NUMERICAL EXPERIMENTS WITH ISOMETRIC MAPPING AND BACK-PROJECTION FOR CONDITIONING COÖRDINATE FAMILIES IN A SIMPLE SOBOLEV SPACE

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Abstract. Numerically stable Galerkin procedures can be constructed by ensuring that the families of trial- and test-functions are well conditioned in the respective Hilbert spaces between which the operator is an (appropriate) isomorphism. We explain an idea for constructing a family that may be well conditioned in a given Sobolev space of nonzero fractional order from a family that is well conditioned in the corresponding zero-order space, by using a naturally occurring isometric operator followed by projection back onto the original subspace. Effectively, the construction results in "preconditioning matrices," to be used in transforming the original Galerkin matrix to produce new ones which may be of much smaller condition number. The underlying geometric setting must be sufficiently simple, so that the Sobolev structures can be "manipulated numerically." While we have not yet proven the well-conditioning of the constructed families, the use of the scheme is illustrated numerically in applications to the approximate solution of a first-kind integral equation arising in two-dimensional acoustic scattering, where a pronounced stabilizing effect is observed.

Contents

1. Introduction.

Frequently, an application of mathematics to the physical sciences leads to the formulation of an operator problem

$$Lu = g, \tag{1.1}$$

in which $L : H_1 \to H_2$ is an isomorphism carrying the Hilbert space H_1 onto the Hilbert space H_2 (so that (1.1) is well posed) and g is a given element of H_2; problem (1.1) may also appear in its

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equivalent "variational form," involving bounded sesquilinear and linear forms. The Galerkin method is commonly employed in the construction of approximate solutions for such operator problems. For example, finite- and boundary-element methods are elaborate applications of the Galerkin method for solving partial differential equations and boundary-integral equations.

Evidently, however, it has not been uncommon for implementations of the Galerkin method to be numerically unstable. Here, we say that a Galerkin procedure is *numerically stable* if the sequence of condition numbers of the Galerkin matrices (sometimes called "stiffness matrices") is bounded. In some cases, an asymptotic bound of the form $O(h_N^{-t})$ can be established on these condition numbers, as in, *e.g.*, [9] and [10]; here, h_N denotes the N^{th} mesh size of a discretization scheme (with $h_N \to 0$ as $N \to \infty$) and t is some *positive* number depending on other circumstances in the problem. The actual numerical work then reveals that these upper bounds do in fact accurately describe the behavior of the condition numbers for diminishing mesh sizes, *i.e.*, that the procedure is indeed unstable.

In many applications, especially in small-scale problems when the data are known very precisely, sufficient accuracy of the computed approximate solution can be achieved in spite of such a numerical instability. In other instances, notably when an iterative method must be employed for solution of the linear systems, it is imperative to maintain reasonably small matrix condition numbers. Presently, this reduction of the system condition numbers to acceptable levels is realized through some *ad hoc* numerical strategy employed after the Galerkin-system matrices have been formed; this is usually referred to as "preconditioning," and is an indispensable first step in the usual iterative methods of solution, where the error at each stage is essentially governed by the condition number of the system matrix; *cf.*, *e.g.*, BRAESS [4].

In this note, however, we consider the possibility of achieving numerical stability through the design of the Galerkin procedure itself, in particular, by studied selection of the trial- and test-functions that essentially determine the procedure. More specifically, when the underlying spaces H_1 and H_2 are appropriate and certain other prerequisites are fulfilled, by a suggestive heuristic argument we motivate a systematic routine for generation of trial- and test-functions that "should" afford numerical stability. (Ideas for numerical-stabilization constructions of this sort arise by studying the central features of the mathematical operation of the Galerkin method in its abstract form.) While we have not presently established practically usable sufficient conditions under which our construction is guaranteed to produce well-conditioned families, we do present numerical evidence obtained in implementations of the scheme that strongly indicates the resultant stabilizing effect.

Even though there may be other desirable features required in the design of a Galerkin procedure for (1.1), here we restrict attention to the numerical-stability question. For example, in large-scale applications it is desirable—and may be essential—that the system matrices be *sparse*, *i.e.*, contain many zero-elements. In the case of boundary-operator problems, this sparseness is now usually achieved by a strategy commonly known as "compression," whereby one uses a judicious selection of the trial- and test-functions to produce many "small" matrix elements, which are then set to zero in the system-solution process. Thus, the matrix used in the computation is a perturbation of the actual one, so it is clear that the success of such a procedure hinges on the size of the condition number of the original Galerkin matrix. That is, numerical stability is also a prerequisite for the success of compression. We plan to discuss the combined issues of numerical stability and compression in a future note.

We conclude this introduction by summarizing the developments in the succeeding sections. In Section 2 we recall the formulation of the Galerkin method and the simple condition sufficient for

numerical stability that is developed in [5], which states that one should choose families of trial- and test-functions that are "well conditioned" in the respective spaces between which the particular operator L is an (appropriate) isomorphism. Section 3 is devoted to a summary of the well-known construction of a scale of Hilbert spaces intermediate between two given spaces, one of which is compactly embedded in the other, along with construction of their antiduals; this is just a recounting of developments given in AUBIN [1]. Working within the setting of Section 3, we describe in Section 4 our idea for systematic generation of numerically stable Galerkin schemes by a certain construction of families of trial- and test-functions from appropriate "initial" families of such functions. The procedure involves mapping the initial functions under a selected isometric isomorphism followed by a projection back onto the initial subspace. Resulting from the construction are matrices to be used in transforming the "original" Galerkin matrices by pre- and postmultiplication, yielding a new sequence of Galerkin matrices that is conjectured to be well conditioned. From an inspection of the recipe, it becomes evident that we must be able to "manipulate numerically" the inner-product structures of the scale of Hilbert spaces in any implementation of the scheme, so that these must be relatively simple, *e.g.*, Sobolev spaces associated with an interval in $\mathbb{R}$, a circle, a plane rectangle, or a spherical surface. In the remaining Sections 5–7, we restrict attention to Sobolev spaces of functions defined in a bounded subinterval (a, b) of the reals $\mathbb{R}$. In Section 5 we recall the definitions of the so-called B-splines and, by relying on results of SCHOENBERG [11], [12], show that these well-known functions generate families of coördinate functions that are well conditioned in $H^0(a, b) \equiv L_2(a, b)$. Consequently, we are motivated to use such splines as our initial family in the construction of Section 4. In Section 6 we use the previous results to apply our idea in constructing families of coördinate functions that we conjecture to be well conditioned in various Sobolev spaces on (a, b) of positive and negative fractional order, *e.g.*, spaces of "periodic" functions and spaces of functions with traces vanishing at both a and b. Finally, Section 7 contains a description of the two numerical applications we have made by employing the families generated in Section 6. There, we construct approximate solutions of a two-dimensional boundary-operator problem involving the scattering of time-harmonic acoustic waves by a sound-soft obstacle of arbitrary smooth shape. One of these applications is only cited, since the details are available in [6].

As we noted, our idea was originally most appropriate for application to problems set in a Sobolev space of functions on a geometrically simple set, say, an interval in $\mathbb{R}$. But in [6] we show how to reformulate a large class of boundary-integral equations set on a manifold in $\mathbb{R}^3$ as corresponding integral-operator problems posed in a space of functions defined in a square in the plane and having vanishing traces on the boundary of the square. Then the underlying geometry becomes sufficiently simple that the Sobolev structures can be manipulated numerically, so permitting the implementation of the ideas explained here in the higher-dimensional case of greater physical importance.

2. Sufficient conditions for numerical stability of a Galerkin procedure.

In this section we review the formulation of a Galerkin procedure for convergent approximate solution of an operator problem in Hilbert space and recall the simple condition guaranteeing the numerical stability of such a procedure that is derived in [5]. This sufficient condition suggests a design strategy leading to the constructions described in Sections 4–6 and the examples of solution-approximation outlined in Section 7. The strategy consists simply in the construction and use of trial- and test- families that are "well conditioned" in the Hilbert spaces between which the operator is an isomorphism.

Let $L : H_1 \to H_2$ be an isomorphism between the Hilbert spaces H_1 and H_2, so that the fundamental operator problem for L is well posed:

$$\textit{Given} \quad g \in H_2, \quad \textit{find} \quad u_g \in H_1 \quad \textit{such that} \qquad Lu_g = g. \tag{OP}$$

The Galerkin method is probably the most well known and commonly applied scheme for the (numerical) construction of a sequence of operators converging strongly (*i.e.*, pointwise) on H_2 to the inverse $L^{-1} : H_2 \to H_1$; with such a construction in hand, it is clear that the unique solution u_g of (OP) can be convergently approximated in the norm of H_1. However, in addition to the convergence question, the issues of numerical stability and "problem size" must be confronted in any implementation of the Galerkin method. In fact, for many purposes it is useful to distinguish two stages in the design and execution of a Galerkin procedure, (1) the "abstract," or "basis-free," formulation, followed by (2) the "numerical-implementation" stage.

The abstract formulation. This step is determined entirely by the selection of two sequences $\left(M_N^1\right)_{N=1}^{\infty}$ and $\left(M_N^2\right)_{N=1}^{\infty}$ of subspaces of H_1 and H_2, respectively. We suppose that M_N^1 and M_N^2 are of equal and finite dimension d_N for each N and that $\left(M_N^k\right)_{N=1}^{\infty}$ is *ultimately dense* in H_k for $k = 1, 2$, *i.e.*, that

$$\lim_{N\to\infty} \operatorname{dist}\left(u, M_N^k\right) = \lim_{N\to\infty} \min_{v\in M_N^k} \left\|u - v\right\|_{H_k} = \lim_{N\to\infty} \left\|u - P_N^k u\right\|_{H_k} = 0 \qquad \text{for every} \quad u \in H_k,$$

in which P_N^k denotes the orthoprojector onto M_N^k in H_k. Then the operator-sequence $\left(P_N^2 L P_N^1\right)_{N=1}^{\infty}$ converges strongly to L, so it is natural to inquire (i) whether the operators $L_N : M_N^1 \to M_N^2$ given by

$$L_N := P_N^2 L \big| M_N^1, \qquad N = 1, 2, \ldots, \tag{2.1}$$

are invertible for all $N \geq$ some N_0 and, if that is the case, (ii) whether the resultant operator-sequence $\left(L_N^{-1} P_N^2\right)_{N=N_0}^{\infty}$ converges strongly to L^{-1}. Put another way, we ask (i) whether the N^{th} *Galerkin subsidiary problem*

$$\textit{Given} \quad g \in H_2, \quad \textit{find} \quad u_g^N \in M_N^1 \quad \textit{such that} \qquad L_N u_g^N = P_N^2 g \tag{OP$_N$}$$

is well posed for all $N \geq$ some N_0 and, if so, (ii) whether the sequence $\left(u_g^N = L_N^{-1} P_N^2 g\right)_{N=N_0}^{\infty}$ of solutions converges in H_1 to the unique solution u_g of (OP), for any data-element $g \in H_2$. When this goes through, naturally we say that the Galerkin procedure determined by $\left(M_N^1\right)_{N=1}^{\infty}$ and $\left(M_N^2\right)_{N=1}^{\infty}$ is *convergent*. With the ultimate denseness of $\left(M_N^1\right)_{N=1}^{\infty}$ in H_1, a necessary and sufficient condition for convergence of the Galerkin procedure is the *eventual uniform invertibility* of the operators L_N, *i.e.*, the existence of an integer N_0 and a positive number α such that

$$\left\|L_N u_N\right\|_{H_2} \geq \alpha \left\|u_N\right\|_{H_1} \qquad \text{for all} \quad u_N \in M_N^1 \qquad \text{whenever} \quad N \geq N_0. \tag{2.2}$$

There are various necessary and sufficient conditions known to guarantee (2.2), as well as sufficient conditions that are applicable to one class of operators or another. In any case, the question of convergence is completely settled in this abstract-formulation stage. In the remainder of this section, we shall suppose that (2.2) obtains.

The numerical-implementation stage. To construct (a numerical approximation to) the solution u_g^N of the N^{th} Galerkin subsidiary problem (OP_N), we first select *bases* for the trial- and test-subspaces. That is, for each N let $\left(\epsilon_n^N\right)_{n=1}^{d_N}$ and $\left(\varepsilon_n^N\right)_{n=1}^{d_N}$ be linearly independent collections chosen from M_N^1 and M_N^2, respectively; we shall refer to the families $\mathcal{F}_1 := \left(\left(\epsilon_n^N\right)_{n=1}^{d_N}\right)_{N=1}^{\infty}$ and $\mathcal{F}_2 := \left(\left(\varepsilon_n^N\right)_{n=1}^{d_N}\right)_{N=1}^{\infty}$ as *coördinate families* (in H_1 and H_2, respectively). Of course, there are many ways in which to choose these collections of finite-dimensional bases, and (once it is assured that (2.2) holds) the numerical performance of the Galerkin procedure depends on their intelligent selection. That is, these choices will determine both the numerical stability and the "effective numerical-problem size" in the implementation.

It is clear that the solution u_g^N of (OP_N) is given by

$$u_g^N = \sum_{n=1}^{d_N} \xi_n^N \epsilon_n^N, \qquad \text{for} \quad N \geq N_0, \tag{2.3}$$

in which the coefficients $\left(\xi_n^N\right)_{n=1}^{d_N}$ are determined by the linear system

$$\sum_{n=1}^{d_N} \left(L\epsilon_n^N, \varepsilon_m^N\right)_{H_2} \xi_n^N = \left(g, \varepsilon_m^N\right)_{H_2}, \qquad m = 1, \ldots, d_N; \tag{2.4}$$

the system is uniquely solvable, owing to (2.2) and the linear independence of $\left(\epsilon_n^N\right)_{n=1}^{d_N}$ and $\left(\varepsilon_n^N\right)_{n=1}^{d_N}$. We say that the Galerkin procedure is *numerically stable* iff the sequence of respective $\ell_2^{d_N}$-condition numbers of the matrices in (2.4) is bounded, with $\ell_2^{d_N}$ denoting the usual d_N-dimensional complex unitary space. We want to control the growth of these condition numbers by a construction of the trial- and test-functions that is based on a known criterion for numerical stability. To recall from [5] this simplest stability condition, let $\mathcal{L}_N : \ell_2^{d_N} \to \ell_2^{d_N}$ denote the operator in $\ell_2^{d_N}$ that is induced by the matrix in (2.4) with respect to the canonical basis, so that the $\ell_2^{d_N}$-condition number of the matrix is just the product $\left\|\mathcal{L}_N\right\| \left\|\mathcal{L}_N^{-1}\right\|$ of operator norms. Now, it is easy to check that $\mathcal{L}_N$ can be factored as

$$\mathcal{L}_N = U_N^{2\,\star} L_N U_N^1, \tag{2.5}$$

in which the "prolongation" operator $U_N^1 : \ell_2^{d_N} \to H_1$ is given by

$$U_N^1 \xi := \sum_{n=1}^{d_N} \xi_n^N \epsilon_n^N, \qquad \text{for} \quad \xi \equiv \left(\xi_n^N\right)_{n=1}^{d_N} \in \ell_2^{d_N}, \tag{2.6}$$

$U_N^2 : \ell_2^{d_N} \to H_2$ is defined similarly in terms of $\left(\varepsilon_n^N\right)_{n=1}^{d_N}$, and (the "restriction" operator) $U_N^{2\,\star} : H_2 \to \ell_2^{d_N}$ is the adjoint of U_N^2. We observe that the linear independence of the sets of coördinate-functions guarantees that the U_N^1 and U_N^2 are injective; by $\left(U_N^1\right)^{-1}$ and $\left(U_N^2\right)^{-1}$ we indicate the inverses of these operators on their respective ranges M_N^1 and M_N^2. Now it is easy to get from (2.5) the upper bound

$$\left\|\mathcal{L}_N\right\| \left\|\mathcal{L}_N^{-1}\right\| \leq \left\{\left\|U_N^2\right\| \left\|\left(U_N^2\right)^{-1}\right\|\right\} \left\{\left\|L_N\right\| \left\|L_N^{-1}\right\|\right\} \left\{\left\|U_N^1\right\| \left\|\left(U_N^1\right)^{-1}\right\|\right\}, \qquad \text{for} \quad N \geq N_0, \tag{2.7}$$

for the condition number of the N^{th} Galerkin matrix, as a product of the condition numbers of three operators. The sequence $\left(\|L_N\|\,\|L_N^{-1}\|\right)_{N=N_0}^{\infty}$ has the upper bound $\|L\|/\alpha$, by virtue of the boundedness of L and (2.2). Therefore, (2.7) implies that numerical stability will obtain if the coördinate families $\mathcal{F}_1$ and $\mathcal{F}_2$ are *well conditioned* in H_1 and H_2, respectively, in the sense of the following definition:

Definition 2.1. Let $\mathcal{F} := \left(\left(b_n^N\right)_{n=1}^{d_N}\right)_{N=1}^{\infty}$ be a coördinate family in the Hilbert space H. Let the prolongation operators $U_N : \ell_2^{d_N} \to H$ be defined analogously as in (2.6), with b_n^N replacing ϵ_n^N. We say that $\mathcal{F}$ is *well conditioned in* H iff the sequence $\left(\|U_N\|\,\|(U_N)^{-1}\|\right)_{N=1}^{\infty}$ of condition numbers is bounded.

With this terminology, we can state that *our designing for numerical stability is based on* (2.7), *assuming that* (2.2) *holds, through the selection/construction of coördinate families well conditioned in the appropriate Hilbert spaces* H_1 *and* H_2.

Besides affording an alternate means for handling the condition numbers associated with a coördinate family that is frequently convenient, the next observation indicates that the process of computing the H-orthoprojections onto the sequence of subspaces $\left(\mathcal{M}_N := \operatorname{span}\left\{b_n^N\right\}_{n=1}^{d_N}\right)_{N=1}^{\infty}$ by using the respective basis-families $\left(b_n^N\right)_{n=1}^{d_N}$ is numerically stable iff the family $\mathcal{F}$ is well conditioned in H:

Lemma 2.1. *Let* $\left(b_n^N\right)_{n=1}^{d_N}$ *be a linearly independent collection in the Hilbert space* H, *and denote the associated prolongation operator by* $U_N : \ell_2^{d_N} \to H$. *Then* $\|U_N\|\,\|(U_N)^{-1}\|$ *is the square root of the* $\ell_2^{d_N}$*-condition number of the* H*-Gram matrix* $\left\{\left(b_n^N, b_m^N\right)_H\right\}_{n,m=1}^{d_N}$ *of* $\left(b_n^N\right)_{n=1}^{d_N}$.

Proof. Let $\mathcal{B}_N : \ell_2^{d_N} \to \ell_2^{d_N}$ denote the operator induced with respect to the canonical basis of $\ell_2^{d_N}$ by the Gram matrix of $\left(b_n^N\right)_{n=1}^{d_N}$, so that the $\ell_2^{d_N}$-condition number of the latter matrix is just $\|\mathcal{B}_N\|\|\mathcal{B}_N^{-1}\|$. Then it is easy to see that $\mathcal{B}_N = U_N^\star U_N$ and $\mathcal{B}_N^{-1} = (U_N)^{-1}\left(U_N^\star \,|\, \mathcal{M}_N\right)^{-1} = (U_N)^{-1}\left((U_N)^{-1}\right)^\star$, so

$$\|\mathcal{B}_N\| = \|U_N^\star U_N\| = \|U_N; \ell_2^{d_N}, H\|^2$$

and

$$\|\mathcal{B}_N^{-1}\| = \|(U_N)^{-1}\left((U_N)^{-1}\right)^\star\| = \|(U_N)^{-1}; \mathcal{M}_N, \ell_2^{d_N}\|^2,$$

from which the conclusion of the Lemma follows. □

There are other important aspects of the conditioning of a coördinate family that should be kept in mind when designing a Galerkin procedure. For example, the following simple result sheds light on the reason for various instabilities occurring in practice.

Lemma 2.2. *Let* $\mathcal{F}$ *be a coördinate family in the Hilbert space* H_1 *generating a sequence of subspaces that is ultimately dense in* H_1. *Suppose further that* H_1 *is densely and compactly embedded in a second Hilbert space* H_2. *Then* $\mathcal{F}$ *can be well conditioned in at most one of* H_1 *and* H_2.

Proof. This follows from Proposition 4.3 of [5]. □

3. Construction of intermediate Hilbert spaces.

Let $\big(H_0, (\,\cdot\,,\,\cdot\,)_0\big)$ and $\big(H_+, (\,\cdot\,,\,\cdot\,)_+\big)$ be two Hilbert spaces with $H_+ \subset H_0$, H_+ dense in H_0, and the natural injection map $\iota : H_+ \to H_0$ compact. In this commonly occurring setting, AUBIN [1] gives a construction of Hilbert spaces H^s intermediate between H_+ and H_0, along with their antiduals. For ready reference, we include here a summary of the main line of this construction and the principal results. In Section 6, we work out two examples, taking H_0 to be $H^0(a,b) \equiv L_2(a,b)$ (in each case) and then identifying H_+ as one or another subspace of $H^1(a,b)$.

Much of the general development can also be found in, *e.g.*, BEREZANSKIĬ [3], albeit from a somewhat different point of view.

