SPECIFICATION OF THE AHP HIERARCHY AND RANK REVERSAL

by

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ABSTRACT

Developed by Dr. Saaty in the early 1970’s, the Analytic Hierarchy Process is a multi-criteria decision making tool. The initial step in applying AHP is to accurately decompose a decision problem into a decision hierarhcy, avoiding both the overspecification (including irrelevant criteria/alternatives) and underspecification (omitting relevant criteria/alternatives).

Aull-Hyde and Duke (2006) introduced the concept of a minimal possible priority weight, the smallest priority weight for any alternative/criterion among n alternatives/criteria. They suggested using the minimal priority weight to detect an over-specified hierarchy. If the priority weight associated with a specific alternative/criterion is within 10% of the corresponding minimal possible weight, the alternative/criterion should be considered for omission from the decision hierarhcy. However, they assumed perfect consistency when determining the minimal possible priority weight.

The first focus of the thesis is to extend their methodology for the case of an inconsistent pairwise comparison matrix. For the case of a 3x3 pairwise comparison matrix, the minimal possible priority weight is shown to be a unique function of the consistency ratio. For higher dimension pairwise comparison matrices, the concept of a consistency ratio set is used to group potential pairwise comparison matrixes

according to their consistency ratios. Within each set, we propose a representative matrix for that set and use its smallest priority weight as the minimal weight for the entire set. Moreover, we numerically show that the minimal priority weight is a decreasing function of the consistency ratio, indicating that higher levels of inconsistency will generate smaller minimal prority weights.

The second focus of the thesis is to investigate any potential link between over-specified hierarchies and the rank reversal phenomenon, via Monte Carlo simulation. The analysis reveals that, as expected, the risk of rank reversal (in matrices having an acceptable level of inconsistency and are at risk for over-specification) increases dramatically as the number of decision alternatives increases. Given that a pairwise comparison matrix, with an acceptable level of inconsistency, exhibits rank-reversal, the likelihood that the associated hierarchy is at risk for over-specification is no more than 5%. This result indicates that no strong link exists between an over-specified hierarchy and rank reversal phenomenon.

Chapter 1

INTRODUCTION

Decision makers frequently face the challenge of evaluating multiple decision alternatives from the perspective of numerous criteria in order to select the single 'best' alternative. Multi-attribute decision making models can assist decision makers in navigating this type of decision making process. Several specific multi-attribute-decision-making methodologies exist: Multi-attribute utility theory, goal programming, the Analytic Hierarchy Process and the Analytic Network Process. This paper focuses on the Analytic Hierarchy Process (AHP). Specifically, this paper presents new insights about the AHP that can potentially improve the value of information generated from the AHP.

Thomas Saaty developed the AHP in the early 1970's. The impetus for its development occurred during the late 1960s when he directed research in the area of arms control and disarmament for the U.S. State Department. Despite the unsurpassed talent of his work group, he was dissatisfied with the outcome of the group's work (Saaty, 1996). Saaty continued to be troubled by the lack of a 'practical and systematic methodology' for setting priorities to assist in decision making. This pervasive issue motivated his development of the Analytic Hierarchy Process, a methodology currently used by academics, businesses, and governmental agencies

(Duke and Aull-Hyde, 2002; Al Khalil, 2002; Chen, 2006; Liu and Hai, 2005; Wong and Li, 2008, etc).

The initial step in applying the AHP methodology is to decompose a multi-criteria problem into a decision hierarchy. This hierarchy graphically depicts the major criteria, the minor criteria (also known as sub-criteria) and all decision alternatives. Figure 1.1 shows a three level hierarchy with 2 criteria and 2 alternatives. When applying the AHP, an important, if not crucial, step is the design of a decision hierarchy that accurately specifies those criteria, *and only those criteria*, viewed as being essential in evaluating all possible decision alternatives. The hierarchy should not include irrelevant criteria (known as over-specification); the hierarchy should not omit relevant criteria (known as under-specification). Over-specification or under-specification of a hierarchy can generate biased results.

Therefore, the first step in applying AHP -- specifying an accurate hierarchy -- is crucial. Given the very scant research literature on how to specify a hierarchy, Aull-Hyde and Duke (2006) proposed an iterative methodology that identifies potentially irrelevant criteria. If the methodology identifies a potentially irrelevant criterion, the decision maker can re-evaluate the validity of this criterion and re-specify the hierarchy accordingly. This thesis extends their original work into the inconsistent situation.

The term "irrelevant alternative/criterion" has two meanings: (1) in the first part of the thesis, it generally means the alternative/criterion that should not be included into the hierarchy; and (2) in the second part of the thesis, it particularly means the alternative/criterion that is a copy or near copy of existing alternatives/criteria.

Since Saaty introduced the AHP in the late 1970s, the method has not been without controversy. Long time debate about the rank reversal phenomena of AHP has, at times, cast doubt on the credibility of the method (Belton and Gear 1983, Dyer 1983, 1985, 1990a, 1990b). Rank reversal refers to a reversal of the rank of decision alternatives when an alternative is either added or deleted. A second focus of this thesis is to investigate a potential link between a misspecified hierarchy and occurrence of rank reversal. Preliminary evidence suggests that, for specific cases, the inclusion of an irrelevant criterion or an irrelevant decision alternative in the hierarchy may lead to rank reversal.

This goal of this thesis is two-fold: (1) to propose a methodology that detects a potentially over-specified (i.e., inclusion of irrelevant criterion/criteria) hierarchy and (2) to investigate a potential link between an over-specified hierarchy and the phenomenon of rank reversal. The thesis is organized as follows. Chapter 2 reviews the AHP methodology and provides some examples in which rank reversal occurs. Chapter 3 reviews the research literature on both the practical applications of AHP and the long-running debate over the rank reversal phenomena of AHP. Chapter 4 presents the proposed methodologies to identify a potentially over-specified hierarchy. Chapter 5 will try to find the link between over-specified hierarchy and rank reversal by conducting simulation. Chapter 6 will provide a summary, general discussion and suggestions for further research.

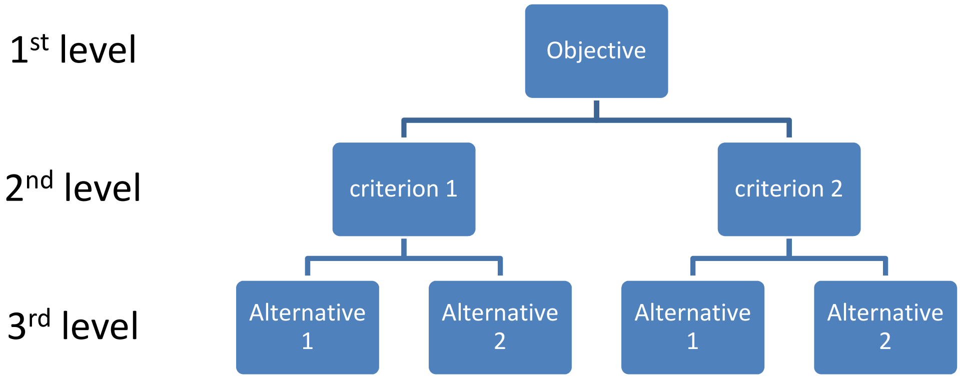

Figure 1.1 An example of a single level hierarchy with 2 criteria and 2 alternatives

Chapter 2

THE ANALYTIC HIERARCHY PROCESS

2.1 Mathematical Methodologies of AHP

Developed by Saaty (1980), the Analytic Hierarchy Process (AHP) is a popular multi-attribute decision making tool that has been successfully applied to a wide variety of decision making situations. It is a compositional methodology that synthesizes a decision maker's preference judgments for each of the decision alternatives, under each criterion within a decision hierarchy, to generate a quantitative measure of the decision maker's relative preference for each decision alternative. AHP not only enables a single decision maker to select a most suitable alternative, but the methodology can also be extended to group decision making (Saaty, 2007).

The AHP methodology involves the following steps:

Step1: Decompose a decision problem into a decision hierarchy, including all criteria, sub-criteria (if applicable) and decision alternatives.

Step 2: Conduct pairwise comparisons of all decision alternatives under each criterion, based on Saaty's 9-point preference rating scale,

Step 3: Derive the local priority weights using the eigenvector method (or other approximation methods.)

Step 4: Synthesize the local priority weights to generate the overall preference weights for each decision alternative.

Step 5: Check the inconsistency level of the decision maker's pairwise comparisons. If the level of inconsistency is unacceptable, then the decision maker should revise the pairwise comparisons described in Step 2.

In Step 1, specification of the decision hierarchy must reflect all criteria that are relevant to selection of a single decision alternative as well as all relevant decision alternatives. Aull-Hyde and Duke (2006) summarized three methods to specify the hierarchy: (1) specification by an external expert (or a team of experts) who is not the decision maker, (2) specification based on relevant criteria/alternatives reported in the research literature, and (3) specification by the decision maker. An ill-specified hierarchy will result in either over-specification or under-specification. An over-specified hierarchy includes irrelevant criteria and/or decision alternatives; an under-specified hierarchy omits relevant criteria and/or decision alternatives. Both cases generate biased results. This thesis proposes a method that provides feedback to a decision maker that the hierarchy may be over-specified. The proposed method in this thesis does not address the issue of an under-specified hierarchy.

To conduct pairwise comparisons of all relevant criteria/decision alternatives, Saaty proposed the formation of an n x n pairwise comparison matrix A = $\{a_{ij})$ where the a_{ij} values are derived according to the 9-point scale (Saaty, 1980, p54) presented in Table 2. 1. The pairwise comparison matrix assumes the following form:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \tag{2.1.1}$$

The matrix A is a positive reciprocal matrix where element a_{ij} is interpreted as the ratio of the preference of alternative i over the preference of alternative j. The value of element a_{ij} is the reciprocal of that of a_{ji}: $a_{ji} = {}^{1}\!/_{a_{ij}}$. All diagonal elements assume the value 1. If all pairwise comparison ratings are consistent, then the a_{ij} elements in matrix A will satisfy

$$a_{ij} \cdot a_{jk} = a_{ik} \text{ for all i, j, k.} \tag{2.1.2}$$

The key component of AHP is the derivation of relative priority weights $\{w_i\}_{i=1}^{n}$ given the pairwise comparison matrix A. The i^{th} element of the **w** vector represents the final priority weight given to criterion/alternative i. Note that $3w_i = 1$. Saaty (1980) proposed the eigenvector method, which uses the normalized principle eigenvector of A, to derive the relative priority weight vector **w**.

If A is consistent (i.e., satisfies equation 2.1.2.), a_{ij} can be expressed as ${}^{w_i}\!/_{w_j} = a_{ij}$, for all i and j (Taha, 2006). Thus, if A is consistent, then

$$A \cdot W = \begin{pmatrix} \frac{w_1}{w_1} & \frac{w_1}{w_2} & \cdots & \frac{w_1}{w_n} \\ \frac{w_2}{w_1} & \frac{w_2}{w_2} & \cdots & \frac{w_2}{w_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{w_n}{w_1} & \frac{w_n}{w_2} & \cdots & \frac{w_n}{w_n} \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = n \cdot \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} \tag{2.1.3}$$

This system of equations indicates that n, the total number of criteria/alternatives, is the eigenvalue of the pairwise comparison matrix A and the vector of priority weights $\{w_i\}_{i=1}^{n}$ is the right eigenvector that corresponds to the eigenvalue n. Saaty (Saaty, 1980, p167) proved that n is the *maximal* eigenvalue of matrix A. Thus,

$$A \cdot W = n \cdot W = \lambda_{\max} \cdot W, \quad W = (w_1, w_2, \cdots, w_n) \tag{2.1.4}$$

Eigenvalues and eigenvectors are continuous functions of the matrix elements, which implies that relatively small changes in the a_{ij} will result in relatively small changes in the eigenvalue and eigenvector. Thus, if the matrix A exhibits a small degree of inconsistency, then **Aw** can be approximated by **n**w, or **Aw** . n**w**. For this reason, Saaty (1980) proposed the use of the right normalized principle eigenvector as the priority weight vector **w**. To simplify calculations, Saaty also proposed several approximation methods to obtain the relative priority weight vector

w of which the additive normalized method is the simplest. Using the simple additive normalized method, the relative priority weight vector is computed as follows:

$$a'_{ij} = {a_{ij}} \Big/ {\sum_{i=1}^{n} a_{ij}} \quad , \qquad i, j = 1, 2, \ldots , n \tag{2.1.5}$$

$$w_i = \frac{1}{n} \sum_{j=1}^{n} a'_{ij} , \qquad i = 1, 2, \ldots , n \tag{2.1.6}$$

As stated earlier, **Aw** ≈ n**w** if the matrix A exhibits a 'small degree' of inconsistency. A key issue in using AHP is to quantitatively define a 'small degree' of inconsistency. If the matrix A of pairwise comparisons is consistent (i.e., equation 2.1.2 holds), then the maximal eigenvalue is simply n, the dimension of the pairwise comparison matrix A (which is the number of criteria/decision alternatives). If A is not consistent (also termed 'inconsistent'), Saaty (Saaty, 1980, 170) proved that the maximal eigenvalue is no smaller than n, or $\lambda_{\max} \geq n$. Because slight changes in the a_{ij} elements of the pairwise comparison matrix A result in slight changes in the eigenvalue and eigenvector, the difference between the maximal eigenvalue $\lambda_{\max}$ and n, $\lambda_{\max} - n$, can be used as the indicator of the level of inconsistency within matrix A.

Saaty defined the following terms to develop a quantitative measure of the degree of inconsistency within a pairwise comparison matrix A. Define the consistency index as CI = $\frac{\lambda_{max} - n}{n - 1}$ and the random index RI = $\frac{1.98(n-2)}{n}$. CI measures the degree of inconsistency of A (i.e., the difference between 8_{max} and n, adjusted for the dimension of the matrix A). RI is the corresponding measure of the degree of inconsistency of a pairwise comparison matrix of dimension n whose elements are randomly generated. The consistency ratio CR, defined as CR = CI/RI, is the ratio of CI to RI. CR is the quantitative measure, proposed by Saaty (1980), of the degree of inconsistency of a pairwise comparison matrix. Saaty (1980) suggested that a pairwise comparison matrix A with a corresponding CR of no more than 10% would qualify as having an ‘acceptable small degree’ of inconsistency. Thus, for a pairwise comparison matrix A with CR ≤ .10, **Aw** suffices as an approximation of n**w** and, therefore, use of the right normalized principle eigenvector **w** serves as an acceptable measure of the relative priority weights associated with the pairwise comparison matrix A. If a given pairwise comparison matrix A has a CR value exceeding 10%, if possible, the decision maker is advised to re-evaluate his/her pairwise comparison ratings.

