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Resonances and Near field heat transfer of finite structures

S. T. Chui,[1,*] Zhifang Lin,[2,3] and Jian Zi[2,3]

[1]*Bartol Research Institute and Dept of Physics and Astronomy, University of Delaware, Newark, DE 19716, USA*
[2]*Surface Physics Laboratory, Department of Physics, Fudan University, Shanghai 200433, China*
[3]*Key Laboratory of Micro and Nano Photonic structures (Ministry of Education), Fudan University, Shanghai, China*

We describe a formulation for near field heat transfer for a finite size system so that the heat conductance can be expressed as sums of contributions from the resonances of the combined structure of the "receiver" and the "source". Our work opens the door to investigate near field heat transfer between finite systems and in particular metamaterials whose resonances have been well studied. We illustrated our results with an analytically tractable example of energy transfer between two split ring resonantors separated by a distance d on top of each other. When the cuts of the two rings are opposite each other, the heat conductance is smaller than when the cuts of the two rings are on top of each other . This result can only come from a finite system calculation.

When two objects are very close together, the heat transfer rate between them can be higher than that expected from black body radiation[1]. The theoretical study of this phenomenon , the near field heat transfer (NFHT), has so far been restricted to infinite systems along the lines suggested by Rytov[2, 3]. The only study of finite systems that we are aware of is that of spheres and cylinders[4–6]. The spherical and cylindrical surfaces are special cases, however. For ordinary finite systems, the electric current perpendicular to and at the boundary is zero. The spherical and cylindrical surfaces differ topologically from the usual finite systems and have no boundary. In this paper we described a formulation for near field heat transfer for finite size systems. This uses an approach[7, 9–12] we recently developed to solve Maxwell's equation and applied to study resonances of fluctuations[13]. Instead of only studying the electromagnetic fields[2] we also focused on the charge densities and currents created by the fluctuating electric fields. We find that the heat conductance can be expressed as sums of contributions from the resonances of the combined structure of the "receiver" and the "source". Our work opens the door to investigating near field heat transfer between finite size systems and in particular metamaterials whose resonances have been well studied. We illustrated our results with an analytically tractable example of energy transfer between two narrow wire split ring resonantors separated by a distance d on top of each other. When the cuts of the two rings are opposite each other, the heat conductance is smaller than that when the cuts of the two rings are on top of each other . We now describe our results in detail.

Due to the quantum zero point motion and thermal effects there is a fluctuating electromagnetic field, $\eta_\alpha(r,\omega)$ in an object α, so that their mean square averages is given by

$$<\eta^*_\alpha(\mathbf{r},\omega)\cdot\eta_\beta(\mathbf{r}'\omega')>=e^2\delta_{\alpha,\beta}\delta(\omega-\omega')\delta(\mathbf{r}-\mathbf{r}') \quad (1)$$

for a constant e^2 given by[3]:

$$e^2=\epsilon''\coth(\hbar\omega/kT)2\hbar/|\epsilon|^2; \quad (2)$$

ϵ'' is the imaginary part of the dielectric constant. Microscopically this fluctuating electric field can come from the fluctuation of the motion of the lattice atomic cores. e^2 exhibit different frequency dependence for different material at different temperatures. For dielectrics, ϵ'' is a constant at low angular frequencies ω. For metals with conductivity σ, $\epsilon''=\sigma/\omega$.

To address the heat transfer we calculate the work done by the fluctuating field of the source on the fluctuatiing current and charge densities $\mathbf{j}$ and n in the "receiver" with our recently developed formulation[7, 9–12] of solving Maxwell's equations. The physical quantities are expressed not on a mesh but in terms of a complete orthonormal set of basis functions $\mathbf{X}_{j,\alpha}(r)$ labelled by the index j in object α. We take the external driving field as the fluctuating field:

$$E^{ext}(\omega)=\eta.$$

The noise electric field can be expanded in the basis as $\mathbf{E}^{ext}(r)=\sum E^{ext}_{X,j,\alpha}\mathbf{X}_{j,\alpha}(r)$. The current density can be similarly expanded. The root mean squared of the expansion coefficient is given by

$$<E^{ext}_{Yj\alpha}E^{ext}_{Xi\beta}>=e^2<X_j|Y_i>=e^2_\alpha\delta_{j,i}\delta_{X,Y}\delta_{\alpha,\beta}. \quad (3)$$

