

MEMORANDUM

RM-4464-PR

JULY 1966

ON MEMBRANE EQUILIBRIA

N. Z. Shapiro and L. S. Shapley

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PREFACE

This Memorandum reports recent results of continuing RAND research on the theoretical and computational aspects of ideal chemical systems [1-13]. This research is companion to studies on the use of mathematical models to investigate the chemistry of physiological subsystems [14-21]. Specifically, the Memorandum deals with theoretical properties of the type of ideal chemical system suggested by problems involving semipermeable membranes in biological systems.

The discussions, results, and informal proofs should be intelligible to mathematically literate biochemists, physiologists, research physicians, etc. Some of the formal proofs, however, require more mathematical background.

SUMMARY

This study mathematically investigates equilibrium properties of certain two-phase chemical systems. The different types of equilibrium compositions and the conditions that govern them are determined. The effects of linear variation in the chemical inputs are studied; e.g., the effects on a multisubstance fluid, separated by a semipermeable membrane, to which another fluid is slowly added. Among other results, it is shown that the relative sizes of the two phases will always change monotonically, but that the absolute sizes may fluctuate.

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* JOSS is the trademark and service mark of The RAND Corporation for its computer program and services using that program.

CONTENTS

PREFACE	iii
SUMMARY	v
ACKNOWLEDGMENTS	vii
SYMBOL GLOSSARY	xi
Section	
I. INTRODUCTION	1
II. THE CHARACTERISTIC FUNCTION	9
III. TYPES OF EQUILIBRIUM COMPOSITION	11
IV. NUMERICAL COMPUTATION	21
V. GAS-LIQUID SYSTEMS	23
VI. INTRODUCTION TO VARIABLE SYSTEMS	27
VII. CLASSIFICATION OF VARIABLE SYSTEMS	36
VIII. MONOTONICITY THEOREM	40
IX. LINEAR RESPONSES	46
REFERENCES	55

SYMBOL GLOSSARY

b_i	Input for X_i , number of moles of X_i in system.
b_i^0	Total number of moles of X_i in $S(0)$.
$b_i(t) = b_i^0 + t\Delta b_i$	Total number of moles of X_i in $S(t)$.
$\bar{b} = \sum_{i=1}^n b_i + q + q'$	Total number of moles in system.
$\bar{b}(t) = q(t) + q'(t) + \sum_{i=1}^n b_i(t)$	Total number of moles in $S(t)$.
c_i	Gibbs Parameter for X_i^I .
c'_i	Gibbs Parameter for X_i^{II} .
Δb_i	Rate at which number of moles of X_i in $S(t)$ is changed per unit change in t .
Δq	Rate at which number of moles of impermeable material in Phase I of $S(t)$ is changed per unit change in t .
$\Delta q'$	Rate at which number of moles of impermeable material in Phase II of $S(t)$ is changed per unit change in t .
F	Free energy.
g	Characteristic function.
i	Index of permeable species.

k_i	Equilibrium constant for the reaction $X_i^I \rightleftharpoons X_i^{II}$.
n	Number of permeable species.
q	Number of moles of impermeable material in Phase I.
q'	Number of moles of impermeable material in Phase II.
q^0	Number of moles of impermeable material in Phase I of $S(0)$.
$q^{0'}$	Number of moles of impermeable material in Phase II of $S(0)$.
$q(t) = q^0 + t\Delta q$	Number of moles of impermeable material in Phase I of $S(t)$.
$q'(t) = q^{0'} + t\Delta q'$	Number of moles of impermeable material in Phase II of $S(t)$.
p_i	Partial pressure of X_i .
p_i^0	Vapor pressure of X_i .
R	Gas constant.
$S(t)$	Variable system at t .
s	Variable ranging from 0 to \bar{b} .
s_0	Root of the characteristic function.
T	Temperature.
t	Used in connection with variable systems.
X_i	Name of the i^{th} permeable species.

X_i^I

X_i in Phase I.

X_i^{II}

X_i in Phase II.

x_i

Number of moles of X_i^I at equilibrium.

x_i'

Number of moles of X_i^{II} at equilibrium.

$x_i(t)$

Number of moles of X_i^I in equilibrium composition of $S(t)$.

$x_i'(t)$

Number of moles of X_i^{II} in equilibrium composition of $S(t)$.

$$\hat{x}_i = \frac{x_i}{\bar{x}}$$

Mole fraction of X_i^I at equilibrium.

$$\hat{x}_i' = \frac{x_i'}{\bar{x}'}$$

Mole fraction of X_i^{II} at equilibrium.

$$\hat{x}_i(t) = \frac{x_i(t)}{\bar{x}(t)}$$

Mole fraction of X_i^I in equilibrium composition of $S(t)$.

$$\hat{x}_i'(t) = \frac{x_i'(t)}{\bar{x}'(t)}$$

Mole fraction of X_i^{II} in equilibrium composition of $S(t)$.

$$\bar{x} = \sum_{i=1}^n x_i + q$$

Total number of moles in Phase I at equilibrium.

$$\bar{x}' = \sum_{i=1}^n x_i' + q'$$

Total number of moles in Phase II at equilibrium.

$$\bar{x}(t) = q(t) + \sum_{i=1}^n x_i(t)$$

Total number of moles in
Phase I in equilibrium
composition of S(t).

$$\bar{x}'(t) = q'(t) + \sum_{i=1}^n x_i'(t)$$

Total number of moles in
Phase II in equilibrium
composition of S(t).

I. INTRODUCTION

The archetype of the mathematical system explored in this Memorandum is a multisubstance fluid partitioned by a single semipermeable membrane, although it is exemplified by a variety of other chemical equilibrium problems (e.g., a gas over a liquid). More explicitly, the Memorandum studies two-phase, constant-pressure-temperature, chemical-equilibrium systems that satisfy the following assumptions:*

- 1) There are no charged species;
- 2) There are no intra-phase reactions--i.e., the only possible "reactions" are the migrations of molecules from one phase to another phase;
- 3) The substances are miscible in all proportions.

Although such simple systems rarely occur in physiological applications, their study is important for several reasons. First, they permit a more thorough mathematical analysis than more complex systems. Second, much of the behavior of the latter can, at least qualitatively, be explained in terms of the behavior of the simpler systems--which, from one point of view, are zeroth-order approximations to many physiological systems. Finally, despite their apparent

* See below for an exact definition of these systems.

simplicity, the systems we study display a wealth of complex and interesting behavior.

We call the type of chemical equilibrium systems studied in this Memorandum primitive chemical equilibrium systems, or more briefly, primitive systems. A primitive system will involve two types of chemical substances: impermeable substances, whose molecules cannot pass through the membrane, and permeable substances, whose molecules can pass through the membrane.

Note that for primitive systems the only practical effect of the presence of impermeable substances is on the mole fractions of the permeable substances. For this reason, there is no loss of generality in aggregating all the impermeable substances to a single impermeable substance. (This would not necessarily be the case if assumptions 1 and 2 above did not hold.) We shall deal therefore with a system containing n permeable substances, named $X_1, \dots, X_i, \dots, X_n$, and one unnamed impermeable substance. And since we can always say that the amount of impermeable material present happens to be zero, the fact that we have specified an impermeable substance does not preclude us from studying problems in which there are no impermeable substances.

We designate the two fluids separated by the membrane as Phase I and Phase II. Note that one phase might lie above or to the side of the other phase; or the membrane might be a surface enclosing a volume, the two phases being the inside and outside of the membrane; or the membrane might be several such closed surfaces and the two phases might be the collective interiors and the exterior. An example is the blood plasma and the interiors of the red blood cells.

When the substance X_i occurs in Phase I, we shall call it X_i^I ; when it occurs in Phase II we shall call it X_i^{II} . Thus, the migration of X_i across the membrane is represented by the "chemical reaction"



Let x_i and x'_i be the number of moles of X_i^I and X_i^{II} , respectively; let q and q' be the number of moles of impermeable material in Phase I and Phase II, respectively; and let \bar{x} and \bar{x}' be the total number of moles in Phase I and Phase II, respectively. That is,

$$\bar{x} = q + \sum_{i=1}^n x_i ; \quad (2)$$

and

$$\bar{x}' = q' + \sum_{i=1}^n x'_i . \quad (2')$$

Let \hat{x}_i and \hat{x}'_i be the mole fractions of X_i^I and X_i^{II} , respectively.* That is,

$$\hat{x}_i = \frac{x_i}{\bar{x}} \quad \text{for } i=1, \dots, n ; \quad (3)$$

and

$$\hat{x}'_i = \frac{x'_i}{\bar{x}'} \quad \text{for } i=1, \dots, n . \quad (3')$$

Let b_i be the total number of moles of X_i in the system. That is,

$$x_i + x'_i = b_i \quad \text{for } i=1, \dots, n . \quad (4)$$

We call b_i the input for X_i . Let \bar{b} be the total number of input moles. That is,

* If $\bar{x} = 0$ then \hat{x}_i is undefined, and if $\bar{x}' = 0$ then \hat{x}'_i is undefined.

$$\bar{b} = q + q' + \sum_{i=1}^n b_i . \quad (5)$$

Equations (2), (2'), (4), and (5) show that

$$\bar{x} + \bar{x}' = \bar{b} . \quad (6)$$

We assume that the system is in equilibrium when its free energy, as defined below, is minimized with respect to the variables $x_1, \dots, x_n, x'_1, \dots, x'_n$, constrained by Eqs. (4) above and the following:

$$x_i \geq 0 \quad \text{for} \quad i=1, \dots, n ; \quad (7)$$

and

$$x'_i \geq 0 \quad \text{for} \quad i=1, \dots, n . \quad (7')$$

Finally, we assume that the free energy is given by an equation of the form:

$$\begin{aligned} F = & \sum_{i=1}^n x_i \left(c_i + \log \hat{x}_i \right) + q \log \frac{q}{\bar{x}} \\ & + \sum_{i=1}^n x'_i \left(c'_i + \log \hat{x}'_i \right) + q' \log \frac{q'}{\bar{x}'} . \end{aligned} \quad (8)$$

Here c_i is the Gibbs parameter for X_i^I , and c_i' is the Gibbs parameter for X_i^{II} , and \hat{x}_i , \hat{x}_i' , \bar{x} , and \bar{x}' are defined in terms of the x_i and x_i' by means of Eqs. (2), (2'), (3), and (3').

