SPECTRAL AND COMBINATORIAL PROPERTIES OF FRIENDSHIP GRAPHS, SIMPLICIAL ROOK GRAPHS, AND EXTREMAL EXPANDERS

by

Jason R. Vermette

A dissertation submitted to the Faculty of the University of Delaware in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics

Spring 2015

SPECTRAL AND COMBINATORIAL PROPERTIES OF FRIENDSHIP GRAPHS, SIMPLICIAL ROOK GRAPHS, AND EXTREMAL EXPANDERS

by

Jason R. Vermette

Approved: ______________________________

Louis F. Rossi, Ph.D.
Chair of the Department of Mathematical Sciences

Approved: ______________________________

George H. Watson, Ph.D.
Dean of the College of Arts & Sciences

Approved: ______________________________

James G. Richards, Ph.D.
Vice Provost for Graduate and Professional Education

I certify that I have read this dissertation and that in my opinion it meets the academic and professional standard required by the University as a dissertation for the degree of Doctor of Philosophy.

Signed: ____

Sebastian M. Cioabă, Ph.D.
Professor in charge of dissertation

I certify that I have read this dissertation and that in my opinion it meets the academic and professional standard required by the University as a dissertation for the degree of Doctor of Philosophy.

Signed: ____

Robert S. Coulter, Ph.D.
Member of dissertation committee

I certify that I have read this dissertation and that in my opinion it meets the academic and professional standard required by the University as a dissertation for the degree of Doctor of Philosophy.

Signed: ____

Felix G. Lazebnik, Ph.D.
Member of dissertation committee

I certify that I have read this dissertation and that in my opinion it meets the academic and professional standard required by the University as a dissertation for the degree of Doctor of Philosophy.

Signed: ____

Steven K. Butler, Ph.D.
Member of dissertation committee

ACKNOWLEDGEMENTS

I would like to thank Andrew Lang for sparking my interest in the theoretical side of mathematics during my time as an undergraduate.

Thanks to my dissertation committee for taking the time to carefully read this thesis and offer advice and suggestions to improve it.

Thanks to my adviser, Sebi Cioabă, for his patience and instruction for the past several years. His guidance was integral to my success as a graduate student. I would also like to thank him for his extremely useful comments on earlier versions of this thesis.

Thanks most of all to my wife, Sharayah, for her endless support throughout my graduate career. I would certainly not have made it this far without her.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

ABSTRACT

Algebraic combinatorics is the area of mathematics that uses the theories and methods of abstract and linear algebra to solve combinatorial problems, or conversely applies combinatorial techniques to solve problems in algebra. In particular, spectral graph theory applies the techniques of linear algebra to study graph theory. Spectral graph theory is the study of the eigenvalues of various matrices associated with graphs and how they relate to the properties of those graphs. The graph properties of diameter, independence number, chromatic number, connectedness, toughness, hamiltonicity, and expansion, among others, are all related to the spectra of graphs. In this thesis we study the spectra of various families of graphs, how their spectra relate to their properties, and when graphs are determined by their spectra. We focus on three topics (Chapters 2–4) in spectral graph theory. The wide range of these topics showcases the power and versatility of the eigenvalue techniques such as interlacing, the common thread that ties these topics together.

In Chapter 1, we review the basic definitions, notations, and results in graph theory and spectral graph theory. We also introduce powerful tools for determining the structure of a graph and its subgraphs using eigenvalue interlacing. Finally, we discuss which graph properties can be deduced from the spectra of graphs, and which graphs are determined by their spectra.

In Chapter 2, we use eigenvalue interlacing to determine whether the friendship graphs and, more generally, graphs with exactly four distinct eigenvalues, are determined by their spectra. We show that friendship graphs are determined by their adjacency spectra with one exception, settling a conjecture in the literature. We also characterize the graphs with all but two eigenvalues equal to ± 1 and determine which of these graphs are determined by their spectra.

In Chapter 3 we study the simplicial rook graphs. We first determine several general properties of these graphs. Next, we determine the spectrum of some subfamilies of the simplicial rook graphs, find partial spectra for all simplicial rook graphs, and confirm several conjectures in the literature on the spectra of simplicial rook graphs, including the fact that the simplicial rook graphs have integral spectra.

In Chapter 4 we study the second largest adjacency eigenvalue of regular graphs. We determine upper bounds on the number of vertices in a regular graph with various given valencies and second eigenvalues, confirming or disproving many conjectures in the literature. In many cases we find the graphs, often unique, that meet these bounds. These graphs are called extremal expanders. We discuss a linear programming bound that guarantees that distance-regular graphs with certain parameters, if they exist, are extremal expanders.

In Chapter 5 we discuss open problems and questions for further research on the topics covered in Chapters 2–4.

Chapter 1

INTRODUCTION

1.1 Algebraic Combinatorics

Combinatorics is the branch of discrete mathematics that studies finite or countable discrete structures. Combinatorics has many areas of active research in the mathematical community, including graph theory, design theory, enumerative combinatorics, and extremal combinatorics. Algebraic combinatorics is the area of mathematics that uses the theories and methods of abstract and linear algebra to solve combinatorial problems, or conversely applies combinatorial techniques to solve problems in algebra. For general reference in combinatorics, see [20, 42, 63, 90].

In particular, spectral graph theory applies the techniques of linear algebra to study graph theory. Spectral graph theory is the study of the eigenvalues of various matrices associated with graphs and how they relate to the properties of those graphs. In general it is possible that two nonisomorphic graphs have the same spectrum (that is, multiset of eigenvalues), but the spectrum of a graph does give a large amount of information about the graph. For example, the spectrum of the adjacency matrix of a graph determines its number of vertices, edges, and closed walks of any fixed length, whether the graph is bipartite, whether it is regular, and, if it is regular, its girth. The spectrum of the Laplacian matrix of a graph determines its number of vertices, edges, connected components, and spanning trees, as well as whether the graph is regular, and, if it is regular, its girth. The graph properties such as diameter, independence number, chromatic number, connectedness, toughness, hamiltonicity, and expansion, among others, are all related to the spectrum. For an excellent reference on what is known about the spectra of graphs (and for definitions of the properties mentioned above), see [17]. In this thesis we study the spectra of various families of graphs, how their spectra relate to their properties, and when graphs are determined by their spectra. We will focus on three topics (Chapters 2–4).

In this chapter, we review the basic definitions, notations, and results in graph theory and spectral graph theory and describe tools for finding information about the structure of a graph and its subgraphs using eigenvalue interlacing. Finally, we discuss which graph properties can be determined using the spectra of graphs, and which graphs are determined by their spectra.

In Chapter 2, we study the friendship graphs and, more generally, graphs with exactly four distinct eigenvalues. The friendship graphs are known to be determined by their Laplacian and signless Laplacian spectra, and it has been conjectured that they are determined by their adjacency spectra. We use eigenvalue interlacing to show that friendship graphs are determined by their adjacency spectra with one exception. Then we characterize the set of graphs with exactly two eigenvalues not equal to ± 1 (the friendship graphs are among them), and determine which of these graphs are determined by their spectra.

In Chapter 3 we study the simplicial rook graphs. We determine several general properties of the simplicial rook graphs and relate these to their spectra. We find a partial spectrum of the simplicial rook graphs and determine the full spectrum of some sub-families. We also confirm several conjectures in the literature on the spectra of simplicial rook graphs, including the fact that the simplicial rook graphs have integral spectra.

In Chapter 4 we study the second largest adjacency eigenvalue of regular graphs. For various fixed k and λ, we determine upper bounds on the number of vertices in a k-regular graph with second eigenvalue at most λ, confirming or disproving many conjectures in the literature. In most cases we find the graphs, called extremal expanders, that meet these bounds. We discuss a linear programming bound that guarantees that when distance-regular graphs with certain parameters exist, they are extremal expanders.

In Chapter 5 we discuss open problems and questions for further research on the topics covered in Chapters 2–4.

Of course, all of the work done by me was under the supervision and advisement of Sebastian Cioabă.

1.2 Definitions, Notation, and Basics of Graph Theory

Definition 1.1. A *graph* G is a pair $(V(G), E(G))$, where $V(G)$ is a set of points (called *vertices*) and $E(G)$ is a set of pairs of vertices (called *edges*). If $\{x, y\} \in E(G)$ then we say x and y are *adjacent* and we write $x \sim y$; otherwise x and y are called *nonadjacent* and we write $x \not\sim y$. A vertex is said to be incident with the edges containing it (and vice versa). The *order* of a graph is $|V(G)|$.

To visualize graphs, we draw the vertices as dots, and when two vertices are adjacent, we draw a line connecting them. For example, if $G = P_3 = (\{x, y, z\}, \{\{x, y\}, \{y, z\}\})$ and $H = C_5 = (\{u, v, x, y, z\}, \{\{u, v\}, \{v, x\}, \{x, y\}, \{y, z\}, \{u, z\}\})$, then G and H can be visualized as the graphs in Figure 1.1 (the graphs P_n and C_n

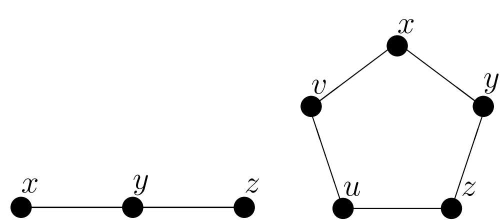


Figure 1.1: The Graphs P_3 and C_5.

will be defined in general in Definition 1.9). Implicit in our definition of graphs is that no edge appears twice in $E(G)$ (that is, $E(G)$ is strictly a set, not a multiset) and $\{x, x\} \notin E(G)$ for any $x \in V(G)$ (that is, the edges are also strictly sets, not multisets). This means that no vertex is adjacent to itself, and no pair of vertices are adjacent more than once.

Definition 1.2. Let G be a graph. A *clique* in G is a subset $S \subset V(G)$ such that every pair of vertices in S are adjacent. The *clique number* of G, denoted $\omega(G)$, is the size of a largest clique in G. An *independent set* (or *coclique*) in G is a subset $T \subset V(G)$

such that no two vertices in T are adjacent. The *independence number* of G, denoted $\alpha(G)$, is the size of a largest independent set in G.

For example, in Figure 1.1 the set $\{x, y\}$ is a clique in both P_3 and C_5, while the set $\{x, z\}$ is an independent set in both graphs. Note that cliques and cocliques can contain more than two vertices; however, there are none larger than two vertices in P_3 or C_5, which implies that $\alpha(P_3) = \omega(P_3) = \alpha(C_5) = \omega(C_5) = 2$.

Definition 1.3. A *subgraph* H of G is a graph H such that $V(H) = S \subseteq V(G)$ and $E(H) \subseteq E(G)$. A subgraph is called *induced* if for every $x, y \in V(H)$ we have $\{x, y\} \in E(H)$ if and only if $\{x, y\} \in E(G)$. If H is an induced subgraph of G, we say that G induces H (or that S induces H) and we write $H = G|_S$.

Seen another way, an induced subgraph is obtained from a graph by deleting some vertices and deleting only those edges that were incident with deleted vertices; all other edges are kept. For example, in Figure 1.1 we see that P_3 is a subgraph of C_5, and in fact P_3 is induced by C_5.

Definition 1.4. Let G be a graph, $S, T \subset V(G)$, and H, K subgraphs of G. We define $E(S, T) = \{\{u, x\} \in E(V) \mid u \in S,\ x \in T\}$, the set of edges from S to T. We define $E(H, K) = E(V(H), V(K))$.

Definition 1.5. A *walk* in a graph G is a sequence of vertices in $V(G)$ such that consecutive vertices are adjacent. The *length* of a walk is the number of edges traversed. A walk is called *closed* if the first and last vertex are the same. A *path* is a walk in which no vertex occurs more than once, except that the first vertex may also be the last, in which case the path is called *closed*. A closed path is also called a *cycle*. A graph G is *connected* if there is a path from x to y for every $x, y \in G$. For $x, y \in V(G)$, the *distance* between x and y, denoted $\operatorname{dist}(x, y)$, is the length of the smallest path containing both x and y. If no such path exists we say $\operatorname{dist}(x, y) = \infty$. By convention we say $\operatorname{dist}(x, x) = 0$. We say G is *connected* if $\operatorname{dist}(x, y)$ is finite for all $x, y \in V(G)$. Otherwise G is called *disconnected*. If G is disconnected and we partition $V(G)$ into sets $X_1, \ldots, X_k$ such that $G|_{X_i}$ is connected for each i and $x \not\sim y$ for $x \in X_i$ and $y \in X_j$ when $i \neq j$, then we call the sets $X_1, \ldots, X_k$ the *connected components* of G. If G is connected, the *diameter* of G is $\operatorname{diam}(G) = \max_{x,y \in V(G)} \operatorname{dist}(x, y)$, the largest distance between any two vertices in G. For $v \in V(G)$ and $S \subset V(G)$, the *distance* between v and S is $\operatorname{dist}(v, S) = \min_{u \in S} \operatorname{dist}(u, v)$. For $S, T \subset V(G)$, the *distance* between S and T is $\operatorname{dist}(S, T) = \min_{u \in S} \operatorname{dist}(u, T) = \min_{u \in S, x \in T} \operatorname{dist}(u, x)$.

For example, in Figure 1.1 (u, v, x, y) is a path of length 3 in C_5 (the edges traversed are $\{u, v\}$, $\{v, x\}$, and $\{x, y\}$), while (u, v, x, y, z, u) is a cycle of length 5 in C_5. In the graphs P_3 and C_5 we find that $\operatorname{dist}(x, y) = 1$ and $\operatorname{dist}(x, z) = 2$. Similarly, in C_5 we find that $\operatorname{dist}(u, \{v, y\}) = 1$ and $\operatorname{dist}(u, \{x, y\}) = 2$. Both P_3 and C_5 are connected and have diameter 2.

Definition 1.6. Let G be a graph, $x \in V(G)$, $X \subset V(G)$, and H a subgraph of G. We define the *neighborhood* of x by $\Gamma(x) = \{v \in V(G) \mid v \sim x\}$. The elements of $\Gamma(x)$ are called the *neighbors* of x. We define $\Gamma_k(x) = \{v \in V(G) \mid \operatorname{dist}(v, x) = k\}$ and $\Gamma_{\geq k}(x) = \{v \in V(G) \mid \operatorname{dist}(v, x) \geq k\}$, the sets of vertices at distance exactly k and at least k, respectively, from x. The elements of $\Gamma_k(x)$ are called the *distance-k neighbors* of x. We define $\Gamma_k(X) = \{v \in V(G) \mid \operatorname{dist}(v, X) = k\}$ and $\Gamma_{\geq k}(X) = \{v \in V(G) \mid \operatorname{dist}(v, X) \geq k\}$. We define $\Gamma_k(H) = \Gamma_k(V(H))$ and $\Gamma_{\geq k}(H) = \Gamma_{\geq k}(V(H))$.

Note that $\Gamma(x) = \Gamma_1(x)$. For example, in Figure 1.1 the graphs P_3 and C_5 we have $\Gamma(y) = \{x, z\}$, and in C_5 we have $\Gamma_2(y) = \{u, v\}$, $\Gamma_1(\{u, v\}) = \{x, z\}$, and $\Gamma_1(\{x, z\}) = \{u, v, y\}$.

Definition 1.7. Let G be a graph. Then the *complement* of G, denoted $\overline{G}$, is the graph with vertex set $V(G)$ and for all $x, y \in V(G)$, $\{x, y\} \in E(\overline{G})$ if and only if $\{x, y\} \notin E(G)$.

For example, the complements of P_3 and C_5 in Figure 1.1 are given in Figure 1.2.

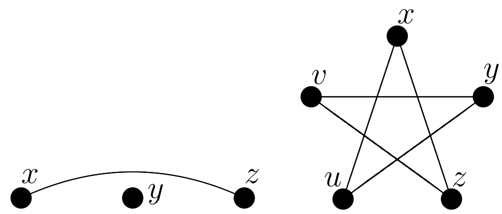


Figure 1.2: The complements of P_3 and C_5.

Definition 1.8. A graph G is said to be *bipartite* if we can partition $V(G)$ into disjoint sets X and Y such that each of X and Y are independent sets (so the only edges are those from a vertex in X to a vertex in Y).

For example, in Figure 1.1 P_3 is bipartite because we can partition $V(P_3)$ into the sets $\{x, z\}$ and $\{y\}$, and clearly every edge has one vertex in $\{x, z\}$ and one vertex in $\{y\}$.

Definition 1.9. The *complete graph* K_n is a graph with n vertices, every pair of which are adjacent. The *complete bipartite graph* $K_{m,n}$ is a graph with vertices partitioned into independent sets X and Y with $|X| = m$ and $|Y| = n$ such that every vertex in X is adjacent to every vertex in Y. The *cycle graph* C_n is a graph with n vertices such that the only edges in C_n are those in a cycle of length n. The *path graph* P_n is a graph with n vertices such that the only edges in P_n are those in a path of length $n - 1$. The *empty graph* E_n is a graph with n vertices and no edges. A circulant graph $\mathrm{Ci}_n(a_1, a_2, \ldots, a_k)$ is a graph on n vertices that can be labeled $v_1, v_2, \ldots, v_n$ so that for every $i \in \{1, 2, \ldots, n\}$, $v_i \sim v_j$ if and only if $j = i \pm a_\ell \pmod n$ for some $\ell \in \{1, 2, \ldots, k\}$. The *complete m-partite graph* $K_{a_1,\ldots,a_m}$ is a graph with vertices partitioned into independent sets X_i with $|X_i| = a_i$ such that each vertex in X_i is adjacent to every vertex in X_j for all $j \neq i$. A complete m-partite graph is also called a *complete multipartite graph*. The complete m-partite graph $K_{2,2,\ldots,2}$ is also called the *cocktail party graph* $CP(m)$.

We see that in Figure 1.1 P_3 is indeed a path graph on 3 vertices, and C_5 is a cycle graph on 5 vertices. We see also that $P_3 \cong K_{1,2}$. For more examples of the graphs

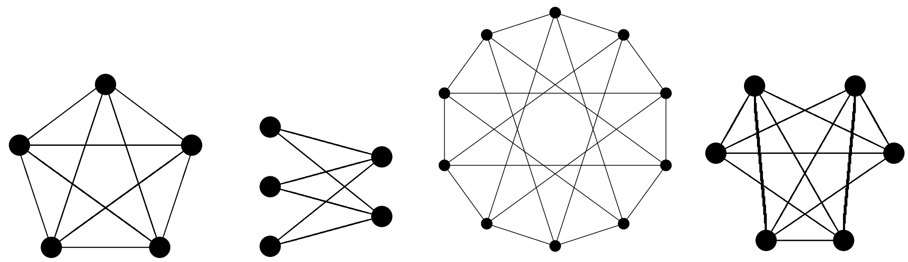


Figure 1.3: The graphs K_5, $K_{3,2}$, $\mathrm{Ci}_{10}(1,4)$, and $K_{2,2,2} \cong CP(3)$.

in Definition 1.9, see Figure 1.3. Here we point out that if we relabel the vertices of the graphs in Figure 1.1, they are still known as P_3 and C_5, since they are still the "same" graph. Of course, we must now more precisely define what we mean by the "same" graph.

Definition 1.10. Graphs G and H are *isomorphic*, denoted $G \cong H$, if there exists a bijection $f : V(G) \to V(H)$ such that $\{x, y\} \in E(G)$ if and only if $\{f(x), f(y)\} \in E(H)$.

Seen another way, G and H are isomorphic if each is simply a relabeling of the vertices of the other. For example, the graph C_5 as labeled in Figure 1.1 is isomorphic to $\overline{C_5}$ as labeled in Figure 1.2 under the bijection $f : V(C_5) \to V(\overline{C_5})$ defined as follows: $f(u) = u$, $f(v) = x$, $f(x) = z$, $f(y) = v$, and $f(z) = y$. Less formally, we might simply observe that both C_5 and $\overline{C_5}$ are cycles on 5 vertices. Similarly, C_5 with vertices labeled in any way is still C_5.

