Dynamics of Random Continued Fractions

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Abstract

We study a generalization of a continued fraction of Ramanujan with random coefficients. A study of the continued fraction is equivalent to an analysis of the convergence of certain stochastic difference equations and the stability of random dynamical systems. We determine the convergence properties of stochastic difference equations and so divergence of their corresponding continued fractions.

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1 Introduction

For the sequence of random variables $a := (a_n)_{n=1}^{\infty}$, denote the continued fraction $\mathcal{S}_1(a)$ by

$$\mathcal{S}_1(a) = \cfrac{1^2 a_1^2}{1 + \cfrac{2^2 a_2^2}{1 + \cfrac{3^2 a_3^2}{1 + \ddots}}} \tag{1.1}$$

We are interested in necessary and sufficient conditions for the divergence (alternatively convergence) of $\mathcal{S}_1(a)$. Special cases of the above continued fraction for particular choices of a have been determined in [3, 2]. A more general analysis has been detailed for arbitrary deterministic sequences [4]. In the present work we focus exclusively on the properties of $\mathcal{S}_1$ with random parameters. This

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includes the possibility of *partially* random sequences, for example, when every n-th element of a is random, but the other's are deterministic.

To evaluate $\mathcal{S}_1$, we study the recurrence for the classical convergents p_n/q_n to the fraction $\mathcal{S}_1$. For a general continued fraction of the form

$$\mathcal{S}_\eta(\gamma) = \eta_0 + \cfrac{\gamma_1}{\eta_1 + \cfrac{\gamma_2}{\eta_2 + \cfrac{\gamma_3}{\eta_3 + \ddots}}}$$

these are defined by the truncated continued fraction: $p_{-1} = 1,\ p_0 = \eta_0,\ q_{-1} = 0,\ q_0 = 1$ and

$$\begin{aligned}
\mathcal{S}_\eta(\gamma) &\approx \frac{p_1}{q_1} = \frac{\eta_1 p_0 + \gamma_1 p_1}{\eta_1 q_0} = \eta_0 + \frac{\gamma_1}{\eta_1} && \text{first order,} \\
&\approx \frac{p_2}{q_2} = \frac{\eta_2 p_1 + \gamma_2 p_0}{\eta_2 q_1 + \gamma_2 q_0} = \eta_0 + \cfrac{\gamma_1}{\eta_1 + \cfrac{\gamma_1}{\eta_2/\gamma_2}} && \text{second order,} \\
&\approx \ldots \\
&\approx \frac{p_n}{q_n} = \frac{\eta_n p_{n-1} + \gamma_n p_{n-2}}{\eta_n q_{n-1} + \gamma_n q_{n-2}} = \eta_0 + \cfrac{\gamma_1}{\eta_1 + \cfrac{\gamma_1}{\cfrac{\eta_2}{\ddots + \cfrac{\ddots}{\eta_n/\gamma_n}}}} && \text{n'th order.}
\end{aligned}$$

A simple induction argument establishes the general recurrence for the numerator and denominator p_n and q_n shown above, namely

$$p_n = \eta_n p_{n-1} + \gamma_n p_{n-2} \quad \text{and} \quad q_n = \eta_n q_{n-1} + \gamma_n q_{n-2}.$$

For the continued fraction $\mathcal{S}_1(a)$ we have

$$q_n = q_{n-1} + n^2 \alpha_n q_{n-2} \quad \text{where} \quad \alpha_n := a_n^2. \tag{1.2}$$

We will use α_n and a_n^2 interchangeably throughout. The p_n terms of the classical convergents also satisfy Eq.(1.2).

Following [2, 4], it is helpful to consider the renormalized sequence (v_n) where

$$v_n := \frac{q_n}{\Gamma(n + 3/2) a_n^{(n+1)}}. \tag{1.3}$$

A standard identity [6, Eq.(1.2.10)] for the separation of the convergents to $\mathcal{S}_1$ yields

$$\begin{aligned} \frac{p_{cn}}{q_{cn}} - \frac{p_{cn-1}}{q_{cn-1}} &= \frac{(-1)^{cn-1}(cn)!^2}{q_{cn}q_{cn-1}} \prod_{j=1}^{cn} a_j^2 \\ &= \frac{(-1)^{cn-1}(cn)!^2}{q_{cn}q_{cn-1}} \left(\prod_{j=1}^{c} \alpha_j \right)^n . \end{aligned} \tag{1.4}$$

In terms of the renormalized sequence (v_n), this is

$$\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} = \frac{(-1)^{n-1}}{v_n v_{n-1} a_n^{(n+1)} a_{n-1}^n} \left(\prod_{j=1}^{n} \alpha_j \right) \left\{ 1 + O\left(\frac{1}{n}\right) \right\}. \tag{1.5}$$

From the above preliminary analysis it is immediately clear that, for $|a_n| = |a_m| = b \neq 0$ for all $n, m \in \mathbb{N}$, the continued fraction $\mathcal{S}_1$ diverges – that is, the convergents separate – if

$$(v_n) \text{ is bounded.} \tag{1.6}$$

The case of cyclic and arbitrary deterministic sequences of parameters a_n has been treated in [4]. To tie the present theory to more classical results we briefly discuss the case of random real parameters in Section 3. In Section 4 we broaden our scope to general random sequences. Before proceeding with the analysis, however, we motivate this study in Section 2 with some numerical experiments of specific examples.

2 Numerical Motivation

For different cases of the parameters a_n in the continued fraction $\mathcal{S}_1$ we plot in the complex plane odd and even iterates of the recurrence

$$v_n = \frac{2}{a_n(2n+1)} \left(\frac{a_{n-1}}{a_n}\right)^n v_{n-1} + \frac{4n^2}{(2n-1)(2n+1)} \left(\frac{a_{n-2}}{a_n}\right)^{(n-1)} v_{n-2}. \tag{2.1}$$

which follows directly from the rescaling Eq.(1.3). Our examples focus on the case $|a_n| = b$ for all n, and, in particular (without loss of generality) $|a_n| = 1$. As a point of reference we reproduce in Fig. 1-2 the dynamics for periodic (a_n) with cycle length $1, 2,$ 3, and 4, and each a_n being a root of unity. These cases have been studied at length in [4]. It appears from these simulations that the sequence (v_n) is bounded for even length cycles, hence $\mathcal{S}_1$ *diverges*. This has been confirmed in [4] for these parameter values. Odd length cycles display a richer variety of behaviors, not all convergent, as shown in Fig. 2(b)-(c).

A remarkable fact is that, even if the sequences (a_n) are chosen at random, as long as the magnitude of the iterates is constant, the odd and even iterates demonstrate a surprising amount of structure. This is shown in by Fig. 3. We explain this remarkable regularity in the following sections.

