ACCURATE DISCRETISATION OF A NONLINEAR MICROMAGNETIC PROBLEM

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Abstract. In this paper we propose a finite element discretization of the Maxwell-Landau-Lifchitz-Gilbert equations governing the electromagnetic field in a ferromagnetic material. Our point of view is that it is desirable for the discrete problem to possess conservation properties similar to the continuous system. We first prove the existence of a new class of Liapunov functions for the continuous problem, and then for a variational formulation of the continuous problem. We also show a special continuous dependence result. Then we propose a family of mass-lumped finite element schemes for the problem. For the resulting semi-discrete problem we show that magnetization is conserved and that semi-discrete Liapunov functions exist. Finally we show the results of some computations that show the behavior of the fully discrete Liapunov functions.

1. Introduction. In this paper we propose a finite element discretization of the Maxwell-Landau-Lifchitz-Gilbert equations governing the electromagnetic field in a ferromagnetic material. Our point of view is that it is desirable for the discrete problem to possess conservation properties similar to the continuous system (see for example [17]). The first part of our paper recalls the standard conservation and energy decay properties of the Maxwell-Landau-Lifchitz-Gilbert equations, and in addition we prove the existence of a new class of Liapunov functions which can be viewed as another set of conserved quantities for these equations. We also prove a special continuous dependence result. The remainder of the paper is devoted to outlining a finite element method that possesses the conservation, energy decay and Liapunov function properties of the continuous system.

To obtain a simple model problem for ferromagnetic calculations, we suppose that there is a bounded cavity $\Omega \subset \mathbb{R}^3$ with a perfectly conducting outer surface Γ. We assume that both Ω and Γ are simply connected. Within the cavity is a ferromagnetic material occupying a bounded sub-domain $\Omega_M \subset \Omega$. For simplicity we assume that outside the ferromagnet (i.e. in $\Omega \setminus \overline{\Omega}_M$) is vacuum.

In order to model the electromagnetic behavior of the ferromagnetic material, the basic Maxwell system must be augmented by an equation describing the influence of the ferromagnet. We outline the equations next (see for example [12, 5, 18]). The electromagnetic field in Ω is described by four vector functions of position and time: $\boldsymbol{E}$, the electric field, $\boldsymbol{H}$ the magnetic field, $\boldsymbol{B}$ the magnetic induction and $\boldsymbol{M}$ the magnetization. The magnetic variables are related as follows:

$$\boldsymbol{B} = \mu_0 \left(\boldsymbol{H} + \boldsymbol{M}\right) \text{ in } \Omega, \tag{1.1}$$

where μ_0 is the magnetic permeability of free space.

The standard Maxwell equations are satisfied throughout Ω so that

$$\epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} - \nabla \times \boldsymbol{H} + \sigma \boldsymbol{E} = -\boldsymbol{J}, \tag{1.2}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \times \boldsymbol{E} = 0. \tag{1.3}$$

Here ϵ_0 is the permeability of free space, σ the conductivity and $\boldsymbol{J}$ is the applied current density. The conductivity σ vanishes in free space but may be non-zero in the ferromagnet. A spatially dependent permittivity ϵ would also be simple to implement using the method presented in this paper.

To complete the system of equations, we need an equation for the magnetization $\boldsymbol{M}$. This is provided by the Landau-Lifchitz-Gilbert equation (denoted LLG in this paper). This states that

$$\frac{\partial \boldsymbol{M}}{\partial t} = |\gamma| \boldsymbol{G}(\boldsymbol{H}, \boldsymbol{M}) \times \boldsymbol{M} + \alpha \frac{\boldsymbol{M}}{|\boldsymbol{M}|} \times \frac{\partial \boldsymbol{M}}{\partial t}, \tag{1.4}$$

or equivalently (see for example [10])

$$\frac{\partial \boldsymbol{M}}{\partial t} = \frac{|\gamma|}{1+\alpha^2} \left(\boldsymbol{G}(\boldsymbol{H}, \boldsymbol{M}) \times \boldsymbol{M} + \alpha \frac{\boldsymbol{M}}{|\boldsymbol{M}|} \times \left(\boldsymbol{G}(\boldsymbol{H}, \boldsymbol{M}) \times \boldsymbol{M}\right) \right), \tag{1.5}$$

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where γ is the gyro-magnetic factor, α is the damping constant, and $\boldsymbol{G}(\boldsymbol{H},\boldsymbol{M})$ is the total magnetic field in the ferromagnet.

This field arises from a number of contributions because the magnetic field $\boldsymbol{H}$ can be augmented by an effective field $\boldsymbol{H}_l(\boldsymbol{M})$ deriving from a local magnetic energy $\mathcal{E}(\boldsymbol{M})$:

$$\boldsymbol{G}(\boldsymbol{H},\boldsymbol{M}) \;=\; \boldsymbol{H}+\boldsymbol{H}_l(\boldsymbol{M}) \quad \text{and} \quad \frac{1}{2}\mu_0\int_{\Omega_M}\boldsymbol{M}\cdot\boldsymbol{H}_l(\boldsymbol{M})d\boldsymbol{x} \;=\; -\,\mathcal{E}(\boldsymbol{M})\,. \tag{1.6}$$

Here $\mathcal{E}(\boldsymbol{M})$ denotes a suitable positive definite quadratic form so that $\boldsymbol{H}_l$ is linear with respect to $\boldsymbol{M}$. In this paper, in particular for numerics, we will deal with the two most frequently retained terms in the literature (see for instance [18],[3],[10]), namely:

Energy of anisotropy: The crystal structure of the ferromagnet introduces preferred directions for the magnetization. Here we consider the simplest case of a uniaxial crystal. In this case there is a preferred direction, called the easy axis, in the direction of a unit vector $\boldsymbol{p}$. Let $P(\boldsymbol{M})$ denote the projection of $\boldsymbol{M}$ on the plane perpendicular to $\boldsymbol{p}$ so that

$$P(\boldsymbol{M})=\boldsymbol{M}-(\boldsymbol{p}\cdot\boldsymbol{M})\boldsymbol{p}.$$

then the energy of anisotropy is

$$\mathcal{E}_{an}(\boldsymbol{M},t)=\int_{\Omega_M}K|P(\boldsymbol{M}(\boldsymbol{x},t))|^2\,d\boldsymbol{x}.$$

Exchange energy: This energy gives cohesion to the magnetization or, from a mathematical point of view, it ensures that $\boldsymbol{M}\in H^1(\Omega_M)$. We take the simple form

$$\mathcal{E}_{ex}(\boldsymbol{M},t)=\int_{\Omega_M}A\left(\sum_i\left|\frac{\partial\boldsymbol{M}(\boldsymbol{x},t)}{\partial x_i}\right|^2\right)d\boldsymbol{x}.$$

On $\Gamma_M=\partial\Omega_M$, the boundary of the ferromagnet, it is necessary to assume a boundary condition. Following [13], we use

$$\frac{\partial\boldsymbol{M}}{\partial n} \;=\; 0 \quad \text{on } \partial\Omega_M\,, \tag{1.7}$$

where n is the unit outward normal to Ω_M.

Hence, in this paper, we use

$$\mathcal{E}(\boldsymbol{M},t) \;=\; \int_{\Omega_M}K|P(\boldsymbol{M}(\boldsymbol{x},t))|^2\,d\boldsymbol{x}+\int_{\Omega_M}A\left(\sum_i\left|\frac{\partial\boldsymbol{M}(\boldsymbol{x},t)}{\partial x_i}\right|^2\right)d\boldsymbol{x} \tag{1.8}$$

where A and K are constants characterizing the material.

Thus, at least formally, the effective field is given by

$$\boldsymbol{H}_l(\boldsymbol{M})=-\frac{2K}{\mu_0}P(\boldsymbol{M})+\frac{2A}{\mu_0}\Delta\boldsymbol{M}.$$

and

$$\boldsymbol{G}(\boldsymbol{H},\boldsymbol{M})=\boldsymbol{H}-\frac{2K}{\mu_0}P(\boldsymbol{M})+\frac{2A}{\mu_0}\Delta\boldsymbol{M}.$$

REMARK 1.1. *The coefficients A, K and α and the unit vector p, are all constants in time but may depend on space (the only assumption is then that α, A and K are non negative and uniformly bounded). For the sake of simplicity of the presentation they are assumed, in this paper, to be true constants in time and space.*

REMARK 1.2. *We have not included a static magnetic field in the expression for $\boldsymbol{G}$ since such a field can be included in the magnetic field $\boldsymbol{H}$. Thus the magnetic field $\boldsymbol{H}$ includes both the demagnetizing field* *and the static field.* In this model, the ferromagnet occupies exactly the region of space where $\boldsymbol{M} \neq 0$ so that

$$\Omega_M = \{\boldsymbol{x} \in \Omega \mid M(\boldsymbol{x}, t) \neq 0 \text{ for all } t\}.$$

This makes sense since a first consequence of the LLG equation is that the norm of $\boldsymbol{M}$ is independent of time. To see this, at least formally, we take the dot product of the LLG equation (1.4) with $\boldsymbol{M}$ and conclude that

$$\frac{1}{2}\frac{\partial}{\partial t}|\boldsymbol{M}(\boldsymbol{x}, t)|^2 = \frac{\partial \boldsymbol{M}}{\partial t}(\boldsymbol{x}, t) \cdot \boldsymbol{M}(\boldsymbol{x}, t) = 0.$$

Thus, for almost every $\boldsymbol{x} \in \Omega$, we have the following conservation of the norm of $\boldsymbol{M}$:

$$|\boldsymbol{M}(\boldsymbol{x}, t)| = |\boldsymbol{M}(\boldsymbol{x}, 0)|. \tag{1.9}$$

Because of this pointwise conservation of the norm of $\boldsymbol{M}$, the extent of the ferromagnet is determined by the initial distribution of magnetization.

REMARK 1.3. *In most works (see for example [18] or [3]) the norm of $\boldsymbol{M}$ is also assumed to be constant in space. Throughout this paper, we shall assume that $|\boldsymbol{M}| \in L^\infty(\Omega_M)$.*

Using the equations (1.1), (1.2), (1.3) and (1.4) and eliminating $\boldsymbol{B}$ gives us the following system which we call the Maxwell-LLG system. The electromagnetic field $(\boldsymbol{E}, \boldsymbol{H}, \boldsymbol{M})$ satisfies

$$\epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} - \nabla \times \boldsymbol{H} + \sigma \boldsymbol{E} = -\boldsymbol{J} \tag{1.10}$$

$$\mu_0 \frac{\partial \boldsymbol{H}}{\partial t} + \nabla \times \boldsymbol{E} = -\mu_0 \frac{\partial \boldsymbol{M}}{\partial t} \tag{1.11}$$

$$\frac{\partial \boldsymbol{M}}{\partial t} = |\gamma| \boldsymbol{G}(\boldsymbol{H}, \boldsymbol{M}) \times \boldsymbol{M} + \frac{\alpha}{|\boldsymbol{M}|} \boldsymbol{M} \times \frac{\partial \boldsymbol{M}}{\partial t} \tag{1.12}$$

in Ω where

$$\boldsymbol{G}(\boldsymbol{H}, \boldsymbol{M}) = \boldsymbol{H} + \boldsymbol{H}_l(\boldsymbol{M}).$$

For simplicity, we have assumed that the boundary of the overall domain Ω is perfectly conducting. Thus

$$\boldsymbol{\nu} \times \boldsymbol{E} = 0 \text{ on } \Gamma = \partial\Omega \tag{1.13}$$

where $\boldsymbol{\nu}$ is the unit outward normal to Ω.

Finally, we assume that $\boldsymbol{E}$, $\boldsymbol{H}$ and $\boldsymbol{M}$ are specified at time $t = 0$. Thus

$$\boldsymbol{E}(t=0) = \boldsymbol{E}_0, \quad \boldsymbol{H}(t=0) = \boldsymbol{H}_0, \text{ and } \boldsymbol{M}(t=0) = \boldsymbol{M}_0 \tag{1.14}$$

where $\boldsymbol{E}_0$, $\boldsymbol{H}_0$ and $\boldsymbol{M}_0$ are given functions. For physical reasons, these initial fields must satisfy the constraint that

$$\nabla \cdot (\boldsymbol{H}_0 + \boldsymbol{M}_0) = 0 \text{ in } \Omega \text{ and } \boldsymbol{\nu} \cdot (\boldsymbol{H}_0 + \boldsymbol{M}_0) = 0 \text{ on } \Gamma.$$

This ensures that $\boldsymbol{B}$ is divergence free.

REMARK 1.4. *Adding a magnetostatic contribution to the LLG equation (like the demagnetising field, see [3]) is redundant here because of the coupling with Maxwell's equations. Moreover it must be pointed out that this coupling allows us to model a conducting ferromagnet via $\sigma \neq 0$. It is not clear that this would be possible by working with a demagnetising field.*

The Maxwell-LLG system (1.10)–(1.14) has been studied first by Visintin [18], who established existence of weak solutions in the three dimensional case, and more recently by Carbou and Fabrie who consider regular solutions local in time [7]. Concerning the numerical analysis of this system, closely related works include [10] and [14] where the exchange contribution is neglected, and [11] and [19] where Maxwell's equations are taken under the magnetostatic limit. In these works energy decay is always ensured but in the first two only the conservation of the norm of $\boldsymbol{M}$ is inherited by the numerical method.

The outline of this paper is as follows. In section 2, we give the main properties of the continuous problem, namely the conservation of the norm of $\boldsymbol{M}$ and the decay of the electromagnetic energy, but also, in addition to these known results, we define new Liapunov functions associated with the nonlinear problem. In section 3, we introduce a weak formulation adapted to the continuous problem in the sense that a variational version of each previous result is given. Moreover we show for this formulation a continuous dependence result. In section 4, we explain how a certain class of finite element methods can be used to approximate the Maxwell-LLG equations while preserving energy decay, the norm of $\boldsymbol{M}$ and the Liapunov functions as shown in section 5. Finally a fully discrete scheme is described and we provide some numerical results to illustrate the method in section 6.

2. Mathematical properties of the continuous problem. Here we shall present some derivations which lead to certain a priori estimates and suitable Liapunov functions associated to the nonlinear Cauchy problem (1.10)– (1.12).

We define the space of vector functions $W^{1,\infty}(\Omega_M)$ by

$$W^{1,\infty}(\Omega_M) \;=\; \left\{ \boldsymbol{u} \in \left(L^\infty(\Omega_M)\right)^3 \mid \forall i,\, \frac{\partial \boldsymbol{u}}{\partial x_i} \in \left(L^\infty(\Omega_M)\right)^3 \right\}.$$

For ease of notation we shall write $V = W^{1,\infty}(\Omega_M)$ equipped with the norm

$$\|\boldsymbol{u}\|_V \;=\; \|\boldsymbol{u}\|_\infty + \sum_{i=1}^{3} \left\| \frac{\partial \boldsymbol{u}}{\partial x_i} \right\|_\infty .$$

It is clear that this space is an algebra. Indeed it is known (see [4] p.155) that $H^1(\Omega) \cap L^\infty(\Omega)$ is an algebra such that

$$u, v \in H^1(\Omega) \cap L^\infty(\Omega)\,, \quad \frac{\partial}{\partial x_i}(uv) \;=\; \frac{\partial u}{\partial x_i} v + u \frac{\partial v}{\partial x_i}\,.$$

It is straightforward to see that for any $\boldsymbol{u}, \boldsymbol{v} \in V$,

$$\|\boldsymbol{u}\boldsymbol{v}\|_V \;\le\; 2\|\boldsymbol{u}\|_V \|\boldsymbol{v}\|_V .$$

We also recall that

$$\begin{aligned} H(\mathrm{curl};\Omega) &= \{\boldsymbol{u} \in (L^2(\Omega))^3 \mid \nabla \times \boldsymbol{u} \in (L^2(\Omega))^3\}, \\ H_0(\mathrm{curl};\Omega) &= \{\boldsymbol{u} \in H(\mathrm{curl};\Omega) \mid \boldsymbol{\nu} \times \boldsymbol{u} = 0 \text{ on } \partial\Omega\}. \end{aligned}$$

In this section we shall assume that there exists regular enough solutions to the Maxwell-LLG problem, namely

$$\begin{aligned} \boldsymbol{E} &\in C^1\left(\mathbb{R}_+, (L^2(\Omega))^3\right) \cap L^\infty\left(\mathbb{R}_+, H_0(\mathrm{curl};\Omega)\right), \\ \boldsymbol{H} &\in C^1\left(\mathbb{R}_+, (L^2(\Omega))^3\right) \cap L^\infty\left(\mathbb{R}_+, H(\mathrm{curl};\Omega)\right), \\ \boldsymbol{M} &\in C^1\left(\mathbb{R}_+, (L^\infty(\Omega))^3\right) \cap L^\infty\left(\mathbb{R}_+, V\right). \end{aligned}$$

We have already seen one classical conservation result (see [10] or [18] for complete proofs):

$$|\boldsymbol{M}(\boldsymbol{x},t)| = |\boldsymbol{M}(\boldsymbol{x},0)|\,, \text{ a.e. } \boldsymbol{x} \in \Omega_M.$$

Another classical result is the decay of the total electromagnetic energy (when $\boldsymbol{J} = 0$). This can easily be derived from the Maxwell-LLG system, but we shall derive it from a new and more general result concerning Liapunov functions for the Maxwell-LLG system.

