ON THE EXISTENCE OF TWO-DIMENSIONAL, LOCALIZED, ROTATING, SELF-SIMILAR VORTICAL STRUCTURES

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Abstract. We prove that a Gaussian monopole, also known as the Lamb-Oseen vortex, is the only localized, rotating, self-similar solution to the two-dimensional, incompressible Navier-Stokes equations where level sets of vorticity and corotating streamfunction coincide. Our definition of self-similarity is restricted to the natural linear combination of space, time and viscous diffusion. We arrive at this conclusion by analytically determining the azimuthal Fourier modes for all possible solutions to this problem and then proving that the amplitude of all but the first (axisymmetric) is zero. Since coherent vortex multipoles are observed to be in a state where lines of vorticity and corotating streamfunction correspond, this casts doubt on the existence of any self-similar asymptotic structure other than the monopole.

Key words. Navier-Stokes equation, vorticity dynamics, coherent structures

AMS subject classifications. 35Q30, 76D05, 86A05, 86A10

1. Introduction.

A fundamental problem in two-dimensional, high-Reynolds number flows is the characterization of coherent vortical structures. The term *coherent structure* refers to a localized region of vorticity that retains its properties, such as physical appearance or certain integral quantities, for an extended or an infinite period of time. It has long been observed that the vorticity fields of initially energetic disorganized two-dimensional flows rapidly relax into a collection of isolated coherent vortical objects in a process known as the "inverse cascade" [13, 23, 24, 25, 34]. In this paper, we study the mathematics underlying the inverse cascade by searching for self-similar attractors to a broad category of coherent objects. In doing so, we reduce the problem to a system of two linear partial differential equations with an unusual nonlinear constraint: The level sets of the solutions must coincide. By characterizing these attractors, it is possible to draw inferences about the ultimate fate of two-dimensional decaying turbulence.

The understanding of two-dimensional, high Reynolds number flows is crucial in many geophysical, aerospace and industrial applications. Vorticity in the form of monopoles, dipoles and other multipoles commonly form in the oceans near coastlines through boundary interactions and jet instabilities. Once created, vortical structures in the oceans, sometimes referred to as "gyres," can survive in the mostly irrotational ocean for years before interacting with a coastline or another structure [24, 34]. In impulsively started free jets and wall jets, transient monopoles form above the outlet and remain as coherent structures for extensive time periods [33, 41]. These represent a few of many important applications where coherent vortical structures play a central role. This article establishes that two-dimensional, self-similar (using the natural scaling presented in §3), multipolar, localized, rotating, vortical structures do not exist. This result, together with the linear stability of the Lamb-Oseen vortex established by Bernoff & Lingevitch [3], suggests that all self-similar multipolar structures, while they may be long-lived on an intermediate timescale, evolve toward monopolar attractors.

Numerical simulations of two-dimensional decaying turbulence indicate that under a broad range of circumstances, initially energetic flows relax rapidly into a system

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of coherent vortical structures separated by vast expanses of nearly irrotational fluid. McWilliams observed that these structures evolve toward collections of monopoles and dipoles [25] while the results of Gama & Frisch as well as Legras, Santangelo & Benzi indicate the existence of other more exotic structures such as vortex tripoles [13, 23]. Interestingly, none of these investigations simulate the Navier-Stokes equations. McWilliams, Legras et. al. as well as Gama & Frisch use hyperviscosity (artificial diffusivities of the form $(-1)^{p+1}\nu_p\nabla^{2p}\omega$, $p > 1$) in their spectral calculations. Since viscosity plays a role in small scale mixing during the axisymmetrization process, hyperviscosity is not likely to provide insight into the long-time relaxation of coherent vortical structures. Also, Gama & Frisch simulated the Navier-Stokes Kuramoto-Sivashinsky system that induces a large-scale instability to the flow [13]. Finally, Carnevale et. al. proposed a scaling theory of the density of systems of isolated vortices [5].

Rather than studying the inverse cascade problem directly, some investigators have dedicated their efforts toward the understanding of large regular arrays of vortex structures. Fine et. al. have investigated the formation of lattices of vortex monopoles, called vortex crystals, in magnetized electron columns where electron densities are analogous to vorticity in the Euler equations with an extremely low dissipation [11, 16]. Like the others, they find that disorganized distributions of electron densities rapidly relax into robust arrays of coherent maxima embedded in a low background field. Taken as a whole, a localized, crystal lattice can be considered a single coherent structure. Regardless of issues surrounding the form or expression of the dissipation or forcing in any of the experiments noted above, they capture the large-scale formation and relaxation of coherent structures from initially random, broadband vorticity fields. On a similar front, Chow, et. al. have derived exact, globally smooth, steady solutions to the 2D Euler equations from solitary wave theory [8, 9]. These solutions are periodic in either one or both cardinal directions. Some correspond to arrays of oppositely-signed vortex monopoles while others have a more intricate structure. While these exact solutions bear some resemblance to the "vortex lattice" experiments discussed above, the solutions if considered coherent structures would extend to infinity since they are periodic arrays.

