2}} y_{[nt]+1}^{(d-1)}$$

$$= \frac{1}{n} \sum_{i=1}^{j} Y_n^{(d-1)} \left(\frac{i}{n} \right) + n \left(t - \frac{j}{n} \right) \frac{1}{n^{d-1/2}} y_j^{(d-1)}, \quad \frac{j-1}{n} \le t \le \frac{j}{n}$$

It can be proved using induction that

$$\frac{Y_n^{(d)}}{\sigma} \to X_{n-1}.$$

Example 3 The near random walk process

$$y_j = \left(1 - \frac{\beta}{n}\right) y_{j-1} + \epsilon_j, \quad (j = 1, \ldots, n),$$

converges weakly to the OU process $X(t)$ where $\{\epsilon_j\}$ is assumed to be i.i.d.$(0, \sigma^2)$ and

$$dX(t) = -\beta X(t)dt + dw(t) \Rightarrow X(t) = e^{-\beta t}X(0) + e^{-\beta t}\int_0^t e^{\beta s}dw(s).$$

4.2 Tanaka's Conjecture

The following content is based on [26]. Consider the integrated process

$$(1 - L)^d y_j = \epsilon_j, \quad j = 1, \ldots, T$$

where L is the lag operator such that $Ly_j = y_{j-1}$, d is a positive integer, $y_0 = 0$ and $\{\epsilon_j\}$ is an i.i.d. sequence with mean 0 and variance σ^2. The process $\{y_j\}$ is also known as unit root process which is a nonstationary time series model. The authors in [43] and [12] showed that when $d = 1$ the least squares estimate (LSE) of the autoregressive coefficient of the the process converges in distribution to a functional of stochastic integrals of Brownian motion. It was shown in [8] and [38] by the functional central limit theorem that for $d > 1$, the statistic

$$\hat{\rho} = \frac{\sum_{t=1}^T y_{t-1}y_t}{\sum_{t=1}^T y_{t-1}^2} \tag{4.1}$$

converges asymptotically to a functional of stochastic integrals of integrated Brownian motion. Specifically,

$$T(\hat{\rho} - 1) \Rightarrow \begin{cases} \frac{(W^2(1)-1)/2}{\int_0^1 W^2(t)dt}, & d = 1, \\ \frac{X_{d-1}^2(1)/2}{\int_0^1 X_{d-1}^2(t)dt}, & d > 1 \end{cases}$$

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where $W(t)$ is the Brownian motion and $X_d(t)$ is the d-fold integrated Brownian motion defined recursively as

$$X_d(t) = \int_0^t X_{d-1}(s)ds, \quad t \geq 0, \ d \geq 1,$$

for all positive integer d and $X_0(t) = W(t)$. As pointed out in [39], $\hat{\rho}$ in equation (4.1) can also be interpreted as the least squares estimator (LSE) of the coefficient ρ of the following model:

$$y_j = \rho y_{j-1} + v_j, \quad (1-L)^{d-1}v_j = \epsilon_j, \quad j = 1, \ldots, T$$

The limiting distribution of the LSE is of interest for statistical inference. For the case $d = 1$, the LSE is the Dickey-Fuller statistic. The analytic form of the density function of its limiting distribution is known to be difficult and earlier researches approximate the distribution by Monte Carlo simulations and by numerical inversion of its Laplace transform. For $d = 2$ and 3, Tanaka in [38] computed the Laplace transform of the limiting distribution using Girsanov theorem. For $d \geq 4$, the Laplace transform is too complicated to compute and an analytic form of the density function for a general d is difficult to find. However, it has been noticed in [38] that

$$E\left[\frac{X_d^2(1)/2}{\int_0^1 X_d^2(t)dt}\right] = d + 1$$

for $d = 1, 2$. Thus, he naturally conjectured that for any positive integer d, it holds that

$$E\left[\frac{X_d^2(1)/2}{\int_0^1 X_d^2(t)dt}\right] = d + 1.$$

In the following context, we provide a method to compute expectation of the above type of functional using the Karhunen-Loève (KL) expansion of $X_d(t)$ and thus prove the conjecture.

4.3 Application of Karhunen-Loève Expansion of Integrated Brownian Motion

As discussed in Chapter 2 and 3, the eigenfunctions of the m-times integrated Brownian motion $X_d(t)$ satisfy the following Sturm-Liouville problem:

$$\lambda f^{(2d+2)}(t) = (-1)^{d+1} f(t) = (i)^{2d+2} f(t)$$

with boundary conditions

$$f^{(k)}(0) = f^{(d+1+k)}(1) = 0$$

for $k = 0, 1, \ldots, d$. Thus, the eigenfunctions are the nontrivial functions of the form

$$f(t) = \sum_{j=0}^{2d+1} c_j e^{\alpha_j t} \tag{4.2}$$

with $\alpha_j = \lambda^{-1/(2d+2)} i \omega_j$ and $\omega_j = \exp(\frac{j\pi}{d+1} i)$ satisfying the boundary conditions. The eigenvalues λ's are determined by setting the determinant of the following matrix

$$M = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \omega_0 & \omega_1 & \cdots & \omega_{2d+1} \\ \vdots & \vdots & \ddots & \vdots \\ \omega_0^d & \omega_1^d & \cdots & \omega_{2d+1}^d \\ \omega_0^{d+1} e^{\alpha_0} & \omega_1^{d+1} e^{\alpha_1} & \cdots & \omega_{2d+1}^{d+1} e^{\alpha_{2d+1}} \\ \vdots & \vdots & \ddots & \vdots \\ \omega_0^{2d+1} e^{\alpha_0} & \omega_1^{2d+1} e^{\alpha_1} & \cdots & \omega_{2d+1}^{2d+1} e^{\alpha_{2d+1}} \end{pmatrix}$$

to be zero.