Definition 1.11. Let G and H be graphs. The *graph union* $G \cup H$ is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. If $V(G)$ and $V(H)$ are disjoint, we call $G \cup H$ a *disjoint union* and write it as $G + H$. The graph kG is the disjoint union of k copies of G. If $V(G)$ and $V(H)$ are disjoint, the *graph join* $G \nabla H$ is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{\{x, y\} \mid x \in V(G),\ y \in V(H)\}$.

We see that the cocktail party graph $CP(k)$ is the complement of the disjoint union kK_2. See Figures 1.4 and 1.5 for more examples of a graph union, disjoint union, and join.

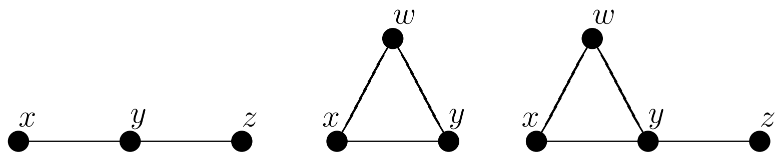


Figure 1.4: The graphs P_3, C_3, and $P_3 \cup C_3$.

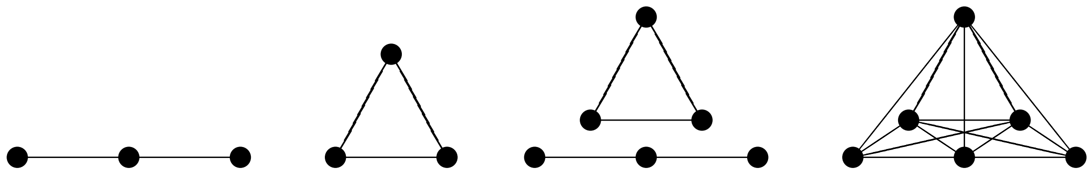


Figure 1.5: The graphs P_3, C_3, $P_3 + C_3$, and $P_3 \nabla C_3$.

Definition 1.12. The *graph Cartesian product* of two simple graphs G and H, denoted $G\Box H$, is the graph with vertex set $V(G) \times V(H)$ such that for any $u, v \in V(G)$ and $x, y \in V(H)$, the vertices (u, x) and (v, y) are adjacent in $G\Box H$ if and only if either $u = v$ and $x \sim y$ in H or $x = y$ and $u \sim v$ in G. The Cartesian product of k simple graphs $G_1, \ldots, G_k$, denoted $\prod_{i=1}^{k} G_i$ is the graph with vertex set $V(G_1) \times \ldots \times V(G_k)$ such that $(x_1, \ldots, x_k) \sim (y_1, \ldots, y_k)$ if and only if there exists i such that $x_i \sim y_i \in G_i$ and $x_j = y_j$ for $j \neq i$. The *t-fold Cartesian product* $G\Box_t H$ denotes the graph obtained by performing the graph Cartesian product t times: $G\Box_0 H = G$, $G\Box_1 H = G\Box H$, and $G\Box_t H = (G\Box_{t-1} H)\Box H$ for $t > 1$.

Another way to view $G\Box H$ is as follows: replace each vertex in G by a copy of H. A vertex in a copy of H is adjacent to the same vertex in another copy of H if there was an edge in G between the vertices that those two copies of H replaced. See Figure 1.6 for an example.

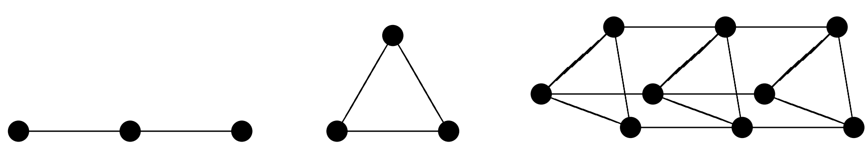


Figure 1.6: The graphs P_3, C_3, and $P_3\Box C_3$.

Definition 1.13. The *n-cube* Q_n the n-fold Cartesian product $K_1 \square_n K_2$.

See Figure 1.7 for a picture of Q_n for the first few values of n.

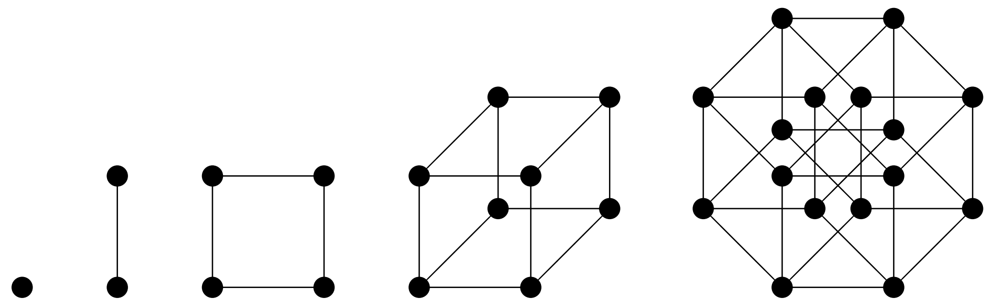


Figure 1.7: The graphs Q_0, Q_1, Q_2, Q_3, and Q_4.

When a graph is made up of smaller graphs with some edges between them (for example, as in the case of the graph join or graph Cartesian product), it is sometimes convenient to draw the graph in blocks. We use large open circles to denote multiple vertices, and label the circles according to the subgraph induced by those vertices. Solid lines between two large circles, or between a large circle and a vertex, indicate that every possible edge is present (for example, in a graph join). Dashed or dotted lines between large circles that induce isomorphic copies of the same graph represent a matching between corresponding vertices (for example, in a graph Cartesian product) or every edge except a matching between corresponding vertices, respectively. See Figure 1.8 for examples.

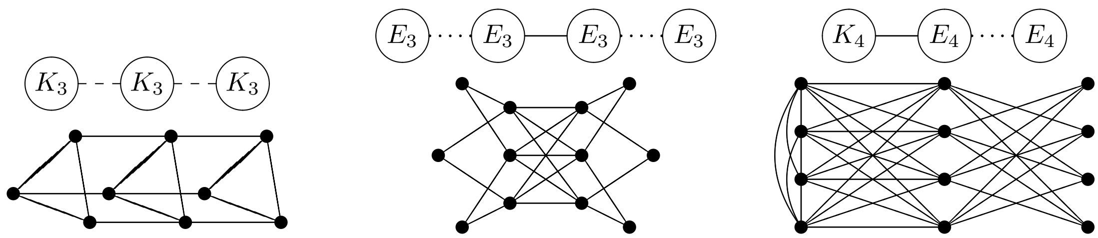


Figure 1.8: Block and standard representation of graphs.

Definition 1.14. Let G be a graph and $u, v \in V(G)$. The *degree* of v, denoted $\deg(v)$, is the number of vertices adjacent to v (that is, $\deg(v) = |\Gamma(v)|$). We denote by $\deg(u,v)$ the number of vertices adjacent to both u and v (that is, $\deg(u,v) = |\Gamma_1(\{u,v\})|$). The *average degree* of G, denoted $\overline{d}(G)$, is the average of the degrees of the vertices in G. The *minimum degree* of G, denoted $\delta(G)$, is equal to $\min_{x\in V(G)} \deg(x)$. The *maximum degree* of G, denoted $\Delta(G)$, is equal to $\max_{x\in V(G)} \deg(x)$. If $\deg(u) = \overline{d}(G) = k$ for all $u \in V(G)$, then we say G is *k-regular*. The *valency* of a k-regular graph is k.

For example, in the graph $P_3 \cup C_3$ as labeled in Figure 1.4, we have $\deg(y) = 3$, $\deg(w,x) = 1$, and $\overline{d}(P_3 \cup C_3) = 2$. It is a straightforward counting exercise to verify that the sum of degrees of the vertices in a graph is twice the number of edges in the graph. Any disjoint union of cycle graphs is 2-regular, and it is straightforward to see that conversely any 2-regular graph is a disjoint union of cycle graphs. The complete graph K_n is $(n-1)$-regular, and the complete bipartite graph $K_{n,n}$ is n-regular.

Definition 1.15. The *girth* of a graph G is the size of the smallest cycle contained in G. If G contains no cycles, we say that the girth of G is infinite.

For example, the complete graph K_n has girth 3 for $n \geq 3$, the n-cube Q_n has girth 4 if $n \geq 2$, and the cycle graph C_n has girth n. The Petersen graph (see Figure 1.9) has girth 5, the Heawood graph (see Figure 1.11) has girth 6, and the Tutte–Coxeter graph (see Figure 1.12) has girth 8.

The girth of a graph gives a lower bound on the number of vertices in the graph. Trivially a graph G with girth g must have at least g vertices (since G contains a cycle on g vertices). This implies that there is a smallest graph (or graphs) with girth g for each g. Indeed, the cycle C_g is obviously the smallest graph with girth g. When G is not 2-regular, G must be larger, but there is still a smallest graph (or graphs) with girth g. Let $n(k,g)$ denote the number of vertices in a smallest k-regular graph with girth g. A (k,g)-cage is a k-regular graph with girth g on $n(k,g)$ vertices. The following lower bound on $n(k,g)$ due to Tutte [94] will be useful.

Lemma 1.1. *Define* $n_l(k,g)$ *by*

$$n_l(k,g) = \begin{cases} \frac{k(k-1)^{(g-1)/2}-2}{k-2} & \text{if } g \text{ is odd,} \\ \frac{2(k-1)^{g/2}-2}{k-2} & \text{if } g \text{ is even.} \end{cases}$$

Then $n(k,g) \geq n_l(k,g)$.

Definition 1.16. Let G be a k-regular graph on n vertices. If there exist integers λ and μ such that $\deg(x,y) = \lambda$ for all $x, y \in V(G)$ such that $x \sim y$ and $\deg(x,y) = \mu$ for all $x, y \in V(G)$ such that $x \not\sim y$, then G is called *strongly regular* with parameters (n, k, λ, μ).

For example, the graph C_5 is strongly regular with parameters $(5, 2, 0, 1)$, the Petersen graph (see Figure 1.9) is strongly regular with parameters $(10, 3, 0, 1)$, and the

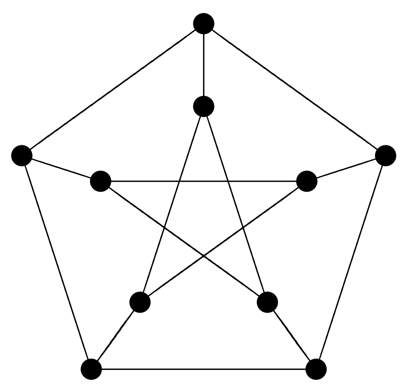


Figure 1.9: The Petersen graph.

rook graph $R(2,3)$ (see Figure 1.10 and Section 3.1) is strongly regular with parameters $(9, 4, 1, 2)$.

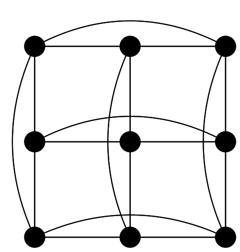


Figure 1.10: The Rook graph $R(2,3)$.

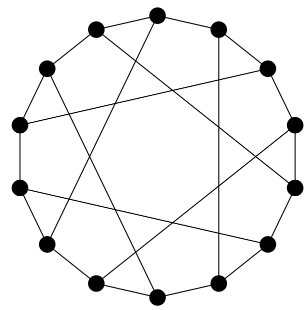


Figure 1.11: The Heawood graph.

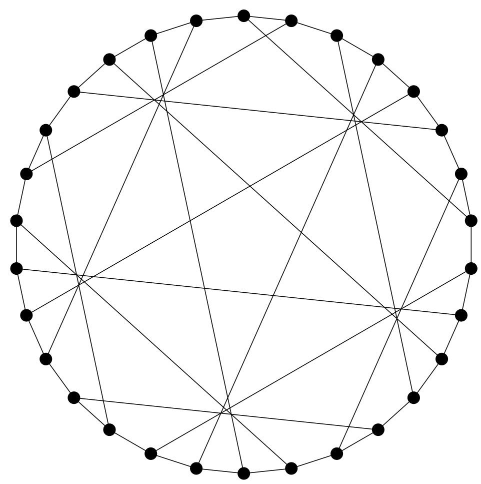


Figure 1.12: The Tutte–Coxeter graph.

Definition 1.17. A k-regular graph G is called *distance-regular* if, for every $x \in V$, any y in $\Gamma_i(x)$ has the same number c_i of neighbors in $\Gamma_{i-1}(x)$ and the same number b_i of neighbors in $\Gamma_{i+1}(x)$ (and the numbers c_i and b_i do not depend on the choice of x). The *intersection array* of a distance-regular graph with diameter d is $\{b_0 = k, b_1, \ldots, b_{d-1}; c_1 = 1, c_2, \ldots, c_d\}$.

We note that, because a distance-regular graph is regular with valency k for some k, the vertices in $\Gamma_i(x)$ also have the same number $a_i = k - b_i - c_i$ of neighbors in $\Gamma_i(x)$. A strongly regular graph is distance-regular with diameter 2. The Heawood graph (see Figure 1.11) is a distance-regular graph with intersection array $\{3, 2, 2; 1, 1, 3\}$ (see [9]). The Tutte-Coxeter graph (see Figure 1.12) is distance-regular with intersection array $\{3, 2, 2, 2; 1, 1, 1, 3\}$ (see [14, Theorem 7.5.1]). We give a few examples of infinite families of distance-regular graphs below.

Definition 1.18. Let $X = \{1, 2, \ldots, n\}$. The *Hamming graph* $H(m, n)$ is the graph whose vertices are the elements of X^m, where two vertices are adjacent when they differ in exactly one coordinate.

The Hamming graph $H(m, 2)$ is isomorphic to the m-cube Q_m (see Figure 1.7). Note also that $H(2, 3)$ is isomorphic to the Rook graph $R(2, 3)$ in Figure 1.10. The Hamming graphs $H(m, n)$ are distance-regular with valency $m(n-1)$, parameters $c_i = i$, $b_i = (n-1)(m-i)$, and diameter m (see, for example, [14, Section 9.2] and [17, Section 12.4.1]).

Definition 1.19. The *Johnson graph* $J(m, n)$ is the graph whose vertices are the n-element subsets of $\{1, 2, \ldots, m\}$, where two vertices are adjacent when they have $n-1$ elements in common.

The Johnson graph $J(5, 2)$ is isomorphic to the complement of the Petersen graph (see Figure 1.9). The Johnson graphs are distance-regular with valency $n(m-n)$, parameters $c_i = i^2$, $b_i = (n-i)(m-n-i)$, and diameter $\min\{n, m-n\}$ (see, for example, [14, Section 9.1] and [17, Section 12.4.2]).

Definition 1.20. A *permutation* of the set $\{1, 2, \ldots, m\}$ is a reordering of the elements of the set. We denote by $\mathcal{S}_m$ the set of all permutations of the set $\{1, 2, \ldots, m\}$.

Definition 1.21. [63, p. 233–240] A *Steiner triple system* $\mathrm{STS}(m)$ is a set of 3-element subsets of $\{1, 2, \ldots, m\}$ (called *triples*) such that every 2-element subset of $\{1, 2, \ldots, m\}$ (called a *pair*) is contained in exactly one triple. A *Kirkman triple system* $\mathrm{KTS}(m)$ is an $\mathrm{STS}(m)$ with the additional property that the set of triples can be partitioned into subsets such that each element in $\{1, 2, \ldots, m\}$ is contained in exactly one triple in each subset of triples. This additional property is called *parallelism*, and these subsets of triples are called the *parallel classes* of the $\mathrm{KTS}(m)$.

For example, both an STS(9) and a KTS(9) are given by the set

$$\begin{aligned}\{\{1,2,3\},\{4,5,6\},\{7,8,9\},\\ \{1,4,7\},\{2,5,8\},\{3,6,9\},\\ \{1,5,9\},\{2,6,7\},\{3,4,8\},\\ \{1,6,8\},\{2,4,9\},\{3,5,7\}\}.\end{aligned}$$

Each line above is a parallel class in the KTS(9). Indeed, we can easily verify that each number 1–9 appears once per line. Steiner and Kirkman triple systems exist only for certain values of m, but they have been shown to always exist for those values [57, 80]:

Proposition 1.2. *A Steiner triple system* STS(m) *exists if and only if* $m \equiv 1, 3 \pmod 6$. *A Kirkman triple system* KTS(m) *exists if and only if* $m \equiv 3 \pmod 6$.

By a simple counting argument, both an STS(m) and a KTS(m) contain $\binom{m}{2}/3 = \frac{1}{6}m(m-1)$ triples. By definition each pair in $\{1, 2, \ldots, m\}$ is contained in exactly one triple, which implies that each element is contained in $(m-1)/2$ triples.

1.3 Spectra of Graphs

We first state some basic results from linear algebra that will be helpful. For general reference see [4, 61]. For a matrix M, we denote by $M_{(i,j)}$ the entry in the i-th row and j-th column of M. The multiset of eigenvalues of a matrix is called the spectrum of the matrix.

Definition 1.22. The *identity matrix* I_n is the $n \times n$ matrix with diagonal entries 1 and all other entries 0. The matrix $J_{m,n}$ is the $m \times n$ matrix with every entry 1. We denote $J_n = J_{n,n}$ and $\mathbf{1}_n = J_{1,n}$ (the all ones vector). The matrix $O_{m,n}$ is the $m \times n$ matrix with every entry 0. We denote $O_n = O_{n,n}$ and $\mathbf{0}_n = O_{1,n}$ (the zero vector). When the size is implicit, we often write simply I, J, O, $\mathbf{1}$, and $\mathbf{0}$ for these matrices. For any matrix M, the transpose $M^\top$ is the matrix with (i, j)-entry equal to $M_{(j,i)}$. A matrix M is called *symmetric* if $M^\top = M$.

Proposition 1.3. ([61, p. 106]) *A real, symmetric $n \times n$ matrix has n real eigenvalues (including multiplicities), and there is a set of n orthonormal eigenvectors for these eigenvalues.*

Proposition 1.4. *The eigenvalues of M^k are $\{\lambda^k \mid \lambda$ is an eigenvalue of $M\}$. The eigenvalues of $M - kI$ are $\{\lambda - k \mid \lambda$ is an eigenvalue of $M\}$.*

Definition 1.23. A real matrix M is called *positive semidefinite* if $x^\top M x \geq 0$ for all real vectors x. Equivalently, M is positive semidefinite if all of the eigenvalues of M are nonnegative.

Proposition 1.5. *For any real, symmetric matrix M, the matrices $M^\top M$ and $MM^\top$ are positive semidefinite. The matrices $M^\top M$ and $MM^\top$ have the same rank and the same nonzero eigenvalues (including multiplicity).*

Definition 1.24. The trace $\mathrm{Tr}(M)$ of a square matrix M is the sum of the diagonal entries of M.

Proposition 1.6. *The sum of the eigenvalues of a matrix M equals $\mathrm{Tr}(M)$.*

Since all of the diagonal entries of the adjacency matrix of any graph are 0, Proposition 1.6 gives the following corollary:

Corollary 1.7. *For any graph G, the sum of the eigenvalues of G equals 0.*

For a real, symmetric matrix M we denote by $\lambda_1(M) \geq \lambda_2(M) \geq \cdots \geq \lambda_n(M)$ the eigenvalues of M.

Theorem 1.8 (Ky Fan Inequality [40])**.** *Let A and B be real, symmetric matrices of size n. Then for any $1 \leq k \leq n$,*

$$\sum_{i=1}^{k} \lambda_i(A+B) \leq \sum_{i=1}^{k} \lambda_i(A) + \sum_{i=1}^{k} \lambda_i(B).$$

By Proposition 1.6, taking $k = n$ in Theorem 1.8 obviously gives equality. Combining this with the inequality obtained by taking $k = n-1$ in Theorem 1.8, we obtain the following corollary:

Corollary 1.9. *Let A and B be real symmetric matrices of size n. Then $\lambda_n(A+B) \geq \lambda_n(A) + \lambda_n(B)$.*

For the remainder of this section we will give basic results of spectral graph theory and give the spectra of some common graphs. For a general reference on spectral graph theory, see [17]. There are several matrices that are associated with a given graph. When we discuss a matrix associated with a graph, we assume the rows and columns of the matrix are indexed by the vertices of the graph.