Several comments concerning the equation for the free energy function are in order:

- 1) The right-hand side is equal to what is classically called the Gibbs Free Energy divided by the constant factor RT , where R is the gas constant and T is the absolute temperature.
- 2) Its form assumes that the system is "ideal"--to the extent that the so-called mole fraction activity coefficients are independent of composition. While not true in practice, this assumption is often sufficiently accurate over a wide range of compositions.
- 3) At first glance, the parameters c_i and c_i' would seem to be necessarily the same. However, on further reflection, this is seen to be true only when the membrane is chemically symmetric and inert, when the pressures and temperatures on both sides are equal, and when the compositions of the two phases are sufficiently similar that the mole fraction activity coefficients are equal. None of these assumptions need hold. Furthermore, if we take as our model of a primitive system a liquid-gas system (instead of two liquids separated by a semipermeable

membrane), then there is absolutely no reason to suppose that $c_i = c'_i$.

- 4) Equation (8) as written defines the free energy only if each $x_i > 0$, each $x'_i > 0$, and $q > 0$ and $q' > 0$. The free energy must also be defined when one or more of these quantities are zero. This can be done; for a discussion of this and related matters, see Ref. 8, Theorem 8.3. \

It turns out [11] that the behavior of a primitive system depends on the Gibbs parameters only to the extent that it depends on the differences, $c_i - c'_i$. Thus, if we define $k_i = e^{c_i - c'_i}$, then a primitive system can be specified by the number of permeable substances, n ; the inputs, b_1, \dots, b_n, q, q' ; and the constants, k_1, \dots, k_n .

For equilibrium compositions in which $\bar{x} > 0$ and $\bar{x}' > 0$, we must have:*

$$\frac{\hat{x}'_i}{\hat{x}_i} = k_i \quad \text{for } i=1, \dots, n. \quad (9)$$

Equation (9) expresses the law of mass action for the "reaction" (1). This motivates us to refer to the k_i as

*See Ref. 8, p. 44, Theorem 10.3.

equilibrium constants. (The k_i are sometimes called "partition coefficients".) The equilibrium constants are more convenient parameters for discussing the behavior of chemical equilibrium systems than are the Gibbs parameters.

Conversely, any composition $x_1, \dots, x_n, q, x'_1, \dots, x'_n, q'$ which satisfies $\bar{x} > 0, \bar{x}' > 0$, the law of mass balance Eq. (4), and the law of mass action Eq. (9) is an equilibrium composition of the system in question.*

Note that $0 < k_i < \infty$. From Eq. (9), it would appear that if we allowed $k_i = 0$, then we might regard X_i as being impermeable material in Phase I; and that if we allowed $k_i = \infty$, then we might regard X_i as being impermeable material in Phase II. The precise sense in which this is true is treated for general chemical equilibrium systems in Ref. 11. In this Memorandum, we continue to assume $0 < k_i < \infty$, and to treat impermeable species in a special way.

*Note that since we have defined equilibrium in terms of the free energy function, this statement requires proof. Also, note that it says nothing about equilibrium compositions for which $\bar{x} = 0$ or $\bar{x}' = 0$. See Ref. 8, p. 44, Theorem 10.3, for a proof.

II. THE CHARACTERISTIC FUNCTION

Assume that $x_1, \dots, x_n, q, x'_1, \dots, x'_n, q'$ form an equilibrium composition and that $\bar{x} > 0$ and $\bar{x}' > 0$. If in Eq. (9) we replace \hat{x}_i and \hat{x}'_i by their values from Eqs. (3) and (3') and apply Eq. (4), then

$$x_i = \frac{b_i \bar{x}}{\bar{x} + k_i \bar{x}'} \quad (10)$$

and

$$x'_i = \frac{k_i b_i \bar{x}'}{\bar{x} + k_i \bar{x}'} \quad (10')$$

If we replace x_i in Eq. (2) by its value from Eq. (10), then

$$\frac{q}{\bar{x}} + \sum_{i=1}^n \frac{b_i}{\bar{x} + k_i \bar{x}'} = 1 \quad (11)$$

If we use Eq. (6), then Eq. (11) can be rewritten as $g(\bar{x}) = 0$ where

$$g(s) = \frac{q}{s} + \sum_{i=1}^n \frac{b_i}{s + k_i (\bar{b} - s)} - 1 \quad (12)$$

We call g the characteristic function of the system. We have shown that, if $\bar{x} > 0$ and $\bar{x}' > 0$, and if the x_i and x_i' define an equilibrium composition, then \bar{x} is a root of the characteristic function. The characteristic function plays an important role in the theory of primitive systems.

Now drop the assumption that $\bar{x} > 0$ and $\bar{x}' > 0$. Assume first that $\bar{x} = 0$. Then each $x_i = 0$, and hence, by Eq. (4), each $x_i' = b_i$. Inspection of Eqs. (10) and (10') shows that they are satisfied; similarly, they are satisfied if $\bar{x}' = 0$. It follows as above from Eqs. (10) and (10') that $g(\bar{x}) = 0$ for these cases. Thus, we have shown that for any primitive chemical equilibrium system Eqs. (10) and (10') are satisfied and \bar{x} is a root of the characteristic function.

III. TYPES OF EQUILIBRIUM COMPOSITION

This section states and proves our principal results concerning the types of equilibrium compositions possible in a primitive chemical equilibrium system, the conditions for each type to occur, the properties of the characteristic function of a primitive system, and the relationships between the characteristic function and the equilibrium compositions. This requires the definition of two conditions.

$$\text{Condition I: Either } q > 0 \quad \text{or} \quad \sum_{i=1}^n \frac{b_i}{k_i} > \bar{b} .$$

$$\text{Condition II: Either } q' > 0 \quad \text{or} \quad \sum_{i=1}^n k_i b_i > \bar{b} .$$

Observe that if Condition I is false, then we must have $q = 0$ and $\sum_{i=1}^n \frac{b_i}{k_i} \leq \bar{b}$. Similarly, if Condition II is not satisfied, then $q' = 0$ and $\sum_{i=1}^n k_i b_i \leq \bar{b}$.

THEOREM 1

Consider a primitive chemical equilibrium system with n permeable species having inputs b_1, \dots, b_n and equilibrium constants k_1, \dots, k_n , and with an impermeable

species having input q in Phase I and input q' in Phase II. Let $\bar{b} = \sum_{i=1}^n b_i + q + q'$.

- (i) The characteristic function $g(s)$ of the system is defined and convex* on the interval $0 < s < \bar{b}$.
- (ii) If both Condition I and Condition II are satisfied, then g has exactly one root, s_0 , in that interval, and the system has exactly one equilibrium composition, given by

$$x_i = \frac{b_i s_0}{s_0 + k_i (\bar{b} - s_0)} \quad \text{for } i=1, \dots, n; \quad (14)$$

and

$$x'_i = \frac{k_i b_i (\bar{b} - s_0)}{s_0 + k_i (\bar{b} - s_0)} \quad \text{for } i=1, \dots, n. \quad (15)$$

Furthermore,

$$\bar{x} = s_0, \quad (16)$$

* A function is convex if its graph is such that it lies below all its chords. For example, a function having an everywhere-positive second derivative is convex. More detailed discussions of convexity appear in Ref. 23, pp. 149-156.

$$\bar{x}' = \bar{b} - s_0, \quad (17)$$

$$\hat{x}_i = \frac{b_i}{s_0 + k_i(\bar{b} - s_0)} \quad \text{for } i=1, \dots, n, \quad (18)$$

and

$$\hat{x}'_i = \frac{k_i b_i}{s_0 + k_i(\bar{b} - s_0)} \quad \text{for } i=1, \dots, n. \quad (19)$$

(iii) If either Condition I or Condition II is satisfied, but not both, then g has no roots in the interval $0 < s < \bar{b}$.

(iv) If Condition I is satisfied but not Condition II, then the system has exactly one equilibrium composition, given by $x_i = b_i$ and $x'_i = 0$, for $i=1, \dots, n$.

(v) If Condition II is satisfied but not Condition I, then the system has exactly one equilibrium composition, given by $x_i = 0$ and $x'_i = b_i$, for $i=1, \dots, n$.

(vi) If neither Condition I nor Condition II is satisfied, then $q = q' = 0$, and $b_i = 0$ for all i , $i=1, \dots, n$, for which $k_i \neq 1$. Conversely, if $q = q' = 0$ and $b_i = 0$ for all i , $i=1, \dots, n$, for which $k_i \neq 1$, then neither Condition I nor Condition II is satisfied.

In this case, the system, with trivial exceptions,^{*} has an infinite number of equilibrium compositions, given by $x_i = tb_i$ and $x'_i = (1 - t)b_i$, for $i=1, \dots, n$, where each t , $0 \leq t \leq 1$, gives a different equilibrium composition.