We study the behavior of the orthonomal eigenfunctions of $X_d(t)$ for any positive integer d at $t = 1$ and show that this is the key for evaluating the expectation. First, we introduce a lemma regarding the upper half of a discrete Fourier matrix. The lemma contains some interesting facts on the discrete Fourier matrix. Its proof is placed at the end of the article.

Lemma 4 *Let* $\omega = \exp\left(\frac{i\pi}{d+1}\right)$ *to be the* $(2d+2)$-*th root of unity. Define* $\tilde{M}$ *to be the* $(d+1) \times (2d+2)$ *matrix with entries*

$$\tilde{M}_{jk} = \omega^{(j-1)(k-1)},$$

for $j = 1, \ldots, d+1$, $k = 1, \ldots, 2d+2$, *i.e. the submatrix of matrix M consisting of its first $d+1$ rows. Let $\boldsymbol{c} = [c_0, c_1, \ldots, c_{2d+1}]'$ be a vector in the null space of $\tilde{M}$, that is,*

$$\tilde{M}\boldsymbol{c} = 0.$$

The vector $\boldsymbol{c}$ satisfies the following properties:

(1) $\displaystyle\sum_{\substack{j,k=0 \\ |j-k| \neq d+1}}^{2d+1} \frac{c_j c_k}{\omega^j + \omega^k} = 0.$

(2) $\displaystyle\sum_{\substack{j,k=0 \\ |j-k| \neq d+1}}^{2d+1} (-1)^{j+k} \frac{c_j c_k}{\omega^j + \omega^k} = 0.$

(3) $\displaystyle\sum_{j,k=0}^{2d+1} (-1)^{j+k} c_j c_k - (2d+2) \sum_{|j-k|=d+1} (-1)^{j+k} c_j c_k = 0.$

Theorem 28 *For the d-th integrated Brownian motion $X_d(t)$, which has a Karhunen-Loève expansion*

$$X_d(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k} e_k(t) \xi_k \;,$$

its orthonomal eigenfunctions satisfy that

$$e_k^2(1) = 2d + 2$$

for every positive integer k.

Proof. The vector $\boldsymbol{c} = [c_0, c_1, \ldots, c_{2d+1}]'$ are nontrivial solutions of the matrix equation $M\boldsymbol{c} = 0$, i.e. $\boldsymbol{c}$ is in the null space of matrix M. Let $\bar{\boldsymbol{c}} = [\bar{c}_0, \bar{c}_1, \ldots, \bar{c}_{2d+1}]'$ with $\bar{c}_i = (-1)^i e^{\alpha_i} c_i$. Then $M\boldsymbol{c} = 0$ can be represented by two equations

$$\begin{cases} \tilde{M}\boldsymbol{c} &= 0, \\ \tilde{M}\bar{\boldsymbol{c}} &= 0. \end{cases}$$

Since $f(t)$ is an eigenfunction, for any k we may write

$$e_k^2(1) = \frac{f^2(1)}{\int_0^1 f^2(t)dt}.$$

Thus we only have to show that

$$f^2(1) = (2d+2) \int_0^1 f^2(t)dt. \tag{4.3}$$

Plugging equation (4.2) into (4.3) yields

$$\sum_{j,k=0}^{2d+1} c_j c_k e^{\alpha_j + \alpha_k} = (2d+2) \sum_{\substack{j,k=0 \\ |j-k|\neq d+1}}^{2d+1} \frac{c_j c_k}{\alpha_j + \alpha_k} (e^{\alpha_j + \alpha_k} - 1) + (2d+2) \sum_{|j-k|=d+1} c_j c_k.$$

$$\tag{4.4}$$

Rearranging the terms in equation (4.4) and substituting $c_j e^{\alpha_j}$ with $(-1)^j \bar{c}_j$, we have

$$(2d+2) \sum_{\substack{j,k=0 \\ |j-k|\neq d+1}}^{2d+1} \frac{c_j c_k}{\alpha_j + \alpha_k} - (2d+2) \sum_{\substack{j,k=0 \\ |j-k|\neq d+1}}^{2d+1} (-1)^{j+k} \frac{\bar{c}_j \bar{c}_k}{\alpha_j + \alpha_k}$$

$$+ \sum_{j,k=0}^{2d+1} c_j c_k e^{\alpha_j + \alpha_k} - (2d+2) \sum_{|j-k|=d+1} c_j c_k = 0. \tag{4.5}$$

As both $\boldsymbol{c}$ and $\bar{\boldsymbol{c}}$ are in the null space of $\tilde{M}$, by the Lemma 4, we obtain

$$\sum_{\substack{j,k=0 \\ |j-k|\neq d+1}}^{2d+1} \frac{c_j c_k}{\alpha_j + \alpha_k} = \frac{1}{\lambda^{-1/(2d+2)}i} \sum_{\substack{j,k=0 \\ |j-k|\neq d+1}}^{2d+1} \frac{c_j c_k}{\omega_j + \omega_k} = 0$$

$$\sum_{\substack{j,k=0 \\ |j-k|\neq d+1}}^{2d+1} (-1)^{j+k} \frac{\bar{c}_j \bar{c}_k}{\alpha_j + \alpha_k} = \frac{1}{\lambda^{-1/(2d+2)}i} \sum_{\substack{j,k=0 \\ |j-k|\neq d+1}}^{2d+1} (-1)^{j+k} \frac{\bar{c}_j \bar{c}_k}{\omega_j + \omega_k} = 0$$

and

$$\sum_{j,k=0}^{2d+1} c_j c_k e^{\alpha_j + \alpha_k} - (2d+2) \sum_{|j-k|=d+1} c_j c_k = 0.$$

Hence, we have proven equation (4.5) and therefore $e_k^2(1) = 2d+2$.