Many groups have studied isolated vortices in more detailed laboratory and computational experiments centered on the structure itself rather than large scale interactions between coherent structures. Ikeda found that perturbed, slightly elliptical vortices would evolve toward tripolar vortices under certain conditions [15]. Similarly, Swenson came to the same conclusions while studying modon-with-rider solutions to barotropic equations [34]. Van Heijst and various collaborators have performed detailed experiments of tripolar vortices in stratified tanks in stationary and rotating frames [12, 18, 19, 39]. This group has also studied the stability of a tripole model consisting of point vortices. Polvani & Carton have studied tripolar solutions to the Euler equations and Navier-Stokes equations with hyperviscosity, and investigate their formation through a variety of mechanisms [29]. Orlandi & van Heijst simulated tripoles arising from dipole collisions [28]. Morel and Carton have studied a wide range of two-dimensional, multipolar objects that can arise from perturbations of an axisymmetric state known as a shielded vortex [26]. In each case, they studied structures consisting of central regions of vorticity surrounded by two or more satellites with oppositely signed vorticity, and found some to be more robust than others. Carton & Legras studied the full development and evolution of a tripole from its generation as a shielded axisymmetric vortex until the central core vortex erodes and is stripped away by the satellites [7]. Voropayev studied vortex dipoles and quadrupoles resulting from dipole collisions, and derived a self-similar equation for a vortex quadrupole [40]. Rossi, Lingevitch & Bernoff found that monopoles subjected to a moderate amplitude quadrupolar perturbation rapidly evolve toward a tripolar configuration whereas a weak perturbation will evolve back into a monopole [31]. The strength of the perturbation plays an important role because the satellites marginally create islands within the flow trajectories that preclude all but diffusive mixing between regions of oppositely signed vorticity. In these experiments where the central region of vorticity is relatively strong, they observed a steady erosion of the satellites as the tripole evolves though it is not possible to conclude whether or not the asymptotic structure will be a tripole or a monopole. More recently, Crowdy has determined a class of localized vortex solutions to the Euler equations [10]. Also, Kloosterziel and Carnevale studied isolated vortical structures using a projection method and found a variety of long-lived vortex multipoles [17].

Other progress has been made toward understanding general properties of coherent vortical structures. Turkington applied variational and asymptotic techniques to studying the evolution of inviscid states characterized by regions of positive vorticity and zero vorticity [36, 37, 38]. In particular, he focused on the limit as the initial vorticity approaches a system of singular vortices (i.e., point vortices). Turkington's work also included solid boundaries rather than unbounded or periodic solutions as others have studied. In this article, we restrict ourselves to unbounded flows, but include viscous diffusion. Ting & Klein have written a monograph on the asymptotic properties of viscous vortical flows [35]. In particular, they find that the Lamb-Oseen monopole is "optimal" among two-dimensional structures in the sense that it approximates the core structure of a coherent vortex to an extra order of accuracy in their two-time analysis. More recent work includes that of Bassom & Gilbert and Le Dizes. Bassom and Gilbert demonstrated that vortical structures tend toward axisymmetry in the weak sense, meaning that if a non-axisymmetric perturbation of the vorticity field is integrated against a test function, the result decays [2]. Le Dizes analyzed non-axisymmetric vortices from perturbed Gaussian monopoles subject to an external multipole straining field, or from unstrained monopoles with steep edge gradients. The latter study focuses on the formation and growth of these structures rather than their fate [21].

While not addressing the inverse cascade problem directly, research into the large-time properties of solutions to Navier-Stokes contributes to our understanding of many questions surrounding the fate of coherent structures. For instance, Carpio proves under quite general conditions that all solutions of the two-dimensional Navier-Stokes equations behave asymptotically like solutions of the heat equation in the sense that $\|h(t) - u(t)\|_q$ tends to zero faster than $u(t)$ for large q. Here h is the solution of the heat equation and u the solution of the Navier-Stokes equations with the same initial divergence-free velocity distribution. This is consistent with our result since the Lamb-Oseen monopole does indeed satisfy the heat equation [6]. On a similar tack, Oliver and Titi have calculated lower bounds on higher derivatives of Navier-Stokes under the assumption that solutions to Navier-Stokes are close to solutions to the heat equation, asserting that some sort of large-time structure persists in decaying flow [27].

The observed formation and evolution of the multipolar vortices guides this analysis into properties of an asymptotic state. In the laboratory and numerical experiments noted above, investigators observe that concentrated regions of vorticity rapidly reorganize, usually into an axisymmetric state, due to purely inviscid mechanisms. After this rapid reorganization, vortical structures can undergo shear-diffusion mixing wherein gradients of vorticity are amplified across streamlines. If Re is the Reynolds number expressing the dimensionless ratio of inertial to viscous forces in a fluid, shear-diffusion mixing occurs on the $\mathrm{Re}^{1/3}$ timescale, much more rapidly than the action of diffusion which one would expect to evolve on the Re timescale [3, 30]. Thus, it is natural to expect that an asymptotic structure would exist only in a state where level sets of vorticity and streamfunction coincide, inhibiting shear-diffusion. Here, it is important to distinguish between an asymptotic structure and a coherent structure. A coherent structure, as discussed earlier, may have very lively dynamics and exist for a long but finite time while an asymptotic structure is a coherent structure that exists forever. The Lamb-Oseen vortex is an example of an asymptotic state.