2.2 An AHP Example

Saaty (1980, p46) presented the following example to illustrate the concepts and methodologies of the AHP. There are three major users of energy in the

U.S.: Household users (C_1), transportation (C_2), and power generating plants (C_3). Suppose we wish to allocate weights to each of these energy users according to their overall contribution to social welfare. We feel that three criteria should be used to define 'contribution to social welfare': Contribution to economic growth, contribution to environmental quality and contribution to national security. The corresponding hierarchy is illustrated in Figure 2.1. The pairwise comparison matrices, as well as final priority weights, are shown in Tables 2.1 and 2.2.

From Tables 2.2 and 2.3, the CR value is less than 10% for all four pairwise comparison matrices. In fact, the pairwise comparison matrix for the three criteria is consistent because CR = 0. Thus, inconsistency is not an issue.

Also from Table 2.2, $w_{crit} = (0.65, 0.13, 0.22)$ is the vector of relative priority weights for the three criteria. We see that economic growth is 5 (0.65/0.13) times as important as environmental impact and about 3 times as important as national security.

Table 2.3 lists the relative priority weight vectors for each group of users with respect to each criterion. Denote these three vectors as $\mathbf{w}_{c1}$, $\mathbf{w}_{c2}$ and $\mathbf{w}_{c3}$. The final *composite* relative priority weight vector $\mathbf{w}$ for the three groups of users is computed as $\mathbf{w} = 0.65\ \mathbf{w}_{c1} + .13\ \mathbf{w}_{c2} + 0.22\ \mathbf{w}_{c3} = (0.62, 0.26, 0.12)$. We can conclude that the social contribution of household users is nearly three times greater than that of transportation and nearly six times greater than that of power generating plants.

2.3 Examples of Rank Reversal in AHP

AHP has been criticized due to the possibility of rank reversal when using the methodology. Rank reversal occurs when the addition or deletion of a decision alternative results in a change of the relative rankings of the remaining alternatives. First observed by Belton and Gear (1983), the issue of rank reversal has generated a long-running debate in the research literature regarding the credibility of AHP. Two famous examples of rank reversal occur in the literature. These examples are presented below.

Hochbaum (2006) considered the 4 x 4 pairwise comparison matrix below. After deleting alternative 4, the ranks of the remaining three alternatives change.

$$\begin{pmatrix} 1 & 1.2 & 1.5 & 6 \\ 1/1.2 & 1 & 1.2 & 8 \\ 1/1.5 & 1/1.2 & 1 & 7 \\ 1/6 & 1/8 & 1/7 & 1 \end{pmatrix} \Longrightarrow \begin{pmatrix} 1 & 1.2 & 1.5 \\ 1/1.2 & 1 & 1.2 \\ 1/1.5 & 1/1.2 & 1 \end{pmatrix} \qquad (2.3.1)$$

The priority weights for the initial 4 x 4 comparison matrix are (0.335, 0.347, 0.273, 0.046). After the deletion of alternative 4, the weights are (0.373, 0.356, 0.271). After the deletion of alternative 4, the ranks of alternatives 1 and 2 change.

The second example by Belton and Gear (1983) is the example most cited in the research literature. Assume three decision criteria, each of equal importance and three decision alternatives, namely A, B and C. The pairwise comparison matrices for the Belton and Gear example are given in Table 2.4 below.

Given that the three criteria are equally important, the final composite weights for alternatives A, B, and C are (0.4512, 0.4697, 0.0791). Belton and Gear now introduce a new alternative D which is a copy of B. When incorporating this additional 'duplicate' alternative into the hierarchy, the final composite weights for A, B, C, and D are (0.3654, 0.2889, 0.0568, 0.2889). Prior to adding alternative D, B is preferred over A (i.e., $A \prec B$). Upon adding alternative D, A is preferred over B (i.e., $A \succ B$).

Table 2.1 The AHP pairwise comparison 9-point rating scale

aij value	Meaning
aij = 1	criteria i and j equally important
aij = 3	criterion i slightly more important than criterion j,
aij = 5	criterion i moderately more important than criterion j,
aij = 7	criterion i strongly more important than criterion j,
aij = 9	criterion i extremely more important than criterion j,
aij = 2, 4, 6, 8	represent intermediate values of importance between the two criteria.

Table 1.2 Pairwise comparisons for criteria (λ_{max}=3, CR = 0)

	Econ Growth	Environmental Impact	National Security	**Weights**
Econ Growth	1	5	3	**0.65**
Env. Impact	1/5	1	3/5	**0.13**
National Security	1/3	5/3	1	**0.22**

Table 2.3 Pairwise comparison matrices for each user group under each criterion and corresponding priority weights

Criterion	Alternative	C_1	C_2	C_3	Weights	CR
Economic	C_1	1	3	6	0.65	0
Growth	C_2	1/3	1	2	0.23	
	C_3	1/6	1/2	1	0.12	
Environmental	C_1	1	2	7	0.59	0.02
Impact	C_2	1/2	1	5	0.33	
	C_3	1/7	1/5	1	0.08	
National	C_1	1	2	3	0.54	0.02
Security	C_2	1/2	1	2	0.30	
	C_3	1/3	1/2	1	0.16	

Table 2.4 Pairwise Comparison Matrices for the Belton and Gear Example

	Alternative	A	B	C	Weights
Criterion 1	A	1	1/9	1	1/11
	B	9	1	9	9/11
	C	1	1/9	1	1/11
Criterion 2	A	1	9	9	9/11
	B	1/9	1	1	1/11
	C	1/9	1	1	1/11
Criterion 3	A	1	8/9	8	8/18
	B	9/8	1	9	9/18
	C	1/8	1/9	1	1/18

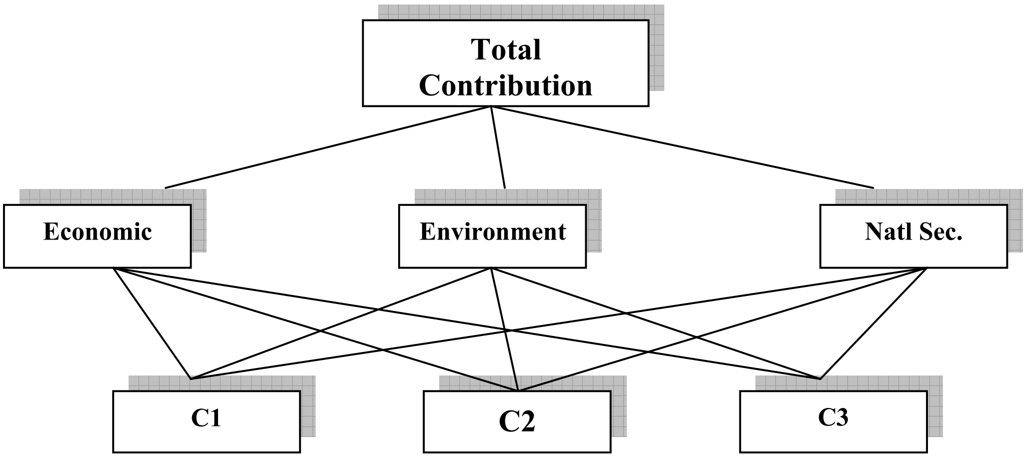

Figure 2.1 The Decision Hierarchy

Chapter 3

LITERATURE REVIEW

3.1 Previous Methods to Establish the Decision Hierarchy

The Analytic Hierarchy Process has been successfully applied to a wide variety of decision situations, such as project management (AL-Harbi, 2001; Al Khalil, 2002), quality control (Lee and Kozar, 2006), resource allocation (Cheng and Li, 2001; Hsu, Wu and Li, 2008), land preservation (Aull-Hyde and Duke, 2002), and multiple evaluation applications (Chen, 2006; Liu and Hai, 2005; Wong and Li, 2008). In general, the AHP methodology can be applied to any decision making process that involves hierarchy structuring, scale measurement, and synthesis.

As discussed in Chapter 2, applying the AHP methodology is a five-step process that consists of: (1) constructing a decision hierarchy, (2) generating pairwise comparisons, (3) deriving local priority weights, (4) synthesizing the local priority weights to generate global priority weights and (5) checking pairwise comparison matrices for acceptable inconsistency levels. The first step -- hierarchy construction-- is crucial. Decision makers must avoid both over-specification and under-specification of the hierarchy; otherwise, results will be biased.

The research literature provides no universal method for decision makers to generate a decision hierarchy or check the credibility of a hierarchy. Three methods currently exist for specifying a hierarchy (Aull-Hyde and Duke, 2006): (1) specification by an expert (or group of experts) who neither participates in the decision making process, nor has any vested interest in the outcome of the decision making process, (2) specification based on criteria/alternatives deemed to be relevant in research literature, and (3) specification by the decision maker

Specification by an external expert. The decision maker may consult an expert or survey a group of experts to construct the decision hierarchy (Chan, Kwok and Duffy, 2004; Cheng and Li, 2001; Liu and Hai, 2005; Kurttila, Pesonen, Kangas and Kajanus, 2000).

Cheng and Li (2001) used AHP to identify key information to more efficiently allocate resources for a construction project. Two experts involved in the construction project were asked to determine the major criteria of the decision hierarchy. They acknowledge that "although opinions from two experts may only provide a very rough picture, it is still appropriate in this exploratory study." Liu and Hai (2005) applied AHP to supplier selection in supply chain management. They specified selection criteria and sub-criteria based on the opinions of 60 managers and supervisors.

Specification based on the previous research literature. The decision hierarchy is typically constructed based on previous research literature (Scholl, Manthey, Helm and Steiner, 2005; Chen, 2006; Wang and Li, 2008; Duke and Aull-

Hyde, 2002; AL-Harbi, 2001; Al Khalil, 2002; Hsu, Wu and Li, 2008; R. Banuelas and J. Antony, 2007; Lee and Kozar, 2006). Duke and Aull-Hyde (2002) used AHP to identify public preferences for land preservation. Based on previous empirical studies by Kline, Wichelns and Rosenberger (1998), the preference criteria used in their study were ‘preserves agriculture’, ‘improves environment’, ‘controls development’ and ‘enhances open space’. Chen (2006) used AHP to select a convention site location. Based on a review of relevant research literature, the preference criteria used to specify the hierarchy were meeting and accommodation facilities, costs, site environment, local support, and extra conference opportunities. Sub-criteria were also selected based on past reported studies.

Specification by the decision maker. In this case, decision makers should have a deep understanding of the overall objective and sufficient knowledge to decompose the problem into a series of criteria and sub-criteria. Given that this approach is usually the most informal and subjective; hierarchies specified solely by the decision maker may be more likely to be ill-specified.

In summary, there is no universal method to specify an AHP decision hierarchy. Decision makers can select criteria and sub-criteria based on the previous studies, experts’ opinions or their own specifications. Regardless of how the hierarchy is constructed, the decision maker should take steps to ensure that the hierarchy is neither over-specified nor under-specified. Aull-Hyde and Duke (2006) proposed an iterative method to detect potentially over-specified hierarchy under consistency. The first goal of this thesis is to extend their method into inconsistent situation.

3.2 The Controversy over Rank Reversal in AHP

AHP has suffered serious criticism due to its potential for rank reversal. As illustrated in Chapter 2, Belton and Gear (1983) pointed out that adding a copy of an existing alternative into the decision hierarchy can potentially change the rank of previous alternatives. They suggested that normalizing the local priority weights by the maximal element of each local priority column, instead of the column sum, would eliminate the rank reversal phenomenon. Belton and Gear's example essentially generated a protracted debate about the validity of AHP. Dyer (1983, 1985) presented another rank reversal example to further question the credibility of AHP. He argued that adding an alternative that was even a near-copy of an existing alternative could also cause rank reversal.

In response to Dyer's challenge, Saaty (1983, 1987) argued that if relative priority weights for two alternatives were within 10% of each other, then the decision maker would likely be indifferent towards the two alternatives. In this case, Saaty suggested the deletion of one alternative from the hierarchy in order to eliminate rank reversal. In support of Saaty's view, Harker and Vargas (1987) argued that, as in the Belton and Gear (1983) example, adding a copy of an existing alternative contributed no new information to the process. Thus, their counterexample was not meaningful. As for Dyer's example, Harker and Vargus (1987) suggested that the use of a 'super' pairwise comparison matrix (also known as a 'supermatrix') would derive the correct priorities and avoid rank reversal. However, the debate was not over.

Dyer (1990a, 1990b) disagreed and insisted that "*When the principle of hierarchy composition is assumed, the results produced by the AHP are arbitrary*" (Dyer, 1990a). Based on the previous rank reversal examples, Dyer argued that Saaty's 10% rule regarding the 'near-copy' alternative was ungrounded and open to doubt. Saaty (1990) relied to Dyer's criticisms by emphasizing the legitimacy of rank reversal within AHP and argued that Dyer's views about AHP were based on traditional multi-attribute utility theory (MAUT), which assumes decision alternatives are independent. However, unlike MAUT, "*AHP was a new and logical theory, but certainly not arbitrary*" (Saaty, 1990). In support of AHP, Harker and Vargas (1990) argued that the concepts of independence within AHP and MAUT were different as well as subjective. "*The reason why rank can reverse in the AHP with relative measurement is clear. It is because the alternatives depend on what alternatives are considered, hence, adding or deleting alternatives can lead to change in final rank*" (Harker & Vargas, 1990).

The legitimacy of AHP continued to be debated in the research literature. Saaty (2000) stated that both rank reversal and rank preservation may occur in real-world decision making situations. He therefore offered two methods of determining priority weights: 'Distributive' and 'Ideal'. The distributive method normalizes the local priority weights by their respective column sum; the distributive method has the potential for rank reversal. The ideal method, proposed by Belton and Gear (1983), normalizes the local priority weights by the largest element within the column; the ideal method preserves rank.

Following the decade-long debate of the legitimacy of rank reversal within AHP, many researchers focused their efforts on modifications to the initial AHP methodologies that would eliminate the rank reversal phenomenon. As mentioned before, Belton and Gear (1983) first proposed a modification that Saaty later termed the "Ideal" mode in order to preserve rank. Barzilai and Golany (1994) argued that no normalization could preserve rank and instead proposed an aggregation rule. Tiantaphyllou (2001) presented two new cases of rank reversal that occurred in AHP but not in multiplicative AHP. Wang and Elhag (2006) proposed that the new local priority column associated with a new decision alternative be normalized by the sum of its first n elements. They argued that this normalization method can preserve rank. Wang and Luo (2009) suggested that rank reversal is a universal phenomenon that not only occurs within AHP, but also appears in many other decision making tools, such as the Borda-Kendall method, the simple additive weighting method and the cross-efficiency evaluation method, among others.

Despite more than 25 years of debate, no consensus exists within the research literature on the validity of rank reversal within AHP. For the purposes of this thesis, the question of whether or not rank reversal is a valid concern in AHP is secondary. A primary goal of this thesis is to provide evidence of a link between the rank reversal phenomenon and an over-specified decision hierarchy.