We suppose that H_0 is a pivot space, *i.e.*, we identify H_0 with its antidual, so that we can identify the antidual of H_+ as the completion H_- of H_0 with respect to the inner product

$$(u_0, v_0) \mapsto (u_0, v_0)_- := (\iota^\star u_0, \iota^\star v_0)_+, \qquad u_0,\, v_0 \in H_0,$$

where $\iota^\star : H_+ \to H_0$ denotes the adjoint of ι. The sesquilinear form $(u_0, u_+) \mapsto (u_0, \iota u_+)_0$ on $H_0 \times H_+$ is then bounded when regarded as densely defined in $H_- \times H_+$; the bounded extension of this form to all of $H_- \times H_+$, the *antiduality pairing*, is indicated by $\langle\,\cdot\,,\,\cdot\,\rangle_0$. The operator $\iota^\star$ is clearly an isometry with range dense in H_+ when it is regarded as densely defined in H_- (with domain H_0), and so extends to an isometric isomorphism of H_- onto H_+; the inverse of the extension is denoted by $J_+ : H_+ \to H_-$ and termed the *antiduality operator*. It is easy to check that the inner products and the antiduality pairing are connected by

$$\big(u_-, J_+ u_+\big)_{H_-} = \big\langle u_-, u_+ \big\rangle_0 = \big(J_+^{-1} u_-, u_+\big)_{H_+}, \qquad \text{for} \quad u_- \in H_-, \quad u_+ \in H_+. \tag{3.1}$$

With this review of the initial setting, we can describe the construction of the spaces intermediate between H_+ and H_0. More precisely, the result of the construction is a scale of Hilbert spaces $\big(H^s\big)_{-1 \le s \le 1}$ such that $H^{s_1} \supset H^{s_2}$ with compact embedding for $s_1 < s_2$, while $H^{-1} = H_-$, $H^0 = H_0$, and $H^1 = H_+$; moreover, H^{-s} is a realization of the antidual of H^s for $s > 0$, with the antiduality pairing on $H^{-s} \times H^s$ obtained as the appropriate restriction/extension of $\langle\,\cdot\,,\,\cdot\,\rangle_0$.

The scale of Hilbert spaces is constructed with the help of a complete set of eigenpairs of J_+^{-1}. In fact, since the embedding of H_+ into H_0 is compact, the version of the Riesz-Fredholm Theorem given by AUBIN [1] asserts that there exists a collection $\big\{(\lambda_n, \hat{\mathrm{e}}_n)\big\}_{n=1}^{\infty}$ of eigenpairs of J_+^{-1}, *i.e.*, such that

$$J_+^{-1} \hat{\mathrm{e}}_n = \lambda_n \hat{\mathrm{e}}_n, \qquad n = 1,\, 2,\, 3,\, \dots\ ,$$

with the sequence $\big(\lambda_n\big)_{n=1}^{\infty}$ positive, nonincreasing, converging to zero, and containing all of the eigenvalues of J_+^{-1}, repeated according to multiplicity, while the family of respective eigenfunctions $\big(\hat{\mathrm{e}}_n\big)_{n=1}^{\infty}$ forms an orthogonal basis for H_+. It is easy to see that $\big(\hat{\mathrm{e}}_n\big)_{n=1}^{\infty}$ is also an orthogonal basis for both H_0 and H_-; clearly, we may suppose that the basis is *orthonormal* in H_0. Thus, for each n we have $\big\|\hat{\mathrm{e}}_n\big\|_0 = 1$, but $\big\|\hat{\mathrm{e}}_n\big\|_+ = \dfrac{1}{\sqrt{\lambda_n}}$ and $\big\|\hat{\mathrm{e}}_n\big\|_- = \sqrt{\lambda_n}$.

Definition 3.1. For $s \ge 0$, let $\big(H^s, (\,\cdot\,,\,\cdot\,)_s\big)$ be the inner-product space formed by the linear manifold $H^s \subset H_0$ comprising all those $u \in H_0$ for which

$$\|u\|_s := \left\{ \sum_{n=1}^{\infty} \frac{1}{\lambda_n^s} \big|(u, \hat{\mathrm{e}}_n)_0\big|^2 \right\}^{\frac{1}{2}} < \infty \tag{3.2}$$

and the corresponding inner product $(\,\cdot\,,\,\cdot\,)_s$ obtained from $\|\cdot\|_s$ by polarization, in the usual manner.

Since the operation $u \mapsto \left(\frac{1}{\sqrt{\lambda_n^s}}(u, \hat{e}_n)_0\right)_{n=1}^{\infty}$ is then easily shown to be an isometric isomorphism from H^s onto ℓ_2, one concludes that H^s is a Hilbert space. Obviously, $H^0 = H_0$ and the "new" inner product for H_0 coincides with the original one (so there is no ambiguity in the notation), while $H^1 = H_+$.

Definition 3.2. For $s < 0$, let H^s be the Hilbert space formed by completing the inner-product space $\big(H_0, (\,\cdot\,,\,\cdot\,)_s\big)$, with $(\,\cdot\,,\,\cdot\,)_s$ obtained again by polarization from the new norm for H_0 given in (3.2).

We list consequences of these definitions; the first three results are either easily verified directly or are established in [1], so we merely present the statements with little comment on their proofs.

Inner product in H^s: For any s in $[-1, 1]$, the inner product for H^s is given by

$$\big(u, v\big)_s = \sum_{n=1}^{\infty} \frac{1}{\lambda_n^s} \langle u, \hat{e}_n\rangle_0 \overline{\langle v, \hat{e}_n\rangle_0}, \qquad u,\, v \in H^s; \tag{3.3}$$

of course, the antiduality-pairing values in (3.3) reduce to H_0-inner-product values when u and v are in H_0.

Fourier expansion in H^s: By (3.3) and the completeness of the eigenfunctions in H^s, the sequence $\big(\lambda_n^{s/2}\hat{e}_n\big)_{n=1}^{\infty}$ gives an orthonormal basis for H^s, again for $s \in [-1, 1]$. The Fourier expansion in this basis turns out to be simply

$$u = \sum_{n=1}^{\infty} \langle u, \hat{e}_n\rangle_0 \hat{e}_n, \qquad \text{for} \quad u \in H^s. \tag{3.4}$$

Antiduality operator and its inverse: For $s > 0$, the antiduality operator $J_s : H^s \to H^{-s}$ and its inverse have the representations

$$J_s u = \sum_{n=1}^{\infty} \frac{1}{\lambda_n^s} (u, \hat{e}_n)_0 \hat{e}_n \qquad \text{for} \quad u \in H^s, \tag{3.5}$$

$$J_s^{-1} u = \sum_{n=1}^{\infty} \lambda_n^s \langle u, \hat{e}_n\rangle_0 \hat{e}_n \qquad \text{for} \quad u \in H^{-s}. \tag{3.6}$$

Factorization of the antiduality operator and its inverse: Now we introduce isometric isomorphisms acting between H_0 and the spaces H^s of positive and negative order; the presence of these operators is already apparent in the forms of the inner products (3.3). It turns out that these isometries provide factorizations of the antiduality operators and their inverses; *cf.* the factorizations established with the help of the spectral theorem in the presentation of Berezanskiĭ [3]. We shall use the isometries defined here to construct coördinate families well conditioned in an H^s-space with $s \neq 0$ from a family that is well conditioned in H_0.

For any $s \in [-1, 1]$ we write

$$I_s u := \sum_{n=1}^{\infty} \lambda_n^{s/2} (u, \hat{e}_n)_0 \hat{e}_n \qquad \text{for} \quad u \in H_0. \tag{3.7}$$

Obviously, I_0 is just the identity operator on H_0. In the general case, we find

Lemma 3.1. *(i.) Let $s \in [-1, 1]$. Then (3.7) defines an isometric isomorphism $I_s : H_0 \to H^s$ with inverse $I_s^{-1} : H^s \to H_0$ given by*

$$I_s^{-1} u := \sum_{n=1}^{\infty} \frac{1}{\lambda_n^{s/2}} \langle u, \hat{e}_n \rangle_0 \hat{e}_n \qquad \text{for} \quad u \in H^s. \tag{3.8}$$

(ii.) For $s \in (0, 1]$, the antiduality operator $J_s : H^s \to H^{-s}$ and its inverse have the factorizations

$$J_s = I_{-s} I_s^{-1}, \tag{3.9}$$

$$J_s^{-1} = I_s I_{-s}^{-1}. \tag{3.10}$$

Proof. The proof consists of routine checking. □

One can define in a similar manner isometries acting between any two spaces H^s and H^t; *cf.* AUBIN [1]. We have no need of the more general construction here, since our starting point will always be a family well conditioned in the current H_0-space.

4. The fundamental construction: isometric mapping and back-projection.

We maintain the setting and intermediate-space construction of the preceding Section 3. We wish to explain our approach to the systematic production of a coördinate family that is well conditioned in some (perhaps fractional-order) space H^s. The idea is based on the following nearly obvious statement, which suggests how one might fashion a family well conditioned in one space out of a family well conditioned in another.

Lemma 4.1. *Let $B : H_1 \to H_2$ be an isomorphism of the Hilbert space H_1 onto the Hilbert space H_2. If $\mathcal{F} := \left(\left(b_n^N \right)_{n=1}^{d_N} \right)_{N=1}^{\infty}$ is a well-conditioned coördinate family in H_1, then $B\mathcal{F} := \left(\left(Bb_n^N \right)_{n=1}^{d_N} \right)_{N=1}^{\infty}$ is a well-conditioned coördinate family in H_2. The condition numbers of the prolongation operators are exactly preserved if B is also an isometry, since then the H_2-Gram matrices of the collections in $B\mathcal{F}$ coincide, respectively, with the H_1-Gram matrices of the collections in $\mathcal{F}$.*

Proof. Since the prolongation operators for the family $B\mathcal{F}$ are given by $BU_N : \ell_2^{d_N} \to H_2$, where the $U_N : \ell_2^{d_N} \to H_1$ are the prolongation operators for $\mathcal{F}$, the first statement is clear. The second statement is immediate. □

For clarity, it is helpful to enumerate the developments explicitly.

(1.) From this point, we shall suppose that $s \in [-1, 1]$ is nonzero and we have in hand a coördinate family $\mathcal{F} := \left(\left(b_n^N \right)_{n=1}^{d_N} \right)_{N=1}^{\infty}$ belonging to $H^{|s|}$ that is well conditioned in the zero-order space H^0 and generates ultimately dense subspaces in H^0. Apparently, such families are readily available in a number of settings encountered in the applications. For example, in Section 5 we present families of coördinate functions constructed from the well-known B-splines and show that they are indeed well conditioned in the current zero-order space. In fact, one might conjecture that most of the families of coördinate functions now commonly used in finite- and boundary-element procedures are well conditioned in the pertinent H^0-space.

(2.) We suppose further that, because of the mapping properties of the operator L figuring in the problem to which we are applying the Galerkin method, we want instead a coördinate family that is well conditioned in H^s.

(3.) We observe from Lemma 2.2 that $\mathcal{F}$ cannot already be well conditioned in H^s. However, we can appeal to Lemma 4.1 if we can find an isomorphism carrying H^0 onto H^s. In fact, the operator I_s defined in (3.7) possesses the desired property—and is even an isometry. Therefore, we have found in $I_s\mathcal{F} := \left(\left(I_s b_n^N \right)_{n=1}^{d_N} \right)_{N=1}^{\infty}$ a coördinate family well conditioned in the desired space H^s. This image-family will generate ultimately dense subspaces in H^s since $\mathcal{F}$ has that property in H^0.

(4.) On the other hand, in an actual application we must use (3.7) for computation of the values of the $I_s b_n^N$. In the first place, this means that we will need to construct the collection $\left\{ (\lambda_n, \hat{\mathrm{e}}_n) \right\}_{n=1}^{\infty}$ of eigenpairs of the operator J_+^{-1} (*cf.* Section 3), so that, as indicated in the Introduction, the underlying geometry should be rather simple. Secondly, repeated use of (3.7) for function-value computations will prove too expensive, so we are motivated to search for an acceptable approximation to the $I_s b_n^N$ that does not destroy the property of well-conditioning. It seems reasonable to explore the use of approximations of the form $\mathcal{P}_N I_s b_n^N$, in which $\mathcal{P}_N$ denotes a projector "back" onto the original (algebraic) subspace $\mathcal{M}_N := \operatorname{span} \left\{ b_n^N \right\}_{n=1}^{d_N}$, say, the orthoprojector $\mathcal{P}_N^t$ reckoned in the inner product of H^t, for an appropriately selected $t \in [-1, 1]$. The first natural choice here might be $t = s$, but we have chosen to concentrate on the value $t = 0$, since in our case (as we already noted in Section 2) the process of successive orthoprojection onto the subspaces $\left(\mathcal{M}_N \right)_{N=1}^{\infty}$ that is performed

by using the respective basis-families $\left(b_n^N\right)_{n=1}^{d_N}$ is numerically stable, under the original assumption that $\mathcal{F}$ is well conditioned in H^0.

(5.) Thus, in the present work our newly constructed coördinate families will be of the special form $\mathcal{P}_N^0 I_s \mathcal{F} := \left(\left(\mathcal{P}_N^0 I_s b_n^N\right)_{n=1}^{d_N}\right)_{N=1}^{\infty}$, in which $\mathcal{P}_N^0$ denotes the orthoprojector onto the subspace $\mathcal{M}_N := \text{span}\left\{b_n^N\right\}_{n=1}^{d_N}$ in the H^0-inner product. The conjecture is that, perhaps under some reasonable additional hypotheses, $\mathcal{P}_N^0 I_s \mathcal{F}$ is well conditioned in the desired space H^s. At this point, we have not yet identified any easily checked sufficient conditions under which the back-projection step can be proven to preserve the well conditioning. Specifically, we have been working to find simple conditions guaranteeing that the operators

$$\mathcal{P}_N^0 I_s \,\big|\, \mathcal{M}_N : \mathcal{M}_N^0 \to \mathcal{M}_N^s, \qquad N = 1, 2, \ldots,$$

are uniformly bounded and uniformly invertible, with $\mathcal{M}_N^t$ denoting the subspace $\mathcal{M}_N$ equipped with the H^t-inner product; with the well conditioning of the original coördinate family in H^0, it is clear that this would suffice to assure the well conditioning of the new family in H^s. (It is easy to show that each of these operators is injective.) Consequently, in this note we are reporting on various other issues relating to the construction, and intend to deal in a separate article with the all-important matter of establishing the preservation of stability under the projection mapping.

In any event, the numerical evidence cited later in simple applications clearly shows the marked reduction in condition numbers resulting from the use of this scheme; however, we do not now know whether the observed reductions constitute genuine numerical stability or simply a beneficial diminution of the condition numbers to a state of "mild instability."

(6.) There are some points to be checked to determine that the construction makes sense whether s is positive or negative. For example, suppose that $s > 0$. since we have required that $\mathcal{M}_N \subset H^{|s|}$, we find that $I_{-s} b_n^N = I_s^{-1} b_n^N \in H^0$, so $\mathcal{P}_N^0 I_{-s} b_n^N$ makes sense. Alternately, again because $\mathcal{M}_N \subset H^{|s|}$, one can show that $\mathcal{P}_N^0$ regarded as densely defined in H^{-s} and mapping into H^{-s} possesses a bounded extension to all of H^{-s}, verifying again that $\mathcal{P}_N^0 I_{-s} b_n^N$ makes sense.

(7.) Now, with the use of the projector, the new coördinate functions are linear combinations of the original ones, so that the coördinate subspaces themselves remain unchanged, and no new convergence proof is required; the procedure simply prescribes a change-of-basis for each subspace. In an application to a Galerkin procedure, the Galerkin matrices constructed from the new trial- and test-functions will be obtained by matrix transformations (pre- and postmultiplication) acting on the Galerkin matrices constructed from the original trial- and test-functions. That is, for each N we can write

$$\mathcal{P}_N^0 I_s b_n^N = \sum_{m=1}^{d_N} A_{nm}^N b_m^N, \qquad n = 1, \ldots, d_N, \tag{4.1}$$

in which the coefficients A_{nm}^N are determined for $n = 1, \ldots, d_N$ by

$$\sum_{m=1}^{d_N} \left(b_m^N, b_l^N\right)_0 A_{nm}^N = \left(I_s b_n^N, b_l^N\right)_0, \qquad l = 1, \ldots, d_N. \tag{4.2}$$

To effect these computations, one must compute the Gram-matrix elements $\left(b_m^N, b_l^N\right)_0$ and the righthand-side inner products $\left(I_s b_n^N, b_l^N\right)_0$; in turn, recalling (3.7), the latter require the values of the generalized Fourier coefficients $\left(b_l^N, \hat{e}_p\right)_0$. Again, because of the hypothesis on $\mathcal{F}$, the processes of solving the (symmetric) systems in (4.2) are numerically stable.

Sometimes the special features of a particular setting will permit significant savings in work here. For example, there may be some translation-invariance to be exploited and/or the Gram matrices may be Toeplitz matrices. Also, one might be able to perform some of the integrations explicitly, so avoiding the need for numerical quadratures to approximate those integrals.

Finally, considering for simplicity the case of the Bubnov-Galerkin method, when the operator L acts entirely in H^s and the trial- and test-subspaces are chosen to coincide, in view of (4.1) it is now clear that the elements of the old and new N^{th} Galerkin matrices are related according to

$$\left(L\mathcal{P}_N^0 I_s b_n^N, \mathcal{P}_N^0 I_s b_m^N\right)_s = \sum_{j=1}^{d_N}\sum_{k=1}^{d_N} A_{nj}^N \left(Lb_j^N, b_k^N\right)_s \overline{A_{mk}^N}, \qquad m, n = 1, \ldots, d_N. \tag{4.3}$$

i.e., by a transformation using the matrix A^N obtained from the projection-calculation. (In the examples, the form of the operator L will permit computation of the H^s-inner products here by reduction to H^0-inner products.) Similarly, the new righthand-side vectors in the Galerkin systems are obtained by a single matrix multiplication acting on the original vectors. This shows that one might be able to employ the present stabilizing device by modification of an already-existing code, avoiding the need to reprogram the entire computation from scratch.

5. Initial coördinate families: examples.

In this section, we recall a well-known construction of coördinate functions, from the polynomial spline-functions obtained as the convolution powers $\chi^{[n]} \equiv \chi^{*n}$ of the characteristic function $\chi \equiv \chi_{[0,1]}$ of the interval $[0,1]$. These splines are usually termed *B-splines*, since they can be used to construct bases for various linear spline-spaces; *cf.*, *e.g.*, SCHOENBERG [11], [12]. Basic information about the convolution powers of χ is also given by AUBIN [1], [2]; for easy reference, we repeat here the definitions and explicit forms established there.

The convolution powers themselves are defined by $\chi^{[1]} \equiv \chi^{*1} := \chi$ and

$$\chi^{[k+1]}(s) := \chi^{*(k+1)}(s) := \int_{-\infty}^{\infty} \chi(s-\sigma)\chi^{*k}(\sigma)\, d\sigma, \qquad s \in \mathbb{R}, \qquad \text{for} \quad k = 1,\, 2, \ldots . \tag{5.1}$$

Then $\chi^{[k+1]}$ is of class C^{k-1} on $\mathbb{R}$ if $k \geq 1$, with support comprising precisely the interval $[0, k+1]$. For $l = 0, \ldots, k$, the restriction of $\chi^{[k+1]}$ to the interval $[l, l+1]$ coincides with a (shifted) polynomial α_k^l of degree k, *i.e.*, we have

$$\chi^{[k+1]}(s) = \sum_{l=0}^{k} \alpha_k^l(s-l)\chi(s-l) \qquad \text{for} \quad s \in \mathbb{R}, \quad s \neq 1, \ldots, k, \tag{5.2}$$

while, at the subinterval-endpoints indicated, the (limiting) values are

$$\chi^{[k+1]}(l) = \alpha_k^l(0) \qquad \text{for} \quad l = 1, \ldots, k \tag{5.3}$$

(and, of course, $\chi^{[k+1]}(0) = \alpha_k^0(0) = 0$, $\chi^{[k+1]}(k+1) = \alpha_k^k(1) = 0$). The polynomials α_k^l are given by

$$\alpha_k^l(s) := \sum_{j=0}^{k} a_k(l,j)\frac{s^j}{j!}, \tag{5.4}$$

with the coefficients defined as

$$a_k(l,j) := \sum_{q=0}^{l} (-1)^q \binom{k+1}{q} \frac{(l-q)^{k-j}}{(k-j)!}, \qquad \text{for} \quad 0 \leq l, j \leq k. \tag{5.5}$$

Recursion relations amongst the α_k^l and the $a_k(l,j)$ are developed in [1], [2], along with other connections. Figure 1 shows the first seven powers, $\chi^{[k+1]}$ for $k = 0, \ldots, 6$.