Proof of Theorem 1:

If any $b_i = 0$, then $x_i = x'_i = 0$ for every equilibrium composition. Thus, if we ignore every species for which $b_i = 0$ and appropriately decrease n (the number of permeable species), then the characteristic function is unchanged, the satisfaction of Conditions I and II is unchanged, and none of the statements of the theorem are affected. We may accordingly assume that each $b_i > 0$. Also, the theorem is trivially true if $n = 0$, so we will assume that $n > 0$.

The characteristic function as defined by Eq. (12) is obviously defined on $0 < s \leq \bar{b}$ if $q > 0$, and on $0 \leq s \leq \bar{b}$ if $q = 0$. Its first two derivatives are easily calculated:

^{*} If every $b_i = 0$, or if $n = 0$, then there is a unique equilibrium composition.

$$\frac{dg}{ds} = -\frac{q}{s^2} - \sum_{i=1}^n \frac{b_i(1 - k_i)}{[s(1 - k_i) + k_i\bar{b}]^2} ; \quad (20)$$

$$\frac{d^2g}{ds^2} = \frac{2q}{s^3} + 2 \sum_{i=1}^n \frac{b_i(1 - k_i)^2}{[s(1 - k_i) + k_i\bar{b}]^3} . \quad (21)$$

The consequences will be detailed in several Lemmas. First, it is apparent from Eq. (21) and from the definition of g , Eq. (12), that

Lemma 1

The characteristic function, g , is continuous and convex on $0 < s < \bar{b}$. It is strictly convex everywhere on this interval unless each $k_i = 1$ and $q = 0$, in which case it is constant and equal to $-\frac{q'}{\bar{b}}$.

By examining the behavior of g near 0, it follows that

Lemma 2

If Condition I is satisfied, then $g(\epsilon) > 0$ for all sufficiently small positive ϵ . If Condition I is not satisfied, then g is defined at 0, and $g(0) \leq 0$.

By examining the behavior of g and its derivative at \bar{b} it follows that

Lemma 3

If Condition II is satisfied, then $g(\bar{b} - \epsilon) < 0$ for all sufficiently small positive ϵ . If Condition II is not satisfied, then $g(\bar{b}) = 0$ and $\frac{dg}{ds}_{s=\bar{b}} \leq 0$.

It then follows from Lemmas 1, 2, and 3, that

Lemma 4

If Conditions I and II both hold, then g has exactly one root in the interval $0 < s < \bar{b}$; if exactly one of these two conditions hold, then g has no roots in that interval; if neither condition holds, then g vanishes identically and, hence, every $k_i = 1$.

Lemma 5

The equilibrium solutions of the system in which $\bar{x} > 0$ and $\bar{x}' > 0$ are in one-to-one correspondence with the roots of g , in $0 < s < \bar{b}$. This correspondence is defined by Eqs. (14) and (15) and it satisfies Eqs. (16), (17), (18), and (19).

To prove Lemma 5, the discussion of Sec. II shows that every equilibrium composition with $\bar{x}, \bar{x}' > 0$ satisfies Eqs. (14) through (19) for some root, s_0 . Conversely, given any root, s_0 , we may define x_i and x_i' by means of Eqs. (14) and (15). Equations (16), (17), (18), and (19)

will then be satisfied, and also $x_i > 0$, $x'_i > 0$, $x_i + x'_i = b_i$, and $\hat{x}'_i = k_i \hat{x}_i$. Thus, the x_i and x'_i are positive and satisfy the laws of mass balance and of mass action for the system. Hence, the x_i and x'_i as defined from s_0 constitute an equilibrium composition. Since $s_0 = \bar{x}$ (Eq. (16)), the correspondence is one-to-one.

Lemma 6

If Conditions I and II are both satisfied, then the only equilibrium composition is the one defined by Lemma 5, associated with the unique (Lemma 4) root of g in $0 < s < \bar{b}$.

To prove Lemma 6, we need only show that the system can have no equilibrium composition with $\bar{x} = 0$ or $\bar{x}' = 0$. But the only equilibrium composition with $\bar{x} = 0$ is given by $x_i = 0$ and $x'_i = b_i$, and the only equilibrium composition in which $\bar{x}' = 0$ is that for which $x'_i = 0$ and $x_i = b_i$. But if the system had more than one equilibrium composition, it would have infinitely many equilibrium compositions* and, hence, infinitely many equilibrium compositions with $\bar{x} > 0$ and $\bar{x}' > 0$, contradicting Lemmas 4 and 5.

*See Ref. 8, p. 38, Lemma 9.3.

Lemma 7

If Condition I is satisfied, but not Condition II, then there is exactly one equilibrium composition. It is given by $x_i = b_i$ and $x'_i = 0$.

To prove this, observe that by Lemmas 4 and 5 every equilibrium composition must have $\bar{x} = 0$ or $\bar{x}' = 0$. That is, the only possible equilibrium compositions are those given by $x_i = 0$ and $x'_i = b_i$, or $x_i = b_i$ and $x'_i = 0$. Exactly one of these two compositions must be an equilibrium composition, because the system possesses at least one equilibrium composition,* and if there were two there would have to be infinitely many, as in the proof of Lemma 6. If $q > 0$ then, since $\bar{x} = q + \sum_{i=1}^n b_i > 0$, the equilibrium composition must be given by $x_i = b_i$ and $x'_i = 0$, as asserted in the lemma. To show that this is also the case when $q = 0$, let us modify the system, producing a new system $\mathcal{J}(q)$ by changing q . Thus the original system is $\mathcal{J}(0)$. Since $\mathcal{J}(0)$ satisfies Condition I but not II, so must $\mathcal{J}(q)$ satisfy Condition I but not II for $q > 0$. But, for $q > 0$, we have already shown that $\mathcal{J}(q)$ has the unique equilibrium composition $x_i(q) = b_i$, $x'_i(q) = 0$. Hence

*See Ref. 8, p. 42, Theorem 9.9.

$$\lim_{q \rightarrow 0^+} x_i(q) = b_i \quad i=1, \dots, n$$

and

$$\lim_{q \rightarrow 0^+} x'_i(q) = 0 \quad i=1, \dots, n.$$

By the continuity results of Ref. 11, it follows that $x_i = b_i$ and $x'_i = 0$ define the equilibrium composition for $f(0)$, as required.

Lemma 8

If Condition II but not Condition I is satisfied, then the system has exactly one equilibrium composition, given by $x'_i = 0$ and $x_i = b_i$.

The proof of Lemma 8 is similar to the proof of Lemma 7, or follows from Lemma 7 by reversing the roles of Phases I and II and replacing the k_i by their reciprocals.

All that remains to complete the proof of Theorem 1 is to observe that if $q = q' = 0$ and every $k_i = 1$ then, clearly neither Condition I nor Condition II can hold, and to discuss the nature of the solutions in this case. By Lemmas 4 and 5, the compositions described in the theorem for this case are equilibrium compositions for $0 < t < 1$.

But the equilibrium compositions are known to form a closed set.* Thus, the compositions described are equilibrium compositions for $0 \leq t \leq 1$. That there are no other equilibrium compositions also follows from Theorem 9.2 in Ref. 8.

* See Ref. 8, p. 38, Theorem 9.2.

IV. NUMERICAL COMPUTATION

The practical, numerical computation of the equilibrium compositions of a primitive system, given its equilibrium constants and inputs, can be accomplished either by means of digital computer programs designed for the solution of general chemical equilibrium programs (e.g., Refs. 1, 5, 10), or with the special techniques described in this section.

A procedure for the determination of the equilibrium composition of a primitive system would first determine the values of the x_i and the x'_i . From these, other quantities such as the mole fractions, \hat{x}_i , \hat{x}'_i ; volumes; concentrations in other scales, etc., could be calculated. The x_i and the x'_i may be calculated from \bar{x} and \bar{x}' through Eqs. (10) and (10') of Sec. II. Since $\bar{x}' = \bar{b} - \bar{x}$, the calculation reduces to a technique for calculating \bar{x} . To do this, one can evaluate the validity of Condition I and the validity of Condition II of Sec. III, and then use Theorem 1 to determine that $\bar{x} = 0$, or $\bar{x} = \bar{b}$, or \bar{x} is an indeterminate between 0 and \bar{b} , or that \bar{x} is the unique root of the characteristic function in the interval $0 < s < \bar{b}$. The characteristic function being convex on $0 < s < \bar{b}$ guarantees that

Newton's method^{*} will converge to the root as long as we begin with an initial value in the interval $0 < s < \bar{b}$.

Although $g(s) = 0$ can be rewritten as a polynomial equation, this is of little practical use except for very small systems. For example, the primitive system treated in Ref. 21 has a characteristic function whose root can be determined by solving a quadratic equation, and thus can be solved analytically.

^{*} See, for example, Ref. 22, pp. 81-82, or any good numerical analysis textbook.

V. GAS-LIQUID SYSTEMS

In this section, to illustrate the application of our results to other than membrane phenomena, we consider a gas-liquid system of n substances, X_1, \dots, X_n , at a pressure P and temperature T .

We assume that under these conditions each substance will be distributed between a single liquid phase and a single gas phase. Thus, we are assuming that the substances, in both liquid and solid form, are completely miscible in all proportions, and that no solid precipitates are formed. We also assume that the substances are involved in no chemical reactions other than evaporation and condensation. We shall show that if this type of system satisfies certain further assumptions it is a primitive chemical equilibrium system. Thus all our results concerning primitive systems will be applicable to this type of gas-liquid system.

We designate the liquid phase Phase I, and the gas phase Phase II, and let x_i , x'_i , \hat{x}_i , \hat{x}'_i , b_i , \bar{x} , \bar{x}' , etc., have their usual meanings.

