Tractable Forms of the Bond Pricing Equation

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Technical Report No. 2003-10

DEPARTMENT
OF
MATHEMATICAL SCIENCES

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Abstract

So far, a small number of analytically tractable single-factor models have been devised for the well-known Bond Pricing Equation (BPE). In this paper, new tractable models are formulated in a systematic manner. First, the BPE is transformed to a standard canonical form in which only one coefficient function appears. In some interesting cases, this single coefficient function is identically zero, leaving nothing more to solve than the classical heat equation. In many cases, the canonical form allows a general solution by standard mathematical techniques such as separation of variables and Laplace transforms. In other cases, the general solution of the BPE is reduced to a single inverse Laplace Transform.

KEY WORDS: bond pricing equation, short-rate, canonical form

1. INTRODUCTION

Over the last 25 years there has been great interest in the modelling of the term structure of interest rates. The value of the interest rate derivatives, such as bonds and swaps, naturally depends on the interest rates.

It can be shown (see e.g [1]) that when the short-term interest rate, r, follows a stochastic differential equation of the form

$$dr = u(r,t)dt + w(r,t)dX,$$

where dX is an increment in a Wiener process and t is time, then the price of a zero coupon bond $V(r,t;T)$ with expiry at $t=T$ will satisfy the partial differential equation PDE

$$\frac{\partial V}{\partial t} + \frac{w^2}{2}\frac{\partial^2 V}{\partial r^2} + (u - \lambda w)\frac{\partial V}{\partial r} - rV \;=\; 0, \tag{1.1}$$

subject to $V(r,T)=1$, where $\lambda(r,t)$ is the market price of risk. We shall refer to equation (1.1) as the Bond Pricing Equation (BPE). Many of the proposed models describing the dynamics of the short-rate take the form $u(r,t)=\alpha+\beta r$ and $w(r,t)=\sigma r^\gamma$ where $\alpha,\beta,\sigma,\gamma$ are constants. These include those of Vasicek [2] ($\gamma=0$), Cox Ingersoll and Ross [3] ($\gamma=1/2$), Brennan and Schwartz

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[4] ($\gamma = 1$) and Dothan [5] ($\gamma = 0$). The empirical work of Chan, Karolyi, Longstaff and Sanders [6] on such models, found that the most successful in capturing the dynamics of the short-rate were those that allowed the volatility of interest rate changes to be highly sensitive to the level of the interest rate, in particular, those with $\gamma \geq 1$. Their unconstrained estimate of γ was 1.5. They also found only weak evidence of a long-run level of mean reversion, suggesting that the short-rate may revert to a short-rate mean which may be time-dependent. However Ait-Sahalia [7] showed that the linearity of the drift appeared to be the main source of misspecification.

Interest rate models such as those of Vasicek [2] and Cox, Ingersoll and Ross [3] are popular, as they lead to analytic solutions of (1.1). Hence they can be used to provide models for the term structure of interest rates by means of the relation

$$y = \frac{-\log V}{T-t}, \qquad \text{for } r \text{ fixed.}$$

However, as Chan *et al* [6] found, many of these well-known models perform poorly in their ability to capture the actual behaviour of the short rate. Interest rate models such as those of Ho and Lee [8]

$$dr = \eta(t)dt + c\ dX, \qquad c \text{ constant },$$

Hull and White [9]

$$dr = (\eta(t) - \gamma r)dt + c\ dX, \qquad c \text{ constant },$$

and Goard [10]

$$dr = [c^2 r(a(t) - qr)]dt + cr^{3/2}dX, \qquad c, q \text{ constant },$$

incorporate time-dependent parameters, which have the added advantage of allowing yield-curves to be fitted.

Another very popular method of modelling interest rates is the Heath, Jarrow and Morton (HJM) forward rate model (see e.g [1]). Rather than modelling the short-term interest rate, this model derives the whole forward-rate curve from which it is a simple matter to fit the yield curve. However, in some places of the world, such as areas of India and Pakistan, derivatives on bonds/interest rates do not exist due to regulatory restrictions. As HJM require market derivative data for calibration, investors are forced to restrict themselves to only using the short rate and hence the modelling of the short rate is still very important.

This paper lays emphasis on presenting new solutions to the BPE where the forms of the coefficients are not as restricted as many of the other well-known one-factor models. We look for specific forms for the risk neutral drift and volatility that lead to tractable forms of (1.1) in the case where the volatility is a function of the interest rate alone. In all of our examples, our short-rate follows the realistic volatility of the form cr^2 or $cr^{3/2}$. As not many solutions are known to (1.1) when the drift function depends on both r and t, we concentrate primarily on this case and present new solutions in Section 2.2. We also take a brief look at the simpler case where the drift function is independent of time in Section 2.3.

2. NEW TRACTABLE SOLUTIONS

This Section is organised as follows: In Section 2.1 we reduce the BPE to its canonical form. Then Section 2.2 concentrates on the case where the drift function depends on r, the interest rate and t, time, and constructs new analytic solutions to the BPE using separation of variables (Section 2.2.2), Laplace Transforms (Section 2.2.3) and by reducing it to the heat equation (Section 2.2.4). In Section 2.3 we briefly look at the case where the drift function only depends on r.

2.1 Reduction to Canonical Form

It is well-known (e.g. [11], [12]) that for any parabolic equation

$$\frac{\partial^2 v}{\partial x^2} + \alpha(x,y)\frac{\partial v}{\partial x} + \beta(x,y)\frac{\partial v}{\partial y} + \gamma(x,y)v \;=\; 0, \tag{2.1}$$

there exists a point transformation of the form $x_1 = x_1(x,y), x_2 = x_2(x,y)$, and $z = H(x,y)v$, such that (2.1) becomes

$$\frac{\partial^2 z}{\partial x_1^2} + \frac{\partial z}{\partial x_2} + Q(x_1,x_2)z \;=\; 0, \tag{2.2}$$

for some function $Q(x_1,x_2)$. This canonical form is much more convenient for analysis, since it includes only one model-dependent adjustable coefficient function. We shall find special cases of the coefficient function $Q(x_1,x_2)$ that allow either a full general solution of (2.2) or reduction to an inverse Laplace Transform. The BPE can be written as

$$\frac{\partial^2 V}{\partial r^2} + 2b(r,t)\frac{\partial V}{\partial r} + \frac{2}{w^2}\frac{\partial V}{\partial t} - \frac{2r}{w^2}V \;=\; 0, \tag{2.3}$$

where

$$b(r,t) \;=\; \left(\frac{u-\lambda w}{w^2}\right),$$

noting that $w = w(r) \neq 0$, $u = u(r,t)$ and $\lambda = \lambda(r,t)$. Hence by substituting the point transformations

$$r_1 \;=\; r_1(r,t),\; r_2 \;=\; r_2(r,t) \text{ and } z \;=\; H(r,t)V$$

into

$$\frac{\partial^2 z}{\partial r_1^2} + \frac{\partial z}{\partial r_2} + Q(r_1,r_2)z \;=\; 0, \tag{2.4}$$

and then equating the coefficients of the derivative terms of the resultant equation with those of equation (2.3), we obtain the transformations that reduce (2.3) to (2.4) as

$$\begin{aligned} t &= mr_2 + a_1 \\ \int \frac{1}{w(r)}dr &= \sqrt{\tfrac{m}{2}}\, r_1 + a_2 \\ z &= H(r,t)V, \text{ where} \\ H(r,t) &= w^{-1/2}c(t)\exp\left(\int b(r,t)dr\right), \end{aligned} \tag{2.5}$$

where a_1, a_2 and $m(\neq 0)$ are arbitrary constants and $c(t)$ is an arbitrary function of t. Then $Q(r_1,r_2)$ in (2.4) is given by

$$\begin{aligned} Q(r_1,r_2) \;=\; -mr - \tfrac{mww'}{2}\left(b(r,t) - \tfrac{w'}{2w}\right) \\ -\tfrac{w^2m}{2}\left[\tfrac{\partial b}{\partial r} - \tfrac{1}{2}\left(\tfrac{ww''-(w')^2}{w^2}\right) + \left(b(r,t) - \tfrac{w'}{2w}\right)^2\right] \\ -m\left(\tfrac{c'(t)}{c(t)} + \tfrac{\partial}{\partial t}\int b(r,t)dr\right). \end{aligned} \tag{2.6}$$

2.2 The Drift Function Depends on Time

In this section we assume $\frac{\partial u}{\partial t}$ and hence $\frac{\partial b}{\partial t}$ are non-zero. We note firstly that if b and w were such that Q could be written in the form $Q(r_1,r_2) = q_0(r_2) + q_1(r_2)r_1 + q_2(r_2)r_1^2$ then (2.4) could be transformed to a constant coefficient linear PDE (see [11]).