2.1. Stationary states and associated Liapunov functions. Despite energy decay, the long time convergence of solutions to the Maxwell-LLG equations is still an open question. The only known result has been obtained by Joly, Komech, Vacus (see [9]) under restrictive assumptions (see also [6], [2] for weak convergence results). For this reason it seems to be interesting to get as much information as possible about the behavior of solutions for long time. The goal of this section is to introduce new Liapunov functions for the Maxwell-LLG system.

We suppose that the triplet $(\widetilde{\boldsymbol{E}}, \widetilde{\boldsymbol{H}}, \widetilde{\boldsymbol{M}}) \in H_0(\mathrm{curl};\Omega) \times H(\mathrm{curl};\Omega) \times V$ is a stationary state of the Maxwell-LLG equations (i.e. the functions satisfy the Maxwell-LLG system with zero time derivative). Note that if $\boldsymbol{J} = 0$ this implies that

$$\sigma \widetilde{\boldsymbol{E}} - \nabla \times \widetilde{\boldsymbol{H}} = 0 \text{ in } \Omega. \tag{2.1}$$

Another important property of the steady state that we shall use is that $\widetilde{\boldsymbol{M}}$ and $\widetilde{\boldsymbol{H}}$ must satisfy

$$\left| \boldsymbol{G}(\widetilde{\boldsymbol{H}}, \widetilde{\boldsymbol{M}}) \times \widetilde{\boldsymbol{M}} \right| = 0, \text{ a.e. } \boldsymbol{x} \in \Omega_M.$$

To establish this property it suffices to note that from the LLG equation, via Pythagorus's theorem,

$$\left| \frac{\partial \boldsymbol{M}}{\partial t} \right|^2 + \left| \frac{\alpha}{|\boldsymbol{M}|} \boldsymbol{M} \times \frac{\partial \boldsymbol{M}}{\partial t} \right|^2 = \gamma^2 \left| \boldsymbol{G}(\boldsymbol{H}, \boldsymbol{M}) \times \boldsymbol{M} \right|^2 .$$

With this observation we make the following definition:

Definition 2.1. *Let $(\widetilde{\boldsymbol{E}}, \widetilde{\boldsymbol{H}}, \widetilde{\boldsymbol{M}})$ be a stationary state. We denote by λ the "indicator" defined by*

$$\lambda = \frac{\boldsymbol{G}(\widetilde{\boldsymbol{H}}, \widetilde{\boldsymbol{M}}) \cdot \widetilde{\boldsymbol{M}}}{|\widetilde{\boldsymbol{M}}|^2} \quad \textit{in } \Omega_M.$$

The main properties of the indicator λ are:

1. The indicator λ is well defined if $|\widetilde{\boldsymbol{M}}|$ is bounded below. Suppose there is a constant M_- such that $|\boldsymbol{M}(\boldsymbol{x}, 0)| \geq M_- > 0$ for a.e. $\boldsymbol{x} \in \Omega_M$ then the indicator λ is in $L^1(\Omega_M)$:
$$\begin{aligned} M_-^2 \int_{\Omega_M} |\lambda| d\boldsymbol{x} \leq \int_{\Omega_M} |\lambda|\, |\widetilde{\boldsymbol{M}}|^2 d\boldsymbol{x} &= \int_{\Omega_M} |\boldsymbol{G}(\widetilde{\boldsymbol{H}}, \widetilde{\boldsymbol{M}}) \cdot \widetilde{\boldsymbol{M}}|\, dA \\ &\leq \|\boldsymbol{G}\|_{H^{-1}(\Omega_M)} \|\widetilde{\boldsymbol{M}}\|_{H^1(\Omega_M)} < \infty. \end{aligned}$$
2. Almost everywhere
$$\boldsymbol{G}(\widetilde{\boldsymbol{H}}, \widetilde{\boldsymbol{M}}) = \lambda \widetilde{\boldsymbol{M}} ,$$
since almost everywhere
$$\boldsymbol{G}(\widetilde{\boldsymbol{H}}, \widetilde{\boldsymbol{M}}) \times \widetilde{\boldsymbol{M}} = 0 \quad , \quad \boldsymbol{G}(\widetilde{\boldsymbol{H}}, \widetilde{\boldsymbol{M}}) \cdot \widetilde{\boldsymbol{M}} = \lambda |\widetilde{\boldsymbol{M}}|^2 . \tag{2.2}$$

Now we can prove the following theorem:

Theorem 2.1. *Suppose $\boldsymbol{J} = 0$. Given $(\widetilde{\boldsymbol{E}}, \widetilde{\boldsymbol{H}}, \widetilde{\boldsymbol{M}})$ a stationary state and λ the associated indicator, then the time dependent function*

$$\begin{aligned} V_\lambda(\boldsymbol{E}, \boldsymbol{H}, \boldsymbol{M}) = \frac{1}{2} \int_\Omega \left(\epsilon_0 \left| \boldsymbol{E} - \widetilde{\boldsymbol{E}} \right|^2 + \mu_0 \left| \boldsymbol{H} - \widetilde{\boldsymbol{H}} \right|^2 \right) d\boldsymbol{x} + \frac{1}{2} \int_{\Omega_M} \mu_0 \lambda \left| \boldsymbol{M} - \widetilde{\boldsymbol{M}} \right|^2 d\boldsymbol{x} \\ + \mathcal{E}(\boldsymbol{M} - \widetilde{\boldsymbol{M}}) \end{aligned}$$

is a strict Liapunov function. More precisely,

$$\frac{d}{dt} V_\lambda(\boldsymbol{E}, \boldsymbol{H}, \boldsymbol{M}) = -\frac{\mu_0}{|\gamma|} \int_\Omega \frac{\alpha}{|\boldsymbol{M}|} \left| \frac{\partial \boldsymbol{M}}{\partial t} \right|^2 d\boldsymbol{x} - \int_\Omega \sigma |\boldsymbol{E}|^2 \, d\boldsymbol{x} . \tag{2.3}$$

This result is interesting for several reasons:

1. If we consider the special case of $\boldsymbol{J}=0$ and $\widetilde{\boldsymbol{E}}=\widetilde{\boldsymbol{H}}=\widetilde{\boldsymbol{M}}=0$ in Ω, then we have a solution of the static equations Maxwell-LLG equations. Taking the indicator $\lambda=0$ we see that

$$V_0(\boldsymbol{E},\boldsymbol{H},\boldsymbol{M}) \;=\; \frac{1}{2}\int_\Omega \left(\epsilon_0\,|\boldsymbol{E}|^2+\mu_0\,|\boldsymbol{H}|^2\right)\,d\boldsymbol{x}+\mathcal{E}(\boldsymbol{M}).$$

   This is the electromagnetic energy of the field. Thus if we define

$$W(\boldsymbol{E},\boldsymbol{H},\boldsymbol{M})=V_0(\boldsymbol{E},\boldsymbol{H},\boldsymbol{M})$$

   we obtain the classical dissipation of energy

$$\frac{d}{dt}W(\boldsymbol{E},\boldsymbol{H},\boldsymbol{M})\le 0.$$

   Note that any other choice of a constant value for λ would give an equivalent energy differing from our choice by a constant.
2. The decay of V_λ is at the same rate as the energy decay:

$$\frac{d}{dt}V_\lambda(\boldsymbol{E},\boldsymbol{H},\boldsymbol{M}) \;=\; \frac{d}{dt}W(\boldsymbol{E},\boldsymbol{H},\boldsymbol{M})\,. \tag{2.4}$$

   This follows because the right hand side of (2.3) is independent of λ.
3. If $(\boldsymbol{E},\boldsymbol{H},\boldsymbol{M})$ converges to $(\widetilde{\boldsymbol{E}},\widetilde{\boldsymbol{H}},\widetilde{\boldsymbol{M}})$ as $t\to\infty$, then, for the associated indicator λ,

$$V_\lambda(\boldsymbol{E},\boldsymbol{H},\boldsymbol{M}) \;\to\; 0\,.$$

The identity (2.4) can be integrated over the time to give a necessary condition for convergence:

Corollary 2.2. *Let $(\boldsymbol{E},\boldsymbol{H},\boldsymbol{M})$ be a solution of (1.10)-(1.12) with initial data $(\boldsymbol{E}_0,\boldsymbol{H}_0,\boldsymbol{M}_0)$ and suppose that $\boldsymbol{J}=0$. If $(\boldsymbol{E},\boldsymbol{H},\boldsymbol{M})$ converges to $(\widetilde{\boldsymbol{E}},\widetilde{\boldsymbol{H}},\widetilde{\boldsymbol{M}})$, then necessarily*

$$W(\boldsymbol{E}_0,\boldsymbol{H}_0,\boldsymbol{M}_0)-W(\widetilde{\boldsymbol{E}},\widetilde{\boldsymbol{H}},\widetilde{\boldsymbol{M}}) \;=\; V_\lambda(\boldsymbol{E}_0,\boldsymbol{H}_0,\boldsymbol{M}_0)\,.$$

A paper is in preparation where these ideas are used to study the asymptotic behavior of solutions to (1.10) – (1.12). The discretization that we shall describe also guarantees the decay of discrete Liapunov functions, and is part of the reason why we can call the method an accurate discretization.

Proof. (of Theorem 2.1.) Let $(\boldsymbol{E},\boldsymbol{H},\boldsymbol{M})$ be a solution to Maxwell-LLG equations associated to initial data $(\boldsymbol{E}_0,\boldsymbol{H}_0,\boldsymbol{M}_0)$ and $(\widetilde{\boldsymbol{E}},\widetilde{\boldsymbol{H}},\widetilde{\boldsymbol{M}})$ be some stationary state. Let us now consider the difference between them and denote

$$\boldsymbol{N}_E=\boldsymbol{E}-\widetilde{\boldsymbol{E}}\,,\quad \boldsymbol{N}_H=\boldsymbol{H}-\widetilde{\boldsymbol{H}}\,,\quad \text{and } \boldsymbol{N}_M=\boldsymbol{M}-\widetilde{\boldsymbol{M}}\,. \tag{2.5}$$

It is convenient to define $\boldsymbol{N}_M$ in all of Ω by extending $\boldsymbol{N}_M$ on Ω_M by zero. We also define $\alpha'=\alpha/|\boldsymbol{M}|$. Since Maxwell's equations are linear and $\partial\widetilde{\boldsymbol{M}}/\partial t=0$, it is straightforward to check (using (2.1) that the triplet $(\boldsymbol{N}_E,\boldsymbol{N}_H,\boldsymbol{N}_M)$ is a solution to the following problem:

$$\epsilon_0\frac{\partial\boldsymbol{N}_E}{\partial t}+\sigma\boldsymbol{N}_E-\nabla\times\boldsymbol{N}_H=0, \tag{2.6}$$

$$\mu_0\frac{\partial\boldsymbol{N}_H}{\partial t}+\nabla\times\boldsymbol{N}_E=-\mu_0\frac{\partial\boldsymbol{N}_M}{\partial t}, \tag{2.7}$$

$$\frac{\partial\boldsymbol{N}_M}{\partial t}=|\gamma|\boldsymbol{G}(\boldsymbol{H},\boldsymbol{M})\times\boldsymbol{M}+\alpha'\boldsymbol{M}\times\frac{\partial\boldsymbol{N}_M}{\partial t}\,. \tag{2.8}$$

We now rewrite the right hand side of the last equation as a function of $\boldsymbol{N}_H$, $\boldsymbol{N}_M$ and $\widetilde{\boldsymbol{M}}$ only. First we have

$$\begin{aligned}\boldsymbol{G}(\boldsymbol{H},\boldsymbol{M})&=\boldsymbol{H}+\boldsymbol{H}_l(\boldsymbol{M})\\&=\left[\boldsymbol{H}-\widetilde{\boldsymbol{H}}\right]+\boldsymbol{H}_l\left(\boldsymbol{M}-\widetilde{\boldsymbol{M}}\right)+\left[\widetilde{\boldsymbol{H}}+\boldsymbol{H}_l\left(\widetilde{\boldsymbol{M}}\right)\right]\\&=\boldsymbol{N}_H+\boldsymbol{H}_l\left(\boldsymbol{N}_M\right)+\boldsymbol{G}(\widetilde{\boldsymbol{H}},\widetilde{\boldsymbol{M}})\\&=\boldsymbol{N}_H+\boldsymbol{H}_l\left(\boldsymbol{N}_M\right)+\lambda\widetilde{\boldsymbol{M}}\,.\end{aligned} \tag{2.9}$$

Besides,

$$\boldsymbol{M} \times \frac{\partial \boldsymbol{N}_M}{\partial t} \;=\; \boldsymbol{N}_M \times \frac{\partial \boldsymbol{N}_M}{\partial t} + \widetilde{\boldsymbol{M}} \times \frac{\partial \boldsymbol{N}_M}{\partial t} \ .$$

Thus

$$\frac{\partial \boldsymbol{N}_M}{\partial t} \;=\; |\gamma| \left[(\boldsymbol{N}_H + \boldsymbol{H}_l(\boldsymbol{N}_M)) \times \boldsymbol{M} + \lambda \widetilde{\boldsymbol{M}} \times \boldsymbol{M} \right] + \alpha' \left[\boldsymbol{N}_M \times \frac{\partial \boldsymbol{N}_M}{\partial t} + \widetilde{\boldsymbol{M}} \times \frac{\partial \boldsymbol{N}_M}{\partial t} \right] \ ,$$

and finally we obtain

$$\begin{aligned} \frac{\partial \boldsymbol{N}_M}{\partial t} = |\gamma| \Big[& (\boldsymbol{N}_H + \boldsymbol{H}_l(\boldsymbol{N}_M)) \times \boldsymbol{N}_M + (\boldsymbol{N}_H + \boldsymbol{H}_l(\boldsymbol{N}_M)) \times \widetilde{\boldsymbol{M}} \\ & + \lambda \widetilde{\boldsymbol{M}} \times \boldsymbol{N}_M \Big] + \alpha' \left[\boldsymbol{N}_M \times \frac{\partial \boldsymbol{N}_M}{\partial t} + \widetilde{\boldsymbol{M}} \times \frac{\partial \boldsymbol{N}_M}{\partial t} \right] \ . \end{aligned} \tag{2.10}$$

Now on the one hand we take the dot product of (2.6) (resp. (2.7)) by $\boldsymbol{N}_E$ (resp.$\boldsymbol{N}_H$) to get, by adding and integrating,

$$\frac{1}{2} \frac{d}{dt} \left[\int_\Omega \epsilon_0 \, |\boldsymbol{N}_E|^2 + \mu_0 \, |\boldsymbol{N}_H|^2 \right] d\boldsymbol{x} + \int_\Omega \sigma |\boldsymbol{N}_E|^2 \, d\boldsymbol{x} \;=\; -\mu_0 \int_\Omega \boldsymbol{N}_H \cdot \frac{\partial \boldsymbol{N}_M}{\partial t} \, d\boldsymbol{x} \ , \tag{2.11}$$

while on the other hand a simple expansion (using (1.8) and (2.9)) leads to

$$\begin{aligned} & \frac{d}{dt} \left[\mathcal{E}(\boldsymbol{N}_M) + \int_\Omega \frac{\mu_0}{2} \lambda |\boldsymbol{N}_M|^2 d\boldsymbol{x} \right] + \int_\Omega \mu_0 \frac{\alpha'}{|\gamma|} \left| \frac{\partial \boldsymbol{N}_M}{\partial t} \right|^2 d\boldsymbol{x} \\ & = \mu_0 \int_\Omega \left(\lambda \boldsymbol{N}_M - \boldsymbol{H}_l(\boldsymbol{N}_M) + \frac{\alpha'}{|\gamma|} \frac{\partial \boldsymbol{N}_M}{\partial t} \right) \cdot \frac{\partial \boldsymbol{N}_M}{\partial t} \, d\boldsymbol{x} \ . \end{aligned} \tag{2.12}$$