Let the vapor pressure of X_i at the temperature T be p_i^0 . We assume that each p_i^0 is positive and finite; i.e., we assume that each X_i is neither completely volatile nor

completely non-volatile. This is analogous to the condition, for a membrane system, that there are no impermeable substances. We assume that the liquid obeys Raoult's law-- which we take to be that the partial pressure, p_i , of X_i^I is equal to the product of the mole fraction, \hat{x}_i , of X_i^I and the vapor pressure, p_i^0 , of X_i :

$$p_i = p_i^0 \hat{x}_i . \quad (30)$$

We assume that the gas obeys Dalton's law of partial pressures--which we take to be that the partial pressure, p_i , of X_i^{II} is equal to the product of the mole fraction, \hat{x}_i' , of X_i^{II} and the total pressure, P :

$$p_i = P \hat{x}_i' . \quad (31)$$

Since at equilibrium X_i must have the same partial pressure in liquid as in gas, we have

$$\frac{\hat{x}_i'}{\hat{x}_i} = \frac{p_i^0}{P} . \quad (32)$$

Thus, the system is a primitive system with the equilibrium constant, k_i , of X_i given by

$$k_i = \frac{p_i^0}{P} . \quad (33)$$

As an example of the application of our general results to gas-liquid systems, let us find the conditions for a non-unique solution. According to Theorem 1 of Sec. III, there will be a non-unique equilibrium composition if and only if $k_i = 1$ for every i for which $b_i \neq 0$, or, applying Eq. (33), if and only if $p_i^0 = P$ for every i for which $b_i \neq 0$. In other words, there will be a non-unique equilibrium composition at pressure P and temperature T precisely if, for every substance present in any positive amount, the vapor pressure is P when the temperature is T ; i.e., precisely if the boiling point of every substance present, at the pressure P , is T .

As another example, let us find the circumstances under which a liquid phase can exist at equilibrium without the formation of a gas phase; i.e., the circumstances required for either a non-unique solution or a unique solution with no gas phase present. Applying Theorem 1 of Sec. III

shows that these are precisely the circumstances under which Condition II fails--that is (because $\hat{x}_i = \frac{b_i}{\bar{b}}$):

$$\sum_{i=1}^n k_i \hat{x}_i \leq 1 ;$$

or, replacing k_i by its value from Eq. (33) and using Eq. (30),

$$\sum_{i=1}^n p_i \leq P . \quad (34)$$

Thus, we have obtained the familiar condition that a liquid phase can exist at equilibrium without the formation of a gas phase if and only if the sum of the partial pressures of its constituents does not exceed the hydrostatic pressure.

Similarly, we can obtain the condition for a gas phase to exist at equilibrium without the formation of a liquid phase:

$$\sum_{i=1}^n \frac{p_i}{p_i^0} \leq 1 .$$

VI. INTRODUCTION TO VARIABLE SYSTEMS

Previous sections have investigated the properties of a fixed primitive chemical equilibrium system. This and the following sections study the manner in which the equilibrium properties of a variable system vary when the inputs are varied. By a variable system we mean specifically a primitive system with n permeable species and fixed equilibrium constants, but with inputs that vary linearly with a parameter t which may or may not denote time. In any case, we emphasize that we are not studying kinetic effects-- at each value of t , the system is allowed to come to equilibrium.

Thus, the amount of X_i input is given by a function

$$b_i(t) = b_i^0 + t\Delta b_i .$$

Similarly, the amount of impermeable material in Phase I is

$$q(t) = q^0 + t\Delta q ,$$

and the amount of impermeable material in Phase II is

$$q'(t) = q'^0 + t\Delta q' .$$

We assume that $\Delta b_i \geq 0$, $\Delta q \geq 0$, $\Delta q' \geq 0$, and $t \geq 0$. Although some of our results are valid without this assumption, it is more convenient to make it throughout.

Thus, each $t \geq 0$ presents a primitive chemical equilibrium system, which we shall call $S(t)$. If $\Delta q = \Delta q' = 0$ and $\sum_{i=1}^n \Delta b_i = 1$, then $S(t)$ may be interpreted as the result of adding t moles of a mixture of the permeable substances with mole fractions $\Delta b_1, \dots, \Delta b_n$ to $S(0)$. If $\sum_{i=1}^n \Delta b_i \neq 1$, then the same type of interpretation holds, except that the Δb_i are no longer mole fractions and t is no longer measured in moles. Finally, if $\Delta q \neq 0$ or $\Delta q' \neq 0$, we may think of adding a mixture of both impermeable and permeable substances to $S(0)$, taking care to distinguish between impermeable material destined for Phase I and impermeable material destined for Phase II.

This section concludes with a sequence of examples of variable primitive chemical equilibrium systems (leaving general results to later sections).

Example 1

Consider an inert symmetric membrane permeable to H_2O but not to some macromolecule, and capable of maintaining a hydrostatic pressure gradient. Suppose we place one mole of the macromolecule on one side (Phase I) of the

membrane, fix the hydrostatic pressure of both phases so that the pressure of Phase I is higher than the pressure of Phase II, hold both phases at the same temperature, and put t moles of H_2O into the system. Thus, we have $q^0 = 1$, $\Delta q = q^{0'} = \Delta q' = 0$, $n = 1$, $X_1 = H_2O$, $b_1^0 = 0$, $\Delta b_1 = 1$, and $k_1 > 1$ (because of the hydrostatic pressure gradient). We see that $S(t)$ satisfies Condition I of Sec. III for all $t \geq 0$, and satisfies Condition II if and only if $t > \frac{1}{k-1}$ moles.

It follows from Theorem 1 that $S(t)$ has a unique equilibrium composition for all $t \geq 0$, and that for $0 \leq t \leq \frac{1}{k-1}$, $x_1' = 0$ (i.e., all the H_2O will be drawn to the side of the membrane having the macromolecule). But for $t > \frac{1}{k_1-1}$, $x_1 = \frac{1}{k_1-1}$ and $x_1' = t - \frac{1}{k_1-1}$; i.e., after $\frac{1}{k_1-1}$ moles of H_2O have been added, all additional H_2O will be drawn to the side not containing the macromolecule.

Example 2

Let $n = 2$, $b_1^0 = 0$, $\Delta b_1 = 1$, $b_2^0 = 1$, $\Delta b_2 = 0$, $q^0 = \setminus$
 $\Delta q = q^{0'} = \Delta q' = 0$. Let $k_2 < 1 < k_1$. In words, $S(t)$ has no impermeable substances, and has two permeable substances, X_1 and X_2 ; the amount of X_2 is fixed at one mole,

but $S(t)$ has t moles of X_1 . We will first describe several realizations of such a system:

- 1) Consider a gas-liquid system of the type described in Sec. V, with two substances X_1 and X_2 , each at the temperature T . The vapor pressure of X_1 exceeds that of X_2 , and the total pressure, P , of the system is held at a fixed value between the vapor pressure of X_1 and the vapor pressure of X_2 .
- 2) Consider two substances, X_1 and X_2 , both permeable to an inert symmetric membrane which is both rigid (i.e., capable of supporting a pressure difference) and a heat insulator (i.e., capable of supporting a temperature difference).^{*} Since k_1 and k_2 depend in different ways on the temperatures and pressures, it is quite possible that $k_1 > 1 > k_2$.
- 3) Consider a metabolically active, non-symmetric membrane which "pumps" X_1 (which might be H_2O),

^{*}Such membranes can be obtained from the same scientific supply houses which furnish frictionless pistons and completely ionizable electrolytes.

from Phase I to Phase II and which "pumps" X_2
(which might be glucose*) from Phase II to
Phase I.

We can apply Theorem 1 to this variable system to see
that, for all $t \geq 0$, it will have a unique solution. For

$$t \leq \frac{1-k_2}{k_1-1} ,$$

Phase II will be vacuous. For

$$\frac{1-k_2}{k_1-1} < t < \frac{k_1}{k_2} \frac{1-k_2}{k_1-1} ,$$

neither phase will be vacuous. For

$$t \geq \frac{k_1}{k_2} \frac{1-k_2}{k_1-1} ,$$

Phase I will be vacuous. Figure 1 shows a plot of the
composition of the system versus t , for the case $k_1 = \frac{3}{2}$,
 $k_2 = \frac{3}{4}$.

*Note that we must then consider only those values
of t for which no glucose precipitate will be formed.