2.2.1 Preliminary Simplifications. In order to solve equation (2.4) by separation of variables or by Laplace Transforms we choose Q to be of the form

$$Q(r_1,r_2) = F(r_1) + G(r_2)$$

and for simplicity take $m = 2$ in (2.6). We thus require for $w = w(r)$

$$Q(r_1,r_2) = -2r + \frac{ww''}{2} - \frac{(w')^2}{4} - 2\frac{c'(t)}{c(t)} - \zeta, \tag{2.7}$$

and $b(r,t)$ to satisfy

$$w^2\frac{\partial b}{\partial r} + w^2 b^2 + 2\frac{\partial}{\partial t}\int b(r,t)dr \;=\; \zeta, \tag{2.8}$$

where ζ is the sum of a function of r and a function of t. For now we let $\zeta = 0$ and will consider a more general form of ζ in Section 2.2.4. Differentiating (2.8) with respect to r we get

$$\frac{\partial}{\partial r}\left(\frac{w^2}{2}\left[\frac{\partial b}{\partial r} + b^2\right]\right) + \frac{\partial b}{\partial t} \;=\; 0. \tag{2.9}$$

Then letting $b(r,t) = B^{-1}\frac{\partial B}{\partial r}$, (2.9) becomes

$$\frac{\partial}{\partial r}\left(-\frac{w^2}{2}\frac{B_{rr}}{B}\right) = \frac{\partial}{\partial t}\left(\frac{\partial}{\partial r}\ln B\right),$$

which on integrating with respect to r becomes

$$-\frac{w^2}{2}B_{rr} = B_t + g(t)B, \tag{2.10}$$

where g is an arbitrary function of t. Now letting $B(r,t) = k(t)U(r,t)$, where $k(t) \;=\; \exp\left(-\int g(t)dt\right)$, equation (2.10) simplifies to

$$\frac{\partial^2 U}{\partial r^2} + \frac{2}{w(r)^2}\frac{\partial U}{\partial t} \;=\; 0.$$

From (2.5) we have

$$\begin{aligned} r_1 &= \int \frac{1}{w(r)}dr, \\ r_2 &= \tfrac{t}{2} \end{aligned} \tag{2.11}$$

where we have set $a_1 = a_2 = 0, m = 2$, and

$$\begin{aligned} V(r,t) &= \tfrac{z}{H(r,t)} \\ &= \tfrac{(w(r))^{1/2}z}{c(t)U(r,t)}. \end{aligned} \tag{2.12}$$

As $b \;=\; B^{-1}\frac{\partial B}{\partial r} \;=\; U^{-1}\frac{\partial U}{\partial r}$, then without loss of generality we let $g(t) = 0$ and hence B satisfies

$$\frac{\partial^2 B}{\partial r^2} + \frac{2}{w(r)^2}\frac{\partial B}{\partial t} \;=\; 0. \tag{2.13}$$

From (2.12) then

$$V(r,t) = \frac{(w(r))^{1/2}z}{c(t)B(r,t)}, \tag{2.14}$$

which needs to satisfy the final condition $V(r,T) = 1$, where T is the expiry time of the bond, so that

$$V(r,T) = 1 = \frac{(w(r))^{1/2}}{c(T)B(r,T)}z(r_1, T/2). \tag{2.15}$$

In the remainder of this section we will concentrate on constructing new analytic solutions to the BPE (1.1) by solving (2.4) subject to (2.7), (2.13) (where $b = B^{-1}\frac{\partial B}{\partial r}$) and (2.15).

2.2.2 Solution by Separation of Variables. Letting $z = X(r_1)T(r_2)$, from (2.4) and (2.7) we get

$$T(r_2) = \beta c(2r_2)e^{-\mu r_2}, \tag{2.16}$$

and that

$$\frac{X''}{X} - 2r + \frac{w''w}{2} - \frac{(w')^2}{4} = \mu, \tag{2.17}$$

where β is constant and μ is the separation constant.

For convenience we let $\bar{t} = T - t$ so that (2.13) becomes

$$\frac{\partial^2 B}{\partial r^2} - \frac{2}{w(r)^2}\frac{\partial B}{\partial \bar{t}} = 0, \tag{2.18}$$

which from (2.15) we solve subject to

$$B(r,0) = \beta(w(r))^{1/2}e^{-\mu T/2}X(r_1). \tag{2.19}$$

We now consider two specific forms of $w(r)$.

Example-1: $\mathbf{w(r) = r^2}$: From (2.17) we have

$$X'' + (-\mu + 2/r_1)X = 0,$$

noting that from (2.11) $r_1 = -\frac{1}{r}$. Hence (see e.g [13])

$$X(r_1) = e^{\sqrt{\mu}r_1}r_1\left[c_1\Phi\left(1+\frac{1}{\sqrt{\mu}};2;-2\sqrt{\mu}r_1\right) + c_2\Psi\left(1+\frac{1}{\sqrt{\mu}};2;-2\sqrt{\mu}r_1\right)\right],$$

where c_1 and c_2 are arbitrary constants and Φ, Ψ represent the Kummer-M and Kummer-U functions respectively (see e.g [14]). A solution to equation (2.4) with $w(r) = r^2$ then is

$$\begin{aligned} z(r_1,r_2) &= X(r_1)T(r_2) \\ &= c(2r_2)e^{-\mu r_2}e^{\sqrt{\mu}r_1}r_1\left[c_1\Phi\left(1+\frac{1}{\sqrt{\mu}};2;-2\sqrt{\mu}r_1\right)\right. \\ &\quad \left. + \; c_2\Psi\left(1+\frac{1}{\sqrt{\mu}};2;-2\sqrt{\mu}r_1\right)\right] \quad \text{for} \quad \mu 0. \end{aligned} \tag{2.20}$$

From (2.18) and (2.19) we now need to solve

$$\frac{\partial^2 B}{\partial r^2} - \frac{2}{r^4}\frac{\partial B}{\partial \bar{t}} = 0, \tag{2.21}$$

subject to

$$B(r,0) = e^{-\mu T/2}e^{-\sqrt{\mu}/r}\left[c_1\Phi\left(1+\frac{1}{\sqrt{\mu}};2;\frac{2\sqrt{\mu}}{r}\right) + \; c_2\Psi\left(1+\frac{1}{\sqrt{\mu}};2;\frac{2\sqrt{\mu}}{r}\right)\right], \tag{2.22}$$

for $\mu 0$. Taking Laplace Transforms of (2.22) with respect to $\bar{t}$, we get

$$r^4 \frac{d^2 \tilde{B}}{dr^2} - 2p\tilde{B} = -2B(r,0) \tag{2.23}$$

where $\tilde{B}(r,p) = \mathcal{L}\{B(r,\bar{t})\}$. The solution to the corresponding homogeneous equation of (2.23) is

$$\tilde{B}_c = r\left[A(p)e^{\sqrt{2p}/r} + D(p)e^{-\sqrt{2p}/r}\right], \tag{2.24}$$

where $A(p)$ and $D(p)$ are arbitrary functions of the transform variable p. Writing $\tilde{B}_c = r\left(\tilde{B}_{c_1} + \tilde{B}_{c_2}\right)$, where $\tilde{B}_{c_1} = A(p)e^{a\sqrt{p}}$ and $\tilde{B}_{c_2} = D(p)e^{-a\sqrt{p}}$ (with $a = \sqrt{2}/r$), Table 2.1 outlines possible invertible choices for $\tilde{B}_{c_2}$ and $\tilde{B}_{c_1}$ and their respective Inverse Laplace Transforms (see Oberhettinger and Badi (1973)).

As an example suppose we choose $A(p) = 0$ and $D(p) = c_4 + c_3 p^{-1/2}$. Then

$$B_c = r\left[\frac{c_3}{\sqrt{\pi\bar{t}}} exp(-\frac{1}{2r^2\bar{t}}) + \frac{c_4}{r\sqrt{2\pi}\bar{t}^{3/2}} exp(-\frac{1}{2r^2\bar{t}})\right], \quad \text{where} \quad \bar{t} = T - t. \tag{2.25}$$

To find $\tilde{B}_p$, a particular solution to (2.23) we use variation of parameters with the two independent solutions of the corresponding homogeneous equations of (2.23).
After a little simplification we find

$$\begin{aligned} &\tilde{B}_p(r,p) \\ &= -\frac{re^{-\sqrt{2p}/r}e^{-\mu T/2}}{\sqrt{2p}} \int_0^r \frac{e^{(\sqrt{2p}-\sqrt{\mu})/r}}{r^3}\left[c_1\Phi\left(1+\frac{1}{\sqrt{\mu}};2;\frac{2\sqrt{\mu}}{r}\right) + c_2\Psi\left(1+\frac{1}{\sqrt{\mu}};2;\frac{2\sqrt{\mu}}{r}\right)\right] dr \\ &+ \frac{re^{\sqrt{2p}/r}e^{-\mu T/2}}{\sqrt{2p}} \int_0^r \frac{e^{-(\sqrt{2p}+\sqrt{\mu})/r}}{r^3}\left[c_1\Phi\left(1+\frac{1}{\sqrt{\mu}};2;\frac{2\sqrt{\mu}}{r}\right) + c_2\Psi\left(1+\frac{1}{\sqrt{\mu}};2;\frac{2\sqrt{\mu}}{r}\right)\right] dr. \end{aligned} \tag{2.26}$$

To find the Laplace inverse of $\tilde{B}_p$ we define

$$h(r) = \frac{e^{-\sqrt{\mu}/r}}{r^3}\left[c_1\Phi\left(1+\frac{1}{\sqrt{\mu}};2;\frac{2\sqrt{\mu}}{r}\right) + c_2\Psi\left(1+\frac{1}{\sqrt{\mu}};2;\frac{2\sqrt{\mu}}{r}\right)\right] \tag{2.27}$$

and also rewrite $\tilde{B}_p$ as

$$\tilde{B}_p = re^{-\mu T/2}(\tilde{B}_{p_1} + \tilde{B}_{p_2}),$$

where

$$\tilde{B}_{p_2} = -\frac{e^{-\sqrt{2p}/r}}{\sqrt{2p}} \int_0^r e^{\sqrt{2p}/r} h(r)dr, \quad \text{and} \quad \tilde{B}_{p_1} = \frac{e^{\sqrt{2p}/r}}{\sqrt{2p}} \int_0^r e^{-\sqrt{2p}/r} h(r)dr.$$

