A Simple Proof of Convergence for an Edge Element Discretization of Maxwell's Equations

Peter Monk
Department of Mathematical Sciences
University of Delaware
Newark, DE 19716

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University of Delaware
Newark, Delaware

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PETER MONK

Abstract. The time harmonic Maxwell's equations for a lossless medium are neither elliptic or definite. Hence the analysis of numerical schemes for these equations presents some unusual difficulties. In this paper we give a simple proof, based on the use of duality, for the convergence of edge finite element methods applied to the cavity problem for Maxwell's equations. The cavity is assumed to be a general Lipschitz polyhedron, and the mesh is assumed to be regular but not quasi-uniform.

1. Introduction. In this paper we are going to give a simple proof of convergence of edge finite element approximations to the cavity problem for Maxwell's equations. We start by describing this boundary value problem. Let Ω be a bounded Lipschitz smooth polyhedron in $\mathbb{R}^3$ with boundary $\Gamma = \partial\Omega$ and unit outward normal $\boldsymbol{\nu}$. We suppose that the boundary Γ consists of a single connected component and that Ω is connected and simply connected. In fact these topological assumptions are not necessary, but we introduce them to shorten the proofs. The results here can be modified to allow more boundary components and non simply connected domains (see for example [2], [6]).

We wish to approximate the electric field $\boldsymbol{E} = \boldsymbol{E}(\boldsymbol{x})$ that satisfies Maxwell's equations

$$\nabla \times (\nabla \times \boldsymbol{E}) - k^2\boldsymbol{E} = \boldsymbol{F} \text{ in } \Omega, \tag{1.1a}$$

$$\boldsymbol{\nu} \times \boldsymbol{E} = 0 \text{ on } \Gamma. \tag{1.1b}$$

Here $\boldsymbol{F}$ is a given function related to the imposed current sources and the parameter k is the wave-number assumed to be real and positive. Equation (1.1b) specifies a standard perfect conducting boundary condition on the boundary of Ω.

The problem can be posed variationally using the space

$$H_0(\text{curl};\Omega) = \left\{\boldsymbol{u} \in \left(L^2(\Omega)\right)^3 \mid \nabla \times \boldsymbol{u} \in \left(L^2(\Omega)\right)^3,\ \boldsymbol{\nu} \times \boldsymbol{u} = 0 \text{ on } \Gamma\right\}.$$

In particular suppose $\boldsymbol{F} \in H_0(\text{curl};\Omega)'$ were $H_0(\text{curl};\Omega)'$ is the dual space of $H_0(\text{curl};\Omega)$ with respect to the $\left(L^2(\Omega)\right)^3$ inner product. Then following the usual Galerkin strategy we arrive at the problem of finding $\boldsymbol{E} \in H_0(\text{curl};\Omega)$ such that

$$\int_\Omega \nabla \times \boldsymbol{E} \cdot \nabla \times \boldsymbol{\phi} - k^2\boldsymbol{E} \cdot \boldsymbol{\phi}\, dV = \int_\Omega \boldsymbol{F} \cdot \boldsymbol{\phi}\, dV, \quad \forall \boldsymbol{\phi} \in H_0(\text{curl};\Omega). \tag{1.2}$$

Because k is real, we can assume that any solution of this problem is real, so all spaces and functions in this paper are real.

Using the Helmholtz decomposition [17] which states that

$$H_0(\text{curl};\Omega) = \tilde{X} \oplus \nabla H_0^1(\Omega) \tag{1.3}$$

where

$$\tilde{X} = \left\{\boldsymbol{u} \in H_0(\text{curl};\Omega) \mid \int_\Omega \boldsymbol{u} \cdot \nabla p\, dV = 0, \quad \forall p \in H_0^1(\Omega)\right\}$$

problem (1.2) can be reduced to a problem on $\tilde{X}$. More precisely, let $\boldsymbol{E} = \boldsymbol{u} + \nabla p$ with $\boldsymbol{u} \in \tilde{X}$ and $p \in H_0^1(\Omega)$. Choosing $\boldsymbol{\phi} = \nabla\xi$ in (1.2) shows that p satisfies

$$-k^2 \int_\Omega \nabla p \cdot \nabla \xi\, dV = \int_\Omega \boldsymbol{F} \cdot \nabla\xi\, dV$$

for all $\xi \in H_0^1(\Omega)$. This uniquely determines p as the solution of a Poisson problem. Once p is determined, we choose $\boldsymbol{\phi} \in \tilde{X}$ in (1.2) and find that $\boldsymbol{u} \in \tilde{X}$ satisfies

$$\int_\Omega \nabla \times \boldsymbol{u} \cdot \nabla \times \boldsymbol{\phi} - k^2\boldsymbol{u} \cdot \boldsymbol{\phi}\, dV = \int_\Omega \boldsymbol{F} \cdot \boldsymbol{\phi}\, dV, \quad \forall \boldsymbol{\phi} \in \tilde{X}. \tag{1.4}$$

The compact embedding of $\tilde{X}$ into $(L^2(\Omega))^3$ [25, 9] and the Fredholm alternative can then be used to show that for any $\boldsymbol{F} \in H_0(\text{curl};\Omega)'$, problem (1.2) has a unique solution $\boldsymbol{E} \in H_0(\text{curl};\Omega)$ depending continuously on the data $\boldsymbol{F}$ provided k is not an interior Maxwell eigenvalue for Ω. For the remainder of the paper we assume $k > 0$ is not such an eigenvalue.