Definition 1.25. Let G be a graph. The *adjacency matrix* $A = A(G)$ of G is the matrix with $A_{(x,y)} = 1$ if $x \sim y$ and $A_{(x,y)} = 0$ if $x \not\sim y$. The *degree matrix* $D = D(G)$ of G is the diagonal matrix with $D_{(x,x)} = \deg(x)$. The *Laplacian matrix* $L = L(G)$ satisfies $L = D - A$. The *signless Laplacian matrix* $|L| = |L(G)|$ satisfies $|L| = D + A$. The *normalized Laplacian* $\mathcal{L} = \mathcal{L}(G)$ satisfies $\mathcal{L} = D^{-1/2}LD^{-1/2}$.

In this thesis we will focus on the adjacency matrix, but each matrix contains information about the graph. For example, one can prove that the nullity of $L(G)$ is the number of connected components in G, and one can prove by induction that the x, y entry in $A(G)^k$ is the number of walks of length k in G beginning at the vertex x and ending at the vertex y.

Each of the graphs in Definition 1.25 is a real, symmetric matrix, and hence have n real eigenvalues (including multiplicities) if G has n vertices. The eigenvalues of the adjacency, Laplacian, signless Laplacian, and normalized Laplacian matrices of a graph are called the adjacency, Laplacian, signless Laplacian, and normalized Laplacian eigenvalues of the graph. We will refer to the adjacency eigenvalues of a graph as simply the eigenvalues of the graph. We will refer to the adjacency spectrum as simply the spectrum of the graph. When we write the spectrum of a graph, we use exponents to denote the multiplicities. We denote by $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$ the adjacency eigenvalues of G. We denote by $\lambda_{\min}(G)$ and $\lambda_{\min-1}(G)$ the smallest and second smallest eigenvalues of G (so $\lambda_{\min}(G) = \lambda_n(G)$ and $\lambda_{\min-1}(G) = \lambda_{n-1}(G)$ if G has n vertices).

For example, the adjacency matrices of P_3 and C_5 (as labeled in Figure 1.1) are

$$A(P_3) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

and

$$A(C_5) = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Thus we find that the spectrum of P_3 is $\{\sqrt{2}^1, 0^1, -\sqrt{2}^1\}$, and the spectrum of C_5 is $\{2^1, \frac{1}{2}(\sqrt{5}-1)^2, \frac{1}{2}(-\sqrt{5}-1)^2\}$ (recall that exponents on eigenvalues indicated their multiplicity). The spectra of many of the graphs defined in Section 1.2 are given below. The results are well known (see, for example, [17, Section 1.4])

Proposition 1.10. *The spectrum of* K_n *is* $\{(n-1)^1, -1^{n-1}\}$. *The spectrum of* $K_{m,n}$ *is* $\{\sqrt{mn}^1, 0^{m+n-2}, -\sqrt{mn}^1\}$. *The spectrum of* C_n *is* $\left\{2\cos\left(\frac{2\pi j}{n}\right)^1 \mid j = 0, 1, \ldots, n-1\right\}$. *The spectrum of* P_n *is* $\left\{2\cos\left(\frac{\pi j}{n+1}\right)^1 \mid j = 1, \ldots, n\right\}$. *The spectrum of* E_n *is* 0^n.

Definition 1.26. Let M be a matrix with eigenvalues $\mu_1, \ldots, \mu_n$. The *spectral radius* of M, denoted $\rho(M)$, is equal to $\max\{|\mu_1|, \ldots, |\mu_n|\}$. The spectral radius of a graph G, denoted $\rho(G)$, is equal to $\rho(A(G))$.

Clearly $\rho(G) = \max\{\lambda_1(G), -\lambda_n(G)\}$. However, it turns out $\rho(G) = \lambda_1(G)$ due to the following corollary, which follows immediately from the Perron-Frobenius Theorem for irreducible matrices (see [17, Section 2.2] and [45, Section 8.8]).

Corollary 1.11. *The spectral radius of a graph is the largest eigenvalue of the graph.*

This also implies that if s is an eigenvalue of a graph G, then G has an eigenvalue $r > 0$ such that $r \geq |s|$.

The spectral radius of a graph is related to the average degree.

Proposition 1.12. ([17, Proposition 3.1.2]) *Let G be a connected graph. If G is k-regular, then $\rho(G) = k$. Otherwise we have $\delta(G) < \overline{d}(G) < \rho(G) < \Delta(G)$.*

In particular, we obtain the following corollary.

Corollary 1.13. *If G is a k-regular graph then $\lambda_1(G) = k$.*

The next two Propositions relate graph properties to the spectrum.

Proposition 1.14. ([17, Proposition 1.3.3]) *A graph with diameter d has at least $d+1$ distinct adjacency eigenvalues.*

The Hoffman ratio bound relates the independence number of regular graphs to their smallest eigenvalue (see, for example, [17, Theorem 3.5.2]).

Proposition 1.15. *If G is a connected, k-regular graph on n vertices, then*

$$\alpha(G) \leq n\frac{-\lambda_n(G)}{k - \lambda_n(G)}.$$

The eigenvalues of a graph Cartesian product $G\Box H$ can be given in terms of the eigenvalues of G and H (see, for example, [17, Section 1.4.6]).

Definition 1.27. If A is an $m \times n$ matrix and B is a $p \times q$ matrix, then the *Kronecker product* $A \otimes B$ is the $mp \times nq$ block matrix given by

$$\begin{pmatrix} A_{(1,1)}B & \cdots & A_{(1,n)}B \\ \vdots & \ddots & \vdots \\ A_{(m,1)}B & \cdots & A_{(m,n)}B \end{pmatrix}.$$

Proposition 1.16. *Let G and H be graphs on n and m vertices, respectively. The adjacency matrix of $G\Box H$ is $A(G\Box H) = A(G) \otimes I_m + I_n \otimes A(H)$. The eigenvalues of $G\Box H$ are $\lambda_i(G) + \lambda_j(H)$ for $1 \leq i \leq n$ and $1 \leq j \leq m$.*

Corollary 1.17. *The spectrum of Q_n is $\left\{(n-2i)^{\binom{n}{i}} \mid i = 0, 1, \ldots, n\right\}$.*

Proposition 1.18. *The spectrum of a disconnected graph is the multiset sum of the spectra of the subgraphs induced by connected components.*

That is, if G has connected components $X_1, \ldots, X_k$ and λ is an eigenvalue of $G|_{X_i}$ with multiplicity a_i for each i, then λ is an eigenvalue of G with multiplicity $a_1 + \cdots + a_k$. Proposition 1.18 leads immediately to following corollary, since the set of connected components of a disjoint union $G + H$ is the union of the connected components of G and those of H.

Corollary 1.19. *Let G and H be graphs on n and m vertices, respectively. The adjacency matrix of $G + H$ is*

$$\begin{pmatrix} A(G) & O_{n,m} \\ O_{m,n} & A(H) \end{pmatrix}.$$

The spectrum of $G + H$ is the multiset sum of the spectra of G and H.

Proposition 1.20. ([17, Section 1.2.3]) *If G is a connected, k-regular graph on n vertices with eigenvalues $k = \lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$, then its complement $\overline{G}$ has eigenvalues $n-k-1 \geq -1-\lambda_n(G) \geq -1-\lambda_{n-1}(G) \geq \cdots \geq -1-\lambda_3(G) \geq -1-\lambda_2(G)$.*

Proposition 1.20 follows from the fact that $A(\overline{G}) = J - A(G) - I$, and the fact that the eigenvector for k is $\mathbf{1}$ while the eigenvectors for $\lambda_2, \ldots, \lambda_n$ are orthogonal to $\mathbf{1}$.

Proposition 1.21. ([14, Section 9.2] and [17, Section 12.4.1]) *The adjacency eigenvalues of the Hamming graph $H(m,n)$ are $(n-1)m - ni$ with multiplicity $\binom{m}{i}(n-1)^i$ for $i = 0, 1, \ldots, m$.*

Proposition 1.22. ([14, Section 9.1] and [17, Section 12.4.2]) *The adjacency eigenvalues of the Johnson graph $J(m,n)$ are $(n-i)(m-n-i)-i$ with multiplicity $\binom{m}{i} - \binom{m}{i-1}$ for $i = 0, 1, \ldots, n$.*

1.4 Vertex Partitions, Quotient Matrices, and Eigenvalue Interlacing

In this section we give powerful tools for determining the structure of graphs with given spectrum. The results in this section involve eigenvalue interlacing and will be extremely useful throughout this thesis. For general reference see [17, Sections 2.3–2.5].

Definition 1.28. Let M be a real, symmetric matrix with rows and columns indexed by $X = \{1, 2, \ldots, n\}$, and let $\mathcal{P} = \{X_1, \ldots, X_m\}$ be a partition of X. The *characteristic matrix* S of $\mathcal{P}$ is the $n \times m$ matrix with $S_{(i,j)} = 1$ if $i \in X_j$ and $S_{(i,j)} = 0$ if $i \notin X_j$. The *cardinality matrix* K of $\mathcal{P}$ is the $m \times m$ diagonal matrix with $K_{(i,i)} = n_i$, where $n_i = |X_i|$. Let the rows and columns of M be indexed by $\mathcal{P}$, so that

$$M = \begin{pmatrix} M_{1,1} & \cdots & M_{1,m} \\ \vdots & \ddots & \vdots \\ M_{m,1} & \cdots & M_{m,m} \end{pmatrix},$$

where $M_{i,j}$ denotes the submatrix of M indexed by rows in X_i and columns in X_j. Then the *quotient matrix* of M with respect to $\mathcal{P}$ is the $m \times m$ matrix Q with $Q_{(i,j)} = q_{i,j}$, where $q_{i,j}$ is the average row sum of $M_{i,j}$. If the row sum of every row in each $M_{i,j}$ is equal to $q_{i,j}$, then we say the partition $\mathcal{P}$ is equitable.

From Definition 1.28 we immediately obtain:

Proposition 1.23. *If M is a real, symmetric matrix with $\mathcal{P}$, S, K, Q, and $M_{i,j}$ as in Definition 1.28, then we have $S^\top MS = KQ$ and $S^\top S = K$. If the partition is equitable then we additionally have $MS = SQ$.*

The fact that $S^\top S = K$ is straightforward. The (i, j)-entry in $S^\top MS$ is the sum of the entries in $M_{i,j}$, while the (i, j)-entry in KQ is $n_i q_{i,j}$, which is also the sum of the entries in $M_{i,j}$. If $i \in X_k$, then the (i, j)-entry of SQ is $q_{k,j}$, while the (i, j)-entry of MS is one of the row sums in $M_{k,j}$. If $\mathcal{P}$ is an equitable partition of M, then every row sum in $M_{k,j}$ is $q_{k,j}$, so in this case $MS = SQ$. We note also that if the partition is equitable, then $M_{i,j}\mathbf{1} = q_{i,j}\mathbf{1}$ for all $i, j \in \{1, 2, \ldots, m\}$. These results imply the following useful proposition.

Proposition 1.24. *If M is a real, symmetric matrix with $\mathcal{P}$, S, K, Q, and $M_{i,j}$ as in Definition 1.28, then for each eigenvector v of Q with eigenvalue λ, Sv is an eigenvector of M with eigenvalue λ. Then M has two types of eigenvalues:*

(i) *the eigenvalues of Q, with eigenvectors constant on X_j for all j, and* (*ii*) *the remaining eigenvalues, with eigenvectors summing to 0 on X_j for each j. These remaining eigenvalues are unchanged if blocks $M_{i,j}$ are replaced by blocks $M_{i,j} + c_{i,j}J$ for some constants $c_{i,j}$.*

Indeed, $Qv = \lambda v$ implies $MSv = SQv = S\lambda v = \lambda Sv$. To see that (*i*) is true, note that $(Sv)_i$ is equal to v_j for all $i \in X_j$. To see that (*ii*) is true, note that the remaining eigenvectors must be orthogonal to those from (*i*), which proves they sum to 0 on each X_j. Adding a multiple of the all ones matrix to a block does not change these eigenvalues precisely because their eigenvectors sum to 0 on each X_j.

If the matrix M above is the adjacency or Laplacian matrix of a graph G and X is the set $V(G)$, then the partition $\mathcal{P}$ is equitable precisely when every vertex in X_i has the same number of neighbors $q_{i,j}$ in X_j. For example, if G is a distance-regular graph with valency k, diameter d, and parameters b_i, c_i, and $a_i = k - b_i - c_i$ as in Definition 1.17, then for any x in $V(G)$ we can partition the vertices according to their distance from x, that is, by $\mathcal{P} = \{X_0, X_1, \ldots, X_d\}$ where $X_i = \Gamma_i(x)$ (this is called the *distance-partition* of G with respect to x). We see that each vertex in X_i has the same number $q_{i,j}$ of neighbors in X_j. Indeed, if $|i - j| \geq 2$, then $q_{i,j} = 0$, and we clearly have $q_{i,i} = a_i$, $q_{i,i+1} = b_i$, and $q_{i,i-1} = c_i$. This yields the following lemma.

Lemma 1.25. *The quotient matrix Q with respect to the distance partition of a distance-regular graph with intersection array $\{b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d\}$ is the tridiagonal matrix*

$$Q = \begin{pmatrix} a_0 = 0 & b_0 = k & 0 & \cdots & 0 \\ c_1 = 1 & a_1 & b_1 & \ddots & \vdots \\ 0 & c_2 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & a_{d-1} & b_{d-1} \\ 0 & \cdots & 0 & c_d & a_d \end{pmatrix}.$$

From Proposition 1.24 and Lemma 1.25 it follows that the eigenvalues of the matrix Q given above are also eigenvalues of the distance-regular graph G, and the remaining eigenvalues of G have eigenvectors which sum to 0 on each part of the distance partition.

Definition 1.29. For a real, symmetric matrix M and a nonzero vector u, the *Rayleigh quotient* of u with respect to M is

$$\mathrm{Ray}(M, u) = \frac{u^\top M u}{u^\top u}.$$

We have the following inequalities for the Rayleigh quotient, called the Rayleigh inequalities:

Proposition 1.26. ([17, Section 2.4]) *If M is a real, symmetric $n \times n$ matrix and $u_1, \ldots, u_n$ is a set of orthonormal eigenvectors such that $Mu_i = \lambda_i(M)u_i$ for $i = 1, \ldots, n$, then for a vector u we have*

(i) $\mathrm{Ray}(M, u) \geq \lambda_i(M)$ *if* $u \in \mathrm{span}\{u_1, \ldots, u_i\}$, *and*

(ii) $\mathrm{Ray}(M, u) \leq \lambda_i(M)$ *if* $u \in \mathrm{span}\{u_1, \ldots, u_{i-1}\}^\perp$.

In either case, equality implies u is an eigenvector of M for the eigenvalue λ_i.

Definition 1.30. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_m$ be two sequences of real numbers with $m < n$. The second sequence is said to *interlace* the first if $\lambda_i \geq \mu_i \geq \lambda_{n-m+i}$ for each $i = 1, \ldots, m$.

Using the Rayleigh inequalities (Proposition 1.26), one can prove the following two useful propositions about eigenvalue interlacing.

Proposition 1.27. *If M is a real, symmetric matrix with quotient matrix Q with respect to some partition $\mathcal{P}$, then the eigenvalues of Q interlace those of M.*

This result is due to Haemers (see [48, 49]). Note that for Proposition 1.27 we do not require that the partition is equitable. If M is the adjacency matrix of a graph G, then Proposition 1.27 implies that the eigenvalues of the quotient matrix of any partition of $V(G)$ interlace those of G. Note that if the partition $\mathcal{P} = \{X_1, \ldots, X_m\}$ is not equitable, then the (i, j)-entry $q_{i,j}$ of Q is the average number of neighbors in X_j of the vertices in X_i.

Definition 1.31. A *principal submatrix* of a matrix M is a matrix obtained by deleting some rows and the corresponding columns from M.

Proposition 1.28 (Cauchy Interlacing)**.** *If N is a principal submatrix of M, then the eigenvalues of N interlace those of M.*

Note that the adjacency matrix of an induced subgraph is a principal submatrix of the adjacency matrix of the graph. Indeed, one simply deletes the rows and columns in the adjacency matrix indexed by the vertices that were removed to obtain the induced subgraph. Proposition 1.28 often gives a large amount of information about the structure of a graph G with given spectrum, since the spectra of any induced subgraphs of G must interlace the spectrum of G. We give the following lemma as an example.

Lemma 1.29. *A graph G with $\lambda_2(G) < 0$ or $\lambda_{\min}(G) > -\sqrt{2}$ is a disjoint union of complete graphs.*

Proof. If G has no connected component with at least 3 vertices, then G is a disjoint union of copies of complete graphs of orders 1 and 2, and we are done. Otherwise, if G is not a disjoint union of complete graphs, then G must induce a subgraph isomorphic to $K_{1,2}$. Indeed, since G has a connected component with at least 3 vertices that is not a complete graph, that component contains a subgraph on three vertices isomorphic to $K_{1,2}$. Then Proposition 1.28 implies $\lambda_2(G) \geq \lambda_2(K_{1,2}) = 0$ and $\lambda_{\min}(G) \leq \lambda_{\min}(K_{1,2}) = -\sqrt{2}$, which contradicts that $\lambda_2(G) < 0$ or $\lambda_{\min}(G) > -\sqrt{2}$. □

We note that in the case where $\lambda_2(G) < 0$ we can further say that G is a complete graph, since a graph with k connected components has at least k positive eigenvalues by Corollaries 1.11 and 1.19.

The following Lemma due to Petrović [78] (see also [14, page 89]) is proved in a similar (but slightly more involved) way:

Lemma 1.30. *Let G be connected graph. Then G has exactly one positive eigenvalue if and only if G is a complete multipartite graph.*

1.5 Which Graphs are Determined by Their Spectra?

Since we have seen that the spectrum of a graph gives a large amount of information about the structure of a graph, it is natural to ask specifically what properties of a graph are determined by the spectrum, and whether the spectrum has enough information to recover the graph. For general graphs, this question has a different answer depending on whether we mean the adjacency, Laplacian, signless Laplacian, or some other spectrum of the graph.

Definition 1.32. If G and H are graphs with the same spectrum, then G and H are called *cospectral*. If G is a graph such that every graph H cospectral with G is isomorphic to G, then G is said to be *determined by its spectrum*, and we say that G is DS for short.

Definition 1.32 is used to refer to the spectra of many of the matrices associated with a graph G. We will use it to refer to adjacency spectra, unless specifically mentioned otherwise.

For general reference on what is known about which graphs are determined by their spectra, see [31, 32]. The question of which graphs are determined by their spectra originated in 1956 in a paper by Günthard and Primas [46] on Hückel's theory from chemistry. At that time, it was believed that every graph was DS. However, there is a pair of nonisomorphic graphs on 5 vertices first noted by Collatz and Sinogowitz [29] (see Figure 1.13). The graphs in Figure 1.13 each have spectrum $\{2^1, 0^3, -2^1\}$. Indeed,

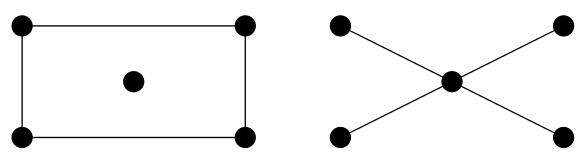


Figure 1.13: Two nonisomorphic graphs with spectrum $\{2^1, 0^3, -2^1\}$.

the graphs are $K_1 + C_4 \cong K_1 + K_{2,2}$ and $K_{1,4}$, so the stated spectrum follows by Proposition 1.10 and Corollary 1.19. It is obvious that the graphs are not isomorphic, since one is connected and the other is not. Thus not all graphs are determined by their spectra (though a quick search of graphs on 1–4 vertices shows that the graphs in Figure 1.13 are the smallest nonisomorphic cospectral graphs).

The question still remains, which graphs are DS? Further, one can ask: are most graphs DS? Schwenk [84] proved that most trees (that is, connected graphs without cycles) are not DS. It is not known whether most graphs in general are DS or not, but Haemers [31] conjectures that most graphs are DS. Recently the question of which graphs are DS has been an active area of research in spectral graph theory.