If we now let

$$B_e(r,\bar t) = \mathfrak{L}^{-1}\left\{\frac{e^{\frac{\sqrt{2p}}{r}}}{\sqrt{2p}}\int_o^r e^{-\frac{\sqrt{2p}}{r_3}}h_e(r_3)dr_3\right\}, \text{ and}$$

$$B_o(r,\bar t) = \mathfrak{L}^{-1}\left\{\frac{e^{\frac{\sqrt{2p}}{r}}}{\sqrt{2p}}\int_o^r e^{-\frac{\sqrt{2p}}{r_3}}h_o(r_3)dr_3\right\} \tag{2.28}$$

where

$$h_e = \frac{1}{2}(h(r)+h(-r)), \text{ the even part of } h, \text{ and}$$
$$h_o = \frac{1}{2}(h(r)-h(-r)), \text{ the odd part of } h,$$

then

$$B_p(r,\bar t) = re^{\frac{-\mu T}{2}}\left\{B_e(r,\bar t)+B_o(r,\bar t)+B_e(-r,\bar t)-B_o(-r,\bar t)\right\}. \tag{2.29}$$

We note firstly that

$$\mathfrak{L}^{-1}\left\{e^{\frac{p}{r}}\int_0^r e^{\frac{-p}{r_3}}h_e(r_3)dr_3\right\} = \int_0^r \mathfrak{L}^{-1}\left(e^{-p\left(\frac{1}{r_3}-\frac{1}{r}\right)}h_e(r_3)\right)dr_3$$
$$= \int_0^r \delta\left(\bar t-\left(\frac{1}{r_3}-\frac{1}{r}\right)\right)h_e(r_3)dr_3.$$

With the substitution $r_4 = -\frac{1}{r_3}$, we get

$$\int_0^r \delta\left(\bar t-\left(\frac{1}{r_3}-\frac{1}{r}\right)\right)h_e(r_3)dr_3 = \int_{-\infty}^{-\frac{1}{r}}\delta\left(\bar t+r_4+\frac{1}{r}\right)h_e\left(-\frac{1}{r_4}\right)\frac{1}{r_4^2}dr_4$$
$$= h_e\left([\bar t+\tfrac{1}{r}]^{-1}\right)\left(\frac{1}{\bar t+\frac{1}{r}}\right)^2.$$

Hence

$$\mathfrak{L}^{-1}\left\{\frac{e^{\frac{\sqrt{p}}{r}}}{\sqrt{p}}\int_0^r e^{\frac{-\sqrt{p}}{r_3}}h_e(r_3)dr_3\right\} = \frac{1}{\sqrt{\pi\bar t}}\int_0^\infty e^{-u^2/(4\bar t)}h_e\left([u+\frac{1}{r}]^{-1}\right)\left(\frac{1}{u+\frac{1}{r}}\right)^2du$$

so that

$$B_e(r,\bar t) = \frac{1}{\sqrt{2\pi\bar t}}\int_0^\infty e^{-u^2/(2\bar t)}h_e\left([u+\frac{1}{r}]^{-1}\right)\left(\frac{1}{u+\frac{1}{r}}\right)^2du.$$

Similarly it can be shown that

$$B_o(r,\bar t) = \frac{1}{\sqrt{2\pi\bar t}}\int_0^\infty e^{-u^2/(2\bar t)}h_o\left([u+\frac{1}{r}]^{-1}\right)\left(\frac{1}{u+\frac{1}{r}}\right)^2du.$$

Hence we have from (2.29) that

$$\begin{aligned} B_p(r,\bar t) = \quad & re^{-\frac{\mu T}{2}}\left\{\frac{1}{\sqrt{2\pi\bar t}}\int_0^\infty e^{-u^2/(2\bar t)}h\left([u+\frac{1}{r}]^{-1}\right)\left(\frac{1}{u+\frac{1}{r}}\right)^2du\right. \\ + & \frac{1}{\sqrt{2\pi\bar t}}\int_0^\infty e^{-u^2/(2\bar t)}h\left([u-\frac{1}{r}]^{-1}\right)\left(\frac{1}{u-\frac{1}{r}}\right)^2du \\ - & \left.\frac{\sqrt{2}}{\sqrt{\pi\bar t}}\int_0^\infty e^{-u^2/(2\bar t)}h_o\left([u-\frac{1}{r}]^{-1}\right)\left(\frac{1}{u-\frac{1}{r}}\right)^2du\right\}. \end{aligned} \tag{2.30}$$

So a solution to (2.23), with $B(r,0)$ as in (2.22) is

$$B(r,\bar{t}) \;=\; B_c(r,\bar{t}) + B_p(r,\bar{t}), \tag{2.31}$$

where $B_c = \mathfrak{L}^{-1}\{\tilde{B}_c\}$ which can be found from (2.24) and Table 2.1, (an example is given in (2.25)) and B_p is given in (2.30). Hence a solution of the BPE ((1.1)), with the final condition $V(r,T)=1$, is

$$V(r,t) = \frac{r}{c(t)B(r,t)} z\left(-\frac{1}{r}, \frac{t}{2}\right),$$

where

$$z(r_1,r_2) = c(2r_2)e^{-\mu r_2}e^{\sqrt{\mu}r_1}r_1\left[c_1\Phi\left(1+\frac{1}{\sqrt{\mu}};2;-2\sqrt{\mu}r_1\right) + c_2\Psi\left(1+\frac{1}{\sqrt{\mu}};2;-2\sqrt{\mu}r_1\right)\right],$$

$r_1 = -\frac{1}{r}$, $r_2 = \frac{t}{2}$ and where $B(r,\bar{t})$ is given by (2.31) noting that $\bar{t} = T - t$.

The stochastic differential equation for the spot risk-neutral rate in this example takes the form

$$dr = r^4 b(r,t)dt + r^2 dX,$$

where $b(r,t) = B^{-1}\partial B/\partial r$.

We note that with the various solutions for B, we can obtain many possible forms for the stochastic differential equation for r. Figure 2.1 shows our solution of the BPE for various parameters, the corresponding yield curve and a simulation of the risk-neutral interest rate over the life of the bond.

Example-2: $\mathbf{w(r) = 2r^{3/2}}$: In this case from equation (2.7) we have

$$\begin{aligned} Q(r_1,r_2) &= F(r_1) + G(r_2), \\ &= -2r + \frac{ww''}{2} - \frac{(w')^2}{4} - 2\frac{c'(t)}{c(t)}, \\ &= -\frac{11}{4}r - 2\frac{c'(t)}{c(t)}, \end{aligned} \tag{2.32}$$

and from (2.11), (2.16) and (2.17), $r_1 = \frac{-1}{\sqrt{r}}$, $T(r_2) = \beta c(2r_2)e^{-\mu r_2}$ and $X(r_1)$ satisfies

$$X'' + \left[-\frac{11}{4}\frac{1}{r_1^2} - \mu\right]X = 0, \tag{2.33}$$

where μ is the separation constant. The general solution of (2.33) (see e.g [13]) is

$$X(r_1) = r_1^{1/2}\left[c_1 J_{\sqrt{3}}(\sqrt{-\mu}r_1) + c_2 J_{-\sqrt{3}}(\sqrt{-\mu}r_1)\right], \mu < 0$$

where J_ν is the Bessel function of the first kind (see e.g [14]) and c_1, c_2 are arbitrary constants. Hence a solution to equation (2.4) is

$$\begin{aligned} z(r_1,r_2) &= X(r_1)T(r_2) \\ &= \beta c(2r_2)e^{-\mu r_2}r_1^{1/2}\left[c_1 J_{\sqrt{3}}(\sqrt{-\mu}r_1) + c_2 J_{-\sqrt{3}}(\sqrt{-\mu}r_1)\right]. \end{aligned} \tag{2.34}$$

From (2.18) and (2.19) we now need to solve

$$\frac{\partial^2 B}{\partial r^2} - \frac{1}{2r^3}\frac{\partial B}{\partial \bar{t}} = 0, \tag{2.35}$$

subject to

$$B(r,0) = r^{1/2} e^{-\mu T/2}\left[c_1 J_{\sqrt{3}}\left(-\sqrt{-\frac{\mu}{r}}\right) + c_2 J_{-\sqrt{3}}\left(-\sqrt{-\frac{\mu}{r}}\right)\right], \quad \text{for} \quad \mu < 0. \tag{2.36}$$

Taking Laplace Transforms of (2.35) with respect to $\bar{t}$, so that $\mathcal{L}\{B(r,\bar{t})\} = \tilde{B}(r,p)$ we get

$$r^3\frac{d^2\tilde{B}}{dr^2} - \frac{p}{2}\tilde{B} = -\frac{B(r,0)}{2}. \tag{2.37}$$

The solution to the corresponding homogeneous equation of (2.37) is

$$\tilde{B}_c(r,p) = r^{1/2}\left[A(p)J_1\left(-i\sqrt{\frac{2p}{r}}\right) + D(p)Y_1\left(-i\sqrt{\frac{2p}{r}}\right)\right],$$

where $A(p)$ and $D(p)$ are arbitrary functions of the transform variable p and J_ν and Y_ν are Bessel functions of the first and second kind respectively. As an example if we let $A(p) = c_3 K_1(a\sqrt{p})$ and $D(p) = c_4 K_1(a\sqrt{p})$, $a = -i$, where K_1 is the modified Bessel function of order 1, then

$$\tilde{B}_c(r,p) = r^{1/2}\left[c_3 K_1(a\sqrt{p})J_1\left(-\sqrt{\frac{2p}{r}}i\right) + c_4 K_1(a\sqrt{p})Y_1\left(-\sqrt{\frac{2p}{r}}i\right)\right],$$

and hence (see [15]),

$$B_c(r,\bar{t}) = \frac{r^{1/2}}{2\bar{t}} e^{-\frac{[-1+2/r]}{4\bar{t}}}\left[c_3 J_1\left(-\frac{1}{2\bar{t}}\sqrt{\frac{2}{r}}\right) + c_4 Y_1\left(-\frac{1}{2\bar{t}}\sqrt{\frac{2}{r}}\right)\right].$$

To find $\tilde{B}_p$, a particular solution to (2.37) we use variation of parameters with the two independent solutions of the corresponding homogeneous equation.