In this paper, we explore the existence of asymptotic states where, in a frame rotating with the steady vortical structure, lines of constant vorticity and constant corotating streamfunction coincide. The corotating streamfunction is a streamfunction in a rigidly rotating reference frame. Within this corotating frame, vorticity is not convected across corotating streamlines, thus the self-similar structure evolves slowly through viscous diffusion. We investigate whether these conditions permit solutions other than a Gaussian distribution of vorticity (ie the Lamb-Oseen monopole) and find that *the Lamb-Oseen monopole is the only solution with these properties.* This proof is limited to one self-similar scaling, though this scaling, also used by Ting & Klein and Voropayev, is the natural one that gives rise to the Lamb-Oseen monopole [35, 40]. The final result of this work is stated as follows.

THEOREM 1.1. *Let $\omega(\vec{x},t)$ be a function of two space dimensions and time with the following properties:*

(i) *ω is a self-similar solution to the Navier-Stokes equation on R^2,*

$$\partial_t\omega + \left(\begin{bmatrix} \partial_y\psi \\ -\partial_x\psi \end{bmatrix} \cdot \nabla\right)\omega = \frac{1}{\mathrm{Re}}\nabla^2\omega,$$
$$\nabla^2\psi = -\omega,$$

satisfying

$$\omega(\vec{\eta},t) = \frac{1}{t}f(\vec{\eta}),$$
$$\vec{\eta} = \frac{\vec{x}\mathrm{Re}^{\frac{1}{2}}}{t^{\frac{1}{2}}}.$$

(ii) *ω is localized so that $\omega \to 0$ as $|\vec{x}|^2 \to \infty$.*

(iii) *There is a number $\widehat{\Omega}$ such that level sets of ω and $\psi + \frac{1}{2}\frac{\widehat{\Omega}}{t}|\vec{x}|^2$ coincide.*

Then, ω has the form

$$\omega = \frac{C_1}{4\pi(C_2 + \frac{t}{\mathrm{Re}})}\exp\left(-\frac{|\vec{x}|^2}{4(C_2 + \frac{t}{\mathrm{Re}})}\right)$$

where C_1 and C_2 are arbitrary constants.

This theorem is a direct result of classical analysis of the underlying partial differential equations constrained by the coincidence of level sets of the solutions. In fact, we were surprised that we did not find a similar result in the literature either as a general property of partial differential equations or as a result of relevance to coherent structures and mixing in fluids.

If one assumes that asymptotic vortical structures are self-similar, this theorem, together with the linear stability of the Lamb-Oseen vortex [3], strongly suggests that the multipoles observed in laboratory and numerical experiments are evanescent and will eventually relax into Gaussian monopoles. In other words, this is evidence that the only possible rotating attractor is the Lamb-Oseen vortex. For instance, multipoles are commonly observed in computations and experiments to have a central core with oppositely-signed satellites. The assumptions of Theorem 1.1 apply to structures of this form and many others as well. Reflecting upon related work suggesting the existence of a steady viscous tripole, it should be noted that Orlandi & van Heijst used periodic boundary conditions for their numerical experiments so these structures would not satisfy condition (ii). Also, Kloosterziel & Carnevale reported the existence of a steady tripole using their projection method. However, this remarkable structure is not self-similar, but rather it retains the same spatial distribution while its amplitude decays [17].

2. Governing equations. We are seeking self-similar solutions to the 2D, incompressible Navier-Stokes equations expressed in terms of vorticity:

$$\partial_t \omega + \left(\begin{bmatrix} \partial_y \psi \\ -\partial_x \psi \end{bmatrix} \cdot \nabla \right) \omega = \frac{1}{\mathrm{Re}} \nabla^2 \omega, \tag{2.1}$$

$$\nabla^2 \psi = -\omega. \tag{2.2}$$

Here, ψ is the stream function ($u = \psi_y$ and $v = -\psi_x$), and u and v are the horizontal and vertical components of the velocity field, respectively. With this structure,

$$\nabla \cdot \begin{bmatrix} u \\ v \end{bmatrix} = 0,$$

so the velocity field is automatically incompressible. Level curves of ψ, called streamlines, are everywhere tangential to the velocity field so that material elements in the flow remain on these streamlines. The domain is unbounded. Equation (2.1) should be considered nondimensionalized, and Re is assumed to be large. If the rotation rate of the structure is Ω (we shall see later that Ω actually decreases like $1/t$), we seek a vorticity field ω and a stream function $\psi = \psi^{\mathrm{rot}} - \frac{1}{2}\Omega|\vec{x}|^2$ satisfying (2.1) and (2.2). The function ψ^{rot} is called the corotating streamfunction and represents material paths in a rotating reference frame. We seek solutions such that level sets of ψ^{rot} and ω coincide as observed in the previously noted laboratory and numerical experiments, so that vorticity is not mixed along corotating streamlines:

$$\partial_t \omega + \left(\Omega \begin{bmatrix} -y \\ x \end{bmatrix} \cdot \nabla \right) \omega = \frac{1}{\mathrm{Re}} \nabla^2 \omega, \tag{2.3}$$

$$\nabla^2 \psi^{\mathrm{rot}} = -(\omega - \Omega). \tag{2.4}$$

Of course, viscosity will act to diffuse vorticity across corotating streamlines. Next, we seek self-similar solutions to Equation (2.3).