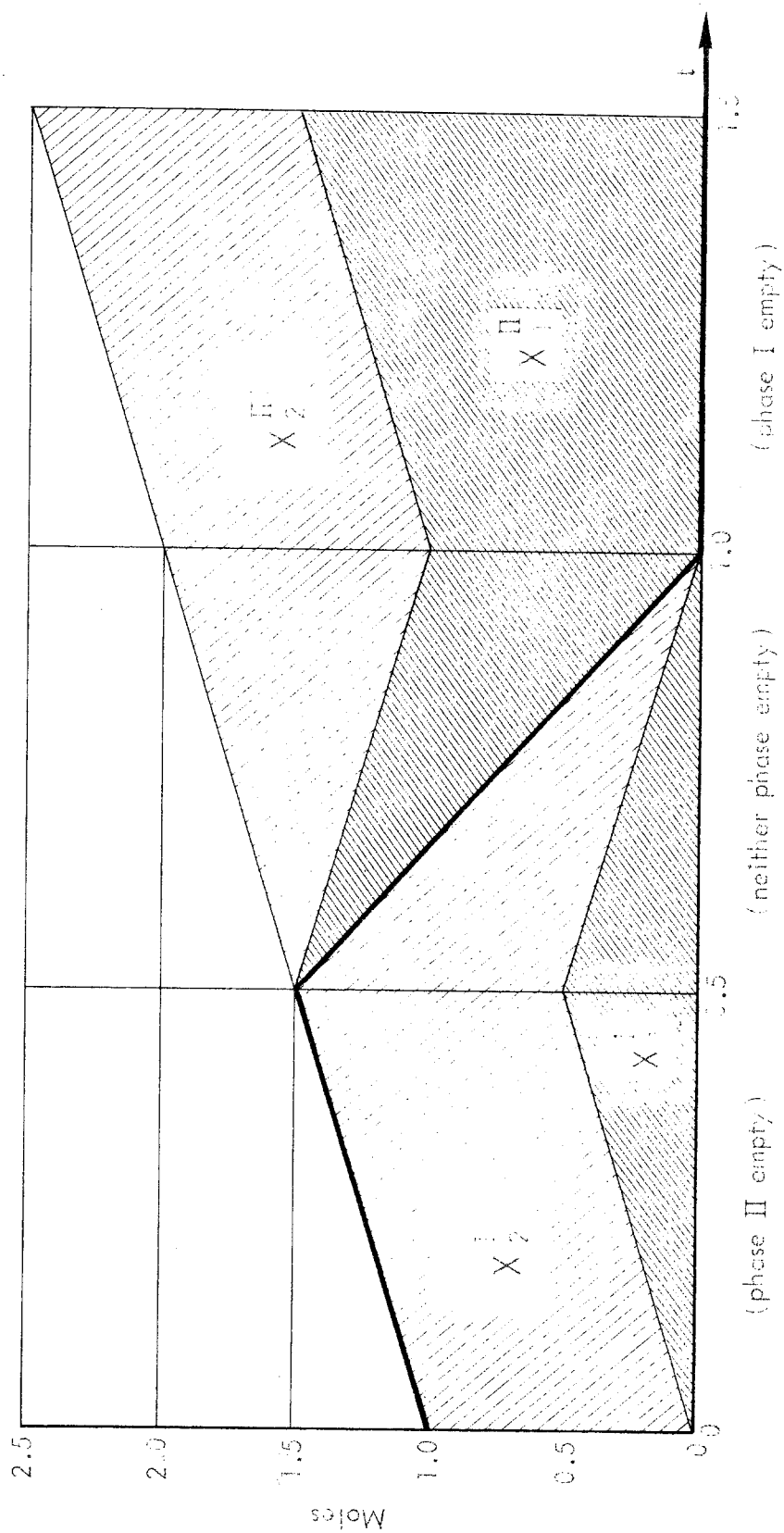


Fig. 1---Variation of composition in the system of Example 2

$$(n = 2, q^0 = \Delta q = q^{0'} = \Delta q' = b_1^0 = \Delta b_2 = 0, b_2^0 = \Delta b_1 = 1, k_1 = \frac{3}{2}, k_2 = \frac{3}{4})$$

Example 3

Consider the result of equilibrating .04 moles of methane and .96 moles of normal octane with t moles of a mixture of normal hexane (with a mole fraction of .96) and normal eicosane (with a mole fraction of .04) at a temperature of 40°C and a pressure of 120 mm of Hg, assuming that Raoult's law and Dalton's law of partial pressures are satisfied. We have $n = 4$, $X_1 = \text{methane}$, $X_2 = \text{normal hexane}$, $X_3 = \text{normal octane}$, $X_4 = \text{normal eicosane}$, $q^0 = q^0' = \Delta q = \Delta q' = 0$, $b_1^0 = .04$, $b_2^0 = 0$, $b_3^0 = .96$, $b_4^0 = 0$, $\Delta b_1 = 0$, $\Delta b_2 = .96$, $\Delta b_3 = 0$, $\Delta b_4 = .04$. According to Ref. 24,* the vapor pressures at 40°C are given by $p_1^0 = 217549.9$ mm Hg, $p_2^0 = 279.4409$ mm Hg, $p_3^0 = 31.10092$ mm Hg, $p_4^0 = .0000257$ mm Hg. Equation (33) shows that $k_1 = 1812.916$, $k_2 = 2.32867$, $k_3 = .259174$, $k_4 = 2.1 \cdot 10^{-7}$. Note that k_1 is so large and k_4 is so small that methane and normal eicosane virtually play the role of impermeable species.

Theorem 1 shows that $S(t)$ has a unique solution for all $t \geq 0$. Figure 2 shows the computed values of \bar{x} and \bar{x}'

*Ref. 24, p. 336.

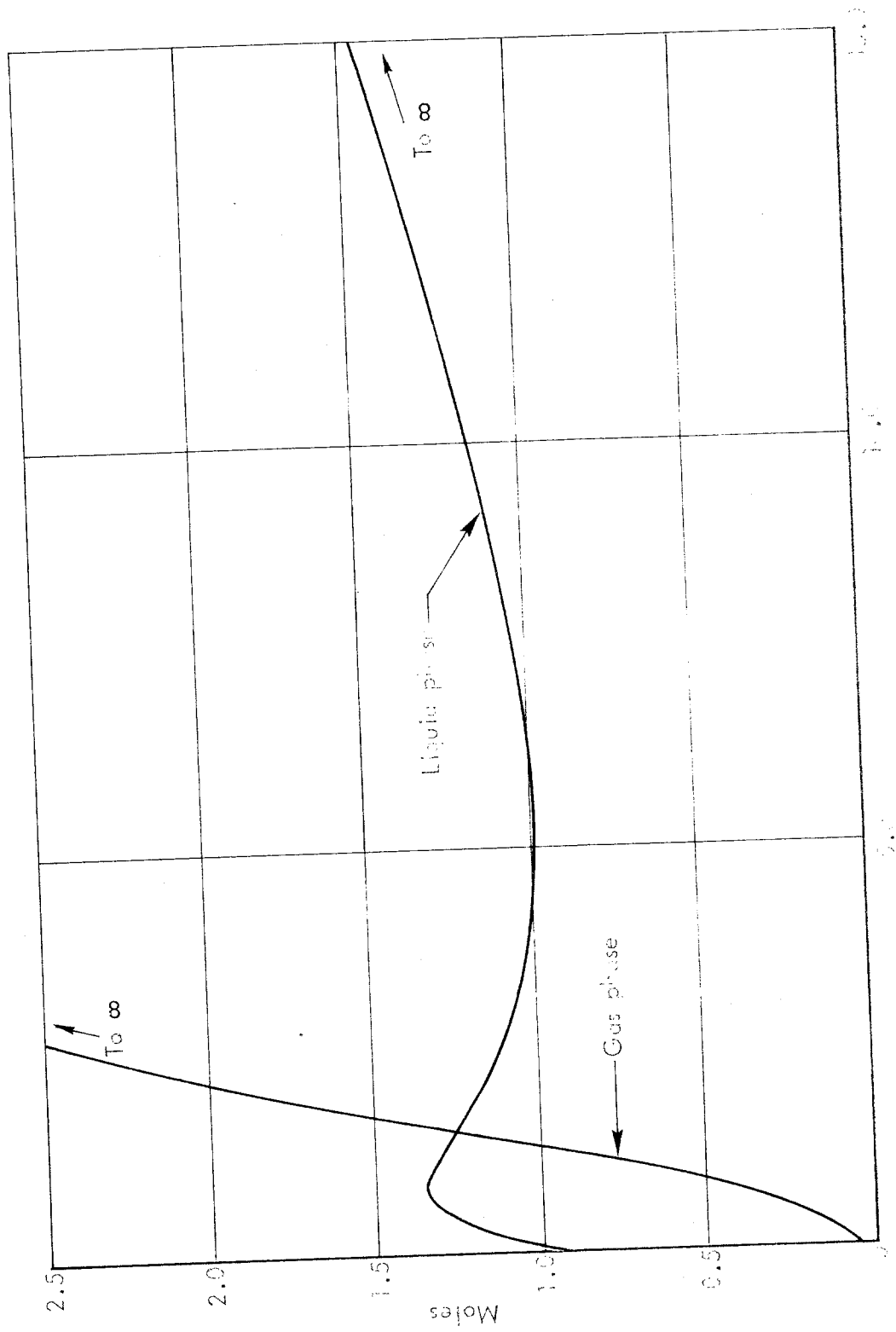


Fig. 2.—Computed sizes of the liquid and gas phases formed when 0.04 moles of methane, 0.96 moles of normal octane, 0.96 moles of normal hexane and 0.04 moles of normal eicosane are equilibrated at 10°C and 120 mm Hg.

(the size of the liquid and gas phases in total number of moles) plotted against t . Note that the complex behavior of \bar{x} shown in Fig. 2 does not represent an artifact such as computational round-off error. Intuitively, one would not anticipate this complicated double reversal in the size of the liquid phase as components are added to such a simple system, involving no intraphase reactions and obeying the most ideal of all possible laws for a two-phase chemical system.

Example 4

Reference 21 treated a primitive system with two permeable species and with an impermeable species representing protein, as a tutorial model of human renal function. In the treatment, Phase I represented plasma and Phase II represented urine. Far from being simple, inert, and symmetric, the "membrane" dividing the two phases represented the aggregated whole of the complex secretion and reabsorption processes involved in human urine production and control.

VII. CLASSIFICATION OF VARIABLE SYSTEMS

This section discusses the behavior of the variable system $S(t)$ for $t \geq 0$ in terms of the four possible types of behavior for a primitive system set forth in Theorem 1 of Sec. III.