In summary, this thesis provides two contributions to the research literature. First, the thesis extends Aull-Hyde and Duke's (2006) methodology, which detects hierarchies that are at risk for over-specification, to inconsistent situation.

Second, the thesis investigates Saaty’s (1983, 1987) suggested link between an over-specified hierarchy and the phenomenon of rank reversal.

Chapter 4

PROPOSED METHODOLOGY

As outlined in Chapter 3 above, no definitive method exists to specify a hierarchy for implementing AHP. To exclude important criteria (under-specification) generates biased results. And, to include irrelevant criteria (over-specification) generates biased results. Aull-Hyde and Duke (2006) proposed a methodology to detect a potentially over-specified hierarchy. In this chapter, we extend their method to inconsistent case.

An over-specified hierarchy contains at least one decision alternative (or at least one criterion) that is irrelevant to the problem. Applying AHP to an over-specified hierarchy will generate a very low relative priority weight for the irrelevant alternative/criterion. Aull-Hyde and Duke (2006) suggested a methodology to detect a potentially irrelevant alternative/criterion. A specific alternative/criterion should be flagged if its computed relative priority weight is within 10% of the minimum possible weight that can be assigned to any alternative/criterion. The following sections present the methodology for deriving this minimum priority weight as a function of the dimension of the pairwise comparison matrix and its inconsistency level. As suggested by Aull-Hyde and Duke (2006), this minimum priority weight can then be compared to the computed priority weight; small differences in these two weights likely identify an irrelevant alternative/criterion. A second objective of this thesis is to investigate a possible link between the phenomenon of rank reversal and a potentially over-specified hierarchy.

4.1 Definitions and Introduction

Aull-Hyde and Duke (2006) defined a minimal priority weight $w_{min}(n)$ as the smallest possible priority weight for a pairwise comparison matrix of dimension n.

Let A denote a consistent n x n pairwise comparison matrix with relative priority weight vector $\mathbf{w} = \{w_i\}$. Define $w_{min}(n)$ such that $w_{min}(n) \leq w_i$ for i = 1, …, n. $w_{min}(n)$ is the minimum priority weight for n decision alternatives and is computed as (Aull-Hyde and Duke (2006)):

$$w_{\min}(n) = \frac{1}{1 + 9 \cdot (n-1)} \tag{4.1.1}$$

The minimal possible weight is calculated by keeping (n-1) alternatives absolutely important (rating as 9) than the remaining one. $w_{min}(n)$ is a decreasing function of n (the number of available alternatives). Essentially, the minimal possible weight in equation (5.1.1) is the smallest possible weight that AHP guarantees for any alternative or criterion. For example, $w_{min}(3) = 0.0526$, meaning that for any given 3x3 pairwise comparison matrix, under perfect consistency, the priority weight assigned to each alternative will be at least 5.26%.

Equation (4.1.1) holds for pairwise comparison matrices that are consistent. In the following sections, we calculate $w_{min}(n)$ for inconsistency pairwise comparison matrices. These calculations are presented for two situations: (1) n = 3 (i.e., exactly three decision alternatives, generating a 3 x 3 pairwise comparison matrix) and (2) n ≥ 4 (i.e., four or more decision alternatives, generating a pairwise comparison matrix having dimension of at least 4).

4.2 Calculation of Minimal Weights for n = 3

If n = 3 situation, in order to calculate a minimal possible weight for an inconsistent pairwise comparison matrix, we assume that one of the three alternatives (denoted as alternative C) is α times less important than the remaining two alternatives, A and B. This assumption extends Aull-Hyde and Duke's (2006) model to a more general case. If the value α is at least 7 (on Saaty's 9-point rating scale), then we can say that alternatives A and B are, at the very least, strongly more important than alternative C. Given that we are assuming that the pairwise comparison matrix is inconsistent, this matrix assumes the following form:

$$\begin{pmatrix} 1 & m & \alpha \\ 1/m & 1 & \alpha \\ 1/\alpha & 1/\alpha & 1 \end{pmatrix} \tag{4.2.1}$$

If the pairwise comparison matrix is consistent, then m = 1. Otherwise, m ≠ 1 and alternative A is m times important than B. In this latter case, the matrix is inconsistent.

Given that the pairwise comparison matrix assumes the form of equation (4.2.1), the characteristic equation and principle eigenvector are, respectively, (Farkas, 2000):

$$\lambda_{\max}^3 - 3\lambda_{\max}^2 - (m + 1/m - 2) = 0 \tag{4.2.2}$$

$$\lambda_{\max} = m^{\frac{1}{3}} + 1/m^{\frac{1}{3}} + 1 \tag{4.2.3}$$

Given equations (4.2.2) and (4.2.3) above, we see that if the 3 x 3 pairwise comparison matrix is consistent (m = 1), then the principle eigenvector is 3 and the remaining two eigenvectors are 0, as expected. Equation (4.2.2) is now used to determine an expression for $w_{min}(3)$. Equation (4.2.3) can be used to determine a numerical range on the parameter m such that the pairwise comparison matrix can be termed as 'acceptably inconsistency' according to Saaty's criterion (Saaty, 1980, p54).

Theorem 4.2.1: For a 3 x 3 pairwise comparison matrix of the form 4.2.1 that displays an acceptable level of inconsistency, the range on m is such that $0.3382 \leq m \leq 2.9568$.

Proof: As described in Chapter 2, if a pairwise comparison matrix is acceptably inconsistent, then CR = CI/RI should be no larger than 0.1. Thus,

$$C.R. = \frac{C.I.}{R.I.} = \left(\frac{\lambda_{\max} - n}{n-1}\right) \Big/ \left(\frac{1.98(n-2)}{n}\right) \leq 0.1 \tag{4.2.4}$$

Setting n = 3 in equation (4.2.4), generates the following boundaries on the principle eigenvalue:

$$3 \leq \lambda_{\max} \leq 3.132 \tag{4.2.5}$$

Combining equations (4.2.5) and (4.2.3) generates a range on m of [0.3382, 2.9568]. As expected, the lower and upper bounds are reciprocals. □

Theorem 4.2.2: For a 3x3 pairwise comparison matrix of the form (4.2.1), the normalized minimal possible weight is a function of the consistency ratio CR.

Proof: Using equation (4.2.4), λ_{max} can be written as

$$\lambda_{\max} = 1.32c + 3 \tag{4.2.6}$$

where c is the value of consistency ratio CR. Thus, λ_{max} is expressed as a function of CR. From the characteristic equation (4.2.2),

$$m + \frac{1}{m} + (3\lambda_{\max}^2 - \lambda_{\max}^3 - 2) = 0 \tag{4.2.7}$$

Equation (4.27) is quadratic in m; hence,

$$\begin{cases} m_1 = (-\Delta + \sqrt{\Delta^2 - 4}) / 2 \\ m_2 = (-\Delta - \sqrt{\Delta^2 - 4}) / 2 \end{cases} \quad \Delta = (3\lambda_{\max}^2 - \lambda_{\max}^3 - 2) \tag{4.2.8}$$

Thus, m can be expressed as a function of λ_{max}. Based on equation (4.2.6), m is also a function of the consistency ratio CR = c. Now, the principle eigenvector is calculated by the following system of linear equations:

$$\begin{pmatrix} 1-\lambda_{\max} & m & \alpha \\ 1/m & 1-\lambda_{\max} & \alpha \\ 1/\alpha & 1/\alpha & 1-\lambda_{\max} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_{\min} \end{pmatrix} = 0 \tag{4.2.9}$$

Based on Saaty's (1980, p98, equation 3) results, the normalized minimal weights can be calculated as:

$$w_{\min} = ((1-\lambda_{\max})^2 - 1) \Big/ \left(\alpha \cdot m + 2\cdot\alpha\cdot(\lambda_{\max}-1) + \frac{\alpha}{m} - 1 + (1-\lambda_{\max})^2\right)$$

(4.2.10)

From equation (4.2.10), w_{min} can be expressed as a function of the principle eigenvalue λ_{max} and the parameter m. Because λ_{max} and m are single variable functions of the consistency ratio c, w_{min} can be expressed as a function of the consistency ratio CR = c. □

We note two observations regarding the above proof.

Observation 1: From equation (4.2.8), it can easily be shows that $m_1m_2=1$, which implies that m1 and m2 are reciprocals.

Observation 2: From equation (4.2.10), given that m and 1/m are reciprocals; both m and 1/m generate the same minimal weight. This observation can be interpreted in the following way: Suppose B is α times more important than alternative C. If A is m times important than B or if B is m times important than A, either of these two cases generates the same minimal possible weight.

For the case of $n = 3$, Table 4.1 gives the minimum weight, computed from equation 4.2.10, for values of CR = c that are deemed 'acceptably inconsistent' (i.e., CR $\leq$ .10) and for $\alpha = 7, 8, 9$. (Recall that α is a pairwise comparison rating from Saaty's 9-point scale. Values of at least 7 imply that one alternative is at least strongly preferred over another.)

For example, if alternatives A and B are deemed to be strongly more important than alternative C ($\alpha = 7$), and the consistency ratio of the corresponding pairwise comparison matrix is 0.05, then alternative C will be assigned a priority weight of no less than 0.0647. Because A and B are strongly more important than C, then C could potentially be treated as an irrelevant/non-important alternative.

Obviously, the minimal weight is a decreasing function of the consistency ratio. The higher the C.R. level, the more inconsistent the comparisons are. The more inconsistency gives the decision maker more freedom to make the comparison. Thus, more values will be used as potential priority weight, indicating the lower bound (minimal possible weight) will be smaller.

From Table 4.1, the relationships between the consistency ratio and the minimal possible weight are nearly linear. A least-squares regression of the data in Table 4.1 will generate the following linear relationships with $R^2 \geq 0.99$:

$$\begin{aligned} w_{\min,a=7}(c) &= 0.0667 - 0.0392 \cdot c \\ w_{\min,a=8}(c) &= 0.0588 - 0.0344 \cdot c \\ w_{\min,a=9}(c) &= 0.0526 - 0.0313 \cdot c \end{aligned} \tag{4.2.11}$$

These equations are illustrated in Figure 4.1.

4.3 Calculation of Minimal Weights for n ≥ 4

When the number of alternatives (criteria) is larger than 3, several unique pairwise comparison matrixes can have the same consistency ratio CR. Thus, we introduce the concept of a consistency level set.

Definition 4.3.1: A *consistency level set* is a set of potential pairwise comparison matrixes that share the same consistency ratio CR = c.

By introducing the concept of a consistency-level set, we can group potential pairwise comparison matrices according to their inconsistency levels (as measured by CR = c). In order to simplify the calculations, we declare a 'representative' pairwise comparison matrix within each consistency-level set. We then use the minimal weight generated by the 'representative' pairwise comparison matrix as the minimal weight for all matrices in the consistency-level set. We assume the following form for the 'representative' pairwise comparison matrix:

$$\begin{pmatrix} 1 & m & 1 & \cdots & a \\ \frac{1}{m} & 1 & 1 & \cdots & a \\ 1 & 1 & 1 & \cdots & a \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{a} & \frac{1}{a} & \frac{1}{a} & \cdots & 1 \end{pmatrix} \tag{4.3.1}$$

The above representative pairwise comparison matrix assumes the simplest form of all matrices within a consistency-level set. (n-1) alternatives are α times more important than the remaining alternative.

Meanwhile, only one perturbation term m exists. If the pairwise comparison matrix is consistent, m = 1. Otherwise, m > 1, indicating alternative A is m times more important than alternative B, or m < 1, indicating B is m times more important than A. The characteristic polynomial is shown in the following theory (Farkas, 2000).

Theorem 4.3.1 For an n x n pairwise comparison matrix of the form in equation (4.3.1), the characteristic equation is:

$$\lambda^{n-3}[\lambda^3 - n \cdot \lambda^2 - (n-2) \cdot (m + \frac{1}{m} - 2)] = 0 \tag{4.3.2}$$

and the principle eigenvector satisfies the following equation:

$$\lambda_{\max}{}^{3} - n \cdot \lambda_{\max}{}^{2} - (n-2) \cdot (m + \frac{1}{m} - 2) = 0 \tag{4.3.3}$$

Theorem 4.3.2 For an n x n pairwise comparison matrix of the form in equation (4.3.1), if the matrix displays an acceptable level of inconsistency (i.e., CR = c < 10%), then the parameter m is bounded on the following interval:

$$[\frac{A-\sqrt{A^2-4}}{2}, \frac{A+\sqrt{A^2-4}}{2}],$$
$$A = 2 + \frac{0.198 \cdot (n-1)}{n} \cdot [n + \frac{0.198 \cdot (n-1) \cdot (n-2)}{n}]^2 \tag{4.3.4}$$

Proof: If a pairwise comparison matrix is acceptably inconsistency, then CR = c ≤ .10. From equation (4.2.4), the principle eigenvalue satisfies:

$$n \le \lambda_{\max} \le n + \frac{0.198 \cdot (n-1) \cdot (n-2)}{n} \tag{4.3.5}$$

Rewrite equation 4.3.3 as:

$$m + \frac{1}{m} = 2 + \frac{\lambda_{\max}{}^{2} \cdot (\lambda_{\max} - n)}{n-2} \tag{4.3.6}$$

The right-hand side of the above equation is an increasing function of the principle eigenvalue $\lambda_{\max}$. The left-hand side is a convex function of the perturbation value m; the left-hand side obviously has a minimal value of 2.

$$2 \le m + \frac{1}{m} \le 2 + \frac{0.198 \cdot (n-1)}{n} \cdot [n + \frac{0.198 \cdot (n-1) \cdot (n-2)}{n}]^2 \tag{4.3.7}$$

Thus, boundaries on m can be obtained by solving the above inequality. □

From equation (4.3.7), if n = 4, m is bounded on the interval [0.2212, 4.5207]. If n = 5, m is bounded on the interval [0.1516, 6.5969]. Obviously, the upper bound and lower bounds are reciprocal. Moreover, the upper bound is an increasing function of n – the dimension of the pairwise comparison matrix.

Theorem 4.3.4: For a 4 x 4 pairwise comparison matrix of the form given in equation (4.3.1), the normalized minimal possible weight can be expressed as a function of the consistency ratio CR = c.