3. Rescaling. An example of a self-similar solution is the Lamb-Oseen monopole

$$\omega(\vec{x}, t) = \frac{1}{4\pi(t/\mathrm{Re})} \exp\left(-\frac{|\vec{x}|^2}{4(t/\mathrm{Re})} \right). \tag{3.1}$$

(See [20, 32] for further discussion and properties of this example.) Like solutions are obtained by translating (3.1) in time or space, or multiplying by a constant. If Re is infinite, then there are entire classes of solutions of the form

$$\psi = \lambda\omega \tag{3.2}$$

which constitute steady solutions where the vorticity field is constant on streamlines and there is no viscous diffusion across streamlines. A classic example of these is the Lamb dipole [20]. In the Lamb-Oseen monopole, we know that the net circulation at a fixed distance from the origin will decay like $\frac{1}{t}$ so that we expect certain monopolar solutions, particularly the tripole, to exhibit similar behavior. Thus, we anticipate $\Omega = \widehat{\Omega}/t$ where $\widehat{\Omega}$ is a constant, and $\psi^{\mathrm{rot}} = \widehat{\psi}^{\mathrm{rot}}/t$.

We seek a similarity variable

$$\vec{\eta} = \begin{bmatrix} \eta_x \\ \eta_y \end{bmatrix}$$

to replace $\vec{x}$ and t:

$$\begin{aligned} \frac{\vec{x}\mathrm{Re}^{\frac{1}{2}}}{t^{\frac{1}{2}}} \rightarrow \vec{\eta} = \begin{bmatrix} \eta_x \\ \eta_y \end{bmatrix}&, \\ \nabla_{\vec{x}} &= \frac{\mathrm{Re}^{\frac{1}{2}}}{t^{\frac{1}{2}}}\nabla_{\vec{\eta}}, \\ \nabla^2_{\vec{x}} &= \frac{\mathrm{Re}}{t}\nabla^2_{\vec{\eta}}, \\ \frac{\partial}{\partial t} &= -\frac{\vec{\eta}}{2t}\cdot\nabla_{\vec{\eta}}. \end{aligned} \tag{3.3}$$

This is the same rescaling used by Ting & Klein and also Cannone & Planchon, and represents a natural recombination of the relevant variables [4, 35]. We further assume that there is a solution of the form

$$\omega(\vec{\eta}, t) = \frac{1}{t} f(\vec{\eta}). \tag{3.4}$$

Thus, Equation (2.3) reduces to an equation for f alone,

$$\widehat{\Omega}(\eta_y\partial_{\eta_x} f - \eta_x\partial_{\eta_y} f) = f + \frac{1}{2}(\vec{\eta}\cdot\nabla_{\vec{\eta}})f + \nabla^2_{\vec{\eta}} f \tag{3.5}$$

If $\vec{\eta}$ is expressed in polar coordinates, we obtain

$$-\widehat{\Omega}\partial_\theta f = f + \left(\frac{r}{2} + \frac{1}{r}\right)\partial_r f + \partial_{rr} f + \frac{1}{r^2}\partial_{\theta\theta} f. \tag{3.6}$$

The coincidence of f and $\widehat{\psi}^{\mathrm{rot}}$ (or ω and ψ^{rot}) is not built into the above equations. Even if one can find solutions to these equations, it remains to be seen if they satisfy the coincidence constraint.

To summarize, we are studying solutions ω with the following properties:

(i) ω solves (2.1-2.2) AND

(ii) Level sets of ω and ψ^{rot} coincide where ψ^{rot} is defined in (2.4), but $\widehat{\Omega}$ is a parameter that can be chosen freely AND

(iii) ω is a self-similar solution described by (3.3) and (3.4). Given (i) and (ii) above, this is equivalent to saying that f solves (3.6).

The last two conditions are not necessary to describe or rescale the problem, but they will become necessary in §4 and §5. During these steps in the analysis, it is sufficient for their Fourier series to converge absolutely so that we can interchange summations. Also, we will need to expand the azimuthal Fourier series coefficients for f in a Taylor series in r near the origin. These restrictions are fairly general, and the regularity of solutions to the two-dimensional vorticity equations has been established beyond these requirements for a broad category of initial conditions [14].

4. Properties of unconstrained self-similar solutions. In this section, we examine the properties of solutions to (3.6) and their corresponding streamfunctions. We seek families of solutions with the minimal amount of regularity discussed in the previous section. Once found, we investigate the possibility that a subset of these solutions can satisfy the nonlinear coincidence constraint.

4.1. Fourier series solution for f. If we express the azimuthal portion of f in a Fourier series,

$$f(r,\theta) = a_0(r) + \sum_{n=1}^{\infty} a_n(r)e^{in\theta} + \mathrm{CC}, \tag{4.1}$$

where CC represents the complex conjugate portion of the complex Fourier series, and substitute this expression into (3.6), we obtain an ordinary differential equation (ODE) for the coefficients

$$r^2 a_n'' + (r + \frac{1}{2}r^3)a_n' + (\lambda_n r^2 - n^2)a_n = 0, \qquad \lambda_n = 1 + in\widehat{\Omega}. \tag{4.2}$$

Since f is twice differentiable in the θ direction, the Fourier series for f converges absolutely.