Typically it is much easier to show that a graph is not DS than to show that a graph is DS. Indeed, to show that a graph is not DS one needs only to find a single example of a nonisomorphic cospectral graph. Conversely, to show that a graph G is DS one must prove that among all graphs, no nonisomorphic graph has the same spectrum as G. Below we give a few results on which graphs are DS. Each can be found in [31].

Proposition 1.31. *The following properties are determined by each of the adjacency, Laplacian, and signless Laplacian spectra of a graph G:*

(i) *The number of vertices.*

(ii) *The number of edges.*

(iii) *Whether G is regular.*

(iv) *Whether G is regular with any fixed girth.*

The following properties are determined by the adjacency spectrum of a graph G:

(v) *The number of closed walks of any fixed length.*

(vi) *Whether G is bipartite.*

The following properties are determined by the Laplacian spectrum of a graph G:

(vii) *The number of connected components.*

(viii) *The number of spanning trees.*

One can use Proposition 1.31 to prove a graph is DS with respect to some matrix by excluding from consideration all graphs that do not share the properties that are determined by that matrix.

Proposition 1.32. *If G is a regular graph, then G is DS if and only if $\overline{G}$ is DS.*

Proposition 1.33. *The path graph P_n is determined by its spectrum. The complete graph K_n, complete bipartite graph $K_{n,n}$, and cycle graph C_n are determined by their spectra, as are their complements. Any disjoint union of complete graphs is determined by its spectrum.*

A useful tool for finding graphs cospectral to a given graph G is called Godsil-McKay switching, or GM-switching for short. The method is given in the following Theorem from [44].

Theorem 1.34. *Let G be a graph and let $\mathcal{P} = \{C_1, C_2, \ldots, C_k, D\}$ be a partition of $V(G)$. Suppose $\mathcal{P} \setminus D$ is an equitable partition of $V(G) \setminus D$ and for any $v \in D$ and $1 \leq i \leq k$, v has exactly 0, $|C_i|/2$, or $|C_i|$ neighbors in C_i. Let G' be the graph obtained as follows. For each $v \in D$ and $1 \leq i \leq k$ such that v has $|C_i|/2$ neighbors in C_i, delete these $|C_i|/2$ edges and add edges from v to the other $|C_i|/2$ vertices in C_i. Then G and G' are cospectral.*

The process of adding and deleting edges in the above theorem is called Godsil-McKay-switching, or GM-switching for short, and the sets $C_1, \ldots, C_k$ are called a GM-switching sets. Once a switching set is found, switching is taking the vertices that are adjacent to half of C_i and changing them so they are adjacent to the other half of C_i. The theorem follows from the fact that the adjacency matrix of G' is equal to $Q^\top A(G)Q$ for a particular regular orthogonal matrix Q. It is possible that the graph G' is isomorphic to G, in which case nothing is gained. However, in the case that G' is not isomorphic to G, Theorem 1.34 implies that neither G nor G' are determined by their spectrum.

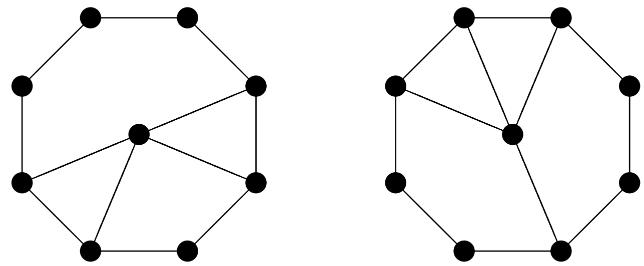


Figure 1.14: A pair of nonisomorphic cospectral graphs on 9 vertices obtained by GM-switching.

Typically, it is difficult to find switching sets with $k > 1$. However, when $k = 1$ it is often not difficult. For example, Figure 1.14 shows a particular graph before and after GM-switching. Theorem 1.34 implies that the graphs in Figure 1.14 have the same spectrum. The switching set $C = C_1$ is the set of all vertices except the center vertex. The set C induces C_8, a regular graph, and the vertex not in C is adjacent to half of the vertices in C. We can see that the graphs are not isomorphic because, for example, one graph induces a cycle on four vertices, while the other does not. This example was originally given in [44].

Figure 1.15 shows another pair of cospectral graphs obtained by using GM-

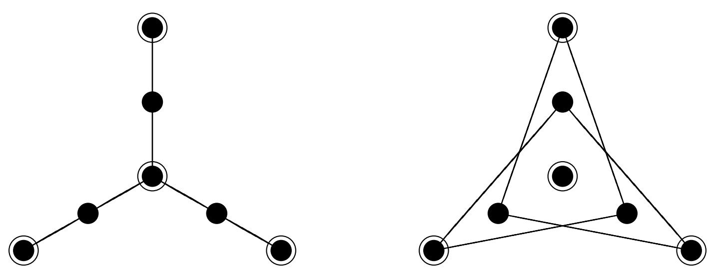


Figure 1.15: A pair of nonisomorphic cospectral graphs on 7 vertices obtained by GM-switching.

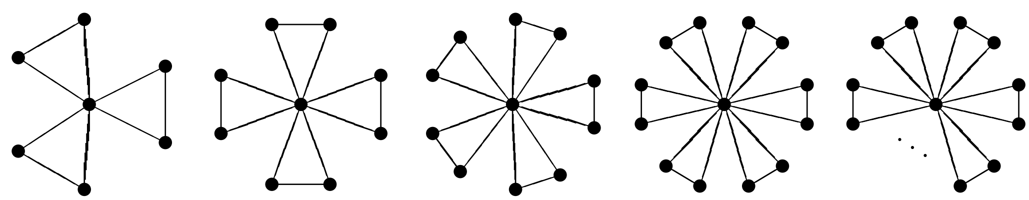
switching. The switching set $C = C_1$ consists of the circled vertices. Clearly C induces a 0-regular graph, and every vertex not in C is adjacent to half of C. One of these graphs is connected and one is not, so they are another pair of nonisomorphic cospectral graphs.

Chapter 2

THE FRIENDSHIP GRAPH AND GRAPHS WITH 4 DISTINCT EIGENVALUES

2.1 The Friendship Graph

The friendship graph F_k (also called the Dutch windmill graph, or k-fan) is the graph consisting of k edge-disjoint triangles that meet in a single vertex (see Figure 2.1). The graph F_k can also be realized as the join $K_1 \nabla kK_2$ (see Definition 1.11),



F_3 F_4 F_5 F_6 F_k

Figure 2.1: The Friendship graph F_k for several values of k.

that is, the cone over kK_2. The friendship graph was made famous by the Friendship Theorem due to Erdős, Rényi and Sós [38], and independently Wilf [97]:

Theorem 2.1 (Friendship Theorem)**.** *In a group of people such that every pair of people have exactly one friend in common, there must be one person who is a friend to all the others.*

If the people are vertices in a graph, and two people are adjacent if and only if they are friends, then the Friendship Theorem states that in a graph such that every pair of vertices has exactly one common neighbor, there is a vertex x adjacent to every other vertex. Then, since each vertex has exactly one common neighbor with x, such a graph must be isomorphic to F_k for some k. Thus the friendship graphs are the only graphs with the property that every pair of vertices has exactly one common neighbor. In addition to the proofs in [38] and [97], the friendship theorem has been independently proved by Longyear and Parsons [64], Brunat [18], Huneke [56], Mertzios and Unger [70], and Bataineh [6].

Clearly F_k has $2k+1$ vertices and $3k$ edges. Let x be the vertex in F_k which is adjacent to every other vertex. To find the spectrum of the adjacency matrix A_k of F_k, consider the partition $\{\{x\}, V(F_k) \setminus \{x\}\}$. Then, labeling the vertices according to the partition, we have

$$A_k = \begin{pmatrix} 0 & \mathbf{1}_{2k}^\top \\ \mathbf{1}_{2k} & R_{2k} \end{pmatrix},$$

where R_{2k} is the adjacency matrix of kK_2 (so the eigenvalues of R_{2k} are $1^k, -1^k$). Clearly the partition $\{\{x\}, V(F_k) \setminus \{x\}\}$ is equitable with quotient matrix

$$Q = \begin{pmatrix} 0 & 2k \\ 1 & 1 \end{pmatrix}.$$

By Proposition 1.24 the eigenvalues $\frac{1}{2} \pm \frac{1}{2}\sqrt{1+8k}$ of Q are also eigenvalues of A_k. Also, Proposition 1.24 implies that the remaining eigenvalues of A_k are unchanged by subtracting multiples of J from some blocks. Thus they are the eigenvalues of the matrix

$$A_k' = \begin{pmatrix} 0 & \mathbf{0}_{2k}^\top \\ \mathbf{0}_{2k} & R_{2k} \end{pmatrix}$$

that are not eigenvalues of the corresponding quotient matrix

$$Q' = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

The matrix A_k' has eigenvalues 0^1, 1^k, and -1^k, while Q' has eigenvalues 0 and 1. Thus the eigenvalues of A_k that are not eigenvalues of Q are 1^{k-1} and -1^k. We have proved:

Proposition 2.2. *The spectrum of A_k is $\{\frac{1}{2} \pm \frac{1}{2}\sqrt{1+8k},\ 1^{k-1},\ -1^k\}$.*

In 2010, Belardo, Borovićanin, Huang, and Wang [7] showed that F_k is determined by the spectrum of its signless Laplacian matrix. They also note that the results of Gui, Liu, and Zhang [47] imply that F_k is determined by its Laplacian spectrum. Belardo, Borovićanin, Huang, and Wang made the following conjecture:

Conjecture 2.3. *The friendship graph F_k is determined by its adjacency spectrum.*

Conjecture 2.3 has caused some activity in the last few years on the spectral characterization of F_k. Combining Propositions 1.28 and 2.2, we have the following corollary.

Corollary 2.4. *If G is a graph cospectral to the friendship graph, then for any induced subgraph H of G on n vertices, we have $\lambda_2(H) \leq 1$ and $\lambda_{n-1}(H) \geq -1$.*

Das [35] claimed to have a proof of Conjecture 2.3, but Abdollahi, Janbaz, and Oboudi [1] noted that the proof contained a mistake. Corollary 2.4 implies that for any graph H on n vertices, if either $\lambda_2(H) > 1$ or $\lambda_{n-1}(H) < -1$ then H cannot be induced by F_k. However, the supposed proof of Conjecture 2.3 in [35] applies Corollary 2.4 to exclude subgraphs that are not necessarily induced. In addition, Abdollahi, Janbaz, and Oboudi [1] proved that Conjecture 2.3 holds for graphs in certain special cases:

Proposition 2.5. *Suppose G is a graph cospectral to F_k and one of the following holds:*

(*i*) *G is connected and planar.*

(*ii*) *G is connected and does not contain C_5 as a subgraph.*

(*iii*) *G has two adjacent vertices of degree 2.*

(*iv*) *$\overline{G}$ is disconnected.*

Then G is isomorphic to F_k.

In Section 2.2, we prove that Conjecture 2.3 holds for connected graphs. In Section 2.3, we classify all graphs with all but two eigenvalues equal to ± 1, a family of graphs that includes the friendship graphs. These results immediately imply that Conjecture 2.3 is true if $k \neq 16$, and there is exactly one counterexample if $k = 16$.

Portions of the remainder of this chapter represent joint work with Sebastian Cioabă, Willem Haemers, and Wiseley Wong on the paper "The graphs with all but two eigenvalues equal to ± 1" in *Journal of Algebraic Combinatorics* **41** (2015), 887–897 [26]. In particular, Wong gave a partial proof of Theorem 2.10 and an alternate proof of Corollary 2.8, although the proofs contained here are independent of them. Haemers gave the original proofs to Lemmas 2.7, 2.12, 2.13 and 2.19, Proposition 2.11, and Corollary 2.14, and a nearly completed a proof of Theorem 2.15. I improved the proof of Theorem 2.15 given by Haemers and fixed some gaps and mistakes, and in this thesis I added many details. Haemers also noted Corollaries 2.16 and 2.18, and Theorem 2.17, which follow directly from the other results. The remaining results are mine.

2.2 Connected Graphs Cospectral to the Friendship Graph

In this section we prove that any connected graph with the same adjacency spectrum as F_k must be isomorphic to F_k.

Any graph G cospectral to F_k must have the same number of vertices, edges, and triangles as F_k by Proposition 1.31. That is, G must have $2k+1$ vertices, $3k$ edges, and k triangles. If G is connected, the diameter of G is at most 3 by Proposition 1.14, since its spectrum has only 4 distinct eigenvalues.

The fact that all but exactly two eigenvalues of F_k are either 1 or -1 and the other two have absolute value more than 1 implies that $A_k^2 - I$ is positive semi-definite and has rank 2. Indeed, it is straightforward to verify that all but exactly two eigenvalues of $A_k^2 - I$ are 0 and the other two are positive. This also implies that any principal submatrix of $A_k^2 - I$ is positive semi-definite and has rank at most 2, which leads to the following pair of very useful lemmas.

Lemma 2.6. *If G is a graph cospectral to F_k, then one connected component of G has minimum degree at least 2 and all other components are isomorphic to K_2.*

Proof. Note that G cannot have two connected components with minimum degree at least 2, since G has only one eigenvalue more than 1. Suppose u is a vertex of degree

1 in G. We will show that the component containing u is isomorphic to K_2, which completes the proof. Let v be the neighbor of u and suppose that v has another neighbor w. Note that v is the only common neighbor of u and w. Then the 2×2 principal submatrix of $A^2 - I$ corresponding to u and w equals

$$S = \begin{pmatrix} 0 & 1 \\ 1 & \deg(w) - 1 \end{pmatrix}.$$

We have $\det S = -1 < 0$, but $A^2 - I$ is positive semi-definite. This is a contradiction, so v must have degree 1. □

Lemma 2.7. *Suppose u and v are distinct vertices in a graph G cospectral to F_k, and each neighbor of u is also a neighbor of v. Then $\deg(v) \geq \deg(u) + 3$.*

Proof. Note that u and v have exactly $\deg(u)$ common neighbors. Then the 2×2 principal submatrix of $A^2 - I$ corresponding to u and v equals

$$S = \begin{pmatrix} \deg(u) - 1 & \deg(u) \\ \deg(u) & \deg(v) - 1 \end{pmatrix}.$$

If $\deg(v) \leq \deg(u) + 2$, then

$$\begin{aligned} \det S &= (\deg(u) - 1)(\deg(v) - 1) - \deg(u)^2 \\ &\leq (\deg(u) - 1)(\deg(u) + 1) - \deg(u)^2 \\ &= \deg(u)^2 + \deg(u) - \deg(u) - 1 - \deg(u)^2 = -1 < 0, \end{aligned}$$

which is a contradiction. □

Lemma 2.6 gives the following corollary:

Corollary 2.8. *If G is a connected graph cospectral to F_k, then the minimum degree of G is 2, and there exists $u \in V(G)$ such that $\deg(u) = 2$.*

Proof. The former statement follows directly from Lemma 2.6. The latter statement follows from the fact that G has $2k + 1$ vertices and $3k$ edges, so the average degree in G must be less than 3. □

Corollary 2.4 immediately implies the following:

Corollary 2.9. *If G is a graph cospectral to F_k, then none of the graphs in Figure 2.2 can be an induced subgraph of G.*

Proof. For each graph it is straightforward to verify that either the second eigenvalue is more than 1 or the second least eigenvalue is less than -1. $\square$

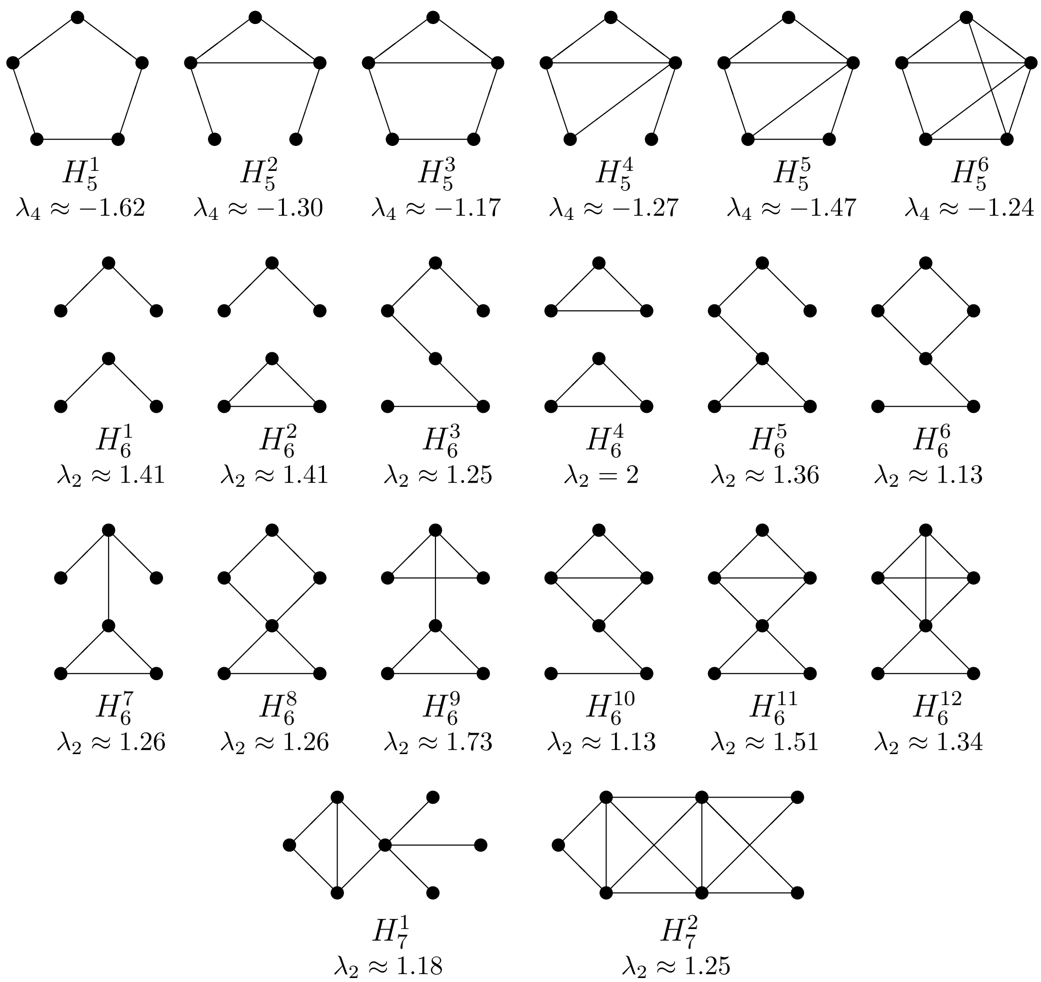


Figure 2.2: Graphs with $\lambda_2 > 1$ or $\lambda_{\min -1} < -1$.

We now have the necessary tools to prove the following theorem, which improves upon Proposition 2.5 and brings us closer to a proof of Conjecture 2.3.

Theorem 2.10. *If G is a connected graph cospectral to F_k, then G is isomorphic to F_k.*

Proof. It is easily verified that the theorem is true for $k < 3$, so we assume $k \geq 3$. Let u be a vertex of degree 2 in G. Let u' and u'' be the neighbors of u, $U = \{u, u', u''\}$, and $X = V(G) \setminus U$. We consider cases on whether or not u' and u'' are adjacent and whether they have common neighbors in X.

Case 1: u' and u'' are adjacent and have no common neighbors in X.

In this case we see that Corollary 2.9 implies that one of u' and u'' has no neighbors in X. Indeed, if $x, x' \in X$ such that x is a neighbor of u' and x' is a neighbor of u'', then the set $\{u, u', u'', x, x'\}$ induces one of graphs H_5^2 or H_5^3, depending on whether $x \sim x'$. Without loss of generality we suppose that u'' has no neighbors in X. Next we use Corollary 2.9 to show that each $x \in X$ has at most one neighbor in X. Indeed, suppose that x has two neighbors x' and x'' in X. First suppose $x \sim u'$. If neither x' nor x'' is adjacent to u', then $\{u, u', u'', x, x', x''\}$ induces H_6^7 or H_6^9, depending on whether $x' \sim x''$. If exactly one of x' and x'' are adjacent to u', then $\{u, u', x, x', x''\}$ induces H_5^2 or H_5^4, depending on whether $x' \sim x''$. If both x' and x'' are adjacent to u', then $\{u, u', x, x', x''\}$ induces H_5^4 or $\{u, u', u'', x, x', x''\}$ induces H_6^{12}, depending on whether $x' \sim x''$. Next suppose $x \not\sim u'$. Then, since $\operatorname{diam}(G) \leq 3$, at least one of x' and x'' is adjacent to u'. Without loss of generality, we assume $x' \sim u'$. Then we have already seen that x' has only one neighbor in X, so $x' \not\sim x''$. Then $\{u, u', u'', x, x', x''\}$ induces H_6^5 or H_6^8, depending on whether $x'' \sim u'$. Thus we have proved that each $x \in X$ has at most one neighbor in X. By Corollary 2.8 this implies that every vertex in X is adjacent to u' and has exactly one neighbor in X. Then X induces $(k-1)K_2$ and $G \cong F_k$ with u' as the vertex adjacent to all others.