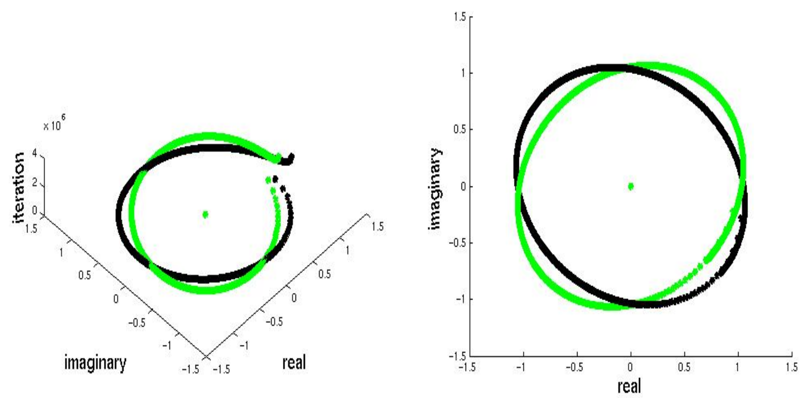

Figure 1: *Dynamics for cycles of length $c = 1$. Shown are the iterates v_n given by Eq.(2.1) with $a_n = \exp(i\pi/2)$ fir all j. Odd iterates are light, even iterates are dark.*

3 Random modulus/fixed phase

We consider parameters a_n of $\mathcal{S}_1$ of the form

$$a_n = b_n e^{i\theta}, \qquad b_n \in \mathbb{R}\ \forall n, \quad \text{and } \theta \text{ fixed.}$$

To our knowledge, results on the convergence/divergence of $\mathcal{S}_1$ are available only for the case where $\theta = 0$ and, for all n, the amplitudes b_n lie on a finite domain that excludes a neighborhood of the origin. This result has its origins in the Seidel-Stern Theorem [6] and is stated next.

Theorem 3.1 (arbitrary real parameters) *The generalized Ramanujan continued fraction $\mathcal{S}_1$ converges whenever all parameters a_n are real and satisfy $0 < m \leq |a_n| \leq M < \infty$.*

Proof. The proof is in [4], but is simple and short enough to warrant repeating here. We write $\mathcal{S}_1$ as a reduced continued fraction $\widehat{\mathcal{S}}_1$ with coefficients $A_i > 0$, that is,

$$\widehat{\mathcal{S}}_1(a) = \cfrac{1}{A_1 + \cfrac{1}{A_2 + \cfrac{1}{A_3 + \ddots}}} \tag{3.1}$$

where

$$A_n = \begin{cases} \frac{n!^2}{(n/2)!^4} 4^{-n} \prod_{j=1}^{n/2} \frac{a_{2j-1}^2}{a_{2j}^2} \; = \; \left(\frac{2}{n\pi} + O\left(\frac{1}{n^2}\right)\right) \prod_{j=1}^{n/2} \frac{a_{2j-1}^2}{a_{2j}^2} & (n \text{ even}) \\ \frac{(((n-1)/2)!)^4}{n!^2} \frac{4^{n-1}}{a_n^2} \prod_{j=1}^{(n-1)/2} \frac{a_{2j}^2}{a_{2j-1}^2} \; = \; \left(\frac{2}{n\pi} + O\left(\frac{1}{n^2}\right)\right) \frac{1}{a_n^2} \prod_{j=1}^{(n-1)/2} \frac{a_{2j}^2}{a_{2j-1}^2} & (n \text{ odd}). \end{cases} \tag{3.2}$$

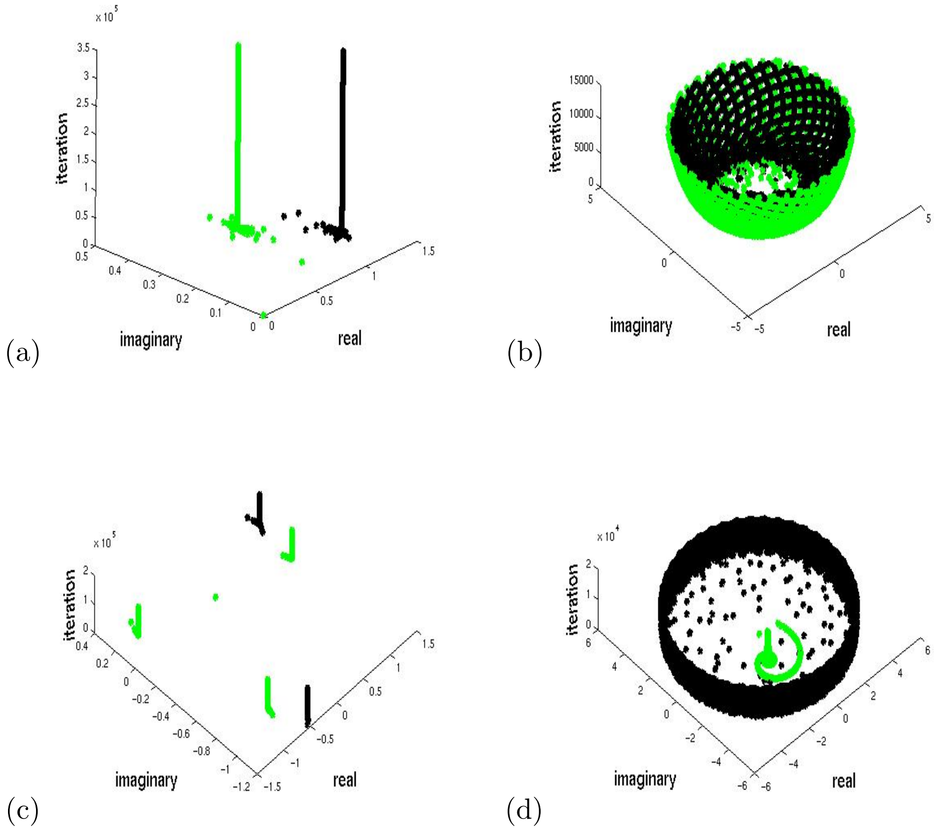

Figure 2: *Dynamics for cycles of length* $c = 2, 3$ *and* 4*. Shown are the iterates* v_n *given by Eq.(2.1) with (a)* $(a_1, a_2) = (\exp(i\pi/4), \exp(i\pi/6))$*, (b)* $(a_1, a_2, a_3) = (\exp(i\pi/4), \exp(i\pi/4), \exp(i\pi/4 + 1/\sqrt{2}))$*, (c)* $(a_1, a_2, a_3) = (\exp(i\pi/2), \exp(i\pi/6), \exp(-i\pi/6))$*, and (d)* $a_1 = a_3 = \exp(i\pi/4)$*,* $a_2 = \exp(i\pi/6)$*,* $a_4 = \exp(i(\pi/6 + 1/2))$ *Odd iterates are light, even iterates are dark.*

