REFORMULATING A BOUNDARY-INTEGRAL EQUATION IN THREE DIMENSIONS AS AN INTEGRAL-OPERATOR PROBLEM IN A PLANE REGION

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Technical Report No. 2001-2

Reformulating a Boundary-Integral Equation in Three Dimensions As an Integral-Operator Problem in a Plane Region †

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Abstract. Motivated by a desire to simplify the design of numerically stable and efficient approximation schemes for boundary-operator problems, we develop a framework in which an integral equation set on the boundary of a domain in $\mathbb{R}^3$ can be systematically reformulated as an integral-operator problem set in a region in the plane; some geometric restrictions are imposed on the shape of the (smooth) boundary. When the plane region is chosen to be a rectangle, the necessary Sobolev-space structures can be handled numerically rather easily in the new simpler geometry, in contrast to the situation on the original boundary. Moreover, familiar trial- and test-functions can then be employed in the construction of approximate solutions of the reformulated problems. We show for two examples how a well-posed problem can be transferred from the domain-boundary setting to the plane-region setting. We describe a numerical implementation of these ideas to a lower-dimensional example involving the approximate solution of a first-kind integral equation associated with the Helmholtz equation that is originally set on the boundary of a domain in $\mathbb{R}^2$.

Contents

1. Introduction.

Frequently, a boundary-value problem in partial differential equations can be reformulated in one way or another as an equivalent operator problem that is well posed in certain (usually, fractional-order) Sobolev spaces of functions or distributions on the boundary $\Gamma := \partial\Omega$ of a domain Ω in $\mathbb{R}^2$ or $\mathbb{R}^3$. Once such a well-posed reformulation has been found, the next task involves the selection or design of a convergent, stable, and easily implemented numerical scheme for approximate solution of the new boundary-operator problem. At that stage, it can be very helpful if one is actually

† This work was supported by the Air Force Office of Scientific Research, Air Force Materiel Command, USAF, under Grant F49620-96-1-0039. The views and conclusions contained herein are those of the author and should not be interpreted as necessarily representing the official policies or endorsements, either expressed or implied, of the Air Force Office of Scientific Research or the U.S. Government.

able to "manipulate numerically" the pertinent Sobolev structures, *i.e.*, if one is able to compute norms, inner products, the action of duality operators, *etc.*, in the Sobolev spaces between which the boundary operator is an isomorphism. For example, supposing that the prerequisites for convergence are fulfilled by the operator and the trial- and test-subspaces, the successful design of a Galerkin procedure that is numerically stable and features "effectively sparse" (or "compressible") system matrices depends entirely on an intelligent selection of the particular basis-functions for the trial- and test-subspaces. Thus, as shown in [4], numerical stability—by which we mean the boundedness of the sequence of condition numbers of the Galerkin-system matrices—will be assured if the families of trial and test basis-functions are *well conditioned* in the respective Sobolev inner products, *i.e.*, if the sequences of Gram matrices of the trial and test basis-functions in the respective Sobolev inner products have bounded condition numbers. Now, in [5] we discuss a scheme that appears, at least in a number of important cases, to produce a family well conditioned in a fractional-order Sobolev space of our choice by the isometric mapping and "back-projection" of a family that is already known to be well conditioned in the associated L_2-space, provided that the underlying geometry is so simple that we can numerically manipulate the Sobolev structures in the sense indicated. For example, this can be carried out for problems set in various Sobolev spaces of functions/distributions in an interval in the line or a rectangle in the plane; two cases in the one-dimensional setting are worked out in [5]. One can also consider working on a sphere, since the Sobolev structures can be handled numerically for that geometry, as well, but some difficulties enter there because of the differing topology.

Thus, we are motivated by the idea that it is easier to design a numerically well-behaved Galerkin scheme for a problem posed in a geometrically simple setting, such as a plane rectangle, than for a problem posed in a relatively complicated geometric setting, such as the boundary of a domain in $\mathbb{R}^3$, just because one can numerically manipulate the structure of Sobolev spaces associated with the simpler geometry. Accordingly, we set up here a framework within which one can systematically study the replacement of an original boundary-integral operator problem by a new integral-operator problem that is set in a region in the plane; in applications, the plane region will be chosen to be a rectangle. Of course, this is to be done in such a way that the desired solution of the original problem can be easily constructed (or approximated) once the solution of the new problem is known (or approximated). As an additional advantage in this approach, well-known families of trial/test functions (splines, wavelets, *etc.*) can be used in constructing approximate solutions in the new geometry. Effectively, we aim at trading a problem that would be solved approximately by a "boundary-element method" for a problem to be solved approximately by a "finite-element method" in a rectangle.

Let us summarize the organization and steps of the development. In Section 2 we specify the geometric restrictions on the original domain in $\mathbb{R}^3$, fix a parameter-domain $\square$ in the plane, and introduce notation for coördinate patches and a partition of unity. Section 3 is concerned with a review of the definitions of the Sobolev spaces for the boundary Γ, along with the definitions and properties of the operators connecting the structures on Γ and on $\square$. The "admissible" integral operators on Γ are introduced in Section 4, where we also explain how such an operator generates a corresponding one in a product space on the region $\square$, and examine relations between the two. Two typical well-posed boundary-operator problems are considered in Section 5, one of the "second kind" and one of the "first kind," in the classical terminology. For each, we show how to formulate a corresponding well-posed operator problem in a product of Sobolev spaces on $\square$, with provision for constructing (an approximation of) the solution of the original problem from (an approximation of) the solution of the new problem. Moreover, we discuss the applicability of the Bubnov-Galerkin

method to the new operator problem in each case. In Section 6 we give some orientation on the Bubnov-Galerkin method in a product space, and expand on some aspects of the application of the method to the reformulated first-kind problem of Section 5. We have begun the numerical work in a simplified two-dimensional setting, to gain some experience in the application of the idea, including the arrangement of the coördinate patches, the use of the partition of unity, and the selection of a "coupling constant" introduced in Section 5 for the first-kind problem. Section 7 contains a description of such an initial application to a first-kind integral equation arising in a two-dimensional acoustic scattering problem. We display or cite some numerical results confirming that the whole apparatus works.

Certain other points should be made clear at the outset.

(1.1) Our "framework for systematic reformulation" is really just a careful exploitation of the classical constructions of the Lebesgue measure and integral for the boundary Γ and of the Sobolev spaces $H^s(\Gamma)$ associated with Γ (for appropriate real s depending on the smoothness of Γ). Both of these constructions employ a covering collection of coördinate patches for the description of the boundary along with a subordinate partition of unity to set up mappings from the boundary into the plane which permit the definitions of the structures on the boundary to be made in terms of structures already in place in the plane, *viz.*, the Lebesgue measure for $\mathbb{R}^2$ and the scale of Sobolev spaces available for $\mathbb{R}^2$. A review of these definitions reveals the presence of certain isometric operators from spaces on the boundary into corresponding spaces on an appropriate subset of the plane, and it is these operators which permit the transfer of problems posed on the boundary to problems posed in the plane.

(1.2) In addition to some smoothness for the boundary Γ, we impose a further geometric restriction by requiring that Γ can be covered with just two coördinate patches, a condition that is not necessary but apparently becomes practically essential when one comes to implement the reformulations numerically. That is, if one supposes an n-patch covering and follows through the developments, then the reformulated problems will emerge in an n-fold product of Hilbert spaces; we have presented the arguments in the case $n = 2$, and so our new problems here are set in a product of two Hilbert spaces.

(1.3) The new unknown in the plane region will vanish in a boundary strip, owing to the use of the partition of unity, so that the new problem can always be set up in some H_0^s-space and its dual (although the H_0^s-space may coincide with the H^s-space if the positive s is small enough). However, wanting to leave some flexibility for the later applications, we permit the spaces in which we set the new problems to be possibly larger.

(1.4) The two example reformulations that we present here involve only integral operators bounded in the usual space $L_2(\Gamma)$, *e.g.*, operators with (at most) a weak singularity. The treatment of, say, operators with more severe singularities in their kernels, such as operators generated by derivatives of double-layer potentials for the Laplace or the Helmholtz equation, is deferred to a subsequent note.

(1.5) We are giving here only the basic ideas underlying the reformulations, and omit mention of a number of analytical issues, such as error estimation and, especially, the effect on the error that is caused by selecting a "small" or "large" overlap of the two coördinate patches. In the program for the lower-dimensional case that we have set up, the size of the overlap is an adjustable parameter, so that we do have the ability there to conduct some numerical experiments for assessment of this effect. We should also point out that we have presently no useful estimates for the permissible size of

the "coupling constant" that is introduced in the reformulation of the first-kind problem examined in Section 5, although we do have some preliminary numerical indications from the few experiments that we have performed.

(1.6) By the same token, we are ignoring some numerical aspects which may become important if large-scale applications are undertaken. For example, there are questions of economy raised not only by the use of overlapping coördinate patches, but also by the fact that a portion of the discretization of the plane region will be "wasted," since the support of the solutions of the new problems will, in the usual implementations that we envision, be contained in a disc.

This article is a companion to [5], where we show how one can work within a simple Sobolev-space structure to generate Galerkin procedures that appear to be numerically stable, by the construction of families of trial- and test-functions that are evidently well conditioned in a selected fractional-order Sobolev space. In fact, the ideas described there provided the initial motivation for seeking a reformulation device of the sort that is the main topic of the present article.

2. Geometry.

Let $\Omega \subset \mathbb{R}^3$ be a bounded regularly open set with boundary $\Gamma := \partial\Omega$ of class C^m, $m \geq 2$. We place some additional restrictions on Γ. First, we suppose that we can find an atlas for Γ comprising two charts, *i.e.*, that we can describe Γ with just two coördinate patches (Γ_+, h_+) and (Γ_-, h_-); one can permit descriptions with greater than two patches, but it will become clear that the resultant numerical schemes are correspondingly more complicated, as explained in the introduction. Thus, Γ_+ and Γ_- are open subsets of Γ with $\Gamma = \Gamma_+ \cup \Gamma_-$, while the maps $h_\pm : \Gamma_\pm \to h_\pm(\Gamma_\pm) \subset \mathbb{R}^2$ are C^m-diffeomorphisms onto open sets in $\mathbb{R}^2$. Moreover, the compositions

$$h_+ \circ h_-^{-1} : h_-\big(\Gamma_+ \cap \Gamma_-\big) \to h_+\big(\Gamma_+ \cap \Gamma_-\big) \qquad \text{and} \qquad h_- \circ h_+^{-1} : h_+\big(\Gamma_+ \cap \Gamma_-\big) \to h_-\big(\Gamma_+ \cap \Gamma_-\big)$$

are C^m-diffeomorphisms between the indicated open subsets of $\mathbb{R}^2$. We also assume that there is a class-C^m partition of unity $\{\varphi^+, \varphi^-\}$ for Γ subordinate to $\{\Gamma_+, \Gamma_-\}$ with the property that there is some open rectangle $\square \subset \mathbb{R}^2$ for which

$$h_+\big(\operatorname{supp}\varphi^+\big) \cup h_-\big(\operatorname{supp}\varphi^-\big) \;\subset\; \square \;\subset\; h_+\big(\Gamma_+\big) \cap h_-\big(\Gamma_-\big).$$

Actually, the plane domain $\square$ need not be a rectangle, but can be merely an open subset of $\mathbb{R}^2$. However, for the reasons already outlined, we shall eventually take $\square$ to be rectangular in applications, and so we shall continue to refer to it as a rectangle.

If Γ is starlike with respect to some enclosed point P, the imposed conditions will be fulfilled. In fact, in that case one can generate coördinate patches with the required properties by first mapping Γ onto a sphere centered at P and then using the two stereographic projection maps into the plane that are associated with the respective poles of the sphere. If Ω is convex, acceptable patches can be constructed even more simply by the direct use of two stereographic projection maps.

We shall describe the developments by continuing to suppose that $\Omega \subset \mathbb{R}^3$, although every statement will have an analog in the situation where $\Omega \subset \mathbb{R}^2$. Thus, the example discussed in Section 7 is set in the latter lower-dimensional geometry.

3. The spaces $H^s(\Gamma)$ and $\mathcal{H}^s(\square)^2$; connecting operators.

Since our reformulation is closely connected with the relations between the definitions of the Sobolev spaces $H^s(\Gamma)$ and the form of the integral over Γ, it is pertinent to recall briefly the constructions. At the same time, we introduce several naturally occurring operators that are indispensable for our transfer of problems set in spaces of functions on Γ to corresponding problems set in spaces of functions on the rectangle $\square$.

For the definitions and fundamental properties of Sobolev spaces (in particular, Sobolev spaces associated with the boundary Γ), antiduality, and constructions of "spaces of negative order," we rely on the developments to be found in, *e.g.*, AUBIN [1], [2], BEREZANSKIĬ [3], GRISVARD [7], and LIONS & MAGENES [10],

The usual surface measure for Γ that is induced by Lebesgue measure λ_3 on $\mathbb{R}^3$ we denote by λ_Γ; the associated spaces $L_p(\Gamma)$ are constructed in the familiar way. In particular, the integral $\int_\Gamma u\, d\lambda_\Gamma$ of an appropriate complex function u over Γ with respect to λ_Γ has meaning; the integral is defined (and computed) with the help of a partition of unity such as $\{\varphi^+, \varphi^-\}$, introduced in the preceding section. In the present case, by recalling the properties listed in Section 2 for the coördinate patches and the partition of unity, we find that

$$\int_\Gamma u\, d\lambda_\Gamma = \int_\square (\varphi^+ u) \circ h_+^{-1}\ \mathfrak{J}h_+^{-1}\, d\lambda_2 + \int_\square (\varphi^- u) \circ h_-^{-1}\ \mathfrak{J}h_-^{-1}\, d\lambda_2 \qquad \text{for} \quad u \in L_1(\Gamma), \tag{3.1}$$

in which $\mathfrak{J}h_\pm^{-1}$ denote the generalized Jacobians; in the special case when the support of the integrand is contained in a coördinate patch, this can be written as, *e.g.*,

$$\int_\Gamma u\, d\lambda_\Gamma = \int_{h_\pm(\Gamma_\pm)} u \circ h_\pm^{-1}\ \mathfrak{J}h_\pm^{-1}\, d\lambda_2 \qquad \text{for} \quad u \in L_1(\Gamma) \quad \text{with} \quad \operatorname{supp} u \subset \Gamma_\pm. \tag{3.2}$$

$H^0(\Gamma)$ denotes the usual (complex) Hilbert space obtained by equipping $L_2(\Gamma)$ with the familiar inner product given by

$$(u, v)_{0,\Gamma} := \int_\Gamma u\overline{v}\, d\lambda_\Gamma \qquad \text{for} \quad u,\, v \in L_2(\Gamma). \tag{3.3}$$

It is helpful to point out an equivalent definition for the space $H^0(\Gamma)$. For this, corresponding to a complex function u on Γ let $Ku := \big(K^+u, K^-u\big)$, in which K^+u and K^-u are constructed on the rectangle $\square$ as the restrictions

$$\left.\begin{aligned} K^+u &:= \left\{ \sqrt{\mathfrak{J}h_+^{-1}}\big(\sqrt{\varphi^+}u\big) \circ h_+^{-1} \right\} \Big| \square \\ K^-u &:= \left\{ \sqrt{\mathfrak{J}h_-^{-1}}\big(\sqrt{\varphi^-}u\big) \circ h_-^{-1} \right\} \Big| \square \end{aligned}\right\}; \tag{3.4}$$

note that the respective (compact) supports satisfy $\operatorname{supp} K^\pm u \subset h_\pm\big(\operatorname{supp}\varphi^\pm\big) \subset \square \subset h_\pm\big(\Gamma_\pm\big)$. Then we again realize $H^0(\Gamma)$ as the set of all complex functions defined a.e. on Γ and such that both K^+u and K^-u belong to $H^0(\square)$. Moreover, it is easily checked that the inner product displayed in (3.3) is also given by

$$(u, v)_{0,\Gamma} = (K^+u, K^+v)_{0,\square} + (K^-u, K^-v)_{0,\square},$$

with $(\,\cdot\,,\,\cdot\,)_{0,\square}$ indicating the inner product in $H^0(\square)$. Consequently, by setting

$$K_0 u := \big(K^+u, K^-u\big) \qquad \text{for} \quad u \in H^0(\Gamma), \tag{3.5}$$

we obtain an isometry $K_0 : H^0(\Gamma) \to H^0(\square)^2$ carrying $H^0(\Gamma)$ onto a (closed) subspace of $H^0(\square)^2$ comprising function-pairs with compact supports in $\square$.

By relying again on the Sobolev structures already in place on $\square$, corresponding steps can be followed in defining the spaces $H^s(\Gamma)$ for $0 < s \leq m$. As we wish to preserve some flexibility in the choice of the spaces in which the eventual reformulations of boundary-operator problems are set, it proves convenient to single out a subspace $\mathcal{H}^s(\square)$ of $H^s(\square)$ containing $H_0^s(\square)$, the closure in $H^s(\square)$ of $C_0^\infty(\square)$. Thus, $\mathcal{H}^s(\square)$ is a Sobolev-Hilbert space of functions on $\square$ satisfying

$$H_0^s(\square) \subset \mathcal{H}^s(\square) \subset H^s(\square) \qquad (0 \leq s \leq m) \tag{3.6}$$

and equipped with the structure inherited from $H^s(\square)$ (so that the natural injections in (3.6) are continuous). If $0 \leq s < 1/2$ this requires that $\mathcal{H}^s(\square) = H^s(\square)$, since we know that $H_0^s(\square) = H^s(\square)$ for such s. For example, we may take $\mathcal{H}^s(\square)$ to be $H_0^s(\square)$ or $H^s_{\rm per}(\square)$, the closure in $H^s(\square)$ of the restrictions to $\square$ of the appropriate doubly-periodic C^∞-functions on $\mathbb{R}^2$; the selection will be dictated by the desire to use one or another family of coördinate functions in the implementation of a Galerkin procedure. The inner product of $H^s(\square)$ we denote by $(\,\cdot\,,\,\cdot\,)_{s,\square}$, while $(\,\cdot\,,\,\cdot\,)_{[s,\square]}$ indicates the usual corresponding inner product for $H^s(\square)^2$.

Definition 3.1. Let $0 \leq s \leq m$. $H^s(\Gamma)$ is the set of all complex functions defined a.e. on Γ and such that both K^+u and K^-u belong to $H^s(\square)$ (in which case they will be in $H_0^s(\square)$, so also in $\mathcal{H}^s(\square)$). $H^s(\Gamma)$ is equipped with the inner product given by

$$(u,v)_{s,\Gamma} := (K^+u, K^+v)_{s,\square} + (K^-u, K^-v)_{s,\square}.$$

The linear operator $K_s : H^s(\Gamma) \to \mathcal{H}^s(\square)^2$ is defined by

$$K_s u := K u = \big(K^+u, K^-u\big) \qquad \text{for} \quad u \in H^s(\Gamma). \tag{3.7}$$

It follows that each $H^s(\Gamma)$ is complete, while K_s is an isometry; the range of K_s is then closed in $\mathcal{H}^s(\square)^2$ and contained in $H_0^s(\square)^2$. It is also important to note that $K_s = K_0 \,\big|\, H^s(\Gamma)$, *i.e.*, that each K_s is simply the appropriate restriction of K_0.

