Boundary Element Methods – An Overview *

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Abstract

Variational methods for boundary integral equations deal with the weak formulations of boundary integral equations. Their numerical discretizations are known as the boundary element methods. This paper gives an overview of the method from both theoretical and numerical point of view. It summaries the main results obtained by the author and his collaborators over the last 30 years. Fundamental theory and various applications will be illustrated through simple examples. Some numerical experiments in elasticity as well as in fluid mechanics will be included to demonstrate the efficiency of the methods.

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1 Introduction

Variational methods for boundary integral equations deal with the weak formulations of boundary integral equations. Their numerical discretizations are generally known as the boundary element methods (BEMs). As the classical integral equation method for numerical solutions to elliptic boundary value problems, central to the BEM is the reduction of boundary value problems to the equivalent integral equations on the boundary. This boundary reduction has the advantage of diminishing the number of space dimension by one and of the capability to handle problems involving infinite domains. The former leads to an appreciable reduction in the number of algebraic equations generated for the solutions, as well as much simplified data representation. On the other hand, it is well known that elliptic boundary value problems may have equivalent formulations in various forms of boundary integral equations. This provides a great variety of versions for BEMs. However, irrespective of the variants of the BEMs and the particular numerical implementation chosen, there is a common mathematical framework into which all these BEMs may be incorporated. This paper addresses to the fundamental issues of this common mathematical framework and is devoted to the mathematical foundation underlying the BEM techniques.

Specifically, this paper will give an expository introduction to the Galerkin-BEM for elliptic boundary value problems from the mathematical point of view. Emphases will be placed upon the variational formulations of the boundary integral equations and the general error estimates for the approximate solutions in appropriate Sobolev spaces. A classification of boundary integral equations will be given based on the Sobolev index. The simple relations between the variational formulations of the boundary integral equations and the corresponding partial differential equations under consideration will be indicated. Basic concepts such as stability, consistency, convergence as well as the condition numbers and ill-posedness will be discussed. Main results obtained by the author and his collaborators over the last 30 years will be summarized. Some numerical experiments will be included to illustrate the fundamental ideas.

BEMs may be considered as application of finite element methods (FEMs) to the boundary integral equations (BIEs) on boundary manifolds. The terminology of BEM originated from the practice of discretizing the boundary manifold of the solution domain for the BIE into boundary elements, resembling the term of finite elements in FEM. As in FEM, the use of the termi-

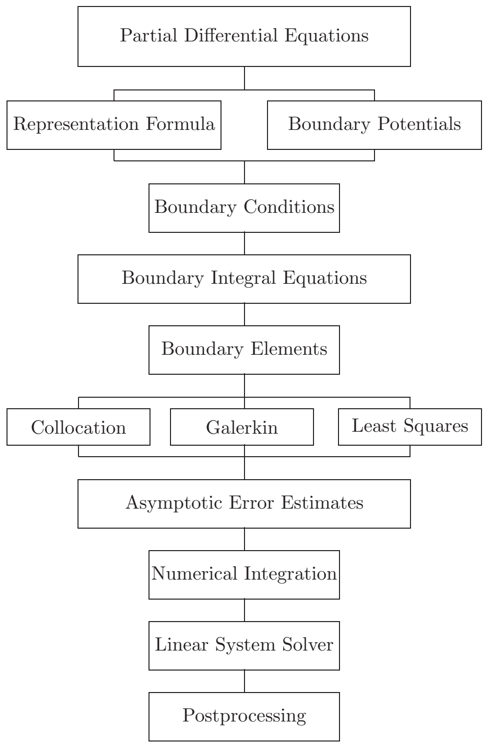

Figure 1: A Flow Chart for Boundary Element Methods

nology boundary elements in two different contexts; the boundary manifolds are decomposed into boundary elements which are geometric objects, while the boundary elements for approximating solutions of BIEs are actually the finite element functions defined on the boundaries. In fact, the term BEM, nowadays denotes any efficient method for the approximate numerical solution of BIEs. Figure 1 is a sketch of the general procedure for approximating the solutions of a boundary value problem via the BEMs. As mentioned earlier, we will only concentrate on the Galerkin-BEMs. For the collocation and least-squares BEMs, we refer to the fundamental papers [26] and [1].

2 An Historic Development

In a celebrated paper [6] by Fichera, solutions of the Dirichlet problems for a large class of elliptic equations of higher order with variable coefficients in the plane were obtained by means of the potential of a simple layer. This procedure, which we termed in [12] the *method of Fichera*, leads to singular integral equations of the first kind. In contrast to the standard *Fredholm method*, solutions of the Dirichlet problems are generally treated by means of the potential of a double layer. The later leads to integral equations of the second kind. Our work 30 years ago was motivated by Fichera's method and it has been a very rewarding experience. Needless to say, we certainly did not expect that the simple idea in [15] has turned out to be of most fruitful in the development of fundamental concepts in the BEMs.

To illustrate the basic idea, we now consider a simple model problem in $\mathbb{R}^2$, the Dirichlet problem for the Laplacian. Let Γ be a smooth closed curve in the plane and let Ω and Ω^c denote its interior and exterior respectively. We consider the boundary value problem:

$$\begin{aligned} -\Delta u &= 0 \quad \text{in} \quad \Omega \quad (\text{or } \Omega^c) && (1)\\ u|_\Gamma &= f \quad \text{on} \quad \Gamma, && (2) \end{aligned}$$

where f is the given data. In the case of the exterior region Ω^c, we append to (1) and (2) the condition at infinity in the form:

$$u = a\, log|x| + O(1), \tag{3}$$

where a is a given constant. The method of Fichera is to seek a solution of the boundary value problem in the form of a simple layer potential

$$u(x) = \int_\Gamma E(x,y)\sigma(y)\, ds_y\ , \quad x \in \mathbb{R}^2 \setminus \Gamma,$$

where σ is the unknown density to be determined and $E(x,y)$ is the fundamental solution for $-\Delta$,