The problem of approximating $\boldsymbol{E}$ by finite elements then reduces to constructing a finite element subspace $X_h \subset H_0(\text{curl};\Omega)$ and computing $\boldsymbol{E}_h \in X_h$ such that

$$\int_\Omega \nabla\times\boldsymbol{E}_h\cdot\nabla\times\boldsymbol{\phi}_h - k^2\boldsymbol{E}_h\cdot\boldsymbol{\phi}_h\,dV = \int_\Omega \boldsymbol{F}\cdot\boldsymbol{\phi}_h\,dV, \quad \forall\boldsymbol{\phi}_h \in X_h. \tag{1.5}$$

The obvious choice of using vector continuous piecewise linear elements is dangerous since, if the domain has re-entrant corners, it is possible to compute finite element solutions that converge to a field that is not a solution of Maxwell's equations [10]. For this simple model problem modifications to the bilinear form to restore convergence are given in [11, 12], but further modifications are needed to handle, for example, discontinuous coefficients.

We prefer to construct X_h using the edge finite elements of Nédélec [20]. These avoid the problem of spurious solutions at the cost of increased complexity. Furthermore these elements can be applied to problems involving discontinuous coefficients (modeling different media) without modification.

Our goal in this paper is to derive estimates for $\boldsymbol{E}-\boldsymbol{E}_h$ in the $H_0(\text{curl};\Omega)$ norm given by

$$\|\boldsymbol{E}-\boldsymbol{E}_h\|_{curl} = \sqrt{\|\boldsymbol{E}-\boldsymbol{E}_h\|^2 + \|\nabla\times(\boldsymbol{E}-\boldsymbol{E}_h)\|^2},$$

where $\|\cdot\|$ is the standard $\left(L^2(\Omega)\right)^3$ norm. Otherwise, for a Hilbert spaces $\mathcal{X}$, we denote the norm by $\|\cdot\|_{\mathcal{X}}$.

There have been three previous results in this direction. In [19], I proved convergence using the ideas of Schatz [22] concerning the compact perturbation of coercive bilinear forms. Due to limitations on the understanding of edge elements and the regularity theory for Maxwell's equations at that time, I had to assume that Ω was convex, and the mesh was quasi-uniform.

In [13], Demkowicz and I applied the theory of collectively compact operators to prove convergence on general Lipschitz polyhedra. We assumed quasi-uniformity of the mesh to provide a certain inverse inequality (which is actually not necessary). Moreover, using the results of [6] our proof extends to include rather general spatially dependent coefficients in the equations (for example piecewise constant coefficients).

Perhaps the most general result to date is due to Boffi and Gastaldi [5]. They use the general convergence theory of Rappaz [15], together with their estimates of Maxwell eigenvalue convergence, to prove convergence on general regular meshes.

Both the work of Boffi and Gastaldi and my own with Demkowicz can be criticized for being too complicated. The goal here is to give a simple proof of convergence not relying on any abstract operator theory.

The main tool we shall use is the improved understanding of edge element interpolation theory and regularity results provided by [2]. We also use results from [3] modified appropriately for a Lipschitz polyhedral domain. Our paper is motivated by the work of Gopalakrishnan and Pasciak [18] who use similar estimates in their analysis of Schwarz methods for Maxwell's equations.

The layout of the paper is as follows. We start by summarizing some of the properties of edge elements. We then derive a weak Garding inequality for the error. After analyzing discrete divergence free vector fields, we use duality theory to prove the desired estimate. The proof is an improved, and simplified, version of the one in [19], but the techniques and approach are very much from [18].

2. Finite Elements and Interpolation. Let τ_h, $h > 0$, be a regular family of tetrahedral finite element meshes on Ω [7]. We shall now briefly summarize the construction of the edge and face finite elements of Nédélec [20], and some of their relevant properties.

Let P_l, $l > 0$ integer, denote the set of polynomials of total degree at most l in x_1, x_2 and x_3. We shall also use the spaces $P_l(e)$ and $P_l(f)$ of polynomials in arc length on an edge e, or surface coordinates on a face f, more precisely $P_l(f) = \{q \mid q = p|_f, p \in P_l\}$. In addition let $\tilde{P}_l$ denote the set of homogeneous polynomials of degree exactly l in x_1, x_2 and x_3. We then define

$$S_l = \left\{\boldsymbol{p} \in (\tilde{P}_l)^3 \mid \boldsymbol{x}\cdot\boldsymbol{p} = 0\right\}$$

and

$$R_l = (P_{l-1})^3 \oplus S_l.$$

With these definitions, the finite element space we shall use is the standard edge element space [20] given by

$$X_h = \{\boldsymbol{u}_h \in H_0(\mathrm{curl};\Omega) \mid \boldsymbol{u}_h|_K \in R_l,\ \forall K \in \tau_h\}.$$

This space has the following unisolvent set of degrees of freedom, defined for sufficiently smooth vector functions $\boldsymbol{u}$ on a tetrahedron K (we give a precise statement of the smoothness requirements later). In particular let

$$M_e = \left\{ \int_e \boldsymbol{u}\cdot\boldsymbol{\tau} q\, ds, \forall q \in P_{l-1}(e)\ \forall \text{ edges } e \text{ of } K \right\}$$

where $\boldsymbol{\tau}$ is a unit tangent to e. Let

$$M_f = \left\{ \int_f \boldsymbol{u}\cdot\boldsymbol{q}\, dA,\ \forall \boldsymbol{q} \in (P_{l-2}(f))^2,\ \forall \text{ faces } f \text{ of } K \right\},$$

$$M_K = \left\{ \int_K \boldsymbol{u}\cdot\boldsymbol{q}\, dV, \quad \forall \boldsymbol{q} \in (P_{l-3}(K))^3 \right\}.$$

Then the degrees of freedom on K are

$$\Sigma_K = M_e \cup M_f \cup M_K.$$

For sufficiently smooth vector fields $\boldsymbol{u}$, these degrees of freedom define an interpolant $r_h\boldsymbol{u}$ element by element. In particular, from [2] we know that this interpolant is well defined provided that there is a $\delta > 0$ and integer $q > 2$ such that for each tetrahedron $K \in \tau_h$

$$\boldsymbol{u} \in (H^{1/2+\delta}(K))^3, \text{ and } \nabla\times\boldsymbol{u} \in (L^q(K))^3.$$

Let

$$H^s(\mathrm{curl};\Omega) = \left\{\boldsymbol{u} \in (H^s(\Omega))^3 \mid \nabla\times\boldsymbol{u} \in (H^s(\Omega))^3\right\}.$$

Using scaling arguments the following estimate is proved in [1].