Case 2: u' and u'' are adjacent and have common neighbors in X.

Let Y be the set of common neighbors of u' and u'' in X, and let $Z = X \setminus Y$. Corollary 2.9 implies that neither u' nor u'' has neighbors in Z. Indeed, if $y \in Y$ and z is a neighbor of u' in Z, then $\{u, u', u'', y, z\}$ induces H_5^4 or H_5^5, depending on whether

$y \sim z$ (note that $u' \sim z$ implies $u'' \not\sim z$ since $z \notin Y$). A similar argument holds if $z \in Z$ is a neighbor of u''. Then, since $\operatorname{diam}(G) \leq 3$, every vertex in Z has a neighbor in Y. Furthermore, we see that Z is an independent set. Indeed, if $\{z, z'\}$ is an edge in Z and y is a neighbor of z' in Y, the set $\{u, u', u'', y, z, z'\}$ induces H_6^{10} or H_6^{11}, depending on whether $y \sim z$. With Corollary 2.8 this implies that every vertex in Z has two neighbors in Y. Since every vertex in Y has two neighbors in U (u' and u''), this implies that $|E(G)| \geq 3+2\,|Y|+2\,|Z| = 3+2(|Y|+|Z|) = 3+2\,|X| = 3+2(2k-2) = 4k-1 > 3k$, a contradiction. Thus Case 2 is not possible.

Case 3: u' and u'' are not adjacent and have no common neighbors in X.

Let V and W be the sets of neighbors of u' and u'' in X, respectively, and let $Z = X \setminus (V \cup W)$. By Corollary 2.8 the sets V and W are not empty. However, Corollary 2.9 implies $|E(V, W)| = 0$, since otherwise there exist $v \in V$ and $w \in W$ with $v \sim w$, so $\{u, u', u'', v, w\}$ induces H_5^1. We find that every vertex in Z is adjacent to every vertex in $V \cup W$. Indeed, let $z \in Z$. Since G is connected, we may assume without loss of generality that $z \sim v$ for some $v \in V$. For each $w \in W$, Corollary 2.9 implies $z \sim w$. Otherwise, we see that $\{u, u', u'', v, w, z\}$ induces H_6^3. Fixing some $w \in W$, we then find that z is adjacent to every $v \in V$, since otherwise $\{u, u', u'', v, w, z\}$ induces H_6^3. Next, Corollary 2.9 implies that Z is an independent set. Indeed, if $\{z, z'\}$ is an edge in Z, $v \in V$ and $w \in W$, then $\{u', v, w, z, z'\}$ induces H_5^4. Since there are k triangles in G, this implies there must be an edge induced in V or W. Without loss of generality we assume $\{v, v'\}$ is an edge in V. Let $w \in W$ and suppose there exists $z \in Z$. Then $\{u', v, v', y, z\}$ induces H_5^4, so Corollary 2.9 implies Z is empty. With Corollary 2.8 this implies that every vertex in V has a neighbor in V and every vertex in W has a neighbor in W. Then every vertex in $V \cup W$ is in a triangle. There is a vertex $x \in V \cup W$ with $\deg(x) = 2$, since otherwise the sum of degrees of the vertices in G is at least $2 + 2k - 1 + 3(2k-3) = 8k - 5 > 6k$ (since $k \geq 3$), a contradiction. Relabeling x as u, we find we are in Case 1 or 2, so we are done.

Case 4: u' and u'' are not adjacent and have common neighbors in X.

Let Y be the set of common neighbors of u' and u'' in X, let V and W be the sets of neighbors of u' and u'' in $X \setminus Y$, respectively, and let $Z = X \setminus (V \cup W \cup Y)$. As in Case 3, Corollary 2.9 implies that $|E(V, W)| = 0$. Corollary 2.9 also implies $|E(V \cup W, Y)| = |E(V)| = |E(W)| = 0$. Indeed, suppose $x \in V \cup W$ and $y \in Y$ are adjacent. Then $\{u, u', u'', x, y\}$ induces H_5^3. If $\{x, x'\}$ is an edge in V or W and $y \in Y$, then $\{u, u', u'', x, x', y\}$ induces H_6^8. By Corollary 2.8, each vertex in $V \cup W$ must have a neighbor in Z. Corollary 2.9 implies that every vertex in Z with a neighbor in $V \cup W$ must be adjacent to every vertex in Y. Indeed, if there exist $z \in Z$, $x \in V \cup W$, and $y \in Y$ such that $z \sim x$ but $z \not\sim y$, then the set $\{u, u', u'', x, y, z\}$ induces H_6^6. Since $\operatorname{diam}(G) \leq 3$ implies that every vertex in Z has a neighbor in $V \cup W \cup Y$, this implies that every vertex in Z has a neighbor in Y. Then Corollary 2.9 implies $|E(Z)| = 0$. Indeed, if $\{z, z'\}$ is an edge in Z and y is a neighbor of z in Y, then $\{u, u', u'', y, z, z'\}$ induces H_6^6 or H_6^8, depending on whether $y \sim z'$. Then Corollary 2.8 implies that every vertex in Z has at least 2 neighbors in Y (unless $|Y| = 1$). Since $V \cup W$ is nonempty (otherwise u' and u'' have precisely the same neighbors, which contradicts Lemma 2.7), there must be a vertex in Z adjacent to every vertex in Y. Since G must have triangles by Proposition 1.31, there must be at least one edge in Y, so $|E(Y)| \geq 1$ and $|Y| \geq 2$. Note that $|Y| + |Z| = 2k - 2 - |V \cup W|$. The above results imply that

$$\begin{aligned}
3k = |E(G)| &= |E(U)| + |E(U, Y)| + |E(U \cup Z, V \cup W)| + |E(Y, Z)| + |E(Y)| \\
&\geq 2 + 2|Y| + 2|V \cup W| + (|Y| + 2(|Z| - 1)) + |E(Y)| \\
&= 2|V \cup W| + |Y| + 2(|Y| + |Z|) + |E(Y)| \\
&\geq 2|V \cup W| + 2 + 2(2k - 2 - |V \cup W|) + 1 \\
&= 4k - 1.
\end{aligned}$$

This implies $k \leq 1$, a contradiction, so Case 4 is not possible. □

If one could prove that a graph cospectral to a friendship graph is necessarily connected, then Theorem 2.10 would prove Conjecture 2.3. However, despite the fact that Lemma 2.6 seems to severely restrict the possible structure of disconnected graphs cospectral to friendship graphs, such a proof could not be found. As we will see in Section 2.3, that is because when $k = 16$ it is not true.

2.3 The Graphs With All But Two Eigenvalues Equal to ± 1

The spectrum of A_k led to several very useful tools for determining the structure of a graph cospectral to F_k. In particular, the fact that F_k has one eigenvalue greater than 1, one eigenvalue less than -1, and the rest of the eigenvalues ± 1 led to Corollary 2.9, considerably reducing the possible induced subgraphs graphs cospectral to F_k, as well as implying that $A_k^2 - I$ has rank 2 and is positive semi-definite, which led to Lemmas 2.6 and 2.7 and Corollary 2.8. In light of these observations, we take a more general approach in section 2.3, and consider all graphs with these properties. That is, we consider all graphs with exactly two eigenvalues $r > 1$ and $s < -1$ different from ± 1. For completeness, we first consider all graphs with all but at most two eigenvalues not equal to ± 1. We will see that such a graph must either be one of the graphs we wish to consider (those with exactly two eigenvalues $r > 1$ and $s < -1$ different from ± 1), or the graph must be a particular disjoint union of complete graphs.

Proposition 2.11. *Let G be a graph with n vertices and adjacency matrix A.*

- (i) *If A has all its eigenvalues equal to ± 1, then $G = \frac{n}{2} K_2$.*
- (ii) *If A has all but one eigenvalue equal to ± 1, then G is a disjoint union of complete graphs with all but one connected components equal to K_2.*
- (iii) *If A has all but two eigenvalues equal to ± 1 and smallest eigenvalue at least -1, then G is a disjoint union of complete graphs with all but two connected components equal from K_2.*
- (iv) *If A has all but two eigenvalues equal to ± 1 and smallest eigenvalue $s < -1$, then the largest eigenvalue of A is $r > 1$.*

Proof. Case (i) follows from the fact that any disjoint union of complete graphs is determined by its spectrum (Proposition 1.33). Case (ii) follows from the same fact and Corollary 1.11. Case (*iii*) follows from the same fact and Lemma 1.29. For Case (*iv*), note that if $s < -1$ then Corollary 1.11 implies $r > 1$. □

Proposition 2.11 illustrates that in order to characterize the graphs with at most two eigenvalues different from ± 1, we need only characterize the graphs with exactly two eigenvalues $r > 1$ and $s < -1$ different from ± 1. We see that Lemmas 2.6 and 2.7 and Corollary 2.9 hold for these graphs with identical proofs. That is, we have:

Lemma 2.12. *If $r > 1$ and $s < -1$ are the only eigenvalues of G different from ± 1, then one connected component of G has minimum degree at least 2 and all other components are isomorphic to K_2.*

Lemma 2.13. *If $r > 1$ and $s < -1$ are the only eigenvalues of G different from ± 1 and u and v are distinct vertices in G such that each neighbor of u is also a neighbor of v, Then $\deg(v) \geq \deg(u) + 3$.*

Corollary 2.14. *If $r > 1$ and $s < -1$ are the only eigenvalues of G different from ± 1, then none of the graphs in Figure 2.2 can be an induced subgraph of G.*

By Lemma 2.12 we can even further reduce our search to only the set $\mathcal{G}$ of connected graphs for which $r > 1$ and $s < -1$ are the only eigenvalues different from ± 1. Then, any graph which has at most two eigenvalues different from ± 1 must be one of cases (*i*)–(*iii*) from Proposition 2.11 or the disjoint union of some isolated edges and a graph in $\mathcal{G}$.

2.4 Characterization of the Graphs in $\mathcal{G}$.

In this section we give a complete characterization of the graphs in $\mathcal{G}$ and their spectra, and give some results that immediately follow from this characterization.

Theorem 2.15. *If G is a graph in $\mathcal{G}$, then G is one of the following graphs:*

(*i*) *the graph $B_1(m)$ with adjacency matrix* $\begin{pmatrix} O_m & J_m - I_m \\ J_m - I_m & O_m \end{pmatrix}$ $(m \geq 3)$ *and spectrum* $\{\pm(m-1),\ 1^{m-1},\ -1^{m-1}\}$, (*ii*) *the graph* $B_2(a,k)$ *with adjacency matrix* $\begin{pmatrix} J_a - I_a & J_{a,2k} \\ J_{2k,a} & R_{2k} \end{pmatrix}$ $(a \geq 1,\ k \geq 2)$ *and spectrum* $\{\frac{a}{2} \pm \frac{1}{2}\sqrt{a^2+8ak-4a+4},\ 1^{k-1},\ -1^{a+k-1}\}$,

(*iii*) *the graph* $B_3(\ell,m)$ *with adjacency matrix* $\begin{pmatrix} R_{2\ell} & J_{2\ell,2m} \\ J_{2m,2\ell} & R_{2m} \end{pmatrix}$ $(\ell \geq m \geq 2)$ *and spectrum* $\{1 \pm 2\sqrt{\ell m},\ 1^{\ell+m-2},\ -1^{\ell+m}\}$,

(*iv*) *the graph* $B_4(m)$ *with adjacency matrix* $\begin{pmatrix} O_{m+1} & N \\ N^\top & O_{m+1} \end{pmatrix}$, *where* $m = 4$ *and* $N = \begin{pmatrix} 1 & \mathbf{1}_4^\top \\ \mathbf{1}_4 & I_4 \end{pmatrix}$ *or* $m = 5$ *and* $N = \begin{pmatrix} J_3 - I_3 & J_3 \\ O_3 & J_3 - I_3 \end{pmatrix}$, *and spectra* $\{\pm 3,\ 1^4,\ -1^4\}$ *or* $\{\pm 4,\ 1^5,\ -1^5\}$, *respectively,*

(*v*) *the graph* $B_5(a,b)$ *with adjacency matrix* $\begin{pmatrix} J_a - I_a & J_{a,b} & \mathbf{1}_a \\ J_{b,a} & J_b - I_b & \mathbf{0}_b \\ \mathbf{1}_a^\top & \mathbf{0}_b^\top & 0 \end{pmatrix}$, *where* $(a,b) = (6,5),\ (4,6),$ *or* $(3,8)$, *and spectra* $\{4 \pm 2\sqrt{10},\ 1^1,\ -1^9\}$, $\{(7 \pm \sqrt{129})/2,\ 1^1,\ -1^8\}$, *or* $\{4 \pm \sqrt{37},\ 1^1,\ -1^9\}$, *respectively,*

(*vi*) *the graph* $B_6(a,m)$ *with adjacency matrix* $\begin{pmatrix} J_a - I_a & J_{a,m} & O_{a,m} \\ J_{m,a} & O_m & J_m - I_m \\ O_{m,a} & J_m - I_m & O_m \end{pmatrix}$, *where* $(a,m) = (3,5)$ *or* $(4,4)$, *and spectra* $\{(1 \pm \sqrt{129})/2,\ 1^5,\ -1^6\}$ *or* $\{1 \pm 2\sqrt{7},\ 1^4,\ -1^6\}$, *respectively.*

We postpone the proof of Theorem 2.15 until Section 2.5. The graphs in $\mathcal{G}$ described in Theorem 2.15 are pictured in Figure 2.3. We see that $\mathcal{G}$ contains three infinite families and seven sporadic graphs. Note that $B_2(1,k)$ is the friendship graph F_k. From the given spectra, the following corollary is immediate:

Corollary 2.16. *No two graphs in* $\mathcal{G}$ *are cospectral.*

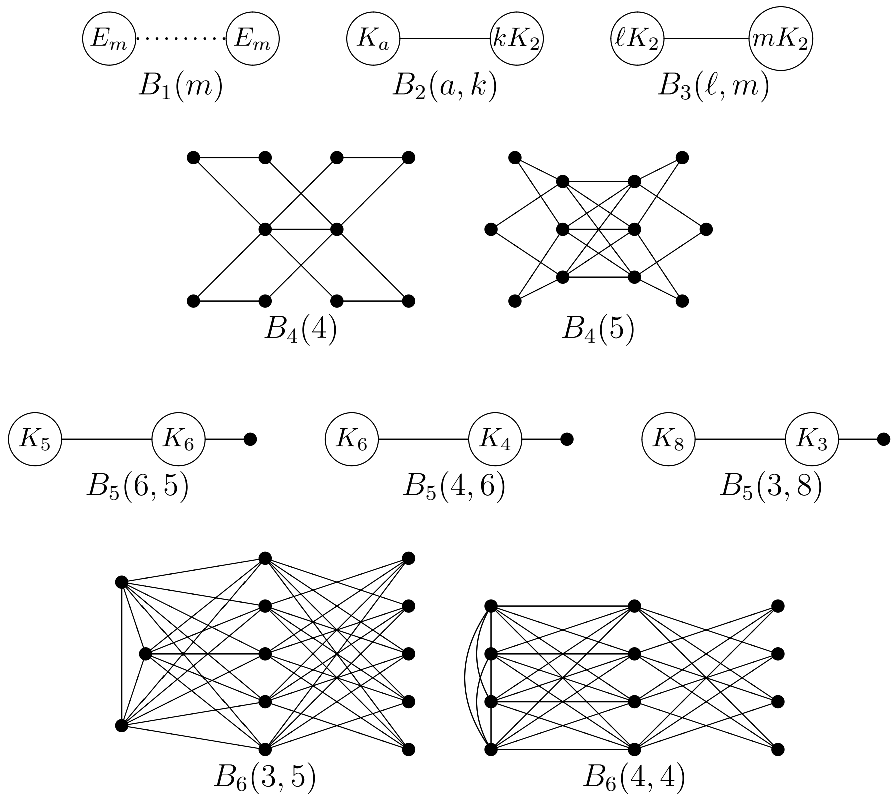


Figure 2.3: The graphs in $\mathcal{G}$.

We can also completely classify which graphs with at most two eigenvalues different from ± 1 are determined by their spectra:

Theorem 2.17. *Suppose G and G' are nonisomorphic cospectral graphs with at most two eigenvalues different from ± 1. Then $G = H + \alpha K_2$ and $G' = H' + \alpha' K_2$, where H and H' are one of the following pairs of graphs in $\mathcal{G}$:*

(i) $H = B_3(\ell, m)$ *and* $H' = B_3(\ell', m')$, *where* $\ell m = \ell' m'$,

(ii) $H = B_3(\ell, m)$ *with* $\ell, m \geq 2$, *and* $H' = B_2(2, k)$ *with* $k = \ell m$,

(iii) $H = B_1(m)$ *and* $H' = B_4(m)$ *with* $m = 4$ *or* 5,

(iv) $H = B_2(1, 16)$ *and* $H' = B_6(3, 5)$ *or* $H = B_2(2, 7)$ *and* $H' = B_6(4, 4)$.

Proof. Recall that by Proposition 1.33 the disjoint union of complete graphs is determined by its spectrum. Thus, by Proposition 2.11 and Lemma 2.12, G and G' must be of the form described above. The components H and H' must share the eigenvalues $r > 1$ and $s < -1$, which easily leads to the stated possibilities for H and H' by Theorem 2.15. □

By taking $\alpha = 0$ in Theorem 2.17, we find the graphs in $\mathcal{G}$ that are not determined by their spectra.

Corollary 2.18. *A graph in $\mathcal{G}$ is determined by its spectrum unless G is one of the following*

(i) $B_2(1, 16)$ *or* $B_2(2, 7)$,

(ii) $B_2(2, k)$, *where k is a composite number,*

(iii) $B_3(\ell, m)$, *where ℓm has a divisor strictly between ℓ and m,*

(iv) $B_4(m)$, *where $m = 4$ or 5.*

Thus we have that the friendship graph $F_k \cong B_2(1, k)$ is determined by its spectrum unless $k = 16$. We see that the friendship graph F_{16} is cospectral with $B_6(3, 5) + 10K_2$ (See Figure 2.4)

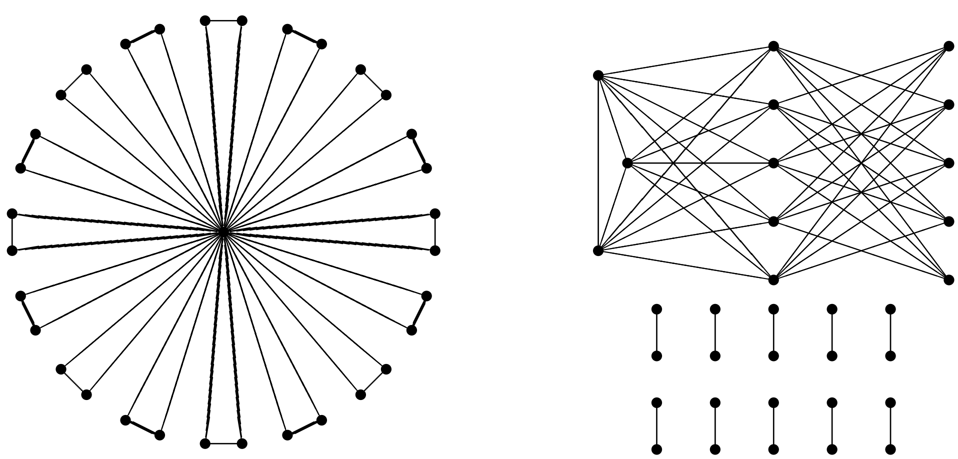


Figure 2.4: The graph F_{16} and its cospectral mate $B_6(3, 5) + 10K_2$.