Proof of Lemma 4. Obviously, when $|j-k| = d+1$, $\omega^j + \omega^k = 0$. Define $\boldsymbol{c}_A = [c_0, c_2, c_4, \ldots, c_{2d}]'$, $\boldsymbol{c}_B = [c_1, c_3, c_5, \ldots, c_{2d+1}]'$. The matrix equation $\tilde{M}\boldsymbol{c} = 0$ can be

rewritten as $\tilde{M}_1 \boldsymbol{c}_A + \tilde{M}_2 \boldsymbol{c}_B = 0$, where

$$\tilde{M}_1 = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2d} \\ 1 & (\omega^2)^2 & (\omega^4)^2 & \cdots & (\omega^{2d})^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & (\omega^2)^d & (\omega^4)^d & \cdots & (\omega^{2d})^d \end{pmatrix}, \tilde{M}_2 = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ \omega & \omega^3 & \omega^5 & \cdots & \omega^{2d+1} \\ \omega^2 & (\omega^3)^2 & (\omega^5)^2 & \cdots & (\omega^{2d+1})^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \omega^d & (\omega^3)^d & (\omega^5)^d & \cdots & (\omega^{2d+1})^d \end{pmatrix}.$$

Thus, we obtain that $\boldsymbol{c}_A = -\tilde{M}_1^{-1} \tilde{M}_2 \boldsymbol{c}_B$. Also, it is easily seen that $\tilde{M}_2 = diag(1, \omega, \ldots, \omega^{2d+1}) \tilde{M}_1$. Since $\tilde{M}_1$ is a discrete Fourier matrix, the entries of its inverse $\tilde{M}_1^{-1}$ can be computed as

$$(\tilde{M}_1^{-1})_{ij} = \frac{1}{d+1} \omega^{-(i-1)(j-1)}, \quad i, j = 1, \ldots, d+1.$$

Let $Y = \tilde{M}_1^{-1} \tilde{M}_2$, direct computation also gives

$$Y_{ij} = \frac{2}{d+1} \cdot \frac{1}{1 - \omega^{2j-2i+1}},$$

$$Y_{ij}^{-1} = \frac{2}{d+1} \cdot \frac{1}{1 - \omega^{2j-2i-1}}, \quad i, j = 1, \ldots, d+1.$$

We formulate the quadratic forms (1) - (3) in terms of matrices and prove them as follows:

(1)

$$\sum_{\substack{j,k=0 \\ |j-k| \neq d+1}}^{2d+1} \frac{c_j c_k}{\omega^j + \omega^k} = \begin{bmatrix} \boldsymbol{c}_A' & \boldsymbol{c}_B' \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{c}_A \\ \boldsymbol{c}_B \end{bmatrix}$$

$$= \boldsymbol{c}_A' A_{11} \boldsymbol{c}_A + \boldsymbol{c}_B' A_{21} \boldsymbol{c}_A + \boldsymbol{c}_A' A_{12} \boldsymbol{c}_B + \boldsymbol{c}_B' A_{22} \boldsymbol{c}_B$$

$$= \boldsymbol{c}_B' \left[Y' A_{11} Y - A_{21} Y - Y' A_{12} + A_{22} \right] \boldsymbol{c}_B$$

where matrices A_{11}, A_{12}, A_{21} and A_{22} are different depending on whether d is even or odd. When d is even, then

$$(A_{11})_{ij} = \frac{1}{\omega^{2(i-1)} + \omega^{2(j-1)}},$$

$$(A_{22})_{ij} = \frac{1}{\omega^{2i-1} + \omega^{2j-1}},$$

$$(A_{12})_{ij} = \begin{cases} 0, & \text{if } |2i - 2j - 1| = d + 1, \\ \frac{1}{\omega^{2(i-1)} + \omega^{2j-1}}, & \text{o.w.} \end{cases}$$

$$(A_{21})_{ij} = (A_{12})_{ji}$$

When d is odd, then

$$(A_{11})_{ij} = \begin{cases} 0, & \text{if } |i - j| = (d + 1)/2, \\ \frac{1}{\omega^{2(i-1)} + \omega^{2(j-1)}}, & \text{o.w.} \end{cases}$$

$$(A_{22})_{ij} = \begin{cases} 0, & \text{if } |i - j| = (d + 1)/2, \\ \frac{1}{\omega^{2i-1} + \omega^{2j-1}}, & \text{o.w.} \end{cases}$$

$$(A_{12})_{ij} = \frac{1}{\omega^{2(i-1)} + \omega^{2j-1}},$$

$$(A_{21})_{ij} = (A_{12})_{ji}$$

From Proposition 4 and Proposition 5 below, we have that

$$Y'A_{11}Y + A_{22} - A_{21}Y - Y'A_{12} = 0$$

and therefore $\displaystyle\sum_{\substack{j,k=0 \\ |j-k| \neq d+1}}^{2d+1} \frac{c_j c_k}{\omega^j + \omega^k} = 0.$