THEOREM 2.1. *If τ_h, $h > 0$, is a regular family of meshes on Ω and if $\boldsymbol{u} \in H^s(\mathrm{curl};\Omega)$, $\frac{1}{2} < s \leq l$, then there is a constant C depending on s but not on h or $\boldsymbol{u}$ such that*

$$\|\boldsymbol{u} - r_h\boldsymbol{u}\|_{curl} \leq Ch^s(\|\boldsymbol{u}\|_{H^s(\Omega)} + \|\nabla\times\boldsymbol{u}\|_{H^s(\Omega)}).$$

We can also define the $H_0(\mathrm{curl};\Omega)$ orthogonal projection $P_h : H_0(\mathrm{curl};\Omega) \to X_h$, such that if $\boldsymbol{u} \in H_0(\mathrm{curl};\Omega)$ then $P_h\boldsymbol{u} \in X_h$ satisfies

$$\int_\Omega \nabla\times(\boldsymbol{u} - P_h\boldsymbol{u})\cdot\nabla\times\boldsymbol{\phi}_h + (\boldsymbol{u} - P_h\boldsymbol{u})\cdot\boldsymbol{\phi}_h\, dV = 0, \quad \forall\boldsymbol{\phi}_h \in X_h. \tag{2.1}$$

This projection satisfies the optimal error estimate

$$\|\boldsymbol{u} - P_h\boldsymbol{u}\|_{curl} = \inf_{\boldsymbol{v}_h\in X_h} \|\boldsymbol{u} - \boldsymbol{v}_h\|_{curl}.$$

If $\boldsymbol{u} \in H^s(\mathrm{curl};\Omega)$, $s > 1/2$, Theorem 2.1 can then be used to provide order estimates for the right hand side of the above equality.

Now let

$$S_h = \left\{p_h \in H_0^1(\Omega) \mid p_h|_K \in P_l \quad \forall K \in \tau_h\right\}.$$

This is just the space of standard continuous, piecewise degree $l-1$ finite elements. We have ([20])

$$\nabla S_h \subset X_h.$$

This provides a large subspace of test functions in X_h. Using this space, we say a function $\boldsymbol{u} \in \left(L^2(\Omega)\right)^3$ is discrete divergence free if

$$\int_\Omega \boldsymbol{u} \cdot \nabla \xi_h \, dV = 0 \quad \forall \xi_h \in S_h.$$

We then have the following discrete Helmholtz decomposition analogous to (1.3(

$$X_h = \tilde{X}_h \oplus \nabla S_h$$

where $\tilde{X}_h$ is the space of discrete divergence free finite elements.

Using the test function $\boldsymbol{\phi}_h = \nabla \xi_h$, $\xi_h \in S_h$ in (2.1) shows that $\boldsymbol{u} - P_h\boldsymbol{u}$ is discrete divergence free since

$$\int_\Omega (\boldsymbol{u} - P_h\boldsymbol{u}) \cdot \nabla \xi_h \, dV = 0, \quad \forall \xi_h \in S_h. \tag{2.2}$$

We shall also need some properties of a subspace of divergence conforming finite element functions in the space

$$H_0(\mathrm{div};\Omega) = \left\{\boldsymbol{u} \in (L^2(\Omega))^3 \mid \nabla \cdot \boldsymbol{u} \in L^2(\Omega), \quad \boldsymbol{\nu} \cdot \boldsymbol{u} = 0 \text{ on } \Gamma\right\}.$$

These finite elements are also found in Nédélec [20]. Let

$$D_l = (P_{l-1})^3 \oplus \tilde{P}_{l-1}\boldsymbol{x}$$

and define $Y_h \subset H_0(\mathrm{div}\,;\Omega)$ by

$$Y_h = \{\boldsymbol{u}_h \in H_0(\mathrm{div};\Omega) \mid \boldsymbol{u}_h|_K \in D_l \quad \forall K \in \tau_h\}.$$

Although we will not need error estimates for this space, we shall need some properties of the interpolant. The degrees of freedom for this space are defined element by element as follows. For a face f in the mesh with normal $\boldsymbol{\nu}_f$, let

$$N_f = \left\{\int_f \boldsymbol{u} \cdot \boldsymbol{\nu}_f q \, dA, \quad \forall q \in P_{l-1}(f) \text{ for each face } f \text{ of } K\right\},$$

and let

$$N_K = \left\{\int_K \boldsymbol{u} \cdot \boldsymbol{q} \, dV, \quad \forall \boldsymbol{q} \in (P_{l-2})^3\right\}.$$

The degrees of freedom on an element K are $\Sigma_K = N_K \cup N_f$. These degrees of freedom define an interpolation operator w_h element by element. This operator is well defined for example on functions $\boldsymbol{u} \in (H^{1/2+\delta}(\Omega))^3$, for some $\delta > 0$.