By Ohm's law, in terms of the resistivity ρ and the current density $\mathbf{j}$, $\rho\mathbf{j}=\mathbf{E}^{tot}$ where the **total local** electric field is a sum of the external field $\mathbf{E}^{ext}$, the electric field due to the currents at other places $\mathbf{E}^{em}$ and fields $\mathbf{E}^s$ localized at the boundary[7, 9–11, 17]. Maxwell's equations in integral form can be written as $\mathbf{E}^{em}=-\mathbf{Z}_0\mathbf{j}$ where the impedance matrix $\mathbf{Z}_0=i\omega\mu_0(\mathbf{L}-\omega^{-2}/\mathbf{C})+1/\sigma$ comes from the electron electron interaction. The inductance $\mathbf{L}$ and the capacitance $\mathbf{C}$ are just representations of the Green's function

$$G^0=e^{i\omega|r-r'|/c}/(4\pi|r-r'|) \quad (4)$$

* chui@udel.edu

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in the orthonormal basis[7, 9–11, 17]. These circuit parameters include the self and the mutual parameters representing the intra-object and inter-object interactions. We impose the boundary condition of no current flow perpendicular to the boundary of the object with a large boundary resistivity ρ_s which we take to approach infinity. The total resistivity ρ is a sum of this boundary term and an intrinsic resistivity $\rho_0 \mathbf{1}$ for current flow in the same object. Because of the boundary resistivity, the current densities j_s normal to the boundary approaches zero. We define boundary electric fields $\mathbf{E}_s = \mathbf{j}_s \rho_s$ as the products of the normal components of the current at the boundaries j_s and ρ_s. They behave like Lagrange multipliers. Their values are determined from the condition that the normal boundary currents become zero. We thus obtain:

$$\mathbf{Z}\mathbf{j} = \mathbf{V}. \tag{5}$$

where $\mathbf{V} = \mathbf{E}^{ext} + \mathbf{E}^s$, $\mathbf{Z} = \mathbf{Z}_0 + \rho_0 \mathbf{1}$.

We calculate the fluctuating current $\mathbf{j}$ that results from $\mathbf{E}^{ext}$ as $j_{i\alpha} = \sum_{j\beta} Z^{-1}_{i\alpha,j\beta} V_{j\beta}$. The resonance of the structure is determined by the poles of the inverse impedance and E^s. This equation contains the near field heat transfer information where the field in object β induces a current in object α. The rate of transfer is determined by the inverse mutual circuit parameters. We describe this next.

Past work on NFHT focus on the Poynting vector from one object to the next. Here we formulate the heat transfer as the difference in the rate of work done at one object by the noise field of the other object. The rate of work done at an object that we called a receiver labelled by α is $P = \int d\mathbf{r} \mathbf{j}_\alpha \cdot \mathbf{E}_\alpha$. In terms of the Fourier transform in time, this is $P(t) = \int d\mathbf{r} d\omega d\omega' \mathbf{j}_\alpha(\omega) \cdot \mathbf{E}_\alpha(\omega') e^{it(\omega+\omega')}$. In terms of the noise (external) electric field of the other object that we called the source labelled by β and the impedance matrix the heat transfer from β to α is

$$< P_{\beta,\alpha} >= \int d\omega \sum_\gamma Y_{\alpha,\gamma,\beta,\beta'} < V_\beta V^*_{\beta'} >, \tag{6}$$

$$Y_{\alpha,\gamma,\beta,\beta'} = Z^{-1}_{\alpha\beta}(\omega) Z_{\alpha,\gamma}(-\omega) Z^{-1}_{\gamma,\beta'}(-\omega),$$

From Eq. (1), we get $< VV^* >= P^a + P^b + P^c$, $P^a =< |\eta|^2 > \delta_{\beta,\beta'}$, $P^b =< \eta^* E^s > +c.c.$ and $P^c =< |E^s|^2 >$.