Let $(a,b) \subset \mathbb{R}$ be a bounded interval. The "mesh size" h_N corresponding to a given positive integer N is

$$h_N := \frac{b-a}{N}.$$

We review the formation of *initial coördinate families* to be used for convergent solution-approximation in problems posed in two settings, a space $H_0^s(a,b)$ (the closure of $C_0^\infty(a,b)$ in $H^s(a,b)$) and a space $H_{\text{per}}^s(a,b)$ (the closure in $H^s(a,b)$ of the restrictions to (a,b) of all the $(b-a)$-periodic functions in $C^\infty(\mathbb{R})$), along with their respective antiduals. By the term "initial," we indicate that these coördinate families turn out to be well conditioned (only) in $H^0(a,b)$, but we intend to generate from them families (that appear to be) well conditioned in a Sobolev space of nonzero order by using the idea of isometric mapping and back-projection, as outlined in the preceding Section 4.

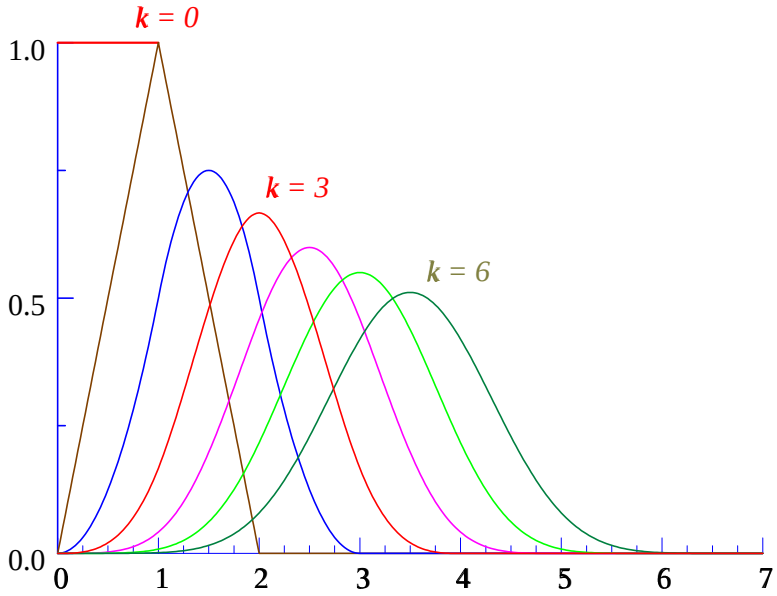

Fig. 1. The convolution powers $\chi^{[k+1]}$ for $k = 0, \ldots, 6$.

Coördinate functions supported in $[a, b]$. Now, with the integers $k \geq 0$ and $N > 0$ satisfying $N - k \geq 1$, the translations/compressions b_n^{Nk} of $\chi^{[k+1]}$ are defined on $\mathbb{R}$ by

$$b_n^{Nk}(s) := \chi^{[k+1]}\Big(\frac{s - (n-1)h_N - a}{h_N}\Big) \qquad \text{for} \quad s \in \mathbb{R}, \qquad \text{for } n \text{ an integer;}$$

usually, we need the b_n^{Nk} only for $n = 1, \ldots, N - k$, but occasionally it is useful to admit any integer n. One can then check that the support of b_n^{Nk} is the interval $[a + (n-1)h_N, a + (n+k)h_N]$ (of length $(k+1)h_N$). Consequently, the support of each function in $\big(b_n^{Nk}\big)_{n=1}^{N-k}$ is contained in $[a, b]$. Corresponding to the selected k, which determines the smoothness of the coördinate functions, we now take the N^{th} subspace $\mathcal{M}_N$ of Section 4 to be $\mathcal{M}_N^k := \text{span}\,\big\{b_n^{Nk}\big\}_{n=1}^{N-k}$ (so that here $d_N = N - k$).

To study the conditioning of the coördinate family $\mathcal{F}_k := \Big(\big(b_n^{Nk}\big)_{n=1}^{N-k}\Big)_{N=k+1}^{\infty}$ in $H^0(a, b)$, we must examine the behavior of the ratio of the largest to the smallest eigenvalues of the Gram matrices $\mathcal{G}_N^k := \big\{\big(b_m^{Nk}, b_n^{Nk}\big)_0\big\}_{m,n=1}^{N-k}$ as $N \to \infty$. Because each function b_n^{Nk} is real and a translate of any other, it is already clear that $\mathcal{G}_N^k$ is an $(N-k) \times (N-k)$ real, symmetric Toeplitz matrix; because the support of b_n^{Nk} is of width $k+1$, $\mathcal{G}_N^k$ is banded, with bandwidth $2k+1$. In fact, the elements of these matrices can be easily calculated; to give the results of this computation, whenever m is an integer we shall use the abbreviation

$$\beta_m^{Nk} := \chi^{[2(k+1)]}(m + k) = \begin{cases} \alpha_{2k+1}^{m+k}(0) = a_{2k+1}(m + k, 0) & \text{for} \quad m = -k+1, \ldots, k+1, \\ 0 & \text{otherwise;} \end{cases} \tag{5.5}$$

here, the explicit expressions follow from (5.2)–(5.4).

Lemma 5.1. *The inner product $\big(b_1^{Nk}, b_m^{Nk}\big)_0$ in $H^0(a, b)$ is given by*

$$\big(b_1^{Nk}, b_m^{Nk}\big)_0 = h_N \chi^{[2(k+1)]}(m + k) = h_N \beta_m^{Nk}, \qquad \textit{for} \quad m = 1, \ldots, N - k, \tag{5.6}$$

so that

$$\left(b_m^{Nk}, b_n^{Nk}\right)_0 = \left(b_1^{Nk}, b_{|n-m|+1}^{Nk}\right)_0 = h_N \beta_{|n-m|+1}^{Nk} \qquad \text{for} \quad m,\, n = 1, \ldots, N-k. \tag{5.7}$$

Proof. Choose an m in $\{\,1, \ldots, k+1\,\}$. We find that

$$\left(b_1^{Nk}, b_m^{Nk}\right)_0 = \int_a^b b_1^{Nk} b_m^{Nk}\, d\lambda_1 = h_N \int_0^N \chi^{[k+1]}(\sigma)\, \chi^{[k+1]}\big(\sigma - (m-1)\big)\, d\sigma;$$

the support of the integrand here is $[m-1, k+1]$, so we can extend the interval of integration to all of $\mathbb{R}$. Thus, we can express the inner product as a value of a convolution:

$$\left(b_1^{Nk}, b_m^{Nk}\right)_0 = h_N \Big(\big(\chi^{[k+1]}\big)^{\check{}} * \chi^{[k+1]} \Big)(m-1),$$

in which $\check{g}(\sigma) := g(-\sigma)$, as usual. Now, with the operator τ_δ of translation by $\delta \in \mathbb{R}$ defined according to $\big(\tau_\delta g\big)(\sigma) := g(\sigma + \delta)$, one can easily check that

$$\begin{aligned} \big(\chi^{*(k+1)}\big)^{\check{}} * \chi^{*(k+1)} &= \check{\chi}^{*(k+1)} * \chi^{*(k+1)} = \big(\check{\chi} * \chi\big)^{*(k+1)} \\ &= \tau_{(k+1)} \Big(\big(\chi * \chi\big)^{*(k+1)} \Big) = \tau_{(k+1)} \chi^{*2(k+1)}; \end{aligned}$$

(5.6) follows from this. Since the first equality in (5.7) is easy to verify, the proof is complete. □

Now we can establish an important conditioning fact:

Proposition 5.1. *Let k be a nonnegative integer. The coördinate family $\mathcal{F}_k$ is well conditioned in $H^0(a,b)$.*

Proof. It is easy to see that the assertion is true for $k = 0$, since each collection $\{b_n^{N0}\}_{n=1}^N$ is an orthogonal set in $H^0(a,b)$, for $N = 1, 2, \ldots$, so that each corresponding Gram matrix $\mathcal{G}_N^0$ is diagonal, with every diagonal element equal to h_N. Therefore, we shall suppose in the remainder of the proof that $k \geq 1$.

We show that the ratio of the largest to the smallest eigenvalues of the Gram matrix $\mathcal{G}_N^k$ is bounded for all $N \geq k+1$, by examining the values of the Rayleigh quotient

$$\mathcal{Q}_N^k(\xi) := \frac{\sum_{m,n=1}^{N-k} \left(b_m^{Nk}, b_n^{Nk}\right)_0 \xi_m \overline{\xi}_n}{\sum_{n=1}^{N-k} |\xi_n|^2} = h_N \frac{\sum_{m,n=1}^{N-k} \chi^{[2(k+1)]}\big(|n-m|+k+1\big) \xi_m \overline{\xi}_n}{\sum_{n=1}^{N-k} |\xi_n|^2}, \qquad \xi \in \ell_2^{N-k},$$

the second equality holding by Lemma 5.1. Now, for any positive integer p let ϕ_p denote the cosine-polynomial given by

$$\begin{aligned} \phi_p(x) &:= \sum_{n=-\infty}^{\infty} \chi^{[p]}\Big(n + \frac{p}{2}\Big) e^{inx} = \sum_{|n| \leq (p/2)-1} \chi^{[p]}\Big(n + \frac{p}{2}\Big) e^{inx} \\ &= \chi^{[2p]}\Big(\frac{p}{2}\Big) + \sum_{1 \leq |n| \leq (p/2)-1} \chi^{[p]}\Big(n + \frac{p}{2}\Big) \cos nx, \qquad x \in \mathbb{R}; \end{aligned}$$

we shall rely on the properties of the ϕ_p established by SCHOENBERG [11], [12], where they are introduced in studying questions of spline-interpolation. (The original definition of ϕ_p in [12] is

given in terms of "central" B-splines; we have translated the definition to involve only the "forward" B-splines being used here; *cf.*, *e.g.*, [12, (1.5)].) Following, *e.g.*, [8], for any $\xi \in \ell_2^{N-k}$ we compute

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\Big|\sum_{n=1}^{N-k}\xi_n e^{\mathrm{i}nx}\Big|^2\phi_{2(k+1)}(x)\,dx = \sum_{m,n=1}^{N-k}\frac{1}{2\pi}\int_{-\pi}^{\pi}\phi_{2(k+1)}(x)e^{-\mathrm{i}(n-m)x}\,dx\,\xi_m\overline{\xi}_n$$

$$= \sum_{m,n=1}^{N-k}\chi^{[2(k+1)]}\big(|n-m|+k+1\big)\,\xi_m\overline{\xi}_n;$$

for the latter equality, we used the fact that $\chi^{[p]}$ is even with respect to $p/2$, *i.e.*, that $\chi^{[p]}\Big(x+\frac{p}{2}\Big) = \chi^{[p]}\Big(-x+\frac{p}{2}\Big)$ for $x \in \mathbb{R}$. Since $(1/2\pi)\int_{-\pi}^{\pi}\Big|\sum_{n=1}^{N-k}\xi_n e^{\mathrm{i}nx}\Big|^2\,dx = \sum_{n=1}^{N-k}|\xi_n|^2$, we get

$$h_N \min_{-\pi\le x\le\pi}\big\{\phi_{2(k+1)}(x)\big\} \le \Omega_N^k(\xi) \le h_N \max_{-\pi\le x\le\pi}\big\{\phi_{2(k+1)}(x)\big\}. \tag{5.8}$$

For $k \ge 1$, the results of SCHOENBERG [12, Lemma 6] show that $\phi_{2(k+1)}$ is strictly decreasing on $[0,\pi]$, with $\phi_{2(k+1)}(0) = 1$ and

$$\phi_{2(k+1)}(\pi) = \Big(\frac{2}{\pi}\Big)^{2(k+1)}\sum_{j=-\infty}^{\infty}\frac{1}{(2j+1)^{2(k+1)}} = 2\Big(\frac{2}{\pi}\Big)^{2(k+1)}\sum_{j=1}^{\infty}\frac{1}{(2j-1)^{2(k+1)}}. \tag{5.9}$$

Since $\phi_{2(k+1)}$ is even and 2π-periodic, it follows that $\phi_{2(k+1)}(0)$ and $\phi_{2(k+1)}(\pi)$ are, respectively, also the maximum and minimum values of the function. Therefore, from the lower and upper bounds on the eigenvalues of $\mathcal{G}_N^k$ obtained from (5.8), we can conclude that the ℓ_2^{N-k}-condition number of $\mathcal{G}_N^k$ is bounded by the reciprocal of the number on the right in (5.9), for all N. □

Proposition 5.1 indicates that we should anticipate a numerical instability if we use one of the families $\mathcal{F}_k$ when we actually need a family well conditioned in a Sobolev space of some order other than zero.

Coördinate functions periodic in $[a,b]$. Instead of functions vanishing outside of a closed subinterval of $[a,b]$, now we generate functions "periodic in $[a,b]$," *i.e.*, with identical values and identical derivative-values at a and b. To this end, still with the integer $k \ge 0$ fixed, suppose now that the integer N is $\ge 2k+1$ and let $\chi_N^{[k+1]}$ denote the periodic extension to all of $\mathbb{R}$ of the restriction $\chi^{[k+1]}|[0,N]$, *i.e.*, of the restricton of $\chi^{[k+1]}$ to the interval $[0,N]$. Thus,

$$\chi_N^{[k+1]}(s) := \sum_{p=-\infty}^{\infty}\chi^{[k+1]}(s-pN) \qquad \text{for} \quad s \in \mathbb{R}.$$

Now we proceed as in the first case to define translations/compressions $\tilde{b}_n^{Nk}$ of $\chi_N^{[k+1]}$ on $\mathbb{R}$ by

$$\tilde{b}_n^{Nk}(s) := \chi_N^{[k+1]}\Big(\frac{s-(n-1)h_N-a}{h_N}\Big) \qquad \text{for} \quad s \in \mathbb{R}, \qquad \text{for } n \text{ an integer;}$$

in this case, only N distinct functions result, since one can readily verify that the $\tilde{b}_n^{Nk}$ are periodic, with period N, in the index n, *i.e.*, one finds that

$$\tilde{b}_{n+pN}^{Nk} = \tilde{b}_n^{Nk} \qquad \text{for every integer } p.$$

Of course, the functions are also periodic in their principal argument, but with period $(b-a)$:

$$\tilde{b}_n^{Nk}\big(s+(b-a)\big) = \tilde{b}_n^{Nk}(s) \qquad \text{for} \quad s \in \mathbb{R},$$

which is also easy to check directly. Moreover, the restrictions of b_n^{Nk} and $\tilde{b}_n^{Nk}$ to $[a,b]$ coincide for $n = 1, \ldots, N-k$:

$$b_n^{Nk}(s) = \tilde{b}_n^{Nk}(s) \qquad \text{for} \quad s \in [a,b] \qquad \text{if} \qquad n = 1, \ldots, N-k, \tag{5.10}$$

but for $n = N-k+1, \ldots, N$ the restriction $\tilde{b}_n^{Nk}\big|[a,b]$ has "split" support $[a, a+(n-N+k)h_N] \cup [b-(N-n+1)h_N, b]$. Now we denote the coördinate subspaces by $\widetilde{\mathcal{M}}_N^k := \operatorname{span}\big\{\tilde{b}_n^{Nk}\big\}_{n=1}^N$; in particular, here we have $d_N = N$.

The conditioning of the coördinate family $\widetilde{\mathcal{F}}^k := \Big(\big(\tilde{b}_n^{Nk}\big)_{n=1}^N\Big)_{N=k+1}^\infty$ in $H^0(a,b)$ depends, as before, on the pertinent Gram matrices, $\widetilde{\mathcal{G}}_N^k := \big\{\big(\tilde{b}_m^{Nk}, \tilde{b}_n^{Nk}\big)_0\big\}_{m,n=1}^N$. Naturally, the elements of these matrices can also be expressed in terms of the β_m^{Nk} defined in (5.5):

Lemma 5.2. *The inner product $\big(\tilde{b}_1^{Nk}, \tilde{b}_m^{Nk}\big)_0$ in $H^0(a,b)$ is given by*

$$\big(\tilde{b}_1^{Nk}, \tilde{b}_m^{Nk}\big)_0 = \begin{cases} \big(b_1^{Nk}, b_m^{Nk}\big)_0 = h_N \chi^{[2(k+1)]}\big(m+k\big) = h_N \beta_m^{Nk} & \text{for} \quad m = 1, \ldots, N-k, \\ \big(b_1^{Nk}, b_{m-N+2}^{Nk}\big)_0 = h_N \beta_{m-N+2}^{Nk} & \text{for} \quad m = N-k+1, \ldots, N; \end{cases} \tag{5.11}$$

the remaining elements of the Gram matrix can be found from

$$\big(\tilde{b}_m^{Nk}, \tilde{b}_n^{Nk}\big)_0 = \big(\tilde{b}_1^{Nk}, \tilde{b}_{|n-m|+1}^{Nk}\big)_0 \qquad \text{for} \quad m, n = 1, \ldots, N. \tag{5.12}$$

Proof. These values are easily computed from those given in Lemma 5.1 and the definitions of the periodic functions. □

It is easy to see that $\widetilde{\mathcal{G}}_N^k$ is an $N \times N$ real, symmetric Toeplitz matrix; its main-diagonal band is of width $2k+1$, just as for the $(N-k) \times (N-k)$ matrix $\mathcal{G}_N^k$, but $\widetilde{\mathcal{G}}_N^k$ also has upper-right and lower-left triangles of side-length k. Moreover, it is essential to observe that $\widetilde{\mathcal{G}}_N^k$ is a *circulant* matrix, *i.e.*, for $n = 2, \ldots, N$, its n^{th} column is obtained by simply permuting the elements of its $(n-1)^{\text{st}}$, moving the last element into first position. Precisely, the matrix $\big\{a_{mn}\big\}_{m,n=1}^N$ is circulant iff the elements a_{mn} satisfy

$$a_{mn} = \begin{cases} a_{N-(n-m)+1,1} & \text{for} \quad m = 1, \ldots, n-1 \\ a_{m-n+1,1} & \text{for} \quad m = n, \ldots, N \end{cases} \qquad \text{for} \quad n = 2, \ldots, N.$$

The circulant property of $\widetilde{\mathcal{G}}_N^k$, a result of the translation-invariance and periodicity properties of the $\tilde{b}_n^{Nk}$, can be readily established from the values given in Lemma 5.2; we omit the details.

For present purposes, the important fact about circulant matrices is that their eigenvalues can easily be computed directly from their elements; *cf.*, *e.g.*, [7]. We can exploit this to show that the families $\widetilde{\mathcal{F}}^k$ are also well conditioned in $H^0(a,b)$. (In passing, it is interesting to note that now the argument of the proof of Proposition 5.1 does not succeed for $k > 1$.)

Proposition 5.2. *Let k be a nonnegative integer. The coördinate family $\widetilde{\mathcal{F}}^k$ is well conditioned in $H^0(a,b)$.*

Proof. Once again we can dispose of the case $k = 0$ just as in the proof of Proposition 5.1 and suppose in the remainder of the argument that $k \geq 1$.

We already noted that each Gram matrix $\widetilde{\mathcal{G}}_N^k$ is a circulant matrix. Therefore, as shown in, *e.g.*, [7], the eigenvalues of $\widetilde{\mathcal{G}}_N^k$ are just the components of the discrete Fourier transform of the first column of the matrix. Thus, recalling Lemma 5.2, these eigenvalues are given by

$$h_N \sum_{p=0}^{N-1} \chi^{[2(k+1)]}(p+k+1) e^{2\pi \mathrm{i} \frac{rp}{N}}$$

$$= h_N \chi^{[2(k+1)]}(k+1) + h_N \sum_{p=1}^{k} \chi^{[2(k+1)]}(p+k+1) e^{2\pi \mathrm{i} \frac{rp}{N}} + h_N \sum_{p=N-k}^{N-1} \chi^{[2(k+1)]}(N-p+k+1) e^{2\pi \mathrm{i} \frac{rp}{N}}$$

$$= h_N \left\{ \chi^{[2(k+1)]}(k+1) + 2 \sum_{p=1}^{k} \chi^{[2(k+1)]}(p+k+1) \cos\left(2\pi \frac{rp}{N}\right) \right\}, \qquad r = 0, \ldots, N-1,$$

that is, recalling the cosine-polynomials ϕ_p cited in the proof of Proposition 5.1, the collection of eigenvalues of $\widetilde{\mathcal{G}}_N^k$ is just

$$\left\{ h_N \phi_{2(k+1)}\left(\frac{2\pi r}{N}\right) \right\}_{r=0}^{N-1}.$$

Accordingly, now we can once again appeal to the positivity properties of the ϕ_p, along with their monotonicity on $[0, \pi]$, established in [12] and summarized in the proof of Proposition 5.1. Thus, the largest eigenvalue of $\widetilde{\mathcal{G}}_N^k$ is h_N for every k and N (and is simple, *i.e.*, of multiplicity one). To identify the smallest eigenvalue, we must pick out the value of $2\pi r/N$ nearest π, and so consider two cases:

▶ N is even: Now the minimum eigenvalue, occurring for $r = N/2$, is $h_N \phi_{2(k+1)}(\pi)$ (this eigenvalue is simple, while there are $(N-2)/2$ distinct eigenvalues of multiplicity two). The ℓ_2^N-condition number of $\widetilde{\mathcal{G}}_N^k$ is given by the reciprocal of the number on the right in (5.9).