The only property of w_h we shall use is the "commuting diagram property" that if $\boldsymbol{u}$ is such that both the interpolants $r_h\boldsymbol{u}$ and $w_h\nabla \times \boldsymbol{u}$ are well defined then

$$\nabla \times r_h\boldsymbol{u} = w_h \nabla \times \boldsymbol{u}. \tag{2.3}$$

This commuting property is part of the discrete deRham diagram whose importance has been pointed out particularly by Boffi and co-workers [4, 5].

Now suppose that $\boldsymbol{u} \in (H^{1/2+\delta}(\Omega))^3$ is such that $\nabla \times \boldsymbol{u} \in Y_h$. Since functions in Y_h are peicewise polynmials of fixed degree, it follows that $\nabla \times \boldsymbol{u} \in (L^q(\Omega))^3$, for any $q \geq 2$ and hence the interpolant $r_h\boldsymbol{u}$ is well defined. Using a scaling argument along the lines of [1] and the equivalence of norms for a piecewise polynomial on the reference element as in [3] we have the following result.

LEMMA 2.2. *Let τ_h be a regular mesh, and suppose $\boldsymbol{u} \in (H^{1/2+\delta}(\Omega))^3$ is such that $\nabla \times \boldsymbol{u} \in Y_h$. Then there exists a constant C independent of h and $\boldsymbol{u}$ such that*

$$\|\boldsymbol{u} - r_h\boldsymbol{u}\| \leq C\left(h^{1/2+\delta}\|\boldsymbol{u}\|_{H^{1/2+\delta}(\Omega)} + h\|\nabla \times \boldsymbol{u}\|\right).$$

One further remark is needed regarding the discrete divergence free space $\tilde{X}_h$. First, we note that

$$\nabla \times X_h \subset Y_h$$

(clearly $\nabla \times X_h \subset H_0(\mathrm{div};\Omega)$ and the piecewise polynomials in $\nabla \times X_h$ are vector functions of degree $l-1$, so in Y_h). Thus, as in [3], we can regard the curl as a bounded operator from X_h into Y_h. In X_h the null-space of the curl operator is denoted $N(\mathrm{curl})$. Let $\boldsymbol{u}_h \in N(\mathrm{curl})$. Since the domain Ω is simply connected and the boundary Γ is connected, the fact that $\nabla \times \boldsymbol{u}_h = 0$ in Ω implies $\boldsymbol{u}_h = \nabla p$ for some $p \in H_0^1(\Omega)$. In addition since $\boldsymbol{u}_h \in X_h$ then $p \in S_h$. Hence $N(\mathrm{curl}) = \nabla S_h$.

The discrete divergence free space $\tilde{X}_h$ is thus given by $\tilde{X}_h = N(\mathrm{curl})^\perp$ where $N(\mathrm{curl})^\perp$ is the orthogonal complement of $N(\mathrm{curl}) \subset X_h$ in the $(L^2(\Omega))^3$ inner product. Now, following [3], let $\nabla_h \times$ denote the discrete adjoint operator for the curl by which we mean that for each $\boldsymbol{w}_h \in Y_h$, $\nabla_h \times \boldsymbol{w}_h \in X_h$ is the unique function such that

$$(\nabla_h \times \boldsymbol{w}_h, \boldsymbol{\psi}_h) = (\boldsymbol{w}_h, \nabla \times \boldsymbol{\psi}_h), \quad \forall \boldsymbol{\psi}_h \in X_h.$$

By a standard theorem from functional analysis (see Theorem 4.6 of [8]) we know that

$$N(\mathrm{curl})^\perp = \nabla_h \times (\nabla \times X_h)$$

so that we have the following result.

LEMMA 2.3. *For each $\boldsymbol{v}_h \in \tilde{X}_h$ there is a function $\boldsymbol{w}_h \in \nabla \times X_h \subset Y_h$ such that $\boldsymbol{v}_h = \nabla_h \times \boldsymbol{w}_h$ or*

$$\int_\Omega \boldsymbol{v}_h \cdot \boldsymbol{\phi}_h \, dV = \int_\Omega \boldsymbol{w}_h \cdot \nabla \times \boldsymbol{\phi}_h \, dV \quad \forall \boldsymbol{\phi}_h \in X_h.$$

This lemma is from [3] where it is pointed out that an alternative way to write the discrete Helmholtz decomposition is as follows. Any function $\boldsymbol{v}_h \in X_h$ may be written

$$\boldsymbol{v}_h = \nabla_h \times \boldsymbol{w}_h + \nabla p_h$$

for some $\boldsymbol{w}_h \in \nabla \times X_h \subset Y_h$ and $p_h \in S_h$. This makes the discrete Helmholtz decomposition look a little more like the continuous one.

3. Error Analysis. This section is devoted to proving our main Theorem 3.4. For convenience we use the notation

$$a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \nabla \times \boldsymbol{u} \cdot \nabla \times \boldsymbol{v} - k^2 \boldsymbol{u} \cdot \boldsymbol{v} \, dV$$

and

$$(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \boldsymbol{u} \cdot \boldsymbol{v} \, dV.$$

At this stage we do not know that $\boldsymbol{E}_h$ exists, but if it does exist we define $\boldsymbol{e}_h = \boldsymbol{E} - \boldsymbol{E}_h$. Then by subtracting (1.5) from (1.2) we obtain the Galerkin error equation

$$a(\boldsymbol{e}_h, \boldsymbol{\psi}_h) = 0, \quad \forall \boldsymbol{\psi}_h \in X_h. \tag{3.1}$$

In particular, choosing $\boldsymbol{\psi}_h = \nabla \xi_h$ for $\xi_h \in S_h$ shows that $\boldsymbol{e}_h$ is discrete divergence free.