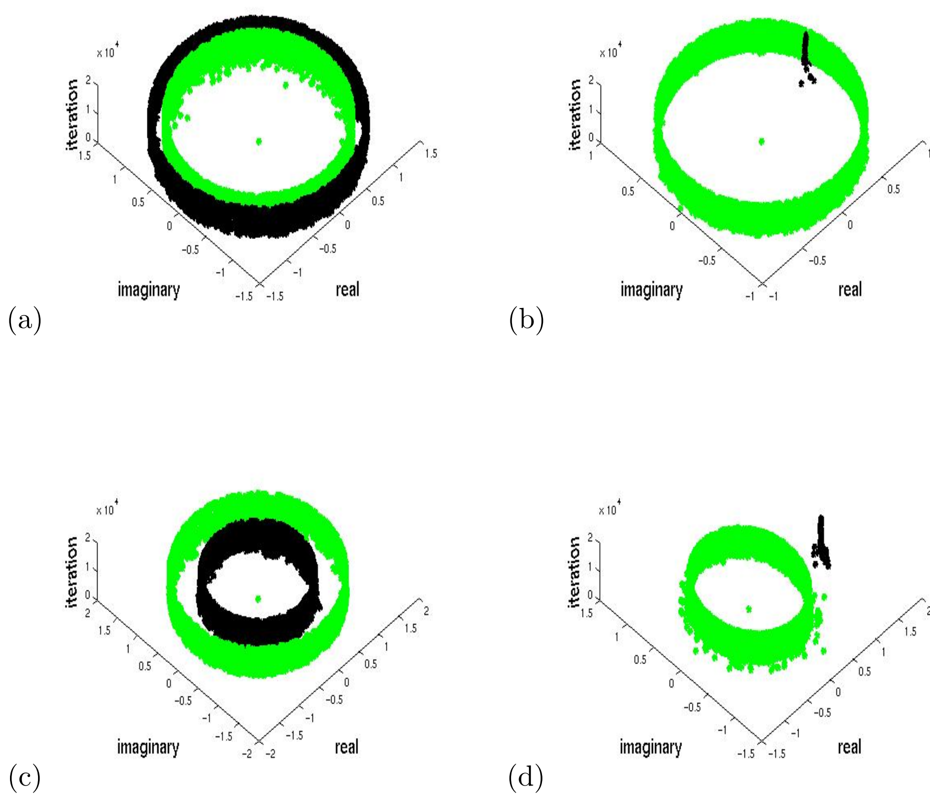

Figure 3: *Dynamics for random cycles. Shown are the iterates* v_n *given by Eq.(2.1) with (a)* $a_n = \exp(i\theta_n),\ \theta_n \sim U[0, 2\pi]$ *(b) one random strand* $\bmod 2$, $a_{2n+1} = \exp(i\pi/6)$, $a_{2n} = \exp(i\theta_n),\ \theta_n \sim U[0, 2\pi]$, *(c) one random strand* $\bmod 3$, $a_{3n+1} = a_{3n+2} = \exp(i\pi/6)$, $a_{3n} = \exp(i\theta_n),\ \theta_n \sim U[0, 2\pi]$, *and (d) one random strand* $\bmod 4$, $a_{4n+1} = a_{4n+2} = a_{4n+3} = \exp(i\pi/6)$, $a_{4n} = \exp(i\theta_n),\ \theta_n \sim U[0, 2\pi]$, *Odd iterates are light, even iterates are dark.*

Note that for real a_n satisfying $0 < m \leq |a_n| \leq M < \infty$ the sum of the coefficients A_n is unbounded. Convergence then follows from the Seidel-Stern Theorem [6], which asserts that a reduced continued fraction converges if and only if $\sum A_i = \infty$. $\square$

Similar results for $\theta \neq 0$ appear to require new tools because the Stern-Seidel Theorem, which equates convergence of $\widehat{\mathcal{S}}_1$ with divergence of the sum $\sum A_i$ above, is not relevant in the case of complex parameters. This appears to be an open problem.

4 General Random Parameters

In this section we pursue a general theory for arbitrary random parameters a_n. Our principal tools draw from a matrix analysis of $\mathcal{S}_1$ based on the renormalized sequence (v_n) defined by Eq.(2.1). Though the phases of the parameters are entirely random, the sequence (v_n) exhibits an odd/even behavior as the figures illustrate. To see why this might be, note that the recurrence is a 2-step backward difference equation. Reformulating Eq.(1.2) in terms of 2×2 matrices yields

(4.1) $$q^{(n)} = Q_n q^{(n-1)} \quad \text{where} \quad Q_n := \begin{bmatrix} 1 & n^2\alpha_n \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad q^{(n)} := \begin{pmatrix} q_n \\ q_{n-1} \end{pmatrix}.$$

The analogous sequence of vectors corresponding to the rescaling v_n is

(4.2) $$v^{(n)} := \begin{pmatrix} v_n \\ v_{n-1} \end{pmatrix}.$$

To show the odd/even behavior inherent in the difference equation we examine the sequence

(4.3) $$v^{(n)} = Y_n v^{(n-1)} \quad \text{where} \quad Y_n := G_n^{-1} Q_n G_{n-1}$$

for

(4.4) $$G_n := \mathrm{Diag}\left(\Gamma\left(n+\frac{3}{2}\right) a_n^{(n+1)}, \Gamma\left(n+\frac{1}{2}\right) a_{n-1}^{n}\right).$$

Define the matrix $\widehat{Y}_n$ by

(4.5) $$\widehat{Y}_n := Y_{2n} Y_{2n-1}.$$

This has the explicit representation

(4.6)
$$\widehat{Y}_n = \begin{bmatrix} \left(\frac{\alpha_{2n-2}}{\alpha_{2n}}\right)^n \left(\frac{1}{\alpha_{2n-2}\alpha_{2n}}\right)^{1/2} \left(\frac{1+4n^2\alpha_{2n}}{4n^2-1/4}\right) & \left(\frac{\alpha_{2n-3}}{\alpha_{2n}}\right)^n \left(\frac{\alpha_{2n-1}}{\alpha_{2n-3}\alpha_{2n}^{1/2}}\right) \left(\frac{(2n-1)^2}{(2n-3/2)(4n^2-1/4)}\right) \\ \left(\frac{\alpha_{2n-2}}{\alpha_{2n-1}}\right)^n \frac{1}{\alpha_{2n-2}^{1/2}} \frac{1}{(2n-1/2)} & \left(\frac{\alpha_{2n-3}}{\alpha_{2n-1}}\right)^{n-1} \left(\frac{(2n-1)^2}{(2n-1)^2-1/4}\right) \end{bmatrix}.$$