2.5 The Proof of the Characterization of $\mathcal{G}$

In this section we prove Theorem 2.15.

Van Dam and Spence [33] classified all bipartite graphs with four distinct eigenvalues. Proposition 8 in [33] gives the bipartite graphs in $\mathcal{G}$, which are the graphs $B_1(m)$ ($m \geq 3$) and $B_4(m)$ ($m = 4$ or 5). Thus it remains only to show that the nonbipartite graphs in $\mathcal{G}$ are exactly the graphs $B_2(a,k)$ ($a \geq 1, k \geq 2$), $B_3(\ell,m)$ ($\ell \geq m \geq 2$), $B_5(6,5)$, $B_5(4,6)$, $B_5(3,8)$, $B_6(3,5)$, or $B_6(4,4)$.

We first show that the nonbipartite graphs listed in Theorem 2.15 have the stated spectra and are thus in $\mathcal{G}$. For the sporadic graphs, it is straightforward to simply compute the spectra. For $B_2(a,k)$ we have adjacency matrix

$$A = \begin{pmatrix} J_a - I_a & J_{a,2k} \\ J_{2k,a} & R_{2k} \end{pmatrix}.$$

Partitioning the vertices according to the blocks of A, we obtain an equitable partition with quotient matrix

$$Q = \begin{pmatrix} a-1 & 2k \\ a & 1 \end{pmatrix}.$$

By Proposition 1.24, the eigenvalues $\frac{a}{2} \pm \frac{1}{2}\sqrt{a^2 + 8ak - 4a + 4}$ of Q are also eigenvalues of A, and the remaining eigenvalues are unchanged by subtracting multiples of J from some blocks of A. Thus the remaining eigenvalues of A are the eigenvalues of

$$A' = \begin{pmatrix} -I_a & O_{a,2k} \\ O_{2k,a} & R_{2k} \end{pmatrix}$$

that are not eigenvalues of the corresponding quotient matrix

$$Q' = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The matrix A' has eigenvalues 1^k, and -1^{a+k}, while Q' has eigenvalues 1 and -1. Thus the eigenvalues of A that are not eigenvalues of Q are 1^{k-1} and -1^{a+k-1}, so $B_2(a,k)$ has the spectrum stated in Theorem 2.15.

For $B_3(\ell, m)$ we have adjacency matrix

$$A = \begin{pmatrix} R_{2\ell} & J_{2\ell,2m} \\ J_{2m,2\ell} & R_{2m} \end{pmatrix}.$$

Partitioning the vertices according to the blocks of A, we obtain an equitable partition with quotient matrix

$$Q = \begin{pmatrix} 1 & 2m \\ 2\ell & 1 \end{pmatrix}.$$

By Proposition 1.24, the eigenvalues $1 \pm 2\sqrt{\ell m}$ of Q are also eigenvalues of A, and the remaining eigenvalues are unchanged by subtracting multiples of J from some blocks of A. Thus the remaining eigenvalues of A are the eigenvalues of

$$A' = \begin{pmatrix} R_{2\ell} & O_{2\ell,2m} \\ O_{2m,2\ell} & R_{2m} \end{pmatrix}$$

that are not eigenvalues of the corresponding quotient matrix

$$Q' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The matrix A' has eigenvalues $1^{\ell+m}$, and $-1^{\ell+m}$, while Q' has eigenvalues 1^2. Thus the eigenvalues of A that are not eigenvalues of Q are $1^{\ell+m-2}$ and $-1^{\ell+m}$, so $B_3(\ell, m)$ has the spectrum stated in Theorem 2.15. Thus all graphs of Theorem 2.15 are in $\mathcal{G}$.

It remains to show that every graph in $\mathcal{G}$ must be one of the stated graphs. Recall that the case of bipartite graphs is already settled in [33]. For the rest of the proof we assume that $G \in \mathcal{G}$ is a nonbipartite graph on n vertices and show that G must be one of the nonbipartite graphs in Theorem 2.15. Let C be a clique in G with maximum size. By Corollary 2.14, G contains no induced odd cycles of length five or more. Indeed, an odd cycle of length 5 is H_5^1, and any cycle of length more than 6 induces H_6^3. Thus, since G is not bipartite, G contains a cycle of length 3 and we have $|C| \geq 3$. If there are more than one cliques of maximum size, we let C be the one for which the number of outgoing edges is minimal. The following lemma, which allows us to partition $V(G)$ into a few manageable sets, is the key to the remainder of the proof.

Lemma 2.19. *The vertices in C can be partitioned into two nonempty subsets X and Y such that the neighborhood of any vertex outside C intersects C in X, Y, or $\emptyset$.*

Proof. If $|C| = n-1$ the result is obvious. Indeed, let x be the vertex in $V(G) \setminus C$. Then $X = \Gamma(x)$, $Y = C \setminus X$, and we are done. Thus we may assume $3 \leq |C| \leq n-2$. Let u and v be distinct vertices outside C such that $U = \Gamma(u) \cap C$ and $V = \Gamma(v) \cap C$ are not empty. We will show that either $U = V$, or $U \cap V = \emptyset$ and $U \cup V = C$, which completes the proof. Note that U and V are proper subsets of C, since otherwise C is not maximal. Suppose that $U \cap V \neq \emptyset$ but $U \not\subset V$. Then there exist vertices $x \in U \cap V$ and $y \in U \setminus V$. Let w be a vertex in $C \setminus U$. Then depending on whether $u \sim v$, $v \sim w$, both, or neither, the set $\{u, v, w, x, y\}$ induces H_5^4, H_5^5, or H_5^6, which contradicts Corollary 2.14. Thus if U and V are not disjoint, then $U \subset V$, and by the same argument $V \subset U$ (just choose $y \in V \setminus U$ and $w \in C \setminus V$). Thus $U \cap V \neq \emptyset$ implies $U = V$. If $U \cap V = \emptyset$, we will show that $C = U \cup V$. Suppose not. That is, suppose there exist vertices $x \in U$, $y \in V$, and $z \in C \setminus (U \cup V)$. Then depending on whether $u \sim v$ the set $\{u, v, x, y, z\}$ induces H_5^2 or H_5^3, which contradicts Corollary 2.14. Then every vertex in C is either in U or V, so $C = V \cup U$, which completes the proof. $\square$

Let $\Gamma X = \Gamma(X) \setminus Y$ and $\Gamma Y = \Gamma(Y) \setminus X$. Note that Lemma 2.19 implies that every vertex in ΓX is adjacent to every vertex in X, and similarly for ΓY and Y. Let Z be the set of vertices not adjacent to any vertex of C. Some of the sets ΓX, ΓY, and Z may be empty, but clearly either ΓX or ΓY is nonempty, since otherwise G would be disconnected or complete. We assume $\Gamma X \neq \emptyset$ and consider three cases: (1) both ΓY and Z are empty, (2) only Z is empty, and (3) Z is nonempty. For convenience we define $a = |X|$, $b = |Y|$, $c = |C| = a + b$, $g = |\Gamma X|$, and $h = |\Gamma Y|$.

Case 1: ΓY and Z are empty

Suppose $b = 1$. Then ΓX contains no edges. Indeed, if $\{u, u'\}$ is an edge in ΓX, then the set $\{u, u'\} \cup X$ is a clique of size $c + 1$, which contradicts the fact that C is maximal. Then the vertex $y \in Y$ and any vertex in ΓX have the same set of neighbors (the set X), which contradicts Lemma 2.13. Therefore $b \geq 2$. Let y and y' be distinct vertices in Y and let $x \in X$. Corollary 2.14 implies that each vertex in ΓX has at most one neighbor in ΓX. Indeed, if $u \in \Gamma X$ has two neighbors u' and u'' in ΓX, then $\{u, u', u'', x, y, y'\}$ induces H_6^{12} or $\{u, u', u'', x, y\}$ induces graph H_5^4, depending on whether or not $u' \sim u''$. By Lemma 2.13, it cannot be the case that $u \in \Gamma X$ has a neighbor in ΓX and $u' \in \Gamma X$ does not, so either every vertex in ΓX has exactly one neighbor in ΓX, or ΓX is an independent set.

If every vertex in ΓX has exactly one neighbor in ΓX, then ΓX induces a disjoint union of edges and $g \geq 2$ is even. Partitioning the vertices of G by $V(G) = \{X, Y, \Gamma X\}$, we find G has adjacency matrix

$$A = \begin{pmatrix} J_a - I_a & J_{a,b} & J_{a,g} \\ J_{b,a} & J_b - I_b & O_{b,g} \\ J_{g,a} & O_{g,b} & R_g \end{pmatrix}.$$

The partition is clearly equitable with quotient matrix

$$Q = \begin{pmatrix} a-1 & b & g \\ a & b-1 & 0 \\ a & 0 & 1 \end{pmatrix}.$$

By Proposition 1.24, every eigenvalue of Q is an eigenvalue of A. Thus, if $G \in \mathcal{G}$, at least one eigenvalue of Q must be 1 or -1. Since $\det(Q + I) = -abg \neq 0$, 1 is not an eigenvalue of Q. Since $\det(Q - I) = -ag(b - 2)$, 1 is only an eigenvalue of Q if $b = 2$. Thus $b = 2$ and we rewrite A as

$$A = \begin{pmatrix} J_a - I_a & J_{a,2k} \\ J_{2k,a} & R_{2k} \end{pmatrix}$$

with $2k = g + b = g + 2 \geq 4$, so $k \geq 2$. Thus $G = B_2(a, k)$ with $a \geq 1$ and $k \geq 2$.

If ΓX has no edges and at least two vertices, then these two vertices have the same neighbors, which contradicts Lemma 2.13. So $g = 1$ and, partitioning the vertices of G as before, we find

$$A = \begin{pmatrix} J_a - I_a & J_{a,b} & \mathbf{1}_a \\ J_{b,a} & J - I_b & \mathbf{0}_b \\ \mathbf{1}_a^\top & \mathbf{0}_b^\top & 0 \end{pmatrix}.$$

Again the partition is equitable, and the quotient matrix is

$$Q = \begin{pmatrix} a-1 & b & 1 \\ a & b-1 & 0 \\ a & 0 & 0 \end{pmatrix}.$$

As before, Proposition 1.24 implies that at least one eigenvalue of Q must be 1 or -1. We have $\det(Q+I) = -ab \neq 0$, so -1 is not an eigenvalue of Q. We have $\det(Q-I) = 2a-(a-2)(b-2)$, so Q has an eigenvalue 1 if and only if $b = 2a/(a-2)+2$ is a positive integer for some positive integer a. The only positive integers a for which $a-2$ divides $2a$ are 3, 4, and 6, so Q has an eigenvalue 1 if and only if $(a,b) = (6,5)$, $(4,6)$, or $(3,8)$. Thus $G = B_5(3,8)$, $B_6(3,5)$, or $B_6(4,4)$.

Case 2: ΓX and ΓY are nonempty, and Z is empty

We first claim that $a \leq 2$ or $b \leq 2$. Suppose not, that is, suppose $a \geq b \geq 3$. Then Corollary 2.14 implies that ΓX is an independent set. Indeed, if $\{u, u'\}$ is an edge in ΓX, y, y', y'' are three distinct vertices in Y, and $x \in X$, then $\{u, u', x, y, y', y''\}$ induces H_6^{12}. So ΓX contains no edges, and by the same argument ΓY has no edges. Then Corollary 2.9 implies that each vertex in ΓX has at least $h-1$ neighbors in ΓY. Indeed, if $u \in \Gamma X$ is not adjacent to either of $v, v' \in \Gamma Y$, then for $x, x' \in X$ and $y, y' \in Y$ we find that the set $\{u, v, v', x, x', y, y'\}$ induces H_7^2. Similarly, every vertex in ΓY has at least $g-1$ neighbors in ΓX. In fact, Lemma 2.13 implies that, unless $g = 1$, every vertex in ΓX has exactly $h-1$ neighbors in ΓY, and similarly unless $h = 1$ every vertex in ΓY has exactly $g-1$ neighbors in ΓX. Indeed, if some $u \in \Gamma X$ is adjacent to every vertex in $\Gamma Y$and u' is another vertex in ΓX, then we have $\Gamma(u') \subseteq \Gamma(u)$ and $\deg(u') \geq \deg(u) - 1$, which contradicts Lemma 2.13. The same argument shows that a vertex in ΓY cannot be adjacent to every vertex of ΓX unless $h = 1$. Lemma 2.13 also implies that no two vertices in ΓX have the same $h-1$ neighbors in ΓY, and similarly for two vertices in ΓY.

This implies that $g = h$ and the subgraph induced by $\Gamma X \cup \Gamma Y$ is either K_2 (only if $g = h = 1$) or a complete bipartite graph with the edges of a perfect matching deleted. In the former case, partitioning the vertices of G by $V(G) = \{X, Y, \Gamma X, \Gamma Y\}$ we find that G has adjacency matrix

$$A = \begin{pmatrix} J_a - I_a & J_{a,b} & \mathbf{1}_a & \mathbf{0}_a \\ J_{b,a} & J_b - I_b & \mathbf{0}_b & \mathbf{1}_b \\ \mathbf{1}_a^\top & \mathbf{0}_b^\top & 0 & 1 \\ \mathbf{0}_a^\top & \mathbf{1}_b^\top & 1 & 0 \end{pmatrix}.$$

Clearly the partition is equitable with quotient matrix

$$Q = \begin{pmatrix} a-1 & b & 1 & 0 \\ a & b-1 & 0 & 1 \\ a & 0 & 0 & 1 \\ 0 & b & 1 & 0 \end{pmatrix}.$$

We see that Q has at least 3 eigenvalues different from ± 1, which contradicts Proposition 1.24. Indeed, since $\det(Q + I) = -3ab \neq 0$, -1 cannot be an eigenvalue of Q. Since $\det(Q - I) = a(b-2) - 2b$, $a > 2$, and $b > 2$, we see that Q has an eigenvalue 1 if and only if $a = 2b/(b-2)$ is an integer greater than 2 for some integer $b > 2$. The only $b > 2$ for which $b - 2$ divides $2b$ are 3, 4, and 6, so Q has an eigenvalue 1 if and only if $(a, b) = (6, 3)$, $(4, 4)$, or $(3, 6)$. However, in each case it is straightforward to verify that none of the other 3 eigenvalues of Q are ± 1. Thus this case is not possible for $G \in \mathcal{G}$.

In the latter case, using the same partition we find that G has adjacency matrix

$$A = \begin{pmatrix} J_a - I_a & J_{a,b} & J_{a,m} & O_{a,m} \\ J_{b,a} & J_b - I_b & O_{b,m} & J_{b,m} \\ J_{m,a} & O_{m,b} & O_m & J - I_m \\ O_{m,a} & J_{m,b} & J_m - I_m & O_m \end{pmatrix},$$

where $m = g = h$. Again the partition is equitable, and we see that the quotient matrix is

$$Q = \begin{pmatrix} a-1 & b & m & 0 \\ a & b-1 & 0 & m \\ a & 0 & 0 & m-1 \\ 0 & b & m-1 & 0 \end{pmatrix}.$$

We again find that Q has at least 3 eigenvalues different from ± 1, which contradicts Proposition 1.24. Indeed, since $\det(Q+I) = -abm^2 \neq 0$, -1 cannot be an eigenvalue of Q. Since $\det(Q-I) = m(2(a+b-4) - (m-4)(a-2)(b-2))$, $m \geq 1$, $a > 2$ and $b > 2$ we see that Q has an eigenvalue 1 if and only if $m = 0$ (a contradiction) or $m = 4+2(a+b-4)/((a-2)(b-2))$ is a positive for some integers $a > 2$ and $b > 2$. The only integer pairs (k, ℓ) with $k, \ell > 0$ such that $k\ell$ divides $2(k+\ell)$ are $(k, \ell) = (1, 1)$, $(1, 2)$, $(2, 1)$, $(2, 2)$, $(3, 6)$, $(4, 4)$, and $(6, 3)$. Letting $(a, b) = (k+2, \ell+2)$, we find that the only integer triples (a, b, m) with $a > 2$, $b > 2$, and $m = 4+2(a+b-4)/((a-2)(b-2))$, so that Q has an eigenvalue 1, are $(3, 3, 8)$, $(3, 4, 7)$, $(4, 3, 7)$, $(4, 4, 6)$, $(5, 8, 5)$, $(6, 6, 5)$, and $(8, 5, 5)$. However, in each case it is straightforward to verify that none of the other 3 eigenvalues of Q are ± 1. Thus this case is also not possible for $G \in \mathcal{G}$, and we have proved that $a \leq 2$ or $b \leq 2$.

Next, we claim that actually $a = b = 2$. First, assume $a > b = 1$. Then ΓX contains no edges, because otherwise C would not be maximal. Indeed, if $\{u, u'\}$ is an edge in ΓX, then the set $\{u, u'\} \cup X$ is a clique of size $c+1$. We also find that each $u \in \Gamma X$ is adjacent to every vertex in ΓY. Otherwise, the set $X \cup \{u\}$ is a clique of size c with fewer outgoing edges than C (the vertex in Y is adjacent to every vertex in ΓY, while u is not), a contradiction. However, if u is adjacent to every vertex of ΓY then u and the vertex in Y have the same neighbors, which contradicts Lemma 2.13. Thus it is not the case that $a > b = 1$ (nor, similarly, $b > a = 1$).

Suppose $a > b = 2$. Since in this case $a \geq 3$, Corollary 2.14 implies that ΓY contains no edges. Indeed, if $\{v, v'\}$ is an edge in ΓY, $x, x', x'' \in X$, and $y \in Y$, then the set $\{v, v', x, x', x'', y\}$ induces H_6^{12}. Corollary 2.14 also implies that no vertex in ΓX has more than one neighbor in ΓX. Indeed, if $x \in X$, $y, y' \in Y$, and $u, u', u'' \in \Gamma X$

with u adjacent to both u' and u'', then depending on whether $u' \sim u''$ or not, the set $\{u, u', u'', x, y, y'\}$ induces H_6^{12} or the set $\{u, u', u'', x, y\}$ induces H_5^4.

By the same argument as the one in the beginning of Case 2, we find that every vertex in ΓX is adjacent to at least $h-1$ vertices in ΓY, otherwise G contains an induced subgraph H_7^2. Then Lemma 2.13 implies that every vertex in ΓX has a neighbor in ΓX. Indeed, if $u \in \Gamma X$ has no neighbors in ΓX, then every neighbor of u is a neighbor of $y \in Y$, but $\deg(y) \le \deg(u) + 2$, a contradiction. Thus every vertex in ΓX has exactly one neighbor in ΓX, so ΓX induces a disjoint union of edges. This implies that every vertex in ΓX is adjacent to every vertex in ΓY. Indeed, if $u \in \Gamma X$ is not adjacent to every vertex in ΓY and $\{u, u'\}$ is an edge in ΓX, then $X \cup \{u, u'\}$ is a clique of size c with fewer outgoing edges than C (since the vertices in Y are adjacent to every vertex in ΓY, but u is not), a contradiction. Then Lemma 2.13 implies $h = 1$, since two vertices in ΓY have precisely the same neighbors. Then, partitioning the vertices of G by $V(G) = \{X, Y \cup \Gamma X, \Gamma Y\}$, we have

$$A = \begin{pmatrix} J_a - I_a & J_{a,2m} & \mathbf{0}_a \\ J_{2m,a} & R_{2m} & \mathbf{1}_{2m} \\ \mathbf{0}_a^\top & \mathbf{1}_{2m}^\top & 0 \end{pmatrix},$$

where $2m = b + g = 2 + g$. The partition is clearly equitable with quotient matrix

$$Q = \begin{pmatrix} a-1 & 2m & 0 \\ a & 1 & 1 \\ 0 & 2m & 0 \end{pmatrix}.$$

Since $\det(Q - I) = 4m \neq 0$, 1 is not an eigenvalue of Q. Since $\det(Q + I) = -2a(2m - 1) \neq 0$, -1 is not an eigenvalue of Q. Thus Q has three eigenvalues not equal to ± 1. By Proposition 1.24, A also has these three eigenvalues, a contradiction. Thus it is not the case that $a > b = 2$, (nor, similarly, $b > a = 2$). Thus we have proved that $a = b = 2$.