▶ N is odd: Here, the minimum eigenvalue is of multiplicity two, corresponding to $r = (N \pm 1)/2$, and has the value $h_N \phi_{2(k+1)}\left(\frac{N \pm 1}{N}\pi\right)$ (the remaining eigenvalues comprise $(N-3)/2$ distinct values, each of multiplicity two). Now the ℓ_2^N-condition number of $\widetilde{\mathcal{G}}_N^k$ is (strictly) bounded by the reciprocal of the number on the right in (5.9). □

A remark analogous to that following Proposition 5.2 is also in order here.

6. Construction of coördinate families: examples.

Here we give two example settings in which we can construct coördinate families according to the program outlined in Section 4, by starting from one of the families shown in the preceding Section 5 to be well conditioned in $H^0(a,b)$. The conjecture is that a family so constructed is well conditioned in the pertinent Sobolev space of selected nonzero order associated with the interval (a,b). We reiterate that, while we have not yet established this well conditioning, the numerical evidence presented in the final Section 7 supports the contention.

Following the choice of a nonzero value $s \in [-1,1]$, the sequence of developments should by now be clear:

(1.) Corresponding to $H_0 = H^0(a,b)$ and a selected Sobolev space H_+ of functions in (a,b) compactly and densely embedded in $H^0(a,b)$, as in Section 3, first find a complete collection of eigenpairs for J_+^{-1}, permitting determination of the action of the isometry I_s for the desired value of s.

(2.) For a chosen "smoothness index" k, act on the corresponding convolution-power coördinate functions b_n^{Nk} with I_s to produce a coördinate family of elements $I_s b_n^{Nk}$ that is well conditioned in the current H^s-space.

(3.) Determine the transformation matrix A^N as in (4.1), (4.2), to be used as described in item 7 of Section 4. Of course, this also permits construction of the actual back-projections $\mathcal{P}_{Nk}^0 I_s b_n^{Nk}$ themselves, if desired (with $\mathcal{P}_{Nk}^0$ denoting the $H^0(a,b)$-orthoprojector onto $\mathcal{M}_N^k$).

In the present section, we shall consider just steps (1) and (2), by showing how to generate the isometric images $I_s b_n^{Nk}$; concerning step (3), we merely recall that the Gram matrices required in the projection computation have already been displayed in Lemma 5.2 and Lemma 5.1 for the respective examples discussed here.

Example 6.1. Spaces $H^s_{\rm per}(a,b)$. For simplicity, we take the interval (a,b) to be $(0,2\pi)$. In the construction of Section 3, we identify H_0 as $H^0(0,2\pi)$ and H_+ as $H^1_{\rm per}(0,2\pi)$; the latter is the subspace of $H^1(0,2\pi)$ comprising those functions u satisfying the "periodicity" condition $u(0) = u(2\pi)$, or, alternately, the closure in $H^1(0,2\pi)$ of the restrictions to $(0,2\pi)$ of all the 2π-periodic functions in $C^\infty(\mathbb{R})$. These choices generate, in the construction of Section 3, spaces H^s denoted now by $H^s_{\rm per}(0,2\pi)$, $-1 \le s \le 1$. To find the explicit forms of the inner product and other structures in $H^s_{\rm per}(0,2\pi)$, we need first to capture the action of the operator J_+^{-1}; it is sufficient to know $J_+^{-1}u$ for, say, $u \in H^0(0,2\pi)$. Accordingly, in view of the form of the inner product in $H^1(0,2\pi)$, the second equality in (3.1) requires here

$$\int_0^{2\pi} \left\{ \left(J_+^{-1}u\right)' \overline{u}_+' + \left(J_+^{-1}u\right) \overline{u}_+ \right\} d\lambda_1 = \int_0^{2\pi} u\, \overline{u}_+\, d\lambda_1 \qquad \text{for} \quad u \in H^0(0,2\pi), \quad u_+ \in H^1_{\rm per}(0,2\pi). \tag{6.1}$$

Since J_+^{-1} will map $H^0(0,2\pi)$ into $H^2_{\rm per}(0,2\pi)$, we find after an integration by parts in (6.1) (and recalling the denseness of $H^1_{\rm per}(0,2\pi)$ in $H^0(0,2\pi)$) that $J_+^{-1}u$ is the solution of a second-order ordinary differential problem with periodic boundary conditions:

$$\left.\begin{aligned} -\left(J_+^{-1}u\right)'' + \left(J_+^{-1}u\right) &= u, \\ \left(J_+^{-1}u\right)(0) &= \left(J_+^{-1}u\right)(2\pi), \\ \left(J_+^{-1}u\right)'(0) &= \left(J_+^{-1}u\right)'(2\pi), \end{aligned}\right\} \qquad \text{for} \quad u \in H^0(0,2\pi). \tag{6.2}$$

Evidently,

$$J_+^{-1}u = \int_0^{2\pi} G_{\rm per}(t,\tau)u(\tau)\,d\lambda_1(\tau) \qquad \text{for every} \quad u \in H^0(0,2\pi), \tag{6.3}$$

in which Green's function $G_{\rm per}$ for the latter boundary-value problem is given by

$$G_{\rm per}(t,\tau) = \frac{\sinh 2\pi}{2(\cosh 2\pi - 1)}\cosh(t-\tau) - \frac{1}{2}\sinh|t-\tau| \qquad \text{for} \quad 0 \le t,\tau \le 2\pi. \tag{6.4}$$

With (6.3), one can check that a complete set of eigenpairs of J_+^{-1} is given by $\left\{\left(\frac{1}{1+p^2}, \hat{e}_p\right)\right\}_{p=-\infty}^{\infty}$, with $\hat{e}_p$ denoting the ordinary complex exponential function normalized in $H^0(0,2\pi)$, $\hat{e}_p(s) := \{2\pi\}^{-\frac{1}{2}} e^{{\rm i}ps}$.

Now all of the constructions of Section 3 can be effected explicitly for the current setting. For present purposes, we need to construct the isometries defined by (3.7), which we shall denote by $\widetilde{I}_s$ in this example; more precisely, we need the images $\widetilde{I}_s\tilde{b}_n^{Nk}$ of the periodic coördinate functions constructed in Section 5. With due regard for the modified indexing used in this example, we find

$$\widetilde{I}_s\tilde{b}_n^{Nk} = \sum_{p=-\infty}^{\infty} \frac{1}{(1+p^2)^{s/2}}(\tilde{b}_n^{Nk}, \hat{e}_p)_0\hat{e}_p \qquad \text{for} \quad n = 1,\dots,N, \quad N \ge 2k+1, \quad k = 1,2,\dots. \tag{6.5}$$

An easy computation produces the Fourier coefficients $(\tilde{b}_n^{Nk}, \hat{e}_p)_0$ explicitly as

$$(\tilde{b}_n^{Nk}, \hat{e}_p)_0 = \frac{\sqrt{2\pi}}{N}\left\{\frac{\sin\frac{p\pi}{N}}{\frac{p\pi}{N}}\right\}^{k+1} \exp\left\{-{\rm i}\frac{p\pi}{N}(2n+k-1)\right\}, \qquad p = \pm1, \pm2, \dots, \quad n = 1,\dots,N, \tag{6.6}$$

for $k = 0, 1, \dots$, and $N \ge 2k+1$; when $p = 0$, the coefficient $(\tilde{b}_n^{Nk}, \hat{e}_0)_0$ has in every case the limiting value for $p \to 0$, or $\sqrt{2\pi}/N$. Therefore, the value of the series in (6.5) can be written

$$\widetilde{I}_s\tilde{b}_n^{Nk}(s) = \frac{1}{N} + \frac{2}{N}\sum_{p=1}^{\infty}\frac{1}{(1+p^2)^{s/2}}\left\{\frac{\sin\frac{p\pi}{N}}{\frac{p\pi}{N}}\right\}^{k+1}\cos p\big(s - \frac{\pi}{N}(2n+k-1)\big), \qquad \text{for} \quad s \in (0,2\pi) \tag{6.7}$$

and for the indicated values of n, k, and N. It is clear that the righthand side of (6.7) gives the periodic extension of $\widetilde{I}_s\tilde{b}_n^{Nk}$ to all of $\mathbb{R}$, and, moreover, for fixed k and N the collection of these periodic extensions of $\{\widetilde{I}_s\tilde{b}_n^{Nk}\}_{n=1}^N$ are all simply translates of, say, the periodic extension of $\widetilde{I}_s\tilde{b}_1^{Nk}$. The latter property can sometimes be exploited to save much labor, especially in the computations involving the orthoprojections $\widetilde{\mathcal{P}}_{Nk}^0\widetilde{I}_s\tilde{b}_n^{Nk}$, which are also mutual translates in the same sense. (Unfortunately, the same circumstance does not obtain in the upcoming Example 6.2.)

In any event, Lemma 2.2 and Proposition 5.2 imply that the family $\left(\left(\widetilde{I}_s\tilde{b}_n^{Nk}\right)_{n=1}^N\right)_{N=2k+1}^{\infty}$ is well conditioned in $H^s_{\rm per}(0,2\pi)$, with the corresponding sequence of condition numbers coinciding with that of the original family in $H^0(0,2\pi)$.

In Figures 2–15 we display graphs of the functions $\tilde{b}_n^{Nk}$, $\widetilde{I}_s\tilde{b}_n^{Nk}$, and $\widetilde{\mathcal{P}}_{Nk}^0\widetilde{I}_s\tilde{b}_n^{Nk}$, with $\widetilde{\mathcal{P}}_{Nk}^0$ denoting the $H^0(a,b)$-orthoprojector onto $\widetilde{\mathcal{M}}_N^k$, for various selected values of s, N, k, and n. There are two figures for each choice of the latter parameters, the first comparing the graphs of $\tilde{b}_n^{Nk}$ and $\widetilde{I}_s\tilde{b}_n^{Nk}$, the second comparing the graphs of $\widetilde{I}_s\tilde{b}_n^{Nk}$ and its purported approximation $\widetilde{\mathcal{P}}_{Nk}^0\widetilde{I}_s\tilde{b}_n^{Nk}$. In the latter figures, the two graphs are practically indistinguishable over most of the interval $[0,2\pi]$; a

magnified view would show that the agreement is poorest near those places where $\widetilde{I}_s \tilde{b}_n^{Nk}$ is changing most rapidly. One should note that the strict "small-support property" of the original coördinate functions is destroyed under the isometric mapping (and under the back-projector), although the transformed functions are, roughly speaking, "small" outside a certain neighborhood of the original support—with that neighborhood expanding with increasing s.

Example 6.2. Spaces $H_0^s(a,b)$. Here, we shall take the interval (a,b) to be of the special form $(-b,b)$ for a given positive number b. In this second example, we identify H_0 as $H^0(-b,b)$ and H_+ as $H_0^1(-b,b)$, the closure in $H^1(-b,b)$ of $C_0^\infty(-b,b)$, the infinitely differentiable complex functions with support in $(-b,b)$. Now the spaces H^s resulting from the construction of Section 3 are denoted for $s>0$ by $H_0^s(-b,b)$; for $s<0$, however, it is customary to write $H^s(-b,b)$ in place of $H_0^s(-b,b)$, and we shall adhere to that notation.

The developments run parallel those of Example 6.1; beginning with the appropriately modified form of (6.1), in place of (6.2) we now come to the requirement

$$\left.\begin{aligned} -\big(J_+^{-1}u\big)'' + \big(J_+^{-1}u\big) &= u, \\ \big(J_+^{-1}u\big)(-b) &= 0, \\ \big(J_+^{-1}u\big)(b) &= 0, \end{aligned}\right\} \qquad \text{for} \quad u \in H^0(-b,b). \tag{6.8}$$

In place of Green's function for the periodic boundary-value problem, now we use Green's function G_0 pertinent to the same ordinary differential operator but with the homogeneous conditions at $\pm b$, which is given by

$$G_0(t_1,t_2) = \frac{\sinh(b+t_-)\sinh(b-t_+)}{2\sinh b\cosh b} \qquad \text{for} \quad -b \le t_1, t_2 \le b, \tag{6.9}$$

with the notation $t_+ := \max\{t_1,t_2\}$ and $t_- := \min\{t_1,t_2\}$. In place of (6.3) we now get

$$J_+^{-1}u = \int_{-b}^{b} G_0(t,\tau)u(\tau)\,d\lambda_1(\tau) \qquad \text{for every} \quad u \in H^0(-b,b). \tag{6.10}$$

In this case, we find that a complete set of eigenpairs of J_+^{-1} is $\left\{\left(\frac{1}{1+(p\pi/2b)^2}, \hat{\epsilon}_p\right)\right\}_{p=1}^{\infty}$, with the eigenfunctions $\hat{\epsilon}_p$, normalized in $H^0(-b,b)$, given by

$$\hat{\epsilon}_p(s) := \frac{1}{\sqrt{b}} \sin\frac{p\pi}{2b}(s+b), \qquad p = 1,\,2,\,\ldots\,. \tag{6.11}$$

Now, we require the images $I_s b_n^{Nk}$ of the coördinate functions b_n^{Nk} defined in Section 5 under the isometry I_s of (3.7). We get

$$I_s b_n^{Nk} = \sum_{p=1}^{\infty} \frac{1}{\{1+(p\pi/2b)^2\}^{s/2}} (b_n^{Nk}, \hat{\epsilon}_p)_0 \hat{\epsilon}_p \qquad \text{for} \quad n = 1,\ldots,N-k, \quad N \ge k+1, \quad k = 1,2,\ldots, \tag{6.12}$$

and a short computation produces the Fourier coefficients of the coördinate functions as

$$\big(b_n^{Nk}, \hat{\epsilon}_p\big)_0 = \frac{h_N}{\sqrt{b}} \left\{ \frac{\sin\frac{p\pi}{2N}}{\frac{p\pi}{2N}} \right\}^{k+1} \sin\frac{p\pi}{2N}(2n+k-1), \qquad p = 1,\,2,\,\ldots, \quad n = 1,\ldots,N-k, \tag{6.13}$$

for $k = 0, 1, \ldots$, and $N \geq k+1$.

Now, on the basis of Lemma 2.2 and Proposition 5.1 we can conclude that the coördinate family $\left(\left(I_s b_n^{Nk} \right)_{n=1}^{N} \right)_{N=k+1}^{\infty}$ is well conditioned in $H_0^s(-b,b)$ if $s > 0$ or well conditioned in $H^s(-b,b)$ if $s < 0$ (and, again, the corresponding sequence of condition numbers coincides with that of the original family in $H^0(-b,b)$).

In Figures 16–27 we display graphs of the functions b_n^{Nk}, $I_s b_n^{Nk}$, and $\mathcal{P}_{Nk}^0 I_s b_n^{Nk}$ (with $\mathcal{P}_{Nk}^0$ denoting the $H^0(a,b)$-orthoprojector onto $\mathcal{M}_N^k$), for various selected values of s, N, k, and n. There are two figures for each choice of the latter parameters, the first comparing the graphs of b_n^{Nk} and $I_s b_n^{Nk}$, the second comparing the graphs of $I_s b_n^{Nk}$ and its purported approximation $\mathcal{P}_{Nk}^0 I_s b_n^{Nk}$. Here we can make comments on the agreement of the graphs in the latter figures and the support properties of the $I_s b_n^{Nk}$ and the $\mathcal{P}_{Nk}^0 I_s b_n^{Nk}$ that are entirely analogous to the observations in the first example. Now, however, the isometric images $I_s b_n^{Nk}$ are not at all mutual translates, a circumstance making more expensive the computation of the coefficients needed for construction of the projections $\mathcal{P}_{Nk}^0 I_s b_n^{Nk}$. In [6] we describe how the latter numbers can be generated in a fairly efficient manner.

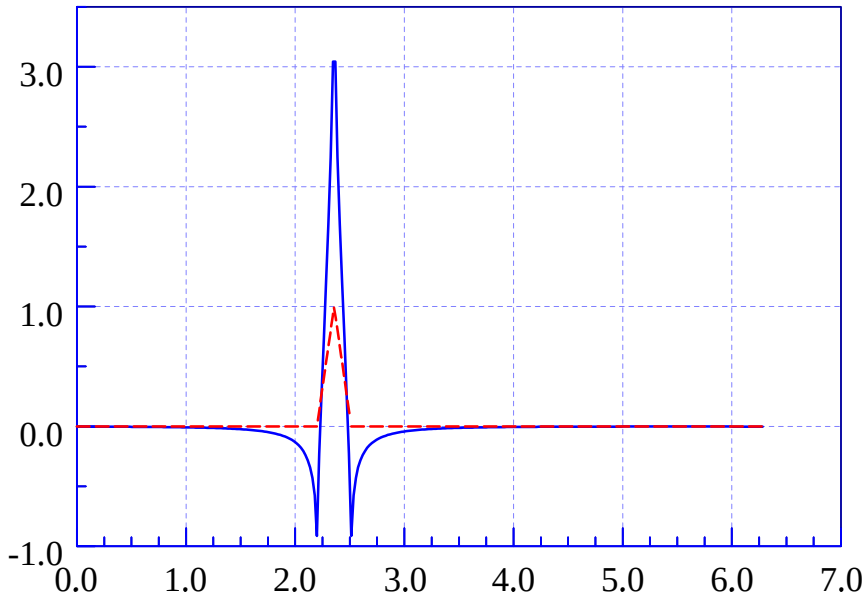

Fig. 2. Graphs of $\tilde{b}_n^{Nk}$ (dotted) and $\widetilde{I}_s\tilde{b}_n^{Nk}$ for $s=-1/2$, $k=1$, $N=40$, and $n=15$.

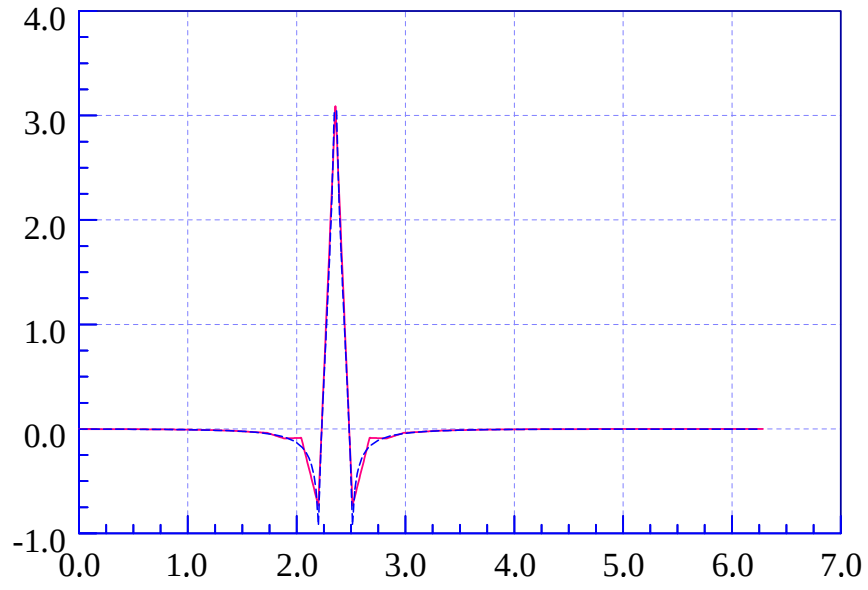

Fig. 3. Graphs of $\widetilde{I}_s\tilde{b}_n^{Nk}$ (dotted) and $\widetilde{\mathcal{P}}_{Nk}^0\widetilde{I}_s\tilde{b}_n^{Nk}$ for $s=-1/2$, $k=1$, $N=40$, and $n=15$.

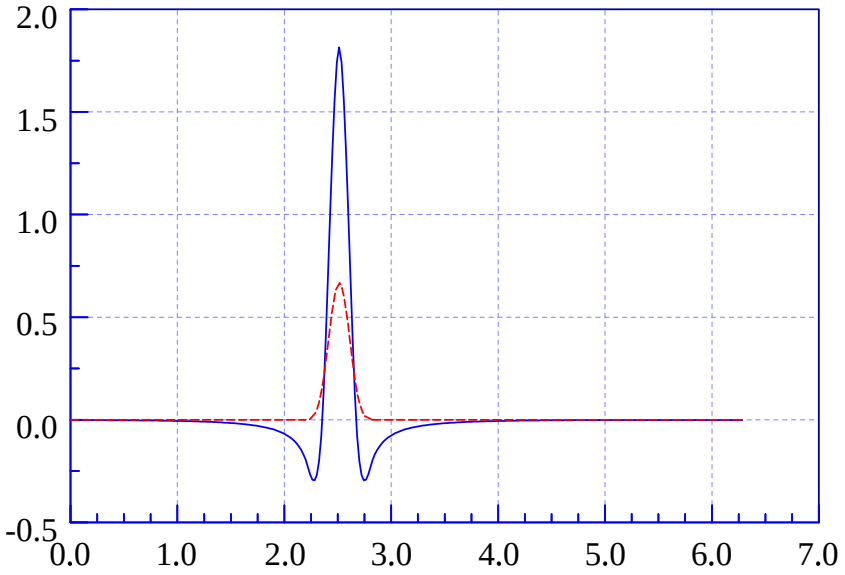

Fig. 4. Graphs of $\tilde{b}_n^{Nk}$ (dotted) and $\widetilde{I}_s\tilde{b}_n^{Nk}$ for $s=-1/2$, $k=3$, $N=40$, and $n=15$.