(2) The proof of (2) is similar to the proof of (1).

$$\sum_{\substack{j,k=0 \\ |j-k| \neq d+1}}^{2d+1} (-1)^{j+k} \frac{c_j c_k}{\omega_j + \omega_k} = [\boldsymbol{c}_A' \boldsymbol{c}_B'] \begin{bmatrix} A_{11} & -A_{12} \\ -A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{c}_A \\ \boldsymbol{c}_B \end{bmatrix}$$

$$= \boldsymbol{c}_A' A_{11} \boldsymbol{c}_A - \boldsymbol{c}_B' A_{21} \boldsymbol{c}_A - \boldsymbol{c}_A' A_{12} \boldsymbol{c}_B + \boldsymbol{c}_B' A_{22} \boldsymbol{c}_B$$

$$= \boldsymbol{c}_B' \left[Y'A_{11}Y + A_{21}Y + Y'A_{12} + A_{22} \right] \boldsymbol{c}_B$$

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By Proposition 4 and Proposition 5 below,

$$Y'A_{11}Y + A_{21}Y + Y'A_{12} + A_{22} = 0.$$

Hence,

$$\sum_{\substack{j,k=0 \\ |j-k| \neq d+1}}^{2d+1} (-1)^{j+k} \frac{c_j c_k}{\omega_j + \omega_k} = 0.$$

(3) Using Proposition 6, we have

$$\sum_{j,k=0}^{2d+1} (-1)^{j+k} c_j c_k = \begin{bmatrix} \boldsymbol{c}'_A & \boldsymbol{c}'_B \end{bmatrix} \begin{bmatrix} J & -J \\ -J & J \end{bmatrix} \begin{bmatrix} \boldsymbol{c}_A \\ \boldsymbol{c}_B \end{bmatrix}$$

$$= \boldsymbol{c}'_A J \boldsymbol{c}_A - \boldsymbol{c}'_B J \boldsymbol{c}_A - \boldsymbol{c}'_A J \boldsymbol{c}_B + \boldsymbol{c}'_B J \boldsymbol{c}_B$$

$$= \boldsymbol{c}'_B \left[Y'JY + JY + Y'J + J \right] \boldsymbol{c}_B$$

$$= 4 \boldsymbol{c}'_B J \boldsymbol{c}_B.$$

When d is even,

$$(2d+2) \sum_{|j-k|=d+1} (-1)^{j+k} c_j c_k = (2d+2) \begin{bmatrix} \boldsymbol{c}'_A & \boldsymbol{c}'_B \end{bmatrix} \begin{bmatrix} 0 & G \\ G' & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{c}_A \\ \boldsymbol{c}_B \end{bmatrix}$$

$$= (2d+2)(\boldsymbol{c}'_B G' \boldsymbol{c}_A + \boldsymbol{c}'_A G \boldsymbol{c}_B)$$

$$= (2d+2) \boldsymbol{c}'_B \left[G'Y + Y'G \right] \boldsymbol{c}_B$$

where matrix G is defined as

$$G_{ij} = \begin{cases} 1, & \text{if } |i - j - \frac{1}{2}| = \frac{1}{2}(d+1), \\ 0, & o.w \end{cases}$$

Matrix multiplication of Y' and G yields,

$$(Y'G)_{ij} = \frac{2}{d+1} \cdot \frac{1}{1 - \omega^{2i-2j+d+1}} = \frac{2}{d+1} \cdot \frac{1}{1 + \omega^{2i-2j}}$$

and

$$(G'Y)_{ij} = \frac{2}{d+1} \cdot \frac{1}{1 + \omega^{2j-2i}}.$$

For any i and j, $(Y'G)_{ij}$ and $(G'Y)_{ij}$ are conjugate of each other and their real parts are both $\frac{1}{d+1}$. Thus, $(2d+2)(G'Y + Y'G) = 4J$.

When d is odd,

$$(2d+2) \sum_{|j-k|=d+1} (-1)^{j+k} c_j c_k = (2d+2) \begin{bmatrix} \boldsymbol{c}'_A & \boldsymbol{c}'_B \end{bmatrix} \begin{bmatrix} H & 0 \\ 0 & H \end{bmatrix} \begin{bmatrix} \boldsymbol{c}_A \\ \boldsymbol{c}_B \end{bmatrix}$$
$$= (2d+2)(\boldsymbol{c}'_A H \boldsymbol{c}_A + \boldsymbol{c}'_B H \boldsymbol{c}_B)$$
$$= (2d+2)\boldsymbol{c}'_B \left[Y'HY + H \right] \boldsymbol{c}_B$$

where H is defined as

$$H_{ij} = \begin{cases} 1, & \text{if } |i - j| = \frac{1}{2}(d+1), \\ 0, & o.w \end{cases}$$

By simple computation

$$(Y'H)_{ij} = \frac{2}{d+1} \cdot \frac{1}{1 - \omega^{2i-2j-m}} = \frac{2}{d+1} \cdot \frac{1}{1 + \omega^{2i-2j+1}}$$
$$(HY^{-1})_{ij} = \frac{2}{d+1} \cdot \frac{1}{1 - \omega^{2j-2i-m-2}} = \frac{2}{d+1} \cdot \frac{1}{1 + \omega^{2j-2i-1}}$$

Since $(Y'H)_{ij}$ and $(HY^{-1})_{ij}$ are conjugate of each other and their real parts are both $\frac{1}{d+1}$, $(Y'H)_{ij} + (HY^{-1})_{ij} = \frac{2}{d+1}$. Thus, $(2d+2)(Y'H + HY^{-1}) = 4J$ and by Proposition 6, $(2d+2)(Y'HY + H) = (2d+2)(Y'H + HY^{-1})Y = 4JY = 4J$.