The determinant of this general $\widehat{Y}_n$ is

$$\det(\widehat{Y}_n) = \left(\frac{\alpha_{2n-2}}{\alpha_{2n}}\right)^{n-1/2} \left(\frac{\alpha_{2n-3}}{\alpha_{2n-1}}\right)^{n-1} \frac{64n^2(2n-1)^2}{(4n-3)(4n-1)^2(4n+1)}.$$

The odd/even behavior is apparent in the α_j terms above. Moreover, the identity

$$\prod_{n=2}^{\infty} \frac{64n^2(2n-1)^2}{(4n-3)(4n-1)^2(4n+1)} = \frac{\pi}{2}$$

follows readily from the Wallis/Stirling formula [1] formula. Hence

$$\lim_{n\to\infty} \det(\mathcal{Y}_n) = \frac{\pi}{2}\beta \quad \text{where} \quad \mathcal{Y}_n = \prod_{j=1}^{n} \widehat{Y}_n. \tag{4.7}$$

and the random variable β is given by

$$\beta := \prod_{n=2}^{\infty} \left(\frac{\alpha_{2n-2}}{\alpha_{2n}}\right)^{n-1/2} \left(\frac{\alpha_{2n-3}}{\alpha_{2n-1}}\right)^{n-1} = \lim_{n\to\infty} \frac{\alpha_2^{1/2}}{\alpha_{2n}^{n-1/2}\alpha_{2n-1}^{n-1}} \prod_{j=1}^{2n-2} \alpha_j. \tag{4.8}$$

Existence of $\lim_{n\to\infty} \det(\mathcal{Y}_n)$ therefore depends on the product Eq.(4.8). Nevertheless, convergence of the determinant is no guarantee of the same for the matrices $\mathcal{Y}_n$. Proving that the matrices converge is the object of the analysis that follows.

Assume, for the moment however, that $\lim_{n\to\infty} \mathcal{Y}_n = \mathcal{Y}^\infty$ where $\mathcal{Y}^\infty$ is a finite complex random matrix. We then have the following generalization of [2, Theorem 4.1] concerning the convergence of odd and even parts of $\mathcal{S}_1(a)$.

Theorem 4.1 (odd and even convergents of random continued fractions) *Let the sequence of complex random variables $(a_n)_{n=1}^\infty$ satisfy*

$$0 \neq \beta := \lim_{n\to\infty} \frac{a_2}{a_{2n}^{2n-1}a_{2n-1}^{2n-2}} \prod_{j=1}^{2n-2} a_j^2 \qquad a.s.$$

For the corresponding continued fraction $\mathcal{S}_1(a)$ defined by Eq.(1.1), let (u_n) be the analog to (v_n) in Eq.(1.3) with q_n replaced by p_n. If the matrix $\mathcal{Y}_n$ defined by Eq.(4.7) converges almost surely to the finite random matrix $\mathcal{Y}^\infty$, then for the standard initial conditions

$$(u_{-1}, u_0, v_{-1}, v_0) = \left(\frac{1}{\sqrt{\pi}}, 0, 0, \frac{2}{a_0\sqrt{\pi}}\right), \tag{4.9}$$

the even and odd parts of $\mathcal{S}_1(a)$ are given by

$$\mathcal{S}_1^{(even)}(a) = \frac{a_0 y_{1,2}^\infty}{2y_{1,1}^\infty}, \quad \textit{and} \quad \mathcal{S}_1^{(odd)}(a) = \frac{a_0 y_{2,2}^\infty}{2y_{2,1}^\infty} \tag{4.10}$$

where $y_{i,j}^{\infty}$ is the i,jth element of $\mathcal{Y}^{\infty}$. These limits are almost surely not equal, thus $\mathcal{S}_1$ diverges almost surely. Indeed, the separation of odd and even limits is given explicitly by

$$\mathcal{S}_1^{(even)}(a) - \mathcal{S}_1^{(odd)}(a) = -\frac{a_0^2\pi}{4a_2 y_{1,1}^{\infty} y_{2,1}^{\infty}}\beta. \tag{4.11}$$

Proof. The first relation Eq.(4.10) is immediate from the definition of the classical convergents. The limits cannot be equal since otherwise we would have

$$\frac{a_0 y_{1,2}^{\infty}}{2y_{1,1}^{\infty}} = \frac{a_0 y_{2,2}^{\infty}}{2y_{2,1}^{\infty}} \qquad \Longrightarrow \qquad y_{1,1}^{\infty} y_{2,2}^{\infty} - y_{1,2}^{\infty} y_{2,1}^{\infty} = 0$$

whence, from Eq.(4.7), $\beta = 0$. But this contradicts the assumption that $\beta \neq 0$ almost surely. To see Eq.(4.11) note that, by Eq.(1.5) and the initial condition $(v_{-1}, v_0) = (0, 2/(a_0\sqrt{\pi})$,

$$\begin{aligned} \mathcal{S}_1^{(even)}(a) - \mathcal{S}_1^{(odd)}(a) &= \lim_{n\to\infty} -\frac{\left(\prod_{j=1}^{2n} \alpha_j\right)}{v_{2n} v_{2n-1} \alpha_{2n}^{n+1/2} \alpha_{2n-1}^{n}}, \\ &= \lim_{n\to\infty} -\frac{a_0^2\pi}{4y_{1,1}^{(n)} y_{2,1}^{(n)}} \frac{\left(\prod_{j=1}^{2n} \alpha_j\right)}{\alpha_{2n}^{n+1/2} \alpha_{2n-1}^{n}}, \end{aligned}$$

where $y_{i,j}^{(n)}$ is the ij-th element of the matrix $\mathcal{Y}_n$ defined by Eq.(4.7). The limit above, together with Eq.(4.8), yields

$$\mathcal{S}_1^{(even)}(a) - \mathcal{S}_1^{(odd)}(a) = -\frac{a_0^2\pi}{4a_2 y_{1,1}^{\infty} y_{2,1}^{\infty}}\beta.$$

□

Remark 4.2 If $\beta = 0$ with probability > 0, then the analysis is indeterminate. Formally from the definition of the classical convergents we have

$$\frac{2}{a_0\sqrt{\pi}} y_{1,1}^{\infty} \mathcal{S}_1^{(even)} = \frac{1}{\sqrt{\pi}} y_{1,2}^{\infty} \quad \text{and} \quad \frac{2}{a_0\sqrt{\pi}} y_{2,1}^{\infty} \mathcal{S}_1^{(odd)} = \frac{1}{\sqrt{\pi}} y_{2,2}^{\infty}.$$