Since each K_s is injective and has closed range, it is left-invertible. In seeking a left inverse for K_0, one is led to consider the operation $\mathcal{K}$ mapping function-pairs U defined on $\square$ to functions defined on Γ according to

$$\mathcal{K}U(\xi) := \left[\sqrt{\frac{\varphi^+(\xi)}{\Im h_+^{-1}\big(h_+(\xi)\big)}}\, U^+\big(h_+(\xi)\big)\right]_{\Gamma_+} + \left[\sqrt{\frac{\varphi^-(\xi)}{\Im h_-^{-1}\big(h_-(\xi)\big)}}\, U^-\big(h_-(\xi)\big)\right]_{\Gamma_-}, \qquad \xi \in \Gamma,$$
$$\text{for} \quad U \equiv (U^+, U^-) : \square \to \mathbb{C}^2; \tag{3.8}$$

here the notation $[\,\cdot\,]_{\Gamma_+}$ indicates, for example,

$$\Big[\varphi^+(\xi)\, U^+\big(h_+(\xi)\big)\Big]_{\Gamma_+} := \begin{cases} \varphi^+(\xi)\, U^+\big(h_+(\xi)\big) & \text{if} \quad \xi \in \operatorname{supp}\varphi^+ \\ 0 & \text{otherwise} \end{cases} \tag{3.9}$$

(recall that $h_+(\operatorname{supp}\varphi^+) \subset \square$), while $[\,\cdot\,]_{\Gamma_-}$ has the similar meaning. In fact, simple manipulations show that

$$\mathcal{K}Ku = u \qquad \text{whenever} \quad u : \Gamma \to \mathbb{C} \tag{3.10}$$

and

$$K\mathcal{K}U = \left((\varphi^+ \circ h_+^{-1})\, U^+ + \left\{ \sqrt{\varphi^+\varphi^-} \sqrt{\frac{\mathfrak{I} h_+^{-1} \circ h_+}{\mathfrak{I} h_-^{-1} \circ h_-}}\; U^- \circ h_- \right\} \circ h_+^{-1}, \right.$$
$$\left. (\varphi^- \circ h_-^{-1})\, U^- + \left\{ \sqrt{\varphi^+\varphi^-} \sqrt{\frac{\mathfrak{I} h_-^{-1} \circ h_-}{\mathfrak{I} h_+^{-1} \circ h_+}}\; U^+ \circ h_+ \right\} \circ h_-^{-1} \right)$$
$$\text{whenever} \quad U \equiv (U^+, U^-) : \square \to \mathbb{C}^2. \tag{3.11}$$

Let $0 \le s \le m$: by appealing to well-known results about the operations on Lebesgue and Sobolev spaces formed from multiplication by a sufficiently regular function and composition with a sufficiently regular transformation, it is clear from (3.11) and Definition 3.1 that the operation $U \mapsto \mathcal{K}U$ takes $H^s(\square)^2$ into $H^s(\Gamma)$ and is bounded in that setting; accordingly, it is legitimate to define $\mathcal{K}_s$ to be the restriction of this operation to $\mathcal{H}^s(\square)^2$:

Definition 3.2. For $0 \le s \le m$, $\mathcal{K}_s : \mathcal{H}^s(\square)^2 \to H^s(\Gamma)$ is defined by

$$\mathcal{K}_s U := \mathcal{K}U \qquad \text{for} \quad U \in \mathcal{H}^s(\square)^2.$$

As indicated, the resultant operators $\mathcal{K}_s$ are bounded. Moreover, we obviously have $\mathcal{K}_s = \mathcal{K}_0 \,\big|\, \mathcal{H}^s(\square)^2$ for $0 < s \le m$.

The composition of K_0 and $\mathcal{K}_0$ produces a third operator $\mathcal{Q}_0$ which is also important for present purposes,

$$\mathcal{Q}_0 := K_0 \mathcal{K}_0 : H^0(\square)^2 \to H^0(\square)^2, \tag{3.12}$$

along with its restrictions

$$\mathcal{Q}_s := \mathcal{Q}_0 \,\big|\, \mathcal{H}^s(\square)^2 = K_s \mathcal{K}_s : \mathcal{H}^s(\square)^2 \to \mathcal{H}^s(\square)^2.$$

We shall show that $\mathcal{Q}_s$ is a projector in $\mathcal{H}^s(\square)^2$ onto the range of K_s (which is a "copy" of $H^s(\Gamma)$ in $\mathcal{H}^s(\square)^2$).

The (Hilbert-space) adjoints of K_s, $\mathcal{K}_s$, and $\mathcal{Q}_s$ we denote by $K_s^\star : \mathcal{H}^s(\square)^2 \to H^s(\Gamma)$, $\mathcal{K}_s^\star : H^s(\Gamma) \to \mathcal{H}^s(\square)^2$, and $\mathcal{Q}_s^\star : \mathcal{H}^s(\square)^2 \to \mathcal{H}^s(\square)^2$, respectively. The important properties of the operators are summarized in

Lemma 3.1. *Let $0 \le s \le m$. Recall the definitions of the bounded operators $K_s : H^s(\Gamma) \to \mathcal{H}^s(\square)^2$, $\mathcal{K}_s : \mathcal{H}^s(\square)^2 \to H^s(\Gamma)$, and $\mathcal{Q}_s : \mathcal{H}^s(\square)^2 \to \mathcal{H}^s(\square)^2$.*

(*i.*) *$\mathcal{K}_s$ is a left inverse of K_s, i.e., we have*

$$\mathcal{K}_s K_s u = u \qquad \textit{for every} \quad u \in H^s(\Gamma); \tag{3.13}$$

in particular, the range $\mathcal{R}(\mathcal{K}_s) = H^s(\Gamma)$.

(*ii.*) *We have $\mathcal{Q}_s^2 = \mathcal{Q}_s$, so that $\mathcal{Q}_s$ is a bounded projector in $H^s(\square)^2$; moreover, the range of $\mathcal{Q}_s$ coincides with the range of K_s:*

$$\mathcal{R}(\mathcal{Q}_s) = \mathcal{R}(K_s) \qquad \Big(= \mathcal{N}(I - \mathcal{Q}_s)\Big), \tag{3.14}$$

so that

$$K_s\mathcal{K}_sU = U \qquad \text{for every} \quad U \in \mathcal{R}\big(K_s\big). \tag{3.15}$$

Further, the null space of $\mathcal{Q}_s$ is given by

$$\mathcal{N}\big(\mathcal{Q}_s\big) = \mathcal{N}\big(\mathcal{K}_s\big) \qquad \Big(= \mathcal{R}\big(I - \mathcal{Q}_s\big)\Big). \tag{3.16}$$

(*iii.*) *$K_s^\star K_s$ is the identity operator on $H^s(\Gamma)$; it follows that $K_sK_s^\star = \mathcal{P}_s$, the orthoprojector onto the range $\mathcal{R}\big(K_s\big)$ in $\mathcal{H}^s(\Box)^2$, while the complete relation between $\mathcal{K}_s$ and $K_s^\star$ is given by $K_s^\star = \mathcal{K}_s\mathcal{P}_s$.*

Proof. (*i*). Equality (3.13) is just (3.10) rewritten for $u \in H^s(\Gamma)$, while the surjectivity of $\mathcal{K}_s$ follows immediately from (3.13).

(*ii*). With (3.13), we get $\mathcal{Q}_s^2 = K_s\mathcal{K}_sK_s\mathcal{K}_s = K_s\mathcal{K}_s = \mathcal{Q}_s$, so that $\mathcal{Q}_s$ is indeed a projector; the equalities $\mathcal{R}\big(\mathcal{Q}_s\big) = \mathcal{N}\big(I - \mathcal{Q}_s\big)$ and $\mathcal{N}\big(\mathcal{Q}_s\big) = \mathcal{R}\big(I - \mathcal{Q}_s\big)$ therefore hold, as they do for any projector. Since $\mathcal{R}\big(\mathcal{K}_s\big)$ is all of $H^s(\Gamma)$, it is clear that the range of $\mathcal{Q}_s = K_s\mathcal{K}_s$ must coincide with the range of K_s. Thus, (3.14) is proven, and now it is obvious that (3.15) just expresses the fact that the projector $\mathcal{Q}_s$ is the identity map on its range. Finally, $\mathcal{N}\big(\mathcal{Q}_s\big) = \mathcal{N}\big(\mathcal{K}_s\big)$ holds since K_s is injective.

(*iii*). From

$$\big(K_s^\star K_s u, v\big)_{s,\Gamma} = \big(K_s u, K_s v\big)_{[s,\Box]} = \big(u, v\big)_{s,\Gamma} \qquad \text{whenever} \quad u, v \in H^s(\Gamma)$$

we see that $K_s^\star K_s$ is the identity operator on $H^s(\Gamma)$. Then $K_sK_s^\star K_sK_s^\star = K_sK_s^\star$, so that $\mathcal{P}_s := K_sK_s^\star$ is a bounded projector in $\mathcal{H}^s(\Box)^2$; since $\mathcal{P}_s$ is clearly self-adjoint, it is an orthoprojector. K_s is injective and has closed range, which implies that its adjoint $K_s^\star$ has closed and dense range, *i.e.*, is surjective; it follows that $\mathcal{R}\big(\mathcal{P}_s\big) = \mathcal{R}\big(K_s\big)$. The final assertion follows directly from the equality $\mathcal{K}_sK_sK_s^\star = K_s^\star$, which holds by (3.13). ∎

Remark. A characterization of the range of K_s is given by

$$\mathcal{R}\big(K_s\big) = \Bigg\{ (U^+, U^-) \in H_0^s(\Box)^2 \;\Bigg|\; \operatorname{supp} U^\pm \subset h_\pm\big(\operatorname{supp}\varphi^\pm\big) \quad \text{and}$$

$$\sqrt{\frac{\varphi^-(\xi)}{\jmath h_+^{-1}\big(h_+(\xi)\big)}}\,U^+\big(h_+(\xi)\big) - \sqrt{\frac{\varphi^+(\xi)}{\jmath h_-^{-1}\big(h_-(\xi)\big)}}\,U^-\big(h_-(\xi)\big) = 0 \quad \text{for every} \quad \xi \in \Gamma_+ \cap \Gamma_-\Bigg\}.$$

Since we do not use this result in the sequel, we omit the simple algebraic manipulations required to check it.

For $0 < s \leq m$, we denote by $\mathcal{H}^{-s}(\Box)^2$ the realization of the antidual of $\mathcal{H}^s(\Box)^2$ which is obtained by regarding the latter as "positive space" and $\mathcal{H}^0(\Box)^2 = H^0(\Box)^2$ as "zero space," or "pivot space." In case $\mathcal{H}^s(\Box)^2$ is taken to be $H_0^s(\Box)^2$, the antidual $\mathcal{H}^{-s}(\Box)^2$ is the space usually indicated by $H^{-s}(\Box)^2$; one can always identify $\mathcal{H}^{-s}(\Box)^2$ as the quotient of the antidual of $H^s(\Box)^2$ *modulo* the annihilator of $\mathcal{H}^s(\Box)^2$, but it usually proves more convenient to resort to other realizations. By $\mathsf{J}_{\Box_s} : \mathcal{H}^s(\Box)^2 \to \mathcal{H}^{-s}(\Box)^2$ we denote the antiduality operator, while the antiduality pairing on $\mathcal{H}^{-s}(\Box)^2 \times \mathcal{H}^s(\Box)^2$ is indicated with $\langle\,\cdot\,,\,\cdot\,\rangle_{[\Box]}$. We shall usually employ without special comment relations such as

$$\big(\Lambda, \mathsf{J}_{\Box_s}U\big)_{[-s,\Box]} = \big\langle\Lambda, U\big\rangle_{[\Box]} = \big(\mathsf{J}_{\Box_s}^{-1}\Lambda, U\big)_{[s,\Box]}, \qquad \text{for} \quad \Lambda \in \mathcal{H}^{-s}(\Box)^2, \quad U \in \mathcal{H}^s(\Box)^2.$$

The spaces $H^{-s}(\Gamma)$ with $0 < s \leq m$ can be introduced in various equivalent ways, but each way derives ultimately from the structure and properties of the spaces already in place on the rectangle $\Box$. For example, one can define $H^{-s}(\Gamma)$ to be the completion of $L_2(\Gamma)$ with respect to the inner product $(u,v) \mapsto (K_0 u, K_0 v)_{[-s,\Box]}$. Instead, we shall rely (as before) on the standard construction of $H^{-s}(\Gamma)$ as the "negative space" corresponding to the "zero space" $H^0(\Gamma)$ and the "positive space" $H^s(\Gamma)$. That is, with $\iota_{\Gamma s}^\star$ denoting the adjoint of the natural injection map $\iota_{\Gamma s} : H^s(\Gamma) \to H^0(\Gamma)$, $L_2(\Gamma)$ is equipped with the new inner product $(\,\cdot\,,\,\cdot\,)_{-s,\Gamma}$ given by

$$(u,v)_{-s,\Gamma} := \big(\iota_{\Gamma s}^\star u, \iota_{\Gamma s}^\star v\big)_{s,\Gamma}, \qquad u,\, v \in L_2(\Gamma);$$

$H^{-s}(\Gamma)$ is then defined to be the completion of $\big(L_2(\Gamma), (\,\cdot\,,\,\cdot\,)_{-s,\Gamma}\big)$. The adjoint $\iota_{\Gamma s}^\star$ extends to an isometric isomorphism of $H^{-s}(\Gamma)$ onto $H^s(\Gamma)$; the inverse of this map is denoted by $J_{\Gamma s} : H^s(\Gamma) \to H^{-s}(\Gamma)$ and termed the "antiduality operator." The sesquilinear form $(u,v) \mapsto (u, \iota_{\Gamma s}^\star v)_{0,\Gamma}$ on $L_2(\Gamma) \times H^s(\Gamma)$ is bounded when regarded as densely defined in $H^{-s}(\Gamma) \times H^s(\Gamma)$; its bounded extension to the latter space is denoted by $\langle\,\cdot\,,\,\cdot\,\rangle_\Gamma$ and called the "antiduality pairing," since every element of the antidual of $H^s(\Gamma)$ is of the form $\langle u_-,\,\cdot\,\rangle_\Gamma$ for a uniquely determined $u_- \in H^{-s}(\Gamma)$. In this way, one identifies $H^{-s}(\Gamma)$ as the antidual of $H^s(\Gamma)$.

Besides the adjoint of an operator in a Hilbert space, we may also use its (Banach-space) transpose, according to convenience in particular instances. We denote by $K_s' : \mathcal{H}^{-s}(\Box)^2 \to H^{-s}(\Gamma)$, $\mathcal{K}_s' : H^{-s}(\Gamma) \to \mathcal{H}^{-s}(\Box)^2$, and $\mathcal{Q}_s' : \mathcal{H}^{-s}(\Box)^2 \to \mathcal{H}^{-s}(\Box)^2$ the respective transposes of the bounded operators K_s, $\mathcal{K}_s$, and $\mathcal{Q}_s$. For example, the defining relation for K_s' is

$$\big\langle K_s' U_-, u_+ \big\rangle_\Gamma = \big\langle U_-, K_s u_+ \big\rangle_{[\Box]} \qquad \text{for each} \quad U_- \in \mathcal{H}^{-s}(\Box)^2, \quad u_+ \in H^s(\Gamma),$$

which is legitimate because $u_+ \mapsto \big\langle U_-, K_s u_+ \big\rangle_{[\Box]}$ is a bounded antilinear functional on $H^s(\Gamma)$. We recall that the adjoint and the transpose of an operator are related through the appropriate antiduality operator(s). For example, we have the equality $K_s^\star = J_{\Gamma s}^{-1} K_s' \mathbf{J}_{\Box s}$ connecting K_s' and $K_s^\star$.

From the properties of, and relations amongst, the three operators K_s, $\mathcal{K}_s$, and $\mathcal{Q}_s$, *e.g.*, those established in the preceding Lemma 3.1, we can immediately deduce dual results for the transposes. For example, we see that K_s' has closed range in $H^{-s}(\Gamma)$ [resp., $\mathcal{K}_s'$ has closed range in $\mathcal{H}^{-s}(\Box)^2$] since the range of K_s [resp., $\mathcal{K}_s$] is closed; then we can conclude that $\mathcal{R}\big(K_s'\big) = H^{-s}(\Gamma)$ because K_s is injective, while $\mathcal{K}_s'$ is injective because $\mathcal{R}\big(\mathcal{K}_s\big) = H^s(\Gamma)$. The remaining needed facts concerning the transpose $\mathcal{Q}_s'$ are collected in

Lemma 3.2. *Let $0 \leq s \leq m$. The operator K_s' is a left inverse of $\mathcal{K}_s'$, i.e., $K_s'\mathcal{K}_s' = I_{H^{-s}(\Gamma)}$, the identity operator on $H^{-s}(\Gamma)$. We have $\mathcal{Q}_s' = \mathcal{K}_s' K_s'$, whence it follows that $\mathcal{Q}_s'$ is a bounded projector in $\mathcal{H}^{-s}(\Box)^2$ with*

$$\mathcal{R}\big(\mathcal{Q}_s'\big) = \mathcal{R}\big(\mathcal{K}_s'\big) \qquad \Big(= \mathcal{N}\big(I - \mathcal{Q}_s'\big)\Big) \qquad \textit{and} \qquad \mathcal{N}\big(\mathcal{Q}_s'\big) = \mathcal{N}\big(K_s'\big) \qquad \Big(= \mathcal{R}\big(I - \mathcal{Q}_s'\big)\Big); \quad (3.17)$$

in particular, we have

$$\mathcal{K}_s' K_s' \Lambda = \Lambda \qquad \textit{for all} \quad \Lambda \in \mathcal{R}\big(\mathcal{K}_s'\big).$$

Proof. The equalities $K_s'\mathcal{K}_s' = I_{H^{-s}(\Gamma)}$ and $\mathcal{Q}_s' = \mathcal{K}_s' K_s'$ follow by transposition from $\mathcal{K}_s K_s = I_{H^s(\Gamma)}$ and $\mathcal{Q}_s = K_s \mathcal{K}_s$, respectively. Then a trivial computation shows that $\big(\mathcal{Q}_s'\big)^2 = \mathcal{Q}_s'$, *i.e.*, that $\mathcal{Q}_s'$ is a projector. The equality $\mathcal{R}\big(\mathcal{Q}_s'\big) = \mathcal{R}\big(\mathcal{K}_s'\big)$ follows from $\mathcal{Q}_s' = \mathcal{K}_s' K_s'$ and the surjectivity of K_s'; the equality $\mathcal{N}\big(\mathcal{Q}_s'\big) = \mathcal{N}\big(K_s'\big)$ follows from $\mathcal{Q}_s' = \mathcal{K}_s' K_s'$ and the injectivity of $\mathcal{K}_s'$. The final assertion

simply expresses the fact that the restriction of the projector $\mathcal{Q}_s'$ to its range is the identity operator on that range. $\square$

We close this section by establishing the self-adjointness of the projector $\mathcal{Q}_0$ in $H^0(\square)^2$ and deriving the implications of that self-adjointness which appear in Lemma 3.5. The latter results are used at various points; for example, they are essential for the actual execution of certain steps in the numerical implementation of Galerkin procedures for our reformulated operator problems.

Lemma 3.3. *The projector $\mathcal{Q}_0$ is self-adjoint in $H^0(\square)^2$, i.e., $\mathcal{Q}_0$ is the orthogonal projector onto $\mathcal{R}\big(K_0\big)$ in $H^0(\square)^2$.*

Proof. From (3.11) we find that the operator $\mathcal{Q}_0$ is described by

$$\mathcal{Q}_0 U := K_0 \mathcal{K}_0 U = \begin{pmatrix} \mathcal{Q}_0^{++} & \mathcal{Q}_0^{+-} \\ \\ \mathcal{Q}_0^{-+} & \mathcal{Q}_0^{--} \end{pmatrix} \begin{pmatrix} U^+ \\ \\ U^- \end{pmatrix} \qquad \text{for} \quad U \equiv (U^+, U^-) \in H^0(\square)^2,$$

in which the elemental operators $\mathcal{Q}_0^{\mu\eta}$, bounded in $H^0(\square)$, are given by

$$\mathcal{Q}_0^{++} u := \big(\varphi^+ \circ h_+^{-1}\big)\, u,$$

$$\mathcal{Q}_0^{+-} u := \left(\sqrt{\varphi^- \varphi^+}\sqrt{\frac{\mathfrak{I} h_+^{-1} \circ h_+}{\mathfrak{I} h_-^{-1} \circ h_-}}\; u \circ h_-\right) \circ h_+^{-1},$$

$$\mathcal{Q}_0^{-+} u := \left(\sqrt{\varphi^- \varphi^+}\sqrt{\frac{\mathfrak{I} h_-^{-1} \circ h_-}{\mathfrak{I} h_+^{-1} \circ h_+}}\; u \circ h_+\right) \circ h_-^{-1},$$

and

$$\mathcal{Q}_0^{--} u := \big(\varphi^- \circ h_+^{-1}\big)\, u, \qquad\qquad \text{for} \quad u \in H^0(\square).$$

To verify that $\mathcal{Q}_0$ is self-adjoint in $H^0(\square)^2$, we must check that $\mathcal{Q}_0^{++}$ and $\mathcal{Q}_0^{--}$ are self-adjoint while $\mathcal{Q}_0^{+-}$ and $\mathcal{Q}_0^{-+}$ are mutually adjoint in $H^0(\square)$, *i.e.*, are connected by $\big(\mathcal{Q}_0^{+-}\big)^\star = \mathcal{Q}_0^{-+}$. But $\mathcal{Q}_0^{++}$ and $\mathcal{Q}_0^{--}$ are simply operators of multiplication by smooth real-valued functions with compact support in $\square$, and so are clearly self-adjoint in $H^0(\square)$. Thus, it remains to examine the relation between $\mathcal{Q}_0^{+-}$ and $\mathcal{Q}_0^{-+}$.