Combining the case of even and odd, we have proved that

$$\sum_{j,k=0}^{2d+1} (-1)^{j+k} c_j c_k - (2d+2) \sum_{|j-k|=d+1} (-1)^{j+k} c_j c_k = 0.$$

Proposition 4 $A_{21}Y + Y'A_{12} = 0$.

Proof. Since the matrices A_{21} and A_{12} are different depending on d even or odd, we divide our proof into two cases. When d is even, the ij-th entry of $A_{21}Y + Y'A_{12}$ is

$$\sum_{\substack{k=1, \\ |2j-2k+1| \neq d+1}}^{d+1} \frac{1}{\omega^{2j-1} + \omega^{2(k-1)}} \cdot \frac{1}{1 - \omega^{2i-2k+1}} + \sum_{\substack{k=1, \\ |2i-2k+1| \neq d+1}}^{d+1} \frac{1}{\omega^{2i-1} + \omega^{2(k-1)}} \cdot \frac{1}{1 - \omega^{2j-2k+1}}$$

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Multiplying each summand by $\omega^{2i-1}+\omega^{2j-1}$, and decomposing each of them by partial fraction, we obtain

$$
(\omega^{2i-1}+\omega^{2j-1})\sum_{\substack{k=1,\\|2j-2k+1|\neq d+1}}^{d+1}\frac{1}{\omega^{2j-1}+\omega^{2(k-1)}}\cdot\frac{1}{1-\omega^{2i-2k+1}}
$$

$$
=\sum_{\substack{k=1,\\|2j-2k+1|\neq d+1}}^{d+1}\frac{\omega^{2k-1}(\omega^{2i}+\omega^{2j})}{(\omega^{2k-1}+\omega^{2j})(\omega^{2k-1}-\omega^{2i})}
$$

$$
=\sum_{\substack{k=1,\\|2j-2k+1|\neq d+1}}^{d+1}\left(\frac{\omega^{2k-1}}{\omega^{2k-1}-\omega^{2i}}-\frac{\omega^{2k-1}}{\omega^{2k-1}+\omega^{2j}}\right)
$$

$$
=\sum_{\substack{k=1,\\|2j-2k+1|\neq d+1}}^{d+1}\left(\frac{1}{1-\omega^{2i-2k+1}}-\frac{1}{1+\omega^{2j-2k+1}}\right)
$$

and

$$
(\omega^{2i-1}+\omega^{2j-1})\sum_{\substack{k=1,\\|2i-2k+1|\neq d+1}}^{d+1}\frac{1}{\omega^{2i-1}+\omega^{2(k-1)}}\cdot\frac{1}{1-\omega^{2j-2k+1}}
$$

$$
=\sum_{\substack{k=1,\\|2i-2k+1|\neq d+1}}^{d+1}\frac{\omega^{2k-1}(\omega^{2i}+\omega^{2j})}{(\omega^{2k-1}-\omega^{2j})(\omega^{2k-1}+\omega^{2i})}
$$

$$
=\sum_{\substack{k=1,\\|2i-2k+1|\neq d+1}}^{d+1}\left(\frac{\omega^{2k-1}}{\omega^{2k-1}-\omega^{2j}}-\frac{\omega^{2k-1}}{\omega^{2k-1}+\omega^{2i}}\right)
$$

$$
=\sum_{\substack{k=1,\\|2i-2k+1|\neq d+1}}^{d+1}\left(\frac{1}{1-\omega^{2j-2k+1}}-\frac{1}{1+\omega^{2i-2k+1}}\right)
$$

$|2j-2k+1|\neq d+1$ is equivalent to $k\neq j-d/2$ and $k\neq j+d/2+1$. Since $1\leq k\leq d+1$, then for $1\leq j\leq md/2$, $k\neq j-d/2$, and for $d/2+1\leq j\leq d+1$, $k\neq j+d/2+1$.

Thus,

$$\sum_{\substack{k=1,\\|2j-2k+1|\neq d+1}}^{d+1}\frac{1}{1-\omega^{2i-2k+1}}+\sum_{\substack{k=1,\\|2i-2k+1|\neq d+1}}^{d+1}\frac{1}{1-\omega^{2j-2k+1}}$$

$$=2\sum_{k=1}^{d+1}\frac{1}{1-\omega^{2i-2k+1}}-\frac{1_{\{1\leq j\leq d/2\}}}{1-\omega^{2i-2j+d+1}}-\frac{1_{\{d/2+1\leq j\leq d+1\}}}{1-\omega^{2i-2j-m-1}}-\frac{1_{\{1\leq i\leq d/2\}}}{1-\omega^{2j-2i+d+1}}-\frac{1_{\{d/2+1\leq i\leq d+1\}}}{1-\omega^{2j-2i-m-1}}$$

$$=2\sum_{k=1}^{d+1}\frac{1}{1-\omega^{2i-2k+1}}-\frac{1_{\{1\leq j\leq d/2\}}}{1+\omega^{2i-2j}}-\frac{1_{\{d/2+1\leq j\leq d+1\}}}{1+\omega^{2i-2j}}-\frac{1_{\{1\leq i\leq d/2\}}}{1+\omega^{2j-2i}}-\frac{1_{\{d/2+1\leq i\leq d+1\}}}{1+\omega^{2j-2i}}$$