Multiplying the equation on the left by $y_{2,1}^{\infty}$ and the right by $y_{1,1}^{\infty}$ and subtracting yields

$$\frac{1}{\sqrt{\pi}}\left(y_{1,1}^{\infty} y_{2,2}^{\infty} - y_{1,2}^{\infty} y_{2,1}^{\infty}\right) = \frac{2}{a_0\sqrt{\pi}} y_{2,1}^{\infty} y_{1,1}^{\infty} \left(\mathcal{S}_1^{(even)} - \mathcal{S}_1^{(odd)}\right).$$

But, if $\beta = 0$, by Eq.(4.7) we have $y_{1,1}^{\infty} y_{2,2}^{\infty} - y_{1,2}^{\infty} y_{2,1}^{\infty} = 0$. and so $y_{2,1}^{\infty} y_{1,1}^{\infty} \left(\mathcal{S}_1^{(even)} - \mathcal{S}_1^{(odd)}\right) = 0$. We cannot determine from this analysis whether the separation of the odd and even convergents is zero as would be the case if $\mathcal{S}_1$ were to converge. □

What remains, then, is to determine whether $\mathcal{Y}_n$ converges as $n \to \infty$. To begin we extract the leading-order behavior. Expanding $\widehat{Y}_n$ in powers of n^{-1} yields

$$\widehat{Y}_n = K_n + \frac{1}{2n} W_n + O\left(n^{-2}\right)$$

where

$$K_n = \begin{bmatrix} \left(\frac{\alpha_{2n-2}}{\alpha_{2n}}\right)^{n-1/2} & 0 \\ 0 & \left(\frac{\alpha_{2n-3}}{\alpha_{2n-1}}\right)^{n-1} \end{bmatrix}, \tag{4.12}$$

and

$$W_n = \begin{bmatrix} 0 & \frac{1}{a_{2n}} \left(\frac{\alpha_{2n-3}}{\alpha_{2n}}\right)^{n} \left(\frac{\alpha_{2n-1}}{\alpha_{2n-3}}\right) \\ \frac{1}{a_{2n-2}} \left(\frac{\alpha_{2n-1}}{\alpha_{2n-2}}\right)^{-n} & 0 \end{bmatrix}. \tag{4.13}$$

Hence,

$$\mathcal{Y}_n = \prod_{j=2}^{n} \left(K_j + \frac{1}{2j} W_j + O(j^{-2}) \right) = \mathcal{U}_n + O(n^{-2}) \tag{4.14}$$

where

$$\mathcal{U}_n := \prod_{j=2}^{n} K_j + \frac{1}{2j} W_j.$$

By induction on n, this factors as

$$\mathcal{U}_n = \left(\prod_{j=2}^{n} K_j \right) \prod_{j=2}^{n} \left(I + \frac{1}{2j} \widehat{W}_j \right), \tag{4.15}$$

where

$$\widehat{W}_n := \left(\prod_{j=2}^{n} K_j \right)^{-1} \left(\prod_{j=2}^{n-1} \widehat{K}_j \right) W_n$$

for

$$\widehat{K}_j = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} K_j \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

As it turns out, $\widehat{W}_n$ has a simple explicit representation:

$$\widehat{W}_n = \frac{1}{a_{2n}} \begin{bmatrix} 0 & \omega_n \\ \omega_n^{-1} & 0 \end{bmatrix}, \quad \text{where} \quad \omega_n = \frac{a_{2n}}{a_2} \prod_{j=1}^{n} \frac{\alpha_{2j-1}}{\alpha_{2j}}. \tag{4.16}$$

To ease the computations, we focus our attention on the rotated product

$$\widehat{U}_n := \left(\prod_{j=2}^{n} K_j\right)^{-1} \mathcal{U}_n \;=\; \prod_{j=2}^{n}\left(I + \frac{1}{2j}\widehat{W}_j\right). \tag{4.17}$$

The justification for this follows next.

Theorem 4.3 *If $\widehat{U}_n \overset{a.s.}{\to} \widehat{U}_\infty$ and $\prod_{j=2}^n K_j \overset{a.s.}{\to} \mathcal{K}_\infty$ where both $\widehat{U}_\infty$ and $\mathcal{K}_\infty$ are nonsingular, then $\mathcal{U}_n \overset{a.s.}{\to} \mathcal{K}_\infty^{-1}\widehat{U}_\infty$ and $\mathcal{Y}_n \overset{a.s.}{\to} \mathcal{Y}^\infty$, a finite random matrix.*

Proof. This follows from Eq.(4.14) and [4, Theorem 5.3]. □

Theorem 4.3, together with Theorem 4.1, yields particularly clean sufficient conditions for the divergence of $\mathcal{S}_1$, but note that we have pushed the question of convergence of $\mathcal{Y}_n$ onto the convergence of $\widehat{U}_n$. We focus next on $\widehat{U}_n$.

Remark 4.4 (parameter qualifications) We briefly summarize our strategy and the accompanying restrictions. Most of the restrictions on the sequences (a_n) come from the invertibility assumption in Theorem 4.1 and that of $\widehat{U}_\infty$ and $\mathcal{K}_\infty$ in Theorem 4.3. The first of these, that $\beta \neq 0$ where β is defined by Eq.(4.8), was discussed in Remark 4.2. The assumption that $\prod_{j=1}^n K_j \overset{a.s.}{\to} \mathcal{K}_\infty$ invertible is equivalent to the condition

$$0 \neq \lim_{n\to\infty} \det\left(\prod_{j=2}^{n} K_j\right) < \infty \qquad \text{a.s.}$$

which amounts to

$$0 \neq \lim_{n\to\infty} \frac{\alpha_2^{1/2}}{\alpha_{2n}^{n-1/2}\alpha_{2n-1}^{n-1}} \prod_{j=1}^{2n-2} \alpha_j < \infty \qquad \text{a.s.} \tag{4.18}$$

From Eq.(4.16), the remaining invertibility assumption, namely that $\widehat{U}_n \overset{a.s.}{\to} \widehat{U}_\infty$ invertible, is equivalent to the condition

$$0 \neq \det \prod_{j=2}^{\infty}\left(I + \frac{1}{2ja_{2j}}\begin{bmatrix} 0 & \omega_n \\ \omega_n^{-1} & 0 \end{bmatrix}\right) < \infty \qquad \text{a.s.,}$$

or, more simply,

$$0 \neq \prod_{j=2}^{\infty}\left(1 - \frac{1}{(2ja_{2j})^2}\right) < \infty \qquad \text{a.s.} \tag{4.19}$$

Conditions Eq.(4.18) and Eq.(4.19) are fundamental to our analysis.