By adding (2.11) and (2.12), we obtain that

$$\begin{aligned} & \frac{d}{dt} \left[\frac{1}{2} \int_\Omega \left(\epsilon_0 \, |\boldsymbol{N}_E|^2 + \mu_0 \, |\boldsymbol{N}_H|^2 + \mu_0 \lambda |\boldsymbol{N}_M|^2 \right) d\boldsymbol{x} + \mathcal{E}(\boldsymbol{N}_M) \right] + \int_\Omega \mu_0 \frac{\alpha'}{|\gamma|} \left| \frac{\partial \boldsymbol{N}_M}{\partial t} \right|^2 d\boldsymbol{x} \\ & + \int_\Omega \sigma |\boldsymbol{N}_E|^2 \, d\boldsymbol{x} \;=\; \mu_0 \int_\Omega \left(\lambda \boldsymbol{N}_M - (\boldsymbol{N}_H + \boldsymbol{H}_l(\boldsymbol{N}_M)) + \frac{\alpha'}{|\gamma|} \frac{\partial \boldsymbol{N}_M}{\partial t} \right) \cdot \frac{\partial \boldsymbol{N}_M}{\partial t} \, d\boldsymbol{x} \ . \end{aligned} \tag{2.13}$$

It now remains to show that the right hand side vanishes. We take the dot product of equation (2.10) successively by

$$\lambda \boldsymbol{N}_M \ , \ -(\boldsymbol{N}_H + \boldsymbol{H}_l(\boldsymbol{N}_M)) \ \text{ and } \ \frac{\alpha'}{|\gamma|} \frac{\partial \boldsymbol{N}_M}{\partial t} \ .$$

Using the triple product notation that

$$(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = \boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w})$$

we get

$$\lambda \boldsymbol{N}_M \cdot \frac{\partial \boldsymbol{N}_M}{\partial t} \;=\; \lambda |\gamma| \left(\boldsymbol{N}_M, \boldsymbol{N}_H + \boldsymbol{H}_l(\boldsymbol{N}_M), \widetilde{\boldsymbol{M}} \right) + \alpha' \lambda \left(\boldsymbol{N}_M, \widetilde{\boldsymbol{M}}, \frac{\partial \boldsymbol{N}_M}{\partial t} \right) \ , \tag{2.14}$$

$$\begin{aligned} -(\boldsymbol{N}_H + \boldsymbol{H}_l(\boldsymbol{N}_M)) \cdot \frac{\partial \boldsymbol{N}_M}{\partial t} \;=\; & -\alpha' \left(\boldsymbol{N}_H + \boldsymbol{H}_l(\boldsymbol{N}_M), \boldsymbol{N}_M, \frac{\partial \boldsymbol{N}_M}{\partial t} \right) \\ & -\lambda |\gamma| \left(\boldsymbol{N}_H + \boldsymbol{H}_l(\boldsymbol{N}_M), \widetilde{\boldsymbol{M}}, \boldsymbol{N}_M \right) - \alpha' \left(\boldsymbol{N}_H + \boldsymbol{H}_l(\boldsymbol{N}_M), \widetilde{\boldsymbol{M}}, \frac{\partial \boldsymbol{N}_M}{\partial t} \right) \ , \end{aligned} \tag{2.15}$$

$$\begin{aligned} \frac{\alpha'}{|\gamma|} \frac{\partial \boldsymbol{N}_M}{\partial t} \cdot \frac{\partial \boldsymbol{N}_M}{\partial t} \;=\; & \alpha' \left(\frac{\partial \boldsymbol{N}_M}{\partial t}, \boldsymbol{N}_H + \boldsymbol{H}_l(\boldsymbol{N}_M), \boldsymbol{N}_M \right) + \alpha' \lambda \left(\frac{\partial \boldsymbol{N}_M}{\partial t}, \widetilde{\boldsymbol{M}}, \boldsymbol{N}_M \right) \\ & + \alpha' \left(\frac{\partial \boldsymbol{N}_M}{\partial t}, \boldsymbol{N}_H + \boldsymbol{H}_l(\boldsymbol{N}_M), \widetilde{\boldsymbol{M}} \right) \ . \end{aligned} \tag{2.16}$$

It then suffices to add and integrate the three identities to get the result. □

3. A weak formulation. Our goal here is to write a weak formulation suitable for finite element discretization. The weak formulation explicitly includes the total magnetic field $\boldsymbol{G}$ as an unknown and so differs from the weak formulation used by Visintin in [18] or Yang and Fredkin in [19]. In trying to use Visintin's formulation we were unable to verify energy decay for the corresponding discrete system. Having $\boldsymbol{G}$ available as a variable makes the proof of energy decay possible. However our formulation does not seem well suited to proving existence which was the aim of Visintin's formulation.

In order to write a variational problem suitable for finite element discretization we need some notation. Let

$$(\boldsymbol{u},\boldsymbol{v})=\int_\Omega \boldsymbol{u}\cdot\boldsymbol{v}\,dV$$

and $\|\boldsymbol{u}\|=\sqrt{(\boldsymbol{u},\boldsymbol{u})}$. We use the usual notation for Sobolev spaces of scalar function so that $H^s(\Omega)$ denotes the space of functions with s square integrable derivatives on Ω equipped with the norm $\|.\|_s$ (so $\|.\|_0=\|.\|$).

The weak formulation is obtained by multiplying each equation by a smooth test function and integrating by parts. If we seek a solution on the time interval $[0,T]$, we are lead to the variational problem of finding

$$\begin{aligned}
\boldsymbol{E}&\in C^1([0,T];(L^2(\Omega))^3)\cap C([0,T];H_0(\mathrm{curl};\Omega_M)),\\
\boldsymbol{H}&\in C^1([0,T];(L^2(\Omega))^3),\\
\boldsymbol{M}&\in C^1([0,T];(L^2(\Omega))^3)\cap C([0,T];(H^1(\Omega_M))^3),\\
\text{and } \boldsymbol{G}&\in C^1([0,T];(L^2(\Omega_M))^3)\cap C([0,T];(H^1(\Omega_M))^3),
\end{aligned}$$

such that

$$\left(\varepsilon_0\frac{\partial\boldsymbol{E}}{\partial t},\boldsymbol{\psi}\right)-(\boldsymbol{H},\nabla\times\boldsymbol{\psi})+(\sigma\boldsymbol{E},\boldsymbol{\psi})=-(\boldsymbol{J},\boldsymbol{\psi}),\forall\boldsymbol{\psi}\in H_0(\mathrm{curl};\Omega)\,,\tag{3.1}$$

$$\left(\mu_0\frac{\partial\boldsymbol{H}}{\partial t},\boldsymbol{\varphi}\right)+(\nabla\times\boldsymbol{E},\boldsymbol{\varphi})=-\left(\mu_0\frac{\partial\boldsymbol{M}}{\partial t},\boldsymbol{\varphi}\right),\forall\boldsymbol{\varphi}\in(L^2(\Omega))^3\,,\tag{3.2}$$

$$\left(\frac{\partial\boldsymbol{M}}{\partial t},\boldsymbol{\xi}\right)\;=\;|\gamma|\,(\boldsymbol{G}\times\boldsymbol{M},\boldsymbol{\xi})+\alpha\left(\frac{\boldsymbol{M}}{|\boldsymbol{M}|}\times\frac{\partial\boldsymbol{M}}{\partial t},\boldsymbol{\xi}\right),\forall\boldsymbol{\xi}\in(H^1(\Omega_M))^3\,,\tag{3.3}$$

$$\begin{aligned}
(\boldsymbol{G},\boldsymbol{\phi})\;&=\;(\boldsymbol{H},\boldsymbol{\phi})-\frac{2}{\mu_0}\left(KP\left(\boldsymbol{M}\right),\boldsymbol{\phi}\right)\\
&\quad-\frac{2}{\mu_0}\sum_{i=1}^{3}\left(A\frac{\partial\boldsymbol{M}}{\partial x_i},\frac{\partial\boldsymbol{\phi}}{\partial x_i}\right),\forall\boldsymbol{\phi}\in(H^1(\Omega_M))^3\,.
\end{aligned}\tag{3.4}$$

We also need suitable initial data for each variable that guarantees $\nabla\cdot(\boldsymbol{H}+\boldsymbol{M})=0$ in Ω and $(\boldsymbol{H}+\boldsymbol{M})\cdot\nu=0$ on $\partial\Omega$. In particular we assume that at the initial time $\boldsymbol{M}(\cdot,0)\in V$. In this paper shall simply assume the existence of a unique solution to the above variational problem in the indicated spaces.

The variational formulation conserves the pointwise norm of $\boldsymbol{M}$ as stated in the following lemma:

LEMMA 3.1. *For almost every $\boldsymbol{x}\in\Omega$,*

$$|\boldsymbol{M}(\boldsymbol{x},t)|=|\boldsymbol{M}_0(\boldsymbol{x},t)|.$$

REMARK 3.1. *In view of our assumption that $\boldsymbol{M}(\cdot,0)\in V$, this lemma and our previous assumptions on $\boldsymbol{M}$ imply that*

$$\boldsymbol{M}\in C^1([0,T];(L^2(\Omega))^3)\cap C([0,T];(H^1(\Omega_M))^3\cap(L^\infty(\Omega_M))^3).$$

Proof. Let $\phi\in C_0^\infty(\Omega_M)$. By taking $\boldsymbol{\xi}=\phi\boldsymbol{M}$ in (3.3), we obtain

$$\left(\frac{\partial\boldsymbol{M}}{\partial t},\phi\boldsymbol{M}\right)\;=\;|\gamma|\,(\boldsymbol{G}\times\boldsymbol{M},\phi\boldsymbol{M})+\alpha\left(\frac{1}{|\boldsymbol{M}|}\boldsymbol{M}\times\frac{\partial}{\partial t}\boldsymbol{M},\phi\boldsymbol{M}\right)\;=\;0.\tag{3.5}$$

This implies

$$\int_\Omega \phi|\boldsymbol{M}|^2 d\boldsymbol{x} \;=\; \int_\Omega \phi|\boldsymbol{M}_0|^2 d\boldsymbol{x}\,, \quad \forall\phi \in C_0^\infty(\Omega_M). \tag{3.6}$$

This shows that $|\boldsymbol{M}(\boldsymbol{x})| = |\boldsymbol{M}_0(\boldsymbol{x})|$, a.e. $\boldsymbol{x} \in \Omega_M$. ∎

3.1. Liapunov functions and energy decay. Our goal here is to prove Theorem 2.1 for the weak formulation. Essentially this is just a matter of checking that the steps in the proof for strong solutions of the Maxwell-LLG system also have a weak analogue.

THEOREM 3.2. *Under the assumptions of Theorem 2.1, the results of Theorem 2.1 holds for the weak Maxwell-LLG system (3.1)-(3.4).*

REMARK 3.2. *As in the continuous case, choosing $\widetilde{\boldsymbol{E}} = \widetilde{\boldsymbol{H}} = \widetilde{\boldsymbol{M}} = 0$ and $\lambda = 0$ implies energy decay.*

Proof. Using the definition of $\boldsymbol{N}_E$, $\boldsymbol{N}_H$ and $\boldsymbol{N}_M$ from (2.5) and defining

$$\boldsymbol{N}_G = \boldsymbol{G} - \boldsymbol{G}(\widetilde{\boldsymbol{H}}, \widetilde{\boldsymbol{M}}),$$

we obtain the following weak analogue of (2.6)-(2.8) using exactly the same argument as at the beginning of the proof of Theorem 2.1:

$$\left(\varepsilon_0 \frac{\partial \boldsymbol{N}_E}{\partial t}, \boldsymbol{\psi}\right) - (\boldsymbol{N}_H, \nabla\times\boldsymbol{\psi}) + (\sigma\boldsymbol{N}_E, \boldsymbol{\psi}) = 0, \forall \boldsymbol{\psi} \in H_0(\mathrm{curl};\Omega)\,, \tag{3.7}$$

$$\left(\mu_0 \frac{\partial \boldsymbol{N}_H}{\partial t}, \boldsymbol{\varphi}\right) + (\nabla\times\boldsymbol{N}_E, \boldsymbol{\varphi}) = -\left(\mu_0 \frac{\partial \boldsymbol{N}_M}{\partial t}, \boldsymbol{\varphi}\right), \forall \boldsymbol{\varphi} \in (L^2(\Omega))^3\,, \tag{3.8}$$

$$\left(\frac{\partial \boldsymbol{N}_M}{\partial t}, \boldsymbol{\xi}\right) \;=\; |\gamma|\,(\boldsymbol{G}\times\boldsymbol{M}, \boldsymbol{\xi}) + \alpha\left(\frac{\boldsymbol{M}}{|\boldsymbol{M}|}\times\frac{\partial \boldsymbol{N}_M}{\partial t}, \boldsymbol{\xi}\right), \forall \boldsymbol{\xi} \in (H^1(\Omega_M))^3\,, \tag{3.9}$$

$$\begin{aligned}(\boldsymbol{N}_G, \boldsymbol{\phi}) \;&=\; (\boldsymbol{N}_H, \boldsymbol{\phi}) - \frac{2}{\mu_0}\left(KP\left(\boldsymbol{N}_M\right), \boldsymbol{\phi}\right)\\ &\quad -\frac{2}{\mu_0}\sum_{i=1}^{3}\left(A\frac{\partial \boldsymbol{N}_M}{\partial x_i}, \frac{\partial \boldsymbol{\phi}}{\partial x_i}\right), \forall \boldsymbol{\phi} \in (H^1(\Omega_M))^3\,.\end{aligned} \tag{3.10}$$

First we compute the right hand side of (3.9). Since $\boldsymbol{M}, \widetilde{\boldsymbol{M}} \in (L^\infty(\Omega))^3$, we can establish that

$$\left(\alpha'\boldsymbol{M}\times\frac{\partial \boldsymbol{N}_M}{\partial t}, \boldsymbol{\xi}\right) = \left(\alpha'\boldsymbol{N}_M\times\frac{\partial \boldsymbol{N}_M}{\partial t}, \boldsymbol{\xi}\right) + \left(\alpha'\widetilde{\boldsymbol{M}}\times\frac{\partial \boldsymbol{N}_M}{\partial t}, \boldsymbol{\xi}\right)$$

for all $\boldsymbol{\xi} \in (L^2(\Omega))^3$. Similarly, using (2.2),

$$\begin{aligned}|\gamma|(\boldsymbol{G}\times\boldsymbol{M}, \boldsymbol{\xi}) &= |\gamma|(\boldsymbol{N}_G\times\boldsymbol{M}, \boldsymbol{\xi}) + |\gamma|(\boldsymbol{G}(\widetilde{\boldsymbol{H}}, \widetilde{\boldsymbol{M}})\times\boldsymbol{M}, \boldsymbol{\xi})\\ &= |\gamma|(\boldsymbol{N}_G\times\boldsymbol{M}, \boldsymbol{\xi}) + |\gamma|(\lambda\widetilde{\boldsymbol{M}}\times\boldsymbol{M}, \boldsymbol{\xi})\end{aligned}$$

for all $\boldsymbol{\xi} \in (L^2(\Omega))^3$. Using these equalities we obtain the weak analogue of (2.6), (2.7) and (2.10) consisting of (3.7) and (3.8) together with

$$\begin{aligned}\left(\frac{\partial \boldsymbol{N}_M}{\partial t}, \boldsymbol{\xi}\right) &= |\gamma|\left(\boldsymbol{N}_G\times\boldsymbol{N}_M + \boldsymbol{N}_G\times\widetilde{\boldsymbol{M}}, \boldsymbol{\xi}\right)\\ &\quad +|\gamma|\left(\lambda\widetilde{\boldsymbol{M}}\times\boldsymbol{N}_M, \boldsymbol{\xi}\right) + \left(\alpha'\boldsymbol{N}_M\times\frac{\partial \boldsymbol{N}_M}{\partial t} + \alpha'\widetilde{\boldsymbol{M}}\times\frac{\partial \boldsymbol{N}_M}{\partial t}, \boldsymbol{\xi}\right)\,,\end{aligned} \tag{3.11}$$

for all $\boldsymbol{\xi} \in (H^1(\Omega_M))^3$. Note that this equation actually holds for all $\boldsymbol{\xi} \in (L^2(\Omega_M))^3$ by a density argument.

Now selecting $\boldsymbol{\psi} = \boldsymbol{N}_E$ and $\boldsymbol{\varphi} = \boldsymbol{N}_H$ in (3.7) and (3.8) respectively, and adding the result gives us precisely (2.11) and (2.12) is obtained exactly as in the proof of Theorem 2.1.

It remains to prove the weak analogue of (2.14)-(2.16). First we choose $\boldsymbol{\xi} = \lambda\boldsymbol{N}_M$ in (3.11). This gives us the integral of (2.14). Next we choose $\boldsymbol{\xi} = \boldsymbol{N}_G$ to get the integral of (2.15). Finally if we choose

$$\boldsymbol{\xi} = \frac{\alpha}{|\gamma||\boldsymbol{M}|}\frac{\partial \boldsymbol{N}_M}{\partial t}$$

which is possible because $|\boldsymbol{M}| \in L^\infty(\Omega)$, we obtain (2.16). ∎

3.2. Continuous dependence result. As discussed in the introduction, Visintin [18] has proved the existence of a weak solution to the Maxwell-LLG equations globally in time. However, for a restricted special case of the system, Alouges and Soyeur [1] have shown non-uniqueness of solutions for certain initial data. It is not known if this troubling problem also occurs for the full Maxwell-LLG equations.