Accordingly, let u and $v \in H^0(\square)$ and consider

$$\big(\mathcal{Q}_0^{+-} u, v\big)_{0,\square} = \int_\square \left(\sqrt{\varphi^- \varphi^+}\sqrt{\frac{\mathfrak{I} h_+^{-1} \circ h_+}{\mathfrak{I} h_-^{-1} \circ h_-}}\; u \circ h_-\right) \circ h_+^{-1}(y)\, \overline{v}(y)\, d\lambda_2(y);$$

since

$$\operatorname{supp}\big(\big(\varphi^+ \circ h_+^{-1}\big)\big(\varphi^- \circ h_+^{-1}\big)\big) = h_+\big(\operatorname{supp}\varphi^+ \cap \operatorname{supp}\varphi^-\big),$$

we can replace the set of integration $\square$ here by the subset $h_+\big(\operatorname{supp}\varphi^+ \cap \operatorname{supp}\varphi^-\big)$. In the resultant integral we make the transformation

$$y = F(x) := h_+ \circ h_-^{-1}(x) \qquad \text{for} \quad x \in h_-\big(\operatorname{supp}\varphi^+ \cap \operatorname{supp}\varphi^-\big),$$

having inverse given by

$$x = F^{-1}(y) = h_- \circ h_+^{-1}(y) \qquad \text{for} \quad y \in h_+\big(\operatorname{supp}\varphi^+ \cap \operatorname{supp}\varphi^-\big);$$

with $\mathcal{J}F$ denoting the Jacobian determinant of the transformation F, this produces

$$\left(\mathcal{Q}_0^{+-}u,v\right)_{0,\square}=\int_{h_-\left(\operatorname{supp}\varphi^+\cap\operatorname{supp}\varphi^-\right)}u(x)\left(\sqrt{\varphi^-\varphi^+}\sqrt{\frac{\mathcal{I}h_+^{-1}\circ h_+}{\mathcal{I}h_-^{-1}\circ h_-}}\,\overline{v}\circ h_+\right)\circ h_-^{-1}(x)\,\left|\mathcal{J}F(x)\right|d\lambda_2(x);$$

we may replace the set of integration $h_-\left(\operatorname{supp}\varphi^+\cap\operatorname{supp}\varphi^-\right)\subset\square$ here by $\square$ itself, since

$$\operatorname{supp}\left(\left(\varphi^+\circ h_-^{-1}\right)\left(\varphi^-\circ h_-^{-1}\right)\right)=h_-\left(\operatorname{supp}\varphi^+\cap\operatorname{supp}\varphi^-\right).$$

Evidently, to establish the desired equality $\left(\mathcal{Q}_0^{+-}u,v\right)_{0,\square}=\left(u,\mathcal{Q}_0^{-+}v\right)_{0,\square}$, we must show that

$$\sqrt{\frac{\mathcal{I}h_+^{-1}\circ h_+}{\mathcal{I}h_-^{-1}\circ h_-}}\circ h_-^{-1}\left|\mathcal{J}F\right|=\sqrt{\frac{\mathcal{I}h_-^{-1}\circ h_-}{\mathcal{I}h_+^{-1}\circ h_+}}\circ h_-^{-1},$$

or

$$\left|\mathcal{J}F\right|=\frac{\mathcal{I}h_-^{-1}\circ h_-}{\mathcal{I}h_+^{-1}\circ h_+}\circ h_-^{-1}\qquad\text{on}\quad h_-\left(\operatorname{supp}\varphi^+\cap\operatorname{supp}\varphi^-\right).$$

But the relation $h_+^{-1}\circ F=h_-^{-1}$, holding on the latter set, gives $\left(\mathcal{I}h_+^{-1}\circ F\right)\left(\mathcal{J}F\right)=\mathcal{I}h_-^{-1}$, so

$$\mathcal{J}F=\frac{\mathcal{I}h_-^{-1}}{\mathcal{I}h_+^{-1}\circ F}=\frac{\mathcal{I}h_-^{-1}}{\mathcal{I}h_+^{-1}\circ h_+\circ h_-^{-1}}=\frac{\mathcal{I}h_-^{-1}\circ h_-}{\mathcal{I}h_+^{-1}\circ h_+}\circ h_-^{-1}\qquad\text{on}\quad h_-\left(\operatorname{supp}\varphi^+\cap\operatorname{supp}\varphi^-\right).$$

As noted, this effectively completes the proof. □

Corollary 3.1. *K_0 and $\mathcal{K}_0$ are mutually adjoint:*

$$\mathcal{K}_0^\star=K_0 \tag{3.18}$$

and

$$K_0^\star=\mathcal{K}_0. \tag{3.19}$$

Proof. From the self-adjointness of $\mathcal{Q}_0$ it follows that $\mathcal{K}_0^\star K_0^\star=\mathcal{Q}_0^\star=\mathcal{Q}_0=K_0\mathcal{K}_0$; then $\mathcal{K}_0^\star K_0^\star K_0=K_0\mathcal{K}_0K_0$, which implies (3.18) because each of $K_0^\star K_0$ and $\mathcal{K}_0K_0$ is the identity operator on $H^0(\Gamma)$ (Lemma 3.1). Similarly, the equality $K_0^\star\mathcal{K}_0^\star K_0^\star=K_0^\star K_0\mathcal{K}_0$ gives (3.19), since $K_0^\star\mathcal{K}_0^\star=\left(\mathcal{K}_0K_0\right)^\star$ is also the identity on $H^0(\Gamma)$. □

We shall couple these facts with the simple observations made in

Lemma 3.4. *For $k=1$ and 2, let H_{k0} and H_{k+} be Hilbert spaces with $H_{k+}\subset H_{k0}$, H_{k+} dense in H_{k0}, and the natural injection map continuous; let H_{k-} be the corresponding negative space, identified as the antidual of H_{k+} with respect to H_{k0} as pivot space. Suppose that $L_0:H_{10}\to H_{20}$ is a bounded linear operator such that the restriction $L_+:=L_0\,|\,H_{1+}$ maps H_{1+} into H_{2+} and is bounded in that setting.*

Then the adjoint $L_0^\star:H_{20}\to H_{10}$ is bounded when regarded as $L_0^\star:\left\{H_{20}\subset H_{2-}\right\}\to H_{1-}$, i.e., as densely defined in H_{2-} and mapping into H_{1-}, and the bounded extension of the latter operator to all of H_{2-} is the transpose $L_+':H_{2-}\to H_{1-}$. In particular, L_+' maps H_{20} into H_{10} and

$$L_+'u_0=L_0^\star u_0\qquad\text{for each}\quad u_0\in H_{20}. \tag{3.20}$$

Proof. It is clear that the remaining assertions will follow once we have established (3.20). Accordingly, let $u_0 \in H_{20}$. Denoting the antiduality pairing on $H_{k-} \times H_{k+}$ by $\langle \cdot, \cdot \rangle_k$ and the inner product for H_{k0} by $(\cdot, \cdot)_{k0}$, we compute

$$\langle L'_+ u_0, u_1 \rangle_1 = \langle u_0, L_+ u_1 \rangle_2 = (u_0, L_0 u_1)_{0,2} = (L_0^\star u_0, u_1)_{0,1} = \langle L_0^\star u_0, u_1 \rangle_1 \qquad \text{for every} \quad u_1 \in H_{1+}.$$

Thus, (3.20) is true. □

Immediately, we get the desired results in

Lemma 3.5. *For $0 < s \leq m$ the following equalities hold:*

$$\left.\begin{aligned} K'_s U_0 &= K_0^\star U_0 = \mathcal{K}_0 U_0 \\ \mathcal{Q}'_s U_0 &= \mathcal{Q}_0^\star U_0 = \mathcal{Q}_0 U_0 \end{aligned}\right\} \qquad \textit{for each} \quad U_0 \in H^0(\square)^2, \tag{3.21}$$

and

$$\mathcal{K}'_s u_0 = \mathcal{K}_0^\star u_0 = K_0 u_0 \qquad \textit{for each} \quad u_0 \in H^0(\Gamma). \tag{3.22}$$

Proof. Apply Lemma 3.4 and (3.20) three times, successively taking L_0 there to be K_0, $\mathcal{Q}_0$, and $\mathcal{K}_0$ (with obvious identifications of the spaces), and appealing to (3.19), the selfadjointness of $\mathcal{Q}_0$, and (3.18), respectively. □

Now, if we should need to compute the action of K'_s, $\mathcal{Q}'_s$, or $\mathcal{K}'_s$ on H^0-functions, we may appeal to (3.21) or (3.22), which says that we should merely compute the corresponding action of $\mathcal{K}_0$, $\mathcal{Q}_0$, or K_0, respectively.

4. Associated integral operators for Γ and $\square$.

Recall that our goal is the replacement of an integral-operator problem set on the boundary Γ of a domain in $\mathbb{R}^3$ by an integral-operator problem set in the rectangle $\square$ in the plane. Accordingly, in this section we shall explain how an integral operator acting in spaces of functions on Γ can be expressed in terms of an integral operator acting in spaces of functions on $\square$. The operators K_s and $\mathcal{K}_s$, developed in the preceding section, are instrumental in this construction, since they provide the connections between spaces on Γ and spaces on $\square$.

We shall consider here only operators that are rather well-behaved, such as operators with kernels having at worst a weak singularity, deferring to a later development the examination of other cases involving, *e.g.*, operators generated by derivatives of double-layer potentials and other singular integral operators.

Besides the boundary Γ itself, the data determining a boundary-integral operator comprise a *kernel* E for Γ, *i.e.*, a complex-valued map $(\xi,\zeta) \mapsto E_\xi(\zeta)$ defined a.e. on $\Gamma \times \Gamma$. As our minimum standing hypothesis, we suppose that we are given a kernel E for Γ such that

$$\left.\begin{array}{l} \textit{the formula} \\ \qquad \mathcal{E}_\Gamma u(\xi) := \displaystyle\int_\Gamma E_\xi u \, d\lambda_\Gamma \quad \textit{for} \quad \textit{a.a. } \xi \in \Gamma, \quad \textit{for each} \quad u \in H^0(\Gamma) \\ \textit{generates a bounded linear operator} \quad \mathcal{E}_\Gamma : H^0(\Gamma) \to H^0(\Gamma). \end{array}\right\} \tag{H.I}$$

Recalling the definition of $H^0(\Gamma)$, the hypothesis (H.I) requires (1) that $K_0\mathcal{E}_\Gamma u$ be defined a.e. on $\square$ and belong to $H^0(\square)^2$ whenever u is a complex measurable function defined a.e. on Γ such that $K_0 u$ belongs to $H^0(\square)^2$, and (2) the existence of a positive M such that $\|K_0\mathcal{E}_\Gamma u\|_{[0,\square]} \le M\|K_0 u\|_{[0,\square]}$ holds for every such u. In various examples we may impose other conditions on the kernel, but any such hypotheses shall be always in addition to (H.I).

Since K_0 is an isometry with left inverse $\mathcal{K}_0$, the definition

$$\mathcal{E}_\square := K_0\mathcal{E}_\Gamma\mathcal{K}_0 \tag{4.1}$$

produces a bounded linear operator $\mathcal{E}_\square : H^0(\square)^2 \to H^0(\square)^2$ whose restriction to $\mathcal{R}\big(K_0\big)$ is a "copy" of $\mathcal{E}_\Gamma$ acting down on $\square$; in all of our reformulated problems, $\mathcal{E}_\square$ will figure as the replacement for $\mathcal{E}_\Gamma$. Clearly, we have

$$\mathcal{N}\big(\mathcal{Q}_0\big) = \mathcal{N}\big(\mathcal{K}_0\big) \subset \mathcal{N}\big(\mathcal{E}_\square\big) \qquad \text{and} \qquad \mathcal{R}\big(\mathcal{E}_\square\big) \subset \mathcal{R}\big(K_0\big) = \mathcal{R}\big(\mathcal{Q}_0\big). \tag{4.2}$$

Moreover, $\mathcal{E}_\Gamma$ can be recovered from $\mathcal{E}_\square$, since (4.1) and (3.13) give

$$\mathcal{E}_\Gamma = \mathcal{K}_0\mathcal{E}_\square K_0. \tag{4.3}$$

In the remainder of this section we shall examine some other useful relations between, and factorizations of, $\mathcal{E}_\Gamma$ and $\mathcal{E}_\square$, and display the explicit form of $\mathcal{E}_\square$, which is essential for a numerical implementation. We shall also consider relations between extensions of $\mathcal{E}_\Gamma$ and $\mathcal{E}_\square$ to larger spaces.

By recalling the form (3.1) of the integral over Γ and accounting for the definitions of the operators $K^{\pm}$, we can display the action of the operator $\mathcal{E}_\Gamma$:

$$\begin{aligned}\mathcal{E}_\Gamma u(\xi) &= \int_\square \left(\varphi^+ E_\xi u\right) \circ h_+^{-1}\, \mathfrak{I} h_+^{-1}\, d\lambda_2 + \int_\square \left(\varphi^- E_\xi u\right) \circ h_-^{-1}\, \mathfrak{I} h_-^{-1}\, d\lambda_2 \\ &= \int_\square \left(\sqrt{\varphi^+} E_\xi\right) \circ h_+^{-1} \sqrt{\mathfrak{I} h_+^{-1}}\, K^+ u\, d\lambda_2 + \int_\square \left(\sqrt{\varphi^-} E_\xi\right) \circ h_-^{-1} \sqrt{\mathfrak{I} h_-^{-1}}\, K^- u\, d\lambda_2, \\ &\qquad\qquad \text{for} \quad \text{a.a. } \xi \in \Gamma, \quad \text{for} \quad u \in H^0(\Gamma). \end{aligned} \tag{4.4}$$

Directly, we shall show that the recipe

$$\begin{aligned}\mathcal{E} U(\xi) &:= \int_\square \left(\sqrt{\varphi^+} E_\xi\right) \circ h_+^{-1} \sqrt{\mathfrak{I} h_+^{-1}}\, U^+\, d\lambda_2 + \int_\square \left(\sqrt{\varphi^-} E_\xi\right) \circ h_-^{-1} \sqrt{\mathfrak{I} h_-^{-1}}\, U^-\, d\lambda_2 \\ &\qquad\qquad \text{for} \quad \text{a.a. } \xi \in \Gamma, \quad \text{for} \quad U \equiv \left(U^+, U^-\right) \in H^0(\square)^2, \end{aligned} \tag{4.5}$$

produces a bounded linear operator $\mathcal{E} : H^0(\square)^2 \to H^0(\Gamma)$, indeed, that

$$\mathcal{E} = \mathcal{E}_\Gamma \mathcal{K}_0; \tag{4.6}$$

once this has been verified, we shall have from (4.4) the important factorization

$$\mathcal{E}_\Gamma = \mathcal{E} K_0. \tag{4.7}$$

We shall prove (4.6) by the direct computation of $\mathcal{E}_\Gamma \mathcal{K}_0 U$ for $U \in H^0(\square)^2$, to show that the expression appearing on the right in (4.5) results. Choosing such a $U \equiv \left(U^+, U^-\right)$ and referring to (3.8) and (3.9) for the definition of $\mathcal{K}_0 U$, we note that the two terms on the right in (3.8) represent functions with supports in Γ_+ and Γ_-, respectively. Therefore, we may use the forms given in (3.2) to get, for a.a. $\xi \in \Gamma$,

$$\begin{aligned}\mathcal{E}_\Gamma \mathcal{K}_0 U(\xi) &= \int_{h_+(\Gamma_+)} E_\xi \circ h_+^{-1} \left[\sqrt{\frac{\varphi^+}{\mathfrak{I} h_+^{-1} \circ h_+}}\, U^+ \circ h_+\right]_{\Gamma_+} \circ h_+^{-1}\, \mathfrak{I} h_+^{-1}\, d\lambda_2 \\ &\qquad + \int_{h_-(\Gamma_-)} E_\xi \circ h_-^{-1} \left[\sqrt{\frac{\varphi^-}{\mathfrak{I} h_-^{-1} \circ h_-}}\, U^- \circ h_-\right]_{\Gamma_-} \circ h_-^{-1}\, \mathfrak{I} h_-^{-1}\, d\lambda_2 \\ &= \int_{h_+(\operatorname{supp}\varphi^+)} E_\xi \circ h_+^{-1} \left\{\sqrt{\frac{\varphi^+}{\mathfrak{I} h_+^{-1} \circ h_+}}\, U^+ \circ h_+\right\} \circ h_+^{-1}\, \mathfrak{I} h_+^{-1}\, d\lambda_2 \\ &\qquad + \int_{h_-(\operatorname{supp}\varphi^-)} E_\xi \circ h_-^{-1} \left\{\sqrt{\frac{\varphi^-}{\mathfrak{I} h_-^{-1} \circ h_-}}\, U^- \circ h_-\right\} \circ h_-^{-1}\, \mathfrak{I} h_-^{-1}\, d\lambda_2, \end{aligned}$$

the latter equality holding because $h_+(\operatorname{supp}\varphi^+)$ and $h_-(\operatorname{supp}\varphi^-)$ lie in $\square$; for the same reason, these sets of integration may each be replaced by $\square$. Therefore, it is now easy to see that this coincides with the expression on the right in (4.5), so that (4.6) is correct.

Other relations between $\mathcal{E}_\Gamma$, $\mathcal{E}_\square$, and $\mathcal{E}$ are useful in certain contexts. For example, from (4.1) and (4.6) we find the counterpart of (4.7):

$$\mathcal{E}_\square = K_0 \mathcal{E}; \tag{4.8}$$

in turn, the latter gives $\mathcal{K}_0\mathcal{E}_\square = \mathcal{K}_0 K_0 \mathcal{E} = \mathcal{E}$, so

$$\mathcal{E} = \mathcal{K}_0 \mathcal{E}_\square, \tag{4.9}$$

which corresponds to (4.6). Now (4.7) and (4.8) yield

$$K_0 \mathcal{E}_\Gamma = K_0 \mathcal{E} K_0 = \mathcal{E}_\square K_0. \tag{4.10}$$

To see how the operator $\mathcal{E}_\square$ is determined explicitly by the kernel E, the coördinate systems for Γ, and the partition of unity, let us display the "matrix form" of $\mathcal{E}_\square$. For this, it appears easiest to proceed from the representation in (4.8), which leads, after some scratchwork, to

$$\mathcal{E}_\square U = \begin{pmatrix} \mathcal{E}_\square^{++} & \mathcal{E}_\square^{+-} \\ \mathcal{E}_\square^{-+} & \mathcal{E}_\square^{--} \end{pmatrix} \begin{pmatrix} U^+ \\ U^- \end{pmatrix} \qquad \text{for} \quad U = \big(U^+, U^-\big) \in H^0(\square)^2, \tag{4.11}$$

in which the elemental operators bounded in $H^0(\square)$ are given by

$$\mathcal{E}_\square^{\mu\eta}\psi(x) := \sqrt{\varphi^\mu \circ h_\mu^{-1}(x)\, \mathcal{J}h_\mu^{-1}(x)} \int_\square E_{h_\mu^{-1}(x)} \circ h_\eta^{-1} \sqrt{\varphi^\eta \circ h_\eta^{-1}\, \mathcal{J}h_\eta^{-1}}\, \psi \, d\lambda_2$$
$$\text{for} \quad \text{a.a. } x \in \square, \quad \psi \in H^0(\square), \quad \text{and} \quad \mu = \pm,\, \eta = \pm. \tag{4.12}$$

Remark 4.1. From the explicit representations given, one notes that the operators $\mathcal{E}_\Gamma$ and $\mathcal{E}_\square$ will inherit certain symmetry properties from symmetry properties of the kernel E. This is in agreement with (4.1) and (4.3), which, with the help of Corollary 3.1, show already that $\mathcal{E}_\Gamma$ is self-adjoint in $H^0(\Gamma)$ iff $\mathcal{E}_\square$ is self-adjoint in $H^0(\square)^2$.