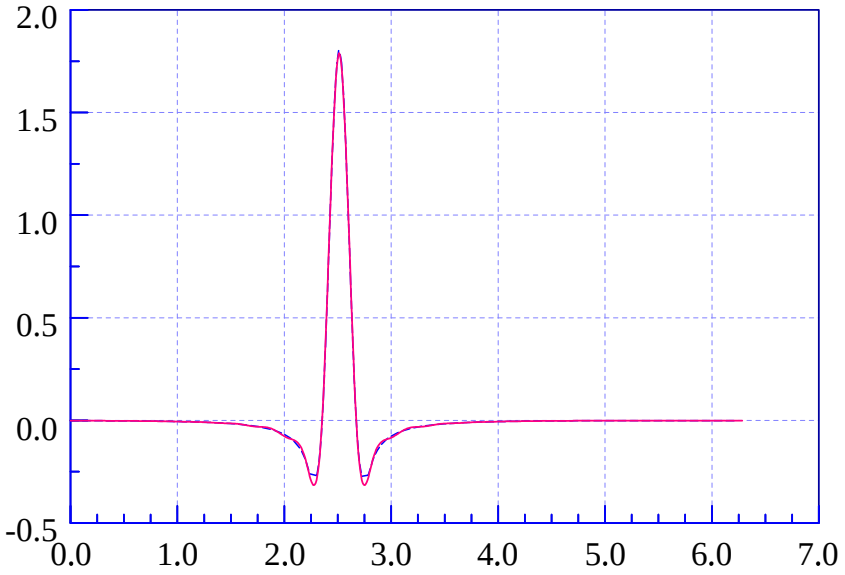

Fig. 5. Graphs of $\widetilde{I}_s\tilde{b}_n^{Nk}$ (dotted) and $\widetilde{\mathcal{P}}_{Nk}^0\widetilde{I}_s\tilde{b}_n^{Nk}$ for $s=-1/2$, $k=3$, $N=40$, and $n=15$.

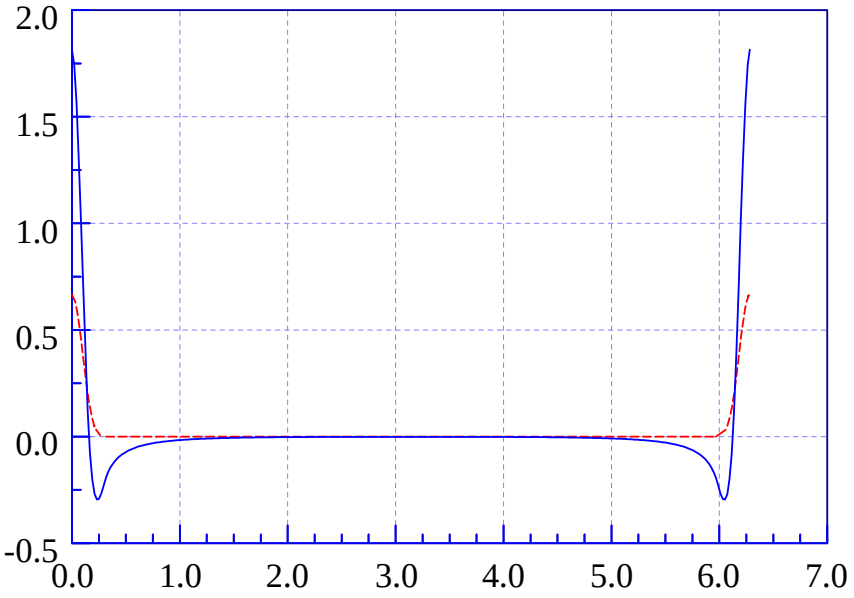

Fig. 6. Graphs of $\tilde{b}_n^{Nk}$ (dotted) and $\widetilde{I}_s\tilde{b}_n^{Nk}$ for $s=-1/2$, $k=3$, $N=40$, and $n=39$.

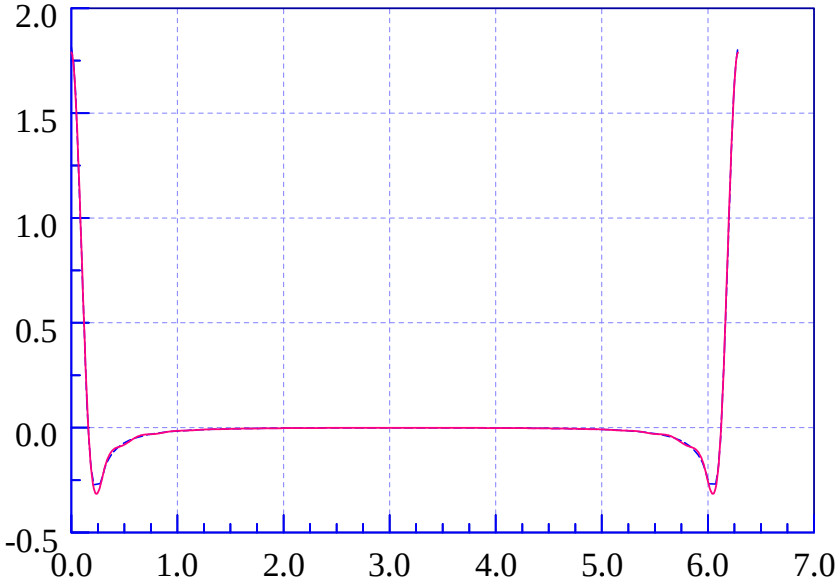

Fig. 7. Graphs of $\widetilde{I}_s\tilde{b}_n^{Nk}$ (dotted) and $\widetilde{\mathcal{P}}_{Nk}^0\widetilde{I}_s\tilde{b}_n^{Nk}$ for $s=-1/2$, $k=3$, $N=40$, and $n=39$.

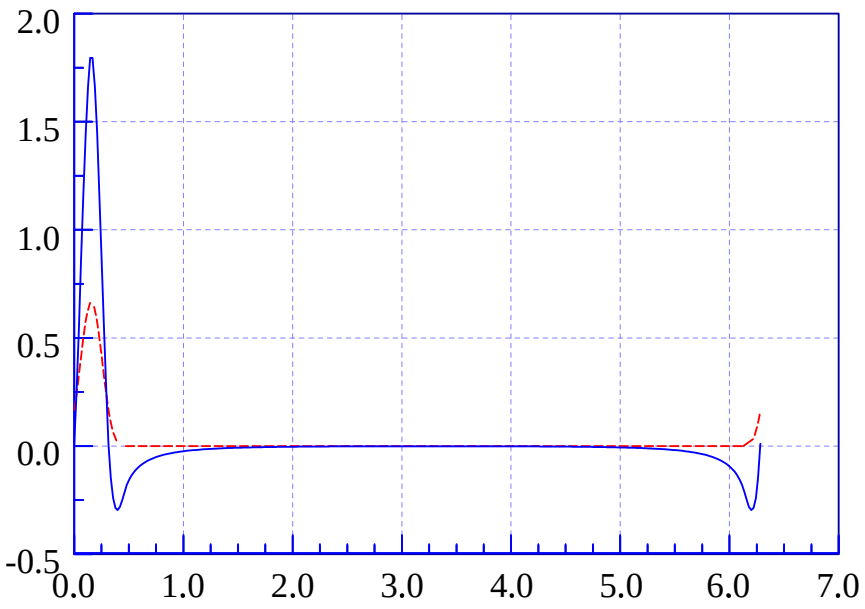

Fig. 8. Graphs of $\tilde{b}_n^{Nk}$ (dotted) and $\widetilde{I}_s\tilde{b}_n^{Nk}$ for $s=-1/2$, $k=3$, $N=40$, and $n=40$.

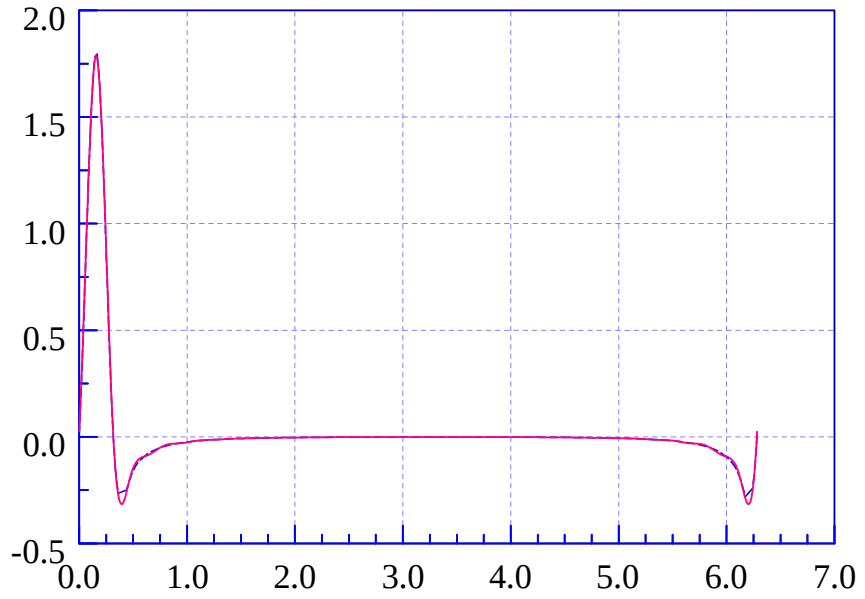

Fig. 9. Graphs of $\widetilde{I}_s\tilde{b}_n^{Nk}$ (dotted) and $\widetilde{\mathcal{P}}_{Nk}^0\widetilde{I}_s\tilde{b}_n^{Nk}$ for $s=-1/2$, $k=3$, $N=40$, and $n=40$.

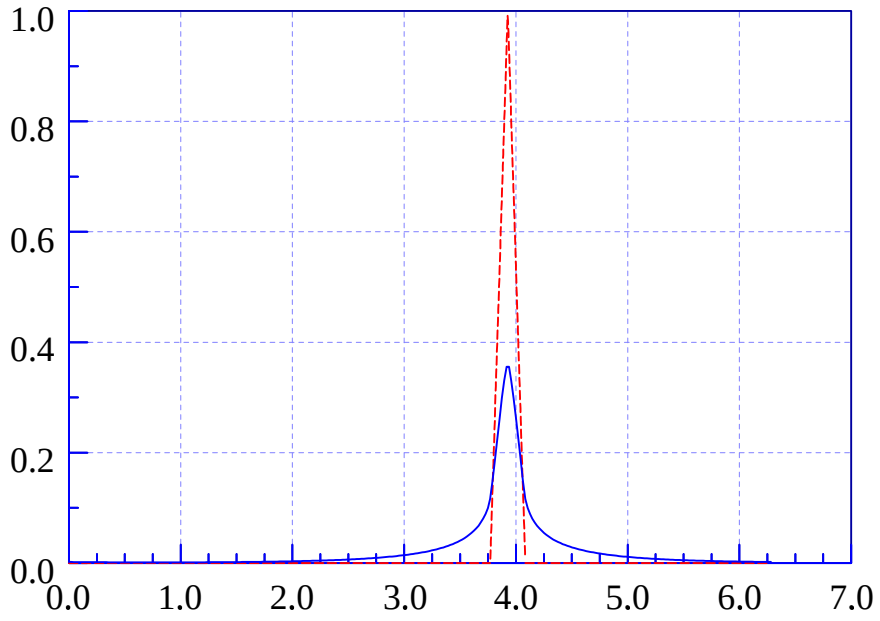

Fig. 10. Graphs of $\tilde{b}_n^{Nk}$ (dotted) and $\widetilde{I}_s\tilde{b}_n^{Nk}$ for $s = 1/2$, $k = 1$, $N = 40$, and $n = 25$.

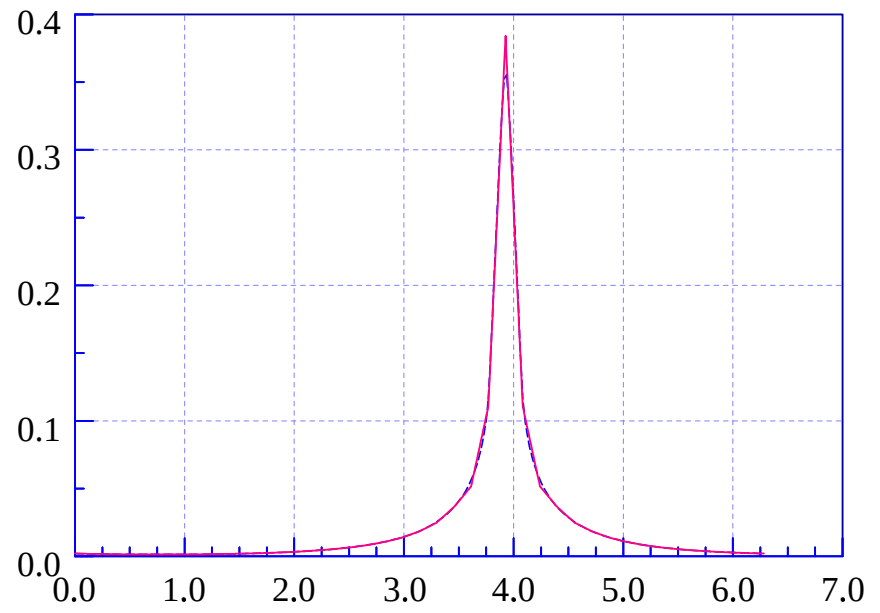

Fig. 11. Graphs of $\widetilde{I}_s\tilde{b}_n^{Nk}$ (dotted) and $\widetilde{\mathcal{P}}_{Nk}^0\widetilde{I}_s\tilde{b}_n^{Nk}$ for $s = 1/2$, $k = 1$, $N = 40$, and $n = 25$.

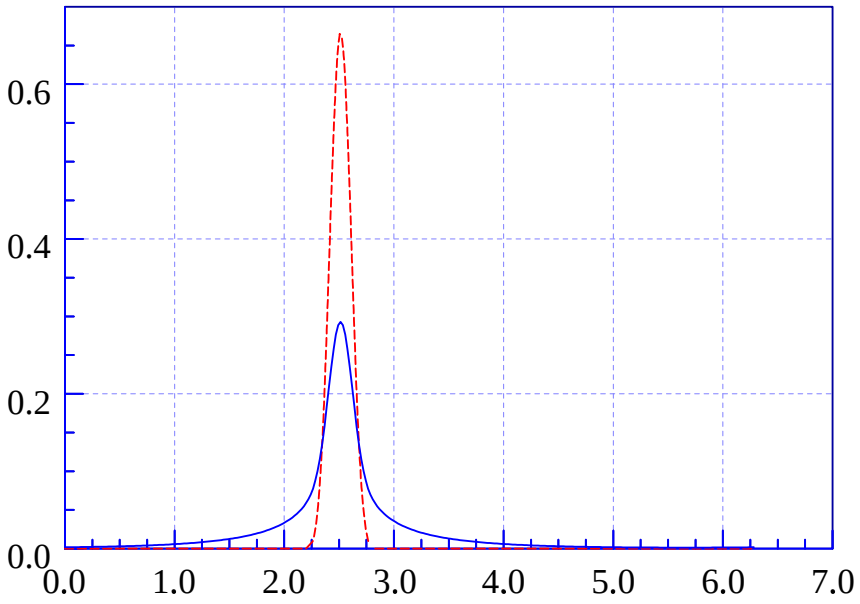

Fig. 12. Graphs of $\tilde{b}_n^{Nk}$ (dotted) and $\widetilde{I}_s\tilde{b}_n^{Nk}$ for $s = 1/2$, $k = 3$, $N = 40$, and $n = 15$.

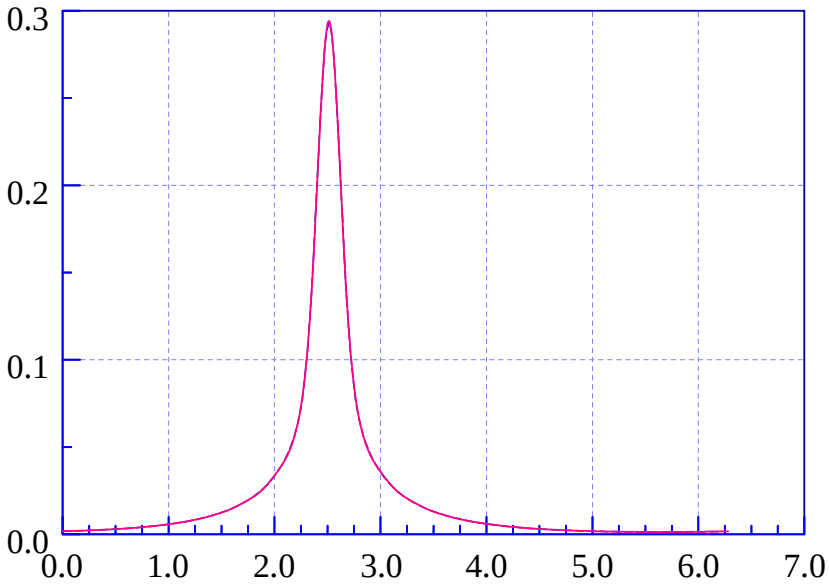

Fig. 13. Graphs of $\widetilde{I}_s\tilde{b}_n^{Nk}$ (dotted) and $\widetilde{\mathcal{P}}_{Nk}^0\widetilde{I}_s\tilde{b}_n^{Nk}$ for $s = 1/2$, $k = 3$, $N = 40$, and $n = 15$.

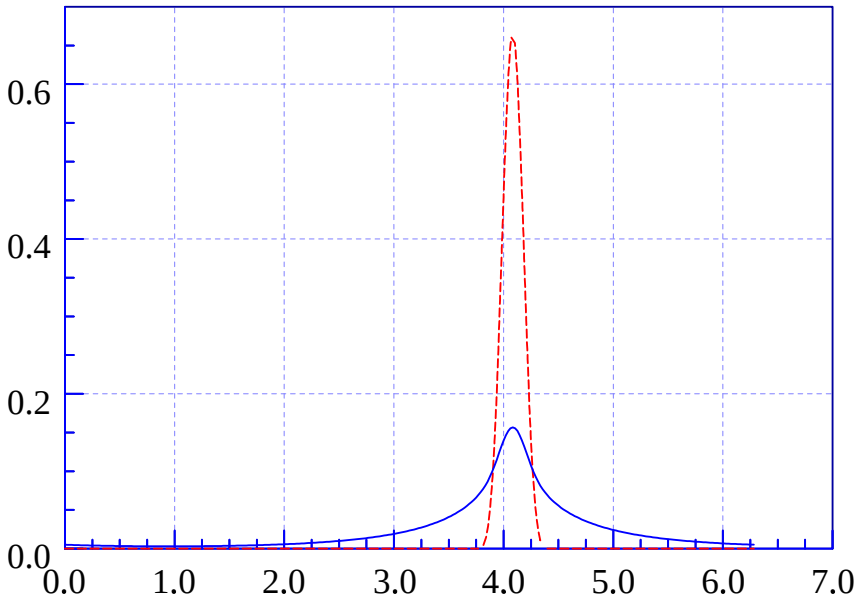

Fig. 14. Graphs of $\tilde{b}_n^{Nk}$ (dotted) and $\widetilde{I}_s\tilde{b}_n^{Nk}$ for $s = 1$, $k = 3$, $N = 40$, and $n = 25$.

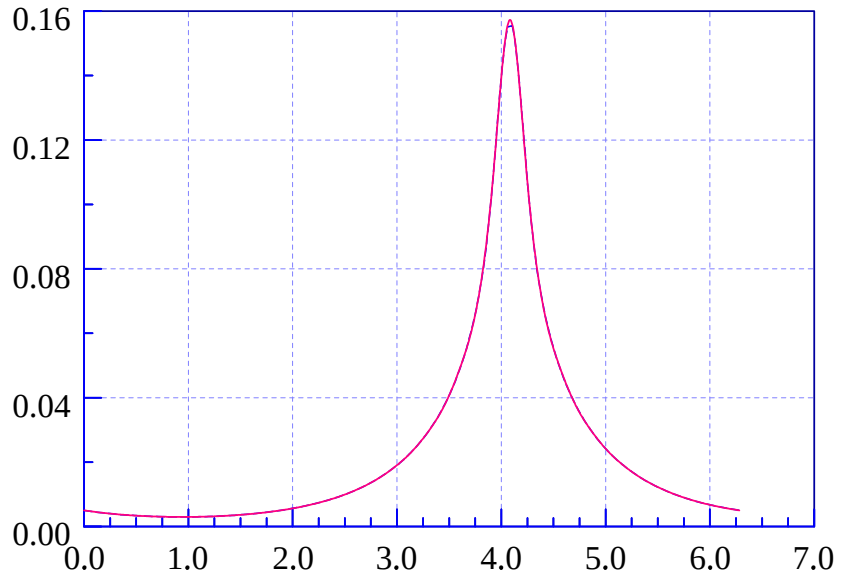

Fig. 15. Graphs of $\widetilde{I}_s\tilde{b}_n^{Nk}$ (dotted) and $\widetilde{\mathcal{P}}_{Nk}^0\widetilde{I}_s\tilde{b}_n^{Nk}$ for $s = 1$, $k = 3$, $N = 40$, and $n = 25$.