In [19] the problem of estimating $\|\boldsymbol{E} - \boldsymbol{E}_h\|_{curl}$ was approached via a classical Garding inequality. Our first lemma is a weaker form of the Garding inequality as used in [18].

LEMMA 3.1. *There is a constant C independent of h, $\boldsymbol{E}$ and $\boldsymbol{E}_h$ such that*

$$\|\boldsymbol{e}_h\|_{curl} \le \|\boldsymbol{E} - P_h \boldsymbol{E}\|_{curl} + C \sup_{\boldsymbol{v}_h \in X_h} \frac{|(\boldsymbol{e}_h, \boldsymbol{v}_h)|}{\|\boldsymbol{v}_h\|_{curl}}. \tag{3.2}$$

Proof. Using a very slight modification of the proof of Lemma 4.4 of [18] we see that by the definition of the curl norm, and the definition of $a(\cdot,\cdot)$ we have

$$\begin{aligned}
\|\boldsymbol{e}_h\|_{curl}^2 &= a(\boldsymbol{e}_h, \boldsymbol{e}_h) + (1+k^2)(\boldsymbol{e}_h, \boldsymbol{e}_h) \\
&= a(\boldsymbol{e}_h, \boldsymbol{E} - P_h \boldsymbol{E}) + a(\boldsymbol{e}_h, P_h \boldsymbol{E} - \boldsymbol{E}_h) + (1+k^2)(\boldsymbol{e}_h, \boldsymbol{e}_h)
\end{aligned}$$

Now using the Galerkin condition (3.1), the definition of the curl norm, and the definition of $\|\cdot\|_{curl}$ we have

$$\begin{aligned}
\|\boldsymbol{e}_h\|_{curl}^2 &= a(\boldsymbol{e}_h, \boldsymbol{E} - P_h \boldsymbol{E}) + (1+k^2)(\boldsymbol{e}_h, \boldsymbol{e}_h) \\
&= (\nabla \times \boldsymbol{e}_h, \nabla \times (\boldsymbol{E} - P_h \boldsymbol{E})) + (\boldsymbol{e}_h, (\boldsymbol{E} - P_h \boldsymbol{E})) + (1+k^2)\left\{(\boldsymbol{e}_h, \boldsymbol{e}_h) - (\boldsymbol{e}_h, (\boldsymbol{E} - P_h \boldsymbol{E}))\right\} \\
&= (\nabla \times \boldsymbol{e}_h, \nabla \times (\boldsymbol{E} - P_h \boldsymbol{E})) + (\boldsymbol{e}_h, (\boldsymbol{E} - P_h \boldsymbol{E})) + (1+k^2)(\boldsymbol{e}_h, P_h \boldsymbol{E} - \boldsymbol{E}_h).
\end{aligned}$$

Hence using the Cauchy-Schwarz inequality, and the boundedness of the projection $P_h : H(\mathrm{curl};\Omega) \to X_h$

$$\begin{aligned}
\|\boldsymbol{e}_h\|_{curl}^2 &\le \|\boldsymbol{E} - P_h \boldsymbol{E}\|_{curl} \|\boldsymbol{e}_h\|_{curl} + (1+k^2) \sup_{\boldsymbol{v}_h \in X_h} \frac{|(\boldsymbol{e}_h, \boldsymbol{v}_h)|}{\|\boldsymbol{v}_h\|_{curl}} \|P_h \boldsymbol{E} - \boldsymbol{E}_h\|_{curl} \\
&= \|\boldsymbol{E} - P_h \boldsymbol{E}\|_{curl} \|\boldsymbol{e}_h\|_{curl} + (1+k^2) \sup_{\boldsymbol{v}_h \in X_h} \frac{|(\boldsymbol{e}_h, \boldsymbol{v}_h)|}{\|\boldsymbol{v}_h\|_{curl}} \|P_h \boldsymbol{e}_h\|_{curl} \\
&\le \|\boldsymbol{E} - P_h \boldsymbol{E}\|_{curl} \|\boldsymbol{e}_h\|_{curl} + (1+k^2) \sup_{\boldsymbol{v}_h \in X_h} \frac{|(\boldsymbol{e}_h, \boldsymbol{v}_h)|}{\|\boldsymbol{v}_h\|_{curl}} \|\boldsymbol{e}_h\|_{curl}.
\end{aligned}$$

This proves the desired estimate with $C = 1 + k^2$. ∎

Our error estimate will be finished if we can estimate the supremum on the right hand side of (3.2). This is done in in Lemma 3.3.

Before we prove this lemma we need to investigate discrete divergence free functions in more detail. For such functions we can construct a nearby exactly divergence free function. This construction was used for example by Girault [16] and myself [19] with an ad-hoc analysis. However the clearest analysis is from Arnold et al. [3].

For a given discrete divergence free function $\boldsymbol{v}_h \in \tilde{X}_h$, let us define $\boldsymbol{v}^h \in H_0(\mathrm{curl};\Omega)$ by

$$\nabla \times \boldsymbol{v}^h = \nabla \times \boldsymbol{v}_h \text{ in } \Omega, \tag{3.3a}$$

$$\nabla \cdot \boldsymbol{v}^h = 0 \text{ in } \Omega. \tag{3.3b}$$

In [3] it is suggested to view $\boldsymbol{v}^h$ as part of the solution of the mixed problem of finding $\boldsymbol{v}^h \in H_0(\mathrm{curl};\Omega)$ and $\boldsymbol{w}^h \in \nabla \times H_0(\mathrm{curl};\Omega)$ such that

$$(\boldsymbol{v}^h, \boldsymbol{\phi}) + (\nabla \times \boldsymbol{\phi}, \boldsymbol{w}^h) = 0, \quad \forall \boldsymbol{\phi} \in H_0(\mathrm{curl};\Omega), \tag{3.4a}$$

$$(\nabla \times \boldsymbol{v}^h, \boldsymbol{\xi}) = (\nabla \times \boldsymbol{v}_h, \boldsymbol{\xi}), \quad \forall \boldsymbol{\xi} \in \nabla \times H_0(\mathrm{curl};\Omega). \tag{3.4b}$$

Both the coercivity condition and Babuška-Brezzi condition for mixed methods are obviously satisfied and so $(\boldsymbol{v}^h, \boldsymbol{w}^h)$ exists.