Hence

$$\begin{aligned}\tilde{B}_p \quad = \quad & \alpha r^{1/2} K_1(-i\sqrt{p})\left\{ Y_1\left(-\sqrt{\frac{2p}{r}}i\right)\int \frac{K_1(-i\sqrt{p})J_1\left(-\sqrt{\frac{2p}{r}}i\right)h(r)dr}{W}\right. \\ & \left. - J_1\left(-\sqrt{\frac{2p}{r}}i\right)\int \frac{K_1(-i\sqrt{p})Y_1\left(-\sqrt{\frac{2p}{r}}i\right)h(r)dr}{W}\right\}\end{aligned} \tag{2.38}$$

where

$$W = r^{1/2}K_1(-i\sqrt{p})\left\{J_1\left(-\sqrt{\frac{2p}{r}}i\right)\frac{\partial}{\partial r}\left[r^{1/2}K_1(-i\sqrt{p})Y_1\left(-\sqrt{\frac{2p}{r}}i\right)\right]\right.$$

$$\left.-Y_1\left(-\sqrt{\frac{2p}{r}}i\right)\frac{\partial}{\partial r}\left[r^{1/2}K_1(-i\sqrt{p})J_1\left(-\sqrt{\frac{2p}{r}}i\right)\right]\right\}, \text{ and}$$

$$h(r) = -\frac{1}{2r^{5/2}}e^{-\mu T/2}\left[c_1 J_{\sqrt{3}}\left(-\sqrt{-\frac{\mu}{r}}\right)+c_2 J_{-\sqrt{3}}\left(-\sqrt{-\frac{\mu}{r}}\right)\right].$$

Although it is not possible to find an exact Inverse Laplace Transform of (2.38), it may be done numerically (see e.g [16]). In any case we have reduced the problem to quadratures. From (2.14) and (2.34) a solution of the BPE (1.1) (with the final condition $V(r,T)=1$) is

$$V(r,t) = \frac{r^{3/4}}{c(t)B(r,t)}z(r_1,r_2),$$

where

$$r_1 = \frac{-1}{r^{1/2}},\ r_2 = \frac{t}{2},\ z(r_1,r_2) = \beta c(2r_2)e^{-\mu r_2}r_1^{1/2}\left[c_1 J_{\sqrt{3}}(\sqrt{-\mu}r_1)+c_2 J_{-\sqrt{3}}(\sqrt{-\mu}r_1)\right],$$

$$B(r,\bar{t}) = \frac{r^{1/2}}{2\bar{t}}e^{-\frac{[-1+2/r]}{4\bar{t}}}\left[c_3 J_1\left(-\frac{1}{2\bar{t}}\sqrt{\frac{2}{r}}\right)+c_4 Y_1\left(-\frac{1}{2\bar{t}}\sqrt{\frac{2}{r}}\right)\right]+\mathfrak{L}^{-1}\{\tilde{B}_p\},$$

where $\tilde{B}_p$ is given in (2.38) and $\bar{t}=T-t$. We note that the stochastic differential equation for the risk-neutral spot rate (when $w(r)=2r^{3/2}$) for this example takes the form

$$dr = 4r^3 b(r,t)dt + 2r^{3/2}dX,$$

where $b(r,t) = B^{-1}\partial B/\partial r$.

2.2.3 Solution by Laplace Transforms. In this section we solve (2.4) by Laplace Transforms. We first make the substitution $\bar{r}_2 = T/2 - r_2$, so that (2.4) becomes

$$\frac{\partial^2 z}{\partial r_1^2} - \frac{\partial z}{\partial \bar{r}_2} + Qz = 0, \tag{2.39}$$

which we solve subject to

$$z(r_1,0) = (w(r))^{-1/2}B(r,T),$$

where $r_1 = \int \frac{1}{w(r)}dr$, $c(t)=1$, and $B(r,t)$ satisfies (2.13).

Example-3: $\mathbf{w(r) = r^2}$: Equation (2.39) with Q as in (2.7) becomes

$$\frac{\partial^2 z}{\partial r_1^2} - \frac{\partial z}{\partial \bar{r}_2} + \frac{2}{r_1} z = 0.$$

Taking Laplace Transforms with respect to $\bar{r}_2$, we get

$$+\frac{d^2 \bar{z}}{dr_1^2} + \bar{z}\left[\frac{2}{r_1} - p\right] = -z(r_1, 0) \tag{2.40}$$

where $\tilde{z}(r_1, p) = \mathcal{L}\{z(r_1, \bar{r}_2)\}$ and $z(r_1, 0) = -r_1 B\left(-\frac{1}{r_1}, T\right)$ with $B(r,t)$ any solution of

$$\frac{\partial^2 B}{\partial r^2} + \frac{2}{r^4}\frac{\partial B}{\partial t} = 0. \tag{2.41}$$

We list some solutions of (2.41) in Table 2.2 (obtained by Laplace Transforms, separation of variables and the classical Lie Symmetry Method (see [11])). Solving the corresponding homogeneous equation of (2.40) we find

$$\bar{z}_c(r_1, p) = A(p)\mathrm{M}\left(\frac{1}{\sqrt{p}}, \frac{1}{2}, 2\sqrt{p}r_1\right) + D(p)\mathrm{W}\left(\frac{1}{\sqrt{p}}, \frac{1}{2}, 2\sqrt{p}r_1\right), \tag{2.42}$$

where $M(\cdot)$ and $W(\cdot)$ are Whittaker functions (see [14] for definition). To find $\bar{z}_p$, the particular solution to (2.40), we can use variation of parameters using the two independent solutions of the corresponding homogeneous equation to (2.40).

Hence

$$z = \mathcal{L}^{-1}\left\{\tilde{z}_c(r_1, p) + \tilde{z}_p(r_1, p)\right\}. \tag{2.43}$$

Hence from (2.13) a solution of the BPE (1.1) is

$$V(r,t) = \frac{r\ z(\frac{-1}{r}, \frac{T-t}{2})}{B(r,t)},$$

where $B(r,t)$ is any solution of (2.41) as outlined in Table 2.2, and z is given in (2.43). In this example the stochastic differential equation for the risk-neutral spot rate takes the form

$$dr = r^4 b(r,t)dt + r^2 dX,$$

where $b(r,t) = B^{-1}\partial B/\partial r$.

Example-4: $\mathbf{w = 2r^{3/2}}$: Equation (2.39) now becomes

$$\frac{\partial^2 z}{\partial r_1^2} - \frac{\partial z}{\partial \bar{r}_2} - \frac{11}{4}\frac{1}{r_1^2} z = 0, \tag{2.44}$$

which we solve subject to

$$z(r_1, 0) = (w(r))^{-1/2} B(r, T),$$

where $B(r,t)$ is any solution of

$$\frac{\partial^2 B}{\partial r^2} + \frac{1}{2r^3}\frac{\partial B}{\partial t} = 0, \tag{2.45}$$

such as those in Table 2.3. Taking the Laplace Transform of (2.44) with respect to $\bar{r}_2$, we get

$$\frac{d^2\bar{z}}{dr_1^2} - \left(p + \frac{11}{4}\frac{1}{r_1^2}\right)\bar{z}(r_1,p) = -z(r_1,0). \tag{2.46}$$

Solving the corresponding homogeneous equation of (2.46), we get

$$\bar{z}_c(r_1,p) = r_1^{\frac{1}{2}}\left(A(p)J_{\sqrt{3}}(\sqrt{p}r_1 i) + D(p)Y_{\sqrt{3}}(\sqrt{p}r_1 i)\right) \tag{2.47}$$

where $A(p)$ and $D(p)$ are arbitrary. To find $\bar{z}_p$, a particular solution to (2.46) we can use variation of parameters using the two independent solutions of the corresponding homogeneous equation to (2.46).

Then

$$z = \mathcal{L}^{-1}\left\{\tilde{z}_c(r_1,p) + \tilde{z}_p(r_1,p)\right\}. \tag{2.48}$$

From (2.14), a solution of the BPE ((1.1)) is

$$V(r,t) = \frac{r^{3/4}z(r_1,r_2)}{B(r,t)}$$

where $B(r,t)$ is any solution of (2.45) such as those outlined in Table 2.3 and $z(r_1, T/2 - r_2)$ with $r_1 = \dfrac{-1}{r^{1/2}}$ and $r_2 = \dfrac{t}{2}$ is given in (2.48). In this example the stochastic differential equation for the risk-neutral spot rate takes the form

$$dr = 4r^3 b(r,t)dt + 2r^{3/2}dX,$$

where $b(r,t) = B^{-1}\partial B/\partial r$.