Consider a variable primitive chemical equilibrium system as defined in Sec. VI and let

$$\bar{b}^0 = \sum_{i=1}^n b_i^0 + q^0 + q^{0'}$$

and

$$\Delta \bar{b} = \sum_{i=1}^n \Delta b_i + \Delta q + \Delta q'$$

In addition, we introduce the following six quantities:

$$\begin{aligned} \sigma^0 &= \begin{cases} \infty & \text{if } q^0 > 0 \\ \sum_{i=1}^n \frac{b_i^0}{k_i} & \text{if } q^0 = 0 \end{cases} & \sigma^{0'} &= \begin{cases} \infty & \text{if } q^{0'} > 0 \\ \sum_{i=1}^n k_i b_i^0 & \text{if } q^{0'} = 0 \end{cases} \\ \Delta \sigma &= \begin{cases} \infty & \text{if } q^0 + \Delta q > 0 \\ \sum_{i=1}^n \frac{\Delta b_i}{k_i} & \text{if } q^0 = \Delta q = 0 \end{cases} & \Delta \sigma' &= \begin{cases} \infty & \text{if } q^{0'} + \Delta q' > 0 \\ \sum_{i=1}^n k_i \Delta b_i & \text{if } q^{0'} = \Delta q' = 0 \end{cases} \\ \tau &= \begin{cases} \frac{\bar{b}^0 - \sigma^0}{\Delta \sigma - \Delta \bar{b}} & \text{if } \Delta \sigma < \infty \\ 0 & \text{if } \Delta \sigma = \infty \end{cases} & \tau' &= \begin{cases} \frac{\bar{b}^0 - \sigma^{0'}}{\Delta \sigma' - \Delta \bar{b}} & \text{if } \Delta \sigma' < \infty \\ 0 & \text{if } \Delta \sigma' = \infty \end{cases} \end{aligned}$$

Note: τ will not be used if $\sigma^0 = \infty$ or if $\Delta\sigma = \Delta\bar{b}$; similarly, τ' will not be used if $\sigma^{0'} = \infty$ or if $\Delta\sigma' = \Delta\bar{b}$.

The σ 's are closely related to Conditions I and II of Sec. III. Thus, Condition I for $S(0)$ is equivalent to the statement $\sigma^0 > \bar{b}^0$. The τ 's represent certain critical values of the parameter t .

Table 1 describes the 13 possible relations between the quantities σ^0 , \bar{b}^0 , $\sigma^{0'}$, $\Delta\sigma$, $\Delta\bar{b}$, and $\Delta\sigma'$. We assert that every variable primitive chemical equilibrium system satisfies one and only one of these sets of four inequalities.

Table 1 also shows the types of behavior of $S(t)$, as t varies in each of the 13 cases. The phrase "I only" means that $S(t)$ has a unique solution and Phase II is vacuous; "II only" means that $S(t)$ has a unique solution and Phase I is vacuous. "Both" means that $S(t)$ has a unique solution and neither phase vanishes. "Ambig" (ambiguous) means that $S(t)$ does not have a unique solution.*

Finally, Table 1 indicates, for each of the 13 cases, the intervals of t in which $S(t)$ has the various types of behavior. (A square bracket means that the endpoint in question is to be included; a curved parenthesis means

* See Sec. III for further properties for each of the four types of behavior.

Table 1

CLASSIFICATION OF VARIABLE PRIMITIVE SYSTEMS

Case	Conditions				Behavior		
1	$a^0 > \bar{b}^0$	$a^{0'} > \bar{b}^0$	$\Delta\sigma \geq \Delta\bar{b}$	$\Delta\sigma' \geq \Delta\bar{b}$	Both [0, ∞)		
2	$a^0 > \bar{b}^0$	$a^{0'} > \bar{b}^0$	$\Delta\sigma < \Delta\bar{b}$	$\Delta\sigma' > \Delta\bar{b}$	Both [0, τ)	II only [τ , ∞)	
3	$a^0 > \bar{b}^0$	$a^{0'} > \bar{b}^0$	$\Delta\sigma > \Delta\bar{b}$	$\Delta\sigma' < \Delta\bar{b}$	Both [0, τ')	I only [τ' , ∞)	
4	$a^0 \leq \bar{b}^0$	$a^{0'} > \bar{b}^0$	$\Delta\sigma \leq \Delta\bar{b}$	$\Delta\sigma' \geq \Delta\bar{b}$	II only [0, ∞)		
5	$a^0 \leq \bar{b}^0$	$a^{0'} > \bar{b}^0$	$\Delta\sigma > \Delta\bar{b}$	$\Delta\sigma' \geq \Delta\bar{b}$	II only [0, τ]	Both (τ , ∞)	
6	$a^0 \leq \bar{b}^0$	$a^{0'} > \bar{b}^0$	$\Delta\sigma > \Delta\bar{b}$	$\Delta\sigma' < \Delta\bar{b}$	II only [0, τ]	Both (τ , τ')	I only [τ' , ∞)
7	$a^0 > \bar{b}^0$	$a^{0'} \leq \bar{b}^0$	$\Delta\sigma \geq \Delta\bar{b}$	$\Delta\sigma' \leq \Delta\bar{b}$	I only [0, ∞)		
8	$a^0 > \bar{b}^0$	$a^{0'} \leq \bar{b}^0$	$\Delta\sigma \geq \Delta\bar{b}$	$\Delta\sigma' > \Delta\bar{b}$	I only [0, τ']	Both (τ' , ∞)	
9	$a^0 > \bar{b}^0$	$a^{0'} \leq \bar{b}^0$	$\Delta\sigma < \Delta\bar{b}$	$\Delta\sigma' > \Delta\bar{b}$	I only [0, τ']	Both (τ' , τ)	II only [τ , ∞)
10	$a^0 \leq \bar{b}^0$	$a^{0'} \leq \bar{b}^0$	$\Delta\sigma \leq \Delta\bar{b}$	$\Delta\sigma' \leq \Delta\bar{b}$	Ambig [0, ∞)		
11	$a^0 \leq \bar{b}^0$	$a^{0'} \leq \bar{b}^0$	$\Delta\sigma \leq \Delta\bar{b}$	$\Delta\sigma' > \Delta\bar{b}$	Ambig {0}	II only (0, ∞)	
12	$a^0 \leq \bar{b}^0$	$a^{0'} \leq \bar{b}^0$	$\Delta\sigma > \Delta\bar{b}$	$\Delta\sigma' \leq \Delta\bar{b}$	Ambig {0}	I only (0, ∞)	
13	$a^0 \leq \bar{b}^0$	$a^{0'} \leq \bar{b}^0$	$\Delta\sigma > \Delta\bar{b}$	$\Delta\sigma' > \Delta\bar{b}$	Ambig {0}	Both (0, ∞)	

that it is to be excluded.) Thus, according to Table 1, if $\sigma^{0'} \leq \bar{b}^0 < \sigma^0$ and $\Delta\sigma < \Delta\bar{b} < \Delta\sigma'$, then $S(t)$ has a unique solution for all $t \geq 0$ and Phase I is vacuous for $t \leq \tau'$, Phase II is vacuous for $t \geq \tau$, and neither phase is vacuous for $\tau < t < \tau'$.

Many readers may find it instructive to derive Table 1 from Theorem 1. We omit this rather long derivation.

VIII. MONOTONICITY THEOREM

In this section we show that some of the quantities associated with a variable system vary monotonically with t . This result is of interest because, as was illustrated in Example 3 of Sec. VI, other significant quantities can behave very non-monotonically.

If $S(t)$ has a unique solution, then let $\bar{x}(t)$, $\bar{x}'(t)$, $x_i(t)$, $\hat{x}_i(t)$, etc., denote the values of \bar{x} , \bar{x}' , x_i , \hat{x}_i , etc., for the unique solution of $S(t)$.

THEOREM 2

If $S(0)$ has a unique solution, then $S(t)$ has a unique solution for all $t \geq 0$, and the ratio $\frac{\bar{x}'(t)}{\bar{x}(t)}$ (admitting ∞ as a possible value) is monotonic in t .

Proof of Theorem 2:

That $S(t)$ has a unique solution for all $t \geq 0$ follows from Table 1. It also follows from Table 1 that the set T of $t > 0$ for which both $\bar{x}(t) > 0$ and $\bar{x}'(t) > 0$ is an interval of the form $(0, \infty)$, $(0, \tau)$, $(0, \tau')$, (τ, ∞) , (τ, τ') , (τ', ∞) , (τ', τ) , or $(0, \infty)$. Furthermore, it follows from Table 1 and from the fact [11] that $\bar{x}(t)$ and $\bar{x}'(t)$ are continuous functions of t that it is sufficient to prove that $\frac{\bar{x}'(t)}{\bar{x}(t)}$ is monotonic in the interval T .

But in T , $\frac{\bar{x}'(t)}{\bar{x}(t)}$ is finite-valued, and continuous. Thus, if this ratio were not monotonic in t , there would have to be a real number r and $t_1, t_2 \in T$ with $t_1 < t_2$ such that

$$\frac{\bar{x}'(t_1)}{\bar{x}(t_1)} = \frac{\bar{x}'(t_2)}{\bar{x}(t_2)} = r .$$

Moreover, we can choose t_1 and t_2 such that $\frac{\bar{x}'(t)}{\bar{x}(t)} = r$ does not hold for any t in (t_1, t_2) . Let $t = \frac{1}{2}(t_1 + t_2)$ and let

$$x_i = \frac{1}{2} \left(x_i(t_1) + x_i(t_2) \right) \quad \text{for } i=1, \dots, n$$

$$x'_i = \frac{1}{2} \left(x_i(t_1) + x_i(t_2) \right) \quad \text{for } i=1, \dots, n$$

$$q = \frac{1}{2} \left(q(t_1) + q(t_2) \right)$$

$$q' = \frac{1}{2} \left(q'(t_1) + q'(t_2) \right) .$$

We will show that $x_1, \dots, x_n, q, x'_1, \dots, x'_n, q'$ is an equilibrium composition for $S(t)$.