We have the following lemma:

LEMMA 3.2. *Let $\boldsymbol{v}_h \in \tilde{X}_h$. Suppose $\boldsymbol{v}^h \in H_0(\mathrm{curl};\Omega)$ satisfies (3.3) then there are constants C and $\delta > 0$ independent of h and $\boldsymbol{v}_h$ and $\boldsymbol{v}^h$ such that*

$$\|\boldsymbol{v}^h - \boldsymbol{v}_h\| \leq Ch^{1/2+\delta}\|\nabla \times \boldsymbol{v}_h\|.$$

Proof. The proof follows [3] checking that their result, proved for convex domains, holds here. From [2] there is an exponent $\delta > 0$ such that $\boldsymbol{v}^h \in \left(H^{1/2+\delta}(\Omega)\right)^3$, and since $\nabla \times \boldsymbol{v}^h = \nabla \times \boldsymbol{v}_h$, we see that $\nabla \times \boldsymbol{v}^h \in (L^q(\Omega))^3$ for all $q > 2$. Hence using Lemma 2.2, $r_h\boldsymbol{v}^h$ is well defined. But then, using the commuting diagram property of edge elements

$$\nabla \times r_h\boldsymbol{v}^h = w_h\nabla \times \boldsymbol{v}^h = w_h\nabla \times \boldsymbol{v}_h = \nabla \times \boldsymbol{v}_h. \tag{3.5}$$

Since $\boldsymbol{v}_h$ is discrete divergence free, by Lemma 2.3 there is a function $\boldsymbol{w}_h \in \nabla \times X_h$ such that

$$(\boldsymbol{v}_h, \boldsymbol{\phi}_h) + (\nabla \times \boldsymbol{\phi}_h, \boldsymbol{w}_h) = 0, \quad \forall \boldsymbol{\phi}_h \in X_h, \tag{3.6a}$$

$$(\nabla \times \boldsymbol{v}_h, \xi_h) = (\nabla \times \boldsymbol{v}_h, \xi_h), \quad \forall \boldsymbol{\xi}_h \in \nabla \times X_h. \tag{3.6b}$$

Thus $(\boldsymbol{v}_h, \boldsymbol{w}_h)$ is nothing else than the mixed finite element approximation to $(\boldsymbol{v}^h, \boldsymbol{w}^h)$ defined by (3.4). Now selecting $\boldsymbol{\phi} = r_h\boldsymbol{v}^h - \boldsymbol{v}_h$ in (3.4a) and $\boldsymbol{\phi}_h = r_h\boldsymbol{v}^h - \boldsymbol{v}_h$ in (3.6a) and using the fact that $\nabla \times \boldsymbol{\phi}_h = 0$ (see (3.5)) we have

$$(\boldsymbol{v}^h - \boldsymbol{v}_h, r_h\boldsymbol{v}^h - \boldsymbol{v}_h) = 0.$$

Thus

$$(\boldsymbol{v}^h - \boldsymbol{v}_h, \boldsymbol{v}^h - \boldsymbol{v}_h) = (\boldsymbol{v}^h - \boldsymbol{v}_h, \boldsymbol{v}^h - r_h\boldsymbol{v}^h) + (\boldsymbol{v}^h - \boldsymbol{v}_h, r_h\boldsymbol{v}^h - \boldsymbol{v}_h).$$

Hence $\|\boldsymbol{v}^h - \boldsymbol{v}_h\| \leq \|\boldsymbol{v}^h - r_h\boldsymbol{v}^h\|$ and using Corollary 2.2 we have

$$\|\boldsymbol{v}^h - \boldsymbol{v}_h\| \leq C\left(h^{1/2+\delta}\|\boldsymbol{v}^h\|_{H^{1/2+\delta}(\Omega)} + h\|\nabla \times \boldsymbol{v}_h\|\right).$$

The a priori estimate $\|\boldsymbol{v}^h\|_{H^{1/2+\delta}(\Omega)} \leq C\|\nabla \times \boldsymbol{v}_h\|$ completes the proof. □

Now we can estimate the troublesome term in (3.2).

LEMMA 3.3. *For all h small enough there exists constants C and δ with $0 < \delta \leq 1/2$ such that*

$$\sup_{\boldsymbol{v}_h \in X_h} \frac{|(\boldsymbol{e}_h, \boldsymbol{v}_h)|}{\|\boldsymbol{v}_h\|_{curl}} \leq Ch^{\delta+1/2}\|\boldsymbol{e}_h\|_{curl}.$$

Proof. This lemma is proved by a duality argument similar to the one in the proof of Lemma 4.3 of [18] and in [19]. Using the continuous Helmholtz decomposition there is a divergence free function $\boldsymbol{e}_0^h \in H_0(\mathrm{curl};\Omega)$ and a scalar $p^h \in H_0^1(\Omega)$ such that

$$\boldsymbol{e}_h = \boldsymbol{e}_0^h + \nabla p^h.$$

Here $p^h \in H_0^1(\Omega)$ satisfies

$$(\nabla p^h, \nabla\xi) = (\boldsymbol{e}_h, \nabla\xi), \quad \forall \xi \in H_0^1(\Omega).$$

Thus, by choosing $\xi = p^h$, we see that $\|\nabla p^h\| \leq \|\boldsymbol{e}_h\|$.