Remark 4.2. In view of (4.1) and (4.3), it is clear that the hypothesis (H.I) can be replaced by the equivalent condition

$$\text{(4.12) } \textit{defines bounded operators} \quad \mathcal{E}_\square^{\mu\eta} : H^0(\square) \to H^0(\square) \quad \textit{for} \quad \mu = \pm,\, \eta = \pm, \tag{H.I$'$}$$

necessary and sufficient for $\mathcal{E}_\square : H^0(\square)^2 \to H^0(\square)^2$ as in (4.11) to be well defined and bounded. It may be useful to keep this in mind, just because it is usually easier to discern and/or analyze the properties of integral operators associated with $\square$ than those associated with Γ. In fact, the developments here suggest another way in which to attack the matter of establishing mapping properties of boundary-integral operators, *viz.*, by transferring the questions to the corresponding ones for operators in spaces of functions on $\square$.

Finally, we shall prepare some mapping results that find application in the study of the example boundary-operator problems discussed in the next sections.

Lemma 4.1. *Let* (H.I) *hold, so that the operator* $\mathcal{E}_\Gamma : H^0(\Gamma) \to H^0(\Gamma)$ *constructed from the kernel* E *is bounded. Let the bounded operator* $\mathcal{E}_\square : H^0(\square)^2 \to H^0(\square)^2$ *be defined as in* (4.1).

(*i.*) $\mathcal{E}_\square$ *is compact iff* $\mathcal{E}_\Gamma$ *is compact.*

(*ii.*) *Suppose that* $0 < s \le m$. *Then* $\mathcal{E}_\Gamma$ *maps* $H^0(\Gamma)$ *into* $H^s(\Gamma)$ *iff* $\mathcal{E}_\square$ *maps* $H^0(\square)^2$ $\big(= \mathcal{H}^0(\square)^2\big)$ *into* $\mathcal{H}^s(\square)^2$.

(*iii.*) *Let* $0 < s \le m$ *and suppose that* $\mathcal{E}_\Gamma$ *maps* $H^0(\Gamma)$ *into* $H^s(\Gamma)$. *Then the densely defined operator* $\mathcal{E}_\Gamma : \big\{L_2(\Gamma) \subset H^{-s}(\Gamma)\big\} \to H^s(\Gamma)$ *is bounded iff the densely defined operator* $\mathcal{E}_\square : \big\{L_2(\square)^2 \subset \mathcal{H}^{-s}(\square)^2\big\} \to \mathcal{H}^s(\square)^2$ *is bounded. Moreover, when these operators are*

bounded, their bounded extensions $\mathcal{E}_\Gamma^s$ *and* $\mathcal{E}_\square^s$ *to all of* $H^{-s}(\Gamma)$ *and* $\mathcal{H}^{-s}(\square)^2$*, respectively, are connected by*

$$\mathcal{E}_\Gamma^s = \mathcal{K}_s \mathcal{E}_\square^s \mathcal{K}_s', \tag{4.13}$$

$$\mathcal{E}_\square^s = K_s \mathcal{E}_\Gamma^s K_s', \tag{4.14}$$

and

$$K_s \mathcal{E}_\Gamma^s = \mathcal{E}_\square^s \mathcal{K}_s'. \tag{4.15}$$

(Under the various hypotheses of the lemma, corresponding statements can be made about $\mathcal{E}$, but we have no need for those here, and so omit them.)

Proof. (*i*). This follows immediately from (4.1) and (4.3).

(*ii*). Suppose that $\mathcal{E}_\Gamma$ maps $H^0(\Gamma)$ into $H^s(\Gamma)$, which means that $K_0 \mathcal{E}_\Gamma u \in H^s(\square)^2$ whenever $u \in H^0(\Gamma)$. Let $U \in H^0(\square)^2$: then $\mathcal{K}_0 U \in H^0(\Gamma)$, so that $\mathcal{E}_\Gamma \mathcal{K}_0 U \in H^s(\Gamma)$, or, as noted, $K_0 \mathcal{E}_\Gamma \mathcal{K}_0 U \in H^s(\square)^2$; the support of $K_0 \mathcal{E}_\Gamma \mathcal{K}_0 U$ is in the (open) rectangle $\square$, so it belongs to $H_0^s(\square)^2 \subset \mathcal{H}^s(\square)^2$. Therefore, by (4.1), $\mathcal{E}_\square$ maps $H^0(\square)^2$ into $\mathcal{H}^s(\square)^2$. The reversed implication is proven in a similar manner, by relying on (4.3).

(*iii*). Let the operator $\mathcal{E}_\Gamma : \{L_2(\Gamma) \subset H^{-s}(\Gamma)\} \to H^s(\Gamma)$ be bounded and denote its bounded extension to all of $H^{-s}(\Gamma)$ by $\mathcal{E}_\Gamma^s$; by (*ii*), we already know that $\mathcal{E}_\square$ maps $L_2(\square)^2$ into $\mathcal{H}^s(\square)^2$. Clearly, $K_s \mathcal{E}_\Gamma^s K_s'$ is then a bounded operator from $\mathcal{H}^{-s}(\square)^2$ into $\mathcal{H}^s(\square)^2$ which, by (3.21.1) and (4.1), is an extension of $\mathcal{E}_\square : \{L_2(\square)^2 \subset \mathcal{H}^{-s}(\square)^2\} \to \mathcal{H}^s(\square)^2$. Therefore, the latter is bounded and its bounded extension is given as in (4.14). This proves one direction of the first statement of (*iii*) and (4.14), while the other direction and (4.13) are proven similarly, by recalling (3.21.2) and (4.3). Finally, (4.15) follows by extending the equality in (4.10), again with the help of (3.21.1). ∎

Remark 4.3. In the reformulation examples to be examined here, an isomorphism condition will always be hypothesized for the operator on Γ, not for the operator on $\square$. This reflects the usual situation in the applications, where the property of bijectivity will have been established for the boundary operator by some other argument, *e.g.*, appeal to a uniqueness theorem available for an underlying boundary-value problem in partial differential equations.

5. Reformulation of boundary-operator problems: examples.

With the developments of the previous paragraphs, we can begin to examine the reformulation of boundary-operator problems. That is, we want to replace an original well-posed problem set in some space(s) of functions on Γ by a new problem well posed in some space(s) on $\square$, in such a way that the solution of the original problem can be easily constructed or approximated once we have constructed or approximated the solution of the new problem. The arguments may be complicated by the fact that an isomorphism between Sobolev spaces on Γ need not correspond to an isomorphism between the corresponding Sobolev spaces on $\square$. Thus, in certain cases one must augment the "directly transformed" operator to achieve a new problem involving an isomorphism.

We shall illustrate the main ideas by examining two settings, corresponding to the classical "second-kind" and "first-kind" cases. Throughout, the basic hypothesis (H.I) is assumed and the operator $\mathcal{E}_\square$ is constructed from $\mathcal{E}_\Gamma$ as in (4.1).

5.1. Example: a problem of the second kind.

Here, we consider the frequently occurring situation in which

$$\mathcal{E}_\Gamma : H^0(\Gamma) \to H^0(\Gamma) \quad \textit{is compact} \qquad \textit{and} \qquad I + \mathcal{E}_\Gamma \quad \textit{is injective.} \tag{H.II}$$

Then $I + \mathcal{E}_\Gamma : H^0(\Gamma) \to H^0(\Gamma)$ is an isomorphism and the basic operator problem is well posed:

$$\textit{given} \quad g \in H^0(\Gamma), \quad \textit{determine} \quad u_g \in H^0(\Gamma) \quad \textit{such that} \qquad \big(I + \mathcal{E}_\Gamma\big) u_g = g. \tag{P.II}_\Gamma$$

In this case, $(\text{P.II})_\Gamma$ can be directly replaced by a well-posed problem in the product space $H^0(\square)^2$. In fact, let $g \in H^0(\Gamma)$. Since K_0 is an isometry from $H^0(\Gamma)$ into $H^0(\square)^2$, the equality $\big(I+\mathcal{E}_\Gamma\big)u_g = g$ holds iff $K_0\big(I + \mathcal{E}_\Gamma\big)u_g = K_0 g$ holds, or, by noting from (4.10) that the two operators are connected by

$$K_0\big(I + \mathcal{E}_\Gamma\big) = \big(I + \mathcal{E}_\square\big)K_0, \tag{5.1}$$

iff $\big(I + \mathcal{E}_\square\big)\widetilde{U}_g = G_0$ holds, in which we have written $\widetilde{U}_g := K_0 u_g$ and $G_0 := K_0 g$.

With this motivation, we pose a problem that we suspect will serve as a replacement for the original:

$$\textit{given} \quad G \in H^0(\square)^2, \quad \textit{determine} \quad U_G \in H^0(\square)^2 \quad \textit{such that} \qquad \big(I + \mathcal{E}_\square\big) U_G = G. \tag{P.II}_\square$$

Let us check that this new problem is well posed, *i.e.*, that the operator $I + \mathcal{E}_\square$ is an isomorphism in $H^0(\square)^2$. Since the compactness of $\mathcal{E}_\square$ follows from that of $\mathcal{E}_\Gamma$, we need show only that $I + \mathcal{E}_\square$ is injective. For this, let $U \in H^0(\square)^2$ with $\big(I + \mathcal{E}_\square\big)U = 0$. By (4.8) we have then $U = -\mathcal{E}_\square U = -K_0 \mathcal{E} U$, so (5.1) gives $K_0\big(I + \mathcal{E}_\Gamma\big)\big(-\mathcal{E}U\big) = \big(I + \mathcal{E}_\square\big)U = 0$, which implies $\big(I + \mathcal{E}_\Gamma\big)\big(\mathcal{E}U\big) = 0$ and so also $\mathcal{E}U = 0$. Therefore, $U = -K_0 \mathcal{E} U = 0$, whence we conclude that $I + \mathcal{E}_\square$ is indeed injective.

Thus, the well-posedness of $(\text{P.II})_\Gamma$ implies that of $(\text{P.II})_\square$. Moreover, if the data g and G are related by $G = K_0 g$, (5.1) shows that the respective solutions u_g and U_G must be connected by $U_G = K_0 u_g$; then also $u_g = \mathcal{K}_0 U_G$.

Now an approximation-of-solution scheme for $(\text{P.II})_\Gamma$ can be derived from one for $(\text{P.II})_\square$. That is, if we take G to be $K_0 g$ and use, say, a Galerkin procedure to generate a sequence $\big(U_G^N\big)_{N=N_0}^\infty$ converging in $H^0(\square)^2$ to the solution U_G of $(\text{P.II})_\square$, then $\big(\mathcal{K}_0 U_G^N\big)_{N=N_0}^\infty$ will provide a sequence from $H^0(\Gamma)$ converging in the latter space to $\mathcal{K}_0 U_G = u_g$.

In this second-kind case, under the hypothesis that the original operator is injective (and so an isomorphism), the reformulated problem comes out directly as well posed, and there is no need to "augment" it. Moreover, in view of the nice form of the operator $I + \mathcal{E}_\square$, the Bubnov-Galerkin method (in which the trial- and test-families coincide) is applicable to $(\text{P.II})_\square$; in such an application, for convergence we need only ensure that the usual requirement of ultimate denseness is fulfilled by the coördinate subspaces (here, in $H^0(\square)^2$), and, for numerical stability, that the trial/test-family is well conditioned in $H^0(\square)^2$. (In Section 6 we outline the implementation of the Bubnov-Galerkin procedure in a product space $H \times H$ and indicate how one can generate an appropriate trial/test-family for the product from such a family in H.) All of this is to be contrasted with the situation in the upcoming case of the "first-kind" problem.

5.2. Example: a problem of the first kind.

Let $0 < s \leq m$. In place of (H.II), now we impose two conditions (H.III) and (H.IV) in addition to (H.I):

$$\left.\begin{array}{l} \mathcal{E}_\Gamma \quad \textit{maps} \quad L_2(\Gamma) \quad \textit{into} \quad H^s(\Gamma) \quad \textit{and the densely defined operator} \\ \qquad\qquad \mathcal{E}_\Gamma : \{L_2(\Gamma) \subset H^{-s}(\Gamma)\} \to H^s(\Gamma) \\ \textit{is bounded.} \end{array}\right\} \qquad \text{(H.III)}$$

The bounded extension of the operator in (H.III) to all of $H^{-s}(\Gamma)$ we denote by $\mathcal{E}^s_\Gamma : H^{-s}(\Gamma) \to H^s(\Gamma)$. Now, according to Lemma 4.1, $\mathcal{E}_\square$ maps $L^2(\square)^2$ into $\mathcal{H}^s(\square)^2$ and the densely defined operator $\mathcal{E}_\square : \{L_2(\square)^2 \subset \mathcal{H}^{-s}(\square)^2\} \to \mathcal{H}^s(\square)^2$ is also bounded; its bounded extension to all of $\mathcal{H}^{-s}(\square)^2$ is denoted by $\mathcal{E}^s_\square$. Relations between $\mathcal{E}^s_\Gamma$ and $\mathcal{E}^s_\square$ are given in (4.13–15). We shall also suppose that

$$\mathcal{E}^s_\Gamma : H^{-s}(\Gamma) \to H^s(\Gamma) \quad \textit{is an isomorphism} \qquad \text{(H.IV)}$$

(recall Remark 4.3). Thus, the basic operator problem for $\mathcal{E}^s_\Gamma$ is well posed:

$$\textit{given} \quad g \in H^s(\Gamma), \quad \textit{determine} \quad u_g \in H^{-s}(\Gamma) \quad \textit{such that} \quad \mathcal{E}^s_\Gamma u_g = g. \qquad (\text{P.I})_\Gamma$$

This is the problem of primary interest; as in the previous example, we want to show

(1) how to replace $(\text{P.I})_\Gamma$ by a new well-posed problem $(\text{P.I})_\square$ involving $\mathcal{E}^s_\square$, whose solution will provide for construction of the solution of the original problem, and

(2) how to generate convergent and numerically stable approximations to the unique solution of the original problem $(\text{P.I})_\Gamma$ by the formulation and execution of a Galerkin procedure for the new problem $(\text{P.I})_\square$.

In the remainder of this section we are concerned with goal (1); the later sections are devoted to goal (2).

For clarity, we shall work our way in several steps to a replacement problem for $(\text{P.I})_\Gamma$. Upon recalling the isometric property of K_s, we observe first that the equality $\mathcal{E}^s_\Gamma u_g = g$ obviously holds iff

$$K_s \mathcal{E}^s_\Gamma u_g = K_s g,$$

or, since $K_s \mathcal{E}^s_\Gamma = \mathcal{E}^s_\square \mathcal{K}'_s$ by (4.15),

$$\mathcal{E}^s_\square \mathcal{K}'_s u_g = K_s g.$$

Since $\mathcal{R}(\mathcal{Q}_s') = \mathcal{R}(\mathcal{K}_s')$ (Lemma 3.2), this gives us an "intermediate" reformulation appearing as

$$\textit{given} \quad g \in H^s(\Gamma), \quad \textit{determine} \quad U_g \in \mathcal{R}(\mathcal{Q}_s') \quad \textit{such that} \qquad \mathcal{E}_\square^s U_g = K_s g; \tag{P.I)$'_\square$}$$

by recalling that $\mathcal{K}_s' K_s'$ is the identity on $\mathcal{R}(\mathcal{Q}_s')$, we conclude that the solution of $(\text{P.I})_\Gamma$ should be subsequently constructed from the solution U_g of $(\text{P.I})'_\square$ by taking $u_g := K_s' U_g$.

We summarize this first step—and ensure that we can later reverse the step—in

Lemma 5.1. *Let hypotheses* (H.I), (H.III), *and* (H.IV) *hold.*

- (*i.*) $\mathcal{E}_\square^s \,|\, \mathcal{R}(\mathcal{Q}_s')$ *is an isomorphism of* $\mathcal{R}(\mathcal{Q}_s')$ *onto* $\mathcal{R}(\mathcal{Q}_s)$, *so that problem* $(\text{P.I})'_\square$ *is uniquely solvable.*
- (*ii.*) *The respective (unique) solutions* u_g *and* U_g *of* $(\text{P.I})_\Gamma$ *and* $(\text{P.I})'_\square$ *are connected by* $u_g = K_s' U_g$ *and* $U_g = \mathcal{K}_s' u_g$. *Moreover, if* $\big(U_g^N\big)_{N=N_0}^\infty$ *is a sequence from* $H^0(\square)^2$ *converging to* U_g *in* $\mathcal{H}^{-s}(\square)^2$, *then the "computable" sequence* $\big(\mathcal{K}_0 U_g^N\big)_{N=N_0}^\infty$ *converges to the desired* u_g *in* $H^{-s}(\Gamma)$.

Proof. (*i*). The equality $\mathcal{E}_\square^s = K_s \mathcal{E}_\Gamma^s K_s'$ of (4.14) implies that $\mathcal{E}_\square^s \,|\, \mathcal{R}(\mathcal{Q}_s')$ maps $\mathcal{R}(\mathcal{Q}_s')$ bijectively onto $\mathcal{R}(\mathcal{Q}_s)$. In fact, we see first that $\mathcal{E}_\Gamma^s K_s'$ carries $\mathcal{R}(\mathcal{Q}_s')$ bijectively onto $H^s(\Gamma)$, since $K_s' \mathcal{K}_s'$ is the identity map on $H^{-s}(\Gamma)$ (Lemma 3.2), so that K_s' is an injection from $\mathcal{K}_s' H^{-s}(\Gamma) = \mathcal{R}\big(\mathcal{K}_s'\big) = \mathcal{R}(\mathcal{Q}_s')$ onto $H^{-s}(\Gamma)$, while we have supposed in (H.IV) that $\mathcal{E}_\Gamma^s$ is bijective from $H^{-s}(\Gamma)$ onto $H^s(\Gamma)$. Finally, K_s obviously takes $H^s(\Gamma)$ bijectively onto $\mathcal{R}(K_s) = \mathcal{R}(\mathcal{Q}_s)$, which shows that the assertion of (*i*) is correct.