Here we shall show that smooth solutions are locally unique by proving a continuous dependence result. This also suggests that a numerical method can be safely used to compute an approximation to smooth solutions of the problem, if they exist.

Suppose that $(\boldsymbol{E}_1, \boldsymbol{H}_1, \boldsymbol{M}_1, \boldsymbol{G}_1)$ and $(\boldsymbol{E}_2, \boldsymbol{H}_2, \boldsymbol{M}_2, \boldsymbol{G}_2)$ are two solutions of the Maxwell-LLG equations such that, at time $t = 0$,

$$|\boldsymbol{M}_1(\boldsymbol{x}, 0)| = |\boldsymbol{M}_2(\boldsymbol{x}, 0)| = |\boldsymbol{M}(\boldsymbol{x})|, \text{ a.e. } \boldsymbol{x} \in \Omega_M.$$

Let

$$\begin{aligned} \boldsymbol{e} &= \boldsymbol{E}_1 - \boldsymbol{E}_2, & \boldsymbol{h} &= \boldsymbol{H}_1 - \boldsymbol{H}_2, \\ \boldsymbol{m} &= \boldsymbol{M}_1 - \boldsymbol{M}_2, & \boldsymbol{g} &= \boldsymbol{G}_1 - \boldsymbol{G}_2. \end{aligned}$$

Then we define

$$E(t) \; = \; \frac{1}{2}\left(\epsilon_0\|\boldsymbol{e}\|^2 + \mu_0\|\boldsymbol{h}\|^2 + \mu_0\|\boldsymbol{m}\|^2\right) + \mathcal{E}(\boldsymbol{m})\,.$$

We prove the following result:

THEOREM 3.3. *Suppose that $\boldsymbol{M}_2/|\boldsymbol{M}|(.,t) \in V$ and $\boldsymbol{G}_2(.,t) \in V$ for each $t \geq 0$. Suppose moreover that there exist constants M_- and M_+ such that*

$$0 \; < M_- \; \leq \; |\boldsymbol{M}(\boldsymbol{x}, 0)| \; \leq \; M_+ \quad , \quad \text{a.e. } \boldsymbol{x} \in \Omega_M.$$

Then, for $0 \leq t \leq T$,

$$E(t) \leq E(0)\exp(\mathcal{C}t)$$

where $\mathcal{C}$ depends only on $\|\boldsymbol{M}_2/|\boldsymbol{M}|\|_V$, $\|\boldsymbol{G}_2\|_V$ and the coefficients of the problem. In addition $\mathcal{C}$ is proportional to A.

REMARK 3.3. *Note that we have used a constant $\mathcal{C}$ that does not depend symmetrically on $\boldsymbol{M}_i$ and $\boldsymbol{G}_i$, $i = 1, 2$. The norm of $\boldsymbol{g}$ does not appear in these estimates. This is not a problem since it can be estimated in terms of norms appearing in our estimate.*

We now introduce two technical results that we shall use in the proof of Theorem 3.3. First we show that the weak version of the LLG equation used in this paper is equivalent to another weak version.

LEMMA 3.4. *If $\boldsymbol{M}_2/|\boldsymbol{M}| \in V$, then, $\forall \boldsymbol{\xi} \in (L^2(\Omega_M))^3$,*

$$\left(\frac{\partial \boldsymbol{M}_2}{\partial t}, \boldsymbol{\xi}\right) = \frac{|\gamma|}{1+\alpha^2}\left[(\boldsymbol{G}_2 \times \boldsymbol{M}_2, \boldsymbol{\xi}) + \alpha\left(\frac{\boldsymbol{M}_2}{|\boldsymbol{M}|} \times (\boldsymbol{G}_2 \times \boldsymbol{M}_2), \boldsymbol{\xi}\right)\right]\,.$$

Proof. This is nothing other than the "weak" form of (1.5). Let $\boldsymbol{\phi} \in (C_0^\infty(\Omega_M))^3$ and let $\boldsymbol{\xi} = \boldsymbol{\phi} \times \frac{\boldsymbol{M}_2}{|\boldsymbol{M}|}$. Then $\boldsymbol{\xi} \in (H^1(\Omega_M))^3$ and we may use it as a test function in (3.3), we obtain

$$\left(\frac{\partial \boldsymbol{M}_2}{\partial t}, \boldsymbol{\phi} \times \frac{\boldsymbol{M}_2}{|\boldsymbol{M}|}\right) \; = \; |\gamma|\left(\boldsymbol{G}_2 \times \boldsymbol{M}_2, \boldsymbol{\phi} \times \frac{\boldsymbol{M}_2}{|\boldsymbol{M}|}\right) + \alpha\left(\frac{\boldsymbol{M}_2}{|\boldsymbol{M}|} \times \frac{\partial \boldsymbol{M}_2}{\partial t}, \boldsymbol{\phi} \times \frac{\boldsymbol{M}_2}{|\boldsymbol{M}|}\right)\,.$$

This yields

$$\begin{aligned} \left(\frac{\boldsymbol{M}_2}{|\boldsymbol{M}|} \times \frac{\partial \boldsymbol{M}_2}{\partial t}, \boldsymbol{\phi}\right) &= |\gamma|\left(\frac{\boldsymbol{M}_2}{|\boldsymbol{M}|} \times (\boldsymbol{G}_2 \times \boldsymbol{M}_2), \boldsymbol{\phi}\right) + \alpha\left(\frac{\boldsymbol{M}_2}{|\boldsymbol{M}|} \times \left(\frac{\boldsymbol{M}_2}{|\boldsymbol{M}|} \times \frac{\partial \boldsymbol{M}_2}{\partial t}\right), \boldsymbol{\phi}\right) \\ &= |\gamma|\left(\frac{\boldsymbol{M}_2}{|\boldsymbol{M}|} \times (\boldsymbol{G}_2 \times \boldsymbol{M}_2), \boldsymbol{\phi}\right) - \alpha\left(\frac{\partial \boldsymbol{M}_2}{\partial t}, \boldsymbol{\phi}\right)\,. \qquad (3.12) \end{aligned}$$

Adding (3.3) to α times (3.12) leads to

$$(1+\alpha^2)\left(\frac{\partial \boldsymbol{M}_2}{\partial t},\boldsymbol{\phi}\right) \;=\; |\gamma|\left(\boldsymbol{G}_2\times\boldsymbol{M}_2,\boldsymbol{\phi}\right)+|\gamma|\alpha\left(\frac{\boldsymbol{M}_2}{|\boldsymbol{M}|}\times\left(\boldsymbol{G}_2\times\boldsymbol{M}_2\right),\boldsymbol{\phi}\right)\;.$$

The terms in $\partial\boldsymbol{M}_2/\partial t$, $\boldsymbol{G}_2\times\boldsymbol{M}_2$ and $\boldsymbol{M}_2\times(\boldsymbol{G}_2\times\boldsymbol{M}_2)/|\boldsymbol{M}|$ are all in $(L^2(\Omega_M))^3$ and hence by a density argument the result holds. ∎

The next result is a straightforward estimate, but it is key to our result since it allows us to estimate terms involving $\boldsymbol{g}$ without using inverse powers of A (which can be very small).

LEMMA 3.5. *For all $\boldsymbol{\xi}\in V$, $\exists C>0$ such that*

$$|(\boldsymbol{g},\boldsymbol{m},\boldsymbol{\xi})| \;\le\; C\,\|\boldsymbol{\xi}\|_V\,E(t)\;,$$

where $(\boldsymbol{g},\boldsymbol{m},\boldsymbol{\xi})$ denotes the mixed product $(\boldsymbol{g},\boldsymbol{m}\times\boldsymbol{\xi})=(\boldsymbol{g}\times\boldsymbol{m},\boldsymbol{\xi})$. More precisely, a possible choice for the constant C is

$$C \;=\; 2\,\max\left(1,\frac{A}{\mu_0},\frac{K}{\mu_0}\right)\;.$$

Proof. For any $\boldsymbol{\phi}\in(H^1(\Omega_M))^3$, using (3.4) for the two sets of solutions, we have

$$(\boldsymbol{G}_1,\boldsymbol{\phi})=(\boldsymbol{H}_1,\boldsymbol{\phi})-\frac{2}{\mu_0}\left(KP\left(\boldsymbol{M_1}\right),\boldsymbol{\phi}\right)-\frac{2}{\mu_0}\sum_{i=1}^{3}\left(A\frac{\partial\boldsymbol{M_1}}{\partial x_i},\frac{\partial\boldsymbol{\phi}}{\partial x_i}\right)\;,$$

$$(\boldsymbol{G}_2,\boldsymbol{\phi})=(\boldsymbol{H}_2,\boldsymbol{\phi})-\frac{2}{\mu_0}\left(KP\left(\boldsymbol{M_2}\right),\boldsymbol{\phi}\right)-\frac{2}{\mu_0}\sum_{i=1}^{3}\left(A\frac{\partial\boldsymbol{M_2}}{\partial x_i},\frac{\partial\boldsymbol{\phi}}{\partial x_i}\right)\;,$$

and we conclude by subtracting that

$$(\boldsymbol{g},\boldsymbol{\phi}) \;=\; (\boldsymbol{h},\boldsymbol{\phi})-\frac{2}{\mu_0}\left(KP(\boldsymbol{m}),\boldsymbol{\phi}\right)-\frac{2}{\mu_0}\sum_{i=1}^{3}\left(A\frac{\partial\boldsymbol{m}}{\partial x_i},\frac{\partial\boldsymbol{\phi}}{\partial x_i}\right)\;. \tag{3.13}$$

Taking $\boldsymbol{\phi}=\boldsymbol{m}\times\boldsymbol{\xi}$ ($\boldsymbol{\phi}\in(H^1(\Omega_M))^3$ since $\boldsymbol{\xi}\in V$), we get

$$(\boldsymbol{g},\boldsymbol{m},\boldsymbol{\xi}) \;=\; (\boldsymbol{h},\boldsymbol{m},\boldsymbol{\xi})-\frac{2}{\mu_0}\left(KP(\boldsymbol{m}),\boldsymbol{m},\boldsymbol{\xi}\right)-\frac{2}{\mu_0}\sum_{i=1}^{3}\left(A\frac{\partial\boldsymbol{m}}{\partial x_i},\boldsymbol{m},\frac{\partial\boldsymbol{\xi}}{\partial x_i}\right)\;. \tag{3.14}$$

Straightforward estimates give

$$\begin{aligned}
|(\boldsymbol{g},\boldsymbol{m},\boldsymbol{\xi})| &\le \|\boldsymbol{\xi}\|_{L^\infty(\Omega_M)}\|\boldsymbol{h}\|\|\boldsymbol{m}\|+\frac{2K}{\mu_0}\|\boldsymbol{\xi}\|_{L^\infty(\Omega_M)}\|\boldsymbol{m}\|^2\\
&\qquad+\frac{2A}{\mu_0}\sum_{i=1}^{3}\left\|\frac{\partial\boldsymbol{\xi}}{\partial x_i}\right\|_{L^\infty(\Omega_M)}\|\boldsymbol{m}\|\left\|\frac{\partial\boldsymbol{m}}{\partial x_i}\right\|\;,\\
&\le \frac{1}{2}\|\boldsymbol{\xi}\|_{L^\infty(\Omega_M)}\left(\|\boldsymbol{h}\|^2+\|\boldsymbol{m}\|^2\right)+\frac{2K}{\mu_0}\|\boldsymbol{\xi}\|_{L^\infty(\Omega_M)}\|\boldsymbol{m}\|^2\\
&\qquad+\frac{A}{\mu_0}\sum_{i=1}^{3}\left\|\frac{\partial\boldsymbol{\xi}}{\partial x_i}\right\|_{L^\infty(\Omega_M)}\left\|\frac{\partial\boldsymbol{m}}{\partial x_i}\right\|^2+\frac{A}{\mu_0}\|\boldsymbol{m}\|^2\sum_{i=1}^{3}\left\|\frac{\partial\boldsymbol{\xi}}{\partial x_i}\right\|_{L^\infty(\Omega_M)}\;,\\
&\le \frac{1}{2}\|\boldsymbol{\xi}\|_{L^\infty(\Omega_M)}\|\boldsymbol{h}\|^2+\frac{1}{2}\|\boldsymbol{\xi}\|_V\max\left(1,\frac{2A}{\mu_0},\frac{2K}{\mu_0}\right)\|\boldsymbol{m}\|^2+2\|\boldsymbol{\xi}\|_V\mathcal{E}(\boldsymbol{m})\;.
\end{aligned}$$

We conclude that

$$|(\boldsymbol{g},\boldsymbol{m},\boldsymbol{\xi})| \;\le\; \max 2\left(1,\frac{A}{\mu_0},\frac{K}{\mu_0}\right)\|\boldsymbol{\xi}\|_V\,E(t)\;.$$

∎

Proof. (of Theorem 3.3) For clarity, the proof is divided into a number of steps:

Step 1:. Computation of $\dfrac{dE(t)}{dt}$.

We note that by the linearity of (3.3) and (3.4)

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\epsilon_0\|\boldsymbol{e}\|^2+\mu_0\|\boldsymbol{h}\|^2\right) &= \epsilon_0(\boldsymbol{e}_t,\boldsymbol{e})-(\boldsymbol{h},\nabla\times\boldsymbol{e})+\mu_0(\boldsymbol{h}_t,\boldsymbol{h})+(\nabla\times\boldsymbol{e},\boldsymbol{h})-(\sigma\boldsymbol{e},\boldsymbol{e})\\ &\le -\mu_0(\boldsymbol{m}_t,\boldsymbol{h})\ ,\end{aligned}\tag{3.15}$$

where we denote by $\boldsymbol{m}_t$ the derivative in time of $\boldsymbol{m}$. Hence

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\epsilon_0\|\boldsymbol{e}\|^2+\mu_0\|\boldsymbol{h}\|^2+\mu_0\|\boldsymbol{m}\|^2\right) &\le \mu_0(\boldsymbol{m},\boldsymbol{m}_t)-\mu_0(\boldsymbol{m}_t,\boldsymbol{h})\\ &\le \mu_0(\boldsymbol{m},\boldsymbol{m}_t)+\mu_0\left[(\boldsymbol{g},\boldsymbol{m}_t)-(\boldsymbol{h},\boldsymbol{m}_t)\right]\\ &\qquad -\mu_0(\boldsymbol{g},\boldsymbol{m}_t)\ .\end{aligned}\tag{3.16}$$

Choosing $\boldsymbol{\phi}=\boldsymbol{m}_t$, we deduce from (3.13) that

$$\begin{aligned}\mu_0\left((\boldsymbol{g},\boldsymbol{m}_t)-(\boldsymbol{h},\boldsymbol{m}_t)\right) &= -2\left(KP(\boldsymbol{m}),\boldsymbol{m}_t\right)-2\sum_{i=1}^{3}\left(A\frac{\partial\boldsymbol{m}}{\partial x_i},\frac{\partial}{\partial t}\frac{\partial\boldsymbol{m}}{\partial x_i}\right)\\ &= -\frac{d}{dt}\left[\int_{\Omega_M}\left(K|P(\boldsymbol{m})|^2+A\sum_{i=1}^{3}\left|\frac{\partial\boldsymbol{m}}{\partial x_i}\right|^2\right)d\boldsymbol{x}\right]\\ &= -\frac{d}{dt}\left[\mathcal{E}\left(\boldsymbol{m}\right)\right]\ .\end{aligned}\tag{3.17}$$

With (3.16), this shows that

$$\frac{d}{dt}\left[\frac{1}{2}\left(\epsilon_0\|\boldsymbol{e}\|^2+\mu_0\|\boldsymbol{h}\|^2+\mu_0\|\boldsymbol{m}\|^2\right)+\mathcal{E}(\boldsymbol{m})\right]\ =\ \mu_0(\boldsymbol{m},\boldsymbol{m}_t)-\mu_0(\boldsymbol{g},\boldsymbol{m}_t)\ ,$$

or equivalently

$$\begin{aligned}\frac{d}{dt}\left(E(t)\right)+\frac{\alpha\mu_0}{|\gamma|}\int_{\Omega_M}\frac{|\boldsymbol{m}_t|^2}{|\boldsymbol{M}|}d\boldsymbol{x} &= \mu_0(\boldsymbol{m},\boldsymbol{m}_t) + \left[-\mu_0(\boldsymbol{g},\boldsymbol{m}_t)+\frac{\alpha\mu_0}{|\gamma|}\int_{\Omega_M}\frac{|\boldsymbol{m}_t|^2}{|\boldsymbol{M}|}d\boldsymbol{x}\right]\\ &= RHS1 + \mu_0 RHS2\ .\end{aligned}\tag{3.18}$$

Step 2:. Estimation of $RHS1=\mu_0(\boldsymbol{m},\boldsymbol{m}_t)$.