Only one solution of (4.2) is regular at the origin and it behaves like r^n there. We shall see later in this section that this particular solution decays in the far field as one would expect for a localized structure. We define $\tilde{a}_n$ to be the solution satisfying (4.2) and

$$\lim_{r\to 0} \frac{\tilde{a}_n}{r^n} = 1, \tag{4.3}$$

and then every regular solution has the form

$$a_n(r) = c_n^0 \tilde{a}_n(r) \tag{4.4}$$

where c_n^0 is a complex number and is the first term of the power series representation of $a_n(r)$. Also, we see that a_0 is proportional to the Lamb-Oseen vortex, given in (3.1). As an aside, the restriction on the far field can be relaxed for the $n = 0$ mode without changing anything that follows in this paper so that these results are applicable to the observed vortex crystal structures. In other words, one could add a constant background vorticity to (3.1). The remainder of this paper, unless otherwise noted, will focus on nonaxisymmetric ($n \neq 0$) modes of f.

Since we seek localized solutions, we must establish that $\tilde{a}_n$ remains bounded as $r \to \infty$ for $n \neq 0$. By transforming (4.2) as

$$\begin{aligned} \frac{r^2}{4} &= s, \\ a_n(r) &= s^{-1/2}e^{-s/2}A(s), \end{aligned} \tag{4.5}$$

we obtain

$$A'' + \left(-\frac{1}{4} + \frac{\lambda_n - 1/2}{s} + \frac{1/4 - n^2/4}{s^2}\right) A = 0. \tag{4.6}$$

Equation (4.6) is the Whittaker equation with solutions that can be written in terms of confluent hypergeometric functions:

$$A(s) = s^{\frac{n+1}{2}} e^{-\frac{s}{2}} M\left(\frac{n}{2} - \lambda_n + 1, n+1, s\right), \tag{4.7}$$

$$A(s) = s^{\frac{n+1}{2}} e^{-\frac{s}{2}} U\left(\frac{n}{2} - \lambda_n + 1, n+1, s\right), \tag{4.8}$$

and returning to the original variables

$$a_n(r) = r^n e^{-\frac{r^2}{4}} M\left(\frac{n}{2} - \lambda_n + 1, n+1, \frac{r^2}{4}\right), \tag{4.9}$$

$$a_n(r) = r^n e^{-\frac{r^2}{4}} U\left(\frac{n}{2} - \lambda_n + 1, n+1, \frac{r^2}{4}\right). \tag{4.10}$$

The latter solution, (4.10), must be discarded because the confluent hypergeometric function U behaves like r^{-n} as $r \to 0$ so for this solution of a_n, $a_n(r)e^{in\theta}$ would not be analytic at the origin. The former solution (4.9) behaves like r^n as $r \to 0$ because $M\left(\frac{n}{2} - \lambda_n + 1, n+1, 0\right) \to 1$. It follows that the first nonzero power series coefficient for the right hand side of (4.9) is 1, so this is precisely $\tilde{a}_n(r)$. The asymptotic behavior of confluent hypergeometric functions is well known, and it is a straightforward exercise to establish that

$$\tilde{a}_n(r) = \frac{\Gamma(n+1)}{\Gamma\left(\frac{n}{2} + 1 - \lambda_n\right)} 2^{n+2\lambda_n} r^{-2\lambda_n} \left[1 + O(r^{-2})\right], \qquad r \to \infty \tag{4.11}$$

(see [1] for example), and therefore

$$\tilde{a}_n(r) \to 0 \text{ as } r \to \infty. \tag{4.12}$$

Thus, (4.9) remains a viable solution satisfying the differential equation (4.2) with the appropriate decay at infinity.

4.2. Fourier series solutions for $\widehat{\psi}^{\text{rot}}$. At the same time, we can express $\widehat{\psi}^{\text{rot}}$ in terms of its Fourier series:

$$\widehat{\psi}^{\text{rot}}(r,\theta) = b_0(r) + \sum_{n=1}^{\infty} b_n(r) e^{in\theta} + \text{CC}. \tag{4.13}$$

Again, the Fourier series for $\widehat{\psi}^{\text{rot}}$ converges absolutely as it did for f. Since

$$\nabla^2 \widehat{\psi}^{\text{rot}} = -f + \widehat{\Omega}, \tag{4.14}$$

then

$$\frac{1}{r}(r\tilde{b}_0')' = -\tilde{a}_0 + \widehat{\Omega} \tag{4.15}$$

$$\frac{1}{r}(r\tilde{b}_n')' - \frac{n^2 \tilde{b}_n}{r^2} = -\tilde{a}_n. \tag{4.16}$$

Again, we seek the special solution $\tilde{b}_n(r)$ that corresponds to $\tilde{a}_n(r)$, understanding that linearity permits the construction of a whole family of solutions $b_n(r)$ by varying c_n^0. As we shall see momentarily, there is another constant corresponding to a homogeneous solution to (4.16) that can also be varied. Also, we seek solutions with sufficient smoothness at the origin,

$$\tilde{b}_n(r) \to r^n \text{ as } r \to 0. \tag{4.17}$$

The general solution to (4.15) is

$$\tilde{b}_0(r) = -\ln r \int_0^r \tilde{a}_0(s) s ds + \int_0^r s \ln s \tilde{a}_0(s) ds + h_0 + l_0 \ln r + \frac{\widehat{\Omega}}{4} r^2. \tag{4.18}$$