2.2.4 Reducing to the Heat Equation

In this section we choose ζ in equation (2.7) and (2.8) to be a function of r so that $Q(r_1,r_2) = 0$, i.e.

$$\zeta(r) = -2r + \frac{ww''}{2} - \frac{(w')^2}{4}. \tag{2.49}$$

With the substitution $\bar{r}_2 = T/2 - r_2$ equation (2.4) becomes the classical heat equation

$$\frac{\partial^2 z}{\partial r_1^2} - \frac{\partial z}{\partial \bar{r}_2} = 0, \tag{2.50}$$

for which many solutions are known and tabulated (see e.g [17]). We solve (2.50) by the method of Laplace Transforms which will incorporate the final condition

$$z(r_1,0) = (w(r))^{-1/2}B(r,T). \tag{2.51}$$

Without loss of generality we have taken $c(t)$ to be 1. With $V(r,t)$ as in (2.14) and $b = B_r/B$ satisfying (2.8), by a similar calculation to that in Section 2.2.1, we find that B now needs to satisfy

$$\frac{w^2}{2}B_{rr} + B_t = \frac{\zeta(r)}{2}B. \tag{2.52}$$

Taking Laplace Transforms of (2.50) with respect to $\bar{r}_2$ we get

$$\frac{\partial^2 \tilde{z}}{\partial r_1^2} - p\tilde{z} \;=\; -z(r_1, 0). \tag{2.53}$$

for which the general solution to the corresponding homogeneous equation for $p > 0$ is

$$\tilde{z}_c \;=\; A(p)e^{\sqrt{p}r_1} + D(p)e^{-\sqrt{p}r_1} \tag{2.54}$$

There are many possible choices for the arbitrary functions $A(p), D(p)$ of the transform variable many of which are listed in Table 2.1 (with $a = r_1$). As an example if $A(p) = 0$ and $D(p) = c_2 + c_1 p^{-1/2}$ we find after inverting (2.54)

$$z_c(r_1, \bar{r_2}) = \frac{c_1}{(\pi\bar{r_2})^{1/2}}\exp\left(\frac{-r_1^2}{4\bar{r_2}}\right) - \frac{c_2 r_1}{\sqrt{\pi\bar{r_2}^3}}\exp\left(\frac{-r_1^2}{4\bar{r_2}}\right). \tag{2.55}$$

To find $\tilde{z}_p$, the particular solution to (2.53), we use variation of parameters with the independent solutions of $\tilde{z}_c$,

$$y_1 = A(p)e^{\sqrt{p}r_1} \quad \text{and} \qquad y_2 = D(p)e^{-\sqrt{p}r_1}.$$

Example-5: $\mathbf{w(r) = r^2}$: Using variation of parameters with $r_1 = -\frac{1}{r}$ we get

$$\tilde{z}_p = \frac{e^{\frac{\sqrt{p}}{r}}}{\sqrt{p}}\int_0^r e^{-\frac{\sqrt{p}}{r_3}}H(r_3)dr_3 - \frac{e^{-\frac{\sqrt{p}}{r}}}{\sqrt{p}}\int_0^r e^{\frac{\sqrt{p}}{r_3}}H(r_3)dr_3 \tag{2.56}$$

where $H(r) = \frac{z(r_1,0)}{2r^2}$. To find $z_p = \mathfrak{L}^{-1}(\tilde{z}_p)$ we let

$$\begin{aligned} z_e(r_1, \bar{r_2}) \;&=\; \mathfrak{L}^{-1}\left\{\frac{e^{\frac{\sqrt{p}}{r}}}{\sqrt{p}}\int_0^r e^{-\frac{\sqrt{p}}{r_3}}H_e(r_3)dr_3\right\} \text{ and} \\ z_o(r_1, \bar{r_2}) \;&=\; \mathfrak{L}^{-1}\left\{\frac{e^{\frac{\sqrt{p}}{r}}}{\sqrt{p}}\int_0^r e^{-\frac{\sqrt{p}}{r_3}}H_o(r_3)dr_3\right\}, \end{aligned}$$

where

$$\begin{aligned} H_e(r) \;&=\; \frac{H(r) + H(-r)}{2}, \text{ the even part of H and} \\ H_o(r) \;&=\; \frac{H(r) - H(-r)}{2}, \text{ the odd part of H} \end{aligned}$$

then

$$z_p(r_1, \bar{r_2}) = z_e(r_1, \bar{r_2}) + z_o(r_1, \bar{r_2}) + z_e(-r_1, \bar{r_2}) - z_o(-r_1, \bar{r_2}).$$

We note firstly that

$$
\begin{aligned}
\mathfrak{L}^{-1}\left\{e^{\frac{p}{r}}\int_0^r e^{-\frac{p}{r_3}}H_e(r_3)dr_3\right\} &= \int_0^r \mathfrak{L}^{-1}\left(e^{-p\left(\frac{1}{r_3}-\frac{1}{r}\right)}H_e(r_3)\right)dr_3 \\
&= \int_0^r \delta\left(\bar{r_2}-\left(\frac{1}{r_3}-\frac{1}{r}\right)\right)H_e(r_3)dr_3.
\end{aligned}
$$

With the substitutions $r_4 = -\frac{1}{r_3}$ and $r_1 = -\frac{1}{r}$, we get

$$
\int_{-\infty}^{r_1} \delta(\bar{r_2}+r_4-r_1)H_e\left(-\frac{1}{r_4}\right)\frac{1}{r_4^2}dr_4,
$$

which evaluates to

$$
H_e\left([\bar{r_2}-r_1]^{-1}\right)\left(\frac{1}{\bar{r_2}-r_1}\right)^2.
$$

Hence,

$$
\begin{aligned}
z_e(r_1,\bar{r_2}) &= \mathfrak{L}^{-1}\left\{\frac{e^{\frac{\sqrt{p}}{r}}}{\sqrt{p}}\int_0^r e^{-\frac{\sqrt{p}}{r_3}}H_e(r_3)dr_3\right\} \\
&= \frac{1}{\sqrt{\pi\bar{r_2}}}\int_0^\infty e^{-\frac{u^2}{4\bar{r_2}}}\mathfrak{L}^{-1}\left[e^{\frac{p}{r}}\int_0^r e^{-\frac{p}{r_3}}H_e(r_3)dr_3\right]du \\
&= \frac{1}{\sqrt{\pi\bar{r_2}}}\int_0^\infty e^{-\frac{u^2}{4\bar{r_2}}}H_e\left([u-r_1]^{-1}\right)\left(\frac{1}{u-r_1}\right)^2 du.
\end{aligned}
$$

Similarly it can be shown that

$$
z_o(r_1,\bar{r_2}) = \frac{1}{\sqrt{\pi\bar{r_2}}}\int_0^\infty e^{-\frac{u^2}{4\bar{r_2}}}H_o\left([u-r_1]^{-1}\right)\left(\frac{1}{u-r_1}\right)^2 du.
$$

Hence

$$
\begin{aligned}
z_p(r_1,\bar{r_2}) &= \frac{1}{\sqrt{\pi\bar{r_2}}}\int_0^\infty e^{-\frac{u^2}{4\bar{r_2}}}H\left([u-r_1]^{-1}\right)\left(\frac{1}{u-r_1}\right)^2 du \\
&+ \frac{1}{\sqrt{\pi\bar{r_2}}}\int_0^\infty e^{-\frac{u^2}{4\bar{r_2}}}H\left([u+r_1]^{-1}\right)\left(\frac{1}{u+r_1}\right)^2 du \\
&- \frac{2}{\sqrt{\pi\bar{r_2}}}\int_0^\infty e^{-\frac{u^2}{4\bar{r_2}}}H_o\left([u+r_1]^{-1}\right)\left(\frac{1}{u+r_1}\right)^2 du.
\end{aligned}
\tag{2.57}
$$

Then

$$
z(r_1,\bar{r_2}) = z_c(r_1,\bar{r_2}) + z_p(r_1,\bar{r_2}), \tag{2.58}
$$

where an example of z_c is given in (2.55) and z_p is given in (2.57). We now need to look for suitable functions $B(r,t)$. From (2.49) and (2.52), $B(r,t)$ needs to satisfy

$$
\frac{r^4}{2}\frac{\partial^2 B}{\partial r^2} + \frac{\partial B}{\partial t} = -rB. \tag{2.59}
$$

By separation of variables we find one possible solution as

$$
\begin{aligned}
B &= X(r)T(t) \\
&= re^{-\mu t}e^{-\frac{\sqrt{2\mu}}{r}}\left[c_3\Phi\left(-\tfrac{1}{\sqrt{2\mu}},0,\tfrac{2\sqrt{2\mu}}{r}\right)+c_4\Psi\left(-\tfrac{1}{\sqrt{2\mu}},0,\tfrac{2\sqrt{2\mu}}{r}\right)\right],
\end{aligned}
\tag{2.60}
$$

where Φ and Ψ are the Kummer-M and Kummer-U functions respectively (see e.g [14]). Hence a solution to the BPE which satisfies the final condition is given as

$$V(r,t) \;=\; \frac{r\ z(-\frac{1}{r},\frac{T-t}{2})}{B(r,t)},$$

where $B(r,t)$ is given by (2.60) and z is given by (2.58). The stochastic differential equation for the risk-neutral rate takes the form

$$dr = r^4 b(r,t)dt + r^2 dX,$$

where $b(r,t) = B^{-1}\partial B/\partial r$.