Proof: Similarly, from equation (4.2.4), the principle eigenvalue can be expressed as a function of the consistency ratio CR = c:

$$\lambda_{\max} = 2.97 \cdot c + 4 \tag{4.3.8}$$

Rewriting equation (4.3.3) as

$$m + \frac{1}{m} + (2 \cdot \lambda_{\max}{}^2 - \frac{\lambda_{\max}{}^3}{2} - 2) = 0 \tag{4.3.9}$$

enables m to be expressed as a function of the principle eigenvalue $\lambda_{\max}$ and, thus, as a function of the consistency ratio CR = c

$$\begin{cases} m_1 = (-\Delta + \sqrt{\Delta^2 - 4}) / 2 \\ m_2 = (-\Delta - \sqrt{\Delta^2 - 4}) / 2 \end{cases} \quad \Delta = (2 \cdot \lambda_{\max}{}^2 - \frac{\lambda_{\max}{}^3}{2} - 2) \tag{4.3.10}$$

The minimal priority weight is calculated by solving the following system of linear equations:

$$\begin{pmatrix} 1-\lambda_{max} & m & 1 & a \\ \frac{1}{m} & 1-\lambda_{max} & 1 & a \\ 1 & 1 & 1-\lambda_{max} & a \\ \frac{1}{a} & \frac{1}{a} & \frac{1}{a} & 1-\lambda_{max} \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_{min} \end{pmatrix} = 0 \qquad (4.3.11)$$

Given Saaty's result (1980, pg 100) the minimal priority weight can be calculated as:

$$w_{min} = \frac{(\lambda_{max}-1)^3 - 3\cdot(\lambda_{max}-1) - (m+\frac{1}{m})}{Q}$$

$$Q = (\lambda_{max}-1)^3 + 3\cdot a\cdot(\lambda_{max}-1)^2 + (4\cdot a - 3 + m\cdot a + \frac{a}{m})\cdot(\lambda_{max}-1) + [m\cdot(2\cdot a-1) + \frac{(2\cdot a-1)}{m} - a]$$

(4.3.12)

The minimal priority weight is a function of the principle eigenvalue λ_{max} and the perturbation term m. Thus, w_{min} is a single-variable function of consistency ratio. □

Theorem 4.3.5: For a 5x5 pairwise comparison matrix of the form given in equation (4.3.1), the normalized minimal possible weight can be expressed as a single-variable function of consistency ratio CR = c.

Proof: The minimal weight is calculated by solving the following system of linear equations:

$$\begin{pmatrix} 1-\lambda_{\max} & m & 1 & 1 & a \\ \frac{1}{m} & 1-\lambda_{\max} & 1 & 1 & a \\ 1 & 1 & 1-\lambda_{\max} & 1 & a \\ 1 & 1 & 1 & 1-\lambda_{\max} & a \\ \frac{1}{a} & \frac{1}{a} & \frac{1}{a} & \frac{1}{a} & 1-\lambda_{\max} \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_{\min} \end{pmatrix} = 0 \tag{4.3.13}$$

Using Maple and assuming $w_1=1$, we can derive the minimal priority weight of the following form:

$$w_{\min} = \frac{1}{(\lambda_{\max} - 1) \cdot a + 1} \tag{4.3.14}$$

Applying equation (4.2.4), we have:

$$\lambda_{\max} = 4.752 \cdot c + 5 \tag{4.3.15}$$

Combining equations (4.3.14) and (4.3.15), we can express the minimal priority weight as a function of the consistency ratio:

$$w_{\min} = \frac{1}{(4.752 \cdot c + 4) \cdot a + 1} \tag{4.3.16}$$

For n = 4, Table 4.2 gives the minimum weight, computed from equation 4.3.12, for values of CR = c that are deemed ‘acceptably inconsistent’ (i.e., CR ≤ 0.10), and for $\alpha = 7, 8, 9$. Values of $\alpha \geq 7$ imply that one alternative is at least strongly preferred over another. Minimum weights for n = 5. Computed from equation 4.3.16, are given in Table 4.3.

Obviously, the minimal priority weight is a decreasing function of the consistency ratio c. The higher the C.R. level, the more inconsistent the comparisons are, giving the decision maker more freedom to make the comparison. Thus, more values will be used as potential priority weight, indicating the lower bound (minimal possible weight) will be smaller. Moreover, the relationship is nearly linear.

For n = 4, the estimated linear relationships are listed below. A least-squares regression of the data in Table 4.2, calculated from equation 4.3.12, generates the following linear relationships, with $R^2 \geq 0.99$:

$$
\begin{aligned}
w_{\min,a=7}(c) &= 0.0455 - 0.0394 \cdot c \\
w_{\min,a=8}(c) &= 0.04 - 0.035 \cdot c \\
w_{\min,a=9}(c) &= 0.0357 - 0.0308 \cdot c
\end{aligned} \tag{4.3.18}
$$

These relationships are illustrated in Figure 4.2.

For n =5, the estimated linear relationships are listed below. A least-squares regression of the data in Table 4.3, calculated from equation 4.3.16, generates the following linear relationships, with $R^2 \geq 0.99$:

$$
\begin{aligned}
w_{\min,a=7}(c) &= 0.0344 - 0.0355 \cdot c \\
w_{\min,a=8}(c) &= 0.0303 - 0.0314 \cdot c \\
w_{\min,a=9}(c) &= 0.027 - 0.0278 \cdot c
\end{aligned} \tag{4.3.19}
$$

These relationships are illustrated in Figure 4.3.

Table 4.2 Minimal weights for n = 3

Consistency Ratio	Wmin (α = 7)	Wmin (α = 8)	Wmin (α = 9)
0	0.0667	0.0588	0.0526
0.01	0.0663	0.0585	0.0523
0.02	0.0659	0.0581	0.0520
0.03	0.0655	0.0577	0.0517
0.04	0.0651	0.0574	0.0513
0.05	0.0647	0.0571	0.0510
0.06	0.0643	0.0567	0.0507
0.07	0.0639	0.0564	0.0504
0.08	0.0635	0.0560	0.0501
0.09	0.0632	0.0557	0.0498
0.1	0.0628	0.0554	0.0495

Table 4.2 Minimal weights for n = 4

C.I. level	Wmin (α=7)	Wmin (α=8)	Wmin (α=9)
0	0.0455	0.0400	0.0357
0.01	0.0450	0.0396	0.0354
0.02	0.0446	0.0393	0.0350
0.03	0.0442	0.0389	0.0347
0.04	0.0438	0.0385	0.0344
0.05	0.0434	0.0382	0.0341
0.06	0.0430	0.0378	0.0338
0.07	0.0426	0.0375	0.0335
0.08	0.0423	0.0372	0.0332
0.09	0.0419	0.0368	0.0329
0.1	0.0415	0.0365	0.0326

Table 4.3 Minimal weights for n = 5

C.I. level	Wmin (α=7)	Wmin (α=8)	Wmin (α=9)
0	0.0345	0.0303	0.0270
0.01	0.0341	0.030	0.0267
0.02	0.0337	0.0296	0.0264
0.03	0.0333	0.0293	0.0261
0.04	0.0330	0.0290	0.0258
0.05	0.0326	0.0287	0.0256
0.06	0.0323	0.0283	0.0253
0.07	0.0319	0.0280	0.0250
0.08	0.0316	0.0277	0.0247
0.09	0.0313	0.0275	0.0245
0.1	0.0309	0.0272	0.0242

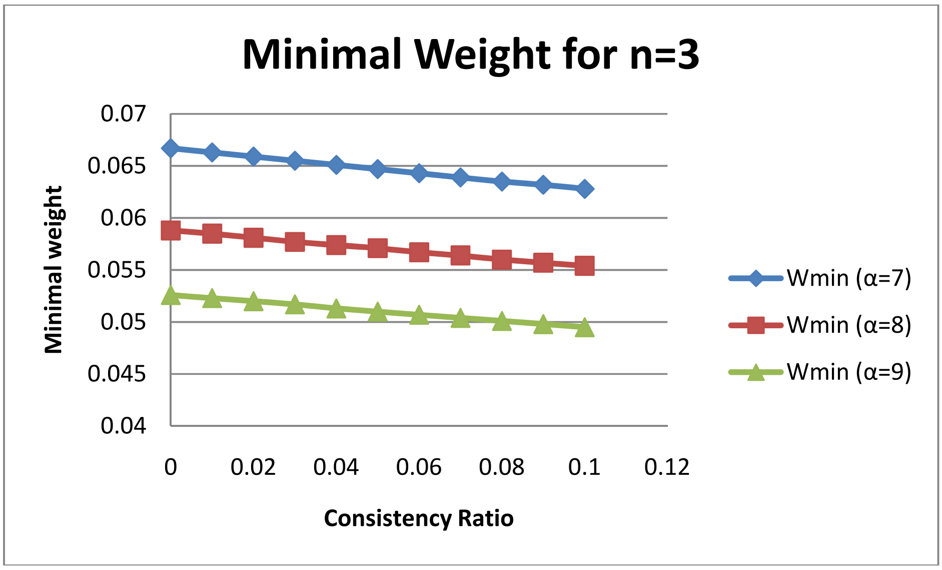

Figure 4.1 Minimal weights for n = 3 and α = 7, 8, 9

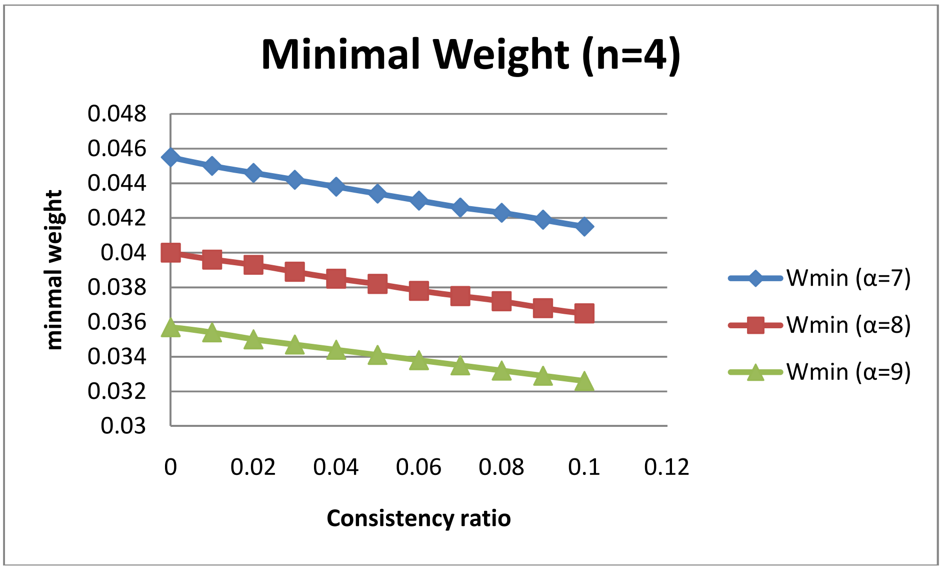

Figure 4.2 Minimal weights for n = 4 and α = 7, 8, 9

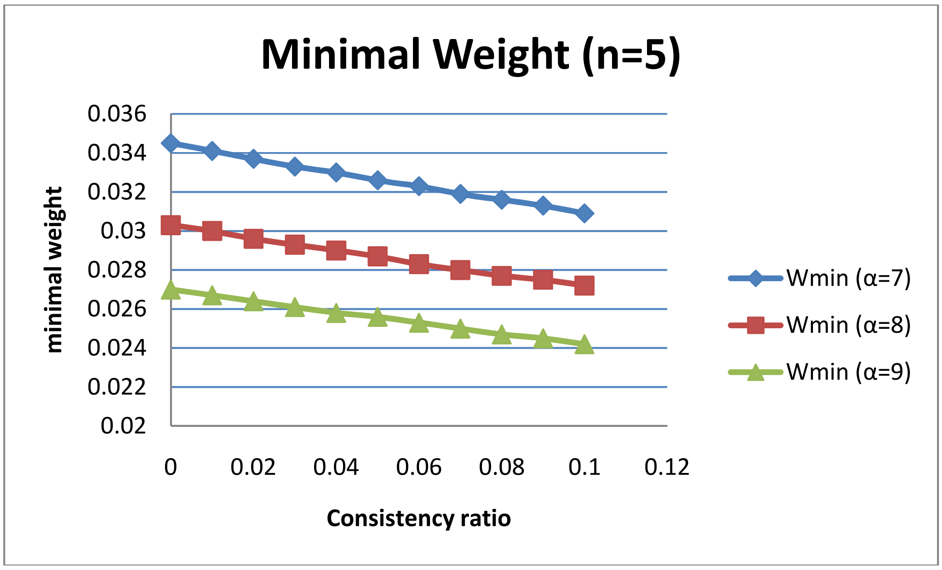

Figure 4.3 Minimal weights for n = 5 and α = 7, 8, 9

Chapter 5

SIMULATION RESULTS

5.1 Simulation Methods

Monte Carlo simulation was used to establish a link, if any, between a potentially misspecified hierarchy and the rank reversal phenomena. The methods described in this section were applied to comparison matrices of dimension n = 3, 4, 5 with a single-level hierarchy from which the best decision alternative is selected.

Using simulation, a set of pairwise comparison matrices was generated such that each matrix in the set (1) exhibited rank reversal, (2) had an acceptable level of inconsistency, and (3) contained at least one a_{ij} value such that $a_{ij} \geq 7$. The last condition is required because, in the case of a misspecified hierarchy, at least one alternative should be, at least, strongly more important than another, indicating that the hierarchy may be risk of over-specification.

For each matrix in this set, the smallest actual priority weight was compared to the absolute smallest possible priority weight for the matrix's level of inconsistency and dimension (computed from equations 4.2.10, 4.3.12, and 4.3.16). If the deviations are 'relatively' small, then a link exists between a potentially misspecified hierarchy and the rank reversal phenomenon. The definition of 'relatively small' will be presented later in this section.

The general simulation methods described above are detailed in the following steps:

Step 1: Randomly generate an n x n pairwise comparison matrix such that each a_{ij} is generated from the intervals [i,i+1] or [1/(i+1), 1/i] (i=1, 2, …, 8) with the equal probability.

Step 2: Ensure that the pairwise comparison matrix generated in step 1 satisfies CR ≤ 0.1 (an acceptable level of inconsistency) and at least one $a_{ij} \geq 7$. If yes, continue to step 3; otherwise return to step 1.

Step 3: Check the pairwise comparison matrix for rank reversal when one alternative is randomly selected for deletion. If rank reversal does not occur, return to Step 1 to generate another pairwise comparison matrix. If rank reversal does occur, record the smallest priority and return to Step 1 to generate another pairwise comparison matrix.

The first and most important step is to generate random pairwise comparison matrices. For an nxn pairwise comparison matrix, n*(n-1)/2 aij values must be generated. In this chapter, we conduct two kinds of simulations, assuming aij to be either non-integer values or integer values. The former one analyzes the possibility of rank reversal theoretically, whereas, the latter one focuses on the risk of rank reversal occurred in practical situations.

If aij is assumed to be non-integer value, it can be drawn from the following continuous intervals with equal probability: (1, 2], (2, 3], (3, 4], (4, 5], (5, 6], (6, 7], (7, 8], (8,9] and (1/2,1], (1/3, 1/2], (1/4, 1/3], (1/5, 1/4], (1/6, 1/5], (1/7, 1/6], (1/8, 1/7], (1/9, 1/8]. This process involved two steps:

Step 1: Generate two random numbers u and α, where $u \sim U\,[0, 1]$ and $\alpha \sim U\,[1, 9]$.