Noting the fact that

$$\Re\left(\frac{1}{1\pm\omega^k}\right)=\frac{1}{2}\quad\text{for any integer }k,$$

and ω^{2i-2j} is conjugate of ω^{2j-2i}. By symmetry, we have

$$\sum_{\substack{k=1,\\|2j-2k+1|\neq d+1}}^{d+1}\frac{1}{1-\omega^{2i-2k+1}}+\sum_{\substack{k=1,\\|2i-2k+1|\neq d+1}}^{d+1}\frac{1}{1-\omega^{2j-2k+1}}$$

$$=2\sum_{k=1}^{d+1}\frac{1}{1-\omega^{2i-2k+1}}-\frac{1_{\{1\leq j\leq d/2\}}}{1+\omega^{2i-2j}}-\frac{1_{\{d/2+1\leq j\leq d+1\}}}{1+\omega^{2i-2j}}-\frac{1_{\{1\leq i\leq d/2\}}}{1+\omega^{2j-2i}}-\frac{1_{\{d/2+1\leq i\leq d+1\}}}{1+\omega^{2j-2i}}$$

$$=d+1-2\cdot\frac{1}{2}=d.$$

Similarly,

$$\sum_{\substack{k=1,\\|2i-2k+1|\neq d+1}}^{d+1}\frac{1}{1+\omega^{2i-2k+1}}+\sum_{\substack{k=1,\\|2j-2k+1|\neq d+1}}^{d+1}\frac{1}{1+\omega^{2j-2k+1}}=d\cdot\frac{1}{2}+d\cdot\frac{1}{2}=d.$$

Thus,

$$\sum_{\substack{k=1,\\|2j-2k+1|\neq d+1}}^{d+1}\frac{1}{\omega^{2j-1}+\omega^{2(k-1)}}\cdot\frac{1}{1-\omega^{2i-2k+1}}+\sum_{\substack{k=1,\\|2i-2k+1|\neq d+1}}^{d+1}\frac{1}{\omega^{2i-1}+\omega^{2(k-1)}}\cdot\frac{1}{1-\omega^{2j-2k+1}}$$

$$=\frac{1}{\omega^{2i-1}+\omega^{2j-1}}\left(\sum_{\substack{k=1,\\|2j-2k+1|\neq d+1}}^{d+1}\left(\frac{1}{1-\omega^{2i-2k+1}}-\frac{1}{1+\omega^{2j-2k+1}}\right)\right.$$

$$\left.+\sum_{\substack{k=1,\\|2i-2k+1|\neq d+1}}^{d+1}\left(\frac{1}{1-\omega^{2j-2k+1}}-\frac{1}{1+\omega^{2i-2k+1}}\right)\right)=0.$$

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When d is odd, the ij-th entry of $A_{21}Y + Y'A_{12}$ is

$$\sum_{k=1}^{d+1} \frac{1}{\omega^{2i-1} + \omega^{2(k-1)}} \cdot \frac{1}{1 - \omega^{2j-2k+1}} + \sum_{k=1}^{d+1} \frac{1}{\omega^{2(k-1)} + \omega^{2j-1}} \cdot \frac{1}{1 - \omega^{2i-2k+1}}.$$

By symmetry,

$$\sum_{k=1}^{d+1} \left(\frac{1}{1 - \omega^{2i-2k+1}} + \frac{1}{1 - \omega^{2j-2k+1}} \right) = \sum_{k=1}^{d+1} \left(\frac{1}{1 + \omega^{2i-2k+1}} + \frac{1}{1 + \omega^{2j-2k+1}} \right) = d + 1.$$

Then

$$\sum_{k=1}^{d+1} \frac{1}{\omega^{2i-1} + \omega^{2(k-1)}} \cdot \frac{1}{1 - \omega^{2j-2k+1}} + \sum_{k=1}^{d+1} \frac{1}{\omega^{2(k-1)} + \omega^{2j-1}} \cdot \frac{1}{1 - \omega^{2i-2k+1}}$$

$$= \frac{1}{\omega^{2i-1} + \omega^{2j-1}} \left(\sum_{k=1}^{d+1} \left(\frac{1}{1 - \omega^{2j-2k+1}} - \frac{1}{1 + \omega^{2i-2k+1}} \right) + \sum_{k=1}^{d+1} \left(\frac{1}{1 - \omega^{2i-2k+1}} - \frac{1}{1 + \omega^{2j-2k+1}} \right) \right)$$

$$= \frac{1}{\omega^{2i-1} + \omega^{2j-1}} \left(\sum_{k=1}^{d+1} \left(\frac{1}{1 - \omega^{2j-2k+1}} + \frac{1}{1 - \omega^{2i-2k+1}} \right) - \sum_{k=1}^{d+1} \left(\frac{1}{1 + \omega^{2i-2k+1}} + \frac{1}{1 + \omega^{2j-2k+1}} \right) \right)$$

$$= 0.$$

Proposition 5 $Y'A_{11} + A_{22}Y^{-1} = 0$.