(*ii*). With $g \in H^s(\Gamma)$, let $u_g \in H^{-s}(\Gamma)$ be the solution of $(\text{P.I})_\Gamma$. Then $\mathcal{K}_s' u_g \in \mathcal{H}^{-s}(\square)^2$ satisfies $\mathcal{E}_\square^s \mathcal{K}_s' u_g = K_s \mathcal{E}_\Gamma^s u_g = K_s g$, so we must have $U_g = \mathcal{K}_s' u_g$. By inverting the latter relation, we get $u_g = K_s' \mathcal{K}_s' u_g = K_s' U_g$, showing that the two solutions are related as claimed. Finally, let $\big(U_g^N\big)_{N=N_0}^\infty$ be a sequence from $H^0(\square)^2$ that converges to U_g in $\mathcal{H}^{-s}(\square)^2$. On the one hand, it is then clear that $\big(K_s' U_g^N\big)_{N=N_0}^\infty$ converges to $K_s' U_g = u_g$ in $H^{-s}(\Gamma)$, while, on the other, we have $K_s' U_g^N = \mathcal{K}_0 U_g^N$ for each N by (3.21.1), so the final assertion is true. ∎

Next, we must replace $(\text{P.I})'_\square$ by a well-posed reformulation set in all of $\mathcal{H}^{-s}(\square)^2$. Since we have $\mathcal{R}\big(\mathcal{Q}_s'\big) = \mathcal{N}\big(I - \mathcal{Q}_s'\big)$, it is clear that such a reformulation is afforded by

$$\left.\begin{aligned} &\textit{given} \quad g \in H^s(\Gamma), \quad \textit{determine} \quad U_g \in \mathcal{H}^{-s}(\square)^2 \quad \textit{satisfying} \\ &\qquad\qquad \mathsf{J}_{\square_s} \mathcal{E}_\square^s U_g = \mathsf{J}_{\square_s} K_s g, \\ &\textit{and} \\ &\qquad\qquad \big(I - \mathcal{Q}_s'\big) U_g = 0; \end{aligned}\right\} \tag{P.I)$''_\square$}$$

we have inserted the antiduality operator $\mathsf{J}_{\square_s} : \mathcal{H}^s(\square)^2 \to \mathcal{H}^{-s}(\square)^2$ here to get the operator $\mathsf{J}_{\square_s} \mathcal{E}_\square^s$ acting entirely in $\mathcal{H}^{-s}(\square)^2$. However, $(\text{P.I})''_\square$ is still not in a form convenient for application of a Galerkin approximation-of-solution procedure. To achieve such a form, we want an equivalent problem involving an isomorphism acting in all of $\mathcal{H}^{-s}(\square)^2$. Evidently, for this it is reasonable first to inquire concerning the properties of operators constructed as linear combinations of the two operators figuring in $(\text{P.I})''_\square$. In fact, we can prove

Theorem 5.1. *Let hypotheses* (H.I), (H.III), *and* (H.IV) *hold. Suppose that* α *is a nonzero complex number. Then* $\mathsf{J}_{\square_s} \mathcal{E}_\square^s + \alpha\big(I - \mathcal{Q}_s'\big)$ *is an isomorphism of* $\mathcal{H}^{-s}(\square)^2$ *onto itself. Moreover, the operator maps* $\mathcal{R}\big(\mathcal{Q}_s'\big)$ *onto* $\mathcal{R}\big(\mathsf{J}_{\square_s} \mathcal{Q}_s\big)$, *so that the unique* $U \in \mathcal{H}^{-s}(\square)^2$ *such that* $\mathsf{J}_{\square_s} \mathcal{E}_\square^s U + \alpha\big(I - \mathcal{Q}_s'\big) U = V$ *lies in* $\mathcal{R}\big(\mathcal{Q}_s'\big)$ *whenever* $V \in \mathcal{R}\big(\mathsf{J}_{\square_s} \mathcal{Q}_s\big)$.

Proof. We suppose that we have proven the following statement and show that the proof of the theorem follows; the proof of the lemma we defer until after the argument.

Lemma 5.2. *$\mathcal{H}^{-s}(\Box)^2$ has the direct-sum decomposition*

$$\mathcal{H}^{-s}(\Box)^2 = \mathcal{R}\big(\mathsf{J}_{\Box s}\mathcal{Q}_s\big) \dot{+} \mathcal{R}\big(I - \mathcal{Q}'_s\big).$$

We show first that $\mathsf{J}_{\Box s}\mathcal{E}^s_\Box + \alpha\big(I - \mathcal{Q}'_s\big)$ is injective. Indeed, suppose that $U \in \mathcal{H}^{-s}(\Box)^2$ and $\mathsf{J}_{\Box s}\mathcal{E}^s_\Box U + \alpha\big(I - \mathcal{Q}'_s\big)U = 0$. It is clear from (4.15) that the operator $\mathcal{E}^s_\Box$ has range in $\mathcal{R}\big(K_s\big) = \mathcal{R}\big(\mathcal{Q}_s\big)$, so $\mathsf{J}_{\Box s}\mathcal{E}^s_\Box U \in \mathcal{R}\big(\mathsf{J}_{\Box s}\mathcal{Q}_s\big)$; since the decomposition of Lemma 5.2 is a direct sum, we can conclude that both $\mathsf{J}_{\Box s}\mathcal{E}^s_\Box U = 0$ and $\alpha\big(I - \mathcal{Q}'_s\big)U = 0$ must hold. The latter equality implies that $U \in \mathcal{N}\big(I - \mathcal{Q}'_s\big) = \mathcal{R}\big(\mathcal{Q}'_s\big)$, so the former shows that $U = 0$, since we already know that the restriction of $\mathcal{E}^s_\Box$ to $\mathcal{R}\big(\mathcal{Q}'_s\big)$ is injective. Therefore, the operator is injective.

To prove that the range of $\mathsf{J}_{\Box s}\mathcal{E}^s_\Box + \alpha\big(I - \mathcal{Q}'_s\big)$ is all of $\mathcal{H}^{-s}(\Box)^2$, we choose any $V \in \mathcal{H}^{-s}(\Box)^2$ and use the decomposition of Lemma 5.2 to write $V = \mathsf{J}_{\Box s}\mathcal{Q}_s V_s + \big(I - \mathcal{Q}'_s\big)V_{-s}$ for some $V_s \in \mathcal{H}^s(\Box)^2$ and $V_{-s} \in \mathcal{H}^{-s}(\Box)^2$. By the mapping property of $\mathcal{E}^s_\Box$ already established in Lemma 5.1.*i*, there exists a unique $\widetilde{U} \in \mathcal{R}\big(\mathcal{Q}'_s\big)$ such that

$$\mathcal{E}^s_\Box \widetilde{U} = \mathcal{Q}_s V_s - \frac{1}{\alpha}\mathcal{E}^s_\Box V_{-s}.$$

Upon setting $U := \widetilde{U} + \frac{1}{\alpha}V_{-s}$ and noting that $\big(I - \mathcal{Q}'_s\big)\widetilde{U} = 0$, we get

$$\mathsf{J}_{\Box s}\mathcal{E}^s_\Box U + \alpha\big(I - \mathcal{Q}'_s\big)U = \mathsf{J}_{\Box s}\mathcal{Q}_s V_s + \big(I - \mathcal{Q}'_s\big)V_{-s} = V,$$

whence we conclude that the operator is surjective.

Finally, Lemma 5.1.*i* certainly implies that $\mathsf{J}_{\Box s}\mathcal{E}^s_\Box + \alpha\big(I - \mathcal{Q}'_s\big)$ maps $\mathcal{R}\big(\mathcal{Q}'_s\big)$ onto $\mathcal{R}\big(\mathsf{J}_{\Box s}\mathcal{Q}_s\big)$, whence the final statement of the theorem is an obvious consequence. □

Returning for the proof of Lemma 5.2, it is convenient first to point out the following elementary result about geometry in Hilbert space; it seems that the statement is not recorded in the standard texts, so we sketch a proof.

Lemma 5.3. *Let the Hilbert space H have the direct-sum decomposition $H = M \dot{+} N$ into the sum of the (closed) subspaces M and N; let Q_M denote the associated operator of projection onto M along N. If $\widetilde{M}$ is a third subspace of H that is mapped bijectively by Q_M onto M, then H also has the direct-sum decomposition $H = \widetilde{M} \dot{+} N$.*

Proof. Since the restriction $Q_M \,\big|\widetilde{M} : \widetilde{M} \to M$ is invertible we can define the operator $Q_{\widetilde{M}}$ acting in H by $Q_{\widetilde{M}} := \big(Q_M \,\big|\widetilde{M}\big)^{-1} Q_M$. A short computation shows that $Q^2_{\widetilde{M}} = Q_{\widetilde{M}}$, so $Q_{\widetilde{M}}$ is a (bounded) projector in H. It is also clear that the range $\mathcal{R}\big(Q_{\widetilde{M}}\big) = \widetilde{M}$ and the null space $\mathcal{N}\big(Q_{\widetilde{M}}\big) = N$, so we can show in the usual way that $\widetilde{M} \cap N = \{0\}$ and $H = \widetilde{M} + N$. □

Finally, we can give the

Proof of Lemma 5.2. According to Lemma 5.3, to establish the direct-sum decomposition claimed for $\mathcal{H}^{-s}(\Box)^2$ we need only verify that $\mathcal{Q}'_s$ maps $\mathcal{R}\big(\mathsf{J}_{\Box s}\mathcal{Q}_s\big)$ bijectively onto $\mathcal{R}\big(\mathcal{Q}'_s\big)$, since we already have $\mathcal{H}^{-s}(\Box)^2 = \mathcal{R}\big(\mathcal{Q}'_s\big) \dot{+} \mathcal{R}\big(I - \mathcal{Q}'_s\big)$. Let us first check that the restriction of $\mathcal{Q}'_s$ to $\mathcal{R}\big(\mathsf{J}_{\Box s}\mathcal{Q}_s\big)$ is injective: if $U \in \mathcal{H}^s(\Box)^2$ and $\mathcal{Q}'_s\mathsf{J}_{\Box s}\mathcal{Q}_s U = 0$ we get

$$\big(\mathcal{Q}_s U, \mathcal{Q}_s U\big)_{[s,\Box]} = \big\langle \mathsf{J}_{\Box s}\mathcal{Q}_s U, \mathcal{Q}_s U\big\rangle_{[\Box]} = \big\langle \mathcal{Q}'_s\mathsf{J}_{\Box s}\mathcal{Q}_s U, U\big\rangle_{[\Box]} = 0.$$

Thus, $\mathcal{Q}_s U = 0$, which implies the injectivity.

Finally, to show that $\mathcal{Q}_s'$ sends $\mathcal{R}\big(\mathsf{J}_{\square_s}\mathcal{Q}_s\big)$ onto $\mathcal{R}\big(\mathcal{Q}_s'\big)$, we observe first that $\mathcal{R}\big(\mathcal{Q}_s^\star\mathcal{Q}_s\big) = \mathcal{R}\big(\mathcal{Q}_s^\star\big)$, since $\mathcal{Q}_s$ has closed range. Therefore,

$$\mathcal{R}\big(\mathsf{J}_{\square_s}\mathcal{Q}_s^\star\mathsf{J}_{\square_s}^{-1}\mathsf{J}_{\square_s}\mathcal{Q}_s\big) = \mathcal{R}\big(\mathsf{J}_{\square_s}\mathcal{Q}_s^\star\big) = \mathcal{R}\big(\mathsf{J}_{\square_s}\mathcal{Q}_s^\star\mathsf{J}_{\square_s}^{-1}\big);$$

since $\mathcal{Q}_s' = \mathsf{J}_{\square_s}\mathcal{Q}_s^\star\mathsf{J}_{\square_s}^{-1}$, this just says that $\mathcal{R}\big(\mathcal{Q}_s'\mathsf{J}_{\square_s}\mathcal{Q}_s\big) = \mathcal{R}\big(\mathcal{Q}_s'\big)$, as we were to show. □

With the help of Theorem 5.1, it is now easy to see that the following operator problem is equivalent to $(\mathrm{P.I})_\square''$, and therefore also a replacement for $(\mathrm{P.I})_\Gamma$ (in the sense described); here, the "coupling constant" α is nonzero and complex:

$$\textit{given}\quad g \in H^s(\Gamma),\quad \textit{determine}\quad U_g \in \mathcal{H}^{-s}(\square)^2 \textit{ such that}\qquad \mathsf{J}_{\square_s}\mathcal{E}_\square^s U_g + \alpha\big(I - \mathcal{Q}_s'\big)U_g = \mathsf{J}_{\square_s}K_s g. \tag{$\mathrm{P.I})_\square$}$$

Moreover, Lemma 5.1.*ii* remains true with "$(\mathrm{P.I})_\square$" replacing "$(\mathrm{P.I})_\square'$" in its statement, *i.e.*, we are still to follow the prescription laid out there when we want to return from the solution U_g (or from an approximate solution U_g^N) of $(\mathrm{P.I})_\square$ to the solution u_g (or, respectively, to an approximate solution u_g^N) of $(\mathrm{P.I})_\Gamma$.

We settle on $(\mathrm{P.I})_\square$ as our replacement for $(\mathrm{P.I})_\Gamma$. Next, we want to show that, under reasonable conditions on the operator $\mathcal{E}_\Gamma^s$ implying that the Bubnov-Galerkin method would be applicable for convergent approximation of the solution of problem $(\mathrm{P.I})_\Gamma$, the number α can be selected to ensure that the Bubnov-Galerkin method is also applicable for convergent approximation of the solution of problem $(\mathrm{P.I})_\square$. Specifically, we shall require the following property of $\mathcal{E}_\Gamma^s$, which is frequently fulfilled in applications:

$$\left.\begin{array}{l} \mathcal{E}_\Gamma^s : H^{-s}(\Gamma) \to H^s(\Gamma) \textit{ has a decomposition} \\ \qquad\qquad \mathcal{E}_\Gamma^s = \mathcal{D}_\Gamma^s + \mathcal{C}_\Gamma^s \\ \textit{in which}\quad \mathcal{C}_\Gamma^s \quad \textit{is compact}\quad \textit{and}\quad \mathcal{D}_\Gamma^s \quad \textit{is definite:}\quad \exists\ \beta_s > 0 \quad \textit{such that} \\ \qquad\qquad \mathrm{Re}\,\langle u, \mathcal{D}_\Gamma^s u\rangle_\Gamma \geq \beta_s \|u\|_{-s,\Gamma}^2 \qquad \text{for every} \quad u \in H^{-s}(\Gamma). \end{array}\right\} \tag{H.V}$$

Now we can establish the coercivity condition (5.2) of the following statement, at least when α is sufficiently small and positive; it is easy to check that this condition guarantees the convergence of the Bubnov-Galerkin method for $(\mathrm{P.I})_\square$, using any family of trial/test subspaces that is ultimately dense in $\mathcal{H}^{-s}(\square)^2$.

Proposition 5.1. *Let hypotheses* (H.I), (H.III), (H.IV), *and* (H.V) *hold. For any sufficiently small positive* α, *there exists* $\gamma_s > 0$ *such that*

$$\mathrm{Re}\,\big(\mathsf{J}_{\square_s}\mathcal{E}_\square^s U + \alpha(I - \mathcal{Q}_s')U, U\big)_{[-s,\square]} \geq \gamma_s \|U\|_{[-s,\square]}^2 \qquad \textit{for every}\quad U \in \mathcal{H}^{-s}(\square)^2. \tag{5.2}$$

Proof. By (4.14), with hypothesis (H.V) we can write

$$\mathcal{E}_\square^s = K_s\mathcal{E}_\Gamma^s K_s' = K_s\mathcal{D}_\Gamma^s K_s' + K_s\mathcal{C}_\Gamma^s K_s',$$

in which $K_s\mathcal{C}_\Gamma^s K_s' : \mathcal{H}^{-s}(\square)^2 \to \mathcal{H}^s(\square)^2$ is clearly compact. Then it suffices to prove (5.2) with $\mathcal{E}_\square^s$ replaced by $K_s\mathcal{D}_\Gamma^s K_s'$, for, once this has been accomplished, the full statement will follow by

extending the result to account for the compact perturbation $K_s\mathcal{C}^s_\Gamma K'_s$ as in [8]. Accordingly, for any $\alpha > 0$ we have the estimate

$$\begin{aligned}
&\mathrm{Re}\,\big(\mathsf{J}_{\square s}K_s\mathcal{D}^s_\Gamma K'_s U + \alpha(I-\mathcal{Q}'_s)U, U\big)_{[-s,\square]} \\
&\qquad = \mathrm{Re}\,\big\langle U, K_s\mathcal{D}^s_\Gamma K'_s U\big\rangle_{[\square]} + \alpha\mathrm{Re}\,\big((I-\mathcal{Q}'_s)U, U\big)_{[-s,\square]} \\
&\qquad = \mathrm{Re}\,\big\langle K'_s U, \mathcal{D}^s_\Gamma K'_s U\big\rangle_{\Gamma} + \alpha\mathrm{Re}\,\big((I-\mathcal{Q}'_s)U, U\big)_{[-s,\square]} \\
&\qquad \geq \beta_s\big\|K'_s U\big\|^2_{-s,\Gamma} + \alpha\big\|U\big\|^2_{[-s,\square]} - \alpha\big|\big(\mathcal{Q}'_s U, U\big)_{[-s,\square]}\big| \\
&\qquad \geq \frac{\beta_s}{\|\mathcal{K}'_s\|^2}\big\|\mathcal{Q}'_s U\big\|^2_{[-s,\square]} + \alpha\big\|U\big\|^2_{[-s,\square]} - \frac{\alpha}{2}\Big\{\big\|U\big\|^2_{[-s,\square]} + \big\|\mathcal{Q}'_s U\big\|^2_{[-s,\square]}\Big\} \\
&\qquad = \frac{\alpha}{2}\big\|U\big\|^2_{[-s,\square]} + \Big\{\frac{\beta_s}{\|\mathcal{K}'_s\|^2} - \frac{\alpha}{2}\Big\}\big\|\mathcal{Q}'_s U\big\|^2_{[-s,\square]} \qquad \text{for every} \quad U \in \mathcal{H}^{-s}(\square)^2;
\end{aligned}$$

here, we used the equality $\mathcal{Q}'_s U = \mathcal{K}'_s K'_s U$ to get the lower bound $\|\mathcal{K}'_s\|\|K'_s U\|_{-s,\Gamma} \geq \|\mathcal{Q}'_s U\|_{[-s,\square]}$. Evidently, we obtain a coercivity inequality of the form (5.2) with $K_s\mathcal{D}^s_\Gamma K'_s$ in place of $\mathcal{E}^s_\square$ whenever the positive α is less than $2\beta_s/\|\mathcal{K}'_s\|^2$. □

Remark 5.1. Numerical experience shows that one should take some care in the selection of the positive α in the construction of any solution-approximation scheme. For example, while the choice of any sufficiently small α will permit the implementation of a Bubnov-Galerkin procedure, one finds that very small values of α will lead to large condition numbers for the system matrices; this is not so surprising, since it is clear that the norm of the inverse of the operator $\mathsf{J}_{\square s}\mathcal{E}^s_\square + \alpha(I-\mathcal{Q}'_s)$ must grow without bound as $\alpha \to 0^+$, because $\mathsf{J}_{\square s}\mathcal{E}^s_\square$ is not invertible. On the other hand, neither is $I - \mathcal{Q}'_s$ invertible, so as α grows large one discovers at best a similar effect on the condition number (and, evidently, perhaps even a failure of invertibility of the Galerkin matrices). Thus, an uninformed choice of α can have the effect of cancelling the beneficial effects of conditioning devices such as the one explained in [5], so it is worthwhile to develop estimates for the permissible values of α in any particular example.

6. Remarks on the Bubnov-Galerkin procedure in a product space.

Since our reformulated problem will always be set in a product Hilbert space, *e.g.*, $H^0(\Box)^2$ or $\mathcal{H}^{-s}(\Box)^2$, we want to record here some simple observations concerning the implementation of Bubnov-Galerkin procedures in product spaces. More precisely, in addition to displaying the actual form taken by such a scheme, we want to indicate how one can generate—in a "natural" manner—a well-conditioned coördinate family for a product space $H \times H$ when such a family is known for H itself.

Let $\big(H, (\,\cdot\,,\,\cdot\,)_H\big)$ be a Hilbert space; the product Hilbert space $\big(H \times H, (\,\cdot\,,\,\cdot\,)_{[H]}\big)$ is equipped with the usual inner product, generated from that of H:

$$(U,V)_{[H]} := (U^+,V^+)_H + (U^-,V^-)_H \qquad \text{for} \quad U \equiv (U^+,U^-),\, V \equiv (V^+,V^-) \in H \times H.$$

Suppose that $\mathcal{F} := \Big(\big(b_n^N\big)_{n=1}^{d_N}\Big)_{N=1}^{\infty}$ is a *coördinate family in* H; this means simply that each of the finite collections $\big\{b_n^N\big\}_{n=1}^{d_N}$ is a linearly independent subset of H, while the dimensions $d_N \to \infty$ as $N \to \infty$. We say that the coördinate family is *well conditioned in* H iff the ℓ_2-condition numbers of the Gram matrices $\mathcal{G}_N := \Big\{\big(b_n^N, b_m^N\big)_H\Big\}_{n,m=1}^{d_N}$, $N = 1,\ 2, \dots$, form a bounded sequence of positive numbers. Recall that the use of trial- and test-families well-conditioned in the respective Hilbert spaces between which the underlying operator is an isomorphism will guarantee numerical stability of the corresponding Galerkin procedure (provided, of course, that we have convergence of the procedure); *cf*. [4], [5]. We write $\mathcal{M}_N := \operatorname{span}\big\{b_n^N\big\}_{n=1}^{d_N}$ for the subspaces of H spanned by the respective finite collections. We say that the sequence $\big(\mathcal{M}_N\big)_{N=1}^{\infty}$ is *ultimately dense in* H iff

$$\lim_{N\to\infty} \operatorname{dist}\big(u, \mathcal{M}_N\big) = \lim_{N\to\infty} \min_{v\in\mathcal{M}_N} \big\|u - v\big\|_H = \lim_{N\to\infty} \big\|u - \mathcal{P}_N u\big\|_H = 0 \qquad \text{for every} \quad u \in H;$$

here, we have denoted by $\mathcal{P}_N$ the orthoprojector onto $\mathcal{M}_N$ in H.