Step 2: If $u < 0.5$, then $\alpha_{ij} = \alpha$; otherwise $\alpha_{ij} = 1/\alpha$.

For validation purposes, 100,000 aij values were generated using the above two-step process. Results presented in Table 5.1 validate the process used to generate the aij values.

If aij is assumed to be integer value, it can be drawn from the following values with equal probability 1/17: 1, 2, 3, 4, 5, 6, 7, 8, 9, 1/2, 1/3, 1/4, 1/5, 1/6, 1/7, 1/8, 1/9. This process also involved two steps:

Step 1: Generate a random number $u \sim U\,[0, 1]$.

Step 2: Assign α_{ij} value according to the value of u:

$\alpha_{ij} = 1/9$, if u belongs to [0, 1/17];

$\alpha_{ij} = 1/8$, if u belongs to (1/17, 2/17];

$\vdots$

$\alpha_{ij} = 9$, if u belongs to (16/17, 1];

The above two steps algorithm will generate required random integer aij values.

5.2 Simulation Results for n = 3, Non-integer aij Value

Using the method described above to generate non-integer aij values, 3x3 pairwise comparison matrices were continuously generated until the number of matrices that exhibited rank reversal reached 20,000. Generation of 20,000 'rank-reversal comparison matrices' was executed under each of the following three conditions: (1) at least one element $\alpha \geq 7$, (2) at least one element $\alpha \geq 8$ and (3) and at least one element $\alpha = 9$. The details of this simulation process are given in Table 5.2.

The results in Table 5.2 indicate that for n = 3, the chance of rank reversal in a matrix of acceptable inconsistency is about 4.6% or about 1 in 22. It should be noted that this result assumes that the aij values need not assume only integer values or their respective reciprocals. Table 5.3 summarizes the deviation between the smallest <u>actual</u> priority weight and the smallest <u>possible</u> priority weight (computed from equation 4.2.10), across all 20,000 generated pairwise comparison matrices.

From table 5.3, we can see that, for each of the three scenarios, about 4% of the actual smallest priority weights are within 10% of their corresponding minimal possible weights. This result indicates that only about 4% of pairwise comparison matrices that exhibit rank reversal (with an acceptable level of inconsistency) may include irrelevant alternatives (i.e., be at risk for an overspecified hierarchy).

Figure 5.1 illustrates the distribution of actual smallest priority weights, for comparison matrices that exhibit rank reversal, for the case of $\alpha \geq 7$.

The distribution in Figure 5.1 is nearly bi-modal, with one mode around 0.07 and the other around 0.12. The reason for this phenomenon can be explained as follows: For a 3x3 pairwise comparison matrix, two situations may generate rank reversal: one alternative is strongly more important than the other two or one alternative is strongly less important than the other two. In the former situation, rank reversal may occur between the two less important alternatives. In the later situation, rank reversal may occur within the two more important alternatives. Under perfect consistency, the pairwise comparison matrixes associated with the two situations are:

$$\underset{(1)}{\begin{pmatrix} 1 & 1 & \alpha \\ 1 & 1 & \alpha \\ 1/\alpha & 1/\alpha & 1 \end{pmatrix}} \qquad \underset{(2)}{\begin{pmatrix} 1 & \alpha & \alpha \\ 1/\alpha & 1 & 1 \\ 1/\alpha & 1 & 1 \end{pmatrix}} \tag{5.4.1}$$

The first pairwise comparison matrix (1) generates a minimal possible priority weight $w_{min}=1/(2*\alpha+1)$. The second pairwise comparison matrix generates a minimal possible priority weight $w_{min}=1/(\alpha+2)$. Assuming $\alpha=7$ in this situation, the two minimal possible priority weights are 0.067 and 0.111, respectively. Thus, the smallest priority weights of rank-reversal comparison matrices, with a $CR \leq 0.1$, should cluster around .067 and around .111, as illustrated in Figure 5.1.

For the case of a 3x3 pairwise comparison matrix ($n = 3$), by assuming non-integer aij values, two observations should be made:

Observation 1: For pairwise comparison matrices that exhibit an acceptable level of inconsistency, rank reversal occurs with a probability of about 4.6%.

Observation 2: For pairwise comparison matrices that DO exhibit rank reversal, no more than 4% of these matrices have an actual minimal priority weight that is within

10% of the absolute minimum priority weight. No more than 12% have an actual minimal priority weight that is within 20% of the absolute minimum priority weight. This observation implies that the link between an over-specified hierarchy (i.e., irrelevant criterion or decision alternative) and the rank-reversal phenomenon is weak, at best.

In n = 3 situation, rank reversal doesn't occur, if aij is assumed to be integer value.

5.3 Simulation Results for n = 4, Non-integer aij Value

As in the case of n = 4, by assuming aij value to be non-integer, 4x4 pairwise comparison matrices were continuously generated until the number of matrices that exhibited rank reversal reached 20,000. Generation of 20,000 'rank-reversal comparison matrices' was executed under each of the following three conditions: (1) at least one element $\alpha \geq 7$, (2) at least one element $\alpha \geq 8$ and (3) and at least one element $\alpha = 9$. The details of this simulation process are given in Table 5.4.

The results in Table 5.4 indicate that for n = 4, by assuming aij to be non-integer values, the chance of rank reversal in a matrix of acceptable inconsistency is about 17.6%. Rank reversal will occur nearly 4 times more frequently than in 3x3 pairwise comparison matrices. It should be noted that this result assumes that the aij values need NOT assume only integer values or their respective reciprocals. Table 5.5 summarizes the deviation between the smallest <u>actual</u> priority weight and the smallest possible priority weight (computed from equation 4.3.12), across all 20,000 generated pairwise comparison matrices.

From the table 5.5, we can see that, for each of the three scenarios, about 4% of the actual smallest priority weights are within 10% of their corresponding minimal possible weights. This result indicates that only about 4% of pairwise comparison matrices that exhibit rank reversal (with an acceptable level of inconsistency) may include irrelevant alternatives (i.e., be at risk for an overspecified hierarchy).

Figure 5.2 shows the distributions of actual smallest priority weight with rank reversal and without rank reversal. It is obvious that the distribution of smallest priority weight with rank reversal is right skewed compared to that without rank reversal, indicating the link between rank reversal and over-specification is weak. On the other hand, more than 10% of smallest priority weights without rank reversal are close to corresponding minimal ones, implying that the idea of minimal priority weight will be useful in detecting the potentially over-specified hierarchy.

For the case of a 4x4 pairwise comparison matrix (n = 4), by assuming aij to be non-integer values, two observations should be made:

Observation 1: For pairwise comparison matrices that exhibit an acceptable level of inconsistency, rank reversal occurs with a probability of about 17.6%.

Observation 2: For pairwise comparison matrices that DO exhibit rank reversal, no more than 5% of these matrices have an actual minimal priority weight that is within 10% of the absolute minimum priority weight. No more than 14% have an actual minimal priority weight that is within 20% of the absolute minimum priority weight.

This observation implies that the link between an over-specified hierarchy (i.e., irrelevant criterion or decision alternative) and the rank-reversal phenomenon is weak, at best.

5.4 Simulation Results for n=4, Integer aij Value

Similarly, if aij is assumed to be integer value, we generate 10,000 rank reversal examples under each of the following three conditions: (1) at least one element $\alpha \geq 7$, (2) at least one element $\alpha \geq 8$ and (3) and at least one element $\alpha = 9$. The details of this simulation process are given in Table 5.6.

The results in Table 5.6 indicate that for n = 4, by assuming aij to be integer value, the chance of rank reversal in a matrix of acceptable inconsistency is about 22%, which is similar with previous result, by assuming aij to be non-integer. Table 5.7 summarizes the deviation between the smallest actual priority weight and the smallest possible priority weight (computed from equation 4.3.12), across all 10,000 generated pairwise comparison matrices.

From the table 5.7, we can see that, for each of the three scenarios, about 7% of the actual smallest priority weights are within 10% of their corresponding minimal possible weights. This result indicates that only about 7% of pairwise comparison matrices that exhibit rank reversal (with an acceptable level of inconsistency) may include irrelevant alternatives (i.e., be at risk for an overspecified hierarchy).

Similarly, for the case of a 4x4 pairwise comparison matrix (n = 4), assuming integer aij values, two observations should be made:

Observation 1: For pairwise comparison matrices that exhibit an acceptable level of inconsistency, rank reversal occurs with a probability of about 22%.

Observation 2: For pairwise comparison matrices that DO exhibit rank reversal, no more than 7% of these matrices have an actual minimal priority weight that is within 10% of the absolute minimum priority weight. No more than 17% have an actual minimal priority weight that is within 20% of the absolute minimum priority weight. This observation implies that the link between an over-specified hierarchy (i.e., irrelevant criterion or decision alternative) and the rank-reversal phenomenon is weak, at best.

5.5 Simulation Results for n=5, Non-integer aij Value

As in the case of n = 5, by assuming non-integer aij values, 5x5 pairwise comparison matrices were continuously generated until the number of matrices that exhibited rank reversal reached 20,000. Generation of 20,000 'rank-reversal comparison matrices' was executed under each of the following three conditions: (1) at least one element $\alpha \geq 7$, (2) at least one element $\alpha \geq 8$ and (3) and at least one element $\alpha = 9$. The details of this simulation process are given in Table 5.8.

The results in Table 5.8 indicate that for n = 5, the chance of rank reversal in a matrix of acceptable inconsistency is about 35.42% or 1 out of 3. Rank reversal will occur nearly 2 times more frequently than in 3x3 pairwise comparison matrices and 8 times more frequently than in 3x3 pairwise comparison matrices. It should be noted that this result assumes that the aij values need NOT assume only integer values or their respective reciprocals. Table 5.9 summarizes the deviation between the smallest actual priority weight and the smallest possible priority weight (computed from equation 4.3.16), across all 20,000 generated pairwise comparison matrices.

From the table 5.7, we can see that, for each of the three scenarios, about 2% of the actual smallest priority weights are within 10% of their corresponding minimal possible weights. This result indicates that only about 2% of pairwise comparison matrices that exhibit rank reversal (with an acceptable level of inconsistency) may include irrelevant alternatives (i.e., be at risk for an overspecified hierarchy).

Figure 5.3 shows the distributions of actual smallest priority weight with rank reversal and without rank reversal. It is obvious that the distribution of smallest priority weight with rank reversal is right skewed compared to that without rank reversal, indicating the link between rank reversal and over-specification is weak. On the other hand, from the distribution, around 5% of smallest priority weights without rank reversal are close to corresponding minimal ones, implying that the idea of minimal priority weight will be useful in detecting the potentially over-specified hierarchy.

For the case of a 5x5 pairwise comparison matrix (n = 5), two observations should be made:

Observation 1: For pairwise comparison matrices that exhibit an acceptable level of inconsistency, rank reversal occurs with a probability of about 35.42%.

Observation 2: For pairwise comparison matrices that DO exhibit rank reversal, no more than 3% of these matrices have an actual minimal priority weight that is within

10% of the absolute minimum priority weight. No more than 9% have an actual minimal priority weight that is within 20% of the absolute minimum priority weight. This observation implies that the link between an over-specified hierarchy (i.e., irrelevant criterion or decision alternative) and the rank-reversal phenomenon is weak, at best.

5.6 Simulation Results for n=5, Integer aij Value

Similarly, if aij is assumed to be integer value, we generate 10,000 rank reversal examples under each of the following three conditions: (1) at least one element $\alpha \geq 7$, (2) at least one element $\alpha \geq 8$ and (3) and at least one element $\alpha = 9$. The details of this simulation process are given in Table 5.6.

The results in Table 5.10 indicate that for n = 5, by assuming aij to be integer value, the chance of rank reversal in a matrix of acceptable inconsistency is about 36.7%. Table 5.11 summarizes the deviation between the smallest <u>actual</u> priority weight and the smallest <u>possible</u> priority weight (computed from equation 4.3.12), across all 10,000 generated pairwise comparison matrices.

From the table 5.11, we can see that, for each of the three scenarios, about 3% of the actual smallest priority weights are within 10% of their corresponding minimal possible weights. This result indicates that only about 3% of pairwise comparison matrices that exhibit rank reversal (with an acceptable level of inconsistency) may include irrelevant alternatives (i.e., be at risk for an overspecified hierarchy).

Similarly, for the case of a 5x5 pairwise comparison matrix (n = 5), assuming integer aij values, two observations should be made:

Observation 1: For pairwise comparison matrices that exhibit an acceptable level of inconsistency, rank reversal occurs with a probability of about 36.7%.

Observation 2: For pairwise comparison matrices that DO exhibit rank reversal, no more than 3% of these matrices have an actual minimal priority weight that is within 10% of the absolute minimum priority weight. No more than 10% have an actual minimal priority weight that is within 20% of the absolute minimum priority weight. This observation implies that the link between an over-specified hierarchy (i.e., irrelevant criterion or decision alternative) and the rank-reversal phenomenon is weak, at best.

5.7 Simulation Summary

By conducting the above simulation process for n = 3, 4, and 5, assuming aij values to be both non-integer and integer, we conclude that (1) rank reversal is more likely to occur as the number of decision alternatives increases and (2) on average, only a small percentage (4%) of matrices that do exhibit rank reversal (and have an acceptable level of inconsistency) have an actual minimal priority weight that is within 10% of the absolute minimum priority weight. Thus, the link between an over-specified hierarchy (i.e., irrelevant criterion or decision alternative) and the rank-reversal phenomenon appears to be weak, at best.