The expression for $\tilde{b}_0$ immediately leads to the observation that $l_0 = 0$ because we seek a streamfunction that is smooth at the origin. Furthermore, we can set $h_0 = 0$ without loss of generality for this problem. To find a general solution for (4.16), we multiply both sides by r^{n+1} and integrate once:

$$\int_0^r \left(s^{n+1} \tilde{b}_n''(r) + s^n \tilde{b}_n'(r) - n^2 s^{n-1} \tilde{b}_n(r) \right) ds = -\int_0^r s^{n+1} \tilde{a}_n(s) ds. \tag{4.19}$$

After integrating by parts on the left side and applying the boundary conditions (4.17).

$$r^{n+1} \tilde{b}_n'(r) - n r^n \tilde{b}_n(r) = -\int_0^r s^{n+1} \tilde{a}_n(s) ds,$$

or

$$\frac{d}{dr} \left(\frac{\tilde{b}_n(r)}{r^n} \right) = -\frac{1}{r^{2n+1}} \int_0^r s^{n+1} \tilde{a}_n(s) ds. \tag{4.20}$$

Again, we integrate from 0 to r to obtain

$$\begin{aligned} \frac{\tilde{b}_n(r)}{r^n} - \lim_{r \to 0} \frac{\tilde{b}_n(r)}{r^n} &= -\int_0^r \frac{1}{t^{2n+1}} \int_0^r s^{n+1} \tilde{a}_n(s) ds dt \\ &= -\int_0^r s^{n+1} \tilde{a}_n(s) \int_s^r \frac{1}{t^{2n+1}} dt ds. \end{aligned} \tag{4.21}$$

The term

$$\tilde{B}_n = \lim_{r \to 0} \frac{\tilde{b}_n(r)}{r^n} \tag{4.22}$$

is a free parameter corresponding to the homogeneous solution to (4.16), and we find

$$\begin{aligned} \tilde{b}_n(r) &= \tilde{B}_n r^n + \frac{1}{2n} \left(r^{-n} \int_0^r s^{n+1} \tilde{a}_n(s) ds - r^n \int_0^r s^{1-n} \tilde{a}_n(s) ds \right) \\ &= r^{-n} \frac{1}{2n} \int_0^r s^{n+1} \tilde{a}_n(s) ds + r^n \left(\tilde{B}_n - \frac{1}{2n} \int_0^r s^{1-n} \tilde{a}_n(s) ds \right). \end{aligned} \tag{4.23}$$

The first term on the right is $O(r^{n+2})$ as $r \to 0$, and as we shall see later is $O(1)$ as $r \to \infty$. The second term is $O\left(r^{n+2}\right)$ as $r \to 0$, but is unbounded as $r \to \infty$ unless

$$\tilde{B}_n = \frac{1}{2n} \int_0^\infty s^{1-n} \tilde{a}_n(s) ds. \tag{4.24}$$

This assures us that the streamfunction remains bounded though possibly non-axisymmetric in the far-field.

To determine $\tilde{B}_n$, we use (4.9) and the integral form of the confluent hypergeometric function:

$$\begin{aligned}
\tilde{B}_n &= \frac{1}{2n}\int_0^\infty s^{1-n}s^n e^{-\frac{s^2}{4}} M\left(\frac{n}{2}-\lambda_n+1, n+1, \frac{s^2}{2}\right) ds \\
&= \frac{1}{2n}\int_0^\infty s e^{-\frac{s^2}{4}} \frac{e^{\frac{s^2}{4}}}{B\left(\frac{n}{2}-\lambda_n+1, \frac{n}{2}+\lambda_n\right)} \int_0^1 e^{\frac{ts^2}{4}} t^{\frac{n}{2}+\lambda_n-1}(1-t)^{\frac{n}{2}-\lambda_n} dt ds \\
&= \frac{1}{2n}\frac{1}{B\left(\frac{n}{2}-\lambda_n+1, \frac{n}{2}+\lambda_n\right)} \int_0^1 t^{\frac{n}{2}+\lambda_n-1}(1-t)^{\frac{n}{2}-\lambda_n} \int_0^\infty s e^{\frac{ts^2}{4}} ds dt \\
&= \frac{1}{2n}\frac{1}{B\left(\frac{n}{2}-\lambda_n+1, \frac{n}{2}+\lambda_n\right)} \int_0^1 t^{\frac{n}{2}+\lambda_n-2}(1-t)^{\frac{n}{2}-\lambda_n} dt \\
&= \frac{1}{2n}\frac{B\left(\frac{n}{2}-\lambda_n+1, \frac{n}{2}+\lambda_n-1\right)}{B\left(\frac{n}{2}-\lambda_n+1, \frac{n}{2}+\lambda_n\right)} \qquad (4.25)
\end{aligned}$$

where $B(x,y)$ is the *Beta function* [22]. The change in the order of integration in (4.25) is justified because $\Re\left(\frac{n}{2}+\lambda_n-2\right) > -1$ and $\Re\left(\frac{n}{2}-\lambda_n\right) > -1$ implying that the integral converges absolutely. Applying the identity

$$B(x, y-1) = \frac{x+y-1}{y-1} B(x,y), \qquad (4.26)$$

(see [22] for example) we conclude that

$$\tilde{B}_n = \frac{1}{2}\frac{1}{\frac{n}{2}+\lambda_n-1}. \qquad (4.27)$$

To understand the behavior of the streamfunction as $r \to \infty$, we study $\tilde{b}_n(r)$ in this limit. Substituting (4.27) into (4.23), we see that

$$\tilde{b}_n = \frac{1}{2n}\left(r^{-n}\int_0^r s^{n+1}\tilde{a}_n(s) ds + r^n \int_r^\infty s^{1-n}\tilde{a}_n(s) ds\right). \qquad (4.28)$$