Corresponding to the coördinate family $\mathcal{F}$ and the sequence $\big(\mathcal{M}_N\big)_{N=1}^{\infty}$ of subspaces we construct a coördinate family $\mathfrak{F}$ and a sequence $\big(\mathfrak{M}_N\big)_{N=1}^{\infty}$ of subspaces for the product $H \times H$: for $N = 1,\ 2, \dots$, the collection $\big\{\mathfrak{b}_n^N\big\}_{n=1}^{2d_N}$ in $H \times H$ is defined by

$$\mathfrak{b}_n^N := \begin{cases} \big(b_n^N, 0\big) & \text{for} \quad n = 1, \dots, d_N \\ \big(0, b_{n-d_N}^N\big) & \text{for} \quad n = d_N + 1, \dots, 2d_N; \end{cases} \tag{6.1}$$

clearly, $\big\{\mathfrak{b}_n^N\big\}_{n=1}^{2d_N}$ is linearly independent, so that the corresponding span $\mathfrak{M}_N := \operatorname{span}\big\{\mathfrak{b}_n^N\big\}_{n=1}^{2d_N}$ is of dimension $2d_N$. In fact, it is not hard to see that

$$\mathfrak{M}_N = \big\{\, (u,v) \,\big|\, u, v \in \mathcal{M}_N \,\big\}.$$

It is also clear that the sequence $\big(\mathfrak{M}_N\big)_{N=1}^{\infty}$ is ultimately dense in $H \times H$ if $\big(\mathcal{M}_N\big)_{N=1}^{\infty}$ is ultimately dense in H, and, moreover, that the coördinate family $\mathfrak{F}$ is well conditioned in the product $H \times H$ if $\mathcal{F}$ is well conditioned in H itself. To see that the latter assertion is true, consider the form of the Gram matrix $\mathfrak{G}_N := \Big\{\big(\mathfrak{b}_n^N, \mathfrak{b}_m^N\big)_{[H]}\Big\}_{n,m=1}^{2d_N}$, which is

$$\mathfrak{G}_N = \begin{pmatrix} \mathcal{G}_N & 0 \\ 0 & \mathcal{G}_N \end{pmatrix}.$$

With this, it is not hard to show that $\mathfrak{G}_N$ and $\mathfrak{G}_N$ have the same sets of eigenvalues (*i.e.*, without regard for multiplicity), from which we conclude that their ℓ_2-condition numbers coincide, as well, since the latter are just the ratios of the largest to the smallest eigenvalues.

We denote the orthoprojector in $H \times H$ onto $\mathfrak{M}_N$ by $\mathfrak{P}_N$.

Now let $\mathfrak{L}$ be an isomorphism of $H \times H$ onto itself. We consider the operator problem

$$\textit{given} \quad G \in H \times H, \quad \textit{find the unique} \quad U_G \in H \times H \quad \textit{such that} \qquad \mathfrak{L}U_G = G. \qquad (\mathrm{P})_{[H]}$$

In particular, assuming that we know the action of $\mathfrak{L}$ through a matrix of operators in H, as

$$\mathfrak{L}U = \begin{pmatrix} \mathfrak{L}^{++} & \mathfrak{L}^{+-} \\ \mathfrak{L}^{-+} & \mathfrak{L}^{--} \end{pmatrix} \begin{pmatrix} U^{+} \\ U^{-} \end{pmatrix}, \tag{6.2}$$

we want to display the form of the Bubnov-Galerkin approximation-of-solution procedure for the problem $(\mathrm{P})_{[H]}$, constructed by using as trial- and test-functions the coördinate family $\mathfrak{F}$. The N^{th} subsidiary problem of that Bubnov-Galerkin procedure appears as

$$\textit{find} \quad U_G^N \in \mathfrak{M}_N \quad \textit{satisfying} \qquad \mathfrak{P}_N \mathfrak{L} U_G^N = \mathfrak{P}_N G;$$

since the latter equality is equivalent to the condition

$$\left(\mathfrak{L}U_G^N, \mathfrak{b}_m^N\right)_{[H]} = \left(G, \mathfrak{b}_m^N\right)_{[H]} \qquad \text{for} \quad m = 1, 2, \ldots, 2d_N, \tag{6.3}$$

by using the basis generating $\mathfrak{M}_N$, we arrive at the numerical formulation for the N^{th} subsidiary problem:

determine $\left(c_n^N\right)_{n=1}^{2d_N}$ *satisfying*

$$\sum_{n=1}^{2d_N} \left(\mathfrak{L}\mathfrak{b}_n^N, \mathfrak{b}_m^N\right)_{[H]} c_n^N = \left(G, \mathfrak{b}_m^N\right)_{[H]} \qquad \textit{for} \quad m = 1, 2, \ldots, 2d_N, \tag{6.4}$$

and construct the N^{th} *Bubnov-Galerkin approximant* U_G^N *by*

$$U_G^N := \sum_{n=1}^{2d_N} c_n^N \mathfrak{b}_n^N. \tag{6.5}$$

Now, we find

$$\left(\mathfrak{L}\mathfrak{b}_n^N, \mathfrak{b}_m^N\right)_{[H]} = \left(\mathfrak{L}^{++}\mathfrak{b}_n^{N+}, \mathfrak{b}_m^{N+}\right)_H + \left(\mathfrak{L}^{+-}\mathfrak{b}_n^{N-}, \mathfrak{b}_m^{N+}\right)_H + \left(\mathfrak{L}^{-+}\mathfrak{b}_n^{N+}, \mathfrak{b}_m^{N-}\right)_H + \left(\mathfrak{L}^{--}\mathfrak{b}_n^{N-}, \mathfrak{b}_m^{N-}\right)_H,$$
$$\text{for} \quad m, n = 1, \ldots, 2d_N,$$

but only one of the four terms on the right is nonzero for each m, n. Accounting for this, we come to the explicit form of the system (6.4) for $\left(c_n^N\right)_{n=1}^{2d_N}$:

$$\sum_{n=1}^{d_N} \left(\mathfrak{L}^{++} b_n^N, b_m^N\right)_H c_n^N + \sum_{n=d_N+1}^{2d_N} \left(\mathfrak{L}^{+-} b_{n-d_N}^N, b_m^N\right)_H c_n^N = \left(G^{+}, b_m^N\right)_H, \quad m = 1, 2, \ldots, d_N, \tag{6.6}$$

$$\sum_{n=1}^{d_N} \left(\mathfrak{L}^{-+} b_n^N, b_{m-d_N}^N\right)_H c_n^N + \sum_{n=d_N+1}^{2d_N} \left(\mathfrak{L}^{--} b_{n-d_N}^N, b_{m-d_N}^N\right)_H c_n^N = \left(G^{-}, b_{m-d_N}^N\right)_H, \quad m = d_N + 1, \ldots, 2d_N.$$

The $2d_N \times 2d_N$ matrix of the system can be represented in block form as

$$\Big\{\big(\mathcal{L}\mathfrak{b}_p^N, \mathfrak{b}_q^N\big)_{[H]}\Big\}_{p,q=1}^{2d_N}$$

$$= \begin{pmatrix} \Big\{\big(\mathcal{L}^{++}b_n^N, b_m^N\big)_H\Big\}_{\substack{m=1,\dots,d_N\\ n=1,\dots,d_N}} & \Big\{\big(\mathcal{L}^{+-}b_{n-d_N}^N, b_m^N\big)_H\Big\}_{\substack{m=1,\dots,d_N\\ n=d_N+1,\dots,2d_N}} \\ \Big\{\big(\mathcal{L}^{-+}b_n^N, b_{m-d_N}^N\big)_H\Big\}_{\substack{m=d_N+1,\dots,2d_N\\ n=1,\dots,d_N}} & \Big\{\big(\mathcal{L}^{--}b_{n-d_N}^N, b_{m-d_N}^N\big)_H\Big\}_{\substack{m=d_N+1,\dots,2d_N\\ n=d_N+1,\dots,2d_N}} \end{pmatrix}.$$

Now let us return to the integral operator $\mathcal{E}_\Gamma$ in $H^0(\Gamma)$ of hypothesis (H.I) in Section 4 and the associated operator $\mathcal{E}_\square$ in $H^0(\square)^2$, defined in (4.1). In the approximate solution of the example problems set up and examined in Section 5, we must apply to $(\mathrm{P.II})_\square$ and to $(\mathrm{P.I})_\square$ the Bubnov-Galerkin procedure sketched here for the generic problem $(\mathrm{P})_{[H]}$, taking $H^0(\square)^2$ and $\mathcal{H}^{-s}(\square)^2$, respectively, for the product space $H \times H$. Thus, we must identify the matrix form of the operators $I + \mathcal{E}_\square$ (for $(\mathrm{P.II})_\square$) and $\mathsf{J}_{\square s}\mathcal{E}_\square^s + \alpha(I - \mathcal{Q}_s')$ (for $(\mathrm{P.I})_\square$), as in (6.2), and compute the collection of inner products appearing on the lefthand side of (6.6), supposing that specific basis functions b_n^N have been selected. By recalling (4.11) and (4.12), this is quite straightforward for the operator $I + \mathcal{E}_\square$, in the case of the second-kind problem of Example 5.1. However, there are some points concerning the first-kind problem of Example 5.2 that warrant comment.

Suppose that a collection $\{\mathfrak{b}_n^N\}_{n=1}^{2d_N}$ has been chosen from $H^0(\square)^2$ as in (6.1), so that the b_n^N are elements of $H^0(\square)$. Taking $\mathcal{L}$ now to be $\mathsf{J}_{\square s}\mathcal{E}_\square^s + \alpha(I - \mathcal{Q}_s')$ and H to be $\mathcal{H}^{-s}(\square)$, we can write

$$\begin{aligned}
\big(\mathsf{J}_{\square s}\mathcal{E}_\square^s\mathfrak{b}_n^N + \alpha(I - \mathcal{Q}_s')\mathfrak{b}_n^N, \mathfrak{b}_m^N\big)_{[-s,\square]} &= \overline{\big(\mathfrak{b}_m^N, \mathsf{J}_{\square s}\mathcal{E}_\square^s\mathfrak{b}_n^N\big)}_{[-s,\square]} + \alpha\big\langle(I - \mathcal{Q}_s')\mathfrak{b}_n^N, \mathsf{J}_{\square s}^{-1}\mathfrak{b}_m^N\big\rangle_{[\square]} \\
&= \overline{\big\langle\mathfrak{b}_m^N, \mathcal{E}_\square^s\mathfrak{b}_n^N\big\rangle}_{[\square]} + \alpha\big\langle(I - \mathcal{Q}_s')\mathfrak{b}_n^N, \mathsf{J}_{\square s}^{-1}\mathfrak{b}_m^N\big\rangle_{[\square]} \\
&= \overline{\big(\mathfrak{b}_m^N, \mathcal{E}_\square\mathfrak{b}_n^N\big)}_{[0,\square]} + \alpha\big\langle(I - \mathcal{Q}_0)\mathfrak{b}_n^N, \mathsf{J}_{\square s}^{-1}\mathfrak{b}_m^N\big\rangle_{[\square]} \\
&= \big(\mathcal{E}_\square\mathfrak{b}_n^N, \mathfrak{b}_m^N\big)_{[0,\square]} + \alpha\big((I - \mathcal{Q}_0)\mathfrak{b}_n^N, \mathsf{J}_{\square s}^{-1}\mathfrak{b}_m^N\big)_{[0,\square]},
\end{aligned}$$

in which we used the basic properties of the antiduality operator $\mathsf{J}_{\square s}$ and its inverse along with those of the antiduality pairing $\langle\cdot,\cdot\rangle_{[\square]}$ as the appropriate extension of the inner product $(\cdot,\cdot)_{[0,\square]}$, as well as the equality $\mathcal{Q}_s'\mathfrak{b}_n^N = \mathcal{Q}_0\mathfrak{b}_n^N$, following from (3.21.2) since $\mathfrak{b}_n^N \in H^0(\square)^2$. Therefore, the matrix of the N^{th} Bubnov-Galerkin system appears as

$$\Big\{\big(\mathsf{J}_{\square s}\mathcal{E}_\square^s\mathfrak{b}_p^N + \alpha(I - \mathcal{Q}_s')\mathfrak{b}_p^N, \mathfrak{b}_q^N\big)_{[-s,\square]}\Big\}_{p,q=1}^{2d_N}$$

$$= \begin{pmatrix} \Big\{\big(\mathcal{E}_\square^{++}b_n^N, b_m^N\big)_{0,\square}\Big\}_{\substack{m=1,\dots,d_N\\ n=1,\dots,d_N}} & \Big\{\big(\mathcal{E}_\square^{+-}b_{n-d_N}^N, b_m^N\big)_{0,\square}\Big\}_{\substack{m=1,\dots,d_N\\ n=d_N+1,\dots,2d_N}} \\ \Big\{\big(\mathcal{E}_\square^{-+}b_n^N, b_{m-d_N}^N\big)_{0,\square}\Big\}_{\substack{m=d_N+1,\dots,2d_N\\ n=1,\dots,d_N}} & \Big\{\big(\mathcal{E}_\square^{--}b_{n-d_N}^N, b_{m-d_N}^N\big)_{0,\square}\Big\}_{\substack{m=d_N+1,\dots,2d_N\\ n=d_N+1,\dots,2d_N}} \end{pmatrix}$$

$$+\alpha \begin{pmatrix} \Big\{\big((I - \mathcal{Q}_0^{++})b_n^N, J_{\square s}^{-1}b_m^N\big)_{0,\square}\Big\}_{\substack{m=1,\dots,d_N\\ n=1,\dots,d_N}} & \Big\{\big(\mathcal{Q}_0^{+-}b_{n-d_N}^N, J_{\square s}^{-1}b_m^N\big)_{0,\square}\Big\}_{\substack{m=1,\dots,d_N\\ n=d_N+1,\dots,2d_N}} \\ \Big\{\big(\mathcal{Q}_0^{-+}b_n^N, J_{\square s}^{-1}b_{m-d_N}^N\big)_{0,\square}\Big\}_{\substack{m=d_N+1,\dots,2d_N\\ n=1,\dots,d_N}} & \Big\{\big((I - \mathcal{Q}_0^{--})b_{n-d_N}^N, J_{\square s}^{-1}b_{m-d_N}^N\big)_{0,\square}\Big\}_{\substack{m=d_N+1,\dots,2d_N\\ n=d_N+1,\dots,2d_N}} \end{pmatrix}, \qquad (6.7)$$

with $J_{\Box_s} : \mathcal{H}^s(\Box) \to \mathcal{H}^{-s}(\Box)$ denoting the antiduality operator. The elemental operators $\mathcal{E}_{\Box}^{\mu\eta}$ are given in (4.12), while the $\mathcal{Q}_0^{\mu\eta}$ are displayed in the proof of Lemma 3.3. Upon inspecting the matrix elements, we see that all inner products are taken in $H^0(\Box)$ and only the computation of the images $J_{\Box_s}^{-1} b_m^N$ requires explanation. In fact, we shall know how to compute the $J_{\Box_s}^{-1} b_m^N$ in the contemplated applications, since we will have selected the spaces $\mathcal{H}^s(\Box)$ in such a way that we can handle numerically the Sobolev structures on them; as previewed in Section 1, the possibility of doing so rests ultimately on the simplicity of the underlying geometry, which is here the plane rectangle $\Box$. However, since the elements $J_{\Box_s}^{-1} b_m^N$ themselves are too cumbersome and costly to deal with repeatedly, in practice we have used not these images but their orthoprojections in $H^0(\Box)$ back onto the original subspace $\mathcal{M}_N$. Similar computations are discussed more fully below in Section 7 and also in [5].

Finally, the inner products forming the righthand side of the N^{th} Bubnov-Galerkin system for $(\text{P.I})_{\Box}$ are (*cf.* (6.4) and (6.6))

$$\left(\mathsf{J}_{\Box_s} K_s g, \mathfrak{b}_m^N\right)_{[-s,\Box]} = \left(K_s g, \mathfrak{b}_m^N\right)_{[0,\Box]} = \begin{cases} \left(K^+ g, b_m^N\right)_{0,\Box} & \text{for} \quad m = 1, \ldots, d_N \\ \left(K^- g, b_{m-d_N}^N\right)_{0,\Box} & \text{for} \quad m = d_N + 1, \ldots, 2d_N. \end{cases} \tag{6.8}$$

7. A numerical implementation for a boundary-integral equation in $\mathbb{R}^2$.

The framework developed in the preceding sections for the reformulation of integral equations set on the boundary of a domain in $\mathbb{R}^3$ can be modified in an obvious manner so that it is applicable for the same purpose in $\mathbb{R}^2$. Accordingly, to gain some experience before implementing the method in the full three-dimensional setting, we have applied the same ideas to generate a Bubnov-Galerkin procedure for constructing approximate solutions of a particular first-kind integral equation on the bounding curve of a domain in $\mathbb{R}^2$; the chosen example arises in studying time-harmonic acoustic scattering by a "soft" obstacle in two dimensions. To perform the computations, we have written a code that will accommodate a fairly large class of smoothly bounded shapes and incorporates a scheme aimed at achieving numerical stability by "conditioning" of the coördinate functions that is explained in [5]; the computer program also served to demonstrate the effectiveness of this conjectured stabilization device. The numerical results, which have been validated by comparison with those of an essentially independent calculation (*viz.*, a "standard" application of the Bubnov-Galerkin method to the same boundary-operator problem, also described in [5]) show that the reformulation approach developed here is successful in producing approximate solutions for the example problem studied, at least for all those Γ for which computations have been performed, which ranged from rotund shapes (including a nonconvex example) to slender ellipses.