Example-6: $\mathbf{w(r) = 2r^{3/2}}$: In order to find z_p for this case we perform a similar calculation to that in the previous example and find that

$$\begin{aligned} z_p(r_1,\bar{r_2}) = \quad & \frac{1}{\sqrt{\pi \bar{r_2}}}\int_0^\infty e^{-\frac{u^2}{4\bar{r_2}}} H(r_1-u)du \\ + \quad & \frac{1}{\sqrt{\pi \bar{r_2}}}\int_0^\infty e^{-\frac{u^2}{4\bar{r_2}}} H(-r_1-u)du \\ - \quad & \frac{2}{\sqrt{\pi \bar{r_2}}}\int_0^\infty e^{-\frac{u^2}{4\bar{r_2}}} H_o(-r_1-u)du. \end{aligned} \tag{2.61}$$

Notation is as in the previous example with $H(r_1) = \frac{z(r_1,0)}{2}$. Then

$$z(r_1,\bar{r_2}) = z_c(r_1,\bar{r_2}) + z_p(r_1,\bar{r_2}) \tag{2.62}$$

where z_c is given in (2.55) and z_p is given in (2.61). To find $B(r,t)$, from (2.49) and (2.52) we find that it needs to satisfy

$$\frac{w^2}{2}\frac{\partial^2 B}{\partial r^2} + \frac{\partial B}{\partial t} \;=\; -\frac{11}{8}rB. \tag{2.63}$$

Some possible solutions to (2.63) are listed in Table 2.4.

Hence a solution to the BPE (1.1) is

$$V(r,t) \;=\; \frac{\sqrt{w}\ z(\frac{-1}{\sqrt{r}},\frac{T-t}{2})}{B(r,t)} \tag{2.64}$$

where the solution for $z(r_1,r_2)$ is outlined in (2.62),with $z(r_1,0)$ satisfying (2.51), and where $B(r,t)$ satisfies (2.63) such as those solutions in Table 2.4. The stochastic differential equation for the risk-neutral spot rate takes the form

$$dr = 4r^3 b(r,t)dt + 2r^{3/2}dX,$$

where $b(r,t) = B^{-1}\partial B/\partial r$.

2.3 The Drift Function is Independent of Time

With w and b functions of r alone, from equation (2.6),

$$Q(r_1) = -2r - b'w^2 + \frac{ww''}{2} - w^2b^2 - \frac{(w')^2}{4}, \tag{2.65}$$

and so using $b = \frac{u}{w^2}$, where we take $\lambda = 0$, we have

$$Q(r_1) = -2r - u' + \frac{2uw'}{w} + \frac{ww''}{2} - \frac{u^2}{w^2} - \frac{(w')^2}{4}. \tag{2.66}$$

Letting $\bar{r}_2 = \frac{T}{2} - r_2$, we could then try and solve

$$\frac{\partial^2 z}{\partial r_1^2} - \frac{\partial z}{\partial \bar{r}_2} + Q(r_1)z = 0, \tag{2.67}$$

for particular u and w function, subject to

$$\begin{aligned} z(r_1, 0) &= \frac{1}{\sqrt{w(r)}} \exp\left(\int b(r)dr\right) \\ &= \frac{1}{\sqrt{w(r)}} \exp\left(\int \frac{u}{w^2}dr\right), \end{aligned}$$

with $r_1 = \int \frac{1}{w(r)}dr$. We note that if for particular functions u and w of r we could write (2.66) as

$$Q(r_1) = \alpha_0 + \alpha_1 r_1 + \alpha_2 r_1^2; \qquad \alpha_0, \alpha_1, \alpha_2 \ \text{ constants},$$

then we could transform (2.67) to a constant coefficient linear PDE (see e.g [11]) before proceeding to solve it. As well, functions u and w for which $Q(r_1) = 0$ satisfy the Airy equation

$$u' = \left(\frac{2w'}{w}\right)u - \left(\frac{1}{w^2}\right)u^2 + \left(\frac{ww''}{2} - \frac{(w')^2}{4} - 2r\right). \tag{2.68}$$

Letting $v = \exp\left(\int \frac{u}{w^2}dr\right)$, equation (2.68) becomes

$$v'' - \frac{1}{w^2}\left(\frac{ww''}{2} - \frac{(w')^2}{4} - 2r\right)v = 0, \tag{2.69}$$

and the initial condition for z becomes

$$z(r_1, 0) = \frac{1}{\sqrt{w(r)}}v(r). \tag{2.70}$$

Example-7: $\mathbf{w(r) = cr^{3/2}}$: From (2.69) we get that $Q(r_1) = 0$ when

$$r^2v'' + \left(\frac{3}{16} + \frac{2}{c^2}\right)v = 0,$$

so that for

$\mathbf{c^2 > 32}$

$$v = Ar^{m_1} + Br^{m_2}, \qquad m_{1,2} = \frac{2c \pm \sqrt{c^2 - 32}}{4c}, \tag{2.71}$$

and so

$$u = \frac{c^2 r^3 [A m_1 r^{m_1 - 1} + B m_2 r^{m_2 - 1}]}{A r^{m_1} + B r^{m_2}}, \tag{2.72}$$

$\mathbf{c^2 < 32}$

$$v = r^{1/2}[A \cos(\alpha \ln r) + B \sin(\alpha \ln r)], \qquad \alpha = \frac{\sqrt{32 - c^2}}{4c}, \tag{2.73}$$

and so

$$u = \frac{c^2 r^2}{2}\left[1 + \frac{\sqrt{32 - c^2}}{2c}\left\{\frac{-A \sin(\alpha \ln r) + B \cos(\alpha \ln r)}{A \cos(\alpha \ln r) + B \sin(\alpha \ln r)}\right\}\right], \tag{2.74}$$

$\mathbf{c^2 = 32}$

$$v = r^{1/2}[A + B \ln r] \tag{2.75}$$

and so

$$u = \frac{16 r^{3/2}\left[(A + 2B) + B r^{1/2} \ln r\right]}{A + B \ln r}. \tag{2.76}$$

Taking Laplace transforms of (2.67), with respect to $\bar{r}_2$, with $Q(r_1) = 0$, requires then that we solve

$$\frac{d^2 \tilde{z}}{dr_1^2} - p\tilde{z} = -z(r_1, 0),$$

where $\tilde{z}(r_1, p) = \mathcal{L}\{z(r_1, \bar{r}_2)\}$ and from (2.70)

$$z(r_1, 0) = \frac{1}{c^{\frac{1}{2}} r^{\frac{3}{4}}} v(r), \tag{2.77}$$

with $r_1 = \frac{-2}{c\sqrt{r}}$. As before we write $\tilde{z} = \tilde{z}_c + \tilde{z}_p$ and find

$$\tilde{z}_c = A(p) e^{\sqrt{p} r_1} + B(p) e^{-\sqrt{p} r_1}, \tag{2.78}$$

where $A(p)$ and $B(p)$ are arbitrary functions of p. Many choices of these functions are possible for which Laplace inverse functions are known (see Table 2.1 with $a = r_1$).

Using variation of parameters

$$\tilde{z}_p = -\frac{e^{\sqrt{p} r_1}}{\sqrt{p}} \int_0^{r_1} e^{-\sqrt{p} r_3} H(r_3) dr_3 + \frac{e^{-\sqrt{p} r_1}}{\sqrt{p}} \int_0^{r_1} e^{\sqrt{p} r_3} H(r_3) dr_3 \tag{2.79}$$

where $H(r_1) = \frac{z(r_1, 0)}{2}$ with the initial condition for z given by (2.77) and with $v(r)$ as in (2.71), (2.73) or (2.75). Performing similar calculations as in previous examples, we find

$$\begin{aligned} z_p(r_1, \bar{r}_2) = \quad & \frac{1}{\sqrt{\pi \bar{r}_2}} \int_0^\infty e^{-\frac{u^2}{4\bar{r}_2}} H(r_1 - u) du \\ + \quad & \frac{1}{\sqrt{\pi \bar{r}_2}} \int_0^\infty e^{-\frac{u^2}{4\bar{r}_2}} H(-r_1 - u) du \\ - \quad & \frac{2}{\sqrt{\pi \bar{r}_2}} \int_0^\infty e^{-\frac{u^2}{4\bar{r}_2}} H_o(-r_1 - u) du, \end{aligned} \tag{2.80}$$

Finally, the solution to the BPE (1.1) can be found by

$$V(r,t) = \frac{c^{1/2}r^{3/4}}{v(r)} z\left(\frac{-2}{c\sqrt{r}}, \frac{T-t}{2}\right).$$

Figures 2.2 and 2.3 show our solution of the BPE for various parameters (Figure 2.2 in the case $c^2 > 32$, Figure 2.3 in the case $c^2 < 32$), the corresponding yield curves and simulations of the risk-neutral interest rate over the life of the bond.