Proof. When d is even, the ij-th entry of $Y'A_{11} + A_{22}Y^{-1}$ is

$$\sum_{k=1}^{d+1} \frac{1}{\omega^{2(j-1)} + \omega^{2(k-1)}} \cdot \frac{1}{1 - \omega^{2i-2k+1}} + \sum_{k=1}^{d+1} \frac{1}{\omega^{2i-1} + \omega^{2k-1}} \cdot \frac{1}{1 - \omega^{2j-2k-1}}$$

By symmetry,

$$\sum_{k=1}^{d+1} \left(\frac{1}{1 - \omega^{2i-2k+1}} + \frac{1}{1 - \omega^{2j-2k-1}} \right) = \sum_{k=1}^{d+1} \left(\frac{1}{1 + \omega^{2i-2k}} + \frac{1}{1 + \omega^{2j-2k}} \right) = d + 1.$$

Then

$$\sum_{k=1}^{d+1} \frac{1}{\omega^{2(j-1)} + \omega^{2(k-1)}} \cdot \frac{1}{1 - \omega^{2i-2k+1}} + \sum_{k=1}^{d+1} \frac{1}{\omega^{2i-1} + \omega^{2k-1}} \cdot \frac{1}{1 - \omega^{2j-2k-1}}$$

$$= \frac{1}{\omega^{2i-1} + \omega^{2(j-1)}} \left(\sum_{k=1}^{d+1} \left(\frac{1}{1 - \omega^{2i-2k+1}} - \frac{1}{1 + \omega^{2j-2k}} \right) + \sum_{k=1}^{d+1} \left(\frac{1}{1 - \omega^{2j-2k-1}} - \frac{1}{1 + \omega^{2i-2k}} \right) \right)$$

$$= \frac{1}{\omega^{2i-1} + \omega^{2(j-1)}} \left(\sum_{k=1}^{d+1} \left(\frac{1}{1 - \omega^{2i-2k+1}} + \frac{1}{1 - \omega^{2j-2k-1}} \right) - \sum_{k=1}^{d+1} \left(\frac{1}{1 + \omega^{2i-2k}} + \frac{1}{1 + \omega^{2j-2k}} \right) \right)$$

$$= 0.$$

When d is odd, the ij-th entry of $Y'A_{11} + A_{22}Y^{-1}$ is

$$\sum_{\substack{k=1, \\ |k-j|\neq(d+1)/2}}^{d+1} \frac{1}{\omega^{2(k-1)} + \omega^{2(j-1)}} \cdot \frac{1}{1 - \omega^{2i-2k+1}} + \sum_{\substack{k=1, \\ |i-k|\neq(d+1)/2}}^{d+1} \frac{1}{\omega^{2i-1} + \omega^{2k-1}} \cdot \frac{1}{1 - \omega^{2j-2k-1}}.$$

By symmetry,

$$\sum_{\substack{k=1, \\ |k-j|\neq(d+1)/2}}^{d+1} \left(\frac{1}{1 - \omega^{2i-2k+1}} + \frac{1}{1 - \omega^{2j-2k-1}} \right) = \sum_{\substack{k=1, \\ |i-k|\neq(d+1)/2}}^{d+1} \left(\frac{1}{1 + \omega^{2i-2k}} + \frac{1}{1 + \omega^{2j-2k}} \right).$$

Then,

$$\sum_{\substack{k=1, \\ |k-j|\neq(d+1)/2}}^{d+1} \frac{1}{\omega^{2(k-1)} + \omega^{2(j-1)}} \cdot \frac{1}{1 - \omega^{2i-2k+1}} + \sum_{\substack{k=1, \\ |i-k|\neq(d+1)/2}}^{d+1} \frac{1}{\omega^{2i-1} + \omega^{2k-1}} \cdot \frac{1}{1 - \omega^{2j-2k-1}}$$

$$= \frac{1}{\omega^{2i-1} + \omega^{2j-2}} \left(\sum_{\substack{k=1, \\ |k-j|\neq(d+1)/2}}^{d+1} \left(\frac{1}{1 - \omega^{2i-2k+1}} - \frac{1}{1 + \omega^{2j-2k)}} \right) \right.$$

$$\left. + \sum_{\substack{k=1, \\ |i-k|\neq(d+1)/2}}^{d+1} \left(\frac{1}{1 - \omega^{2j-2k-1}} - \frac{1}{1 + \omega^{2i-2k}} \right) \right) = 0.$$

Thus, $Y'A_{11} + A_{22}Y^{-1} = 0$.

Proposition 6

$$YJ = J, Y'J = J$$

Proof.

$$(YJ)_{ij} = \frac{2}{d+1} \sum_{k=1}^{d+1} \frac{1}{1 - \omega^{2k-2i+1}} = \frac{2}{d+1} \sum_{k=1}^{d+1} \frac{1}{1 - \omega^{2k-1}} = 1,$$

and

$$(Y'J)_{ij} = \frac{2}{d+1} \sum_{k=1}^{d+1} \frac{1}{1 - \omega^{2i-2k+1}} = \frac{2}{d+1} \sum_{k=1}^{d+1} \frac{1}{1 - \omega^{2k-1}} = 1.$$