We evaluate the integral over ω in Eq. (6) by contour integration in terms of the poles of the matrix Z^{-1} and E^s, which are just the resonance frequencies ω_i of the system. The exponential factor of the Green's function in Eq. (4) and hence the impedances provides for the cutoff of the contributions far in the ω complex plane.

Our formulation can be applied to different metamaterials whose resonances are well studied. Here we illustrate our approach with the example of two thin wire split ring resonantors on top of each other kept at different temperatures T_1 and T_2. The rate of heat transfer is $P_{1,2} - P_{2,1}$. For illustrative purposes we assume that the temperature is high so that the mean square fluctuation of the noise electric field is

$$e^2 \approx F\xi \tag{7}$$

where $F = 2kT/\sigma$, $\xi(\omega) = \sigma^2/|\epsilon\omega|^2$, At high (low) frequencies $\xi \propto 1/\omega^2$ ($\xi = 1$). We first recapitulate the properties of this split ring system[7].

We express the physical quantities of the system as Fourier transforms of the azimuthal angle ϕ : $g_m = \int d\phi g(\phi) e^{im\phi}$ for any function g. Because of the circular symmetry, the impedance matrix Z_m is diagonal in m. In general, in wire structures of lower symmetry, the circuit parameters are no longer "diagonal" and we have to consider the "off-diagonal" elements of the impedance matricies. Because $G^0(r - r')$ is singular for r=r', the off-diagonal elements are much smaller[7]. In our basis the capacitative impedance and hence Z_m is proportional to m^2. Thus Z_m^{-1} decreases rapidly as m is increased. This sets the limit of the number of Fourier modes that needs to be retained. In this paper we focus on the thin wire limit which makes the problem analytically tractable. In that limit at low frequencies $|L_m|$ and $1/|c_m| = 1/|C_m|/m^2$ are independent of m and proportional to $ln(R/a)$ where R is the radius of the ring, a is the core radius of the wire[8], $\omega Z_m \approx i\mu_0 L(\omega^2 - \omega_u^2 m^2)$ where $\omega_u = c/R$. The intrinsic resistivity moves the poles away from the real axis but otherwise it does not affect much the final numerical result and thus will be neglected. For a **single** resonantor with a cut at $\phi = 0$,

$$E^s = \sum_m (E_m^{ext}/Z_m)/A^s \tag{8}$$

where $A^s = \pi \cot(\omega\pi/\omega_u)/(i\mu_0 L\omega_u)$. The resonances fall into two classes, the "odd" and "even" resonances. The even resonances corresponds to the poles of Z_m at angular frequencies $\omega_{em} = m\omega_u$ and wavelengths at $\lambda_{em} = 4\pi R/(2m)$ for integer m=1, 2... The odd resonances corresponds to the poles of E^s and the zeros of A^s at angular frequencies $\omega_{om} = (m + 1/2)\omega_u$ and wavelengths at $\lambda_{om} = 4\pi R/(2m + 1)$ for integer m= 0, 1,... All together the resonance wavelength approaches twice the length of the ring (circumference) divided by an integer in the thin wire limit, the same as for the dipole antenna.

For the combined system of two split rings a distance d apart, there are self and mutual impedances, X_m and X'_m respectively, between the rings. As d approaches zero, $X - X' \propto d^2$. We focus on the small d limit here. We use parity symmetry to make the coupling matrix between rings diagonal by considering, for any physical quantities p, $p_\pm = p_1 \pm p_2$. The diagonal inverse impedances are given by $Z_\pm^{-1} = 1/X_\pm$, $X_\pm = X_m \pm X'_m$. The poles of $Z_\pm^{-1}$ that correspond to half of the the resonance frequencies, $\omega_{m,\pm} \approx m\omega_u$, in the small d limit.