We shall give some description for each of the major steps required in setting up the application. Now, Ω is a bounded domain of class C^2 in $\mathbb{R}^2$ with boundary curve $\Gamma := \partial\Omega$. We require that Γ be starlike with respect to some enclosed point P; to simplify the numerical work somewhat, we have also assumed that Ω is symmetric with respect to a midline. Therefore, we may describe Γ in polar coördinates with pole at P by a radius-function

$$\vartheta \mapsto \varrho_\Gamma(\cos\vartheta), \qquad -\pi < \vartheta \leq \pi.$$

Coördinate patches; partition of unity. By exploiting the hypothesis of starlikeness for Γ, we let $\mathrm{R}_\Gamma : \Gamma \to \Sigma_1$ denote the associated radial-projection mapping given by $\mathrm{R}_\Gamma(\xi) := |\xi|^{-1}\xi$ for $\xi \in \Gamma$, carrying Γ bijectively onto the unit circle Σ_1. We first choose an appropriate pair $\{\Sigma_1^+, \Sigma_1^-\}$ of covering coördinate patches for Σ_1, then set $\Gamma_\pm := \mathrm{R}_\Gamma^{-1}\{\Sigma_1^\pm\}$ to get a pair $\{\Gamma_+, \Gamma_-\}$ of covering coördinate patches for Γ. Specifically, we select $\delta \in (0, \pi/2)$ and, under the usual identification of Σ_1 with $(-\pi, \pi]$, take Σ_1^+ to correspond to the complement of $\{\, \vartheta \,|\, |\vartheta - (-\pi/2)| \leq \pi/2 - \delta \,\}$ and Σ_1^- to correspond to the complement of $\{\, \vartheta \,|\, |\vartheta - \pi/2| \leq \pi/2 - \delta \,\}$. The coördinate function $h_+ : \Gamma_+ \to (-b, b)$ is then taken to be the composition of the radial map $\mathrm{R}_\Gamma \,|\, \Gamma_+ : \Gamma_+ \to \Sigma_1^+$ followed by the stereographic projection map (into the equatorial plane) based on the south pole of Σ_1; the coördinate function $h_- : \Gamma_- \to (-b, b)$ is constructed in the corresponding manner, using $\mathrm{R}_\Gamma \,|\, \Gamma_- : \Gamma_- \to \Sigma_1^-$ followed by the stereographic projection map based on the north pole of Σ_1. Here, the open interval $(-b, b)$ takes the place of the rectangle $\square$; it is easy to check that

$$b = \frac{\cos\delta}{1 - \sin\delta}.$$

In the remainder of this section, "$\square$" stands for $(-b, b)$. Explicitly, one finds that the $h_\pm : \Gamma_\pm \to (-b, b)$ are given by

$$h_\pm(\xi) := \frac{\xi_1}{|\xi| \pm \xi_2}, \qquad \xi \in \Gamma_\pm,$$

with the ξ_k denoting the Cartesian coördinates of $\xi \in \mathbb{R}^2$, while values of the inverses $h_\pm^{-1} : (-b, b) \to \Gamma_\pm$ can be computed from

$$h_\pm^{-1}(s) = \varrho_\Gamma\Big(\frac{2s}{s^2+1}\Big)\Big\{\frac{2s}{s^2+1}\hat{\mathrm{e}}_1 \mp \frac{s^2-1}{s^2+1}\hat{\mathrm{e}}_2\Big\}, \qquad -b < s < b,$$

with the $\hat{\mathrm{e}}_k$ denoting the standard Cartesian unit-basis vectors. In this lower-dimensional case, the $\mathfrak{I}h_\pm^{-1}$ are just the magnitudes $\big|h_\pm^{-1\prime}\big|$ of the derivatives of the inverses $h_\pm^{-1}$.

We need an easily computable partition of unity for Γ subordinate to the covering $\{\Gamma_+, \Gamma_-\}$. For this, we choose $\delta_0 \in (0, \delta)$ along with a positive integer p and set

$$f_p(\theta) := (\theta + \delta_0)^p(\delta_0 - \theta)^p, \qquad -\delta_0 \le \theta \le \delta_0;$$

then, with $a_p := \int_{-\delta_0}^{\delta_0} f_p(\theta)\, d\theta$, by putting

$$F_p(\vartheta) := \frac{1}{a_p}\int_{-\delta_0}^{\vartheta} f_p(\theta)\, d\theta, \qquad -\delta_0 \le \vartheta \le \delta_0,$$

we form a function monotonically increasing from 0 to 1 on $[-\delta_0, \delta_0]$ with its derivatives of order less than or equal to p vanishing at each endpoint, so we can extend F_p to all of $[-\pi/2, \pi/2]$ by setting $F_p(\vartheta) := 0$ for $\vartheta \in [-\pi/2, -\delta_0)$ and $F_p(\vartheta) := 1$ for $\vartheta \in (\delta_0, \pi/2]$, to get a nondecreasing function of class C^p with support $[-\delta_0, \pi/2]$ and equalling unity on $[\delta_0, \pi/2]$. In the obvious manner, we generate from F_p a function φ^+ on Γ with support in Γ_+ and equalling unity on a closed segment containing $\Gamma \setminus \Gamma_-$ in its interior. Therefore, by setting $\varphi^- := 1 - \varphi^+$, it is clear that we get a partition of unity $\{\varphi^+, \varphi^-\}$ for Γ subordinate to the cover $\{\Gamma_+, \Gamma_-\}$.

The parameters δ, δ_0, and p are selected as input-data for the numerical computation. The explicit forms of the various compositions required, such as $h_+ \circ h_-^{-1}$ on $h_-(\Gamma_+ \cap \Gamma_-)$, $\varphi^+ \circ h_+^{-1}$, *etc.*, can now be easily computed; we omit a listing of these. Moreover, now the operators K_0, $\mathcal{K}_0$, and $\mathcal{Q}_0$ can be constructed, on the basis of the definitions given in Section 3; *cf.*, the expressions provided in the proof of Lemma 3.3.

The kernel for Γ and its associated integral operators. The framework set up to this point is applicable for any choice of the kernel E for Γ generating the associated integral operators $\mathcal{E}_\Gamma$ and $\mathcal{E}_\square$. Now, for a positive "wavenumber" κ, we choose the kernel given by

$$S_\xi(\zeta) := \frac{\mathrm{i}}{4}H_0^{(1)}\big(\kappa|\zeta - \xi|\big), \qquad \text{for} \quad \xi,\, \zeta \in \Gamma, \quad \zeta \ne \xi,$$

with $H_0^{(1)}$ denoting, as usual, the Hankel function of first kind and order zero, which generates the integral operator $\mathcal{S}_\kappa : H^0(\Gamma) \to H^0(\Gamma)$ by

$$\mathcal{S}_\kappa u(\xi) := \frac{\mathrm{i}}{4}\int_\Gamma H_0^{(1)}\big(\kappa|\zeta - \xi|\big)\, u(\zeta)\, d\lambda_\Gamma(\zeta) \qquad \text{for} \quad \text{a.a. } \xi \in \Gamma, \quad \text{for each} \quad u \in H^0(\Gamma),$$

as in hypothesis (H.I). The operator $\mathcal{S}_\kappa$ is the so-called direct-value operator corresponding to the single-layer potential for the Helmholtz equation in two dimensions which is based on Γ.

The corresponding operator $(\mathcal{E}_\square =)\ \mathfrak{S}_\kappa : H^0(\square)^2 \to H^0(\square)^2$, defined in terms of $(\mathcal{E}_\Gamma =)\ \mathcal{S}_\kappa$ just as in (4.1), is constructed numerically directly from the kernel S (and from the other already-prepared functions, such as h_μ^{-1} and $\phi^\eta \circ h_\mu^{-1}$) by using (4.11), with the elemental operators $\mathfrak{S}_\kappa^{\mu\eta}$ defined just as in (4.12).

We now choose $s = 1/2$, although we continue to write, *e.g.*, $J_{\Box s}$ in place of $J_{\Box\frac{1}{2}}$ when it is convenient. Then the operator $\mathcal{S}_\kappa$ fulfills the hypothesis (H.III) laid down for Example 5.2, as shown in [9]; $\mathcal{S}^s_\kappa : H^{-\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma)$ denotes the extension of $\mathcal{S}_\kappa$. Provided that the square κ^2 of the wavenumber is not a Dirichlet eigenvalue for the negative Laplacian in Ω, hypotheses (H.IV) and (H.V) also hold, again as shown in [9], so we shall impose that restriction on κ. (Although the latter condition essentially limits the usefulness of the present formulation in acoustics studies to "small" values of κ^2, this is of little concern for our testing program.)

We take $\mathcal{H}^s(\Box)$ to be $H_0^{\frac{1}{2}}(\Box)$; the antidual space is then usually denoted by $H^{-\frac{1}{2}}(\Box)$. The operator corresponding to $\mathcal{S}^s_\kappa$ as in (4.14) and extending $\mathfrak{S}_\kappa$ is denoted by $\mathfrak{S}^s_\kappa : H^{-\frac{1}{2}}(\Box)^2 \to H_0^{\frac{1}{2}}(\Box)^2$.

Of course, we want to solve the problem $(\mathrm{P.I})_\Gamma$, which here appears as

$$\textit{given} \quad g \in H^{\frac{1}{2}}(\Gamma), \quad \textit{determine} \quad u_g \in H^{-\frac{1}{2}}(\Gamma) \quad \textit{such that} \qquad \mathcal{S}^s_\kappa u_g = g,$$

by solving instead $(\mathrm{P.I})_\Box$, which is now

$$\textit{given} \quad g \in H^{\frac{1}{2}}(\Gamma), \quad \textit{determine} \quad U_g \in H^{-\frac{1}{2}}(\Box)^2 \quad \textit{such that} \quad \mathsf{J}_{\Box s}\mathfrak{S}^s_\kappa U_g + \alpha\big(I - \mathcal{Q}'_s\big)U_g = \mathsf{J}_{\Box s}K_s g.$$

When $g = v^\iota|_\Gamma$, the trace on Γ of an "incident-field" velocity potential v^ι, *i.e.*, a function satisfying the Helmholtz equation $\Delta v^\iota + \kappa^2 v^\iota = 0$ in an open set containing the closure of Ω, then the corresponding unique solution u_g has a physical interpretation as the normal derivative on Γ of the "total" acoustic field in the exterior of Ω. (Once this normal derivative is known, it can be used as the density of an exterior single-layer potential to yield the scattered field in the exterior region.) In those problems solved for validation of the programs, we have taken the incident field to be a plane wave with its fronts propagating in a direction that is set as input for the numerical computation, *i.e.*, we put

$$v^\iota(\xi) := e^{\mathrm{i}\kappa\hat{\mathrm{e}}\cdot\xi}, \qquad \xi \in \mathbb{R}^2,$$

in which the unit vector $\hat{\mathrm{e}}$ is chosen as input data. Sample graphs of the resulting approximations to the normal derivative of the total acoustic field can be found in [5], along with a description of the more standard Bubnov-Galerkin procedure yielding the comparison approximations, which agreed (to within errors) with those obtained in the present calculations.

The initial trial/test functions. Our "initial," or "unmodified," trial/test functions are constructed, in a well-known manner described by, *e.g.*, AUBIN [1], [2] (*cf.*, also, [5]), as polynomial spline functions generated from the convolution powers χ^{*k} of the characteristic function χ of the interval $[0, 1]$. That is, $\chi^{*1} := \chi$ and

$$\chi^{*(k+1)}(s) := \int_{-\infty}^{\infty} \chi(s - \sigma)\chi^{*k}(\sigma)\, d\sigma, \qquad s \in \mathbb{R}, \qquad \text{for} \quad k = 1, 2, \ldots .$$

Then $\chi^{*(k+1)}$ is of class C^{k-1} on $\mathbb{R}$ if $k \geq 1$, with support the interval $[0, k + 1]$; the restriction of $\chi^{*(k+1)}$ to each interval $[l, l + 1]$ is a polynomial of degree k, for $l = 0, \ldots, k$.

Now, with the integers $k \geq 0$ and $N > 0$ such that $N - k \geq 1$, the corresponding "mesh size" is defined by

$$h_N := \frac{2b}{N},$$

and then the translations/compressions b_n^{Nk} of $\chi^{*(k+1)}$ are defined in $[-b,b]$ by

$$b_n^{Nk}(s) := \chi^{*(k+1)}\Big(\frac{s-(n-1)h_N+b}{h_N}\Big), \qquad -b \le s \le b, \qquad \text{for} \quad n = 1,\dots,N-k.$$

One can then check that the support of b_n^{Nk} is the interval $[-b+(n-1)h_N, -b+(n+k)h_N]$. Consequently, each function in $\{b_n^{Nk}\}_{n=1}^{N-k}$ has support in the interval $[-b,b]$. In [5] it is shown that each of the coördinate families $\Big(\big(b_n^{Nk}\big)_{n=1}^{N-k}\Big)_{N=k+1}^{\infty}$, for $k = 0, 1, 2, \dots$, is well conditioned in $H^0(\square)$, and so cannot be well conditioned in a Sobolev space on $\square$ of nonzero order, as pointed out in [4]. Let the collection $\{\mathfrak{b}_n^{Nk}\}_{n=1}^{2(N-k)}$ in $H^0(\square)^2$ be constructed from the collection $\{b_n^{Nk}\}_{n=1}^{N-k}$ in $H^0(\square)$ just as in (6.1); we term these the "initial," or "unmodified," coördinate functions.

Corresponding to the selected k, which determines the smoothness of the trial functions, we denote the N^{th} coördinate subspace by $\mathcal{M}_N^k := \operatorname{span}\{b_n^{Nk}\}_{n=1}^{N-k}$; in particular, now $d_N = N-k$. The developments of AUBIN [2] imply that the sequence $(\mathcal{M}_N^k)_{N=k+1}^{\infty}$ has the requisite property of ultimate denseness in $H^0(\square)$, and therefore also in $H^{-\frac{1}{2}}(\square)$. Then the corresponding sequence $(\mathfrak{M}_N^k)_{N=k+1}^{\infty}$ of subspaces, with the $\mathfrak{M}_N^k := \operatorname{span}\{\mathfrak{b}_n^{Nk}\}_{n=1}^{N-k}$ related to the $\mathcal{M}_N^k$ just as in Section 6, is ultimately dense in the product space $H^{-\frac{1}{2}}(\square)^2$.

The Bubnov-Galerkin matrices; plots of condition-number variation with N. Given the discretization integer N, we compute the elements of the $2(N-k)\times 2(N-k)$ N^{th} Bubnov-Galerkin matrix for the problem $(\text{P.I})_{\square}$, as in (6.7), for two choices of the trial/test functions spanning $\mathfrak{M}_N^k$. The first choice, in case (A), comprises the unmodified functions just introduced, while the second, in case (B), derives from the application of a prospective "conditioning transformation" to those unmodified functions; the idea is explained briefly below and more fully in [5]. In each case, the LINPACK condition number of the matrix is computed for a range of increasing values of N; comparison of the variation with N in the two cases is shown in Figures 1–6, for various shapes of the domain Ω and values of κa, where a is a characteristic length of the shape. In all the results shown, we used cubic splines, *i.e.*, we took $k = 3$, and set the value of the parameter α at 0.1. Selection of "large" values for α, say, $\alpha = 1.0$, resulted in very erratic behavior of the condition numbers. We have not indicated the values used for the various technical parameters, such as δ, δ_0, *etc.*

From a review of the figures, numerical instability is evident in all the results of case (A), which are indicated with the label "unmodified coördinate functions." On the other hand, the figures also clearly illustrate the consistent reduction in condition number achieved by the use of the "new," or "preconditioned," coördinate functions in case (B). However, we cannot presently state that the latter behavior represents genuine numerical stability, since we have not yet proven the necessary results on the validity of the scheme.

(A.) Here, we use the unmodified collections $\{\mathfrak{b}_n^{Nk}\}_{n=1}^{2(N-k)}$. In this case, we should compute the elements of the matrix

$$\Big\{\big(\mathsf{J}_{\square_s}\mathfrak{S}_\kappa \mathfrak{b}_n^{Nk} + \alpha(I-\mathcal{Q}_s')\mathfrak{b}_n^{Nk}, \mathfrak{b}_m^{Nk}\big)_{[-s,\square]}\Big\}_{m,n=1}^{2(N-k)},$$

by making the necessary identifications in the general form displayed on the right in (6.7).

Evidently, here we must compute values of the elements $J_{\square_s}^{-1} b_m^{Nk}$ (recall that $s = 1/2$); these calculations can be performed after finding the eigenpairs of the antiduality operator, which we can do explicitly here, since we are working in the "simple" Sobolev spaces $H_0^s(\square)$ and their antiduals, as

explained in [5]. On the other hand, the numerical computations will be relatively time-consuming, so we would like to avoid executing them many times. Accordingly, we employ an approximation, replacing the elements $J_{\square_s}^{-1} b_n^{Nk}$ by their orthogonal projections in $H^0(\square)$ onto $\mathcal{M}_N^k$, which are much more easily computed and handled in the integrations. Denoting by $\mathcal{P}_N^k$ the $H^0(\square)$-orthoprojector onto $\mathcal{M}_N^k$, we can write

$$\mathcal{P}_N^k J_{\square_s}^{-1} b_n^{Nk} = \sum_{m=1}^{N-k} A_{nm}^{Nk}\, b_m^{Nk}, \qquad n = 1, \ldots, N-k, \tag{7.1}$$

in which the coefficients A_{nm}^{Nk} are found from the defining condition

$$\sum_{m=1}^{N-k} (b_m^{Nk}, b_l^{Nk})_{0,\square}\, A_{nm}^{Nk} = (J_{\square_s}^{-1} b_n^{Nk}, b_l^{Nk})_{0,\square}, \qquad l = 1, \ldots, N-k. \tag{7.2}$$

Now, in this case in which the underlying Sobolev space is $H_0^{\frac{1}{2}}(\square)$, the elements of $\left\{J_{\square_s}^{-1} b_n^{Nk}\right\}_{n=1}^{N-k}$ are not merely translates of each other (unlike the situation in a space of periodic functions), so one cannot avoid solving the system in (7.2) for $n = 1, \ldots, N-k$. However, this requires less work than one might anticipate. In fact, the linearly independent functions in $\left\{b_n^{Nk}\right\}_{n=1}^{N-k}$ are translates of each other (and have "small" supports in $(-b, b)$), so their $H^0(\square)$-Gram matrix, figuring in the systems (7.2), is a symmetric, positive-definite (and banded) Toeplitz matrix. Since the inverse of a symmetric, positive-definite $M \times M$ Toeplitz matrix can be computed in $O(M^2)$ operations by using the Trench algorithm, as explained in, *e.g.*, [6],† the calculation of the matrix $A^{Nk} := \left\{A_{nm}^{Nk}\right\}_{n,m=1}^{N-k}$ of coefficients enabling construction of the orthogonal projections is reduced to performing the Trench inversion procedure, involving $O(N^2)$ operations, followed by $N-k$ matrix-vector multiplications, or one matrix-matrix multiplication. Since the coördinate family $\left(\left(b_n^{Nk}\right)_{n=1}^{N-k}\right)_{N=k+1}^{\infty}$ is well-conditioned in $H^0(\square)$ for any $k \geq 0$, the corresponding sequence of $H^0(\square)$-Gram matrices has bounded condition numbers, *i.e.*, this process of computing $H^0(\square)$-orthogonal projections onto the subspaces $\mathcal{M}_N^k$ is numerically stable.

Consequently, in this first case the (approximation of the) N^{th} Bubnov-Galerkin matrix is formed in three steps:

(A.1) computing the elements of each of the matrices

$$\begin{pmatrix} \left\{\left(\mathfrak{S}_\kappa^{++} b_n^{Nk}, b_m^{Nk}\right)_{0,\square}\right\}_{\substack{m=1,\ldots,N-k \\ n=1,\ldots,N-k}} & \left\{\left(\mathfrak{S}_\kappa^{+-} b_{n-(N-k)}^{Nk}, b_m^{Nk}\right)_{0,\square}\right\}_{\substack{m=1,\ldots,N-k \\ n=N-k+1,\ldots,2(N-k)}} \\ \left\{\left(\mathfrak{S}_\kappa^{-+} b_n^{Nk}, b_{m-(N-k)}^{Nk}\right)_{0,\square}\right\}_{\substack{m=N-k+1,\ldots,2(N-k) \\ n=1,\ldots,N-k}} & \left\{\left(\mathfrak{S}_\kappa^{--} b_{n-(N-k)}^{Nk}, b_{m-(N-k)}^{Nk}\right)_{0,\square}\right\}_{\substack{m=N-k+1,\ldots,2(N-k) \\ n=N-k+1,\ldots,2(N-k)}} \end{pmatrix} \tag{7.3}$$

and

† One should beware a typographical error in the symbolic code displayed for the Trench algorithm in [6], which becomes apparent upon comparison with the textual description.

$$\begin{pmatrix} \left\{ \left((I - \mathcal{Q}_0^{++}) b_n^{Nk}, b_m^{Nk} \right)_{0,\square} \right\}_{\substack{m=1,\ldots,N-k \\ n=1,\ldots,N-k}} & \left\{ \left(\mathcal{Q}_0^{+-} b_n^{Nk}, b_{m-(N-k)}^{Nk} \right)_{0,\square} \right\}_{\substack{m=N-k+1,\ldots,2(N-k) \\ n=N-k+1,\ldots,2(N-k)}} \\ \left\{ \left(\mathcal{Q}_0^{+-} b_n^{Nk}, b_{m-(N-k)}^{Nk} \right)_{0,\square} \right\}_{\substack{m=N-k+1,\ldots,2(N-k) \\ n=1,\ldots,N-k}} & \left\{ \left((I - \mathcal{Q}_0^{--}) b_{n-(N-k)}^{Nk}, b_{m-(N-k)}^{Nk} \right)_{0,\square} \right\}_{\substack{m=N-k+1,\ldots,2(N-k) \\ n=N-k+1,\ldots,2(N-k)}} \end{pmatrix}, \tag{7.4}$$

(A.2) postmultiplication of the matrix in (7.4) by the transpose of the matrix $\begin{pmatrix} A^{Nk} & A^{Nk} \\ A^{Nk} & A^{Nk} \end{pmatrix}$, to perform the transformation implied in (7.1), and, finally,

(A.3) coupling the two matrices with the parameter α.