We just use the Cauchy-Schwarz inequality to write

$$\mu_0|(\boldsymbol{m},\boldsymbol{m}_t)|\le\mu_0\|\boldsymbol{m}\|\,\|\boldsymbol{m}_t\|.\tag{3.19}$$

Step 3:. Estimation of $RHS2=-(\boldsymbol{g},\boldsymbol{m}_t)+\dfrac{\alpha}{|\gamma|}\displaystyle\int_{\Omega_M}\frac{|\boldsymbol{m}_t|^2}{|\boldsymbol{M}|}d\boldsymbol{x}$.

Using (3.3), we have

$$\begin{aligned}-\left(\boldsymbol{m}_t,\boldsymbol{g}\right)+\frac{\alpha}{|\gamma|}\left(\boldsymbol{m}_t,\frac{\boldsymbol{m}_t}{|\boldsymbol{M}|}\right) &=\\ &-|\gamma|\left(\boldsymbol{G}_1\times\boldsymbol{M}_1-\boldsymbol{G}_2\times\boldsymbol{M}_2,\boldsymbol{g}\right)\\ &\quad-\alpha\left(\frac{\boldsymbol{M}_1}{|\boldsymbol{M}|}\times\frac{\partial\boldsymbol{M}_1}{\partial t}-\frac{\boldsymbol{M}_2}{|\boldsymbol{M}|}\times\frac{\partial\boldsymbol{M}_2}{\partial t},\boldsymbol{g}\right)\\ &\qquad+\alpha\left(\boldsymbol{G}_1\times\boldsymbol{M}_1-\boldsymbol{G}_2\times\boldsymbol{M}_2,\frac{\boldsymbol{m}_t}{|\boldsymbol{M}|}\right)\\ &\qquad\quad+\frac{\alpha^2}{|\gamma|}\left(\frac{\boldsymbol{M}_1}{|\boldsymbol{M}|}\times\frac{\partial\boldsymbol{M}_1}{\partial t}-\frac{\boldsymbol{M}_2}{|\boldsymbol{M}|}\times\frac{\partial\boldsymbol{M}_2}{\partial t},\frac{\boldsymbol{m}_t}{|\boldsymbol{M}|}\right)\\ &:= T_1 + T_2 + T_3 + T_4.\end{aligned}\tag{3.20}$$

In the following computations the right hand side is controlled term by term. The first term, T_1 can be expanded as

$$\boldsymbol{G}_1 \times \boldsymbol{M}_1 - \boldsymbol{G}_2 \times \boldsymbol{M}_2 \;=\; \boldsymbol{g} \times \boldsymbol{M}_1 - \boldsymbol{G}_2 \times \boldsymbol{m}\;, \tag{3.21}$$

and estimated using Lemma 3.5. We have:

$$|T_1| \;=\; |\gamma|\,|(\boldsymbol{G}_2 \times \boldsymbol{m}, \boldsymbol{g})| \;\le\; C|\gamma|\,\|\boldsymbol{G}_2\|_V\, E(t)\;, \tag{3.22}$$

where $C = 2\max(1, A/\mu_0, K/\mu_0)$.

The last term, T_4, can be expanded using

$$\frac{\boldsymbol{M}_1}{|\boldsymbol{M}|} \times \frac{\partial \boldsymbol{M}_1}{\partial t} - \frac{\boldsymbol{M}_2}{|\boldsymbol{M}|} \times \frac{\partial \boldsymbol{M}_2}{\partial t} \;=\; \frac{\boldsymbol{m}}{|\boldsymbol{M}|} \times \frac{\partial \boldsymbol{M}_2}{\partial t} + \frac{\boldsymbol{M}_1}{|\boldsymbol{M}|} \times \boldsymbol{m}_t\;. \tag{3.23}$$

Hence T_4 may be written:

$$T_4 \;=\; \frac{\alpha^2}{|\gamma|}\left(\frac{\boldsymbol{m}}{|\boldsymbol{M}|} \times \frac{\partial \boldsymbol{M}_2}{\partial t}, \frac{1}{|\boldsymbol{M}|}\boldsymbol{m}_t\right). \tag{3.24}$$

By Lemma 3.4, we have

$$T_4 \;=\; \frac{\alpha^2}{1+\alpha^2}\left(\frac{\boldsymbol{m}}{|\boldsymbol{M}|} \times \boldsymbol{m}_t, \boldsymbol{G}_2 \times \frac{\boldsymbol{M}_2}{|\boldsymbol{M}|} + \alpha \frac{\boldsymbol{M}_2}{|\boldsymbol{M}|} \times \left(\boldsymbol{G}_2 \times \frac{\boldsymbol{M}_2}{|\boldsymbol{M}|}\right)\right). \tag{3.25}$$

Hence

$$|T_4| \;\le\; C_4 \|\boldsymbol{m}\|\,\|\boldsymbol{m}_t\| \tag{3.26}$$

where

$$C_4 = \frac{\alpha^2 \|\boldsymbol{G}_2\|_V}{M_-(1+\alpha^2)}\left\|\frac{\boldsymbol{M}_2}{|\boldsymbol{M}|}\right\|_V \left(1 + \alpha \left\|\frac{\boldsymbol{M}_2}{|\boldsymbol{M}|}\right\|_V\right).$$

It remains now to estimate $T_2 + T_3$. Expanding T_2 we have

$$\begin{aligned} -\left(\frac{\boldsymbol{M}_1}{|\boldsymbol{M}|} \times \frac{\partial \boldsymbol{M}_1}{\partial t} - \frac{\boldsymbol{M}_2}{|\boldsymbol{M}|} \times \frac{\partial \boldsymbol{M}_2}{\partial t}, \boldsymbol{g}\right) = &-\left(\frac{\boldsymbol{M}_1}{|\boldsymbol{M}|} \times \frac{\partial \boldsymbol{M}_1}{\partial t}, \boldsymbol{G}_1\right) + \left(\frac{\boldsymbol{M}_1}{|\boldsymbol{M}|} \times \frac{\partial \boldsymbol{M}_1}{\partial t}, \boldsymbol{G}_2\right) \\ &+ \left(\frac{\boldsymbol{M}_2}{|\boldsymbol{M}|} \times \frac{\partial \boldsymbol{M}_2}{\partial t}, \boldsymbol{G}_1\right) - \left(\frac{\boldsymbol{M}_2}{|\boldsymbol{M}|} \times \frac{\partial \boldsymbol{M}_2}{\partial t}, \boldsymbol{G}_2\right), \end{aligned}$$

and for T_3

$$\begin{aligned} \left(\boldsymbol{G}_1 \times \frac{\boldsymbol{M}_1}{|\boldsymbol{M}|} - \boldsymbol{G}_2 \times \frac{\boldsymbol{M}_2}{|\boldsymbol{M}|}, \frac{\partial}{\partial t}(\boldsymbol{M}_1 - \boldsymbol{M}_2)\right) \;=\; &\left(\boldsymbol{G}_1 \times \frac{\boldsymbol{M}_1}{|\boldsymbol{M}|}, \frac{\partial \boldsymbol{M}_1}{\partial t}\right) - \left(\boldsymbol{G}_1 \times \frac{\boldsymbol{M}_1}{|\boldsymbol{M}|}, \frac{\partial \boldsymbol{M}_2}{\partial t}\right) \\ &- \left(\boldsymbol{G}_2 \times \frac{\boldsymbol{M}_2}{|\boldsymbol{M}|}, \frac{\partial \boldsymbol{M}_1}{\partial t}\right) + \left(\boldsymbol{G}_2 \times \frac{\boldsymbol{M}_2}{|\boldsymbol{M}|}, \frac{\partial \boldsymbol{M}_2}{\partial t}\right). \end{aligned}$$

By adding these two equalities, we see that

$$\begin{aligned} T_2 + T_3 \;\;=\;\; &\alpha\left(\boldsymbol{G}_1 \times \frac{\boldsymbol{M}_2}{|\boldsymbol{M}|}, \frac{\partial \boldsymbol{M}_2}{\partial t}\right) - \alpha\left(\boldsymbol{G}_1 \times \frac{\boldsymbol{M}_1}{|\boldsymbol{M}|}, \frac{\partial \boldsymbol{M}_2}{\partial t}\right) \\ &+\alpha\left(\boldsymbol{G}_2 \times \frac{\boldsymbol{M}_2}{|\boldsymbol{M}|}, \frac{\partial \boldsymbol{M}_1}{\partial t}\right) + \alpha\left(\boldsymbol{G}_2 \times \frac{\boldsymbol{M}_1}{|\boldsymbol{M}|}, \frac{\partial \boldsymbol{M}_1}{\partial t}\right)\;. \end{aligned} \tag{3.27}$$

Then we compute

$$\begin{aligned} |T_2 + T_3| \;&\le\; \alpha\left|\left(\boldsymbol{G}_1 \times \frac{\boldsymbol{m}}{|\boldsymbol{M}|}, \frac{\partial \boldsymbol{M}_2}{\partial t}\right) - \left(\boldsymbol{G}_2 \times \frac{\boldsymbol{m}}{|\boldsymbol{M}|}, \frac{\partial \boldsymbol{M}_1}{\partial t}\right)\right| \\ &\le\; \alpha\left|\left(\boldsymbol{g} \times \frac{\boldsymbol{m}}{|\boldsymbol{M}|}, \frac{\partial \boldsymbol{M}_2}{\partial t}\right) - \left(\boldsymbol{G}_2 \times \frac{\boldsymbol{m}}{|\boldsymbol{M}|}, \boldsymbol{m}_t\right)\right| \\ &\le\; \alpha\left|\left(\boldsymbol{g} \times \frac{\boldsymbol{m}}{|\boldsymbol{M}|}, \frac{\partial \boldsymbol{M}_2}{\partial t}\right)\right| + \alpha\left|\left(\frac{\boldsymbol{G}_2}{|\boldsymbol{M}|} \times \boldsymbol{m}, \boldsymbol{m}_t\right)\right|\;. \end{aligned} \tag{3.28}$$

Since by Lemma 3.4

$$\left(\boldsymbol{g}\times\frac{\boldsymbol{m}}{|\boldsymbol{M}|},\frac{\partial \boldsymbol{M}_2}{\partial t}\right) \;=\; \frac{|\gamma|}{1+\alpha^2}\left(\boldsymbol{g}\times\boldsymbol{m},\boldsymbol{G}_2\times\frac{\boldsymbol{M}_2}{|\boldsymbol{M}|}+\alpha\frac{\boldsymbol{M}_2}{|\boldsymbol{M}|}\times\left(\boldsymbol{G}_2\times\frac{\boldsymbol{M}_2}{|\boldsymbol{M}|}\right)\right)$$

we conclude, via Lemma 3.5, that

$$|T_2+T_3| \;\le\; C_{23}E(t)+C_{24}\|\boldsymbol{m}\|\,\|\boldsymbol{m}_t\|. \tag{3.29}$$

where

$$C_{23}=\frac{2|\gamma|}{1+\alpha^2}\max(1,A/\mu_0,K/\mu_0)\|\boldsymbol{G}_2\|_V\left\|\frac{\boldsymbol{M}_2}{|\boldsymbol{M}|}\right\|_V\left(1+\alpha\left\|\frac{\boldsymbol{M}_2}{|\boldsymbol{M}|}\right\|_V\right), C_{24}=\alpha\frac{\|\boldsymbol{G}_2\|_V}{M_-}.$$

Step 4:. Conclusion of the proof.

From identity (3.18), and by adding the estimates in (3.19), (3.22), (3.26) and (3.29), we have established, provided that $\boldsymbol{M}_2,\boldsymbol{G}_2\in V$, that

$$\frac{1}{2}\frac{d}{dt}E(t)+\frac{\alpha\mu_0}{|\gamma|}\int_{\Omega_M}\frac{|\boldsymbol{m}_t|^2}{|\boldsymbol{M}|}d\boldsymbol{x} \;\le\; C_1\,E(t) \;+\; C_2\,\|\boldsymbol{m}\|\,\|\boldsymbol{m}_t\|. \tag{3.30}$$

where

$$C_1 \;=\; 2|\gamma|\max(\mu_0,A,K)\|\boldsymbol{G}_2\|_V\left(1+\frac{1}{1+\alpha^2}\left\|\frac{\boldsymbol{M}_2}{|\boldsymbol{M}|}\right\|_V\left(1+\alpha\left\|\frac{\boldsymbol{M}_2}{|\boldsymbol{M}|}\right\|_V\right)\right)\;,$$

and

$$C_2 \;=\; \mu_0\left(1+\frac{\alpha^2}{M_-(1+\alpha^2)}\|\boldsymbol{G}_2\|_V\left\|\frac{\boldsymbol{M}_2}{|\boldsymbol{M}|}\right\|_V\left(1+\alpha\left\|\frac{\boldsymbol{M}_2}{|\boldsymbol{M}|}\right\|_V\right)+\alpha\frac{\|\boldsymbol{G}_2\|_V}{M_-}\right).$$

Now we use the arithmetic geometric mean inequality to write

$$\|\boldsymbol{m}\|\,\|\boldsymbol{m}_t\|\le\frac{\delta}{2}\left\|\frac{\boldsymbol{m}_t}{|\boldsymbol{M}|^{1/2}}\right\|^2+\frac{1}{2\delta}\left\||\boldsymbol{M}|^{1/2}\boldsymbol{m}\right\|^2$$

and choose

$$\delta=\frac{\alpha\mu_0}{|\gamma|C_2}.$$

Then (3.30) becomes

$$\frac{1}{2}\frac{d}{dt}E(t)+\frac{\alpha\mu_0}{2|\gamma|}\int_{\Omega_M}\frac{|\boldsymbol{m}_t|^2}{|\boldsymbol{M}|}d\boldsymbol{x} \;\le\; C_1\,E(t) \;+\; \mu_0\frac{C_2^2|\gamma|M_+}{2\alpha\mu_0^2}\,\|\boldsymbol{m}\|^2. \tag{3.31}$$

We have shown that

$$\frac{d}{dt}\left(E(t)\right) \;\le\; \mathcal{C}\,E(t)\;, \tag{3.32}$$

where $\mathcal{C}=\max(C_1,C_2^2|\gamma|M_+/2\alpha\mu_0^2)$. An application of Gronwall's inequality proves the Theorem. □

4. A Numerical Scheme. Here we present some finite element schemes for approximating equations (3.1)–(3.4). Let τ_h be a mesh covering Ω using regular finite elements of maximum diameter h. At this stage the mesh could either consist of tetrahedral or hexahedral elements, but we assume that the boundary of Ω_m coincides with faces of the mesh.

Using the mesh τ_h, we will construct finite element spaces $U_h\subset H_0(\mathrm{curl};\Omega)$, $V_h\subset H(\mathrm{div};\Omega)$ and $W_h\subset(L^2(\Omega_M))^3\cap(L^\infty(\Omega_M))^3\cap(H^1(\Omega_M))^3$. These spaces are such that $\nabla\times U_h\subset V_h$ and $W_h\subset V_h$.

Assuming we can construct suitable spaces U_h, V_h and W_h, the semi-discrete numerical method we propose is to find $(\boldsymbol{E}_h(t), \boldsymbol{H}_h(t), \boldsymbol{M}_h(t), \boldsymbol{G}_h(t)) \in U_h \times V_h \times W_h \times W_h$ such that

$$\varepsilon_0 \left(\frac{\partial \boldsymbol{E}_h}{\partial t}, \boldsymbol{\psi}_h \right)_{U_h} - (\boldsymbol{H}_h, \nabla \times \boldsymbol{\psi}_h)_{V_h} + (\sigma \boldsymbol{E}_h, \boldsymbol{\psi_h})_{U_h} = -(\boldsymbol{J}, \boldsymbol{\psi_h})_{U_h}, \quad \forall \boldsymbol{\psi}_h \in U_h, \tag{4.1}$$

$$\mu_0 \left(\frac{\partial \boldsymbol{H}_h}{\partial t}, \boldsymbol{\varphi}_h \right)_{V_h} + (\nabla \times \boldsymbol{E}_h, \boldsymbol{\varphi}_h)_{V_h} = -\mu_0 \left(\frac{\partial \boldsymbol{M}_h}{\partial t}, \boldsymbol{\varphi}_h \right)_{V_h}, \quad \forall \boldsymbol{\varphi}_h \in V_h, \tag{4.2}$$

and

$$\left(\frac{\partial \boldsymbol{M}_h}{\partial t}, \boldsymbol{\xi}_h \right)_{W_h} = |\gamma| (\boldsymbol{G}_h \times \boldsymbol{M}_h, \boldsymbol{\xi}_h)_{W_h} + \alpha \left(\frac{\boldsymbol{M}_h}{|\boldsymbol{M}_h|} \times \frac{\partial \boldsymbol{M}_h}{\partial t}, \boldsymbol{\xi}_h \right)_{W_h}, \forall \boldsymbol{\xi}_h \in W_h, \tag{4.3}$$

$$(\boldsymbol{G}_h, \boldsymbol{\phi}_h)_{W_h} = (\boldsymbol{H}_h, \boldsymbol{\phi}_h)_{V_h} - \frac{2}{\mu_0} (KP(\boldsymbol{M}_h), \boldsymbol{\phi}_h)_{W_h} - \frac{2}{\mu_0} \sum_{i=1}^{3} \left(A \frac{\partial \boldsymbol{M}_h}{\partial x_i}, \frac{\partial \boldsymbol{\phi}_h}{\partial x_i} \right)_{W_h}, \quad \forall \boldsymbol{\phi}_h \in W_h, \tag{4.4}$$

where $(\cdot,\cdot)_{U_h}$ (respectively $(\cdot,\cdot)_{V_h}$ and $(\cdot,\cdot)_{W_h}$) are suitable discrete inner products depending on the space U_h (respectively V_h and W_h).The compatibility condition that $\nabla \times U_h \subset V_h$ is also assumed to hold.