$$E(x,y) = -\frac{1}{2\pi}\, log\, |x-y|.$$

From the boundary condition (2), we then obtain the BIE

$$V\sigma := \int_\Gamma E(x,y)\sigma(y)\, ds_y = f \quad \text{on} \quad \Gamma, \tag{4}$$

a BIE of the first kind with a weakly singular kernel. Differentiating (4) with respect to the arc length yields

$$\int_\Gamma \frac{\partial E}{\partial s_x}(x,y)\sigma(y)\, ds_y = \frac{\partial}{\partial s_x} f(x) \quad \text{on} \quad \Gamma,$$

which is a singular integral equation with a Cauchy kernel. This singular equation forms the the theoretical basis of the Fichera's method in [6]. We remark that not only are Cauchy kernels hard to handle numerically but they produce certain non-uniqueness which must be accounted for. In [12], we modified the Fichera's approach by seeking a solution in the form

$$u(x) = \int_\Gamma E(x,y)\sigma(y) ds_y + \omega\ , \quad x \in \mathbb{R}^2 \backslash \Gamma \tag{5}$$

with an adding unknown constant ω and consider the modified system

$$\begin{aligned} V\sigma + \omega = f \quad &\text{on} \quad \Gamma \\ \int_\Gamma \sigma\ ds = \Lambda&, \end{aligned} \tag{6}$$

where Λ is a given constant which is equal to $-2\,\pi\, a$ for the exterior problem under condition (3). However, for the interior problem it can be chosen arbitrarily and in particular it can be chosen to be zero. We remark that in this way not only the Cauchy kernel has been eliminated and replaced with a logarithmic one but at the same time the non-uniqueness will be automatically accounted for. Indeed, the following existence and uniqueness theorem of solution in the classical Höder function space $C^{0,\lambda}(\Gamma)$ has been established in [12].

Theorem 2.1 *Given* $(f, \Lambda) \in C^{1,\lambda}(\Gamma) \times \mathbb{R}$*, the system* (6) *has a unique solution pairs* $(\sigma, \omega) \in C^{0,\lambda}(\Gamma) \times \mathbb{R}$*.*

The modified Fichera method has been also extended in [12] to a special class of boundary value problems for the equations of the form

$$\Delta^m u - s\Delta^{m-1} u = 0 \quad \text{in} \quad \Omega \text{ (or } \Omega^c), m = 1 \text{ or } 2.$$

The constant s is given and can be complex. It is in [15], we began our investigation on the weak solutions of the corresponding systems of integral equations for this special class of boundary value problems. In particular, the following crucial result for the simple-layer boundary integral operator V defined by (4) was presented for the first time.

Theorem 2.2 *Under the assumption*

$$\max_{x,\, y \,\in\, \Gamma} |x - y| < 1, \tag{7}$$

the integral operator V satisfies the inequalities

$$\gamma_1 \; ||\sigma||^2_{-1/2} \leqslant \gamma_2 \; ||V\sigma||^2_{1/2} \leqslant \; \langle \sigma, V\sigma \rangle \; \leqslant \gamma_3 \; ||\sigma||^2_{-1/2} \tag{8}$$

for $\sigma \in H^{-1/2}(\Gamma)$, where γ_i's are constants.

Here $\langle \cdot, \cdot \rangle$ denotes the L_2- duality pairing between the standard Sobolev spaces $H^{1/2}(\Gamma)$ and its dual $H^{-1/2}(\Gamma)$, which is the *energy space* of the boundary integral operator V. These inequalities in (8) provide us all the essential properties for the weak solutions of (4). Similar to partial differential equations, the concept of a weak solution of BIEs may be introduced by multiplying the boundary equation (4) by a test function and integrating over Γ leads to a variational form with one difference, that is, in most of the cases, we may not be able to form the integration by parts as in the case of partial differential equations. For the present concrete model, if the given data f is in $H^{1/2}(\Gamma)$, then the unknown function $\sigma \in H^{-1/2}(\Gamma)$ is said to be a *weak solution* of the boundary integral equation (4), provided it satisfies the variational form

$$\langle \chi, V\sigma \rangle \quad = \quad \langle \chi, f \rangle \quad \forall\, \chi \in H^{-1/2}(\Gamma). \tag{9}$$

The existence and uniqueness of the weak solution of (9) follows from the well-known Lax-Milgram Lemma, since V is $H^{-1/2}(\Gamma) -$ *elliptic* from (8) under the assumption of (7).

The assumption of (7) is of course rather restricted at a first glance. However, we may rewrite V in the form

$$\begin{aligned} V\sigma(x) \;\; &:= \;\; -\frac{1}{2\,\pi} \int_\Gamma log|x-y| \; \sigma(y) ds_y \\ &= \;\; -\frac{1}{2\,\pi} \int_\Gamma log \left(\frac{|x-y|}{2d} \right) \sigma(y) ds_y \\ &- \;\; c \int_\Gamma \sigma(y) ds_y \end{aligned}$$

with $c = \frac{1}{2\pi}\, log(2d)$ and $d = \max_{x,\, y \in \Gamma} |x - y|$. This shows that for the general Γ without the assumption (7), V satisfies a Gårding inequality in the form

$$\langle \sigma, V\sigma \rangle \geqq c_0 \; ||\sigma||^2_{-1/2} - c_1 \; ||\sigma||^2_{-1/2-\epsilon} \tag{10}$$

for all $\sigma \in H^{-1/2}(\Gamma)$; $\epsilon > 0$, a constant. As a consequence, the following corresponding result for the modified system (6) has been established in [15]:

Theorem 2.3 . *Given $(f, \Lambda) \in H^{1/2}(\Gamma) \times \mathbb{R}$, there exists a unique solution pair $(\sigma, \omega) \in H^{-1/2}(\Gamma) \times \mathbb{R}$ of the system*

$$\begin{aligned} \langle \chi, V\sigma \rangle + \omega \langle \chi, 1 \rangle &= \langle \chi, f \rangle, \\ \langle \sigma, 1 \rangle \kappa &= \Lambda \kappa \end{aligned}$$

for all $(\chi, \kappa) \in H^{-1/2}(\Gamma) \times \mathbb{R}$.