In passing, we should note that, owing to the symmetry of the kernel S and the midline-symmetry postulated for the shape of the region Ω, there are symmetries in the matrices (7.3) and (7.4) that can be exploited to shorten the numerical computations. In fact, one finds that the matrix in (7.3) is symmetric and its two main-diagonal blocks are identical, while the pairs of main- and secondary-diagonal blocks in the matrix of (7.4) are identical and the blocks themselves are symmetric.

Some erratic behavior of the condition numbers is apparent from the plots in Figures 1–6; we ascribe this to the approximation that was used, since the calculation of the LINPACK condition number is evidently rather sensitive to errors in the matrix elements. One also observes a curious "oscillatory" variation of the condition number with N. The origin of this oscillation is not known, but it is inferred that the plots shown are essentially correct, because of the cited agreement achieved between the approximate solutions of $(\mathrm{P.I})_\Gamma$ by the present method and by the alternate procedure of [5].

(B.) We pointed out that the coördinate family $\left(\left(b_n^{Nk} \right)_{n=1}^{N-k} \right)_{N=k+1}^{\infty}$ is well conditioned in $H^0(\square)$ (for any $k \geq 0$); the family is therefore ill-conditioned in $H^{-\frac{1}{2}}(\square)$ and so unsuitable for construction of a numerically stable Bubnov-Galerkin procedure in the example under study here (*cf.* [4]). This explains the condition-number behavior observed in case (A). In this second computation, we use a construction described in [5] to generate a coördinate family that appears to be well conditioned in $H^{-\frac{1}{2}}(\square)$.

To briefly motivate and explain the idea, we recall that the antiduality operator $J_{\square_s} : H_0^{\frac{1}{2}}(\square) \to H^{-\frac{1}{2}}(\square)$ has a factorization

$$J_{\square_s} = I_{-\frac{1}{2}} I_{\frac{1}{2}}^{-1},$$

in which $I_{\frac{1}{2}} : H^0(\square) \to H_0^{\frac{1}{2}}(\square)$ and $I_{-\frac{1}{2}} : H^0(\square) \to H^{-\frac{1}{2}}(\square)$ are certain isometric isomorphisms; *cf.*, *e.g.*, [3], [5]. In the present case of a sufficiently simple geometry underlying the Sobolev space, we can compute the action of either of these operators for the same reason that we could compute the action of the antiduality operator in case (A), but again we want to avoid repeated evaluation of this action on the original coördinate functions. Now, clearly, the coördinate family $\left(\left(I_{-\frac{1}{2}} b_n^{Nk} \right)_{n=1}^{N-k} \right)_{N=k+1}^{\infty}$ is well conditioned in $H^{-\frac{1}{2}}(\square)$ for any k, by the well-conditioning of the original family in $H^0(\square)$ and the isometric property of $I_{-\frac{1}{2}}$; this is the family that we wish to use as our new coördinate functions for case (B), but to reduce the work we shall again appeal to

the "back-projection" device, as in case (A), and hope that this does not destroy the property of well-conditioning.

However, we shall be inconsistent in approximating values of the images $J_{\Box_s}^{-1} I_{-\frac{1}{2}} b_n^{Nk} \in H_0^{\frac{1}{2}}(\Box)$, which are also required. Since we can write $J_{\Box_s}^{-1} I_{-\frac{1}{2}} = I_{\frac{1}{2}} I_{-\frac{1}{2}}^{-1} I_{-\frac{1}{2}} = I_{\frac{1}{2}}$, these images are just the $I_{\frac{1}{2}} b_n^{Nk}$, which can be computed in the same manner as the $I_{-\frac{1}{2}} b_n^{Nk}$, as described in [5].

Thus, we replace both the $I_{-\frac{1}{2}} b_n^{Nk}$ and $I_{\frac{1}{2}} b_n^{Nk}$ by their $H^0(\Box)$-orthoprojections $\mathcal{P}_N^k I_{-\frac{1}{2}} b_n^{Nk}$ and $\mathcal{P}_N^k I_{\frac{1}{2}} b_n^{Nk}$ onto $\mathcal{M}_N^k$. In effect, this returns us to the original coördinate subspaces, since the new coördinate functions now appear just as linear combinations of the original ones. Having already computed the inverse of the appropriate $H^0(\Box)$-Gram matrix with the Trench algorithm, after computing the requisite inner products $(I_{-\frac{1}{2}} b_n^{Nk}, b_l^{Nk})_{0,\Box}$ and $(I_{\frac{1}{2}} b_n^{Nk}, b_l^{Nk})_{0,\Box}$ for $l, n = 1, \ldots, N-k$, now two matrix-matrix multiplications are required for construction of the matrices $B^{Nk} := \{B_{nm}^{Nk}\}_{n,m=1}^{N-k}$ and $\widetilde{B}^{Nk} := \{\widetilde{B}_{nm}^{Nk}\}_{n,m=1}^{N-k}$ such that

$$\mathcal{P}_N^k I_{-\frac{1}{2}} b_n^{Nk} = \sum_{m=1}^{N-k} B_{nm}^{Nk} b_m^{Nk}, \qquad n = 1, \ldots, N-k, \tag{7.5}$$

and

$$\mathcal{P}_N^k I_{\frac{1}{2}} b_n^{Nk} = \sum_{m=1}^{N-k} \widetilde{B}_{nm}^{Nk} b_m^{Nk}, \qquad n = 1, \ldots, N-k. \tag{7.6}$$

With these transformation matrices and the matrices (7.3) and (7.4) already computed and stored, we produce an approximation to the N^{th} Bubnov-Galerkin matrix in this second case (B) in three steps:

(B.1) premultiplication of the matrix of (7.3) by the block matrix $\begin{pmatrix} B^{Nk} & B^{Nk} \\ B^{Nk} & B^{Nk} \end{pmatrix}$ and postmultiplication by its transpose, to effect the transformation implied in (7.5),

(B.2) premultiplication of the matrix of (7.4) by the same block matrix as in the first step, but postmultiplication by the transpose of $\begin{pmatrix} \widetilde{B}^{Nk} & \widetilde{B}^{Nk} \\ \widetilde{B}^{Nk} & \widetilde{B}^{Nk} \end{pmatrix}$ to perform the transformation following from (7.6),

and

(B.3) coupling of the resultant matrices with the parameter α.

As we pointed out, the evidence in Figures 1–6 strongly indicates that this modification of the coördinate functions imparts numerical stability to the previously unstable computation.

This completes the brief description of our numerical experiment with an application of the original reformulation to a lower-dimensional problem.

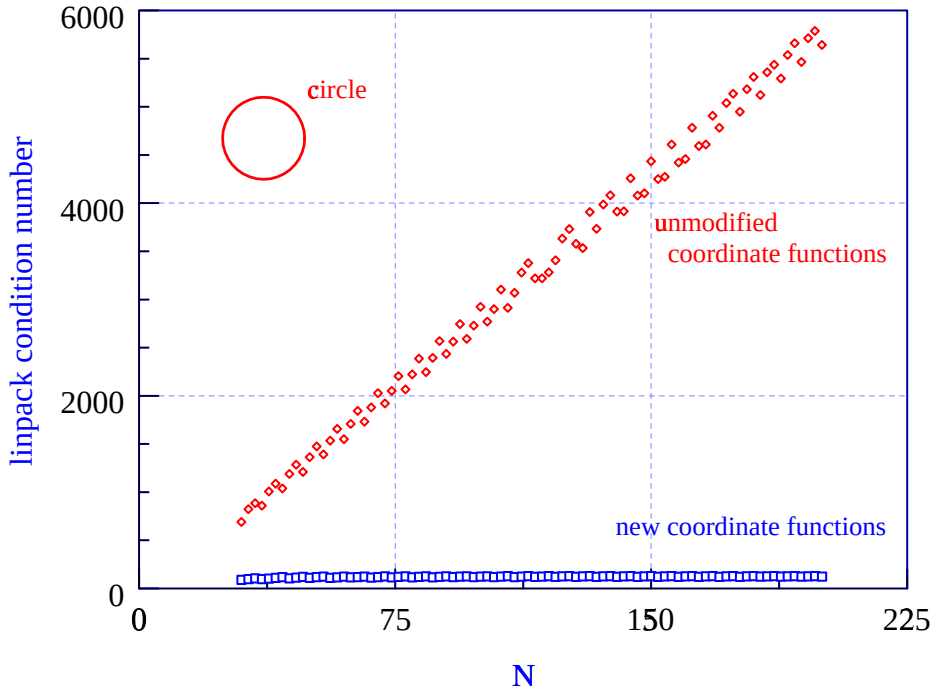

Fig. 1. circle: $\kappa a = 3.0$; $\alpha = 0.1$; $k = 3$

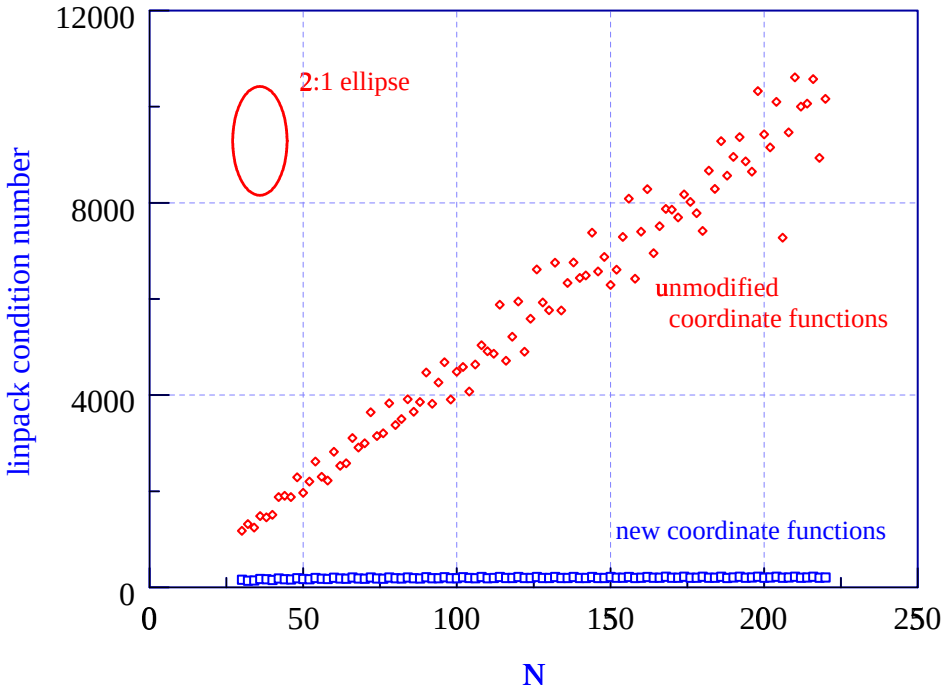

Fig. 2. 2:1 ellipse: $\kappa a = 3.0$; $\alpha = 0.1$; $k = 3$

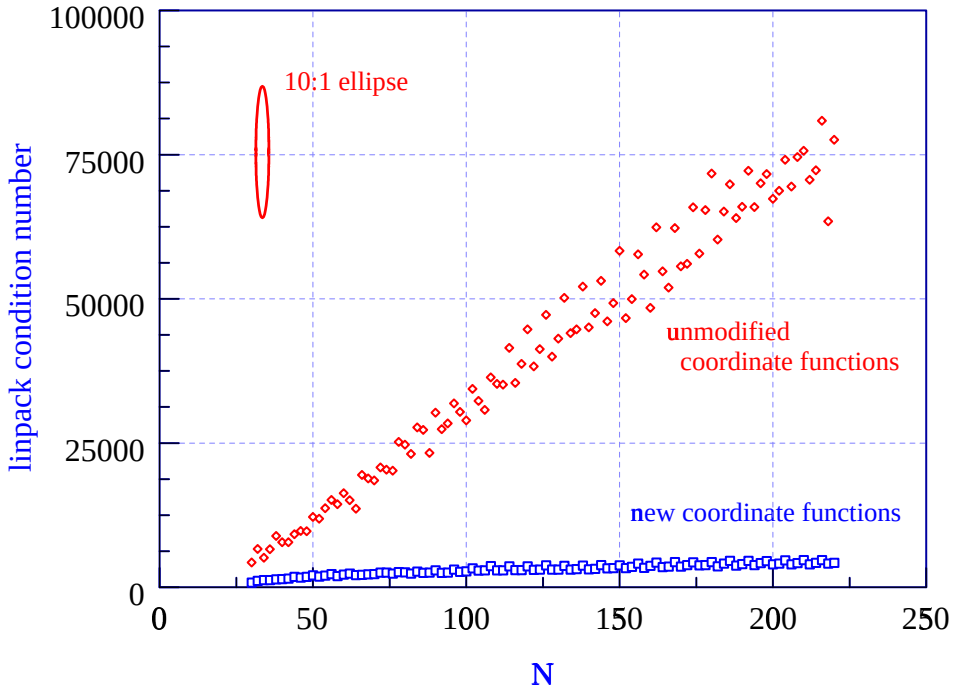

Fig. 3. 10:1 ellipse: $\kappa a = 3.0$; $\alpha = 0.1$; $k = 3$

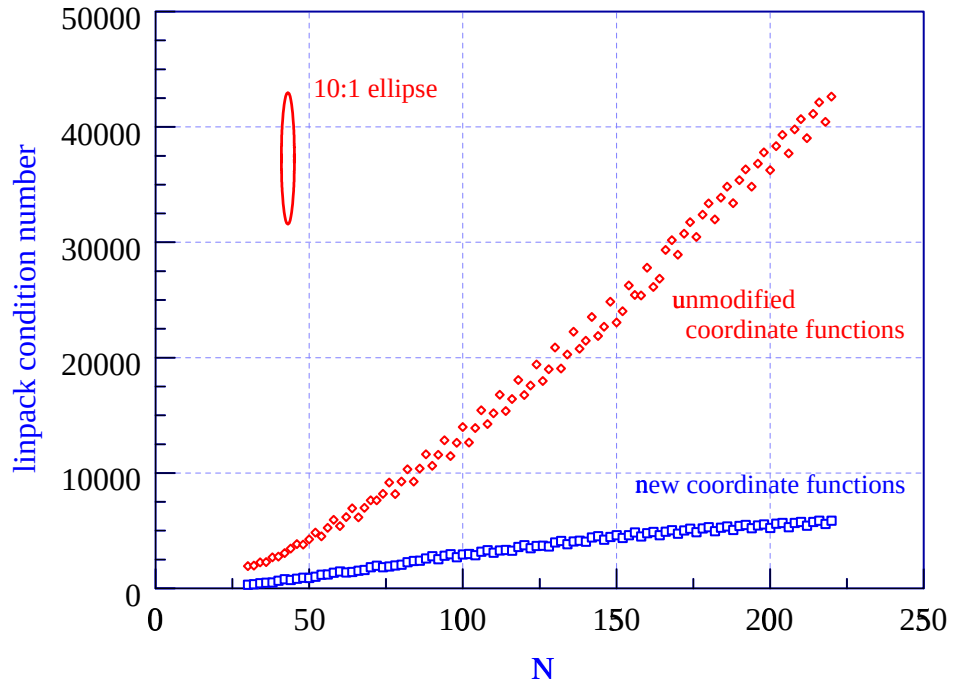

Fig. 4. 10:1 ellipse: $\kappa a = 10.0$; $\alpha = 0.1$; $k = 3$

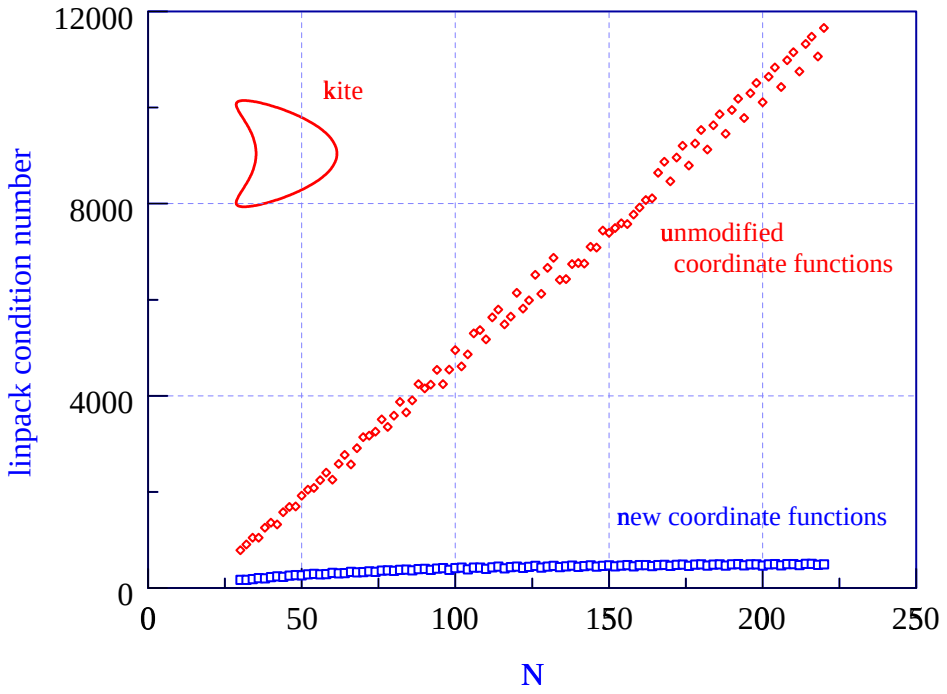

Fig. 5. kite: $\kappa a = 3.0$; $\alpha = 0.1$; $k = 3$

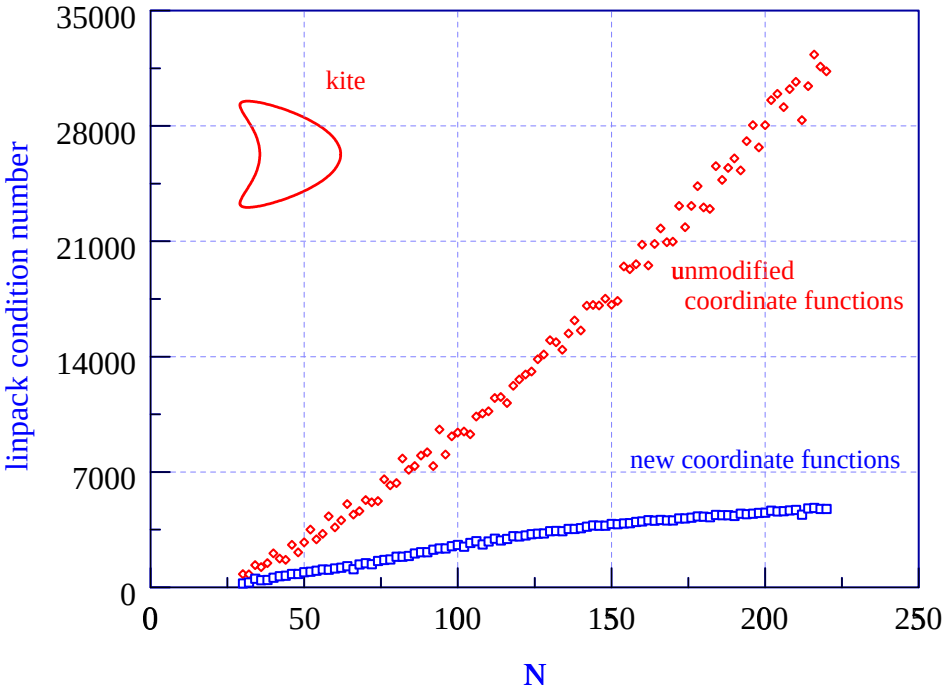

Fig. 6. kite: $\kappa a = 10.0$; $\alpha = 0.1$; $k = 3$

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