4.4 Expectation of the Limiting Distribution

Theorem 29 *Suppose that $X(t)$ is a mean zero Gaussian process, with its KL expansion given by*

$$X(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k} e_k(t) \xi_k.$$

If $e_k^2(1) = c$, then

$$E\left[\frac{X^2(1)/2}{\int_0^1 X^2(t)dt}\right] = c/2.$$

Proof. Denote the Laplace transform of $\mathcal{J} = \int_0^1 X^2(t)dt$ by $\phi(u)$. Then for $u > 0$,

$$
\begin{aligned}
\phi(u) =& E\left[\exp(-u\mathcal{J})\right] = E\left[\exp\left\{-u\sum_{m=1}^{\infty} \lambda_m \xi_m^2\right\}\right] \\
=& \prod_{m=1}^{\infty} E\left[\exp\left\{-u\lambda_m \xi_m^2\right\}\right] \\
=& \prod_{m=1}^{\infty} (1 + 2u\lambda_m)^{-1/2}
\end{aligned}
$$

Making use of the following identity

$$\frac{1}{a} = \int_0^{\infty} e^{-au}du, \quad a > 0,$$

we have

$$E\left[\frac{X^2(1)/2}{\int_0^1 X^2(t)dt}\right]$$

$$=\frac{1}{2}E\left[X^2(1)\int_0^\infty \exp\left\{-u\int_0^1 X^2(t)dt\right\}du\right]$$

$$=\frac{1}{2}E\left[\left(\sum_{k=1}^\infty \sqrt{\lambda_k}e_k(1)\xi_k\right)^2 \int_0^\infty \exp\left\{-u\sum_{m=1}^\infty \lambda_m\xi_m^2\right\}du\right]$$

$$=\frac{1}{2}\int_0^\infty E\left[\left(\sum_{k,j=1}^\infty \sqrt{\lambda_k\lambda_j}e_k(1)e_j(1)\xi_k\xi_j\right)\prod_{m=1}^\infty \exp\left\{-u\lambda_m\xi_m^2\right\}\right]du$$

$$=\frac{1}{2}\int_0^\infty \sum_{k,j=1}^\infty \sqrt{\lambda_k\lambda_j}e_k(1)e_j(1)\cdot E\left[\xi_k\xi_j \prod_{m=1}^\infty \exp\left\{-u\lambda_m\xi_m^2\right\}\right]du$$

$$=\frac{1}{2}\int_0^\infty \sum_{k=1}^\infty \lambda_k e_k^2(1)\cdot E\left[\xi_k^2 \prod_{m=1}^\infty \exp\left\{-u\lambda_m\xi_m^2\right\}\right]du$$

$$=\frac{1}{2}\int_0^\infty \sum_{k=1}^\infty \lambda_k e_k^2(1)\cdot \prod_{m\neq k} E\left[\exp\left\{-u\lambda_m\xi_m^2\right\}\right]\cdot E\left[\xi_k^2 \exp\left\{-u\lambda_k\xi_k^2\right\}\right]du.$$

Here ξ_k^2 has Chi-squared distribution and it's well-known that its moment generating function is

$$E\left[\exp\left\{-t\xi_m^2\right\}\right] = (1+2t)^{-1/2}. \tag{4.6}$$

Differentiating equation (4.6), we obtain

$$E\left[\xi_k^2 \exp\{-t\xi_k^2\}\right] = (1+2t)^{-3/2}.$$

It follows that,

$$E\left[\frac{X^2(1)/2}{\int_0^1 X^2(t)dt}\right]$$

$$=\frac{1}{2}\int_0^\infty \sum_{k=1}^\infty \lambda_k e_k^2(1)\cdot \prod_{m\neq k}(1+2u\lambda_m)^{-1/2}\cdot(1+2u\lambda_k)^{-3/2}du$$

$$=\frac{1}{2}\int_0^\infty \prod_{m=1}^\infty(1+2u\lambda_m)^{-1/2}\cdot \sum_{k=1}^\infty \lambda_k e_k^2(1)\cdot(1+2u\lambda_k)^{-1}du$$

$$=\frac{1}{2}\int_0^\infty \prod_{m=1}^\infty(1+2u\lambda_m)^{-1/2}\cdot \sum_{k=1}^\infty e_k^2(1)\cdot \frac{d}{du}\log\left[(1+2u\lambda_k)^{1/2}\right]du$$

$$=\frac{1}{2}\int_0^\infty \phi(u)\cdot \sum_{k=1}^\infty e_k^2(1)\cdot \frac{d}{du}\log\left[(1+2u\lambda_k)^{1/2}\right]du$$

$$=\frac{c}{2}\int_0^\infty \phi(u)\cdot \sum_{k=1}^\infty \frac{d}{du}\log\left[(1+2u\lambda_k)^{1/2}\right]du$$

$$=\frac{c}{2}\int_0^\infty \phi(u)\cdot \frac{d}{du}\log\left[\prod_{k=1}^\infty(1+2u\lambda_k)^{1/2}\right]du$$

$$=-\frac{c}{2}\int_0^\infty \phi(u)\cdot \frac{d}{du}\log\left[\phi(u)\right]du = -\frac{c}{2}\int_0^\infty d\phi(u)$$

$$=-\frac{c}{2}\left(\lim_{u\to\infty}\phi(u)-\phi(0)\right)=\frac{c}{2}.$$

The Tanaka's conjecture is now a direct corollary of Theorem 29.

Corollary 3 *For any positive integer d,*

$$E\left[\frac{X_d^2(1)/2}{\int_0^1 X_d^2(t)dt}\right]=d+1.$$

Remark. If we let $X(t)$ be the Brownian bridge $B(t)=W(t)-tW(1)$, then clearly $B(1)=0$, and

$$E\left[\frac{B^2(1)/2}{\int_0^1 B^2(t)dt}\right]=0,$$

which can also be verified by its eigenfunction $f_k(t)=\sin(k\pi t)$ using the above approach.

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