Here the bilinear form (or sesquilinear form rather) of the modified system satisfies the Gårding inequality:

$$\left\langle \begin{pmatrix} \chi \\ \kappa \end{pmatrix}, A \begin{pmatrix} \chi \\ \kappa \end{pmatrix} \right\rangle \geq c_0 \left\{ ||\chi||^2_{H^{-1/2}(\Gamma)} + |\kappa|^2 \right\} - c_1 \left\{ ||\chi||^2_{H^{-1/2-\epsilon}(\Gamma)} + |\kappa|^2 \right\},$$

where A is the matrix of operators defined by $A := \begin{pmatrix} V & 1 \\ \langle \cdot, 1 \rangle & 0 \end{pmatrix}$.

It is worthy noting that there is an intimated relation between the Gårding inequality (10) for the boundary integral operator V and that of the bilinear form associated with a related transmission problem for the corresponding partial differential operator P ($= -\Delta$ in the present case). In fact this simple relation for the $-\Delta$ has been systematically extended in [3] to a general class of boundary integral operators associated with strongly elliptic boundary value problems. For such class of boundary integral operators, Gårding's inequality is a consequence of strong ellipticity of the corresponding boundary value problem for the partial differential equation. In the present special case, $P := -\Delta$, the transmission problem then reads to find a function $v \in H^1(\Omega, P) \bigcap H^1_{loc}(\Omega^c, P)$ satisfying

$$Pv = 0 \quad \text{in} \quad \mathbb{R}^2 \setminus \Gamma$$

together with the transmission conditions

$$[\gamma_0 v]_\Gamma = 0, \quad \text{and} \quad [\gamma_1 v]_\Gamma = \sigma \in H^{-1/2}(\Gamma),$$

where we have adapted the notation $[\cdot]_\Gamma$ for the jump of traces of the function v across the boundary Γ. For any $\sigma \in H^{-1/2}(\Gamma)$, the solution can then be represented in the form a simple-layer potential

$$v(x) := \int_\Gamma E(x,y)\sigma(y)ds_y, \quad x \in \mathbb{R}^2 \setminus \Gamma.$$

From the generalized Green's formula, it follows that

$$\langle \sigma, V\sigma \rangle + \int_{\Omega^c} (P\tilde{v})\tilde{v}dx \quad = \quad a_\Omega(v,v) + a_{\Omega^c}(\tilde{v},\tilde{v}), \tag{11}$$

which relates the boundary bilinear form $\langle \cdot, \cdot \rangle$ for V on Γ to the domain bilinear forms $a_\Omega(\cdot,\cdot)$ and $a_{\Omega^c}(\cdot,\cdot)$ for P over Ω and Ω^c, respectively. Here a cut-off function ϕ has been employed in the neighborhood of the boundary Γ such that $\tilde{v} := \phi v$ in order to ensure the existence of the quadratic form over the exterior domain Ω^c, without introducing the weighted Sobolev spaces as in the French school [23],[24] and [20]. It is this simple relation (11) which connects Gårding's inequalities for the partial differential operators to those for the associated boundary integral operators (see [15] and [3]) for the details).

3 Mathematical Foundation

As is well known, Gårding's inequality plays a fundamental role not only for the existence of the variational solutions to the BIEs but also for error estimates of the Galerkin-BEMs. The basic approach presented in [15] has led to the development of fundamental results for the boundary element analysis and has laid down some of the mathematical foundations for the BEMs. In this section we collect the basic mathematical ingredients for the method, and summarize some of the fundamental results obtained by the author together with his collaborators over the last 30 years. The presentation here follows the recent book chapter [18] and details of proofs can be found in [18] as well as in the forthcoming book monograph [19].

We begin with a general boundary integral equation of the form

$$A\sigma = f \quad \text{on} \quad \Gamma. \tag{12}$$

Here $f \in H^{s-\alpha}(\Gamma)$, $s \in \mathbb{R}$, is the given data, and 2α is a fixed constant. (It is assumed that the boundary manifold Γ is sufficiently smooth for the corresponding s and α to be specified.) We first define the *order* of the boundary integral operator A; we say that the order of the boundary integral equation operator A is 2α if the mapping

$$A : H^{s+\alpha}(\Gamma) \mapsto H^{s-\alpha}(\Gamma)$$

for any $s \in \mathbb{R}$ with $|s| \leq s_0$ is continuous where s_0 is some fixed positive constant. We now classify the boundary integral equation (12) according to the order of A. The boundary integral equation (12) is said to be a *first kind boundary integral equation* if the order of A is negative (i.e., $2\alpha < 0$). In case the order is zero ($2\alpha = 0$), and the operator A is of the form $aI + K$, where K is either a Cauchy-singular integral operator or K is compact and $a \neq 0$. The latter defines a *Fredholm integral equation of the second kind* while the former defines a *Cauchy singular integral equation*. In case the order is positive ($2\alpha > 0$), and $A = L + K$, where L is a differential operator and K a possibly hypersingular integral operator. Then (12) is an integro-differential equation, if the order of L is equal to 2α while it is called a *hypersingular integral equation*, if the order of L is less than 2α.

In the example for the Laplace equation (1), there are four basic boundary integral operators, namely

$$\begin{aligned}
V\sigma(x) &:= \int_\Gamma E(x,y)\sigma(y)ds_y \quad \text{(simple-layer integral operator)} \\
K\mu(x) &:= \int_\Gamma \frac{\partial}{\partial n_y} E(x,y)\mu(y)ds_y \quad \text{(double-layer integral operator)} \\
K'\sigma(x) &:= \int_\Gamma \frac{\partial}{\partial n_x} E(x,y)\sigma(y)ds_y \quad \text{(the transpose of } K\text{)} \\
D\mu(x) &:= -\frac{\partial}{\partial n_x}\int_\Gamma \frac{\partial}{\partial n_y} E(x,y)\mu(y)ds_y \quad \text{(hypersingular integral operator)}
\end{aligned}$$

Hence according to the above classification, for solving interior and exterior Dirichlet and Neumann problems, we may arrive at both first and second kind boundary integral equations (12). Here the operator A is defined in terms of the four basic boundary integral operators

$$\begin{aligned}
2\alpha &= -1, \quad A = V \\
2\alpha &= 0, \quad A = \frac{1}{2}I \pm K \quad , \quad A = \frac{1}{2}I \mp K' \\
2\alpha &= +1, \quad A = D.
\end{aligned}$$

Here I stands for the identity operator.