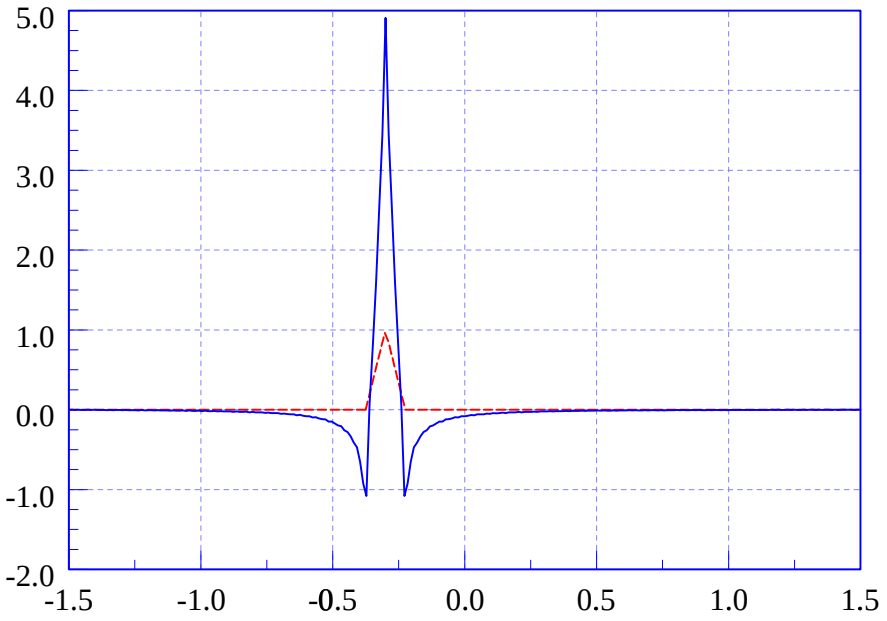

Fig. 16. Graphs of b_n^{Nk} (dotted) and $I_s b_n^{Nk}$ for $s = -1/2$, $k = 1$, $N = 40$, and $n = 16$.

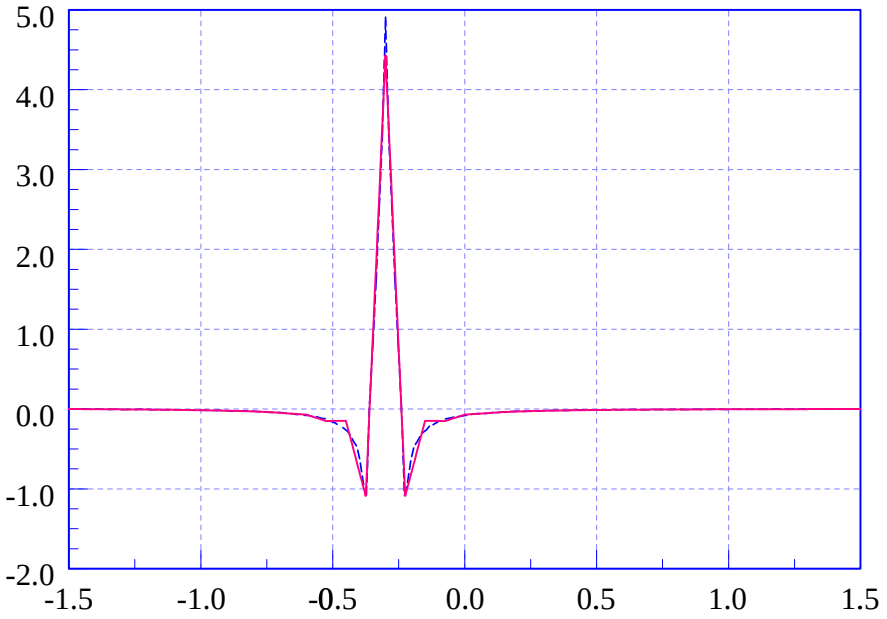

Fig. 17. Graphs of $I_s b_n^{Nk}$ (dotted) and $\mathcal{P}_{Nk}^0 I_s b_n^{Nk}$ for $s = -1/2$, $k = 1$, $N = 40$, and $n = 16$.

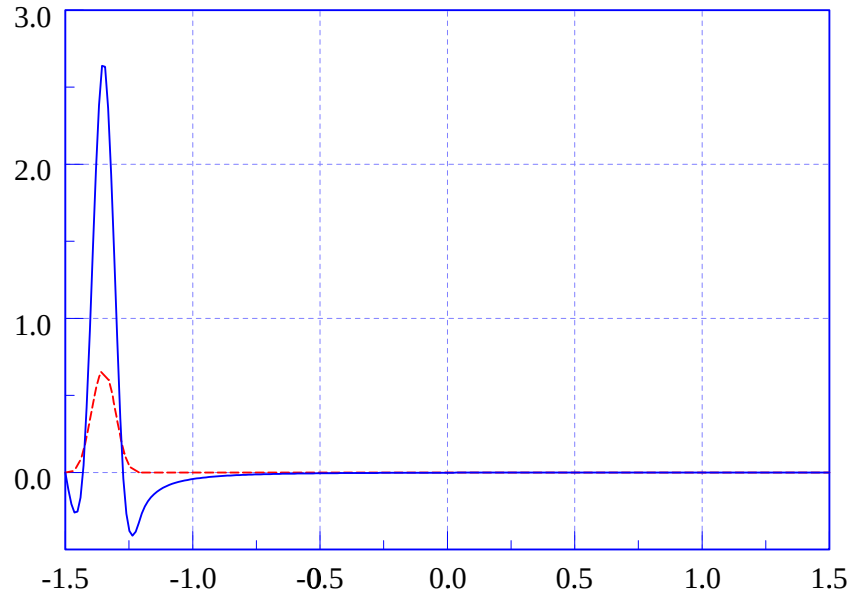

Fig. 18. Graphs of b_n^{Nk} (dotted) and $I_s b_n^{Nk}$ for $s = -1/2$, $k = 3$, $N = 40$, and $n = 1$.

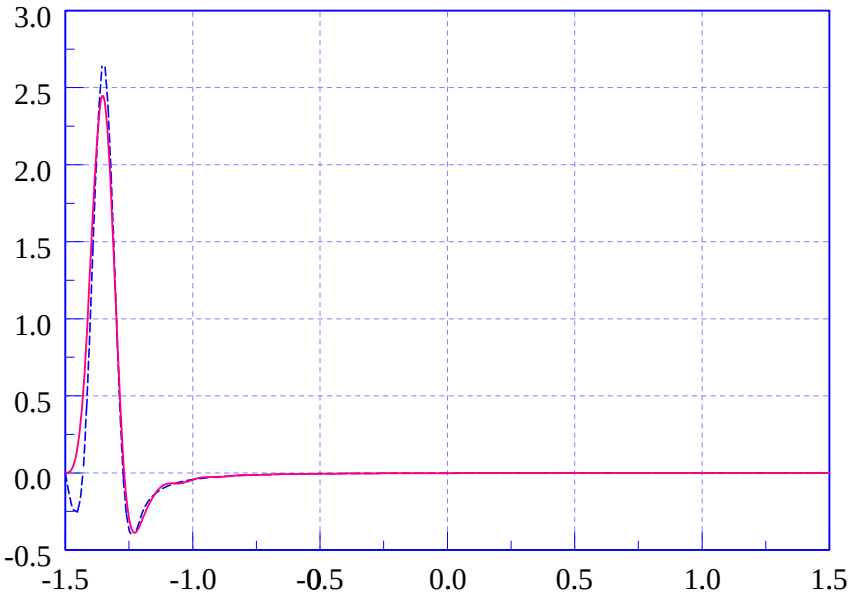

Fig. 19. Graphs of $I_s b_n^{Nk}$ (dotted) and $\mathcal{P}_{Nk}^0 I_s b_n^{Nk}$ for $s = -1/2$, $k = 3$, $N = 40$, and $n = 1$.

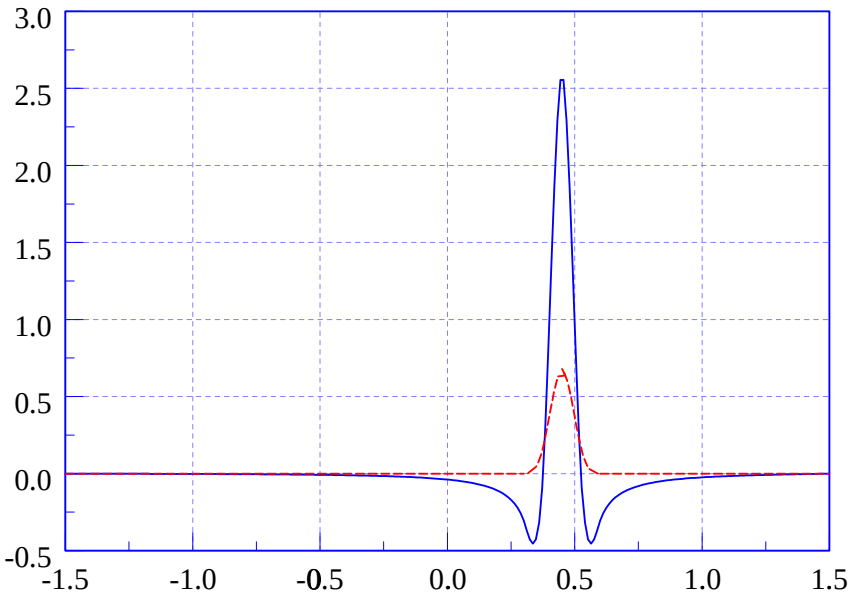

Fig. 20. Graphs of b_n^{Nk} (dotted) and $I_s b_n^{Nk}$ for $s = -1/2$, $k = 3$, $N = 40$, and $n = 25$.

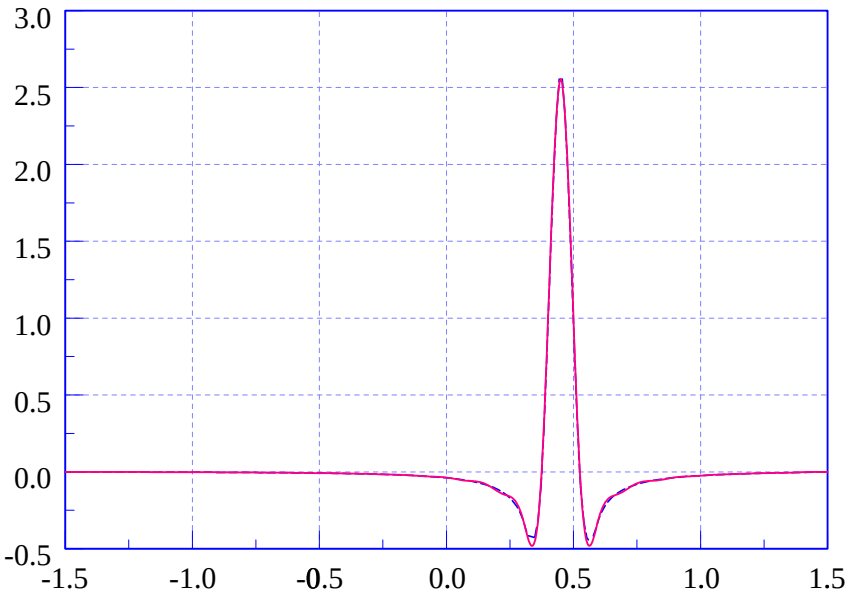

Fig. 21. Graphs of $I_s b_n^{Nk}$ (dotted) and $\mathcal{P}_{Nk}^0 I_s b_n^{Nk}$ for $s = -1/2$, $k = 3$, $N = 40$, and $n = 25$.

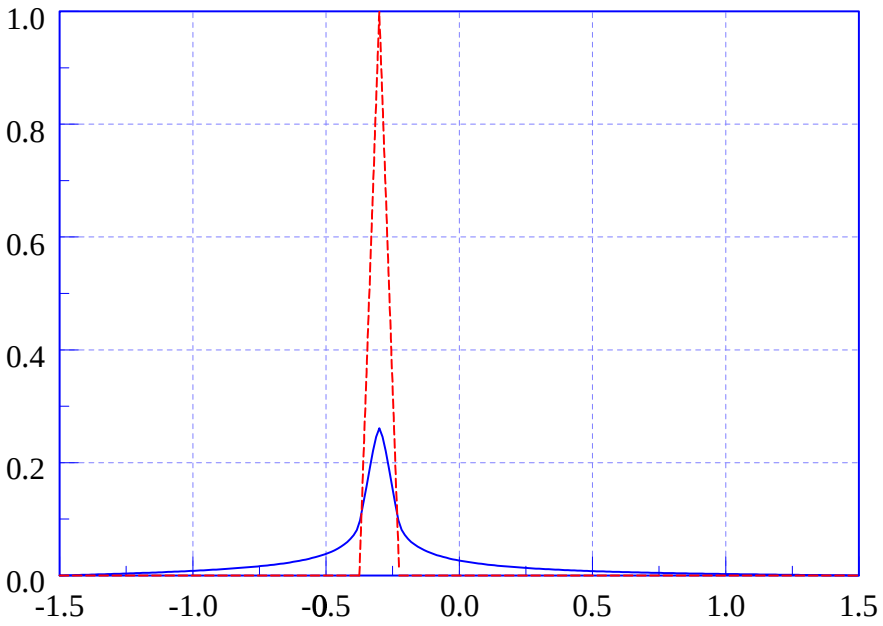

Fig. 22. Graphs of b_n^{Nk} (dotted) and $I_s b_n^{Nk}$ for $s = 1/2$, $k = 1$, $N = 40$, and $n = 16$.

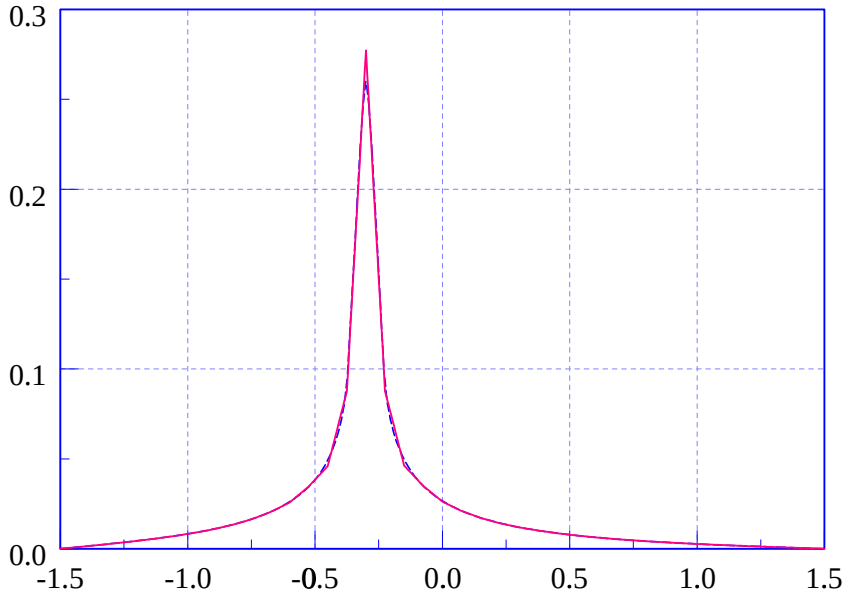

Fig. 23. Graphs of $I_s b_n^{Nk}$ (dotted) and $\mathcal{P}_{Nk}^0 I_s b_n^{Nk}$ for $s = 1/2$, $k = 1$, $N = 40$, and $n = 16$.

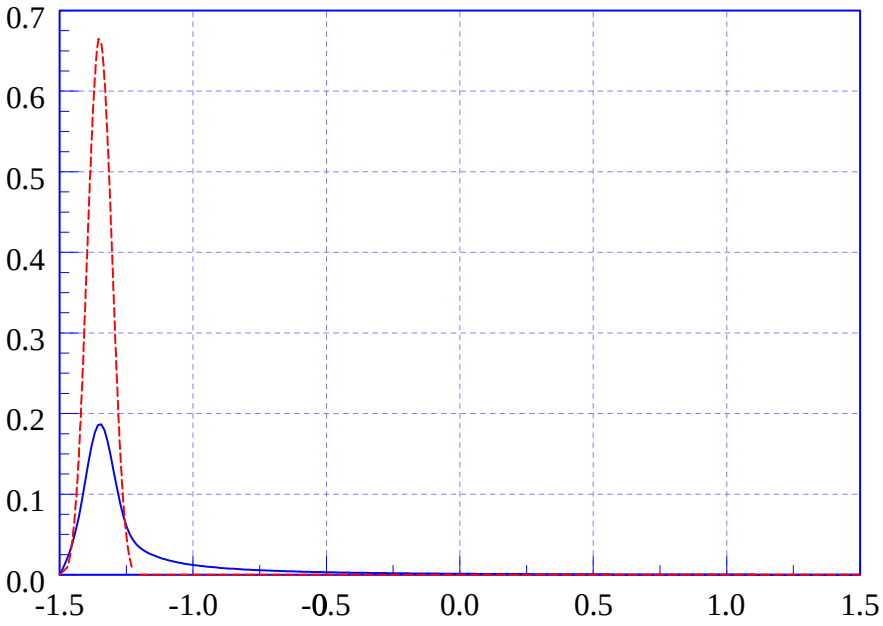

Fig. 24. Graphs of b_n^{Nk} (dotted) and $I_s b_n^{Nk}$ for $s = 1/2$, $k = 3$, $N = 40$, and $n = 1$.

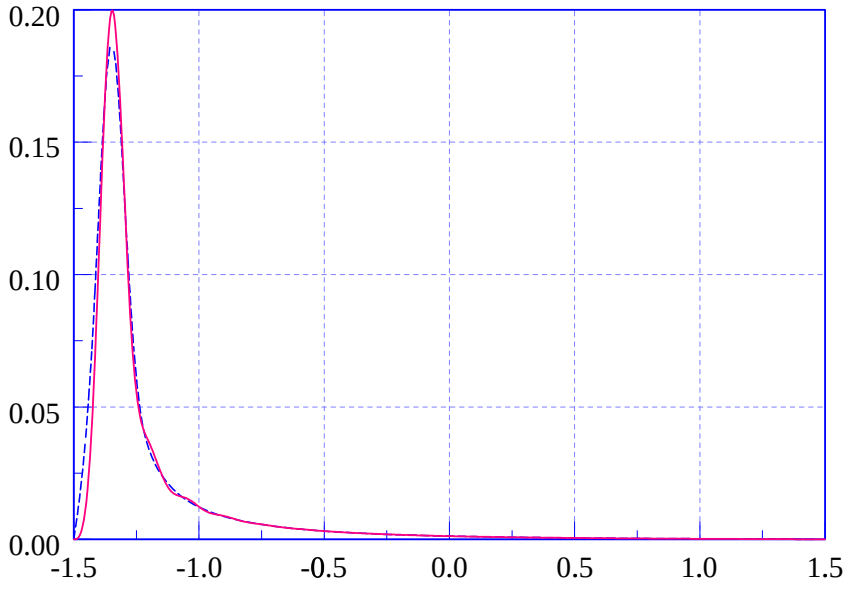

Fig. 25. Graphs of $I_s b_n^{Nk}$ (dotted) and $\mathcal{P}_{Nk}^0 I_s b_n^{Nk}$ for $s = 1/2$, $k = 3$, $N = 40$, and $n = 1$.

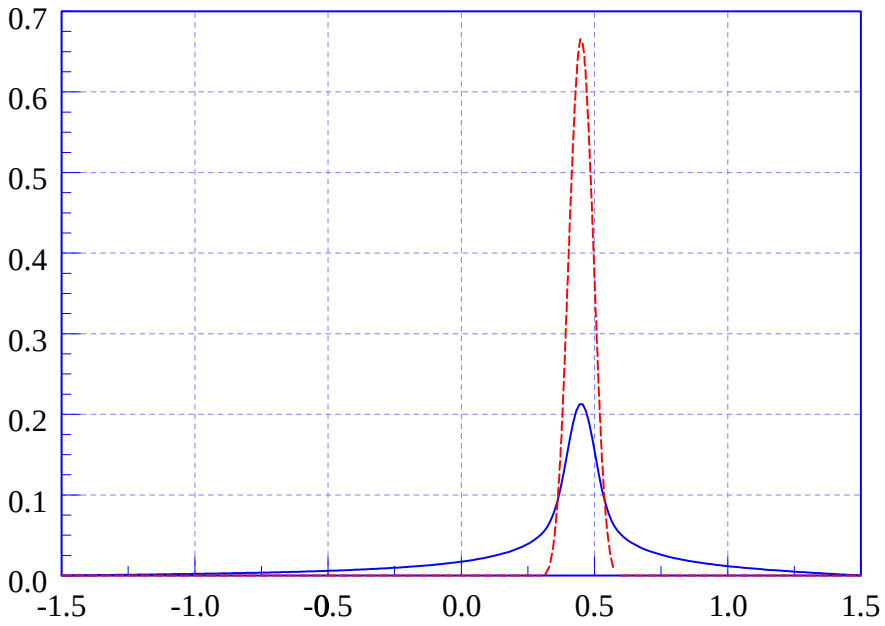

Fig. 26. Graphs of b_n^{Nk} (dotted) and $I_s b_n^{Nk}$ for $s = 1/2$, $k = 3$, $N = 40$, and $n = 25$.