4.1 Stochastic Matrix Analysis

In this section we shall prove following.

Theorem 4.5 (stochastic matrix products) *Let*

$$\epsilon, b > 0, \qquad b - \epsilon > 0 \quad \text{and} \quad b + \epsilon > 1, \tag{4.20}$$

and let (ζ_j) *and* (ζ'_j) *be sequences of zero mean independent random variables that satisfy*

$$\sum_{j=1}^{\infty} \frac{1}{j^{2(b-\epsilon)}} var(\zeta_j) < \infty \quad \text{and} \quad \sum_{j=1}^{\infty} \frac{1}{j^{2(b-\epsilon)}} var(\zeta'_j) < \infty. \tag{4.21}$$

Then the matrix product

$$\prod_{j=1}^{n} \left(I + \frac{1}{(2j)^b} \begin{bmatrix} 0 & \zeta_j \\ \zeta'_j & 0 \end{bmatrix} \right) \tag{4.22}$$

converges almost surely to a finite matrix as $n \to \infty$*. If, in addition,*

$$0 < \left| \prod_{j=1}^{\infty} \left(1 - \frac{\zeta'_j \zeta_j}{(2j)^{2b}} \right) \right| < \infty \qquad a.s.,$$

then the matrix product converges almost surely to an invertible matrix.

To begin we collect some useful facts about the rate of convergence of sequences. We denote the limit of the sequence of random variables (a_n) by a_∞, and denote by $(a_n) \prec (\epsilon_n)$ almost sure convergence of (a_n) when this is provided by $|a_n - a_\infty| = O(\epsilon_n)$ almost surely.

Lemma 4.6 *Let* (a_n) *and* (b_n) *be complex sequences, let* (ϵ_n) *be a positive sequence and let* (z_n) *with* $|z_n| = z \in \mathbb{R}_+ \ \forall\ n = 1, 2, \ldots$ *be any complex number. Suppose that*

$$(a_n) \prec (\epsilon_n) \quad \text{and} \quad (b_n) \prec (\epsilon_n),$$

then

$$(a_n + b_n) \prec (\epsilon_n), \qquad (a_n b_n) \prec (\epsilon_n) \quad \text{and} \quad (z_n a_n) \prec (\epsilon_n)$$

Proof. The first two relations are clear. The last relation follows immediately from $|z_n a_n - z_n a_\infty| = z|a_n - a_\infty|$. □

We state next a fundamental result which will yield, eventually, the conditions Eq.(4.21) for convergence of infinite products of random variables. Recall that a sequence (Σ_n) is a martingale with respect to the sequence (ζ_n) if, for all $n \geq 1$

$$(a) \quad \mathcal{E}(\Sigma_n) < \infty, \quad \text{and} \quad (b) \quad \mathcal{E}(\Sigma_{n+1} \mid \zeta_1, \ldots, \zeta_n) = \Sigma_n.$$

For example, the sequence of partial sums of random complex-valued variables with fixed modulus and random phase uniformly distributed on $[0, 2\pi)$ is a martingale.

Lemma 4.7 (martingale convergence theorem) *Let* (ζ_n) *be a sequence of zero mean random variables. Denote the corresponding martingale of partial sums by* $(\Sigma_n) = \left(\sum_{j=1}^{n} \zeta_j\right)$. *If*

$$\mathcal{E}(|\Sigma_n|^2) < \infty \tag{4.23}$$

then Σ_n *converges almost surely to a finite random variable* Σ_∞.

For the proof of Lemma 4.7 see [5, Theorem 7.8.1-2]. We are now ready to state the main building block for the proof of Theorem 4.5.

Proposition 4.8 *Let* ζ_n *be zero mean independent random variables satisfying*

$$\sum_{j}^{\infty} \frac{var(\zeta_j)}{j^{2(b-\epsilon)}} < \infty \quad for \quad \epsilon, b > 0, \quad and \quad 0 < b - \epsilon. \tag{4.24}$$

Then

$$\sum_{j=1}^{n} \frac{\zeta_j}{j^{b-\epsilon}} \overset{a.s.}{\to} \Sigma_\infty \quad as \quad n \to \infty \tag{4.25}$$

where Σ_∞ *is a finite random variable. Moreover*

$$\left(\sum_{j=1}^{n} \frac{\zeta_j}{j^b} \right) \prec \left(\frac{1}{j^\epsilon} \right). \tag{4.26}$$

Proof. Equation Eq.(4.25) is a slight modification of [5, Ex.7.8.2] and follows immediately from Lemma 4.7. To prove Eq.(4.26), note that by Eq.(4.25) we have

$$\sup_n \left| \sum_{j=1}^{n} \frac{\zeta_j}{j^{b-\epsilon}} \right| < \infty \qquad \text{a.s.} \tag{4.27}$$

This together with Abel's transformation, [8, Eq.(I.2.1)],

$$\sum_{j=m}^{n} \frac{1}{j^\epsilon} \frac{\zeta_j}{j^{b-\epsilon}} = \sum_{j=m}^{n-1} \left(\left(\sum_{k=m}^{j} \frac{\zeta_k}{k^{b-\epsilon}} \right) \left(\frac{1}{j^\epsilon} - \frac{1}{(j+1)^\epsilon} \right) \right) + \frac{1}{n^\epsilon} \sum_{j=m}^{n} \frac{\zeta_j}{j^{b-\epsilon}} \qquad (m < n), \tag{4.28}$$

yield

$$\left| \sum_{j=m}^{n} \frac{\zeta_j}{j^b} \right| \le \frac{1}{n^\epsilon} \sup_k \left| \sum_{j=m}^{k} \frac{\zeta_j}{j^{b-\epsilon}} \right| = O\left(\frac{1}{n^\epsilon} \right) \qquad \text{a.s.} \tag{4.29}$$

□

Lemma 4.9 (product convergence) *Let ζ_j be zero mean independent random variables satisfying Eq.(4.24). Then for $b > 1/2$ the product*

$$\prod_{j=1}^{n}\left(1+\frac{1}{j^b}\zeta_j\right) \tag{4.30}$$

converges almost surely as $n \to \infty$.

Proof. Since the elements ζ_j have zero mean, by Proposition 4.8 ζ_j satisfies

$$\sum_{j=1}^{n}\frac{\zeta_j}{j^b} \overset{a.s.}{\to} \Sigma_\infty \quad \text{as} \quad n \to \infty.$$

Also, since ζ_j have zero mean and satisfy Eq.(4.24), we have

$$\sum_j \left|\frac{\zeta_j}{j^b}\right|^2 < \infty.$$

Thus, by general sufficient criteria for the convergence of infinite complex products [7, the Coriolis Test] the product Eq.(4.30) converges almost surely. □

We are now ready to proceed with the proof of the main result of this section.