Weak formulations for the boundary integral equations are generally different for the first and second kind equations. In the former, the boundary sesquilinear forms are connected with domain sesquilinear forms for the partial differential equations in the interior as well as in the exterior domain, while in the latter, it connects only with the sesquilinear form either for the interior or for the exterior domain, but not both, depending on the direct or indirect approach. For the second kind boundary integral equations, a *premultiplied operator* as in [7] is needed in order to give the appropriate duality pairing in the variational formulations for the boundary integral equations. As we have seen from the model problems, for the boundary integral equation (12) whose sesquilinear form coincides with the variational sesquilinear form of the boundary value problem, the *strong ellipticity* of boundary integral operators introduced in [26], in the form of Gårding inequalities for the corresponding boundary integral operators in the trace space on the boundary manifold, will be a consequence of *strong ellipticity* of the original boundary value problems (see [3]).

To formulate the Galerkin-BEM for the equation (12), let $\mathcal{H} = H^{\alpha}(\Gamma)$ denote the solution space and $\mathcal{H}_h \subset \mathcal{H}$ be a one-parameter family of finite-dimensional subspaces of $\mathcal{H}$. Then given $f = A\sigma \in \mathcal{H}'$ (with $\sigma \in \mathcal{H}$), we may formulate the Galerkin method as to find an element $\sigma_h \in \mathcal{H}_h$ satisfying the Galerkin equation

$$a_{\Gamma}(\sigma_h, \chi_h) := \langle A\sigma_h, \chi_h \rangle = \langle A\sigma, \chi_h \rangle \tag{13}$$

for all $\chi_h \in \mathcal{H}_h$. For the convergence of the Galerkin solutions, we need the basic concepts of *consistency, stability* and *convergence* as in standard numerical approximating schemes. The well-known general principle known as the Lax equivalence theorem states that

$$\textit{consistency} + \textit{stability} \Rightarrow \textit{convergence}$$

which applies to the BEMs without any exception. In fact, Céa's lemma for the BEM below is a classical convergence theorem based on the complementary concepts of consistency and stability. To be more definite, let us state in the following the definitions of consistency and stability for the Galerkin-BEM (13) of the BIE (12).

(I) *Consistency*: Let $A_h : \mathcal{H}_h \subset \mathcal{H} \to \mathcal{H}' \subset \mathcal{H}_h'$ be a family of continuous mappings approximating the operator A. The operators A_h is said to be consistent with A if for every $\upsilon \in \mathcal{H}$ there holds

$$\lim_{h\to 0} ||A_h P_h \upsilon - P_h^* A\upsilon||_{\mathcal{H}'_h} = 0,$$

where P_h is the projection and P_h^* its dual.

(II) *A-prior bound*: For $0 < h < h_0$, there exists a constant $c_0 = c_0(h_0)$ independent of σ and h such that

$$||\sigma_h||_{\mathcal{H}} \leq c_0 ||\sigma||_{\mathcal{H}}.$$

In addition, we need some kind of approximation property for the family of the finite-dimensional subspaces $\mathcal{H}_h$ of $\mathcal{H}$, namely,

(III) *Ap property*: The family of the finite-dimensional subspaces $\mathcal{H}_h$ of $\mathcal{H}$ is said to have the Ap property, if for every $\upsilon \in \mathcal{H}$, there exists a sequence $\upsilon_h \in \mathcal{H}_h \subset \mathcal{H}$ such that

$$||\upsilon - \upsilon_h||_{\mathcal{H}} \to 0 \text{ as } h \to 0^+ .$$

We remark that for the Galerkin-BEM (13), consistency condition (I) is a consequence of the Ap property (III) of the approximate sequences and that (II) is a stability condition for the family of approximate solutions. From condition (II), we see that if $\sigma = 0$, then $\sigma_h = 0$. This means that the corresponding homogeneous equation

$$\langle A\sigma_h, \chi_h \rangle = 0 \quad \text{for all} \quad \chi_h \in \mathcal{H}_h \tag{14}$$

has only the trivial solution. Since (14) is equivalent to a quadratic system of linear equations in terms of a basis of $\mathcal{H}_h$, this implies the unique solvability of the inhomogeneous equation (13) for every h with $0 < h \leq h_0$. Condition (II) also implies that there is a mapping

$$\mathcal{G}_h : \mathcal{H} \ni \sigma \longmapsto \sigma_h \in \mathcal{H}_h \subset \mathcal{H}$$

such that $\mathcal{G}_h$ is uniformly bounded, that is,

$$|||\mathcal{G}_h||| \leq c_0 \tag{15}$$

independent of h. Moreover, we see that $\mathcal{G}_h^2 \sigma = \mathcal{G}_h \sigma_h = \sigma_h = \mathcal{G}_h \sigma$, the second equality following from the unique solvability of (13). Hence $\mathcal{G}_h$ is a projection, the so-called Galerkin projection.

Now from (15), we see that

$$A_h^{-1} := \mathcal{G}_h(P_h^* A)^{-1}$$

is uniformly bounded, provided A^{-1} is bounded. Consequently, with the AP property (III),

$$\|\sigma-\sigma_h\|_{\mathcal{H}} \leq c\|A_h P_h\sigma - A_h P_h \sigma_h\|_{\mathcal{H}'_h} = c\|A_h P_h \sigma - P_h^* A\sigma\|_{\mathcal{H}'_h} \to 0 \quad \text{as} \quad h \to 0^+$$

as expected under Condition (I). Hence as usual, the stability condition (II) plays a fundamental role in the abstract error estimates.