To do this, observe first that

$$\begin{aligned}
 x_i + x'_i &= \frac{1}{2} \left(x_i(t_1) + x_i(t_2) \right) + \frac{1}{2} \left(x'_i(t_1) + x'_i(t_2) \right) \\
 &= \frac{1}{2} \left(x_i(t_1) + x'_i(t_1) \right) + \frac{1}{2} \left(x_i(t_2) + x'_i(t_2) \right) \\
 &= \frac{1}{2} \left(b_i(t_1) + b_i(t_2) \right) \\
 &= b_i(t)
 \end{aligned}$$

and that $q = q(t)$ and $q' = q'(t)$. Thus, the laws of mass balance for $S(t)$ are satisfied.

We also have

$$\begin{aligned}
 \bar{x}' &= \frac{1}{2} \left(\bar{x}'(t_1) + \bar{x}'(t_2) \right) \\
 &= \frac{r}{2} \left(\bar{x}(t_1) + \bar{x}(t_2) \right) \\
 &= r\bar{x} .
 \end{aligned}$$

It is also clear that $\bar{x} > 0$ and $\bar{x}' > 0$. Hence we may write:

$$\begin{aligned}
 \frac{\hat{x}_i'}{\hat{x}_i} &= \frac{\bar{x}}{\bar{x}'} \frac{x_i'}{x_i} = \frac{\bar{x}}{\bar{x}'} \frac{x_i'(t_1) + x_i'(t_2)}{x_i(t_1) + x_i(t_2)} \\
 &= \frac{\bar{x}}{\bar{x}'} \frac{\bar{x}'(t_1)\hat{x}_i'(t_1) + \bar{x}'(t_2)\hat{x}_i'(t_2)}{\bar{x}'(t_1)\hat{x}_i(t_1) + \bar{x}'(t_2)\hat{x}_i(t_2)} \\
 &= \frac{\bar{x}}{\bar{x}'} \frac{r\bar{x}(t_1)k_i\hat{x}_i(t_1) + r\bar{x}(t_2)k_i\hat{x}_i(t_2)}{\bar{x}'(t_1)\hat{x}_i(t_1) + \bar{x}'(t_2)\hat{x}_i(t_2)} \\
 &= \frac{\bar{x}}{\bar{x}'} r k_i = k_i .
 \end{aligned}$$

Thus, the laws of mass action are satisfied.

Therefore, $x_1, \dots, x_n, q, x_1', \dots, x_n', q'$ is an equilibrium composition and, hence, the unique solution to $S(t)$. Therefore, $\bar{x}(t) = \bar{x}$ and $\bar{x}'(t) = \bar{x}'$ so that $\frac{\bar{x}'(t)}{\bar{x}(t)} = \frac{\bar{x}'}{\bar{x}} = r$, a contradiction. QED.

COROLLARY 1

The functions $\frac{x_i'(t)}{x_i(t)}, \frac{x_i(t)}{b_i(t)}, \frac{\bar{x}(t)}{\bar{b}(t)}$ and their re-

ciprocals are also monotonic in t .

Proof of Corollary 1:

From Eq. (9) of Sec. I,

$$\frac{x_i'(t)}{x_i(t)} = k_i \frac{\bar{x}'(t)}{\bar{x}(t)} .$$

From Eq. (10) of Sec. II,

$$\frac{x_i(t)}{b_i(t)} = \frac{\bar{x}(t)}{\bar{x}(t) + k_i \bar{x}'(t)} = \frac{1}{1 + k_i \frac{\bar{x}'(t)}{\bar{x}(t)}}.$$

From Eq. (6) of Sec. I,

$$\frac{\bar{x}(t)}{\bar{b}(t)} = \frac{1}{1 + \frac{\bar{x}'(t)}{\bar{x}(t)}}.$$

The reciprocal of a monotonic function is monotonic. QED.

COROLLARY 2

If for some i , $\Delta b_i = 0$, then for that i , $x_i(t)$ and $x_i'(t)$ are monotonic functions of t .

Proof of Corollary 2:

Because

$$x_i(t) = \frac{b_i^0 + \Delta b_i t}{1 + k_i \left[\frac{\bar{x}'(t)}{\bar{x}(t)} \right]} = \frac{b_i^0}{1 + k_i \left[\frac{\bar{x}'(t)}{\bar{x}(t)} \right]}$$

(see Eq. (10) in Sec. II), and $x_i'(t) = b_i^0 - x_i(t)$.

COROLLARY 3

Either each of the functions $x_i(t)$, $i=1, \dots, n$, and $\bar{x}(t)$ is monotonically increasing, or each of the functions $x_i'(t)$, $i=1, \dots, n$, and $\bar{x}'(t)$ is monotonically increasing.

Proof of Corollary 3:

If $\frac{\bar{x}'(t)}{\bar{x}(t)}$ is monotonically decreasing, then we observe that

$$x_i(t) = \frac{b_i(t)}{1 + k_i \frac{\bar{x}'(t)}{\bar{x}(t)}}.$$

The ratio of a monotonically increasing, non-negative function to a monotonically decreasing positive function is monotonically increasing. Since $\bar{x}(t) = q(t) + \sum_{i=1}^n x_i(t)$, $\bar{x}(t)$ is also monotonically increasing.

If $\frac{\bar{x}'(t)}{\bar{x}(t)}$ is monotonically increasing, then $\frac{\bar{x}(t)}{\bar{x}'(t)}$ is monotonically decreasing and the result follows by reversing the roles of Phase I and Phase II.

Note the illustration of this Corollary in Fig. 2.

IX. LINEAR RESPONSES

The examples of Sec. VI indicate that the x_i are rarely linear functions of t . Under what circumstances will linearity hold?

THEOREM 3

Assume that $S(0)$ has a unique solution with neither phase vacuous; then $x_i(t)$, $i=1, \dots, n$, will all be linear functions of t , $0 \leq t < \infty$, precisely if either:

Case A: There are constants u , u' such that

$$\Delta b_i = ux_i(0) + u'x_i'(0) \quad \text{for } i=1, \dots, n, \quad (40)$$

$$\Delta q = uq^0, \quad (41)$$

$$\Delta q' = u'q^{0'}, \quad (42)$$

in which case:

$$x_i(t) = (1 + tu)x_i(0), \quad (43)$$

$$x_i'(t) = (1 + tu')x_i'(0), \quad (44)$$

$$\bar{x}(t) = (1 + tu)\bar{x}(0) , \quad (45)$$

$$\bar{x}'(t) = (1 + tu)\bar{x}'(0) , \quad (46)$$

$$\hat{x}_i(t) = \hat{x}_i(0) , \quad (47)$$

$$\hat{x}'_i(t) = \hat{x}'_i(0) ; \quad (48)$$

or,

Case B: The constant $r = \frac{\bar{x}'(0)}{\bar{x}(0)}$ is such that

$$\sum_{i=1}^n \frac{\Delta b_i (1 - k_i)}{1 + rk_i} = \frac{\Delta q'}{r} - \Delta q , \quad (49)$$

in which case

$$x_i(t) = \frac{b_i(t)}{1 + rk_i} , \quad (50)$$

$$x'_i(t) = \frac{rk_i b_i(t)}{1 + rk_i} , \quad (51)$$

$$\frac{\bar{x}'(t)}{\bar{x}(t)} = r . \quad (52)$$

Proof of Theorem 3:

Since $S(0)$ has a unique solution, so does $S(t)$ for $t \geq 0$. Assume first that each $x_i(t)$ is a linear function of t . Since $b_i(t)$, $q(t)$, $q'(t)$, $\bar{b}(t)$ are all by definition linear functions of t , then

$$x'_i(t) = b_i(t) - x_i(t) ,$$

$$\bar{x}(t) = \sum_{i=1}^n x_i(t) + q(t) ,$$

$$\bar{x}'(t) = \bar{b}(t) - \bar{x}(t)$$

are also linear. By hypothesis, $\bar{x}(0) > 0$; hence, for any $t \geq 0$,

$$\bar{x}(t) = \frac{1}{2} \left(\bar{x}(0) + \bar{x}(2t) \right) \geq \frac{\bar{x}(0)}{2} > 0 .$$

Similarly, $\bar{x}'(t) > 0$ for $t \geq 0$. We can then assert the law of mass action:

$$\frac{x'_i(t)}{\bar{x}'(t)} = \frac{k_i x_i(t)}{\bar{x}(t)} \quad \text{for } i=1, \dots, n \quad \text{and } t \geq 0 .$$

Hence,

$$x_i'(t)\bar{x}(t) = k_i x_i(t)\bar{x}'(t) \quad \text{for } n=1, \dots, n \quad \text{and } t \geq 0 .$$

(54)

A linear function defined for $t \geq 0$ can be extended linearly to the whole real line. If this is done for $x_i(t)$, $x_i'(t)$, $\bar{x}(t)$, $\bar{x}'(t)$, then Eq. (54) remains valid for all real t , and represents two ways of factoring the same quadratic function of t into linear factors. But since the linear factors correspond to the roots of the quadratic, this factorization is unique except for multiplicative constants and the order of the factors. Thus we have shown that either $\bar{x}(t)$ is proportional to $x_i(t)$ or $\bar{x}(t)$ is proportional to $\bar{x}'(t)$. Since $\bar{x}(t) > 0$ for $t > 0$, we have shown that: either for every i

$$\frac{x_i(t)}{\bar{x}(t)} \quad \text{is a constant, viz.,} \quad \frac{\bar{x}_i(0)}{\bar{x}(0)} ; \quad (55)$$

or

$$\frac{\bar{x}'(t)}{\bar{x}(t)} \quad \text{is a constant, viz.,} \quad \frac{\bar{x}'(0)}{\bar{x}(0)} = r . \quad (56)$$