Proof of Theorem 4.5 Our proof follows the same pattern as that of [2, Theorem 8.1]. We split the matrices in the infinite product into upper and lower triangular pieces and show that the resulting submatrices and their products converge. Let

$$U := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad L := \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

We write

$$\left(I + \frac{1}{(2j)^b}\begin{bmatrix} 0 & \zeta_j \\ \zeta_j' & 0 \end{bmatrix}\right) = \left(I + \frac{1}{(2j)^b}\zeta_j U\right)\left(I + \frac{1}{(2j)^b}\zeta_j' L\right) - \frac{1}{(2j)^{2b}}\zeta_j\zeta_j' UL$$

and define the partial product

$$\Pi_{UL}^n = \prod_{j=1}^{n}\left(\left(I + \frac{1}{(2j)^b}\zeta_j U\right)\left(I + \frac{1}{(2j)^b}\zeta_j' L\right)\right).$$

For $n \in \mathbb{N}$ let

$$\Pi_U^n := \prod_{j=1}^{n}\left(I + \frac{1}{(2j)^b}\zeta_j U\right), \qquad \Pi_L^n := \prod_{j=1}^{n}\left(I + \frac{1}{(2j)^b}\zeta_j' L\right) \tag{4.31}$$

$$\Sigma_n := \sum_{j=1}^{n}\frac{1}{(2j)^b}\zeta_j, \qquad \text{and} \qquad \Sigma_n' := \sum_{j=1}^{n}\frac{1}{(2j)^b}\zeta_j'. \tag{4.32}$$

We interpret Σ_0 and Σ_0' to be zero. By [2, Lemma 8.6] (replace their "$zm_j\omega^j$" by "$\frac{1}{(2j)^b}\zeta_j$" and "$zm_j\omega^{-j}$" by "$\frac{1}{(2j)^b}\zeta_j'$") Π_{UL}^n can be rewritten as

$$\Pi_{UL}^n = \Pi_U^n \Pi_L^n \prod_{j=1}^{n} (I + R_j)\,,$$

where

$$R_n := \frac{1}{(2n)^b}\zeta_n' \begin{bmatrix} -\Sigma_{n-1} - (\Sigma_{n-1})^2 \Sigma_{n-1}' & -(\Sigma_{n-1})^2 \\ \Sigma_{n-1}\Sigma_n' + \Sigma_{n-1}\Sigma_{n-1}' + \Sigma_{n-1}'\Sigma_n' (\Sigma_{n-1})^2 & \Sigma_{n-1} + (\Sigma_{n-1})^2 \Sigma_n' \end{bmatrix}. \tag{4.33}$$

By the definitions of Σ_0 and Σ_0', we have $R_1 := 0$. The partial sums Σ_n and Σ_n' converge almost surely by Proposition 4.8. By induction it can be shown that

$$\Pi_U^n = \begin{bmatrix} 1 & \Sigma_n \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \Pi_L^n = \begin{bmatrix} 1 & \Sigma_n' \\ 0 & 1 \end{bmatrix},$$

thus the sequences of matrices (Π_U^n) and (Π_L^n) converge almost surely . Hence, if $\prod_{j=1}^n (I + R_j)$ converges almost surely, then the sequence (Π_{UL}^n) converges almost surely. We rewrite R_n as

$$R_n = \frac{\zeta_n'}{(2n)^b} \begin{bmatrix} r_{11}^n & r_{12}^n \\ r_{21}^n & r_{22}^n \end{bmatrix} \tag{4.34}$$

where, by Lemma 4.6, Eq.(4.26), and Eq.(4.32)

$$\left(r_{jk}^n\right) \prec \left(\frac{1}{n^\epsilon}\right) \qquad \forall j,k. \tag{4.35}$$

Our strategy is to split R_n into a sequence of absolutely convergent matrices and a convergent scaling of its limit. To this end, let

$$R_n = P_n + \frac{\zeta_n'}{(2n)^b} T \tag{4.36}$$

where

$$P_n := \frac{\zeta_n'}{(2n)^b} \begin{bmatrix} r_{11}^n - r_{11}^\infty & r_{12}^n - r_{12}^\infty \\ r_{21}^n - r_{21}^\infty & r_{22}^n - r_{22}^\infty \end{bmatrix} \quad \text{and} \quad T := \begin{bmatrix} r_{11}^\infty & r_{12}^\infty \\ r_{21}^\infty & r_{22}^\infty \end{bmatrix}. \tag{4.37}$$

If

$$\prod_{j=1}^{n} \left(I + \frac{\zeta_j'}{(2j)^b} T \right) \quad \text{and} \quad \sum_{j=1}^{\infty} |P_n| \tag{4.38}$$

converge, then by [2, Theorem 6.1], the product $\prod_{j=1}^n (I + R_j)$ converges. Convergence of the product in Eq.(4.38) follows exactly as in the proof of [2, Lemma 8.7] and relies on Lemma 4.9. An examination of the eigenvalues of P_n shows that, almost surely, $|P_n| = O(1/n^{b+\epsilon})$. Thus, since by Eq.(4.20) $b + \epsilon > 1$, we have that $\sum_{j=1}^\infty |P_n|$ converges almost surely.

Again, by Proposition 4.8, we have

$$\sum_{j=1}^{n}\left|\frac{\zeta_j\zeta_j'}{j^{2b}}\right| < \infty,$$

whence convergence of the matrix sum

$$\sum_{j=1}^{n}\left|\frac{\zeta_j\zeta_j'}{j^{2b}}UL\right|.$$

This proves the convergence of Eq.(4.22).