The stability condition (II) for the Galerkin method can be replaced by the well-known *Ladyzenskaya-Babuška-Brezzi condition* (BBL-condition), also called *inf-sup condition*, a condition which plays a fundamental role in the study of elliptic boundary-value problems with constraints as well as in the analysis of convergence and stability of FEMs and is most familiar to the researchers in the FEM analysis (see [22] and [2]).

We recall that a sesquilinear form $B(\cdot,\cdot) : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathbb{C}$ on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ is said to satisfy the BBL-condition or *inf-sup condition* if there exists a constant $\gamma_0 > 0$ such that

$$\inf_{0\neq u\in\mathcal{H}_1} \sup_{0\neq v\in\mathcal{H}_2} \frac{|B(u,v)|}{\|u\|_{\mathcal{H}_1}\|v\|_{\mathcal{H}_2}} \geq \gamma_0 \,.$$

For our purpose, we consider the special discrete form of the BBL-condition with both $\mathcal{H}_1$ and $\mathcal{H}_2$ replaced by $\mathcal{H}$ and the sesquilinear $B(\cdot,\cdot)$ form by the boundary sesquilinear form $a_\Gamma(\cdot,\cdot)$. That is, it is based on the definition

(IV) *The BBL-condition*: There exists a constant $\gamma_0 > 0$ such that

$$\sup_{0\neq\chi_h\in\mathcal{H}_h} \frac{|a_\Gamma(v_h,\chi_h)|}{||\chi_h||_{\mathcal{H}}} \;\geq\; \gamma_0\, ||v_h||_{\mathcal{H}} \quad \forall \quad v_h \in \mathcal{H}_h.$$

As in case of the FEM, under the BBI-condition, the following Céa's lemma can be establided.

Theorem 3.1 *If the BBL-condition* (IV) *holds, then the Galerkin equations* (13) *are uniquely solvable for each* $\sigma \in \mathcal{H}$, *and we have the quasi-optimal error estimate*

$$||\sigma - \sigma_h||_{\mathcal{H}} \leq c \inf_{\chi_h\in\mathcal{H}_h} ||\sigma - \chi_h||_{\mathcal{H}},$$

where the constant c *is independent of* σ *and* h.

As in the case of partial differential equations, this simple, yet crucial, estimate in the Céa's lemma shows that the problem of estimating the error between the solution σ and its Galerkin approximations σ_h is reduced to a problem in the approximation theory.

Our final theorem in this section draws the relations between the BBL-condition (I) and the Gårding inequality, namely

Gårding's inequality + uniqueness + Ap property $\Rightarrow$ *BBL-condition*

To this end we need the definition of the Gårding inequality for the boundary integral operator A of (12) in the form:

(V) *The Gårding inequality*: The boundary integral operator A is said to satisfy a Gårding inequality, if there exist a compact operator $C : \mathcal{H} \to \mathcal{H}'$ (the dual of $\mathcal{H}$) and positive constant γ such that the inequality

$$Re\Big\{a_\Gamma(\upsilon, \upsilon) + \langle C\upsilon, \upsilon\rangle_\Gamma\Big\} \geq \gamma||\upsilon||^2_{\mathcal{H}}$$

holds for all $\upsilon \in \mathcal{H}$.

Theorem 3.2 *Suppose that the boundary sesquilinear form $a_\Gamma(\cdot,\cdot)$ satisfies Gårding's inequality and*

$$Ker(a_\Gamma) := \{\sigma_0 \in \mathcal{H} \mid a_\Gamma(\sigma_0, \chi) = 0 \quad \forall \chi \in \mathcal{H}\} = \{0\}.$$

Then a_Γ satisfies the BBL-condition, provided $\mathcal{H}_h$ satisfies the Ap property.

A proof of this theorem is available in [28] and [18].

4 Error Estimates and Ill-posedness

Céa's lemma can be used for obtaining an optimal order of convergence of the Galerkin solutions in the energy norm if further approximation property of the approximate subspaces are provided. If in addition the approximate subspaces $\mathcal{H}_h$ possesses an inverse property below, one may obtain convergence results in stronger norms. On the other hand, the Aubin-Nitsche lemma (see, [16]) is used for determining the accuracy in weaker norms.

In the following, we collect some general results concerning the error estimates for the approximate solutions of (12) by the Galerkin-BEM. In what follows, we assume the following assumptions hold:

(A1) The boundary integral operator

$$A : H^{s+\alpha}(\Gamma) \mapsto H^{s-\alpha}(\Gamma)$$

is a continuous isomorphism for any $s \in R$ with $|s| \leq s_0$.

(A2) The operator A satisfies a Gårding's inequality with $\mathcal{H} = H^{\alpha}(\Gamma)$ being the energy space for the operator A.

(A3) Let $\mathcal{H}_h = S_h^{\ell,m} \subset \mathcal{H}$ with $\ell, m \in \mathbb{N}_0$ and $m \leq \ell - 1$, a regular boundary element space in the sense of [2], that is, it has the properties:

(i) *Approximation property*: Let $t \leq s \leq \ell$ and $t < m + \frac{1}{2}$ for $n = 2$ or $t \leq m$ for $n = 3$. Then there exists a constant c such that for any $\upsilon \in H^s(\Gamma)$, a sequence $\chi_h \in S_h^{\ell,m}$ exists and satisfies the estimate

$$||\upsilon - \chi_h||_{H^t(\Gamma)} \leq ch^{s-t}||\upsilon||_{H^s(\Gamma)}.$$

(ii) *Inverse property*: For $t \leq s < m + \frac{1}{2}$ for $n = 2$ or $t \leq s \leq m$ for $n = 3$, there exists a constant M such that for all $\chi_h \in S_h^{\ell,m}$,

$$||\chi_h||_{H^s(\Gamma)} \leq Mh^{t-s}||\chi_h||_{H^t(\Gamma)}.$$

The following results have been established in [15],[16], [17] and [8].