Let $X = \{x, x'\}$ and $Y = \{y, y'\}$. The argument above that each vertex in ΓX has at most one neighbor in ΓX still holds in this case. We again find that each vertex in ΓX is adjacent to at least $h-1$ vertices in ΓY. Indeed, if $u \in \Gamma X$ and both $v, v' \in \Gamma Y$ are not adjacent to u, then depending on whether or not $v \sim v'$ we have $\{u, v, v', x, x', y\}$ induces H_6^{11} or $\{u, v, v', x, x', y, y'\}$ induces H_7^2. Then, as before, Lemma 2.13 implies that every vertex in ΓX has a neighbor in ΓX, since otherwise every neighbor of a vertex $u \in \Gamma X$ with no neighbors in ΓX is a neighbor of y while $\deg(y) \le \deg(u) + 2$. Thus, again, every vertex in ΓX has exactly one vertex in ΓX, so ΓX induces a disjoint union of edges. Also as before, every vertex of ΓX must be adjacent to every vertex of ΓY, since otherwise we find a clique of size c with fewer outgoing edges than C. By the same arguments as above (but swapping X with Y and ΓX with ΓY), we see that ΓY is also a disjoint union of edges. Thus, partitioning the vertices of G by $V(G) = \{Y \cup \Gamma X, X \cup \Gamma Y\}$, we find that A is as follows:

$$A = \begin{pmatrix} R_{2\ell} & J_{2\ell,2m} \\ J_{2m,2\ell} & R_{2m} \end{pmatrix},$$

where $2\ell = g + 2$ and $2m = h + 2$, so $\ell, m \ge 2$. Without loss of generality $\ell \ge m$, so in this case $G = B_3(\ell, m)$ with $\ell \ge m \ge 2$.

Case 3: Z is not empty.

Since G is connected there exists an edge from a vertex in Z to a vertex in $\Gamma X \cup \Gamma Y$. Without loss of generality we assume there is a vertex $u \in \Gamma X$ with a neighbor $z \in Z$. Then Corollary 2.14 implies that $b = 1$. Suppose not, that is, suppose there are two vertices $y, y' \in Y$, let $x \in X$, and let w be a neighbor of z different from u. If $w \in \Gamma Y$, then the set $\{u, w, x, y, z\}$ induces H_5^1 or H_5^3, depending on whether or not $u \sim w$. If $w \in \Gamma X$, then $\{u, w, x, y, y', z\}$ induces H_6^8 or H_6^{11}, depending on whether or not $u \sim w$. If $w \in Z$, then $\{u, w, x, y, y', z\}$ induces H_6^5 or H_6^9, depending on whether or not $u \sim w$. Thus $b = 1$.

Let $Y' = Y \cup \Gamma X$ and $Z' = \Gamma Y \cup Z$. Let $m = |Y'| = g + b = g + 1 \ge 2$. We see that Y' is an independent set, since every vertex in Y' is adjacent to every vertex in X (so X and an edge in Y' would induce a clique of size $c + 1$).

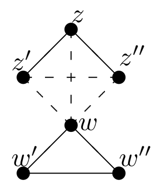


Figure 2.5: A subgraph induced by G if a vertex in Z' has two neighbors in Z'.

Corollary 2.14 implies that each vertex in Z' has at most one neighbor in Z'. Indeed, suppose a vertex $z \in Z'$ has two neighbors $z', z'' \in Z'$ and let $w, w', w'' \in C$. Note that at most one of most one vertex in C has neighbors among $z, z', z'' \in Z'$, since $b = 1$ and $Z' = \Gamma Y \cup Z$. Without loss of generality we assume w' and w'' have no neighbors among z, z', z''. Thus the subgraph induced by $B = \{w, w', w'', z, z', z''\}$ is the graph in Figure 2.5, where the solid edges must be present, the dashed edges may or not be present, and no other edges may be present. If no dashed edges are present, then B induces H_6^2. Suppose one dashed edge is present. If it is $\{w, z'\}$ or $\{w, z''\}$, then B induces H_6^5. If it is $\{w, z\}$, then B induces H_6^7. If it is $\{z', z''\}$, then B induces H_6^4. Suppose two dashed edges are present. If they are $\{w, z'\}$ and $\{w, z''\}$, then B induces H_6^8. If they are $\{w, z\}$ and one of $\{w, z'\}$ or $\{w, z''\}$, then $B \setminus \{w'\}$ induces H_5^2. If they are $\{z', z''\}$ and one of $\{w, z\}$, $\{w, z'\}$, or $\{w, z''\}$, then B induces H_6^9. Suppose three dashed edges are present. If they are $\{w, z\}$, $\{w, z'\}$, and $\{w, z''\}$, then $B \setminus \{w'\}$ induces H_5^4. If they are $\{w, z'\}$, $\{w, z''\}$, and $\{z', z''\}$, then B induces H_6^{11}. If they are $\{w, z\}$, $\{z', z''\}$, and one of $\{w, z'\}$ or $\{w, z''\}$, then again B induces H_6^{11}. If all four dashed edges are present, then B induces H_6^{12}. Thus in all cases a contradiction arises, so each vertex in Z' has at most one neighbor in Z', and since all vertices have degree at least two, each vertex in Z' has a neighbor in Y'. We partition Z' by $Z' = Z_1 \cup Z_2$, where the vertices in Z_2 are adjacent to every vertex in Y', while the vertices in Z_1 are not.

Next we will show that every vertex in Y' has the same degree, and for any pair of vertices in Y', each has exactly one neighbor that is not a neighbor of the other. Let $x \in X$ and $y, y' \in Y'$, and without loss of generality assume $\deg(y) \leq \deg(y')$. We consider the 3×3 principal submatrix S of $A^2 - I$ corresponding to x, y, y'. We have

$$S = \begin{pmatrix} \deg(x) - 1 & a-1 & a-1 \\ a-1 & \deg(y) - 1 & \deg(y, y') \\ a-1 & \deg(y, y') & \deg(y') - 1 \end{pmatrix},$$

since x has $a-1$ neighbors in common with each of y, y' (namely, the other $a-1$ vertices in X). We have $S = (a-1)J + S'$, where

$$S' = \begin{pmatrix} \deg(x) - a & \mathbf{0}_2^\top \\ \mathbf{0}_2 & T \end{pmatrix} \text{ and } T = \begin{pmatrix} \deg(y) - a & \deg(y, y') - a + 1 \\ \deg(y, y') - a + 1 & \deg(y') - a \end{pmatrix}.$$

Note that $\deg(x) > a$ and $\deg(y') \geq \deg(y) \geq \deg(y, y') \geq a$. Corollary 2.14 implies that y' has at most two neighbors in Z' that are not neighbors of y. Indeed, if y' has two adjacent neighbors z, z' that are not neighbors of y, and $x' \in X$, then $\{x, x', y, y', z, z'\}$ induces H_6^{11}. Otherwise, if y' has three neighbors z, z', z'' that are not neighbors of y, and $x' \in X$, then $\{z, z', z''\}$ is an independent set and $\{x, x', y, y', z, z', z''\}$ induces H_7^1. Thus we have

$$\deg(y, y') \leq \deg(y) \leq \deg(y') \leq \deg(y, y') + 2 \leq \deg(y) + 2. \tag{2.1}$$

We see that if T is positive definite, then so are S' (because $\deg(x) - a > 0$) and S (because $(a-1)J$ is positive semi-definite), which contradicts $\operatorname{rank} S \leq \operatorname{rank}(A^2 - I) = 2$. Therefore $\det T = (\deg(y) - a)(\deg(y') - a) - (\deg(y, y') - a + 1)^2 \leq 0$. By (2.1), we have $\deg(y), \deg(y') \in \{\deg(y, y'), \deg(y, y') + 1, \deg(y, y') + 2\}$ and $\deg(y) \leq \deg(y')$. If $\deg(y) = \deg(y, y')$ then every neighbor of y is a neighbor of y' and Lemma 2.13 implies $\deg(y') \geq \deg(y) + 3$, which contradicts (2.1). If $\deg(y) = \deg(y, y') + 2$, then also $\deg(y') = \deg(y, y') + 2$ and $\det T = 3 + 2(\deg(y, y') - a) > 0$, a contradiction. If $\deg(y) = \deg(y, y') + 1$ and $\deg(y') = \deg(y, y') + 2$, then $\det T = 1 + \deg(y, y') - a > 0$, a contradiction. Thus we must have $\deg(y) = \deg(y') = \deg(y, y') + 1$ (in this case $\det T = 0$), so every vertex in Y' has the same degree, and for any pair of vertices in Y', each has exactly one neighbor that is not a neighbor of the other. Further, since Y' is an independent set and every vertex in Y' is adjacent to every vertex in X and Z_2, every vertex in Y' has the same number of neighbors in Z_1 and for any pair of vertices in Y', each has exactly one neighbor in Z_1 that is not a neighbor of the other. The only possibilities are that each vertex in Y' has exactly one neighbor in Z_1 (a different neighbor for each vertex in Y') or each vertex in Y' has exactly one non-neighbor in Z_1 (a different non-neighbor for each vertex in Y').

To see that this is true, suppose that the vertices in Y' have at least two neighbors and at least two non-neighbors in Z_1. Let $y, y' \in Y'$. Then each of y, y' has a neighbor in Z_1 that the other does not. Let $z, z' \in Z_1$ such that $z \sim y$, $z' \sim y'$, $z \not\sim y'$, and $z' \not\sim y$. Since $\deg(y) = \deg(y') = \deg(y, y') + 1$, the rest of the neighbors of y are neighbors of y', and vice versa. Now, each of y and y' must have at least one more neighbor and one more non-neighbor, and they must have them in common, so there exist $v, v' \in Z_1$ such that $y, y' \sim v$ and $y, y' \not\sim v'$. Since $v \in Z_1$, there must exist some $u \in Y'$ such that $u \not\sim v$. Then $u \sim z$, since otherwise y has two neighbors in Z_1 which u does not. Similarly, we must have $u \sim z'$. We have $u \not\sim v'$, since otherwise u has two neighbors that y does not. Since $v' \in Z_1$, there is a vertex $u' \in Y'$ such that $u' \sim v'$. Since none of u, y, y' are adjacent to v', u' cannot have any other neighbor that is not a neighbor of each of u, y, y'. Thus, since $u \not\sim v$, $y \not\sim z'$, and $y' \not\sim z$, we have $u' \not\sim z, z', v$. Then each of u, y, y' has two neighbors which u' does not, a contradiction. We have proved that either each vertex in Y' has exactly one neighbor in Z_1, or each vertex in Y' has exactly one non-neighbor in Z_1. In the former case, we find that each $z \in Z_1$ also has exactly one neighbor in Y' (clearly a different neighbor for each vertex in Z_1, since a vertex in Y' has only one neighbor in Z_1), since otherwise two neighbors of z which are in Y' have the same neighborhood in Z_1, contradiction. Thus in this case the principal submatrix of A corresponding to $Y' \cup Z_1$ is

$$\begin{pmatrix} O_m & I_m \\ I_m & M \end{pmatrix},$$

where M is the adjacency matrix of the subgraph of G induced by Z_1. In the latter case, the same argument (replacing the word neighbor with non-neighbor) shows that each vertex in Z_1 also has exactly one non-neighbor in Y' (again, clearly a different non-neighbor for each vertex in Z_1). Thus in this case the principal submatrix of A corresponding to $Y' \cup Z_1$ is

$$\begin{pmatrix} O_m & J_m - I_m \\ J_m - I_m & M \end{pmatrix},$$

where M is defined as before. In either case we clearly have $|Z_1| = |Y'| = m \geq 2$.

Next we find that Corollary 2.14 implies that adjacent vertices in Z' have the same neighbors in Y'. Indeed, suppose $\{z, z'\}$ is an edge in Z', and suppose there is a vertex $y \in Y'$ adjacent to z but not to z'. Let $x \in X$, and let $y' \in Y'$ be a neighbor of z' (which must exist, since every vertex in Z' has a neighbor in Y'). Then $\{x, y', y, z, z'\}$ induces H_5^1 or H_5^3, depending on whether or not $y' \sim z$. So z and z' have the same set of neighbors in Y', and hence $z, z' \in Z_2$. This implies that vertices in Z_1 have no neighbors in Z', so by Lemma 2.12 each vertex in Z_1 has at least two neighbors in Y' and the principal submatrix of A corresponding to $Y' \cup Z_1$ is

$$\begin{pmatrix} O_m & J_m - I_m \\ J_m - I_m & O_m \end{pmatrix}.$$

Finally, Lemma 2.13 implies that Z_2 is empty. Indeed, if $z \in Z_1$ and $z' \in Z_2$, then every neighbor of z is also a neighbor of z', but $\deg(z') \leq \deg(z) + 2$, a contradiction. Thus such z and z' cannot both exist. Since Z_1 is not empty, this implies Z_2 is empty, and we find that partitioning the vertices of G by $V(G) = \{X, Y', Z'\}$ we have

$$A = \begin{pmatrix} J_a - I_a & J_{a,m} & O_{a,m} \\ J_{m,a} & O_m & J_m - I_m \\ O_{m,a} & J_m - I_m & O_m \end{pmatrix}.$$

Clearly the partition is equitable with quotient matrix

$$Q = \begin{pmatrix} a-1 & m & 0 \\ a & 0 & m-1 \\ 0 & m-1 & 0 \end{pmatrix}.$$

Since $\det(Q+I) = -am(m-1) \neq 0$, -1 is not an eigenvalue of Q. Since $\det(Q-I) = -m(a(m-3)-2(m-2))$, Q has eigenvalue 1 if and only if $a = 2(m-2)/(m-3)$ is an integer greater than 1 ($a > 1$ since $b = 1$) for some integer $m \geq 2$. The only integers $m \geq 2$ such that $m-3$ divides $2(m-2)$ are 4 and 5, so Q has all three eigenvalues not equal to ± 1 unless (a, m) equals $(4, 4)$ or $(3, 5)$. By Proposition 1.24, A also has three eigenvalues not equal to ± 1 unless (a, m) equals $(4, 4)$ or $(3, 5)$. Thus in this case $G = B_6(3, 5)$ or $B_6(4, 4)$. $\square$

Chapter 3

SIMPLICIAL ROOK GRAPHS

3.1 Rook Graphs and Simplicial Rook Graphs

For any chess piece, one can define a graph whose vertices are the tiles of a chess board, with two vertices adjacent if and only if the given chess piece can travel from one to the other by a legal chess move. (see, for example, [96]). Elkies [36] noted that the rook is the only chess piece whose graph formed in this way is regular. The rook graph is the graph defined above such that the chess piece used is the rook. This graph can be extended to higher dimensions and a different number of tiles in each direction. To generalize, the rook graph $R(m,n)$ is the graph whose vertices are the tiles in an m dimensional chessboard with n tiles in each direction (so $R(2,8)$ is the original rook graph), with the same adjacency relation. Seen another way, the graph $R(m,n)$ is the graph whose vertices are m-tuples of nonnegative integers at most n, with two vertices adjacent if and only if they differ in only one coordinate. We observe that $R(m,n)$ is isomorphic to the Hamming graph $H(m,n)$ (indeed, the chess board tile locations can be given as m-tuples, and a rook can travel between two tiles precisely when they differ in exactly one coordinate). Thus the rook graph $R(m,n)$ is regular with valency $m(n-1)$, when $m=2$ the rook graph is strongly regular with parameters $(n^2, 2(n-1), n-2, 2)$, and for $m>2$ the rook graph is distance-regular with intersection array $\{m(n-1), (m-1)(n-1), \ldots, (m-i)(n-1), \ldots, n-1; 1, 2, \ldots, m-1, m\}$, because the Hamming graphs have these properties (see Definition 1.18, [14, Section 9.2] and [17, Section 12.4.1]). In answer to a question on Mathoverflow, Godsil [43] showed that the independence number of the Hamming graph $H(m,n)$ (and hence also the rook graph $R(m,n)$) is n^{m-1}.

Similarly, a simplicial rook graph is a graph whose vertices are the tiles of a simplicial chessboard, where again tiles are adjacent when a rook can travel from one to the other by a legal move. Of course, we must define what we mean by a simplicial chess board, and what a rook's legal move on that board looks like. In the space $\mathbb{R}^m$, the standard basis vectors $\mathbf{e}_1, \ldots, \mathbf{e}_m$ are the vectors such that $\mathbf{e}_i$ equals 1 in the i-th coordinate and 0 elsewhere. The standard simplex in $\mathbb{R}^m$ is the convex hull of the standard basis vectors in $\mathbb{R}^m$, and the n-th dilate of standard simplex is the convex hull of $n\mathbf{e}_1, \ldots, n\mathbf{e}_m$. Then the simplicial rook graph $SR(m, n)$ is the graph whose vertices are the integer lattice points in the n-th dilate of the standard simplex in $\mathbb{R}^m$ (see Figure 3.1) with two vertices adjacent if and only if their difference is a multiple

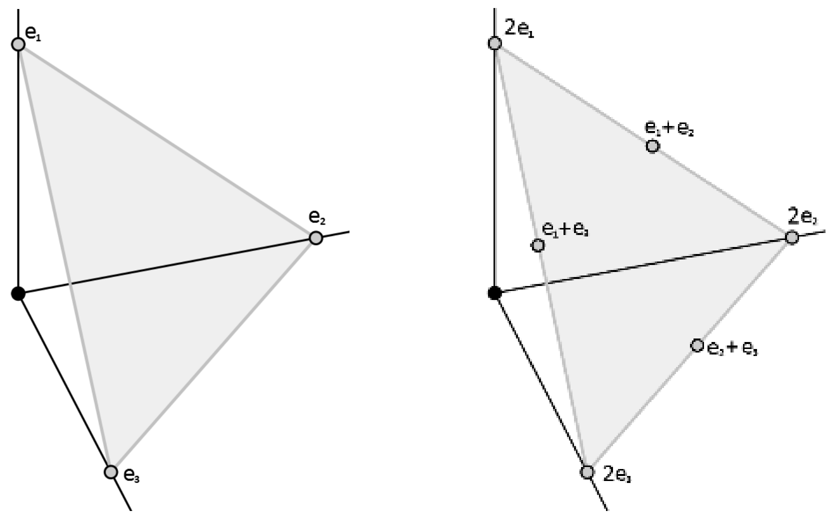


Figure 3.1: The integer lattice points in the n-th dilate of the standard simplex in $\mathbb{R}^3$, $n = 1, 2$.

of $\mathbf{e}_i - \mathbf{e}_j$ for some pair i, j. Seen another way, $SR(m, n)$ is the graph with vertex set $V(m, n) = \{(x_1, x_2, \ldots, x_m) \mid 0 \leq x_i \leq n,\ \sum_{i=1}^{m} x_i = n\}$, the set of m-tuples of nonnegative integers whose coordinates sum to n (see Figure 3.2), such that two vertices are adjacent if and only if they differ in exactly two coordinates.