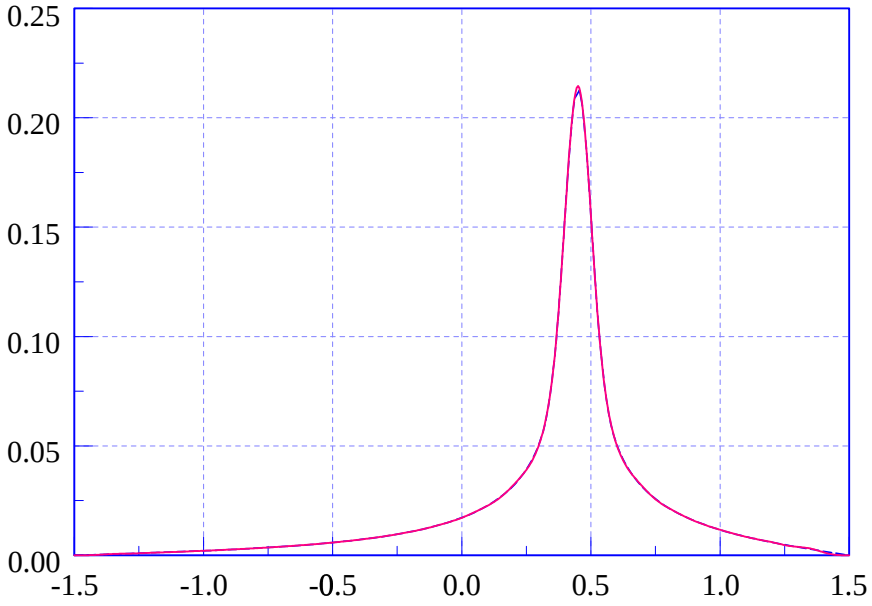

Fig. 27. Graphs of $I_s b_n^{Nk}$ (dotted) and $\mathcal{P}_{Nk}^0 I_s b_n^{Nk}$ for $s = 1/2$, $k = 3$, $N = 40$, and $n = 25$.

7. Numerical implementations in Galerkin procedures: examples.

Now we show in two examples the numerical stabilizing effect that can be achieved by implementing the constructions described in the preceding sections. More precisely, we present numerical results on the approximate solutions of a single boundary-operator problem in two dimensions that have been obtained by two different numerical schemes. The operator problem is a classical first-kind integral equation, involving an integral operator that is compact in the usual L_2-space. Thus, one expects that a "usual" application of the Galerkin method, using coördinate functions well conditioned in the L_2-sense, will result at best in a numerical instability; the computational results bear out this expectation, in each of the two cases. However, the results also show that numerical stability appears to result from the "preconditioning" of the Galerkin procedures that is achieved by using our construction of coördinate functions and accounting for the orders of the Sobolev spaces between which the underlying operator is an isomorphism.

Let Ω denote a bounded, simply connected domain in $\mathbb{R}^2$, with boundary $\Gamma := \partial\Omega$ a smooth, simple curve of class C^2; in the applications, for simplicity in the parametric descriptions, we have also required that Γ be starlike with respect to some point in Ω. By $H^s(\Gamma)$ we indicate the Sobolev space of order s associated with Γ, constructed, say, as in Section 3 on the basis of the compact imbedding of $H^1(\Gamma)$ in $H^0(\Gamma) \equiv L_2(\Gamma)$.

We introduce an integral operator $S_\kappa : H^0(\Gamma) \to H^0(\Gamma)$, acting first in $H^0(\Gamma)$, according to

$$S_\kappa u(\xi) := \frac{\mathrm{i}}{4} \int_\Gamma H_0^{(1)}\big(\kappa|\xi - \zeta|\big)\, u(\zeta)\, d\lambda_\Gamma(\zeta) \qquad \text{for} \quad \text{a.a. } \xi \in \Gamma, \quad \text{for each} \quad u \in H^0(\Gamma);$$

here, the "wavenumber" κ is a given positive number and $H_0^{(1)}$ denotes, as usual, the Hankel function of first kind and order zero. The operator S_κ is the "direct-value operator" corresponding to the single-layer potential based on Γ for the Helmholtz equation in two dimensions. When regarded as $S_\kappa : H^0(\Gamma) \to H^0(\Gamma)$, this operator is compact; however, it is shown in [9] that S_κ maps $H^0(\Gamma)$ into $H^{\frac{1}{2}}(\Gamma)$ (in fact, into $H^1(\Gamma)$) and is bounded when regarded as the densely defined operator $S_\kappa : \big\{H^0(\Gamma) \subset H^{-\frac{1}{2}}(\Gamma)\big\} \to H^{\frac{1}{2}}(\Gamma)$; the bounded extension of the latter we shall denote again simply by $S_\kappa : H^{-\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma)$. Moreover, provided that the square κ^2 of the wavenumber is not a Dirichlet eigenvalue for the negative Laplacian in Ω, it is also shown that $S_\kappa : H^{-\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma)$ is an isomorphism, so that the basic operator problem is well posed:

$$\textit{given} \quad g \in H^{\frac{1}{2}}(\Gamma), \quad \textit{determine} \quad u_g \in H^{-\frac{1}{2}}(\Gamma) \quad \textit{such that} \qquad S_\kappa u_g = g. \qquad (S_\kappa \mathrm{P})$$

We shall suppose that κ satisfies the indicated restriction (although the condition essentially limits the usefulness of the present formulation in acoustics studies to "small" values of κ^2).

When $g = v^\iota|_\Gamma$, the trace on Γ of an "incident-field" velocity potential v^ι, *i.e.*, a function satisfying the Helmholtz equation $\Delta v^\iota + \kappa^2 v^\iota = 0$ in an open set containing the closure of Ω, then the corresponding unique solution u_g of $(S_\kappa \mathrm{P})$ has a physical interpretation as the normal derivative on Γ of the ("total") acoustic field in the exterior of Ω. (Once this normal derivative is known, it can be used as the density of an exterior single-layer potential to construct the scattered field—and therefore also the total acoustic field—in the exterior region.)

While we are presently concerned mainly with the condition numbers of the matrices produced in various implementations of the Galerkin procedure for problem $(S_\kappa \mathrm{P})$, and not the solution for any one choice of g, we have validated our codes by solving particular problems using each of the approaches indicated below and confirming that the results agree (to within error). In those

applications, we have taken the incident field to be a plane wave propagating in a direction that is set as input for the numerical computation. Thus, we have put

$$v^\iota(\xi) = e^{\mathrm{i}\kappa\hat{\mathrm{e}}\cdot\xi}, \qquad \xi \in \mathbb{R}^2,$$

in which the unit vector $\hat{\mathrm{e}}$ is chosen data. For comparison purposes, we provide below plots of the approximations to the respective solutions in those cases, *i.e.*, plots of the approximations to the corresponding (real and imaginary parts of the) normal derivatives of the total fields, for two regions Ω and various incident directions $\hat{\mathrm{e}}$.

Example 7.1. In this first implementation of the Galerkin procedure for $(S_\kappa \mathrm{P})$, we assume for definiteness that the plane boundary curve Γ is starlike with respect to a point $\mathcal{O}$, and introduce, in the usual manner, a parametric description of Γ that is given by a smooth, 2π-periodic mapping $X_\Gamma : \mathbb{R} \to \Gamma$ of class C^2 carrying $[0, 2\pi)$ bijectively onto Γ, employing as parameter the polar angle measured from some ray emanating from $\mathcal{O}$. With this parametrization, the problem $(S_\kappa \mathrm{P})$ can be transferred from its original setting in the spaces $H^{-\frac{1}{2}}(\Gamma)$, $H^{\frac{1}{2}}(\Gamma)$ down to the space $H^{-\frac{1}{2}}_{\mathrm{per}}(0, 2\pi)$, where we choose to cast it in the form

$$\textit{given} \quad g \in H^{\frac{1}{2}}(\Gamma), \quad \textit{determine} \quad \psi_g \in H^{-\frac{1}{2}}_{\mathrm{per}}(0, 2\pi) \quad \textit{such that} \qquad J_{\frac{1}{2}} \mathcal{S}_\kappa \psi_g = J_{\frac{1}{2}}(g \circ X_\Gamma), \quad (\mathcal{S}_\kappa \mathrm{P})$$

in which $J_{\frac{1}{2}} : H^{\frac{1}{2}}_{\mathrm{per}}(0, 2\pi) \to H^{-\frac{1}{2}}_{\mathrm{per}}(0, 2\pi)$ denotes the antiduality operator (an isometric isomorphism) and the isomorphism $\mathcal{S}_\kappa$ carrying $H^{-\frac{1}{2}}_{\mathrm{per}}(0, 2\pi)$ onto $H^{\frac{1}{2}}_{\mathrm{per}}(0, 2\pi)$ is related to S_κ through the parametrization-map X_Γ in the usual way. That is, *e.g.*, on $H^0(0, 2\pi)$ we find

$$\mathcal{S}_\kappa \psi(\vartheta) := \frac{\mathrm{i}}{4} \int_0^{2\pi} H_0^{(1)}\Big(\kappa \big| X_\Gamma(\vartheta) - X_\Gamma(\varphi)\big|\Big) \big| X_\Gamma'(\varphi)\big| \psi(\varphi)\, d\varphi, \quad \text{a.a. } \vartheta \in (0, 2\pi), \quad \text{for } \psi \in H^0(0, 2\pi).$$

Obviously, the solution u_g of the original problem $(S_\kappa \mathrm{P})$ is to be recovered from ψ_g by $u_g := \psi_g \circ X_\Gamma^{-1}$. The operator $J_{\frac{1}{2}}$ has been introduced so that the $H^{-\frac{1}{2}}_{\mathrm{per}}(0, 2\pi)$-inner products in the Galerkin procedure can be written in terms of the antiduality pairing on $H^{-\frac{1}{2}}_{\mathrm{per}}(0, 2\pi) \times H^{\frac{1}{2}}_{\mathrm{per}}(0, 2\pi)$, and so also in terms of the inner product in $H^0(0, 2\pi)$, in view of the regularity of the coördinate functions here.

Since our new problem $(\mathcal{S}_\kappa \mathrm{P})$ is posed in the single space $H^{-\frac{1}{2}}_{\mathrm{per}}(0, 2\pi)$, we can formulate a Bubnov-Galerkin method, in which the trial- and test-families are taken to be identical; the results of [9] imply that the appropriate form of (2.2) holds for the operator $(L =)\ J_{\frac{1}{2}} \mathcal{S}_\kappa$, so we need only note further that the coördinate subspaces here will have the property of ultimate denseness in $H^{-\frac{1}{2}}_{\mathrm{per}}(0, 2\pi)$. Now we can appeal to the construction of Section 6 to generate coördinate functions that appear to be well conditioned in $H^{-\frac{1}{2}}_{\mathrm{per}}(0, 2\pi)$. On the basis of (2.7), we would then anticipate observing numerical stability in the resultant procedure set up as in (2.3), (2.4).

We have executed such computations for various regions Ω and values of k determining the smoothness of the coördinate functions. Numerical results on the condition numbers of the Galerkin matrices are displayed in Figures 28–32, when Ω is a circular disc, a 10:1 ellipse, and a "kite," or "boomerang." In each case, a is a characteristic dimension of the shape, while the condition number is not the ℓ_2-number, but that computed by the routine supplied in the well-known linear-algebra package LINPACK. The plots clearly show the numerical instability associated with the use of the "unmodified" coördinate family $\Big(\big(\tilde{b}_n^{Nk}\big)_{n=1}^{N}\Big)_{N=2k+1}^{\infty}$, along with the significantly reduced condition

numbers found by employing instead the "new" coördinate functions $\left(\left(\widetilde{\mathcal{P}}^0_{Nk}\widetilde{I}_{-\frac{1}{2}}\tilde{b}^{Nk}_n\right)_{n=1}^{N}\right)_{N=2k+1}^{\infty}$, although we cannot now assert unequivocally that this latter behavior is true numerical stability.

Finally, in Figures 33–40 we show the real and imaginary parts of the normal derivative of the total acoustic field on the "soft" obstacle boundary corresponding to plane waves incident from various directions indicated by the value Θ (and the small schematic given with each plot). Here, both Θ and the angular coördinate θ (determining position on the boundary) are measured positive in the standard manner, counterclockwise from the positive axis of abscissas. In the final plot, of Figure 40, the frequency parameter κa has the value 10.

Example 7.2. This second implementation employs the nonstandard parametrization and reformulation described in [6] and leads instead to a new problem posed in the product space $H^{-\frac{1}{2}}(-b,b)^2$, with $H^{-\frac{1}{2}}(-b,b)$ denoting the antidual space of $H_0^{\frac{1}{2}}(-b,b)$; here, b denotes an appropriate positive number. Consequently, to produce coördinate functions we can rely again on the constructions of Section 6, but now using those set up for the space $H_0^s(-b,b)$ and its antidual $H^{-s}(-b,b)$ (with $s>0$). We have also implemented this second approach numerically; the resulting condition-number plots are contained in [6]. The results are qualitatively the same as those already shown here for Example 7.1, *i.e.*, the numerical instabilities associated with the use of the families $\left(\left(b^{Nk}_n\right)_{n=1}^{N-k}\right)_{N=k+1}^{\infty}$ are again pronounced, while the "preconditioned" coördinate functions $\left(\left(\mathcal{P}^0_{Nk}I_{-\frac{1}{2}}b^{Nk}_n\right)_{n=1}^{N-k}\right)_{N=2k+1}^{\infty}$ lead to marked reductions in the condition numbers, appearing once more to signal numerical stability.

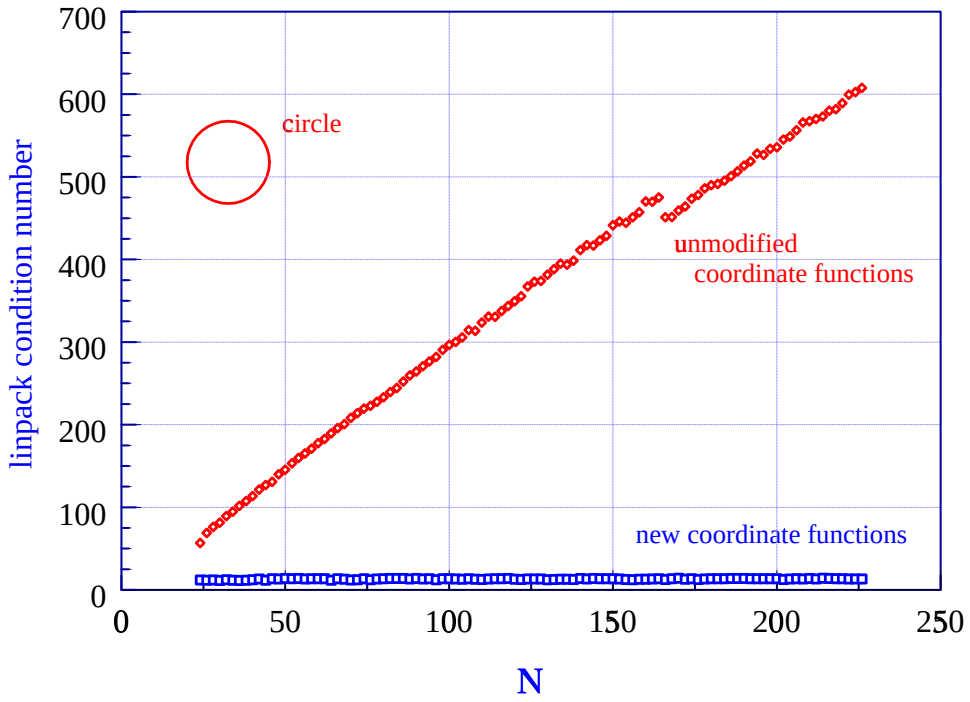

Fig. 28. circle: $\kappa a = 3.0$, $k = 1$.

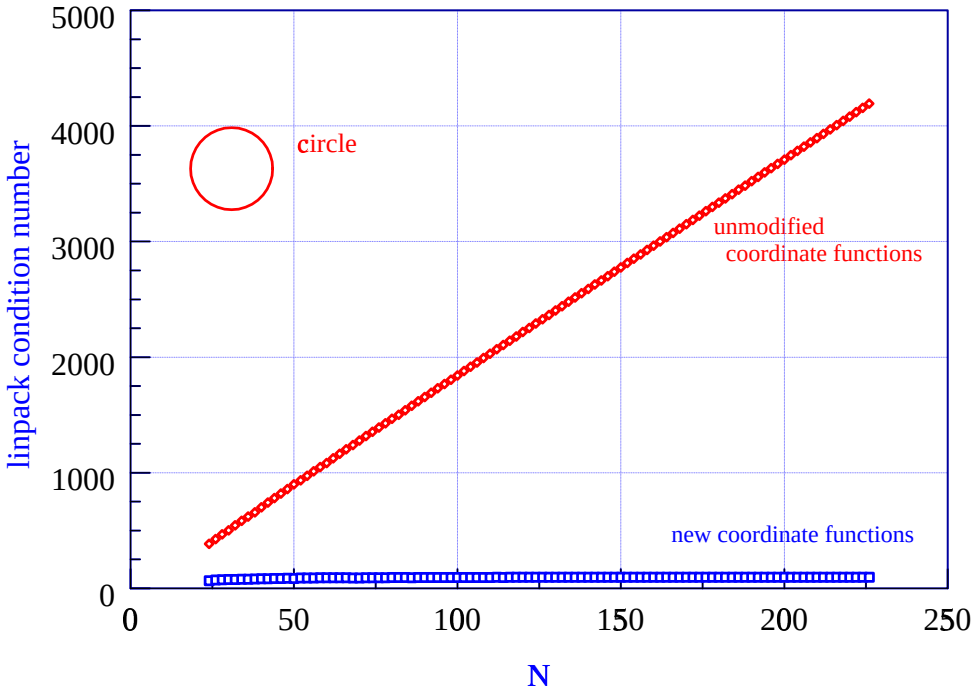

Fig. 29. circle: $\kappa a = 3.0$, $k = 3$.

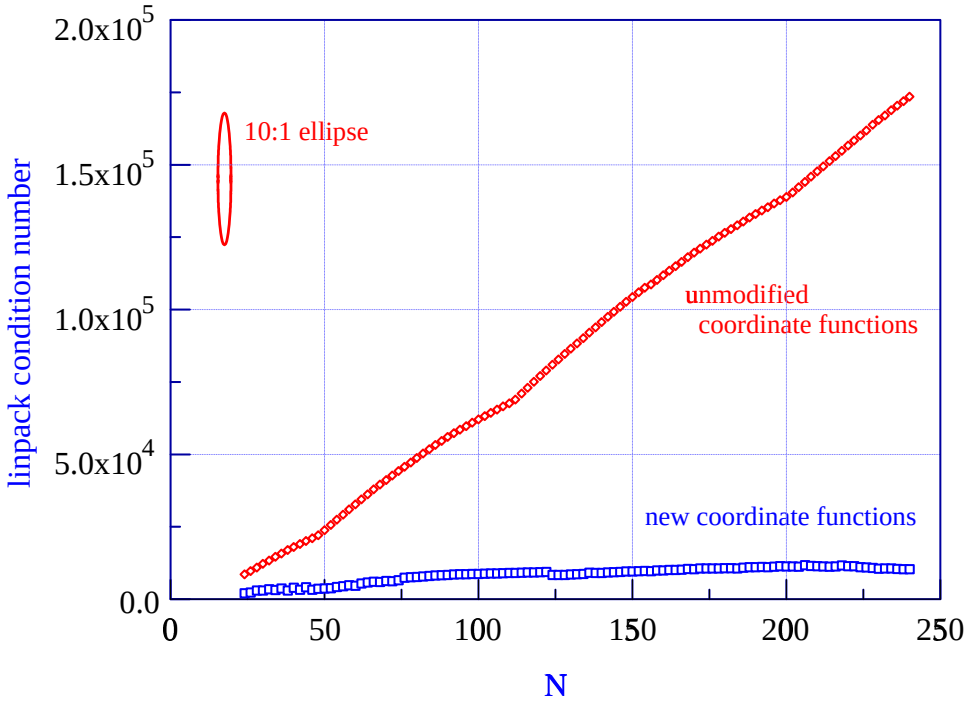

Fig. 30. $10:1$ ellipse: $\kappa a = 3.0$, $k = 1$.