Let $S_h \subset H^1(\Omega_M)$. The space W_h is nothing more than $W_h = S_h|_{\Omega_M} \times S_h|_{\Omega_M} \times S_h|_{\Omega_M}$ where

$$S_h|_{\Omega_M} = \{p_h|_{\Omega_M} \mid p_h \in S_h\}.$$

The key requirement is that the quadrature used to define $(\cdot,\cdot)_{W_h}$ uses exactly all the degrees of freedom of the underlying space (with positive quadrature weights). With this assumption, we shall see that equation (4.3) is satisfied pointwise at the interpolation points of the method.

4.1. Linear Tetrahedral Elements. In our previous paper [14] we used the second family of Nédélec edge elements [16] on tetrahedra to discretize the Maxwell-LLG equations without the exchange term. This family can also be used to discretize the problem treated here. The mesh τ_h consists of regular tetrahedra, and we assume that Ω and Ω_M are exactly covered by the tetrahedra (i.e. the boundary of Ω_M coincides with faces of tetrahedra for each h). The space U_h is the classical Nédélec second family space [16]

$$U_h = \left\{ \boldsymbol{u}_h \in H_0(\mathrm{curl}; \Omega) \mid \boldsymbol{u}_h|_K \in (P_1)^3, \ \forall K \in \tau_h \right\}$$

where P_1 is the set of polynomials of total degree one in x, y and z.

The discrete inner product for this space is defined as follows:

$$(\boldsymbol{u}, \boldsymbol{v})_{U_h} = \sum_{K \in \tau_h} Q_K(\boldsymbol{u} \cdot \boldsymbol{v})$$

where the local quadrature Q_K is given by

$$Q_K(\phi) = \frac{\mathrm{volume}(K)}{4} \sum_{i=1}^{4} \phi(\boldsymbol{a}_i),$$

where $\boldsymbol{a}_i$ is the ith node of K. This choice of discrete inner product gives a block diagonal mass matrix which improves the efficiency of time-stepping. However the quadrature scheme may allow spurious modes for $\boldsymbol{E}_h$, but this remains to be investigated. Of course the use of $(\boldsymbol{u}, \boldsymbol{v})_{U_h} = (\boldsymbol{u}, \boldsymbol{v})$ avoids any possibility of spurious modes at the expense of solving (by conjugate gradients) a well conditioned matrix problem at each timestep.

The space V_h is simply

$$V_h = \left\{ \boldsymbol{v}_h \in (L^2(\Omega))^3 \mid \boldsymbol{v}_h|_K \in (P_1)^3, \ \forall K \in \tau_h \right\}$$

and $(.,.)_{V_h} = (.,.)$.

The space W_h is then

$$W_h = \left\{ \boldsymbol{w}_h \in (H^1(\Omega_M))^3 \mid \boldsymbol{w}_h|_K \in (P_1)^3 \ , \ \forall K \in \tau_h \text{ with } K \subset \Omega_M \right\},$$

and the discrete inner product for W_h is

$$(\boldsymbol{u}, \boldsymbol{v})_{W_h} = \sum_{K \in \tau_h, K \subset \Omega_M} Q_K(\boldsymbol{u} \cdot \boldsymbol{v}).$$

This choice precisely diagonalizes the mass matrix for W_h (if degrees of freedom parallel to the coordinate axes are used at each node in the mesh).

4.2. Hexahedral elements. Here we discuss just one member of a family of hexahedral elements. The two dimensional analogue of these elements is used in this paper for our numerical examples. Here the mesh τ_h consists of hexahedral elements with each edge parallel to one of the three coordinate axes. Let $Q_{l,m,n}$ denote the set of polynomials of degree at most l in x_1, m in x_2 and n in x_3.

The space W_h is a product of quadratic elements. Let

$$W_h = \left\{ \boldsymbol{w} \in (H^1(\Omega_M))^3 \mid \boldsymbol{w}|_K \in Q^3_{2,2,2}, \ \forall K \in \tau_h \text{ with } K \subset \Omega_M \right\},$$

The discrete inner product $(.,.)_{V_h}$ is computed using the tensor product Simpson's rule on each element. The space U_h is chosen to be the cubic edge space of Nédélec [15]:

$$U_h = \{ \boldsymbol{u} \in H_0(\text{curl}; \Omega) \mid \boldsymbol{u}|_K \in Q_{2,3,3} \times Q_{3,2,3} \times Q_{3,3,2}, \ \forall K \in \tau_h \}.$$

The definition of $(.,.)_{U_h}$ involves anisotropic quadrature rules for each component of the vector in U_h. For details see [8]. Finally the space V_h is

$$V_h = \left\{ \boldsymbol{v} \in (L^2(\Omega))^3 \mid \boldsymbol{v}|_K \in Q_{3,2,2} \times Q_{2,3,2} \times Q_{2,2,3}, \ \forall K \in \tau_h \right\}$$

together with the use of an exact inner product $(.,.)_{V_h} = (.,.)$.

5. Properties of the Semi-discrete Scheme. The numerical scheme conserves the magnitude of the magnetization at the quadrature points of the discrete inner product. This is the discrete analogue of (1.9).

LEMMA 5.1. *Let $\boldsymbol{a}_i$ be a quadrature point for the integration scheme used to compute the discrete inner product $(\cdot,\cdot)_{W_h}$. For each time t,*

$$|\boldsymbol{M}_h(\boldsymbol{a}_i, t)| = |\boldsymbol{M}_h(\boldsymbol{a}_i, 0)|$$

In particular, $\|\boldsymbol{M}_h\|_{\infty,\Omega_M}$ is bounded independent of t and h.

Proof. We choose $\boldsymbol{\xi}_h \in W_h$ to interpolate $\boldsymbol{M}_h$ at $\boldsymbol{a}_i$ and interpolate zero at all other quadrature (or interpolation) points. Let w_i be the corresponding quadrature weight in the discrete inner product. Using this $\boldsymbol{\xi}_h$ in (4.3) we conclude that

$$\begin{aligned} w_i \frac{\partial}{\partial t} \boldsymbol{M}_h(\boldsymbol{a}_i) \cdot \boldsymbol{M}_h(\boldsymbol{a}_i) &= \left(\frac{\partial}{\partial t} \boldsymbol{M}_h, \boldsymbol{\xi}_h \right)_{W_h} \\ &= w_i \left(|\gamma| \left(\boldsymbol{G}_h \times \boldsymbol{M}_h \right)(\boldsymbol{a}_i) \cdot \boldsymbol{M}_h(\boldsymbol{a}_i) + \alpha \left(\frac{\boldsymbol{M}_h}{|\boldsymbol{M}_h|} \times \frac{\partial \boldsymbol{M}_h}{\partial t} \right)(\boldsymbol{a}_i) \cdot \boldsymbol{M}_h(\boldsymbol{a}_i) \right) \\ &= 0. \end{aligned}$$

□

The method also conserves the energy of the electromagnetic field, again mimicking the continuous case. This will be a consequence of the existence of a discrete Liapunov function which we discuss next.

Let us assume $\boldsymbol{J} = 0$. We can see that $(\widetilde{\boldsymbol{E}}_h, \widetilde{\boldsymbol{H}}_h, \widetilde{\boldsymbol{M}}_h, \widetilde{\boldsymbol{G}}_h) \in U_h \times V_h \times W_h \times W_h$ is a stationary state of the semi-discrete problem if

$$\begin{aligned} -(\widetilde{\boldsymbol{H}}_h, \nabla \times \boldsymbol{\psi}_h)_{V_h} + (\sigma \widetilde{\boldsymbol{E}}_h, \boldsymbol{\psi}_h)_{U_h} &= 0 \quad , \forall \boldsymbol{\psi}_h \in W_h, \\ (\nabla \times \widetilde{\boldsymbol{E}}_h, \boldsymbol{\phi}_h)_{V_h} &= 0 \quad , \forall \boldsymbol{\phi}_h \in V_h, \\ (\widetilde{\boldsymbol{G}}_h \times \widetilde{\boldsymbol{M}}_h, \boldsymbol{\xi}_h)_{W_h} &= 0 \quad , \forall \boldsymbol{\xi}_h \in W_h. \end{aligned}$$

In addition (4.4) is also satisfied.

The assumption on the discrete inner product $(,\cdot,)_{W_h}$ implies that $\widetilde{\boldsymbol{G}}_h \times \widetilde{\boldsymbol{M}}_h = 0$ at all the interpolation points of W_h. Hence at each interpolation point $\widetilde{\boldsymbol{G}}_h$ and $\widetilde{\boldsymbol{M}}_h$ are parallel, and there is a function $\lambda_h \in S_h|_{\Omega_M}$ such that

$$\widetilde{\boldsymbol{G}}_h = \lambda_h \widetilde{\boldsymbol{M}}_h$$

at the interpolation points of $S_h|_{\Omega_M}$.

Now we can state and prove the semi-discrete version of Theorem 2.1.

THEOREM 5.2. *Suppose $\boldsymbol{J} = 0$ and*

$$(\widetilde{\boldsymbol{E}}_h, \widetilde{\boldsymbol{H}}_h, \widetilde{\boldsymbol{M}}_h, \widetilde{\boldsymbol{G}}_h) \in U_h \times V_h \times W_h \times W_h$$

is a stationary state of the semi-discrete problem with associated indicator λ_h. Let

$$\begin{aligned} V_{\lambda_h,h}(\boldsymbol{E}_h, \boldsymbol{H}_h, \boldsymbol{M}_h) = \frac{1}{2}\Big(&\epsilon_0(\boldsymbol{E}_h - \widetilde{\boldsymbol{E}}_h, \boldsymbol{E}_h - \widetilde{\boldsymbol{E}}_h)_{U_h} + \mu_0(\boldsymbol{H}_h - \widetilde{\boldsymbol{H}}_h, \boldsymbol{H}_h - \widetilde{\boldsymbol{H}}_h)_{V_h}\Big) \\ &+\frac{\mu_0}{2}(\lambda_h(\boldsymbol{M}_h - \widetilde{\boldsymbol{M}}_h), \boldsymbol{M}_h - \widetilde{\boldsymbol{M}}_h)_{W_h} + \mathcal{E}_h(\boldsymbol{M}_h - \widetilde{\boldsymbol{M}}_h)\,, \end{aligned}$$

where

$$\mathcal{E}_h(\boldsymbol{v}_h) = (KP(\boldsymbol{v}_h), P(\boldsymbol{v}_h))_{W_h} + \sum_i \left(A\frac{\partial \boldsymbol{v}_h}{\partial x_i}, \frac{\partial \boldsymbol{v}_h}{\partial x_i}\right)_{W_h}$$

for any $\boldsymbol{v}_h \in W_h$. Then $V_{\lambda_h,h}$ is a strict Liapunov function and

$$\frac{d}{dt}V_{\lambda_h,h}(\boldsymbol{E}_h, \boldsymbol{H}_h, \boldsymbol{M}_h) = -\frac{\mu_0}{|\gamma|}\left(\frac{\alpha}{|\boldsymbol{M}_h|}\frac{\partial \boldsymbol{M}_h}{\partial t}, \frac{\partial \boldsymbol{M}_h}{\partial t}\right)_{W_h} - (\sigma \boldsymbol{E}_h, \boldsymbol{E}_h)_{U_h}.$$

Proof. The proof is essentially to check that the steps in the proof of Theorem 2.1 hold in the semi-discrete variational setting. This is not entirely obvious due to the use of discrete inner products (we use this as an essential part of our proof) and the introduction of $\boldsymbol{G}_h$ as an explicit variable in the problem.

As in the proof of Theorem 2.1, we define (we use the same notation even though the fields are now discrete)

$$\boldsymbol{N}_E = \boldsymbol{E}_h - \widetilde{\boldsymbol{E}}_h,\ \boldsymbol{N}_H = \boldsymbol{H}_h - \widetilde{\boldsymbol{H}}_h,\ \boldsymbol{N}_M = \boldsymbol{M}_h - \widetilde{\boldsymbol{M}}_h,\ \text{ and } \boldsymbol{N}_G = \boldsymbol{G}_h - \boldsymbol{G}(\widetilde{\boldsymbol{H}}, \widetilde{\boldsymbol{M}})_h\,. \tag{5.1}$$

Now using the linearity of (4.1) and (4.2) and the fact that $\boldsymbol{J} = 0$ we can easily check the analogue of (2.11):

$$\frac{1}{2}\frac{d}{dt}\left[(\epsilon_0 \boldsymbol{N}_E, \boldsymbol{N}_E)_{U_h} + (\mu_0 \boldsymbol{N}_H, \boldsymbol{N}_H)_{V_h}\right] + (\sigma \boldsymbol{N}_E, \boldsymbol{N}_E)_{U_h} = -(\mu_0 \boldsymbol{N}_H, \frac{d}{dt}\boldsymbol{N}_M)_{V_h}. \tag{5.2}$$

To simplify notation, we can define the semi-discrete linear magnetic field $\boldsymbol{H}_{l,h} \in W_h$ by

$$\mu_0(\boldsymbol{H}_{l,h}(\boldsymbol{M}_h), \boldsymbol{\xi}_h)_{W_h} = -2\left[(KP(\boldsymbol{M}_h), \boldsymbol{\xi}_h)_{W_h} + \sum_i (A\frac{\partial}{\partial x_i}\boldsymbol{M}_h, \frac{\partial}{\partial x_i}\boldsymbol{\xi}_h)_{W_h}\right], \quad \forall \boldsymbol{\xi}_h \in W_h.$$

Using the definition of $\mathcal{E}_h$ and the definition of $\boldsymbol{H}_{l,h}$ a direct calculation shows that

$$\begin{aligned} &\frac{d}{dt}\left[\mathcal{E}_h(\boldsymbol{N}_M) + \frac{\mu_0}{2}(\lambda_h \boldsymbol{N}_M, \boldsymbol{N}_M)_{W_h}\right] + \left(\frac{\alpha\mu_0}{|\gamma||\boldsymbol{M}_h|}\frac{\partial}{\partial t}\boldsymbol{N}_M, \frac{\partial}{\partial t}\boldsymbol{N}_M\right)_{W_h} \\ &= 2\left(KP(\boldsymbol{N}_M), \frac{\partial}{\partial t}\boldsymbol{N}_M\right)_{W_h} + 2\sum_i \left(A\frac{\partial}{\partial x_i}\boldsymbol{N}_M, \frac{\partial}{\partial t}\frac{\partial}{\partial x_i}\boldsymbol{N}_M\right)_{W_h} \end{aligned}$$

$$+\mu_0\left(\lambda_h \boldsymbol{N}_M, \frac{\partial}{\partial t}\boldsymbol{N}_M\right)_{W_h} + \left(\frac{\alpha\mu_0}{|\gamma||\boldsymbol{M}_h|}\frac{\partial}{\partial t}\boldsymbol{N}_M, \frac{\partial}{\partial t}\boldsymbol{N}_M\right)_{W_h}$$

$$= -\mu_0\left(H_{l,h}(\boldsymbol{N}_M), \frac{\partial}{\partial t}\boldsymbol{N}_M\right)_{W_h} + \mu_0\left(\lambda_h \boldsymbol{N}_M, \frac{\partial}{\partial t}\boldsymbol{N}_M\right)_{W_h}$$

$$+ \left(\frac{\alpha\mu_0}{|\gamma||\boldsymbol{M}_h|}\frac{\partial}{\partial t}\boldsymbol{N}_M, \frac{\partial}{\partial t}\boldsymbol{N}_M\right)_{W_h}. \tag{5.3}$$

This is the semi-discrete analogue of (2.12). Adding (5.2) and (5.3) we obtain the analogue of (2.13):

$$\frac{d}{dt}\left\{\frac{1}{2}\left[(\epsilon_0 \boldsymbol{N}_E, \boldsymbol{N}_E)_{U_h} + (\mu_0 \boldsymbol{N}_H, \boldsymbol{N}_H)_{V_h} + \mu_0(\lambda_h \boldsymbol{N}_M, \boldsymbol{N}_M)_{W_h}\right] + \mathcal{E}_h(\boldsymbol{N}_M)\right\}$$

$$+(\sigma \boldsymbol{N}_E, \boldsymbol{N}_E)_{U_h} + \left(\frac{\alpha\mu_0}{|\gamma||\boldsymbol{M}_h|}\frac{\partial}{\partial t}\boldsymbol{N}_M, \frac{\partial}{\partial t}\boldsymbol{N}_M\right)_{W_h}$$

$$= \mu_0\left[\left(\frac{\alpha}{|\gamma||\boldsymbol{M}_h|}\frac{\partial}{\partial t}\boldsymbol{N}_M, \frac{\partial}{\partial t}\boldsymbol{N}_M\right)_{W_h} - (\boldsymbol{N}_H, \frac{\partial}{\partial t}\boldsymbol{N}_M)_{V_h} - (H_{l,h}(\boldsymbol{N}_M), \frac{\partial}{\partial t}\boldsymbol{N}_M)_{W_h}\right.$$

$$\left. +(\lambda_h \boldsymbol{N}_M, \frac{\partial}{\partial t}\boldsymbol{N}_M)_{W_h}\right]. \tag{5.4}$$

Now we need to verify that the right hand side of this expression vanishes in order to prove the theorem.