Theorem 4.1 (Asymptotic error estimates) *Under the above assumptions, let $m > \alpha - 1/2$ for $n = 2$ or $m \geq \alpha$ for $n = 3$, and $s_0 \geq max\{\ell, |2\alpha - \ell|\}$. Then we have the asymptotic error estimate of optimal order*

$$||\sigma - \sigma_h||_{H^t(\Gamma)} \leq ch^{s-t}||\sigma||_{H^s(\Gamma)}$$

for $2\,\alpha - \ell \leq t \leq s \leq \ell, t \leq m + 1/2$ for $n = 2$ or $t \leq m$ for $n = 3$, and $\alpha \leq s$. Moreover, the condition number of the Galerkin equation (13) *is of $O(h^{-2|\alpha|})$.*

Theorem 4.2 (Ill-posedness) *If the datum f is replaced by its L_2-perturbation f_ϵ, and σ_h^ϵ is the corresponding Galerkin solution, then for $\alpha < 0$, we have the modified error estimate*

$$||\sigma - \sigma_h^\epsilon||_{H^t(\Gamma)} \leq c\{h^{s-t}||\sigma||_{H^s(\Gamma)} + h^{-(t+|2\alpha|)}||f - f_\epsilon||_{H^0(\Gamma)}\}.$$

Consequently if $||f - f_\epsilon||_{H^0(\Gamma)} \leq \epsilon$, *then the choice of* h *given by*

$$h_{opt} = \epsilon^\mu \quad with \quad \mu := \frac{1}{s + |2\alpha|},$$

yields an optimal rate of convergence:

$$||\sigma - \sigma_h^\epsilon||_{H^t(\Gamma)} = O(\epsilon^{(s+t)/(s+|2\alpha|)}) \quad as \quad \epsilon \to 0^+.$$

We note that in Theorem 4.2, of particular interest is the L_2-estimate, when $t = 2$. In this case, our result coincides with those obtained by the Tikhonov regularization method [27] and [21].

In closing this section, we now include some numerical experiments by the author and his collaborators in order to illustrate the applicability of the BEMs from computational point of view. We present the examples in chronological order which in a way it may also indicate the progress of the development of BEMs over the past 30 years.

We begin with a typical exterior boundary value model problem which can be solved by using the BEM. Here the exterior Dirichlet problem for the biharmonic equation is modelled for the viscous flow past an obstacle. The boundary value problem is reduced to system of integral equations of the first kind by the modified Fichera method. In Figure 2, the streamlines of the flow past an ellipse is plotted from the Stokes expansion up to including $O(logRe)^{-1}$ for the Reynolds number $Re = 0.0025$. In Figure 3, we plot the absolute errors of the unknown constant ω (in the modified Fichera method) against the number $(N+1)$ of points for various eccentricities ϵ of the ellipse. Details are given in [10] and [11], which summaries our early work for the period from 1980 to 1985.

The asymptotic error estimates in Theorem 4.1 shows that the condition number of the boundary integral operators is of $O(h^{-2\alpha})$, where 2α denotes the order of the boundary integral operator. The simple-layer boundary integral operator V (for the Laplace equation as well as for the Helmholtz equation) is a continuous operator from $H^{-1/2}(\Gamma)$ into $H^{1/2}(\Gamma)$ and has the order -1; as a consequence, its condition number will be unbounded. For iterative schemes, a good preconditioner must be employed in order to speed-up the convergence. To find a good preconditioner, it has been one of the most active research topics in recent years. Figure 4 summaries our study on various effects of the condition numbers of the simple-layer operator V for the two-dimensional Helmholtz equation for the period from 1995 to 2001.

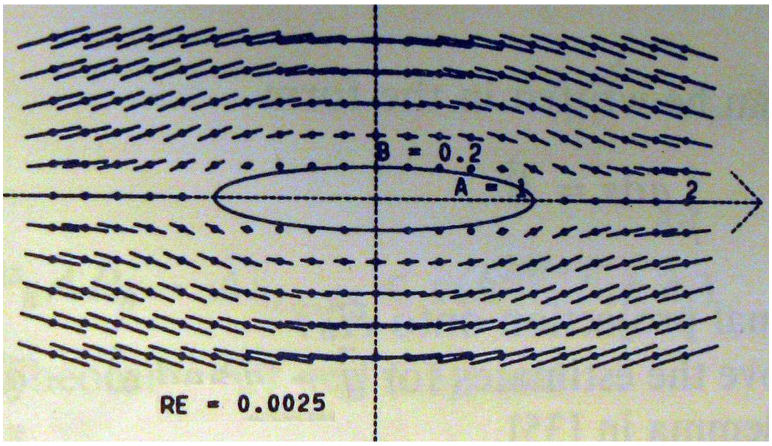

Figure 2: Viscous flow past an obstacle

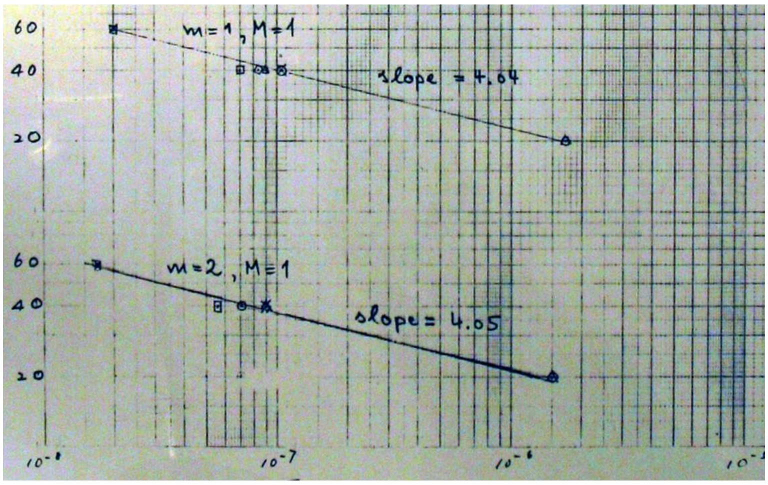

Figure 3: Absolute error estimates

For interested readers, we refer the details to our paper in the special issue of *Advances in Computational Mathematics* [4].