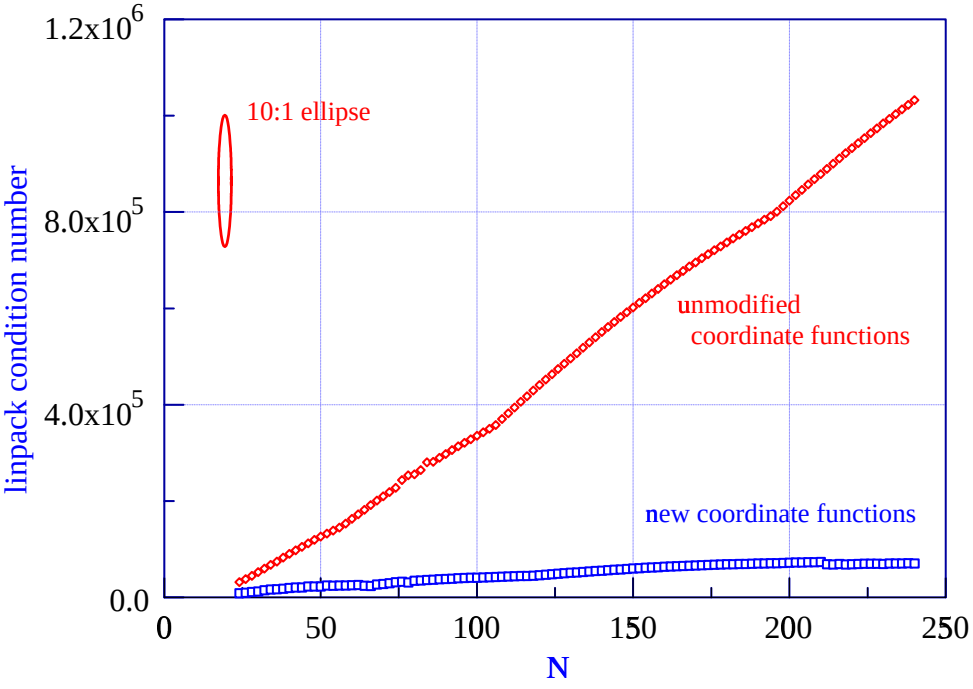

Fig. 31. 10:1 ellipse: $\kappa a = 3.0$, $k = 3$.

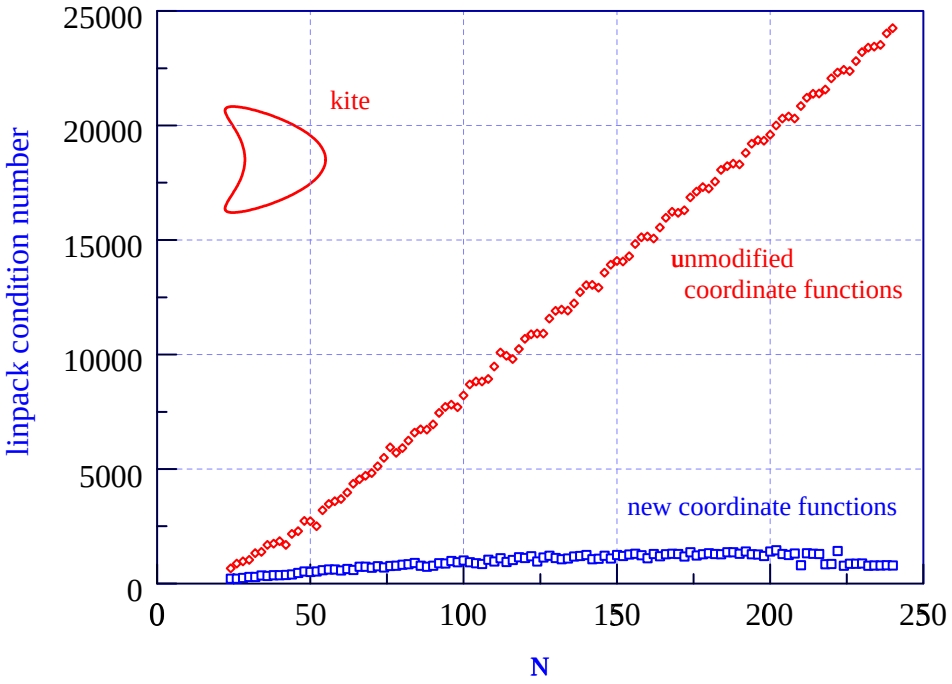

Fig. 32. kite: $\kappa a = 3.0$, $k = 3$.

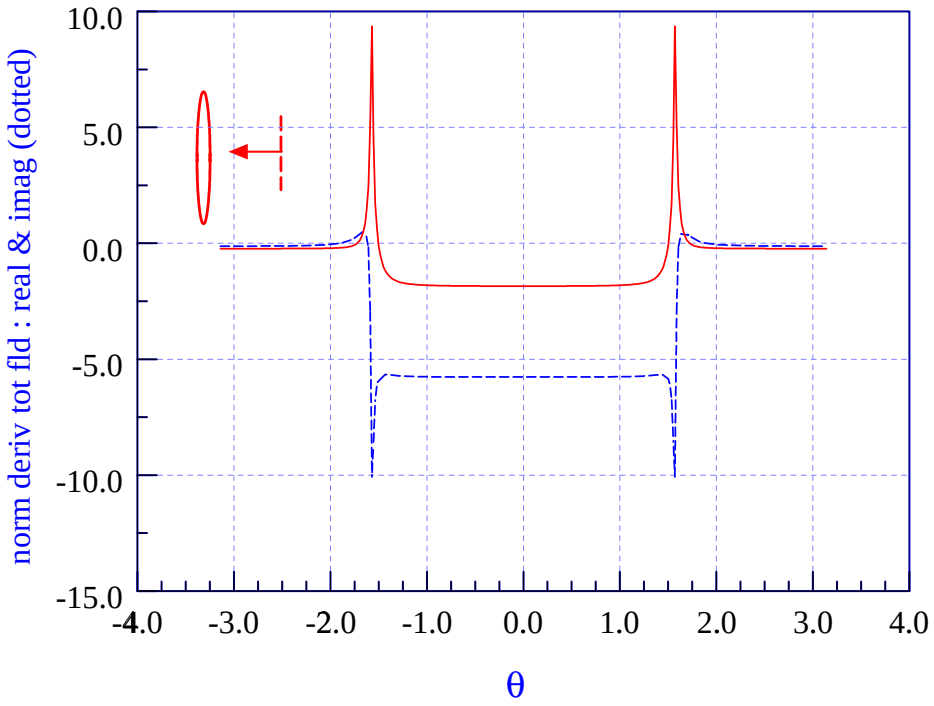

Fig. 33. 10:1 ellipse: $\kappa a = 3.0$, $\Theta = 0$; $N = 250$, $k = 3$.

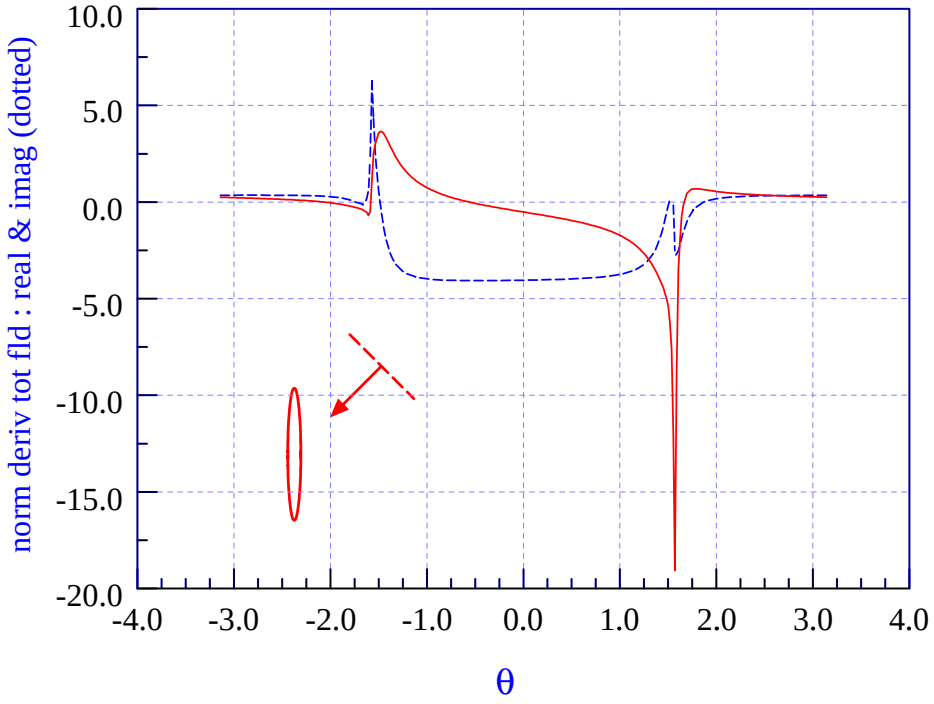

Fig. 34. 10:1 ellipse: $\kappa a = 3.0$, $\Theta = \pi/4$; $N = 250$, $k = 3$.

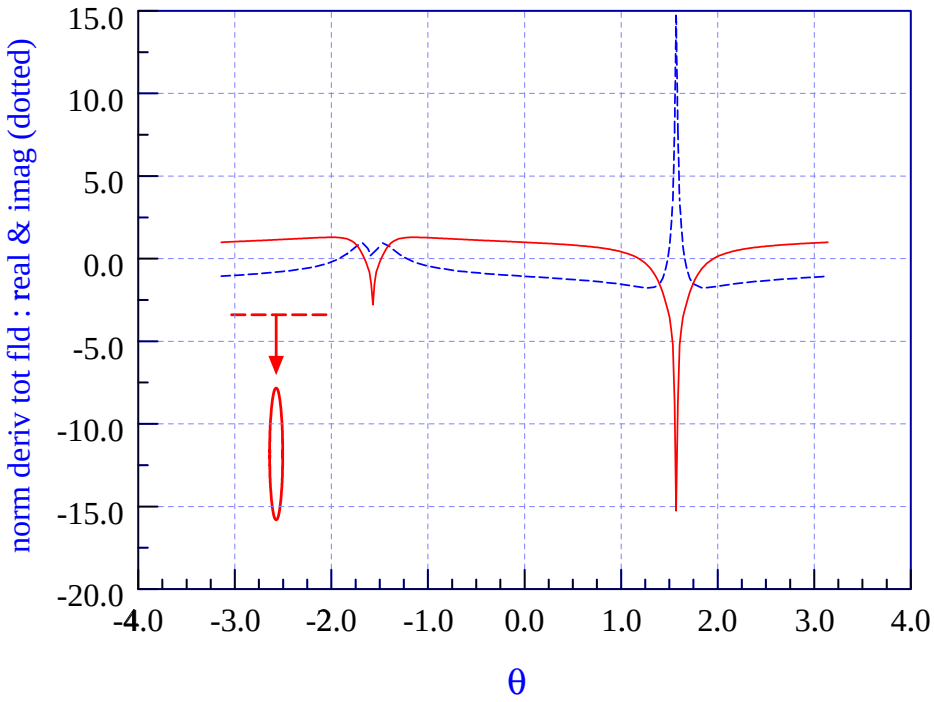

Fig. 35. 10:1 ellipse: $\kappa a = 3.0$, $\Theta = \pi/2$; $N = 250$, $k = 3$.

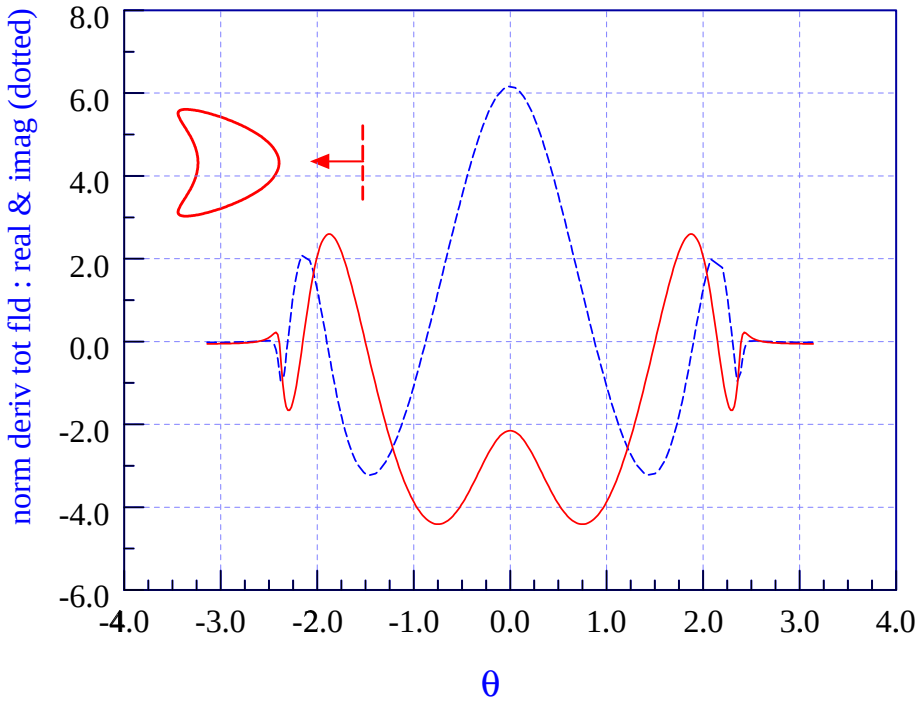

Fig. 36. kite: $\kappa a = 3.0$, $\Theta = 0$; $N = 250$, $k = 3$.

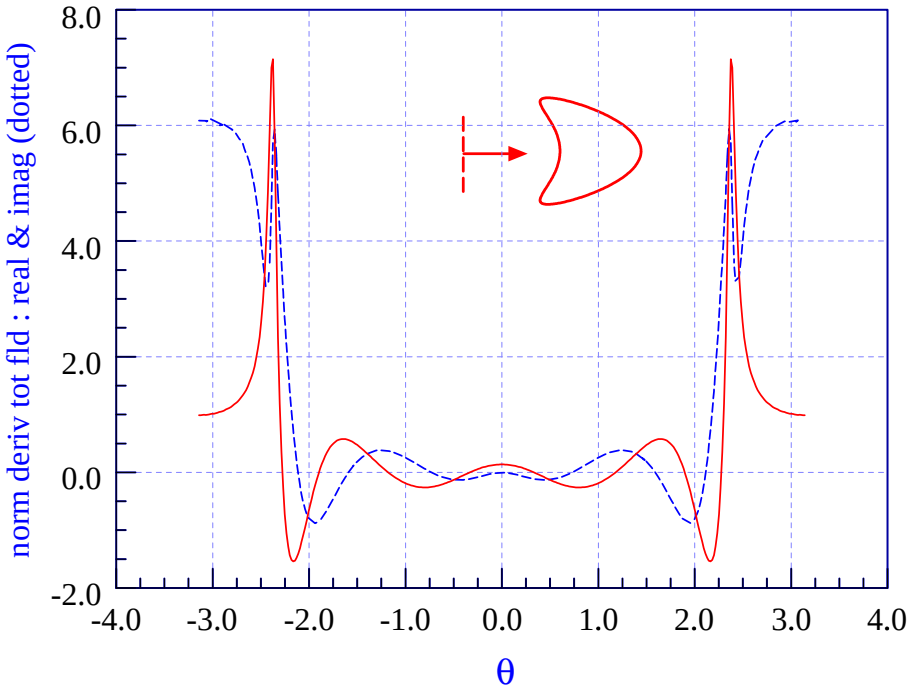

Fig. 37. kite: $\kappa a = 3.0$, $\Theta = \pi$; $N = 250$, $k = 3$.

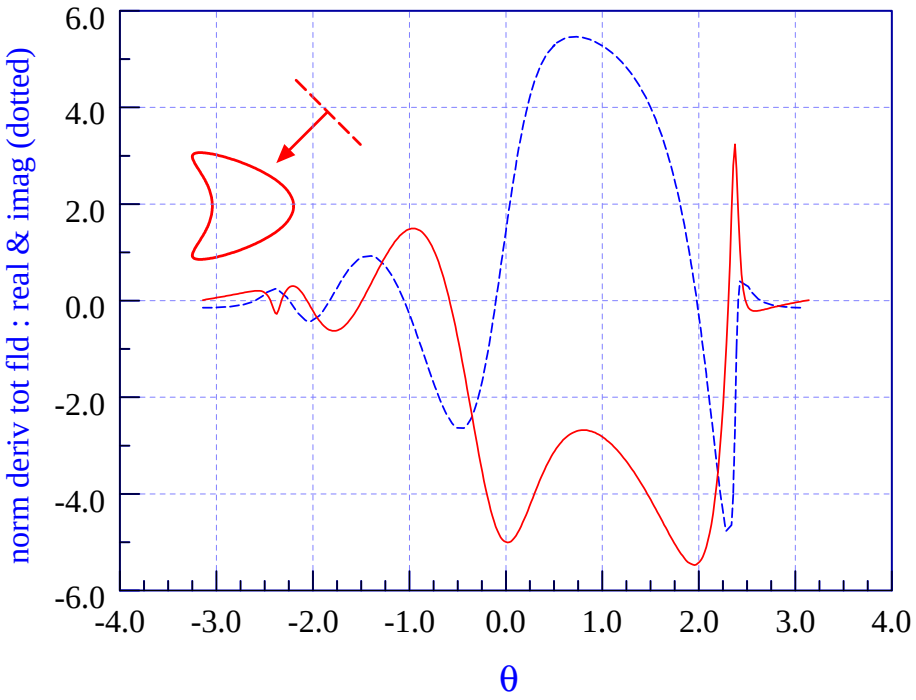

Fig. 38. kite: $\kappa a = 3.0$, $\Theta = \pi/4$; $N = 250$, $k = 3$.

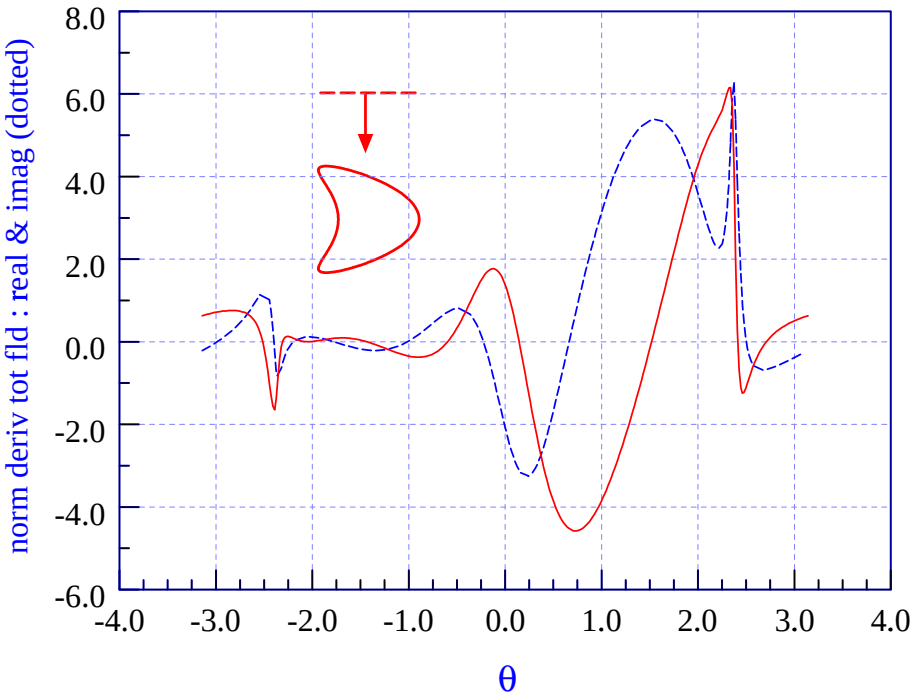

Fig. 39. kite: $\kappa a = 3.0$, $\Theta = \pi/2$; $N = 250$, $k = 3$.

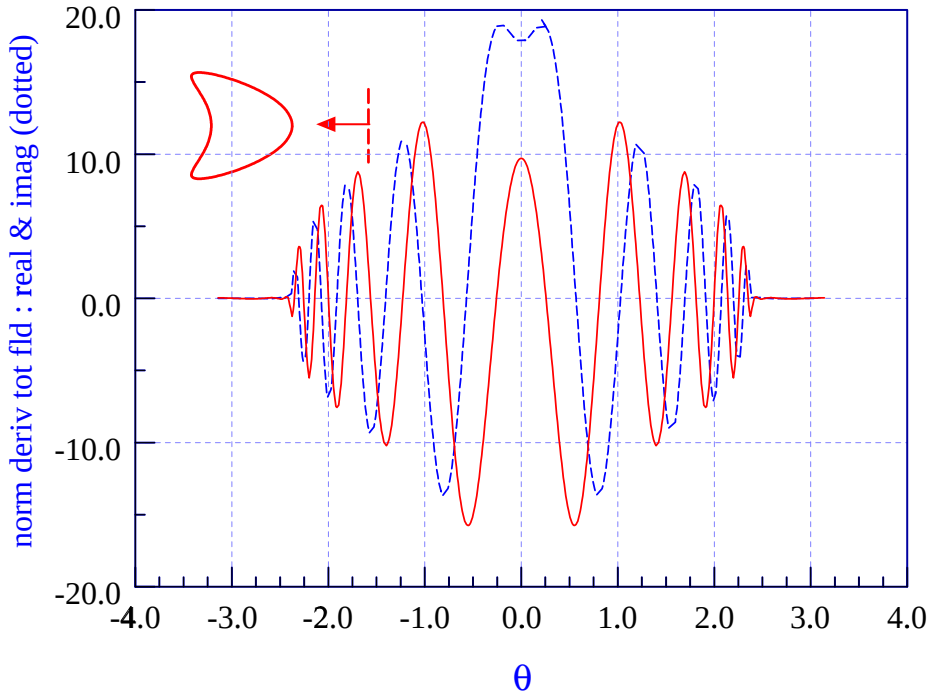

Fig. 40. kite: $\kappa a = 10.0$, $\Theta = 0$; $N = 240$, $k = 1$.

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