The first part of this verification is similar to the proof of (2.14). We note that since (4.3) holds pointwise at the interpolation points of $S_h|_{\Omega_M}$ we can use as a test function any function with well defined point values. In particular $\boldsymbol{\xi}_h = \lambda_h \boldsymbol{N}_M$ is a good test function (even though it is not in W_h). It is here that we use crucially the fact that the semi-discrete scheme is defined using discrete inner-products. Hence, using the fact that $\partial\widetilde{\boldsymbol{M}}_h/\partial t = 0$, we have

$$\left(\frac{\partial \boldsymbol{N}_M}{\partial t}, \lambda_h \boldsymbol{N}_M\right)_{W_h} = |\gamma|\,(\boldsymbol{G}_h \times \boldsymbol{M}_h, \lambda_h \boldsymbol{N}_M)_{W_h} + \alpha\left(\frac{\lambda_h}{|\boldsymbol{M}_h|}\boldsymbol{M}_h \times \frac{\partial \boldsymbol{N}_M}{\partial t}, \boldsymbol{N}_M\right)_{W_h}.$$

Expanding this expression and using the fact that $\widetilde{\boldsymbol{G}}_h \times \widetilde{\boldsymbol{M}}_h = 0$, as well as standard vector identities, we obtain

$$\left(\frac{\partial \boldsymbol{N}_M}{\partial t}, \lambda_h \boldsymbol{N}_M\right)_{W_h} = |\gamma|(\boldsymbol{N}_G \times \widetilde{\boldsymbol{M}}_h, \lambda_h \boldsymbol{M}_h)_{W_h} + \alpha\left(\frac{\lambda_h}{|\boldsymbol{M}_h|}\widetilde{\boldsymbol{M}}_h \times \frac{\partial \boldsymbol{N}_M}{\partial t}, \boldsymbol{N}_M\right)_{W_h}, \tag{5.5}$$

which should be compared to (2.14).

To derive an equation like (2.15) we use the linearity of (4.4), and the fact that $\partial \boldsymbol{N}_M/\partial t \in W_h$ to see that

$$\left(\boldsymbol{N}_H, \frac{\partial \boldsymbol{N}_M}{\partial t}\right)_{V_h} + \left(\boldsymbol{H}_{l,h}(\boldsymbol{N}_M), \frac{\partial \boldsymbol{N}_M}{\partial t}\right)_{W_h} = \left(\boldsymbol{N}_G, \frac{\partial \boldsymbol{N}_M}{\partial t}\right)_{W_h} = \left(\frac{\partial \boldsymbol{N}_M}{\partial t}, \boldsymbol{N}_G\right)_{W_h}.$$

Now using (4.3) we can write (expanding the result slightly)

$$\left(\boldsymbol{N}_H, \frac{\partial \boldsymbol{N}_M}{\partial t}\right)_{V_h} + \left(\boldsymbol{H}_{l,h}(\boldsymbol{N}_M), \frac{\partial \boldsymbol{N}_M}{\partial t}\right)_{W_h}$$

$$= |\gamma|(\boldsymbol{G}_h \times \boldsymbol{M}_h, \boldsymbol{N}_G)_{W_h} + \alpha\left(\frac{1}{|\boldsymbol{M}_h|}\boldsymbol{M}_h \times \frac{\partial \boldsymbol{N}_M}{\partial t}, \boldsymbol{N}_G\right)_{W_h}$$

$$= |\gamma|(\boldsymbol{G}_h \times \boldsymbol{M}_h, \boldsymbol{N}_G)_{W_h} + \alpha\left(\frac{1}{|\boldsymbol{M}_h|}\boldsymbol{N}_M \times \frac{\partial \boldsymbol{N}_M}{\partial t}, \boldsymbol{N}_G\right)_{W_h}$$

$$+ \quad \alpha\left(\frac{1}{|\boldsymbol{M}_h|}\widetilde{\boldsymbol{M}}_h \times \frac{\partial \boldsymbol{N}_M}{\partial t}, \boldsymbol{N}_G\right)_{W_h}. \tag{5.6}$$

To derive the analogue of (2.16) we again note that (4.3) holds for general continuous test functions. Hence choosing

$$\boldsymbol{\xi}_h = \frac{\alpha}{|\gamma|\,|\boldsymbol{M}_h|}\frac{\partial \boldsymbol{N}_M}{\partial t},$$

we obtain

$$\left(\frac{\partial \boldsymbol{N}_M}{\partial t}, \frac{\alpha}{|\gamma|\,|\boldsymbol{M}_h|}\frac{\partial \boldsymbol{N}_M}{\partial t}\right)_{W_h} = \left(\boldsymbol{G}_h \times \boldsymbol{M}_h, \frac{\alpha}{|\gamma|\,|\boldsymbol{M}_h|}\frac{\partial \boldsymbol{N}_M}{\partial t}\right)_{W_h}.$$

Now we can expand the right hand side of this equation and use the fact that $\widetilde{\boldsymbol{G}}_h = \lambda_h \widetilde{\boldsymbol{M}}_h$ at the quadrature points for the discrete W_h inner product to conclude that

$$\begin{aligned}\left(\frac{\partial \boldsymbol{N}_M}{\partial t}, \frac{\alpha}{|\gamma|\,|\boldsymbol{M}_h|}\frac{\partial \boldsymbol{N}_M}{\partial t}\right)_{W_h} &= \left(\boldsymbol{N}_G \times \boldsymbol{N}_M, \frac{\alpha}{|\gamma|\,|\boldsymbol{M}_h|}\frac{\partial \boldsymbol{N}_M}{\partial t}\right)_{W_h} + \left(\boldsymbol{N}_G \times \widetilde{\boldsymbol{M}}_h, \frac{\alpha}{|\gamma|\,|\boldsymbol{M}_h|}\frac{\partial \boldsymbol{N}_M}{\partial t}\right)_{W_h}\\ &\quad + \left(\lambda \widetilde{\boldsymbol{M}}_h \boldsymbol{N}_M, \frac{\alpha}{|\gamma|\,|\boldsymbol{M}_h|}\frac{\partial \boldsymbol{N}_M}{\partial t}\right)_{W_h}.\end{aligned} \tag{5.7}$$

Adding (5.5), (5.6) and (5.7) shows that

$$\begin{aligned}&\mu_0\left[\left(\frac{\alpha}{|\gamma||\boldsymbol{M}_h|}\frac{\partial}{\partial t}\boldsymbol{N}_M, \frac{\partial}{\partial t}\boldsymbol{N}_M\right)_{W_h} - \left(\boldsymbol{N}_H, \frac{d}{dt}\boldsymbol{N}_M\right)_{V_h} - \left(H_{l,h}(\boldsymbol{N}_M), \frac{\partial}{\partial t}\boldsymbol{N}_M\right)_{W_h}\right.\\ &\qquad\left. + \left(\lambda_h \boldsymbol{N}_M, \frac{\partial}{\partial t}\boldsymbol{N}_M\right)_{W_h}\right]\\ &= \mu_0|\gamma|\left[(\boldsymbol{N}_G \times \widetilde{\boldsymbol{M}}_h, \lambda_h \boldsymbol{N}_M)_{W_h} - (\boldsymbol{G}_h \times \boldsymbol{M}_h, \boldsymbol{N}_G)_{W_h}\right].\end{aligned}$$

However we can expand the last term on the right hand side and use standard identities to show that

$$(\boldsymbol{G}_h \times \boldsymbol{M}_h, \boldsymbol{N}_G)_{W_h} = (\boldsymbol{N}_G \times \boldsymbol{M}_h, \boldsymbol{N}_G)_{W_h} + (\widetilde{\boldsymbol{G}}_h \times \boldsymbol{M}_h, \boldsymbol{N}_G)_{W_h} = (\widetilde{\boldsymbol{G}}_h \times \boldsymbol{M}_h, \boldsymbol{N}_G)_{W_h}.$$

Again using the fact that $\widetilde{\boldsymbol{G}}_h = \lambda_h \widetilde{\boldsymbol{M}}_h$ at the quadrature points for the inner product we have (upon expanding and canceling further terms)

$$\begin{aligned}(\boldsymbol{G}_h \times \boldsymbol{M}_h, \boldsymbol{N}_G)_{W_h} &= (\lambda_h \widetilde{\boldsymbol{M}}_h \times \boldsymbol{M}_h, \boldsymbol{N}_G)_{W_h}\\ &= -(\lambda_h \boldsymbol{N}_M \times \boldsymbol{M}_h, \boldsymbol{N}_G)_{W_h}\\ &= -(\lambda_h \boldsymbol{N}_M \times \widetilde{\boldsymbol{M}}_h, \boldsymbol{N}_G)_{W_h}\\ &= (\boldsymbol{N}_G \times \widetilde{\boldsymbol{M}}_h, \lambda_h \boldsymbol{N}_M)_{W_h}.\end{aligned}$$

Using these results shows that the right hand side of (5.4) vanishes and this completes the proof. ∎

In the same way as for the continuous problem, a corollary of this result is that the discrete energy of the system decays. It would now be desirable to prove convergence of the method using the discrete analogue of the continuous dependence result proved for the variational problem. While we believe this to be possible, we have not yet done it.

6. Numerical results. First we give some details of the fully discrete scheme used for calculations. Then examples are provided in the 2-D case. We limit ourselves here to some static examples that illustrate the foregoing theory.

6.1. A discrete scheme. We choose an explicit/implicit time-stepping scheme that guarantees the conservation of the norm of $\boldsymbol{M}_h$ pointwise.

Let $\Delta t > 0$ be the time step. Then we wish to compute

$$(\boldsymbol{E}_h^{(n)}, \boldsymbol{H}_h^{(n+\frac{1}{2})}, \boldsymbol{M}_h^{(n+\frac{1}{2})}, \boldsymbol{G}_h^{(n)}) \in U_h \times V_h \times W_h \times W_h$$

for $n = 0, 1, , \cdots$ such that

$$\boldsymbol{E}_h^{(n)} \approx \boldsymbol{E}_h(t_n), \quad \boldsymbol{H}_h^{(n+\frac{1}{2})} \approx \boldsymbol{H}_h(t_{n+\frac{1}{2}}), \quad \boldsymbol{M}_h^{(n+\frac{1}{2})} \approx \boldsymbol{M}_h(t_{n+\frac{1}{2}}), \quad \boldsymbol{G}_h^{(n)} \approx \boldsymbol{G}_h(t_n),$$

where $t_n = n\Delta t$.

The initial data for $\boldsymbol{E}$ gives $\boldsymbol{E}_h^{(0)}$. The values of $\boldsymbol{H}_h^{(1/2)}$ and $\boldsymbol{M}_h^{(1/2)}$ can then be found by a half time step of a suitable explicit scheme (for example Runge-Kutta). From then on

$$(\boldsymbol{E}_h^{(n+1)}, \boldsymbol{G}_h^{(n+1)}, \boldsymbol{H}_h^{(n+\frac{3}{2})}, \boldsymbol{M}_h^{(n+\frac{3}{2})})$$

is determined from

$$(\boldsymbol{E}_h^{(n)}, \boldsymbol{G}_h^{(n)}, \boldsymbol{H}_h^{(n+\frac{1}{2})}, \boldsymbol{M}_h^{(n+\frac{1}{2})})$$

as follows. First we can determine $\boldsymbol{E}_h^{(n+1)}$.

$$\begin{aligned} &\varepsilon_0 \left(\frac{\boldsymbol{E}_h^{(n+1)} - \boldsymbol{E}_h^{(n)}}{\Delta t}, \boldsymbol{\psi}_h \right)_{U_h} + \frac{1}{2} \left(\sigma \left(\boldsymbol{E}_h^{(n+1)} + \boldsymbol{E}_h^{(n)} \right), \boldsymbol{\psi}_h \right)_{U_h} - \left(\boldsymbol{H}_h^{(n+\frac{1}{2})}, \nabla \times \boldsymbol{\psi}_h \right)_{V_h} \\ &= - \left(\boldsymbol{J}_h^{(n+\frac{1}{2})}, \boldsymbol{\psi}_h \right)_{U_h} \quad \forall \boldsymbol{\psi}_h \in U_h \,. \end{aligned} \tag{6.1}$$

Assuming that the discrete inner product is well chosen, this is a rapid explicit calculation.

The remaining fields are computed from the non-linear system consisting of

$$\mu_0 \left(\frac{\boldsymbol{H}_h^{(n+\frac{3}{2})} - \boldsymbol{H}_h^{(n+\frac{1}{2})}}{\Delta t}, \boldsymbol{\varphi}_h \right)_{V_h} + \left(\nabla \times \boldsymbol{E}_h^{(n+1)}, \boldsymbol{\varphi}_h \right)_{V_h} = -\mu_0 \left(\frac{\boldsymbol{M}_h^{(n+\frac{3}{2})} - \boldsymbol{M}_h^{(n+\frac{1}{2})}}{\Delta t}, \boldsymbol{\varphi}_h \right)_{V_h}, \tag{6.2}$$

for all $\boldsymbol{\varphi}_h \in V_h$,

$$\begin{aligned} \left(\frac{\boldsymbol{M}_h^{(n+3/2)} - \boldsymbol{M}_h^{(n+1/2)}}{\Delta t}, \boldsymbol{\xi}_h \right)_{W_h} &= |\gamma| \left(\boldsymbol{G}_h^{(n+1)} \times \boldsymbol{M}_h^{(n+1)}, \boldsymbol{\xi}_h \right)_{W_h} \\ &+ \alpha \left(\frac{\boldsymbol{M}_h^{(n+1)}}{|\boldsymbol{M}_h|} \times \frac{\boldsymbol{M}_h^{(n+3/2)} - \boldsymbol{M}_h^{(n+1/2)}}{\Delta t}, \boldsymbol{\xi}_h \right)_{W_h}, \end{aligned} \tag{6.3}$$

for all $\boldsymbol{\xi}_h \in W_h$ where $\boldsymbol{M}_h^{(n+1)} = (\boldsymbol{M}_h^{(n+\frac{3}{2})} + \boldsymbol{M}_h^{(n+\frac{1}{2})})/2$,

$$\begin{aligned} \left(\boldsymbol{G}_h^{(n+1)}, \boldsymbol{\phi}_h \right)_{W_h} &= \left(\boldsymbol{H}_h^{(n+1)}, \boldsymbol{\phi}_h \right)_{V_h} - \frac{2}{\mu_0} \left(KP\left(\boldsymbol{M}_h^{(n+1)} \right), \boldsymbol{\phi}_h \right)_{W_h} \\ &- \frac{2}{\mu_0} \sum_{i=1}^{3} \left(A \frac{\partial \boldsymbol{M}_h^{(n+1)}}{\partial x_i}, \frac{\partial \boldsymbol{\phi}_h}{\partial x_i} \right)_{W_h}, \end{aligned} \tag{6.4}$$

for all $\boldsymbol{\phi}_h \in W_h$, where

$$\boldsymbol{H}_h^{(n+1)} = (\boldsymbol{H}_h^{(n+\frac{3}{2})} + \boldsymbol{H}_h^{(n+\frac{1}{2})})/2.$$

In practice we actually replace $\boldsymbol{H}^{n+1/2}$ by an equivalent discretization of $\boldsymbol{B}$ that makes (6.2) explicit. In our code the above two equations are then solved by Newton's method at each time-step which is very time consuming. Due to the use of the discrete inner product, equation (6.3) is satisfied at the quadrature points of the discrete inner product and hence at the interpolation points for W_h.