To complete the proof note that

$$\det \widehat{U}_n = \det \prod_{j=1}^{n}\left(I + \frac{1}{(2j)^b}\begin{bmatrix} 0 & \zeta_j \\ \zeta_j' & 0 \end{bmatrix}\right) = \prod_{j=1}^{n}\det\left(I + \frac{1}{(2j)^b}\begin{bmatrix} 0 & \zeta_j \\ \zeta_j' & 0 \end{bmatrix}\right) = \prod_{j=1}^{n}\left((1 - \frac{1}{(2j)^{2b}}\zeta_j'\zeta_j\right).$$

This product is nonzero if $\left|1 - \frac{1}{(2j)^{2b}}\zeta_j'\zeta_j\right| \geq m > 0 \quad \forall\, j$, in which case $\widehat{U}_n$ converges invertibly. □

4.2 Application to continued fractions

Theorems 4.1, 4.3 and 4.5 together yield sufficient conditions for the divergence of the partial fraction $\mathcal{S}_1$. We specialize these results to the case of continued fractions with parameters (a_n) distributed uniformly on the unit circle in the complex plane as shown in Fig. 3. Following section 4.1, by Eq.(4.16) the random variables ζ_n and ζ_n' in this case are given by

$$\zeta_n := \frac{1}{a_{2n}}\omega_n = \frac{1}{a_2}\left(\prod_{j=1}^{n}\frac{\alpha_{2j-1}}{\alpha_{2j}}\right). \tag{4.39}$$

and

$$\zeta_n' := \frac{1}{a_{2n}}\omega_n^{-1} = \frac{a_2}{\alpha_{2n}}\left(\prod_{j=1}^{n}\frac{\alpha_{2j-1}}{\alpha_{2j}}\right)^{-1}. \tag{4.40}$$

These are also random variables, uniformly distributed on the unit circle. By symmetry it is immediate that $\mathcal{E}(\zeta_n) = \mathcal{E}(\zeta_n') = 0$ and $\mathrm{var}(\zeta_n)$ and $\mathrm{var}(\zeta_n')$ are bounded for all n. Thus, for all $b - \epsilon > 1/2$ with $1 < \epsilon + b$ (in particular, for $b > 3/4$ and ϵ small)

$$\sum_n \frac{1}{n^{2(b-\epsilon)}}\mathrm{var}(\zeta_n) < \infty \quad \text{and} \quad \sum_n \frac{1}{n^{2(b-\epsilon)}}\mathrm{var}(\zeta_n') < \infty.$$

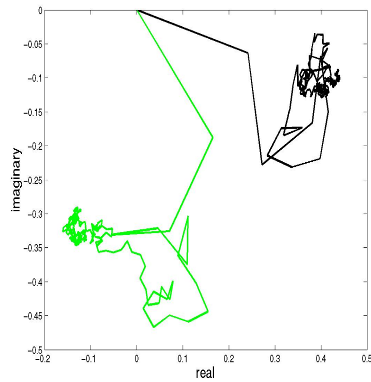

Figure 4: *Sequence of partial sums given by Eq.(4.41) for random* $(a_n) = (\exp(\mathrm{i}\theta_n))$ *for* θ_n *uniformly distributed on* $[0, 2\pi]$ *corresponding to Fig. 3(b). The dark line corresponds to the partial sums of* $\frac{1}{a_{2j}}\omega_j$ *and the light line to the partial sums of* $\frac{1}{a_{2j}}\omega_j^{-1}$.

Define the partial sums by

$$\Sigma_n := \sum_j^n \frac{\zeta_j}{j^b} \quad \text{and} \quad \Sigma'_n := \sum_j^n \frac{\zeta'_j}{j^b} \tag{4.41}$$

By Proposition 4.8 it follows that

$$\lim_{n\to\infty} \Sigma_n \overset{a.s.}{\to} \Sigma^\infty \quad \text{and} \quad \lim_{n\to\infty} \Sigma'_n \overset{a.s.}{\to} \Sigma^{\infty\prime}. \tag{4.42}$$

Here Σ^∞ and $\Sigma^{\infty\prime}$ are finite random variables. The partial sums represent a random walk. In Fig. 4 we show one realization of this random walk with $b = 1$ after 10^4 steps. The continued fraction has $b = 2$, so we have thus proved the divergence of $\mathcal{S}_1$ for the cases illustrated in Fig. 3. By Theorem 4.1 the odd and even parts of $\mathcal{S}_1$ converge to separate limits as is shown in the odd and even iterates of Eq.(2.1) converging to separate orbits.

5 Summary and Open Problems

The analysis of section 4.1 was more general than we needed. This allows for a slightly more general result with regard to the corresponding continued fractions. In [2] the following continued fraction was studied:

$$\mathcal{S}_1(a,b) = \cfrac{1^b a_1^2}{1 + \cfrac{2^b a_2^2}{1 + \cfrac{3^b a_3^2}{1 + \ddots}}}. \tag{5.1}$$

It was shown that this leads to the rescaled sequence $(v_n^{(b)})$, analogous to Eq.(1.3),

$$v_n^{(b)} := \frac{q_n}{\Gamma^{b/2}(n+3/2)a_n^{(n+1)}}. \tag{5.2}$$

The difference equation Eq.(2.1) then becomes

$$v_n^{(b)} = \left(\frac{2}{2n+1}\right)^{b/2} \frac{1}{a_n}\left(\frac{a_{n-1}}{a_n}\right)^n v_{n-1}^{(b)} + \left(\frac{4}{(2n-1)(2n+1)}\right)^{b/2} n^2 \left(\frac{a_{n-2}}{a_n}\right)^{(n-1)} v_{n-2}^{(b)}, \tag{5.3}$$

and the matrix product

$$\mathcal{U}_n^{(b)} = \left(\prod_{j=2}^{n} K_j\right) \prod_{j=2}^{n}\left(I + \frac{1}{(2j)^{b/2}}\widehat{W}_j\right). \tag{5.4}$$

We conclude with the following generalization summarizing our main results.

Theorem 5.1 (summary) *For $\epsilon > 0$, $b > 1$ and $b - \epsilon > 0$, let $a := (a_n)$ be a random sequence of complex, zero mean, independent random variables satisfying*

$$0 \neq \prod_{n=1}^{\infty}\left(1 - \frac{1}{(2n)^{2b}a_{2n}^2}\right) < \infty \quad \textit{and} \quad 0 \neq \lim_{n\to\infty} \frac{a_2}{a_{2n}^{2n-1}a_{2n-1}^{2n-2}} \prod_{j=1}^{2n-2} a_j^2 < \infty \quad \textit{almost surely.}$$

and

$$\sum_{n}^{\infty} \frac{1}{n^{2(b-\epsilon)}} var\left(\frac{1}{a_2}\prod_{j=1}^{n}\frac{a_{2j-1}^2}{a_{2j}^2}\right) < \infty \quad \textit{and} \quad \sum_{n}^{\infty} \frac{1}{n^{2(b-\epsilon)}} var\left(\frac{a_2}{a_{2n}^2}\prod_{j=1}^{n}\frac{a_{2j}^2}{a_{2j-1}^2}\right) < \infty.$$

Then the iterates $v_n^{(b)}$ of the corresponding stochastic difference equation Eq.(5.3) are bounded almost surely and the stochastic Ramanujan continued fraction $\mathcal{S}_1(a,b)$ defined by Eq.(5.1) diverges almost surely with the even/odd parts of $\mathcal{S}_1(a,b)$ converging almost surely to separate limits.

Note that only sufficient conditions for divergence of random continued fractions have been determined with this analysis. The apparently more delicate question of necessary conditions for *convergence* remains open.

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