Table 5.1 Distribution of simulated non-integer a_{ij} values

Interval	Percentage	Interval	Percentage
(1, 2]	6.24%	(1/2, 1]	6.24%
(2, 3]	6.22%	(1/3, 1/2]	6,24%
(3, 4]	6.26%	(1/4, 1/3]	6.22%
(4, 5]	6.22%	(1/5, 1/4]	6.25%
(5, 6]	6.21%	(1/6, 1/5]	6.25%
(6, 7]	6.22%	(1/7, 1/6]	6.23%
(7, 8]	6.23%	(1/8, 1/7]	6.34%
(8, 9]	6.39%	(1/9, 1/8]	6.24%

Table 5.2 Simulation results n=3, non-integer a_{ij} value

	$\alpha \geq 7$	$\alpha \geq 8$	$\alpha = 9$
Total # of matrixes with CR ≤ 0.1	441,676	436,619	442,626
Total # of matrixes with CR ≤ 0.1 and rank reversal	20,000	20,000	20,000

Table 5.3 Comparison of actual to smallest possible weight for n = 3, non-integer a_{ij} value

Percent Deviation from Smallest Possible Weight	$\alpha \geq 7$	$\alpha \geq 8$	$\alpha = 9$
10%	**4.25%**	**3.52%**	**3.69%**
20%	11.78%	11.39%	11.31%
30%	21.38%	20.70%	21.26%
40%	31.61%	31.22%	30.98%
50%	41.86%	40.73%	40.94%
60%	48.46%	47.22%	47.59%
70%	50.88%	49.69%	49.84%
80%	53.86%	52.08%	51.42%
90%	60.43%	58.17%	56.91%
100%	69.12%	69.12%	64.63%

Table 5.4 Simulation results for n = 4, non-integer a_{ij} value

	$\alpha \geq 7$	$\alpha \geq 8$	$\alpha = 9$
Total # of matrixes with C.R. ≤ 0.1	112,416	113,977	114,537
Total # of matrixes with CR ≤ 0.1 and rank reversal	20,000	20,000	20,000

Table 5.5 Comparison of actual to smallest possible weight for n = 4, non-integer a_{ij} value

Percent Deviation from Smallest Possible Weight	$\alpha \geq 7$	$\alpha \geq 8$	$\alpha = 9$
10%	**5.25%**	**3.81%**	**2.73%**
20%	13.67%	11.6%	9.88%
30%	22.81%	20.55%	18.68%
40%	31.50%	29.45%	27.33%
50%	40.18%	37.55%	35.13%
60%	48.61%	45.70%	43.24%
70%	57.07%	54.10%	51.36%
80%	64.77%	61.97%	59.32%
90%	72.30%	69.45%	67.06%
100%	78.41%	75.84%	73.58%

Table 5.6 Simulation results for n = 4, integer a_{ij} value

	$\alpha \geq 7$	$\alpha \geq 8$	$\alpha = 9$
Total # of matrixes with C.R.≤ 0.1	45,905	46,579	47,113
Total # of matrixes with CR ≤ 0.1 and rank reversal	10,000	10,000	10,000

Table 5.7 Comparison of actual to smallest possible weight for n = 4, integer a_{ij} value

Percent Deviation from Smallest Possible Weight	$\alpha \geq 7$	$\alpha \geq 8$	$\alpha = 9$
10%	**7.5%**	**5.81%**	**5.94%**
20%	16.81%	14.40%	14.75%
30%	26.32%	23.92%	23.84%
40%	35.50%	32.68%	32.66%
50%	44.92%	41.95%	41.90%
60%	54.55%	51.12%	52.03%
70%	63.19%	59.79%	61.18%
80%	70.50%	67.76%	69.15%
90%	77.20%	74.64%	75.74%
100%	82.89%	80.64%	81.06%

Table 5.8 Simulation results n=5, non-integer a_{ij} value

	$\alpha \geq 7$	$\alpha \geq 8$	$\alpha = 9$
Total # of matrixes with CR ≤ 0.1	55,924	56,142	57,332
Total # of matrixes with CR ≤ 0.1 and rank reversal	20,000	20,000	20,000

Table 5.9 Comparison of actual to smallest possible weight for n = 5, non-integer a_{ij} value

Percent Deviation from Smallest Possible Weight	$\alpha \geq 7$	$\alpha \geq 8$	$\alpha = 9$
10%	**2.44%**	**1.80%**	**1.68%**
20%	8.75%	6.74%	5.54%
30%	18.21%	15.30%	13.34%
40%	28.56%	25.32%	22.79%
50%	39.17%	35.44%	33.04%
60%	49.35%	45.49%	42.74%
70%	58.52%	54.97%	52.17%
80%	67.31%	63.57%	60.72%
90%	74.23%	70.82%	68.42%
100%	80.32%	77.22%	75.31%

Table 5.10 Simulation results n=5, integer a_{ij} value

	$\alpha \geq 7$	$\alpha \geq 8$	$\alpha = 9$
Total # of matrixes with CR $\leq$ 0.1	27,701	27,655	26,831
Total # of matrixes with CR $\leq$ 0.1 and rank reversal	10,000	10,000	10,000

Table 5.11 Comparison of actual to smallest possible weight for n = 5, integer a_{ij} value

Percent Deviation from Smallest Possible Weight	$\alpha \geq 7$	$\alpha \geq 8$	$\alpha = 9$
10%	**3.2%**	**2.7%**	**2.3%**
20%	10.3%	7.1%	7.9%
30%	18.6%	17.5%	17.3%
40%	27.8%	28%	26.7%
50%	38.7%	39%	37.4%
60%	48.8%	49.1%	48%
70%	58.9%	58.5%	56.4%
80%	66.5%	66.9%	65.5%
90%	73.3%	72.3%	72%
100%	79%	78.8%	78%

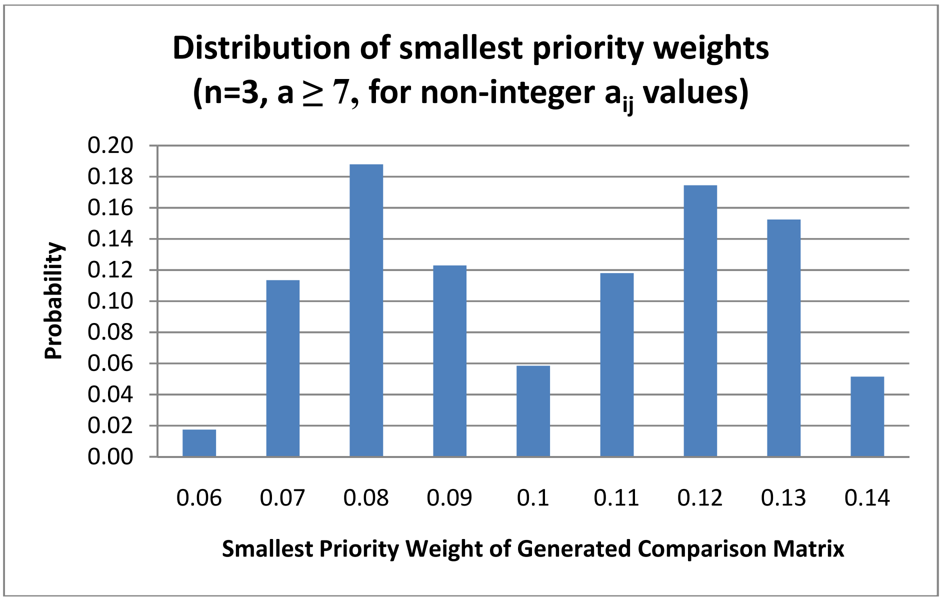

Figure 5.1 Distribution of smallest priority weight when rank reversal occurs, n = 3, a ≥ 7, for non-integer a_{ij} values

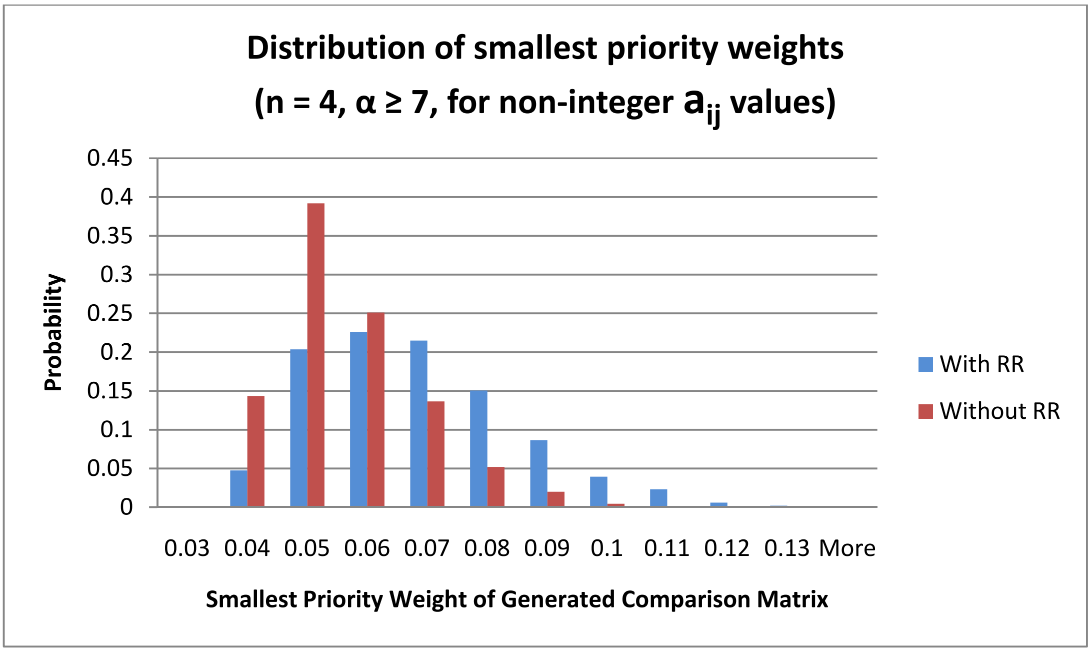

Figure 5.2 Distribution of smallest priority weights, n=4, $\alpha \geq 7$, for non-integer a_{ij} values

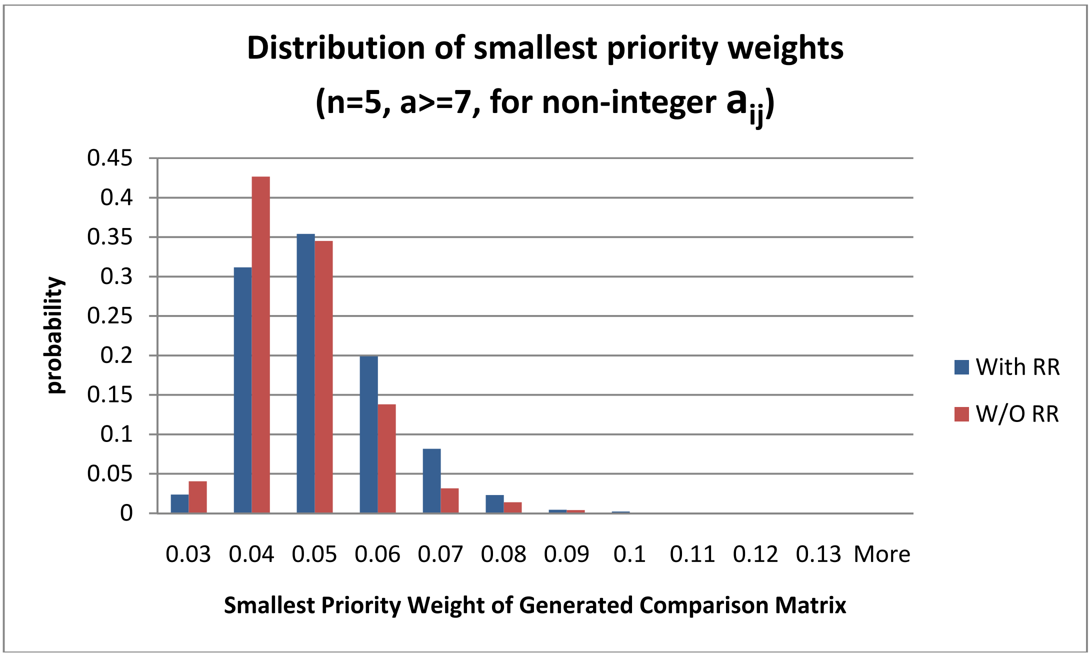

Figure 5.2 Distribution of smallest priority weights, n=5, $\alpha \geq 7$, for non-integer a_{ij} values

Chapter 6

CONCLUSION

This thesis focused on two issues: (1) the calculation of an absolute minimal priority weight for a pairwise comparison matrix that exhibits an acceptable level of inconsistency and (2) investigation of a link, if any, between a potentially over-specified hierarchy and the rank reversal phenomenon.

With respect to the first issue, the thesis extended the work of Aull-Hyde and Duke's (2006) by generating an absolute minimal priority weights for matrices that exhibit an acceptable level of inconsistency. Aull-Hyde and Duke (2006) assumed a perfectly consistent matrix when determining an absolute minimal priority weight. Thus, the methodology proposed by Aull-Hyde and Duke can be implemented without the assumption of perfect consistency.

When determining the minimum possible priority weight for an inconsistent 3x3 pairwise comparison matrix, the minimum possible priority weight can be expressed as a unique function of the consistency ratio (CR as defined by Saaty (1980)). To determine the minimum possible priority weight for an inconsistent 4x4 pairwise comparison matrix, a 'consistency ratio set' was used to group potential pairwise comparison matrixes according to their consistency ratio. The minimal weight generated by the 'representative' pairwise comparison matrix is used as the minimal weight for all matrices in the consistency-level set. The minimal relative priority weight is a decreasing function of the CR value, indicating that matrices with higher levels of inconsistency will have smaller minimal relative priority weights.

This result implies that the ability to detect a misspecified hierarchy, using the procedure proposed by Aull-Hyde and Duke (2006), becomes more difficult as the inconsistency levels of the associated matrices increase.

Given that an absolute minimal relative priority weight had been developed for the case of an inconsistent pairwise comparison matrix, the relationship between a mis-specified hierarchy and the rank-reversal phenomenon was investigated. This analysis revealed that, as expected, the risk of rank reversal (in matrices having an acceptable level of inconsistency and are at risk for over-specification) increases dramatically as the number of decision alternatives increases. Given that a pairwise comparison matrix, with an acceptable level of inconsistency, exhibits rank-reversal, the likelihood that the associated hierarchy is at risk for over-specification is no more than 4%. This result indicates that no strong link exists between an over-specified hierarchy and rank reversal phenomena.

The thesis extends the work of Aull-Hyde and Duke (2006) by calculating a minimal relative priority weight for an inconsistent pairwise comparison matrix. The minimal priority weight can be of practical importance by enabling decision makers to check the credibility of their established hierarchy.

The simulation analysis verified, as expected, that the risk of rank reversal increases as the number of decision alternatives/criteria increases. As the dimension of the pairwise comparison matrix increases, the level of inconsistency also tends to rise. Thus, the risk of rank reversal increases as the level of inconsistency increases.

Future research directions include:

1. Can we find a representative matrix that generates a better (i.e., higher) lower bound on minimal relative priority weight for all matrices in the given consistency ratio set?

2. Does a link between an over-specified hierarchy and rank reversal exist in multiple-level decision hierarchies?

3. What is the risk of rank reversal for any matrix (i.e., no restriction on the aij values) having an acceptable level of inconsistency?

REFERENCES

Alan, H.S., W.Y. Chan and G. Vincent. 2004. Using AHP for determining priority in a safety management system. Industrial Management & Data Systems, Vol. 104, No. 5: 430-445.

Aull-Hyde, R. and J.M. Duke. 2006. Using the AHP iteratively to specify the hierarchy. Working paper, Department of Food and Resource Economics, University of Delaware.

Banuelas, R. and J. Antony. 2007. Application of stochastic analytic hierarchy process within a domestic appliance manufacturer. Journal of the Operations Research Society 58: 29-38.

Barzilai, J. and B. Golany. 1994. AHP rank reversal, Normalization and Aggregation Rules. INFOR, Vol 32, No. 2: 57-64.

Belton, V. and A. E. Gear. 1983. On a short coming of Saaty's method of analytic hierarchies. Omega 11: 228-230.