Using the discrete Helmholtz decomposition we also can write

$$\boldsymbol{v}_h = \boldsymbol{v}_{0,h} + \nabla\xi_h$$

for some $\boldsymbol{v}_{0,h} \in \tilde{X}_h$ and $\xi_h \in S_h$. Since we have already shown that $\boldsymbol{e}_h$ is discrete divergence free, we have

$$(\boldsymbol{e}_h, \boldsymbol{v}_h) = (\boldsymbol{e}_h, \boldsymbol{v}_{0,h}) = (\boldsymbol{e}_0^h, \boldsymbol{v}_{0,h}) + (\nabla p^h, \boldsymbol{v}_{0,h}). \tag{3.7}$$

The first term on the right hand side is estimated by

$$|(\boldsymbol{e}_0^h, \boldsymbol{v}_{0,h})| \le \|\boldsymbol{e}_0^h\|\,\|\boldsymbol{v}_{0,h}\| \le C\|\boldsymbol{e}_0^h\|\,\|\boldsymbol{v}_h\| \tag{3.8}$$

where we have made use of the fact that $\|\nabla\xi_h\| \le \|\boldsymbol{v}_h\|$. Thus we can estimate this term by estimating $\|\boldsymbol{e}_0^h\|$ which we do next.

We define the adjoint variable $z \in H_0(\mathrm{curl};\Omega)$ such that

$$a(\boldsymbol{\phi}, \boldsymbol{z}) = (\boldsymbol{e}_0^h, \boldsymbol{\phi}), \quad \forall \boldsymbol{\phi} \in H_0(\mathrm{curl};\Omega). \tag{3.9}$$

Clearly $\boldsymbol{z}$ is the weak solution in $H_0(\mathrm{curl};\Omega)$ of

$$\nabla\times\nabla\times\boldsymbol{z} - k^2\boldsymbol{z} = \boldsymbol{e}_0^h.$$

and the assumption that k is not an interior Maxwell eigenvalue implies that $\boldsymbol{z}$ is well defined and there is a constant C such that $\|\boldsymbol{z}\|_{curl} \le C\|\boldsymbol{e}_0^h\|$.

Since $\boldsymbol{e}_0^h$ is divergence free, it follows that $\boldsymbol{z}$ is also divergence free (to see this take $\boldsymbol{\phi} = \nabla\xi$ for $\xi \in H_0^1(\Omega)$ in equation (3.9)). Thus we have

$$\nabla\times\boldsymbol{z} \in (L^2(\Omega))^3, \quad \nabla\cdot\boldsymbol{z} = 0 \text{ in } \Omega \text{ and } \boldsymbol{\nu}\times\boldsymbol{z} = 0 \text{ on } \Gamma.$$

By Proposition 3.7 of [2] we have $\boldsymbol{z} \in \left(H^{1/2+\delta}(\Omega)\right)^3$ for some δ with $0 < \delta \le 1/2$ together with the norm bound

$$\|\boldsymbol{z}\|_{H^{1/2+\delta}(\Omega)} \le C\|\boldsymbol{e}_0^h\|.$$

In addition we see that $\nabla\times\boldsymbol{z} \in (L^2(\Omega))^3$ is the weak solution of

$$\begin{aligned} \nabla\times(\nabla\times\boldsymbol{z}) &= k^2\boldsymbol{z} + \boldsymbol{e}_0^h \in (L^2(\Omega))^3, \\ \nabla\cdot(\nabla\times\boldsymbol{z}) &= 0 \text{ in } \Omega, \\ \boldsymbol{\nu}\cdot(\nabla\times\boldsymbol{z}) &= 0 \text{ on } \Gamma. \end{aligned}$$

Thus again by Proposition 3.7 of [2],

$$\nabla\times\boldsymbol{z} \in \left(H^{1/2+\delta}(\Omega)\right)^3$$

with the norm bound

$$\|\nabla\times\boldsymbol{z}\|_{H^{1/2+\delta}(\Omega)} \le C\|\boldsymbol{e}_0^h\|.$$

We conclude that $\boldsymbol{z} \in H^{1/2+\delta}(\mathrm{curl};\Omega)$. Hence the interpolant $r_h\boldsymbol{z}$ is well defined, and we can use Theorem 2.1 to obtain the error estimate

$$\|\boldsymbol{z} - P_h\boldsymbol{z}\|_{curl} \le \|\boldsymbol{z} - r_h\boldsymbol{z}\|_{curl} \le Ch^{1/2+\delta}\|\boldsymbol{e}_0^h\|.$$

Now using (3.9) and the fact that $\boldsymbol{z}$ is divergence free we have

$$\|\boldsymbol{e}_0^h\|^2 = a(\boldsymbol{e}_0^h, \boldsymbol{z}) = a(\boldsymbol{e}_0^h + \nabla p^h, \boldsymbol{z}) = a(\boldsymbol{e}_h, \boldsymbol{z}).$$

Hence by the Galerkin condition (3.1)

$$\|\boldsymbol{e}_0^h\|^2 = a(\boldsymbol{e}_h, \boldsymbol{z} - P_h\boldsymbol{z}) \le C\|\boldsymbol{e}_h\|_{curl}\|\boldsymbol{z} - P_h\boldsymbol{z}\|_{curl} \le Ch^{1/2+\delta}\|\boldsymbol{e}_0^h\|\,\|\boldsymbol{e}_h\|_{curl}.$$