In the first case, we have from Eq. (53) that for every i ,

$$\frac{x'_i(t)}{\bar{x}'(t)} \text{ is a constant, } \frac{x'_i(0)}{\bar{x}'(0)} .$$

Define

$$u = \frac{d}{dt} \frac{\bar{x}(t)}{\bar{x}(0)}$$

and

$$u' = \frac{d}{dt} \frac{\bar{x}'(t)}{\bar{x}'(0)} ;$$

since \bar{x} and \bar{x}' are linear, u and u' are constants. Then

$$\begin{aligned} \Delta b_i &= \frac{d}{dt} b_i(t) = \frac{d}{dt} \left(x_i(t) + x'_i(t) \right) \\ &= \frac{d}{dt} \left(x_i(0) \frac{\bar{x}(t)}{\bar{x}(0)} + x'_i(0) \frac{\bar{x}'(t)}{\bar{x}'(0)} \right) = u x_i(0) + u' x'_i(0) , \end{aligned}$$

which is Eq. (40). Also,

$$\begin{aligned}
 \Delta q &= \frac{d}{dt} q(t) = \frac{d}{dt} \left(\bar{x}(t) - \sum_{i=1}^n x_i(t) \right) \\
 &= \frac{d}{dt} \left(\bar{x}(t) - \sum_{i=1}^n \frac{\bar{x}(t)}{\bar{x}(0)} x_i(0) \right) \\
 &= \left(1 - \sum_{i=1}^n \frac{x_i(0)}{\bar{x}(0)} \right) \frac{d}{dt} \bar{x}(t) \\
 &= u \left(\bar{x}(0) - \sum_{i=1}^n x_i(0) \right) = u \left(q(0) \right) = u q^0,
 \end{aligned}$$

which is Eq. (41). The proof of Eq. (42) is similar.

Then Eqs. (43) through (48) follow by simple manipulations.

In the second case, Eq. (56), we have

$$\begin{aligned}
 \bar{x}(t) &= \frac{r+1}{r+1} \bar{x}(t) = \frac{\bar{x}(t) + \bar{x}'(t)}{r+1} \\
 &= \frac{\bar{b}(t)}{r+1} = \frac{q(t) + q'(t) + \sum_{i=1}^n b_i(t)}{r+1}.
 \end{aligned} \tag{58}$$

Now $\bar{x}(t)$ must be the root of the characteristic function of $S(t)$; that is,

$$\frac{q(t)}{\bar{x}(t)} + \sum_{i=1}^n \frac{b_i(t)}{\bar{x}(t) + k_i \bar{x}'(t)} = 1 .$$

Multiplying both sides of this by $\bar{x}(t)$ gives

$$q(t) + \sum_{i=1}^n \frac{b_i(t)}{1 + rk_i} = \bar{x}(t) .$$

Replacing $x(t)$ by its value from Eq. (58), differentiating with respect to t , and simplifying, gives Eq. (49); then Eqs. (50), (51), and (52) follow easily.

Finally, it is an algebraic exercise to show that, if the condition for either case A or case B holds, then the $x_i(t)$ have the claimed behavior and hence are linear. QED.

Note that Case A of the theorem corresponds simply to expanding the two phases separately, without changing the mole fractions. One type of variable system which satisfies Case B is given by $\Delta q = \Delta q' = 0$ and $\Delta b_i = 0$ for all i for which $k_i \neq 1$. That is, if a system is varied by changing only the inputs of "undriven" substances (i.e., substances for which $k_i = 1$), then the response is linear. Here the $x_i(t)$ and $x_i'(t)$ are constant for the substances whose inputs are not altered.

For systems in which $\Delta q = \Delta q' = 0$ and for which $b_i^o > 0$, for all i , we may simplify Eq. (49). Observe from Eq. (10) that

$$1 + rk_i = \frac{b_i^o}{\hat{x}_i(0)} ;$$

hence,

$$\frac{1 - k_i}{1 + rk_i} = \frac{\hat{x}_i(0)}{b_i^o} (1 - k_i) = \frac{\hat{x}_i(0) - \hat{x}_i'(0)}{b_i^o} ,$$

so Eq. (49) becomes

$$\sum_{i=1}^n \frac{\Delta b_i}{b_i^o} \left(\hat{x}_i(0) - \hat{x}_i'(0) \right) = 0 . \quad (59)$$

REFERENCES

1. White, W. B., S. M. Johnson, and G. B. Dantzig, "Chemical Equilibrium in Complex Mixtures," J. Chem. Phys., Vol. 28, No. 5, May 1958, pp. 751-755 (also: The RAND Corporation, P-1059).
2. Dantzig, G. B., S. M. Johnson, and W. B. White, "A Linear Programming Approach to the Chemical Equilibrium Problem," Mgmt. Sci., Vol. 5, No. 1, October 1958, pp. 38-43.
3. Dantzig, G. B., and J. C. DeHaven, "On the Reduction of Certain Multiplicative Chemical Equilibrium Systems to Mathematically Equivalent Additive Systems," J. Chem. Phys., Vol. 36, No. 10, May 15, 1962, pp. 2620-2627 (also: The RAND Corporation, P-2419).
4. Shapiro, N. Z., Analysis by Migration in the Presence of Chemical Reaction, The RAND Corporation, P-2596, June 1962.
5. Clasen, R. J., The Linear-Logarithmic Programming Problem, The RAND Corporation, RM-3707-PR, June 1963.
6. Shapiro, N. Z., Conditions for a Homogeneous Mixture to be Ideal, The RAND Corporation, RM-3677-PR, June 1963.
7. -----, On the Behavior of a Chemical Equilibrium System When Its Free Energy Parameters Are Changed, The RAND Corporation, RM-4128-PR, May 1964.
8. Shapiro, N. Z., and L. S. Shapley, "Mass Action Laws and the Gibbs Free Energy Function," J. Soc. Indust. Appl. Math., Vol. 13, No. 2, June 1965, pp. 353-375 (also: The RAND Corporation, RM-3935-1-PR).
9. Shapiro, N. Z., A Generalized Technique for Eliminating Species in Complex Chemical Equilibrium Calculations, The RAND Corporation, RM-4205-PR, September 1964.
10. Clasen, R. J., The Numerical Solution of the Chemical Equilibrium Problem, The RAND Corporation, RM-4345-PR, January 1965.
11. Shapiro, N. Z., Variations in the Parameters of a Chemical Equilibrium System, The RAND Corporation, (work in progress).

12. Clasen, R. J., The Fitting of Data by Least Squares to Non-Linearly Parameterized Functions, The RAND Corporation, P-3252, October 1965.
13. Dantzig, G. B., J. H. Folkman, and N. Z. Shapiro, On the Continuity of the Minimum Set of a Continuous Function, The RAND Corporation, RM-4657-PR, February 1966.
14. Dantzig, G. B., J. C. DeHaven, and I. Cooper, et al., "A Mathematical Model of the Human Respiratory System," Perspect. Biol. Med., Vol. 4, No. 3, 1961, pp. 324-376 (also: The RAND Corporation, RM-2519-PR).
15. DeHaven, J. C., E. C. DeLand, N. S. Assali, and W. Manson, "Physicochemical Characteristics of Placental Transfer," San Diego Symposium for Biomedical Engineering, Vol. 2, La Jolla, California, 1962, pp. 217-224 (also: The RAND Corporation, P-2565).
16. DeHaven, J. C., and E. C. DeLand, The Reactions of Hemoglobin and Steady States in the Human Respiratory System: An Investigation Using Mathematical Models and an Electronic Computer, The RAND Corporation, RM-3212-PR, December 1962.
17. Maloney, J., Jr., et al., "Analysis of Chemical Constituents of Blood by Digital Computer," Surgery, Vol. 54, 1963, p. 158 (also: The RAND Corporation, RM-3541-PR).
18. Bradham, G. B., et al., "Laboratory and Digital Computer Study of PH of Blood at Hypothermic Levels," San Diego Symposium for Biomedical Engineering, August 1964.
19. Bradham, G. B., et al., "Isotope Dilution and Thermodynamics in the Study of Intercompartmental Body Fluid Exchange," Surgery Gynec. Obstet., Vol. 119, No. 5, November 1964, pp. 1062-1068.
20. DeLand, E. C., and G. B. Bradham, Fluid Balance and Electrolyte Distribution in the Human Body, The RAND Corporation, RM-4347-PR, January 1965.
21. DeHaven, J. C., and N. Z. Shapiro, Intrinsic Control of Body Fluid and Electrolyte Distribution and Urine Formation, The RAND Corporation, RM-4609-PR, July 1965.

22. Hamming, R. W., Numerical Methods for Scientists and Engineers, McGraw-Hill Book Company, Inc., San Francisco, 1962.
23. Dantzig, G. B., Linear Programming and Extensions, Princeton University Press, Princeton, New Jersey, 1963 (also: The RAND Corporation, R-366-PR).
24. Rossini, F. D., et al., Selected Values of Physical and Thermodynamic Properties of Hydrocarbons and Related Compounds, American Petroleum Institute Research Project 44, Carnegie Press, Pittsburgh, Pennsylvania, 1953.