Chen, C. F. 2006. Applying the Analytical Hierarchy Process (AHP) Approach to Convention Site Selection. Journal of Travel Research, Vol. 45: 167.

Dorit, S. H. and A. Levin. 2006. Methodologies and Algorithms for Group-Rankings Decision. Management Science Vol. 52, No. 9: 1394-1408.

Duke, J. M. and R. Aull- Hyde. 2002. Identifying public preferences for land preservation using the analytic hierarchy process. Ecological Economics 42: 131-145.

Dyer, J. S. and H. V. Ravinder. 1983. Irrelevant alternatives and the Analytical Hierarchy Process. Technical Report, Department of Management, The University of Texas at Austin.

Dyer, J. S. and R. E. Wendell. 1985. A critique of the Analytic Hierarchy Process. Technical Report, Department of Management, The University of Texas at Austin.

Dyer, J. S. 1990a. Remarks on the analytic hierarchy process. Management Science Vol.36, No.3: 249-258.

Dyer, J. S. 1990b. A clarification of "Remarks on the analytic hierarchy process. Management Science Vol.36, No.3: 274-275.

Eddie, W.L., H.L. Cheng. 2001. Information priority-setting for better resource allocation using analytic hierarchy process (AHP). Information Management & Computer Security 9/2: 61-70.

Farkas, A. 2000. The analysis of the Principal Eigenvector of Pair-wise Comparison Matrixes. Working paper, Budapest Tech, Economics department.

Harker, P. T. and L. G. Vargas. 1987. The theory of ratio scale estimation: Saaty's analytic hierarchy process. Management Science Vol.33, No. 11.

Harker, P. T. and L. G. Vargas. 1990. Reply to "Remarks on the analytic hierarchy process" by J.S. Dyer. Management Science Vol.36, No.3: 269-273.

Hsu, P. F., C. R. Wu, and Z. R. Li. 2008. Optimizing resource-based allocation for senior citizen housing to ensure a competitive advantage using the analytic hierarchy process. Building and Environment 43: 90-97.

Kamal M. Al-Subhi AL-Harbi. 2001. Application of the AHP in project management. International Journal of Project Management 19: 19-27.

Kurttila, M., M. Pesonen, J. Kangas and M. Kajanus. 2000. Utilizing the analytic hierarchy process (AHP) in SWOT analysis- a hybrid method and its application to a forest-certification case. Forest Policy and Economics 1: 41-52.

Lee, Y. H. and K. A. Kozar. 2006. Investigating the effect of website quality on e-business success: An analytic hierarchy process (AHP) approach. Decision Support System 42: 1382-1401.

Liu, F. H. and H. L. Hai. 2005. The voting analytic hierarchy process method for selecting supplier. Int. J. Production Economics 97: 308-317.

Mohammed I. Al Khalil. 2002. Selecting the appropriate project delivery method using AHP. International Journal of Project Management 20: 469-474.

Saaty, T. L. 1980. The Analytic Hierarchy Process, New York: McGraw-Hill International.

Saaty, T. L. 1983. Axiomatic foundations of the analytic hierarchy process. Management Science, 32: 841-855.

Saaty, T. L. 1987. Rank generation, preservation, and reversal in the analytic hierarchy process. Decision Science, 18: 157-177.

Saaty, T. L. 1990. An exposition of the AHP in reply to the paper "Remarks on the analytic hierarchy process". Management Science Vol.36, No.3: 259-268.

Saaty, T. L. 2007. Group Decision Making: Drawing out and Reconciling Differences, RWS Publications; 1st edition.

Scholl, A., L. Manthey, R. Helm and M. Steiner. 2005. Solving multi-attribute design problems with analytic hierarchy process and conjoint analysis: An empirical comparison. European Journal of Operational Research, 164: 760-770.

Srdjevic, B. 2005. Combining different prioritization methods in the analytic hierarchy process synthesis. Computer & operations research 32: 1897-1919.

Triantaphyllou, E. 2001. Two new cases of rank reversal when the AHP and some of its additive variants are used that do not occur with the multiplicative AHP. Journal of multi-criteria decision analysis 10: 11-25.

Taha, H. A. 2006. Operations Research: An introduction, Prentice Hall; 8th edition.

Millet, I. and T. L. Saaty. 2000. On the relativity of relative measures-accommodating both rank preservation and rank reversals in the AHP. European Journal of Operational Research 121: 205-212.

Wang, Y. M. and Taha M.S. Elhag. 2006. An approach to avoiding rank reversal in AHP. Decision Support Systems 42: 1474-1480.

Wang, Y. M. and Y. Luo. 2009. On rank reversal in decision analysis. Mathematical and Computing Modeling 49: 1221-1229.

Wong, K. W. and H. Li. 2008. Application of the analytical hierarchy process (AHP) in multi-criteria analysis of the selection of intelligent building systems. Building and Environment 43: 108-125.

APPENDIX A

Matlab Simulation Programs for n=3:

```
function w=mweight(c,a); %c is the C.R. level. a is the max element. w return to the corresponding minimal weight.
lamda=1.32*c+3;
d=3*lamda^2-lamda^3-2;
m=(-d+sqrt(d^2-4))/2;
w=((1-lamda)^2-1)/(a*m+2*a*(lamda-1)+a/m-1+(1-lamda)^2);
end
```

```
function [V, H]=simulate3(n);%V returns to the minimal weights and matrix information, H returns to the smallest priority from matrix without rank reversal. N inputs that total # of rank reversal generated.

t=cputime;
i=0; %the number of rank reversal example;
j=0; %the number of total simulations;
k=0; %the number of total examples satisfying CR<=0.1 and max element >=7;
o=1;
coun=zeros(1,10);
while i<n
   j=j+1;
   u=rand();     %generate random numbers
   a=rand()*8+1;
   if u<=0.5
      a=a;
   else
      a=1/a;
   end
   u=rand();
   b=rand()*8+1;
   if u<=0.5;
      b=b;
   else
```

```
   b=1/b;
end
u=rand();
c=rand()*8+1;
if u<=0.5
   c=c;
else
   c=1/c;
end
c=9;
M=[1,a,b;1/a,1,c;1/b,1/c,1]; %generate pairwise comparison matrix
[v,d]=eig(M);
CR=(d(1,1)-3)/1.32;
if (CR<=0.1)&&(max([a,b,c])>=7)  % check conditions
   k=k+1;
   v1=abs(v(1,1));
   v2=abs(v(2,1));
   v3=abs(v(3,1));
   X=(v1/v2-1)*(a-1);
   Y=(v1/v3-1)*(b-1);
   Z=(v2/v3-1)*(c-1);
   if (X<0)||(Y<0)||(Z<0) %check if rank reversal occurs
      V(i+1,1)=a;
      V(i+1,2)=b;
      V(i+1,3)=c;
      V(i+1,4)=CR;
      V(i+1,5)=min([v1,v2,v3])/(v1+v2+v3); %smallest weight
      V(i+1,6)=V(i+1,5)-mweight(V(i+1,4),max([a,b,c]))*1.1;
      V(i+1,7)=V(i+1,5)-mweight(V(i+1,4),max([a,b,c]))*1.2;
      V(i+1,8)=V(i+1,5)-mweight(V(i+1,4),max([a,b,c]))*1.3;
      V(i+1,9)=V(i+1,5)-mweight(V(i+1,4),max([a,b,c]))*1.4;
      V(i+1,10)=V(i+1,5)-mweight(V(i+1,4),max([a,b,c]))*1.5;
      V(i+1,11)=V(i+1,5)-mweight(V(i+1,4),max([a,b,c]))*1.6;
      V(i+1,12)=V(i+1,5)-mweight(V(i+1,4),max([a,b,c]))*1.7;
      V(i+1,13)=V(i+1,5)-mweight(V(i+1,4),max([a,b,c]))*1.8;
      V(i+1,14)=V(i+1,5)-mweight(V(i+1,4),max([a,b,c]))*1.9;
      V(i+1,15)=V(i+1,5)-mweight(V(i+1,4),max([a,b,c]))*2;
      i=i+1;
   else
      H(o)=min([abs(v(1,1)),abs(v(2,1)),abs(v(3,1))])/(v1+v2+v3);
      o=o+1;
   end
   v1=0; v2=0; v3=0; X=0; Y=0; Z=0;
```

```
    end
    a=0; b=0; c=0; M=zeros(3,3); v=zeros(3,3); d=zeros(3,3); CR=0;
end
for c=1:n
    if V(c,6)<0
        coun(1)=coun(1)+1;
    end
    if V(c,7)<0
        coun(2)=coun(2)+1;
    end
    if V(c,8)<0
        coun(3)=coun(3)+1;
    end
    if V(c,9)<0
        coun(4)=coun(4)+1;
    end
    if V(c,10)<0
        coun(5)=coun(5)+1;
    end
    if V(c,11)<0
        coun(6)=coun(6)+1;
    end
    if V(c,12)<0
        coun(7)=coun(7)+1;
    end
    if V(c,13)<0
        coun(8)=coun(8)+1;
    end
    if V(c,14)<0
        coun(9)=coun(9)+1;
    end
    if V(c,15)<0
        coun(10)=coun(10)+1;
    end
end
coun
k
j
time=cputime-t
end
```

APPENDIX B

Matlab Simulation Programs for n=4:

```
function w=mweight(c,a);
L=2.97*c+4;
D=2*L^2-L^3/2-2;
m=(-D+sqrt(D^2-4))/2;
Q=(L-1)^3+3*a*(L-1)^2+(4*a-3+a*m+a/m)*(L-1)+[m*(2*a-1)+(2*a-1)/m-a];
w=[(L-1)^3-3*(L-1)-(m+1/m)]/Q;
end


function [V, H]=simulate4(n);  %V returns to the minimal weights and matrix information, H returns to the smallest priority from matrix without rank reversal. N inputs that total # of rank reversal generated.

t=cputime;
o=1;
i=0; %count the number of rank reversal;
j=0; %count the total number of simulation;
k=0; %count the total number of example s.t. CR<=0.1 and max element>=7;
S123=0;S234=0;S124=0;S134=0;
coun=zeros(1,10);
while i<n
   j=j+1;
   u=rand();
   a12=rand()*8+1;  %generate random numbers for aij between [1,9]
   if u<=0.5
      a12=a12;
   else
      a12=1/a12;
   end
   u=rand();
   a13=rand()*8+1;
   if u<=0.5
      a13=a13;
   else
```

```
      a13=1/a13;
   end
   u=rand();
   a14=rand()*8+1;
   if u<=0.5
      a14=a14;
   else
      a14=1/a14;
   end
   a14=9;
   u=rand();
   a23=rand()*8+1;
   if u<=0.5
      a23=a23;
   else
      a23=1/a23;
   end
   u=rand();
   a24=rand()*8+1;
   if u<=0.5
      a24=a24;
   else
      a24=1/a24;
   end
   u=rand();
   a34=rand()*8+1;
   if u<=0.5
      a34=a34;
   else
      a34=1/a34;
   end

A=[1,a12,a13,a14;1/a12,1,a23,a24;1/a13,1/a23,1,a34;1/a14,1/a24,1/a34,1]; %generate
random pairwise comparison matrix
   [v,d]=eig(A);   %calculate the eignvalue and eignvector
   CR=(d(1,1)-4)/2.97;  %calculate the consistency ratio
   if (CR<=0.1)&&(max([a12,a13,a14,a23,a24,a34])>=8) % choose random pairwise
comparison matrix satisfying CR requirement and maximal element larger than 7
      k=k+1;     %calculate the number of matrix satisfying the requirements.
      v1=abs(v(1,1));
      v2=abs(v(2,1));
      v3=abs(v(3,1));
      v4=abs(v(4,1));
```

```
A123=[1,a12,a13;1/a12,1,a23;1/a13,1/a23,1];   %delete alternative 4
A124=[1,a12,a14;1/a12,1,a24;1/a14,1/a24,1];   %delete alternative 3
A134=[1,a13,a14;1/a13,1,a34;1/a14,1/a34,1];   %delete alternative 2
A234=[1,a23,a24;1/a23,1,a34;1/a24,1/a34,1];   %delete alternative 1


[v123,d123]=eig(A123); %check alternative 1,2,3 situation
if (d123(1,1)-3)/1.32<=0.1
   w1=abs(v123(1,1));
   w2=abs(v123(2,1));
   w3=abs(v123(3,1));
   X=(v1/v2-1)*(w1/w2-1);
   Y=(v1/v3-1)*(w1/w3-1);
   Z=(v2/v3-1)*(w2/w3-1);
   if (X<0)||(Y<0)||(Z<0) %check if there is rank reversal, if yes, S123=1,
otherwise, 0
      S123=1;
   end
   w1=0; w2=0; w3=0; X=0; Y=0; Z=0;
end
%v123=zeros(3,3); d123=zeros(3,3);

[v124,d124]=eig(A124); %check alternative 1,2,4 situation
if (d124(1,1)-3)/1.32<=0.1
   w1=abs(v124(1,1));
   w2=abs(v124(2,1));
   w4=abs(v124(3,1));
   X=(v1/v2-1)*(w1/w2-1);
   Y=(v1/v4-1)*(w1/w4-1);
   Z=(v2/v4-1)*(w2/w4-1);
   if (X<0)||(Y<0)||(Z<0) %check if there is rank reversal, if yes, S124=1,
otherwise, 0
      S124=1;
   end
   w1=0; w2=0; w4=0; X=0; Y=0; Z=0;
end
%v124=zeros(3,3); d124=zeros(3,3);

[v134,d134]=eig(A134);        %check alternative 1,3,4 situation
if ((d134(1,1)-3)/1.32<=0.1)
   w1=abs(v134(1,1));
   w3=abs(v134(2,1));
   w4=abs(v134(3,1));
```

```
        X=(v1/v3-1)*(w1/w3-1);
        Y=(v1/v4-1)*(w1/w4-1);
        Z=(v3/v4-1)*(w3/w4-1);
        if (X<0)||(Y<0)||(Z<0) %check if there is rank reversal, if yes, S134=1,
otherwise, 0
            S134=1;
        end
        w1=0; w4=0; w3=0; X=0; Y=0; Z=0;
    end
    %v134=zeros(3,3); d134=zeros(3,3);

    [v234,d234]=eig(A234); %check alternative 2,3,4 situation
    if ((d234(1,1)-3)/1.32<=0.1)
        w2=abs(v234(1,1));
        w3=abs(v234(2,1));
        w4=abs(v234(3,1));
        X=(v2/v3-1)*(w2/w3-1);
        Y=(v2/v4-1)*(w2/w4-1);
        Z=(v3/v4-1)*(w3/w4-1);
        if (X<0)||(Y<0)||(Z<0) %check if there is rank reversal, if yes, S234=1,
otherwise, 0
            S234=1;
        end
        w4=0; w2=0; w3=0; X=0; Y=0; Z=0;
    end
    %v234=zeros(3,3); d234=zeros(3,3);

    if (S123==1)||(S234==1)||(S134==1)||(S124==1) %check if given matrix suffers
from rank reversal.
        V(i+1,1)=a12;
        V(i+1,2)=a13;
        V(i+1,3)=a14;
        V(i+1,4)=a23;
        V(i+1,5)=a24;
        V(i+1,6)=a34;
        V(i+1,7)=CR;
        V(i+1,8)=min([v1,v2,v3,v4])/(v1+v2+v3+v4);
        V(i+1,9)=V(i+1,8)-mweight(CR,max([a12,a13,a14,a23,a24,a34]))*1.1;
        V(i+1,10)=V(i+1,8)-mweight(CR,max([a12,a13,a14,a23,a24,a34]))*1.2;
        V(i+1,11)=V(i+1,8)-mweight(CR,max([a12,a13,a14,a23,a24,a34]))*1.3;
        V(i+1,12)=V(i+1,8)-mweight(CR,max([a12,a13,a14,a23,a24,a34]))*1.4;
        V(i+1,13)=V(i+1,8)-mweight(CR,max([a12,a13,a14,a23,a24,a34]))*1.5;
        V(i+1,14)=V(i+1,8)-mweight(CR,max([a12,a13,a14,a23,a24,a34]))*1.6;
```

```
        V(i+1,15)=V(i+1,8)-mweight(CR,max([a12,a13,a14,a23,a24,a34]))*1.7;
        V(i+1,16)=V(i+1,8)-mweight(CR,max([a12,a13,a14,a23,a24,a34]))*1.8;
        V(i+1,17)=V(i+1,8)-mweight(CR,max([a12,a13,a14,a23,a24,a34]))*1.9;
        V(i+1,18)=V(i+1,8)-mweight(CR,max([a12,a13,a14,a23,a24,a34]))*2;
        i=i+1; %calculate the number of rank reversal
      else
        H(o)=min([v1,v2,v3,v4])/(v1+v2+v3+v4);
        o=o+1;
      end
      S123=0; S234=0;
S134=0;S124=0;A123=zeros(3,3);A134=zeros(3,3);A234=zeros(3,3);A124=zeros(3,3
);
   end

a12=0;a13=0;a14=0;a23=0;a24=0;a34=0;CR=0;A=zeros(4,4);v=zeros(3,3);d=zeros(3,
3);
end
for c=1:n
   if V(c,9)<0
      coun(1)=coun(1)+1;
   end
   if V(c,10)<0
      coun(2)=coun(2)+1;
   end
   if V(c,11)<0
      coun(3)=coun(3)+1;
   end
   if V(c,12)<0
      coun(4)=coun(4)+1;
   end
   if V(c,13)<0
      coun(5)=coun(5)+1;
   end
   if V(c,14)<0
      coun(6)=coun(6)+1;
   end
   if V(c,15)<0
      coun(7)=coun(7)+1;
   end
   if V(c,16)<0
      coun(8)=coun(8)+1;
   end
   if V(c,17)<0
```

```
        coun(9)=coun(9)+1;
    end
    if V(c,18)<0
        coun(10)=coun(10)+1;
    end
end
coun
j
k
time=cputime-t
end
```