Analogous to Eq. (8) the boundary fields are given by

$$E_\sigma^s = \sum_m f_{m,\sigma} E_\sigma^{ext}(m). \tag{9}$$

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where $\sigma = \pm$, $f_{m\sigma} = -S_{-\sigma m}/S_{-\sigma}$. When the cuts are on opposite sides

$$S_{\sigma,m} = i\mu_0/[\omega(X_m + \sigma(-1)^m X'_m)], \tag{10}$$

$$2S_s/\pi^2 = c_s \cot(k_s/2)/k_s + c_{-s}[2\cot(k_{-s}) - \cot(k_{-s}/2)]/k_{-s}. \tag{11}$$

$k_s = \omega\pi/\omega_s$, $1/\omega_s^2 = L_s c_s$, $L_s = L + sL'$, $1/c_s = 1/c + s/c'$. The contributions to the power transferred comes from the poles of $f_{m,\sigma}$, which are the zeros $\omega_{\sigma,i}$ of S_σ. The solutions of $S_\pm(\omega_{\pm,n}) = 0$ are

$$\omega_{\pm,n} = (n+p)\omega_{0\pm}, \tag{12}$$

$p = 2(c_+/c_-)^{1/2}/\pi$. In contrast to the single ring case, there is a low lying solution at n=0. The integer n is even (odd) for the positive (negative) parity case. From Eq.(11) and (12), we get

$$\partial_\omega S_\pm(\omega_{\pm,j}) \approx -\pi^2 c_-/(2\omega_{\pm,j}) \tag{13}$$

With these we can look at the heat transfer.

To be specific, we consider the power transfer from ring 1 to ring 2. Now $Z_{2,1}^{-1} = Z_{1,2}^{-1} = (Z_+^{-1} - Z_-^{-1})/2$, $Z_{2,2}^{-1} = Z_{1,1}^{-1} = (Z_+^{-1} + Z_-^{-1})/2$. From Eq. (6) the power transmitted P^a can be written as

$$P_{12}^a = F\sum_m \int d\omega (Z_-^*|Z|_+^{-2} - Z_+^{-1} - Z_-^{-1} + Z_+^*|Z|_-^{-2})\xi/2.$$

By contour integration with L'Hopital's rule , the power tranmitted becomes a sum of contributions over all the poles in the upper half of the complex plane. We get, in the thin wire small d limit, with $c_- >> c_+$, $L_- << L_+$, $L_+/L_- \approx 2/(1 - L'/L)$,

$$P_{12}^a = 2F\pi/[(1 - |L'/L|)^2]/\mu_0 L \sum_{m>0} \xi(m\omega_u)$$

This contribution to the heat conductance is not a function of the location of the cut. We next turn our attention to contributions P^b, P^c that involve the boundary fields.

For P^b. we get from Eq. (6)

$$P^b = 2\sum_{m,j,s=\pm} h_3 S_{s,m}(\omega_{s,j})/S'_s(\omega_{s,j}).$$

$h_3 = \xi\pi[Y_{2,2,1,2}(\omega_{s,j}) + (-1)^m Y_{2,2,1,2}(\omega_{s,j})]$, The power P^c can be similarly obtained.

From symmetry the heat transfer in the other direction, from 2 to 1, is of the same functional form.

When the cuts are on same side, the factor of $(-1)^m$ in EQ. (10) is absent. $S_{\pm,m}/(i\mu_0) = Z_{\pm,m}^{-1}$,

$$S_\pm = c_\pm \pi^2 \cot(k_\pm)/k_\pm. \tag{14}$$

The zero's of $\bar{S}_\pm$ are thus at $\omega_{n,\pm} = (n+1/2)\omega_\pm$, similar to the single ring case. The corresponding residues can be obatined from the derivative

$$\partial_\omega S_\pm = -c_\pm \pi^2/\omega. \tag{15}$$

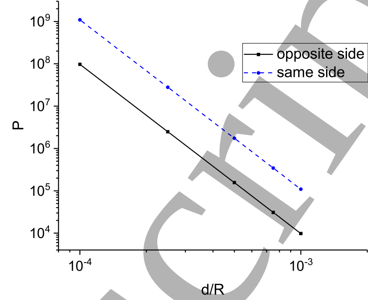

FIG. 1. Log-Log plot of the near field conductivity as a function of the separation d for cases with the cuts on the same side and on opposite sides. The unit is $k_B\Delta T/(\rho_0\omega_u\rho_u)$ where $\rho_u = Z_v t$. Z_v is the resistance of the vacuum, 277; t, the thickness.