Next we verify that the time stepping scheme conserves the magnetization.

Lemma 6.1. *Let $\boldsymbol{a}_i$ be an interpolation point for W_h then for $n \geq 0$,*

$$|\boldsymbol{M}_h^{(n+\frac{1}{2})}(\boldsymbol{a}_i)| = |\boldsymbol{M}_h^{(1/2)}(\boldsymbol{a}_i)|$$

Proof. Similarly to the proof of Lemma 5.1, we choose $\boldsymbol{\xi}_h$ in (6.3) to interpolate $\boldsymbol{M}_h^{(n+\frac{3}{2})} + \boldsymbol{M}_h^{(n+\frac{1}{2})}$ at $\boldsymbol{a}_i$ and vanish at all other points. We conclude that

$$(\boldsymbol{M}_h^{(n+\frac{3}{2})}(\boldsymbol{a}_i) - \boldsymbol{M}_h^{(n+\frac{1}{2})}(\boldsymbol{a}_i)) \cdot (\boldsymbol{M}_h^{(n+\frac{3}{2})}(\boldsymbol{a}_i) + \boldsymbol{M}_h^{(n+\frac{1}{2})}(\boldsymbol{a}_i)) = 0.$$

Hence $|\boldsymbol{M}_h^{(n+\frac{3}{2})}(\boldsymbol{a}_i)|^2 - |\boldsymbol{M}_h^{(n+\frac{1}{2})}(\boldsymbol{a}_i)|^2 = 0$, and the result follows. ☐

Obviously it would be desirable to prove convergence of the fully discrete scheme, together with the existence and behavior of the Liapunov function. This has yet to be completed. Instead we show some numerical results that suggest that the fully discrete Liapunov function behaves as expected.

6.2. Examples of discrete stationary states and Liapunov functions. In this section we shall present some numerical results for a 2D version of the problem and method outlined in the previous sections. The coefficients and basic geometry are from the NIST web site

`http://www.ctcms.nist.gov/~rdm/toc.html#standards`

and mimic permalloy:

$$\begin{aligned}
A &= 1.3 \times 10^{-11} \text{ J/m} \\
K &= 500 \text{ J/m}^3 \\
\alpha &= 1 \\
\gamma &= 2.2 \times 10^5
\end{aligned}$$

The region Ω_M is a 1×2 micron rectangle (the NIST benchmark actually calls for the object to be a wafer 200 nm thick, but in this study we will only do a 2D calculation). The magnetization has constant magnitude at every point in the ferromagnet

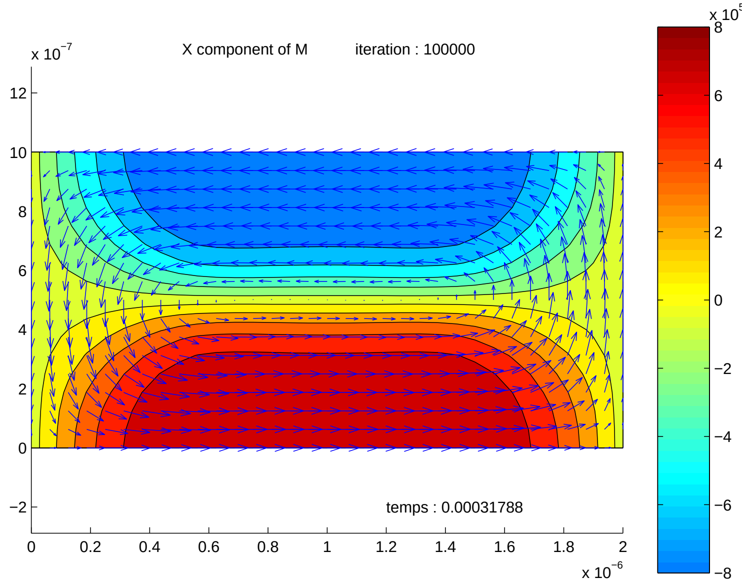
$$|\boldsymbol{M}(\boldsymbol{x}, t)| = 8.0 \times 10^5 \text{ A/m at every } \boldsymbol{x} \in \Omega_M.$$

The 2D problem can be obtained from the full 3D case, formally, by assuming that the ferromagnet occupies an infinite cylinder with axis parallel to the z coordinate axis. Then the fields are assumed to be independent of the z coordinate (depending only on x, y and t). This reduces the problem to two space dimensions. However due to the non-linear nature of the problem the fields cannot be decoupled into TE and TM modes. Thus the 2D simulation proceeds using field vectors with three components. The 2D finite elements are correspondingly obtained from the 3D cubic elements we discussed in section § 4.2.

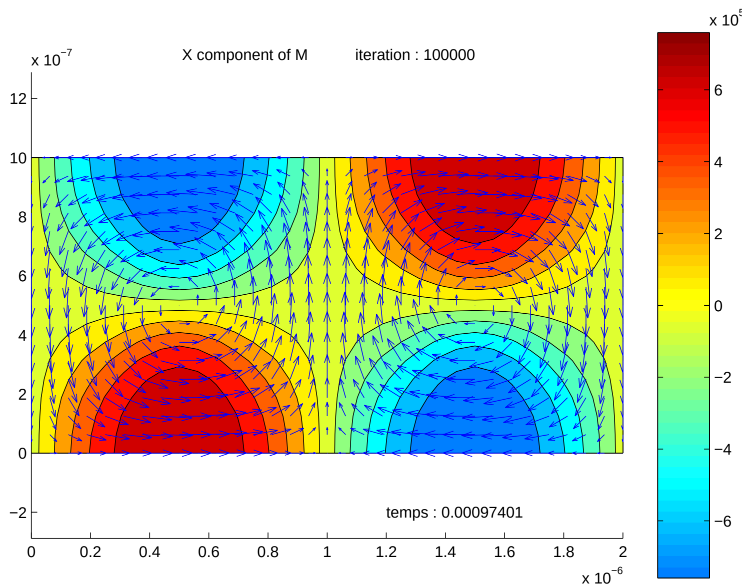
In the two examples presented here, the ferromagnet if covered by a 4×8 grid of square elements. The surrounding air layer is a further two elements thick all around the ferromagnet. This gives $h = 0.125$ microns. Due to explicit time stepping of Maxwell's equations there is a CFL condition that must be satisfied. This is a strong constraint on the time-step and we use $\Delta t = 4.167 \times 10^{-17}$ seconds. One positive effect of the very small time step is that we do not need to perform a large number of Newton steps at each time-step.

In these computations we seek to compute an approximate stationary state of the Maxwell-LLG system, so we compute for many time-steps. Our experience is that convergence to steady state slows as the simulation proceeds. Hence, since we are only interested to find a steady state solution, we progressively increase the gyro-magnetic factor γ when we detect a slow down of convergence. More precisely, our criterion for changing γ is that we try to keep at two Newton steps per time step. If more Newton steps are needed we decrease γ whereas if only one Newton step is needed we increase γ. This corresponds to using a different, non-physical, time step. In the computations shown here we actually integrated for 10^5 time steps.

In Figure 6.1 we show the final results of two computations that differ only in the initial data used. We only show the x and y components of $\boldsymbol{M}$ as arrows and the x component using color contours (this is so that the domains are easily visible) in the ferromagnet. During the progress of the solution we monitor the



a) Single Domain



b) Two Domains

FIG. 6.1. *Here we show the xy components of the magnetization* $\boldsymbol{M}$ *as arrows and the x component as color contours (to facilitate visualizing the domains) at the final time for two different initial conditions.*

electromagnetic energy, the exchange energy term $\mathcal{E}_{ex}$, the energy of anisotropy $\mathcal{E}_{an}$ and the gyro-magnetic factor γ. Results corresponding to the single domain case in Figure 6.1 a) are shown in Figure 6.2.

To test the Liapunov functions we computed the indicator function λ_h for each steady state shown in Figure 6.1, and then using the appropriate initial data we computed the energy and Liapunov function $V_{\lambda_h,h}$ for each λ_h. The results are shown in Figure 6.3. In Figure 6.3a) we are computing the single domain steady state shown in Figure 6.1 a). The total energy is marked W (in red) and is the sum of the components in Figure 6.2. The total energy decreases during the computation. We also show the evolution of the Liapunov

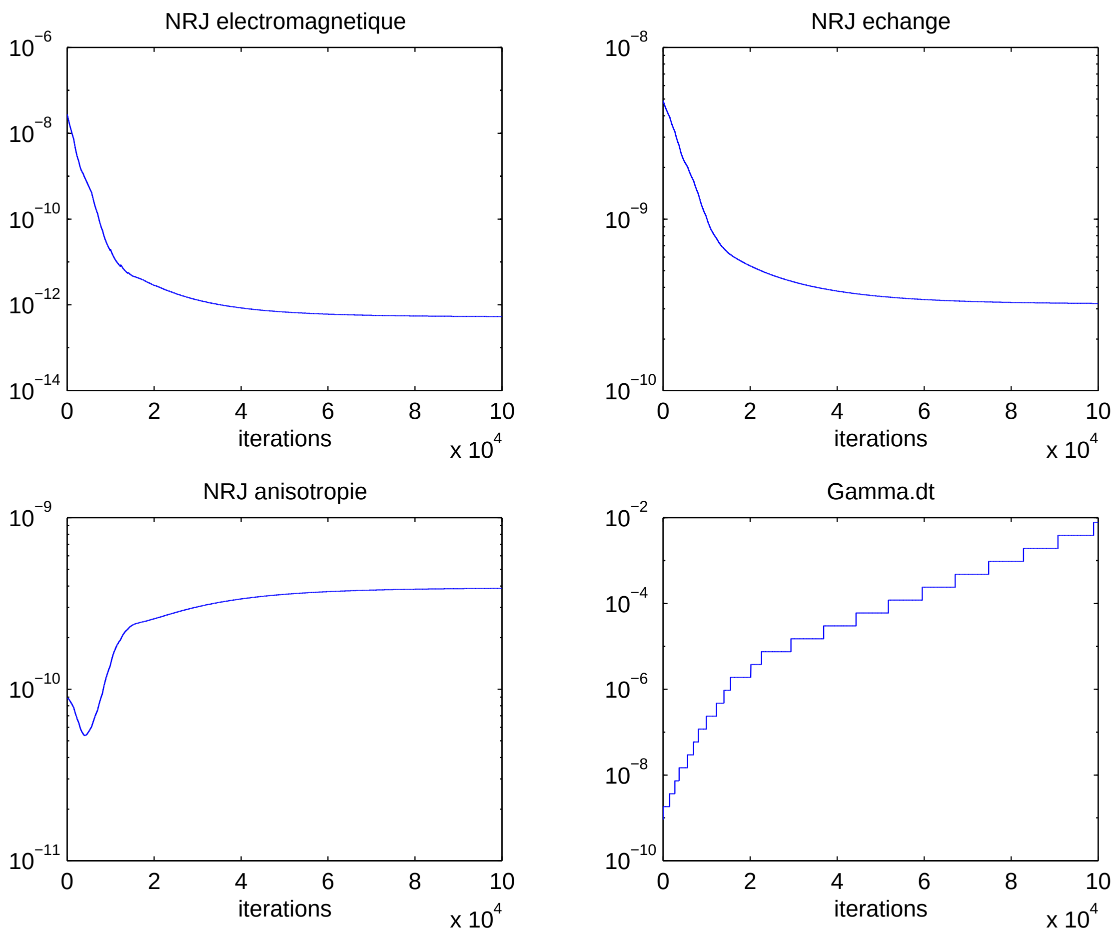


FIG. 6.2. *Here we show the time change of the various components of the energy during the computation of Figure 6.1 a). The top left panel shows the electromagnetic energy (i.e. the energy for $\boldsymbol{E}$ and $\boldsymbol{H}$), the top right shows the exchange energy $\mathcal{E}_{ex}$, the bottom left shows the energy of anisotropy $\mathcal{E}_{an}$ and the bottom right panel shows how γ is increased throughout the computation.*

functions corresponding to the single and double domain static solutions (these are marked $Vld1$ and $Vld2$ respectively). As suggested by Theorem 5.2 the Liapunov functions decay at the same rate as the energy. In this case it is difficult to distinguish the two Liapunov functions. However in Figure 6.3b) we show the same data corresponding to the computation of the double domain static solution in Figure 6.1 b). Again the evolution of the energy and Liapunov functions is similar, but only the Liapunov function for the double domain case ($Vld2$) decreases to zero. The other Liapunov function (which is a Liapunov function for the single domain static solution but not the double domain case) does not decrease to zero.

7. Conclusion. Our limited computational and theoretical study of the finite element method proposed here suggests that it not only conserves the magnitude of the magnetization but also has the correct behavior of energy decay and of the Liapunov functions. It is in this sense that we claim the method to be an "accurate" scheme for computing the solution of the Maxwell-LLG system.

There are still many open questions with the scheme. We would like to prove an error estimate (although it seems unlikely that a long time error estimate will be possible), and we would like to improve the efficiency of the time-stepping so as to handle 3D problems.

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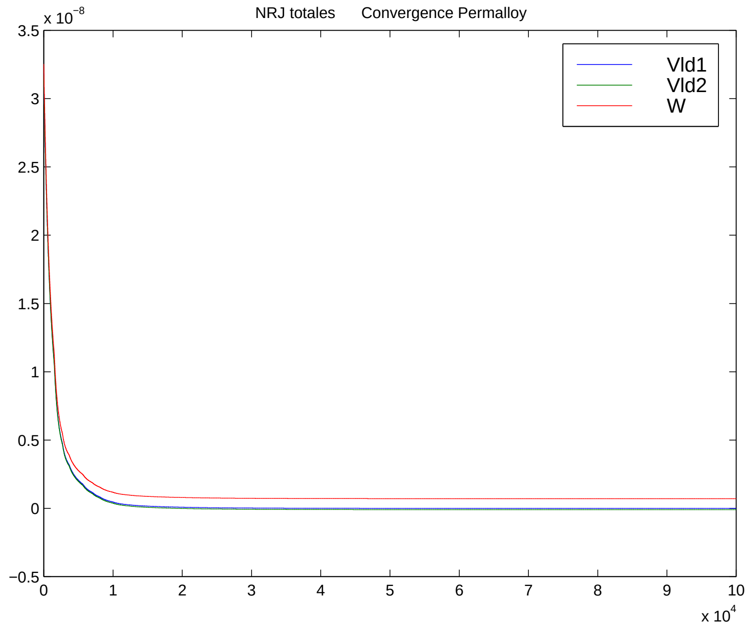
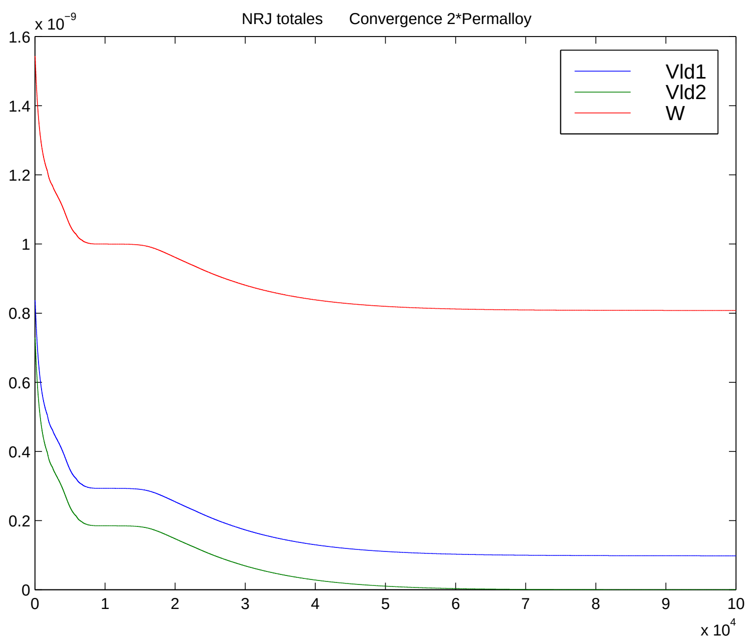


a) Single domain.



a) Double domain.

FIG. 6.3. *Here we show the evolution of the total energy and Liapunov functions for the two examples shown in Figure 6.1. In the top panel we use initial data that results in the single domain case shown in Figure 6.1a). Similarly the lower panel corresponds to Figure 6.1b). In each case W is the total energy, $Vld1$ is the Liapunov function for the single domain and $Vld2$ is the Liapunov function for the double domain. These results verify the predicted rate of decay of the energy and Liapunov functions, and show that the Liapunov functions can distinguish final states (particularly in the lower panel).*

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