Hybrid methods based on the coupling procedure of boundary element and finite element methods have been proven to be one of the most popular and efficient methods in applied mechanics and engineering. Figure5 shows a plate under uniform symmetric tension in the vertical direction. In the center of the plate, an elliptic cutout is located as the notch configuration. Two macro-elements are placed in the center near the elliptic cutout. A BEM is used in the macro-elements with fine grids on the boundaries of the macro-elements, while a global FEM is employed outside the macro-elements with a coarse grid. The mesh points of the macro-elements can be chosen independently of the nodes of the finite element structure so that various independent meshes can be easily connected via *mortar-like* elements on the skeleton. Our method here can also serve as a basic algorithm for coupled preconditioned iterative solution schemes in domain decomposition. Figure5 gives a visualization of normal stress distribution in the direction of the loading for the whole plate. In the far field of the notch we have a constant stress field, while the high stress gradients in the near field of the notch are very accurately approximated within each of the macro-elements by using the BEM. This project took more or less ten years beginning in the early 1990 and ended in 2000. Details of the numerical procedures and theoretical analysis are summarized in [13] and [14].

Figure 6 contains the bistatic radar cross section plots in 3D electromagnetic scattering. The solutions of the Maxwell equations are obtained by solving the well-known magnetic field integral equation (MFIE) for the surface current; MFIE is an integral equation of the second kind with a weakly singular kernel. An efficient numerical algorithm is developed based on the collocation scheme. We approximate the unit sphere by triangular patches. These triangular patches are generated by iterations. Each triangular patch is then divided into four smaller triangles by connecting the midpoints of each sides. The results are in excellent agreement with theoretical results based on the Mie series. This represents part of our activities in electromagnetic scattering for the period from 1995 to 2001. Details are available in the publication [9].

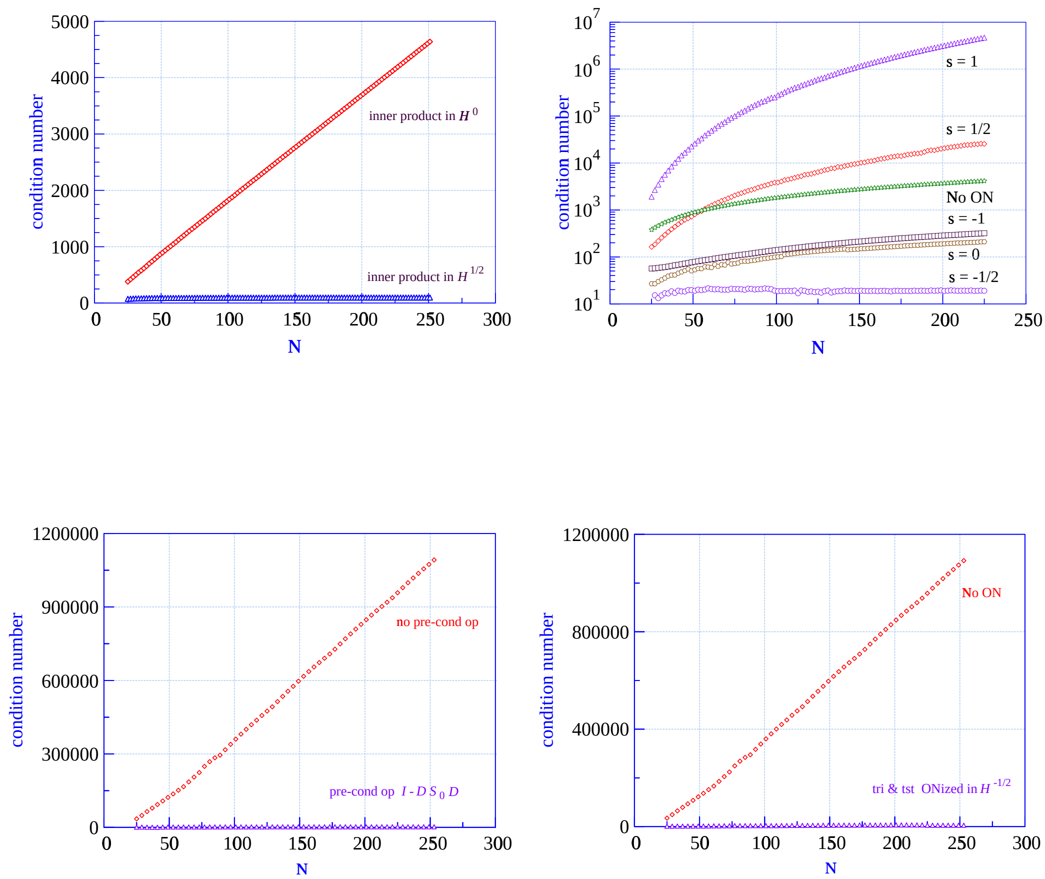

Figure 4: Condition numbers of V with and without a pre-conditioner

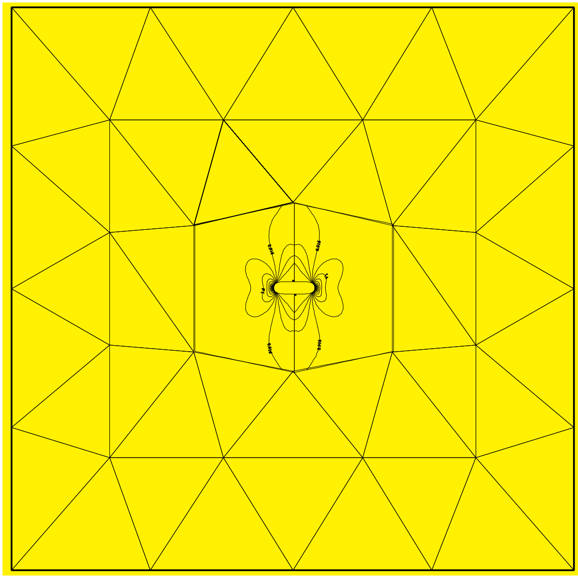

Figure 5: Stress isolines in the macro-elements

5 Concluding Remarks

Over the past 30 years, needless to say, it has been developed so fast in the areas of boundary element research from both computational and mathematical point of views. In the following, we give a quick overview on some of the interesting developments and leave out the detail of references. These are in the areas such as