P^{b1} and P^c can be easily computed. To evaluate P^{b2} , we find that

$$S_\pm(\omega_{-\pm}) = -c_\pm \pi^2 \delta_\pm/k_\pm$$

where $\delta_\pm = (j+1/2)(\omega_\pm/\omega_{-\pm} - 1) = -\delta - \pm$. From Eq. (**??**) we get, to lowest order in δ,

$$P^{b2} \approx 3c_-^2/(c_+\pi^3)\sum_{j,m}(2j+1)/[(j+1/2)^2 - m^2]^4.$$

We have evaluated the conductance numerically for the example[7] with $a/R = 0.025$, $1 - L'/L \approx (200d/R)^2$. A log-log plot of the conductance as a function of the separation is shown in Fig. (1). As the separation d decreases, X approaches X', ω_0 decreases and P increases. As is advertized, when the cuts are on opposite sides, the conductivity is smaller. The difference in conductivities of the two cases come entirely from the boundary fields and is absent in treatments for infinite systems..

We have examined the effect of the upper limit of summation. The results change by 5 per cent when the upper limits are changed from 5 to 20.

In conclusion we showed that the near field heat conductance for a finite size system can be completely characterized by the resonances of the combined structure of the "receiver" and the "source". For the example of the split rings, when the cuts of the two rings are opposite each other the conductance is smaller than that when the cuts of the two rings are on top of each other. This difference comes from the terms P^b, P^c that depends on the boundary fields and thus will be absent in treatments in infinite systems.

When the distance d between the transmitter and the receiver becomes big, the mutual impedance, $\mathbf{Z}^m$, which is proportional to e^{ikd}/d, becomes much smaller than the self impeadnces $\mathbf{Z}^s$. The inverse total impedance can be written as a power series in $\mathbf{Z}^m/\mathbf{Z}^s$. More precisely

$$1/\mathbf{Z} = (\mathbf{Z}^s + \mathbf{Z}^m)^{-1} \approx 1/\mathbf{Z}^s - \frac{1}{\mathbf{Z}^s}\mathbf{Z}^m\frac{1}{\mathbf{Z}^s} + ..$$

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The higher order terms in this power series expansion represents reflection and multiple scattering of the system. If one sums all the terms of the series from the multiple sacttering, one can recover the result that is applicable in the near field limit. In this approximation the poles are from the self impedances. Physically, the noise electric field in the transmitter η_t excites resonant noise currents in the transmitter $\mathbf{j}_t = \frac{1}{\mathbf{Z}^s}\eta_t$. This noise current radiates an electromagnetic field at the receiver $\mathbf{E}_r = \mathbf{Z}^m \mathbf{j}_t$. The field E_r in turn excites noise currents at the receiver $\mathbf{j}_r = \frac{1}{\mathbf{Z}^s}\mathbf{E}_r$. $\mathbf{j}_r$ and $\mathbf{E}_r$ does work in the receiver. The mode of heat transfer is radiative. This result can also be stated in terms of the Poynting vector. For a **finite** size system, there can be resonances in the transmitter and the receiver, which can affect the heat conduction rate. $\mathbf{Z}^s$ controls the emissivity and the absorption rate of the system.

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[21] The dielectric constant is $\epsilon/\epsilon_0 = 1 - \omega_p^2/(\omega^2 + i\gamma\omega)$. For Au, $\omega_p = 1.37 \times 10^{16} rad/s$, $\gamma = 4.08 \times 10^{13} rad/s$.
[22] From p. 39, Eq. (21) of PBM ($\nu = 0$), $\int_0^x xJ_0(x) = xJ_1(x)$. Check: $(xJ_1)' = J_1 + xJ_1' = J_1 + (xJ_0 - J_1)$.
[23] For $x >> 1$, $J_1(x) \approx (2/\pi x)^{1/2}\cos(x - \pi/2 - \pi/4)$. $J_1'(x) \approx -(2/\pi x)^{1/2}\sin(x - \pi/2 - \pi/4)$. We have picked our basis functions so that they satisfy the boundary condition of $J_1'(k_iR) = 0$. Thus $k_ia - \pi/4 \approx (i - 1/2)\pi$ for large i. $J_1(k_ia) \approx (2/\pi k_ia)^{1/2}(-1)^{i-1}$. Now the normalization $c = |JR/2^{1/2}|$.
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