3. CONCLUSION

There exist at the moment a small number of analytically tractable models for the Bond Pricing Equation (BPE). From solutions to this equation we are then able to build yield curves, giving investment return as a function of waiting time to expiry. In this paper we have shown how reducing the BPE to its standard canonical form, in which only one model-dependent adjustable coefficient function is left, and then finding special cases for this coefficient function that allows either a full general solution of the BPE or the reduction to a single inverse Laplace transform, we are able to expand the class of analytically solvable models. In some of our cases, the coefficient function was identically zero, leaving nothing more to solve than the classical heat equation. Unlike many of the current models, most of our solutions to the BPE use short-rate models which incorporate time in the drift function and all our solutions incorporate a realistic $r^{\frac{3}{2}}$ or r^2 dependence on interest rate volatility. The new solutions that we present, naturally satisfy the required final condition, namely that the solutions should uniformly reach the expiry value 1, independent of r. Some solutions to the BPE and their corresponding yield curves have been plotted, along with simulations of the associated short-rate.

While some interest rate data may be fitted to one particular interest rate model, other interest rate data may not. As such, there is always a need for new solvable models to the BPE. It is hoped that the new solvable models, introduced here, may allow a better match to the term structure of some real interest rate data.

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possible choices for $\tilde{B}_{c_1}, \tilde{B}_{c_2}$	Inverse Laplace Transforms
$c_1 e^{-ap^{\frac{1}{2}}}$	$\frac{1}{2}c_1 a\pi^{-\frac{1}{2}}\bar{t}^{-\frac{3}{2}}\exp(-a^2/(4\bar{t}))$
$c_1 p^{\frac{1}{2}} e^{-ap^{\frac{1}{2}}}$	$\frac{1}{4}c_1\pi^{-\frac{1}{2}}(a^2-2\bar{t})\bar{t}^{-\frac{5}{2}}\exp(-a^2/(4\bar{t}))$
$c_1 p^{-\frac{1}{2}} e^{-ap^{\frac{1}{2}}}$	$c_1(\pi\bar{t})^{-\frac{1}{2}}\exp(-a^2/(4\bar{t}))$
$c_1 p e^{-ap^{\frac{1}{2}}}$	$\frac{1}{4}c_1 a\pi^{-\frac{1}{2}}\bar{t}^{-\frac{5}{2}}(a^2/(2\bar{t})-3)\exp(-a^2/(4\bar{t}))$
$c_1 p^{-1} e^{-ap^{\frac{1}{2}}}$	$c_1\mathrm{Erfc}(\frac{1}{2}a\bar{t}^{-\frac{1}{2}})$
$c_1 p^{\frac{3}{2}} e^{-ap^{\frac{1}{2}}}$	$\frac{1}{4}c_1\pi^{-\frac{1}{2}}\bar{t}^{-\frac{5}{2}}(3-3a^2/(2\bar{t})+a^4/(4\bar{t}^2))$
$c_1 p^{-\frac{3}{2}} e^{-ap^{\frac{1}{2}}}$	$2c_1(\bar{t}/\pi)^{\frac{1}{2}}\exp(-a^2/(4\bar{t}))-a\mathrm{Erfc}(\frac{1}{2}a\bar{t}^{-\frac{1}{2}})$
$c_1 p^{\frac{1}{2}n-\frac{1}{2}} e^{-(ap)^{\frac{1}{2}}}$	$c_1(2\bar{t})^{-\frac{1}{2}n}(\pi\bar{t})^{-\frac{1}{2}}\exp(-\frac{1}{4}a/\bar{t})\mathrm{He}_n[(2\bar{t}/a)^{-\frac{1}{2}}]$ where $\mathrm{He}_n(r)$ is the Hermite's polynomial of order n defined as $\mathrm{He}_n(r)=(-1)^n e^{\frac{1}{2}r^2}\frac{d^n}{dr^n}e^{-\frac{1}{2}r^2}$
$c_1 p^{\nu} e^{-ap^{\frac{1}{2}}}$	$2^{-\nu-\frac{1}{2}}c_1\pi^{-\frac{1}{2}}\bar{t}^{-\nu-1}\exp(-a^2/(8\bar{t}))\mathrm{D}_{2\nu+1}[a(2\bar{t})^{-\frac{1}{2}}]$ where $\mathrm{D}_\nu(r)$ is the parabolic cylindrical function defined as $\mathrm{D}_\nu(r)=e^{-\frac{1}{4}r^2}\mathrm{He}_\nu(r),\ \nu=0,1,2,....$
$c_1 e^{-ap^{\frac{1}{2}}}(p^{\frac{1}{2}}+b)^{-1}$	$c_1(\pi\bar{t})^{-\frac{1}{2}}\exp(-a^2/(4\bar{t}))-c_1 b\ \exp(ab+b^2\bar{t})\mathrm{Erfc}(\frac{1}{2}a\bar{t}^{-\frac{1}{2}}+b\bar{t}^{\frac{1}{2}})$
$c_1 p^{-\frac{1}{2}} e^{-ap^{\frac{1}{2}}}(p^{\frac{1}{2}}+b)^{-1}$	$c_1\exp(ab+b^2\bar{t})\mathrm{Erfc}(\frac{1}{2}a\bar{t}^{-\frac{1}{2}}+b\bar{t}^{\frac{1}{2}})$
$c_1 p^{-1}(p^{\frac{1}{2}}+b)^{-1}e^{-ap^{\frac{1}{2}}}$	$c_1 b^{-1}\mathrm{Erfc}(\frac{1}{2}a\bar{t}^{-\frac{1}{2}})-c_1 b^{-1}\exp(ab+b^2\bar{t})\mathrm{Erfc}(\frac{1}{2}a\bar{t}^{-\frac{1}{2}}+b\bar{t}^{\frac{1}{2}})$
$c_1 p^{\frac{1}{2}}(p^{\frac{1}{2}}+b)^{-1}e^{-ap^{\frac{1}{2}}}$	$c_1(\pi\bar{t})^{-\frac{1}{2}}(\frac{1}{2}a/\bar{t}-b)\exp(-\frac{1}{4}a^2/\bar{t})$ $+c_1 b^2\exp(ab+b^2\bar{t})\mathrm{Erfc}(\frac{1}{2}a\bar{t}^{-\frac{1}{2}}+b\bar{t}^{\frac{1}{2}})$
$c_1 p(p^{\frac{1}{2}}+b)^{-1}e^{-ap^{\frac{1}{2}}}$	$c_1\pi^{-\frac{1}{2}}\bar{t}^{-\frac{3}{2}}(b^2\bar{t}-\frac{1}{2}-\frac{1}{2}ab+\frac{1}{4}a^2/\bar{t})\exp(-\frac{1}{4}a^2/\bar{t})$ $-c_1 b^3\exp(ab+b^2\bar{t})\mathrm{Erfc}(\frac{1}{2}a\bar{t}^{-\frac{1}{2}}+b\bar{t}^{\frac{1}{2}})$
$c_1 p^{-\frac{3}{2}}(p^{\frac{1}{2}}+b)^{-1}e^{-ap^{\frac{1}{2}}}$	$c_1 b^{-1}[2(\bar{t}/\pi)^{\frac{1}{2}}\exp(-\frac{1}{4}a^2/\bar{t})-(a+b^{-1})\mathrm{Erfc}(\frac{1}{2}a\bar{t}^{-\frac{1}{2}})$ $+b^{-1}\exp(ab+b^2\bar{t})\mathrm{Erfc}(\frac{1}{2}a\bar{t}^{-\frac{1}{2}}+b\bar{t}^{\frac{1}{2}})]$
$c_1(p^{\frac{1}{2}}+b)^{-2}e^{-ap^{\frac{1}{2}}}$	$c_1(2b\bar{t}^2+ab+1)\exp(ab+b^2\bar{t})\mathrm{Erfc}(\frac{1}{2}a\bar{t}^{-\frac{1}{2}}+b\bar{t}^{\frac{1}{2}})$ $-2c_1 b(\bar{t}/\pi)^{\frac{1}{2}}\exp(-\frac{1}{4}a^2/\bar{t}))$
$c_1 p^{-\frac{1}{2}}(p^{\frac{1}{2}}+b)^{-2}e^{-ap^{\frac{1}{2}}}$	$2c_1(\bar{t}/\pi)^{\frac{1}{2}}\exp(-\frac{1}{4}a^2/\bar{t})$ $-c_1(2b\bar{t}+a)\exp(ab+b^2\bar{t})\mathrm{Erfc}(\frac{1}{2}a\bar{t}^{-\frac{1}{2}}+b\bar{t}^{\frac{1}{2}})$
$c_1 p^{-1}(p^{\frac{1}{2}}+b)^{-2}e^{-ap^{\frac{1}{2}}}$	$c_1 b^{-2}\mathrm{Erfc}(\frac{1}{2}a\bar{t}^{-\frac{1}{2}})-2c_1 b^{-1}(\bar{t}/\pi)\exp(-\frac{1}{4}a^2/\bar{t})$ $+\ c_1(2\bar{t}-b^{-2}\frac{a}{b})\exp(ab+b^2\bar{t})\mathrm{Erfc}(\frac{1}{2}a\bar{t}^{-\frac{1}{2}}+b\bar{t}^{\frac{1}{2}})$

Table 2.1: Possible choices for $\tilde{B}_{c_1}=A(p)e^{a\sqrt{p}}$ and $\tilde{B}_{c_2}=D(p)e^{-a\sqrt{p}}$ in (2.24), (2.54) and (2.78) with their corresponding Laplace inverses.