- *Adaptive methods and error estimators*: Rank 1986, Yu and Wendland 1989, Göhner 1989, Stephan and Suri 1989;

- *Multigrid methods*: Hackbush 1981, Rank 1987, Schippers 1987, Petersdorff and Stephan 1989;

- *Multipole and cluster techniques*: Greengard and Rokhlin 1987, 1997, Hackbush and Nowak 1989, Sauter 1992, Nédélec 2001, Of, Steinbach and Wendland 2001, Darrigrand 2002, Cakoni, Darrigrand and Hsiao 2004;

- *Wavelets*: Dahmen, Prössdorf and Schneider 1993, 1994, Schwab and Petersdorff 1996, 1997, Petersdorff, Schneider and Schwab 1997, Levin, Schneider and Spasojevic 1996, 1997, Micchelli, Xu and Zhao 1977, Schneider 1998,

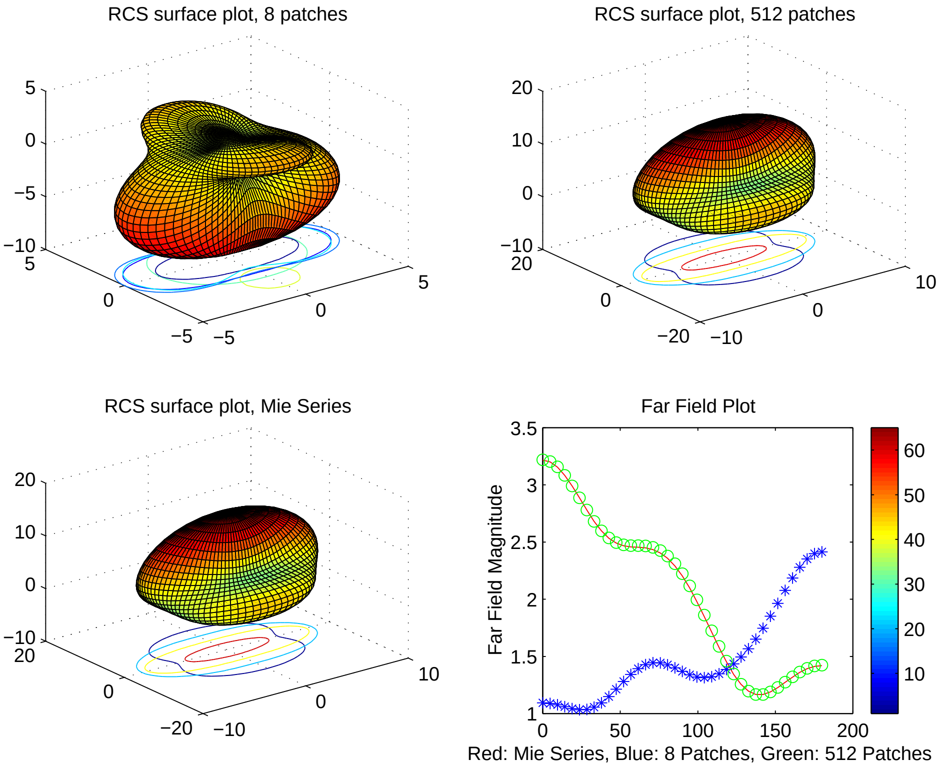

Figure 6: Radar cross-section surfaces in electromagnetic scattering

Lage and Schwab 1999, Tran, Stephan and Zaprianov 1998, Hsiao and Rathsfeld 2002, Eppler and Harbrecht 2004, Kaehler 2004, Dahmen, Harbrecht and Schneider 2004;

- *Coupling with FEM*: Johnson and Nedelec 1980, MacCamy and Marin 1980, Feng, K 1983, Wendland 1986, 1988, 1989, Costablel 1988, Costabel and Stephan 1988, 1990, Han 1988, Gatica and Hsiao 1989, 1990, Hsiao 1990, Porter and Hsiao 1990, Hsiao and Gatica 1992, 1995, Gatica and Wendland 1994, Barrenechea, Gatica and Hsiao 1998, Gatica, Hsiao and Mellado 2001, Gatica and Heuer 2000, 2002, Gatica, Heuer and Stephan 2001, Gatica, Harbrecht and Schneider 2001, Gatica, Gatica, L. F. and Stephan 2003, Gatica, Maischak and Stephan 2003, Stephan 2004, Barrientos, Gatica, Rodriguez and Torrejon 2003, Gonz alez and Meddahi 2004, Gatica and Meddahi 2004

- *Domain/Boundary decomposition and parallelization*: Hsiao and Wendland 1991, 1992, Hsiao, Schnack and Wendland 1999, Hsiao, Heuer, Stephan and Tran 1998, Steinbach, and Wendland 2000, Hsiao, Khoromskij and Wendland 2001;

- *Nonlinear problems*: Ruotsalainen and Wendland 1988, Ruotsalainen and Sarannen 1989, Eggermount and Sarannen 1990, Hsiao 1990, 1996.

- *Time-dependent problems*: Bamberger and Duoung 1986, Costabel, Onishi and Wendland 1987, Arnold and Noon 1988, Hebeker and Hsiao 1989, Hsiao and Saranen 1989, 1990, 1993, Costabel 1990, Li and Yinnian 2003, Celorrio, Hohage and Sayas 2004,

to name a few. As for the future research direction for the BEMs, we believe that the following topics will be most challenging and demanding. These are

(1) Fast BEM algorithms for 3-D problems

(2) Efficient BEM algorithms for problems in acoustic and electromagnetic scattering with high frequency.

With respect to these topics, we refer the readers to some of the most recent contributions [25] and [5].

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