APPENDIX C

Matlab Simulation Programs for n=5:

```
function m=mweight(c,a);
m=1/(1+a*(4.752*c+4));
end

function [V, H]=simulate5(n); %V returns to the minimal weights and matrix information, H returns to the smallest priority from matrix without rank reversal. N inputs that total # of rank reversal generated.
o=1;
t=cputime;
i=0; %count the number of rank reversal;
j=0; %count the total number of simulation;
k=0; %count the total number of example s.t. CR<=0.1 and max element>=7;
S1=0; S2=0; S3=0; S4=0; S5=0;
coun=zeros(1,10);
while i<n
   j=j+1;
   u=rand();
   a12=rand()*8+1;  %generate random numbers for aij between [1/9,9]
   if u<=0.5
      a12=a12;
   else
      a12=1/a12;
   end
   %a12=9;
   u=rand();
   a13=rand()*8+1;
   if u<=0.5
      a13=a13;
   else
      a13=1/a13;
   end
   u=rand();
   a14=rand()*8+1;
   if u<=0.5
```

```
   a14=a14;
else
   a14=1/a14;
end
u=rand();
a15=rand()*8+1;  %generate random numbers for aij between [1/9,9]
if u<=0.5
   a15=a15;
else
   a15=1/a15;
end
u=rand();
a23=rand()*8+1;  %generate random numbers for aij between [1/9,9]
if u<=0.5
   a23=a23;
else
   a23=1/a23;
end
u=rand();
a24=rand()*8+1;  %generate random numbers for aij between [1/9,9]
if u<=0.5
   a24=a24;
else
   a24=1/a24;
end
u=rand();
a25=rand()*8+1;  %generate random numbers for aij between [1/9,9]
if u<=0.5
   a25=a25;
else
   a25=1/a25;
end
u=rand();
a34=rand()*8+1;  %generate random numbers for aij between [1/9,9]
if u<=0.5
   a34=a34;
else
   a34=1/a34;
end
u=rand();
a35=rand()*8+1;  %generate random numbers for aij between [1/9,9]
if u<=0.5
   a35=a35;
```

```
    else
        a35=1/a35;
    end
    u=rand();
    a45=rand()*8+1;  %generate random numbers for aij between [1/9,9]
    if u<=0.5
        a45=a45;
    else
        a45=1/a45;
    end
    a45=9;

A=[1,a12,a13,a14,a15;1/a12,1,a23,a24,a25;1/a13,1/a23,1,a34,a35;1/a14,1/a24,1/a34,1,
a45;1/a15,1/a25,1/a35,1/a45,1];
    [v,d]=eig(A);
    CR=(d(1,1)-5)/4.752;
    if (CR<=0.1)&&(max([a12,a13,a14,a15,a23,a24,a25,a34,a35,a45])>=8)
        k=k+1;
        v1=abs(v(1,1));
        v2=abs(v(2,1));
        v3=abs(v(3,1));
        v4=abs(v(4,1));
        v5=abs(v(5,1));


A5=[1,a12,a13,a14;1/a12,1,a23,a24;1/a13,1/a23,1,a34;1/a14,1/a24,1/a34,1]; %deletin
g alternative 5;
        [V5,D5]=eig(A5);

        if (D5(1,1)-4)/2.97<=0.1
            X1=(v1/v2-1)*(V5(1,1)/V5(2,1)-1);
            X2=(v1/v3-1)*(V5(1,1)/V5(3,1)-1);
            X3=(v1/v4-1)*(V5(1,1)/V5(4,1)-1);
            X4=(v2/v3-1)*(V5(2,1)/V5(3,1)-1);
            X5=(v2/v4-1)*(V5(2,1)/V5(4,1)-1);
            X6=(v3/v4-1)*(V5(3,1)/V5(4,1)-1);
            if (X1<0)||(X2<0)||(X3<0)||(X4<0)||(X5<0)||(X6<0) %Check if rank reversal
occurs, when deleting 5.
                S5=1;
            end
        X1=0; X2=0; X3=0; X4=0; X5=0; X6=0;
        end
        A5=zeros(4,4);
```

```
A4=[1,a12,a13,a15;1/a12,1,a23,a25;1/a13,1/a23,1,a35;1/a15,1/a25,1/a35,1]; %deletin
g alternative 4;
    [V4,D4]=eig(A4);
    if (D4(1,1)-4)/2.97<=0.1
      X1=(v1/v2-1)*(V4(1,1)/V4(2,1)-1);
      X2=(v1/v3-1)*(V4(1,1)/V4(3,1)-1);
      X3=(v1/v5-1)*(V4(1,1)/V4(4,1)-1);
      X4=(v2/v3-1)*(V4(2,1)/V4(3,1)-1);
      X5=(v2/v5-1)*(V4(2,1)/V4(4,1)-1);
      X6=(v3/v5-1)*(V4(3,1)/V4(4,1)-1);
      if (X1<0)||(X2<0)||(X3<0)||(X4<0)||(X5<0)||(X6<0) %Check if rank reversal
occurs, when deleting 4.
        S4=1;
      end
    X1=0; X2=0; X3=0; X4=0; X5=0; X6=0;
    end
    A4=zeros(4,4);


A3=[1,a12,a14,a15;1/a12,1,a24,a25;1/a14,1/a24,1,a45;1/a15,1/a25,1/a45,1]; %deletin
g alternative 3;
    [V3,D3]=eig(A3);
    if (D3(1,1)-4)/2.97<=0.1
      X1=(v1/v2-1)*(V3(1,1)/V3(2,1)-1);
      X2=(v1/v4-1)*(V3(1,1)/V3(3,1)-1);
      X3=(v1/v5-1)*(V3(1,1)/V3(4,1)-1);
      X4=(v2/v4-1)*(V3(2,1)/V3(3,1)-1);
      X5=(v2/v5-1)*(V3(2,1)/V3(4,1)-1);
      X6=(v4/v5-1)*(V3(3,1)/V3(4,1)-1);
      if (X1<0)||(X2<0)||(X3<0)||(X4<0)||(X5<0)||(X6<0) %Check if rank reversal
occurs, when deleting 3.
        S3=1;
      end
    X1=0; X2=0; X3=0; X4=0; X5=0; X6=0;
    end
    A3=zeros(4,4);


A2=[1,a13,a14,a15;1/a13,1,a34,a35;1/a14,1/a34,1,a45;1/a15,1/a35,1/a45,1]; %deletin
g alternative 2;
    [V2,D2]=eig(A2);
```

```
        if (D2(1,1)-4)/2.97<=0.1
            X1=(v1/v3-1)*(V2(1,1)/V2(2,1)-1);
            X2=(v1/v4-1)*(V2(1,1)/V2(3,1)-1);
            X3=(v1/v5-1)*(V2(1,1)/V2(4,1)-1);
            X4=(v3/v4-1)*(V2(2,1)/V2(3,1)-1);
            X5=(v3/v5-1)*(V2(2,1)/V2(4,1)-1);
            X6=(v4/v5-1)*(V2(3,1)/V2(4,1)-1);
           if (X1<0)||(X2<0)||(X3<0)||(X4<0)||(X5<0)||(X6<0) %Check if rank reversal
occurs, when deleting 2.
                S2=1;
            end
        X1=0; X2=0; X3=0; X4=0; X5=0; X6=0;
        end
        A2=zeros(4,4);


A1=[1,a23,a24,a25;1/a23,1,a34,a35;1/a24,1/a34,1,a45;1/a25,1/a35,1/a45,1]; %deletin
g alternative 1;
        [V1,D1]=eig(A1);
        if (D1(1,1)-4)/2.97<=0.1
            X1=(v2/v3-1)*(V1(1,1)/V1(2,1)-1);
            X2=(v2/v4-1)*(V1(1,1)/V1(3,1)-1);
            X3=(v2/v5-1)*(V1(1,1)/V1(4,1)-1);
            X4=(v3/v4-1)*(V1(2,1)/V1(3,1)-1);
            X5=(v3/v5-1)*(V1(2,1)/V1(4,1)-1);
            X6=(v4/v5-1)*(V1(3,1)/V1(4,1)-1);
            if (X1<0)||(X2<0)||(X3<0)||(X4<0)||(X5<0)||(X6<0) %Check if rank reversal
occurs, when deleting 1.
                S1=1;
            end
        X1=0; X2=0; X3=0; X4=0; X5=0; X6=0;
        end
        A1=zeros(4,4);

        if (S1==1)||(S2==1)||(S3==1)||(S4==1)||(S5==1)
            i=i+1;
            V(i,1)=a12;
            V(i,2)=a13;
            V(i,3)=a14;
            V(i,4)=a15;
            V(i,5)=a23;
            V(i,6)=a24;
            V(i,7)=a25;
```

```
        V(i,8)=a34;
        V(i,9)=a35;
        V(i,10)=a45;
        V(i,11)=CR;
        V(i,12)=min([v1,v2,v3,v4,v5])/(v1+v2+v3+v4+v5);
        V(i,13)=V(i,12)-
mweight(V(i,11),max([a12,a13,a14,a15,a23,a24,a25,a34,a35,a45]))*1.1;
        V(i,14)=V(i,12)-
mweight(V(i,11),max([a12,a13,a14,a15,a23,a24,a25,a34,a35,a45]))*1.2;
        V(i,15)=V(i,12)-
mweight(V(i,11),max([a12,a13,a14,a15,a23,a24,a25,a34,a35,a45]))*1.3;
        V(i,16)=V(i,12)-
mweight(V(i,11),max([a12,a13,a14,a15,a23,a24,a25,a34,a35,a45]))*1.4;
        V(i,17)=V(i,12)-
mweight(V(i,11),max([a12,a13,a14,a15,a23,a24,a25,a34,a35,a45]))*1.5;
        V(i,18)=V(i,12)-
mweight(V(i,11),max([a12,a13,a14,a15,a23,a24,a25,a34,a35,a45]))*1.6;
        V(i,19)=V(i,12)-
mweight(V(i,11),max([a12,a13,a14,a15,a23,a24,a25,a34,a35,a45]))*1.7;
        V(i,20)=V(i,12)-
mweight(V(i,11),max([a12,a13,a14,a15,a23,a24,a25,a34,a35,a45]))*1.8;
        V(i,21)=V(i,12)-
mweight(V(i,11),max([a12,a13,a14,a15,a23,a24,a25,a34,a35,a45]))*1.9;
        V(i,22)=V(i,12)-
mweight(V(i,11),max([a12,a13,a14,a15,a23,a24,a25,a34,a35,a45]))*2;
      else
        H(o)=min([v1,v2,v3,v4,v5])/(v1+v2+v3+v4+v5);
        o=o+1;
      end
      S1=0;S2=0;S3=0;S4=0;S5=0;v1=0;v2=0;v3=0;v4=0;v5=0;
  end

a12=0;a13=0;a14=0;a15=0;a23=0;a24=0;a25=0;a34=0;a35=0;a45=0;A=zeros(5,5);CR
=0;
end
for m=1:n
  if V(m,13)<0;
      coun(1)=coun(1)+1;
  end
   if V(m,14)<0;
      coun(2)=coun(2)+1;
   end
   if V(m,15)<0;
```

```
      coun(3)=coun(3)+1;
    end
    if V(m,16)<0;
      coun(4)=coun(4)+1;
    end
    if V(m,17)<0;
      coun(5)=coun(5)+1;
    end
    if V(m,18)<0;
      coun(6)=coun(6)+1;
    end
    if V(m,19)<0;
      coun(7)=coun(7)+1;
    end
    if V(m,20)<0;
      coun(8)=coun(8)+1;
    end
    if V(m,21)<0;
      coun(9)=coun(9)+1;
    end
    if V(m,22)<0;
      coun(10)=coun(10)+1;
   end
end
coun
j
k
time=cputime-t
end
```