We have thus proved that

$$\|\boldsymbol{e}_0^h\| \le Ch^{1/2+\delta}\|\boldsymbol{e}_h\|_{curl} \tag{3.10}$$

Now we estimate the term $(\nabla p^h, \boldsymbol{v}_{0,h})$ in (3.7). Since $\boldsymbol{v}_{0,h}$ is discrete divergence free Lemma 3.2 implies that there is a divergence free function $\boldsymbol{v}_0^h \in H(\text{curl};\Omega)$ with

$$\|\boldsymbol{v}_0^h - \boldsymbol{v}_{0,h}\| \le Ch^{1/2+\delta}\|\nabla \times \boldsymbol{v}_{0,h}\| = Ch^{1/2+\delta}\|\nabla \times \boldsymbol{v}_h\|.$$

Now using the fact that $\boldsymbol{v}_0^h$ is divergence free, and the error estimate above, we have

$$\begin{aligned}(\nabla p^h, \boldsymbol{v}_{0,h}) &= (\nabla p^h, \boldsymbol{v}_{0,h} - \boldsymbol{v}_0^h)\\ &\le Ch^{1/2+\delta}\|\nabla p^h\|\,\|\nabla \times \boldsymbol{v}_h\|.\end{aligned} \tag{3.11}$$

Using (3.10) in (3.8) and using the resulting estimate together with (3.11) in (3.7) proves the desired result. ☐

We now state and prove our main theorem:

THEOREM 3.4. *Let Ω be a simply connected Lipschitz polyhedron with connected boundary Γ. Suppose k is not a Maxwell eigenvalue for Ω. Then if $\boldsymbol{E}$ satisfies (1.2) and $\boldsymbol{E}_h \in X_h$ satisfies (1.5) there is a constant C independent of h, $\boldsymbol{E}$ and $\boldsymbol{E}_h$ and a constant $h_0 > 0$ independent of $\boldsymbol{E}$ and $\boldsymbol{E}_h$ such that for all $0 < h < h_0$,*

$$\|\boldsymbol{E} - \boldsymbol{E}_h\|_{curl} \le \frac{1}{1 - Ch^{1/2+\delta}} \inf_{\boldsymbol{v}_h \in X_h} \|\boldsymbol{E} - \boldsymbol{v}_h\|_{curl}.$$

Here $\delta > 0$ is the exponent in Lemma 3.2.

Remarks:

1. Choosing h small enough that (for example) $Ch^{1/2+\delta} < 1/2$ proves quasi-optimal convergence of the edge element approximation. Furthermore the constant $1/(1 - Ch^{1/2+\delta})$ can be made arbitrarily close to unity. This seems to me to be a surprising result given that the norms are not k dependent.
2. If $\boldsymbol{u} \in H^s(\text{curl};\Omega)$ for some s with $1/2 < s \le l$, then Theorems 2.1 and 3.4 show that for all sufficiently small h there is a constant C such that

   $$\|\boldsymbol{E} - \boldsymbol{E}_h\|_{curl} \le Ch^s.$$

   In general the polyhedral boundary Γ causes singularities in the solution that prevent high global regularity. Nevertheless, as we have seen, we can expect sufficient regularity to guarantee a convergence rate of better than $O(h^{1/2})$.

Proof. Lemma 3.3 shows that

$$\sup_{\boldsymbol{v}_h \in X_h} \frac{|(\boldsymbol{e}_h, \boldsymbol{v}_h)|}{\|\boldsymbol{v}_h\|_{curl}} \le Ch^{1/2+\delta}\|\boldsymbol{e}_h\|_{curl}.$$

Putting this together with (3.2) shows that

$$\|\boldsymbol{e}_h\|_{curl} \le \|\boldsymbol{E} - P_h\boldsymbol{E}\|_{curl} + Ch^{1/2+\delta}\|\boldsymbol{e}_h\|_{curl}.$$

Choosing h small enough that $1 - Ch^{1/2+\delta} > 0$ proves the result. ☐

COROLLARY 3.5. *For any $\boldsymbol{F} \in H_0(curl\,;\Omega)'$, there is an $h_0 > 0$ such that for all $h < h_0$, equation (1.5) has a unique solution.*

Proof. It suffices to prove uniqueness. Let $\boldsymbol{F} = 0$, then since k is not a Maxwell eigenvalue, $\boldsymbol{E} = 0$ in (1.2) and $\boldsymbol{E}_h = 0$ is one solution of the discrete problem. By the above error estimate, for any solution $\boldsymbol{E}_h$ of the discrete problem $\|\boldsymbol{E}_h\|_{curl} \le C \inf_{\boldsymbol{v}_h \in X_h} \|\boldsymbol{v}_h\| = 0$. Hence $\boldsymbol{E}_h = 0$, and uniqueness is proved. ☐

4. Conclusion. The proof we have given rests critically on regularity results for the dual problem, and on the estimate for in Lemma 2.3 for the approximation of a discrete divergence free function by a divergence free function. For smooth coefficients these results still hold. But for general coefficients both results might be difficult to obtain, however it is possible that using arguments like those in [6] the explicit estimates used here could be replaced by uniform convergence estimates based on compactness arguments of the type used by Schatz and Wang [23].

From the point of view of analyzing other elements, the proof here is also valid for second family edge elements on tetrahedra [21] and for the edge finite elements on parallelepipeds in [20]. Extending these results to $h - p$ elements such as those in [24, 14] would require the low regularity interpolation results in Theorem 2.1 and the estimate in Lemma 2.2. As far as I am aware, these are not yet proved.

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