If we fix $R > 0$ and substitute (4.11) into (4.28), we see that

$$\begin{aligned}
\tilde{b}_n(r) = {} & \frac{1}{2n}\frac{\Gamma(n+1)}{\Gamma\left(\frac{n}{2}+1-\lambda_n\right)} 2^{n+2\lambda_n}\left(\frac{1}{2-2\lambda_n+n} - \frac{1}{2-2\lambda_n-n}\right) r^{2-2\lambda_n} + \\
& O\left(r^{-2\lambda_n}\right) + O\left(r^{-n}\right). \qquad (4.29)
\end{aligned}$$

Since $\lambda_n = 1 + in\widehat{\Omega}$, this expression establishes that $\tilde{b}_n \sim O(1)$ as $r \to \infty$.

To summarize, using (4.2) and (4.16) together with physically relevant boundary conditions, we have determined the following when $n \neq 0$:

1. There is a localized family of solutions to (4.2) given by Equation (4.9).
2. The corotating streamfunction Fourier modes corresponding to (4.9) are given by Equation (4.23) and the constant for the homogeneous part is given by (4.27).
3. The corotating streamfunction Fourier modes are $O(1)$ as $r \to \infty$.

4. A countably infinite number of independent free parameters c_n^0 are available to meet the nonlinear constraint corresponding to the coincidence of corotating streamlines and lines of constant vorticity.

Of course, the $n = 0$ modes correspond to the Lamb-Oseen vortex as one might expect.

5. The coincidence of streamlines and vorticity contours. In the previous section, we established the existence of Fourier modes with the appropriate behavior near the origin and at infinity which satisfy the linear constraints on this problem. In this section, we discuss the nonlinear constraint that lines of constant vorticity and corotating streamlines coincide. We examine these constraints as power series near the origin. From Frobenius theory, we know that exactly one solution to (4.2) satisfies the appropriate conditions at the origin. If $n > 0$ and

$$a_n(r) = \sum_{k=0}^{\infty} c_n^k r^{n+2k} \tag{5.1}$$

then we obtain the appropriate recurrence relation

$$c_n^k = -\frac{\frac{1}{2}(2k+n-2)+\lambda_n}{4k(k+n)} c_n^{k-1}, \quad k \geq 1. \tag{5.2}$$

Thus, if n is even (odd), $a_n(r)$ is even (odd). The first coefficient c_n^0 is the free parameter discussed in the previous section.

However, it follows that if

$$b_n(r) = B_n r^n + \sum_{k=0}^{\infty} d_n^k r^{n+2+2k} \tag{5.3}$$

where

$$B_n = c_n^0 \tilde{B}_n \tag{5.4}$$

then we can determine the coefficients d_n^k in terms of c_n^k from (4.23):

$$d_n^k = -\frac{c_n^k}{4(k+1)(n+k+1)}. \tag{5.5}$$

Imposing the coincidence of streamlines and vorticity contours yields nonlinear constraints on $a_n(r)$ and $b_n(r)$. In particular, if the streamlines and vorticity contours coincide, then

$$\nabla\widehat{\psi}^{\mathrm{rot}} \cdot R(\nabla\omega) = \frac{1}{r}\left(-\partial_r\widehat{\psi}^{\mathrm{rot}}\partial_\theta\omega + \partial_\theta\widehat{\psi}^{\mathrm{rot}}\partial_r\omega\right) = 0 \tag{5.6}$$

where R is a standard $\pi/2$ rotation. Expressing $\widehat{\psi}^{\mathrm{rot}}$ and f in terms of their Fourier expansions yields

$$\sum_{n=0}^{\infty}\left\{\left[\sum_{k+l=n}\frac{i}{r}\left(-k a_k b_l' + l a_k' b_l\right)\right] e^{in\theta} + \mathrm{CC}\right\} = 0 \tag{5.7}$$

where $a_{-k} = \overline{a_k}$. Gathering terms in this way is justified because the constituent Fourier series are absolutely convergent. Thus, finding self-similar, viscous, rotating vortical structures is equivalent to finding a set of functions $\{a_n\}$ satisfying (4.9) and $\{b_n\}$ satisfying (4.23) such that

$$\sum_{k+l=n} (-k a_k b_l' + l a_k' b_l) = 0 \tag{5.8}$$

for every n.