$$1).\quad \frac{r}{2\sqrt{\pi}t^{3/2}}e^{-\frac{1}{4t}\left(a^2-\frac{2}{r^2}\right)}\left[\sin\left(\frac{a}{\sqrt{2}rt}\right)\left(a+\frac{\sqrt{2}}{r}\right)+\cos\left(\frac{a}{\sqrt{2}rt}\right)\left(a-\frac{\sqrt{2}}{r}\right)\right] \quad \text{for } a \geq \frac{\sqrt{2}}{r}$$

$$2).\quad \frac{r}{\sqrt{\pi t}}e^{-\frac{1}{4t}\left(a^2-\frac{2}{r^2}\right)}\left[\cos\left(\frac{a}{\sqrt{2}rt}\right)+\sin\left(\frac{a}{\sqrt{2}rt}\right)\right] \text{ for } a \geq \frac{\sqrt{2}}{r}$$

$$3).\quad \frac{r}{\sqrt{\pi t}}e^{-\frac{1}{4t}\left(a^2-\frac{2}{r^2}\right)}\left[\sin\left(\frac{a}{\sqrt{2}rt}\right)\left(\frac{1}{\sqrt{2}rt}+1\right)+a\cos\left(\frac{a}{\sqrt{2}rt}\right)\right] \text{ for } a \geq \frac{\sqrt{2}}{r}$$

$$4).\quad \frac{r}{\sqrt{\pi t}}e^{-\frac{1}{4t}\left(a^2-\frac{2}{r^2}\right)}\left[\cos\left(\frac{a}{\sqrt{2}rt}\right)\left(\frac{1}{1-\sqrt{2}rt}\right)+\frac{a}{2t}\sin\left(\frac{a}{\sqrt{2}rt}\right)\right] \text{ for } a \geq \frac{\sqrt{2}}{r}$$

Note: that in the above solutions we can have $+r$ added to these (when $B(r,0)=r \quad \tilde{B}_p = r/p$)

$$5).\quad e^{-\lambda t}\left[A\sin\left(\frac{\sqrt{2}\sqrt{-\mu}}{r}\right)+B\cos\left(\frac{\sqrt{2}\sqrt{-\mu}}{r}\right)\right] \text{ for } \mu < 0$$

$$6).\quad \alpha r t^{1/2} e^{\frac{1}{2r^2 t}}$$

$$7).\quad \alpha_1 r + \alpha_2$$

$$8).\quad \left(\frac{\alpha_1 r}{\sqrt{t}}+\frac{\alpha_2}{t^{3/2}}\right)e^{\frac{1}{2r^2 t}}$$

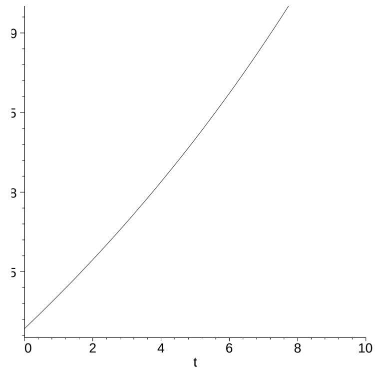
Table 2.2: Solutions to equation (2.41) for $B(r,t)$ by Laplace Transforms, separation of variables and the classical Lie symmetry method.

$$
\begin{gathered}
1).\;\; c_1 r e^{\dfrac{1}{2rt}} + \frac{c_2}{t^2} e^{\frac{1}{2rt}} \\
2).\;\; Q(rt) \text{ where } Q(z) = \alpha \int e^{\dfrac{1}{2z}} dz + \beta \\
3).\;\; e^{-\mu t}\left[A r^{1/2} J_1\left(\frac{\sqrt{2}\sqrt{-\mu}}{\sqrt{r}}\right) + B r^{1/2} Y_1\left(\frac{\sqrt{2}\sqrt{-\mu}}{\sqrt{r}}\right)\right], \text{ for } \mu < 0 \\
4).\;\; \frac{r^{1/2}}{2t} e^{-\dfrac{(a^2 - 2/r)}{4t}} \left[J_1\left(\frac{-a}{\sqrt{2rt}}\right) + Y_1\left(\frac{-a}{\sqrt{2rt}}\right)\right], \text{ for } a \geq -\frac{\sqrt{2}}{r}
\end{gathered}
$$

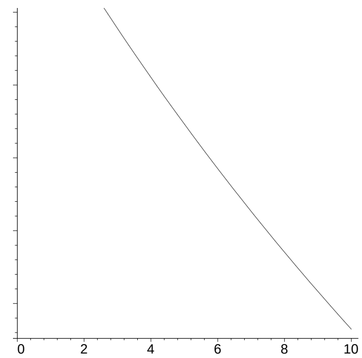
Table 2.3: Some solutions to equation (2.45) for $B(r,t)$.

$$
\begin{gathered}
1).\;\; e^{-\mu t} r^{1/2}\left[c_3 J_{\frac{\sqrt{15}}{2}}\left(-\sqrt{\frac{-2\mu}{r}}\right) + c_4 J_{-\frac{\sqrt{15}}{2}}\left(-\sqrt{\frac{-2\mu}{r}}\right)\right] \\
2).\;\; rte^{\frac{1}{4rt}}[C_1 W(1, \tfrac{\sqrt{7}i}{4}, \tfrac{1}{2rt}) + C_2 M(1, \tfrac{\sqrt{7}i}{4}, \tfrac{1}{2rt})] \\
\text{where } M(\cdot) \text{ and } W(\cdot) \text{ are Whittaker functions} \\
3).\;\; \frac{\sqrt{r}e^{\frac{1}{2rt}}}{t}[C_1 cos(\tfrac{\sqrt{7}}{4} ln(rt^2)) + C_2 sin(\tfrac{\sqrt{7}}{4} ln(rt^2))]
\end{gathered}
$$

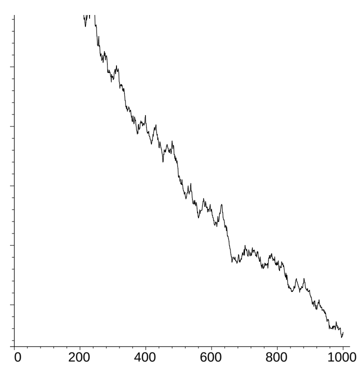
Table 2.4: Some solutions to equation (2.63) for $B(r,t)$.



(i).

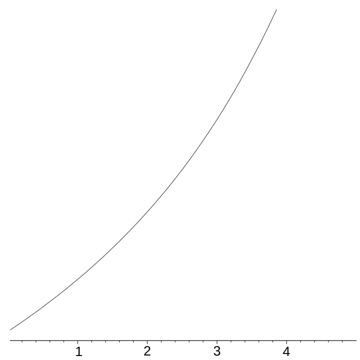


(ii).

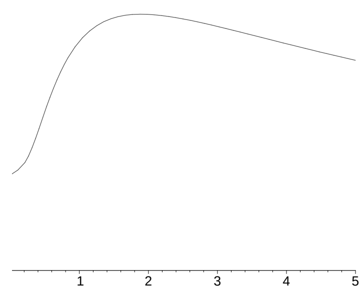


(iii).

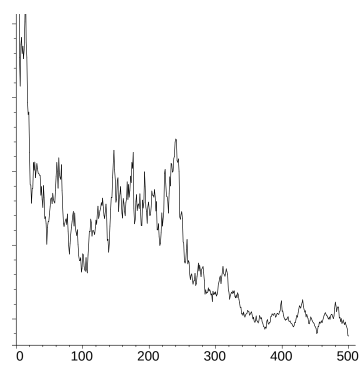
Figure 2.1: Example-1 solutions for (i) the bond price, (ii) the corresponding yield curve and (iii) the simulated interest rate model over the life of the bond, with the parameters $T = 10, \mu = 1, c_1 = 1, c_2 = 0, r = 0.04$ and B_c as in (2.25) with $c_3 = 10, c_4 = 1$.



(i).

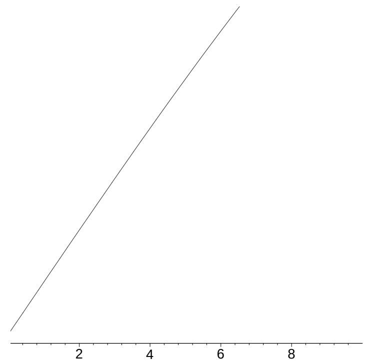


(ii).

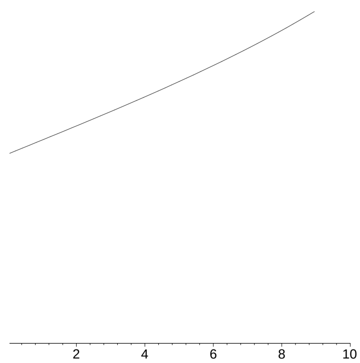


(iii).

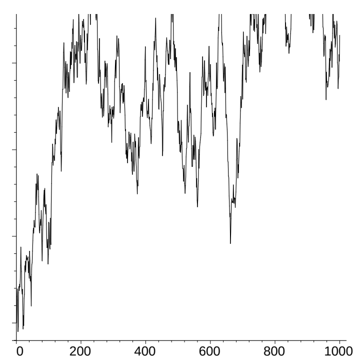
Figure 2.2: Example-7 solutions for (i) the bond price, (ii) the corresponding yield curve and (iii) the simulated interest rate model over the life of the bond, with the parameters $T = 5, c = 6, A = 0.0001, B = -0.5, m_1 = \frac{5}{12}, m_2 = \frac{7}{12}, r = 0.04$.



(i).



(ii).



(iii).

Figure 2.3: Example-7 solutions for (i) the bond price, (ii) the corresponding yield curve and (iii) the simulated interest rate model over the life of the bond, with the parameters $T = 10, c = 0.1, A = 1, B = 0, r = 0.04$.