To classify this set of functions, we focus on the case when $n = 0$ and study (5.8) for each power of r. If $n = 0$, $l = -k$ so we only need focus on odd powers of r. Collecting terms, we obtain

$$\sum_{k=1}^{\infty} \left[k \left(-a_k \overline{b_k}' - a_k' \overline{b_k} + \overline{a_k} b_k' + \overline{a_k}' b_k \right) \right] = 0, \tag{5.9}$$

which can be simplified as

$$\sum_{k=1}^{\infty} k \Im \left[(\overline{a_k} b_k)' \right] = 0. \tag{5.10}$$

Clearly, there is no information about the $k = 0$ mode from this expression.

We observe that the nonzero terms in the power series of $(\overline{a_k} b_k)'$ correspond to odd powers, and that the lowest power of $(\overline{a_k} b_k)'$ is r^{2k-1}. Examining the coefficients term by term, we see that

$$\overline{a_k} b_k = \overline{c_k^0} B_k r^{2k} + \sum_{m=k}^{\infty} \left[\left(\sum_{i=0}^{m-k} \overline{c_k^i} d_k^{m-k-i} \right) + \overline{c_k^{m-k+1}} B_k \right] r^{2m+2}. \tag{5.11}$$

Substituting (5.11) into (5.10), we see that

$$\begin{aligned}
\sum_{k=1}^{\infty} k \Im \left[(\overline{a_k} b_k)' \right] =& \sum_{k=1}^{\infty} k \Im \Bigg\{ 2k \overline{c_k^0} B_k r^{2k-1} + \\
& \sum_{m=k}^{\infty} (2m+2) \left[\left(\sum_{i=0}^{m-k} \overline{c_k^i} d_k^{m-k-i} \right) + \overline{c_k^{m-k+1}} B_k \right] r^{2m+1} \Bigg\} \\
=& \sum_{k=1}^{\infty} 2k^2 \Im \left(\overline{c_k^0} B_k r^{2k-1} \right) + \\
& \sum_{m=1}^{\infty} \left\{ (2m+2) r^{2m+1} \sum_{k=1}^{m} k \Im \left[\left(\sum_{i=0}^{m-k} \overline{c_k^i} d_k^{m-k-i} \right) + \overline{c_k^{m-k+1}} B_k \right] \right\} \\
=& \sum_{k=1}^{\infty} 2k^2 |c_k^0|^2 r^{2k-1} \Im \left(\tilde{B}_k \right) + \\
& \sum_{m=1}^{\infty} \left\{ (2m+2) r^{2m+1} \sum_{k=1}^{m} k \Im \left[\left(\sum_{i=0}^{m-k} \overline{c_k^i} d_k^{m-k-i} \right) + \overline{c_k^{m-k+1}} B_k \right] \right\} \\
=& 0.
\end{aligned} \tag{5.12}$$

We shall prove that $a_n(r) = 0$ for $n \geq 1$ by strong induction. From (5.2), it suffices to prove that $c_n^0 = 0$. (It follows from (5.5) that d_n^0 is also zero.)

If we examine $O(r)$ terms in Equation (5.12), we see that only the first term in the first summation makes any contribution, so that

$$2|c_1^0|^2 r \Im\left(\tilde{B}_1\right) = 0. \tag{5.13}$$

From (4.27), we know that $\Im(\tilde{B}_1) \neq 0$, and therefore $c_1^0 = 0$.

Now, we assume that $c_l^0 = 0$ for all $l < n$. Simplifying (5.12), we have

$$\sum_{k=n}^{\infty} 2k^2 |c_k^0|^2 r^{2k-1} \Im\left(\tilde{B}_k\right) + \\ \sum_{m=n}^{\infty} \left\{ (2m+2) r^{2m+1} \sum_{k=n}^{m} k \Im\left[\left(\sum_{i=0}^{m-k} \overline{c_k^i} d_k^{m-k-i} \right) + \overline{c_k^{m-k+1}} B_k \right] \right\} = 0. \tag{5.14}$$

The first nonzero term occurs at $O(r^{2n-1})$ so that

$$2n^2 |c_n^0|^2 \Im\left(\tilde{B}_n\right) = 0, \tag{5.15}$$

leaving us to conclude once again that $c_n^0 = 0$. Thus, by strong induction, $c_n^0 = 0$ for all $n > 0$, and therefore $\tilde{a}_n(r) = 0$ and $\tilde{b}_n(r) = 0$. Of course, the surviving $n = 0$ component, corresponding to the Lamb-Oseen monopole, automatically satisfies the coincidence constraint (5.6).

This establishes the theorem stated in §1.

6. Conclusion. We have sought out localized, rotating, self-similar solutions to the Navier-Stokes equations under very general circumstances and found that they do not exist except in the form of Gaussians. This result does not imply that coherent multipoles do not exist on an intermediate timescale. To the contrary, such multipoles have been observed in laboratory experiments and in nature. Nor does this result preclude the possibility of a non-self-similar asymptotic structure, but one would think that this is extremely unlikely.

Thus, this result suggests that the observed multipolar structures are evolving toward to a monopolar, asymptotic state if not through a dynamic instability, such as Carton & Legras observed, then through mixing and viscous erosion, such as observed by Rossi, Lingevitch & Bernoff. Finally, this result is not restricted solely to multipolar structures consisting of a central core with oppositely signed satellites as have been studied extensively in the literature. Rather, this result is generally applicable to any localized, rotating vortical object.

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