The graph $SR(1, n)$ is clearly just the isolated vertex $n\mathbf{e}_1$. The graph $SR(2, n)$ is isomorphic to K_{n+1}, since every pair of vertices in $V(2, n) = \{(x_1, x_2) \mid 0 \leq x_i \leq n,\ x_1 + x_2 = n\}$ must differ in exactly two coordinates. Similarly, the graph $SR(m, 1)$

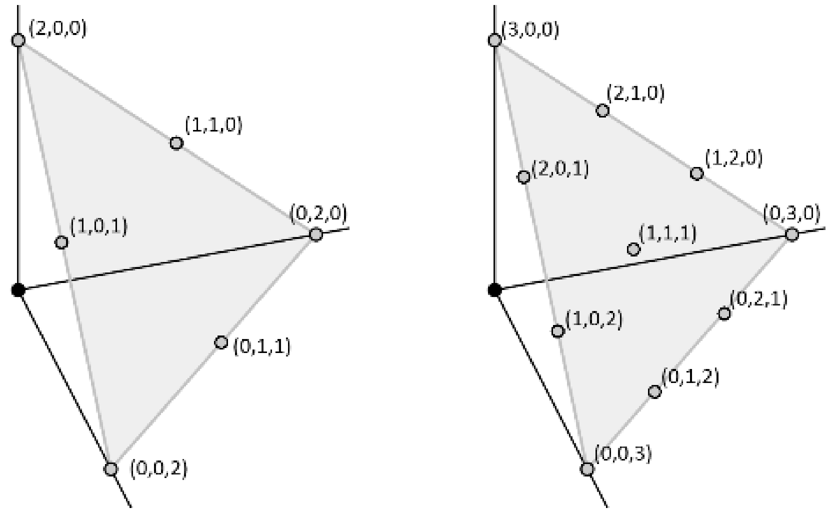


Figure 3.2: Lattice points in the n-th dilate of the standard simplex in $\mathbb{R}^3$ viewed as 3-tuples summing to n, $n = 2, 3$.

is isomorphic to K_m, since every pair of vertices $\mathbf{e}_i$ and $\mathbf{e}_j$ in $V(m,1) = \{\mathbf{e}_i \mid 1 \le i \le n\}$ differ by $\mathbf{e}_i - \mathbf{e}_j$. The graph $SR(m,2)$ is isomorphic to the Johnson graph $J(m+1,2)$ (or equivalently to the triangular graph $T(m+1)$, the line graph of K_{m+1}) under the following bijection: for $1 \le i < j \le m$, the vertices $\mathbf{e}_i + \mathbf{e}_j \in V(m,2)$ correspond the the vertices $\{i,j\} \in V(J(m+1,2))$, while for $1 \le i \le m$ the vertices $2\mathbf{e}_i \in V(m,2)$ correspond to the vertices $\{i, m+1\} \in V(J(m+1,2))$. It is straightforward to verify that this bijection preserves adjacency, so the graphs are isomorphic. As we have seen, for m or n at most 2 the simplicial rook graphs are familiar graphs that are strongly regular, distance-regular, or line graphs. However, as noted by Martin and Wagner [68], for $m, n \ge 3$ simplicial rook graphs are not strongly regular, distance-regular, or line graphs. The simplicial rook graphs $SR(3,4)$ and $SR(4,3)$ are pictured in Figure 3.3.

The simplicial rook graphs were first introduced by Martin and Wagner [68]. Those authors showed that $SR(m,n)$ is a regular graph of valency $n(m-1)$ and has $\binom{n+m-1}{m-1}$ vertices. To see the latter, the definition $V(m,n) = \{(x_1, x_2, \ldots, x_m) \mid 0 \le x_i \le n, \sum_{i=1}^{m} x_i = n\}$ is helpful: the vertices are weak compositions of n into m parts, so there are $\binom{n+m-1}{m-1}$ of them. To see the former, note that for a vertex

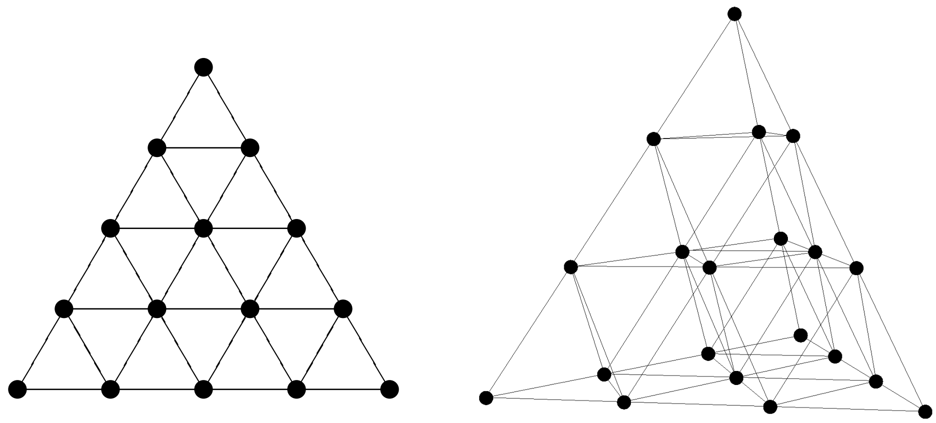


Figure 3.3: The Simplicial Rook graphs $SR(3,4)$ and $SR(4,3)$.
In this figure, a line through multiple vertices indicates that the vertices on that line are all pairwise adjacent.

$x = (x_1, x_2, \ldots, x_m) \in V(m,n)$ there are $x_i + x_j$ vertices that differ from x in only the coordinates i, j for any pair i, j, so x has valency

$$\sum_{1 \le i < j \le m} (x_i + x_j) = (m-1) \sum_{i=1}^{m} x_i = n(m-1).$$

We denote by V_i the set of vertices in $V(m,n)$ with exactly i nonzero coordinates. If $x \sim y$ and the two coordinates where x and y differ are i and j, then we say the edge $\{x, y\}$ is an (i,j)-edge, or that the edge is in the i,j direction, and we say that x and y are (i,j)-neighbors. We denote by $A(m,n)$ the adjacency matrix of $SR(m,n)$.

Portions of the remainder of this chapter represent joint work with Andries Brouwer, Sebastian Cioabă, and Willem Haemers on the paper "Notes on simplicial rook graphs", submitted to the *Journal of Algebraic Combinatorics* [13]. In particular, Haemers gave an outline of the proof of Proposition 3.11 and gave the idea for Propositions 3.17 and 3.18 without proof. Brouwer gave brief (but complete) proofs to Propositions 3.2, 3.5, and 3.24, and Theorem 3.13, to which I have added many details. Brouwer also gave the alternate proof to Proposition 3.11. The proof of Proposition 3.22 is completely due to Brouwer, with very few details added. The conjectured spectra of $SR(m,5)$ and $SR(4,n)$ are also due to Brouwer. The remaining results are mine.

3.2 Independence Number

We denote by $\alpha(m,n)$ the independence number of $SR(m,n)$. The independence number $\alpha(m,n)$ is the number of pairwise non-attacking rooks that can be placed on a simplicial chessboard of dimension $m-1$ with $n+1$ tiles in each direction. Martin asked for the value of $\alpha(3,n)$ on Mathoverflow. An interesting discussion followed, concluding with the proof of the following proposition by Elkies [37]. We remark that after Elkies et al. proved Proposition 3.1 on Mathoverflow, it was pointed out that the value of $\alpha(3,n)$ had already been found by Nivasch and Lev [75] using a combinatorial method and, independently, Blackburn, Paterson, and Stinson [10] using linear programming. However, the proof in [37] is less complicated and independent of the others.

Proposition 3.1. *The independence number of $SR(3,n)$ is $\alpha(3,n) = \left\lfloor \frac{2n}{3} + 1 \right\rfloor$.*

Proof. We begin by noting that in any set of vertices in $SR(3,n)$, one of the coordinates must have an average value of no more than $n/3$. Indeed, the average sum of coordinates is always exactly n, so not all three coordinates can have an average value of more than $n/3$. In an independent set in $SR(3,n)$, no two vertices can agree in any coordinate, since such vertices would differ in the other two coordinates and thus be adjacent. Since the largest number of distinct nonnegative integers whose average is at most $n/3$ is $\left\lfloor \frac{2n}{3} + 1 \right\rfloor$ (namely, the integers $0, 1, \ldots, \lfloor 2n/3 \rfloor$), we see that $\alpha(3,n) \leq \left\lfloor \frac{2n}{3} + 1 \right\rfloor$. To see that equality can always be achieved, we first suppose $n = 3k$ and consider the set

$$S = \{(2k-2i, k+i, i) \mid i = 0, 1, \ldots, k\} \cup \{(2k-2i-1, i, k+1+i) \mid i = 0, \ldots, k-1\}.$$

Clearly S is an independent set. Indeed, no two vertices in S are equal in any coordinate, since each coordinate equals each of the numbers $0, 1, \ldots, 2k$ exactly once. Since $n = 3k$, S is an independent set of size $2k+1 = 2n/3+1 = \left\lfloor \frac{2n}{3} + 1 \right\rfloor$. For $n = 3k+1$, the set S' obtained by adding 1 to the first coordinate of every vertex in S is still an independent set of size $2k+1$, but in this case $2k+1 = \lfloor 2k + 5/3 \rfloor = \left\lfloor \frac{2(3k+1)}{3} + 1 \right\rfloor = \left\lfloor \frac{2n}{3} + 1 \right\rfloor$. For $n = 3k-1$, the set S'' obtained by subtracting 1 from the first coordinate of every vertex in S (and removing $(-1, 2k, k)$, which is not a vertex) is an independent set of size $2k = \lfloor 2k + 1/3 \rfloor = \left\lfloor \frac{2(3k-1)}{3} + 1 \right\rfloor = \lfloor \frac{2n}{3} + 1 \rfloor$. Thus for any value of n we have $\alpha(3, n) = \lfloor \frac{2n}{3} + 1 \rfloor$. $\square$

See Figure 3.4 for maximal independent sets in $SR(3, n)$ when $n = 8, 9, 10$.

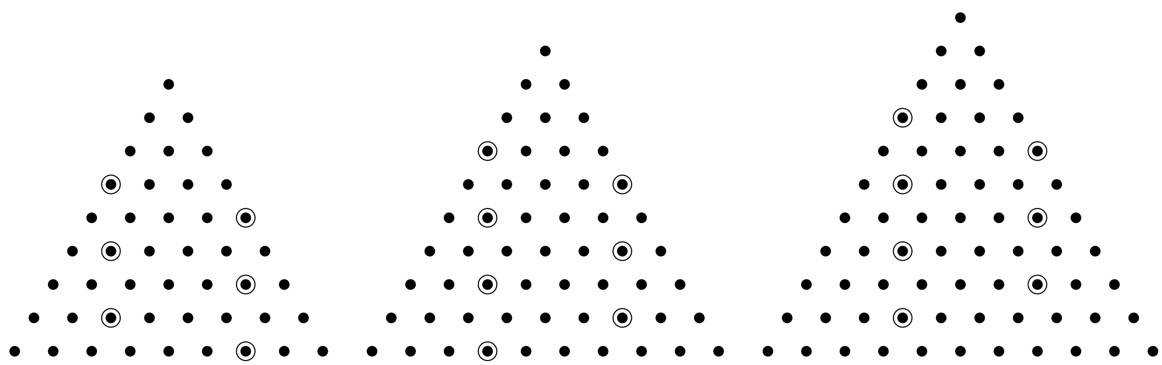


Figure 3.4: Maximal independent sets in $SR(3, 8)$, $SR(3, 9)$, and $SR(3, 10)$.

Martin and Wagner [68] asked the value of $\alpha(m, n)$ for other values of m and n. We give the answer when $n = 3$.

Proposition 3.2. *The independence number of the graph $SR(m, 3)$ is*

$$\alpha(m,3) = \begin{cases} \frac{1}{6}(m+1)(m+2) & \text{for } m \equiv \pm 1 \pmod 6, \\ \frac{1}{6}m(m+3) & \text{for } m \equiv 3 \pmod 6, \\ \frac{1}{6}m(m+2) & \text{for } m \equiv 0, 4 \pmod 6, \\ \frac{1}{6}(m^2+2m-2) & \text{for } m \equiv 2 \pmod 6. \end{cases}$$

Proof. We first prove that

$$\alpha(m,3) \leq m + \left\lfloor \frac{1}{3}\left(m \left\lfloor \frac{1}{2}(m-3) \right\rfloor + 1 \right) \right\rfloor$$

Let S be a maximal independent set in $SR(m, 3)$. To give an upper bound on the size of S, we count edges in a graph $K \cong K_m$ (with vertices labeled $1, 2, \ldots, m$) covered by vertices of S in the following way: A vertex of the form $3\mathbf{e}_i$ (a singleton) covers no edges. A vertex of the form $2\mathbf{e}_i + \mathbf{e}_j$ (a pair) covers the edge $\{i, j\}$. A vertex of the form $\mathbf{e}_i + \mathbf{e}_j + \mathbf{e}_k$ (a triple) covers the edges $\{i, j\}$ and $\{i, k\}$ and $\{j, k\}$. We see that if any edge is covered twice, then the two corresponding vertices are adjacent.

Thus the vertices in S cover each edge in K at most once. We say the singleton $3\mathbf{e}_i$ is located at the vertex i in K and the pair $2\mathbf{e}_i + \mathbf{e}_j$ is located at the vertex i in K and touches the vertex j in K (a triple $\mathbf{e}_i + \mathbf{e}_j + \mathbf{e}_k$ is not located at any vertex in K, but touches i, j, and k). We note that S contains at most m singletons plus pairs. Indeed, for each coordinate i, S contains at most one vertex located at i in K (since any two vertices located at i are adjacent). Furthermore, there can be at most one singleton in S (since all singletons are adjacent). Since a triple covers 3 edges, a pair covers only 1, and a singleton covers 0, we find an upper bound by assuming that there are 1 singleton and $m-1$ pairs in S (as we have seen, each is located at a different vertex, and similarly each pair touches a different vertex and does not touch the vertex at which the singleton is located). The vertex in K at which the singleton is located still has $m-1$ edges uncovered, while the $m-1$ vertices at which pairs are located each have $m-3$ edges uncovered (for each i, one edge is covered by the pair located at i and one edge is covered by the pair that touches i). A triple that touches i covers two edges incident with i (namely, the triple $\mathbf{e}_i + \mathbf{e}_j + \mathbf{e}_k$ touches i and covers the edges $\{i, j\}$ and $\{i, k\}$ incident with i), so at most $\frac{1}{2}(m-3)$ triples can touch each vertex at which a pair is located, while at most $\frac{1}{2}(m-1) = \frac{1}{2}(m-3) + 1$ triples can touch the vertex at which the singleton is located. Since each triple touches three vertices, there can be at most $\frac{1}{3}(m \left\lfloor \frac{1}{2}(m-3) \right\rfloor + 1)$ triples in S. Thus $|S| \leq m + \left\lfloor \frac{1}{3}\left(m \left\lfloor \frac{1}{2}(m-3) \right\rfloor + 1\right) \right\rfloor$, which proves that the given upper bound on $\alpha(m, n)$ holds.

Separating into cases on the value of $m \pmod 6$, it is straightforward to verify that this upper bound is identical to the claimed value of $\alpha(m, n)$, so to complete the proof we need only show that we can always construct an independent set which meets the bound. We will construct the independent sets using Steiner triple systems and Kirkman triple systems.

If $m \equiv \pm 1 \pmod 6$, consider a Steiner triple system STS$(m+2)$, which exists by Proposition 1.2. This system contains $\binom{m+2}{2}/3 = \frac{1}{6}(m+2)(m+1)$ triples containing the integers $1, 2, \ldots, m+1, m+2$. By definition of a Steiner triple system, one of these triples contains both $m+1$ and $m+2$, say $\{x, m+1, m+2\}$ for some $x \in \{1, 2, \ldots, m\}$.

Furthermore, since each pair $\{y, m+1\}$ and $\{y, m+2\}$ must be contained in some triple for each $y \in \{1, 2, \ldots, m\} \setminus \{x\}$, and since each triple containing exactly one of $m+1$ and $m+2$ contains two such pairs, there are $(m-1)/2$ triples containing $m+1$ but not $m+2$, and $(m-1)/2$ triples containing $m+2$ but not $m+1$. Removing each triple containing at least one of $m+1$ or $m+2$, we obtain a system $\mathcal{S}$ of $\frac{1}{6}(m+2)(m+1)-m$ triples containing only the integers $1, 2, \ldots, m$. For each triple $\{i, j, k\} \in \mathcal{S}$, we include the triple $\mathbf{e}_i + \mathbf{e}_j + \mathbf{e}_k$ in S, and consider the edges covered in K as above. For each $y \in \{1, 2, \ldots, m\} \setminus \{x\}$, y was in two of the removed triples, so there are two edges in K incident with y which are not covered (the edges $\{y, z\}$ and $\{y, z'\}$ are not covered if $\{y, z, m+1\}$ and $\{y, z', m+2\}$ were removed from the STS$(m+2)$). Every other edge is covered, since every other pair in $\{1, 2, \ldots, m\}$ is in a triple in S. Thus the graph induced by the uncovered edges is regular of valency 2, and is therefore a union of cycles. We orient the cycles arbitrarily, and for each oriented edge (i, j) in a cycle, we add the pair $2\mathbf{e}_i + \mathbf{e}_j$ to S. By this construction, no edges are covered twice and no two pairs are located at the same vertex in K. Finally, we add $\mathbf{e}_x$ to S. No other vertex in S is located at x (and no pair touches x), so S is still an independent set. S contains $\left(\frac{1}{6}(m+2)(m+1)-m\right)+(m-1)+1 = \frac{1}{6}(m+2)(m+1)$ vertices, so we are done in this case.

If $m \equiv 0, 4 \pmod 6$, let $m' = m+1$ so that $m' \equiv \pm 1 \pmod 6$. Consider the independent set S in $SR(m', 3)$ obtained as above from a Steiner triple system STS$(m'+2)$ containing a triple $\{x, m'+1, m'+2\}$. Note $|S| = \frac{1}{6}(m'+2)(m'+1)$. Then each coordinate $y \in \{1, \ldots, m'\} \setminus \{x\}$ is nonzero in $(m'+1)/2$ vertices in S. Indeed, before the triples containing $m'+1$ and $m'+2$ are removed, every element in $\{1, 2, \ldots, m'+2\}$ is contained in $(m'+1)/2$ triples in STS$(m'+2)$. Each y is contained in two of the removed triples, so $(m'+1)/2-2$ triples in S are nonzero at the coordinate y. Finally, the two pairs $2\mathbf{e}_y + \mathbf{e}_z$ and $2\mathbf{e}_{z'} + \mathbf{e}_y$ (added when considering the oriented cycles) are nonzero at y. Thus each coordinate in $\{1, \ldots, m'\} \setminus \{x\}$ is nonzero for precisely $(m'+1)/2$ vertices in S. Choose a coordinate, delete it, and remove any vertices that did not equal 0 in that coordinate. The resulting set S'

contains $\frac{1}{6}(m'+2)(m'+1)-(m'+1)/2$ vertices and is an independent set in $SR(m,3)$. We have $|S'| = \frac{1}{6}(m'+2)(m'+1)-(m'+1)/2 = \frac{1}{6}(m'-1)(m'+1) = \frac{1}{6}m(m+2)$, so we are done in this case.

If $m \equiv 3 \pmod 6$, consider a KTS(m), which exists by Proposition 1.2 and contains $\binom{m}{2}/3 = \frac{1}{6}m(m-1)$ triples. Let $\mathcal{S}$ be the set of all of the triples in the KTS(m) except those from a single parallel class. For each triple $\{i,j,k\} \in \mathcal{S}$, we include the triple $\mathbf{e}_i+\mathbf{e}_j+\mathbf{e}_k$ in S. The removed class contained $m/3$ triples, so the triples in S cover all but m edges in K. Each $x \in \{1,\ldots,m\}$ was contained in exactly one removed triple. For each triple $\{i,j,k\}$, $i<j<k$, which was removed, we add the pairs $2\mathbf{e}_i+\mathbf{e}_j$, $2\mathbf{e}_j+\mathbf{e}_k$, and $2\mathbf{e}_k+\mathbf{e}_i$ to S. This process adds m pairs to S, no two located at the same vertex in K and no two touching the same vertex in K. Thus S is an independent set on $\left(\frac{1}{6}m(m-1)\right)-\frac{m}{3}+m = \frac{1}{6}m(m+3)$ vertices, so we are done in this case.

If $m \equiv 2 \pmod 6$, let $m' = m+1$ so that $m' \equiv 3 \pmod 6$. Consider the independent set S in $SR(m',3)$ obtained from a KTS(m') as above. Each coordinate in $\{1,\ldots,m'\}$ is nonzero for $(m'-1)/2-1$ triples in S and 2 pairs in S. Thus each coordinate is nonzero for precisely $(m'+1)/2$ vertices in S. Choose a coordinate, delete it, and remove any vertices that were not 0 in that coordinate. The resulting set S' contains $\frac{1}{6}m'(m'+1)-(m'+1)/2$ vertices and is an independent set in $SR(m,3)$. We have $|S'| = \frac{1}{6}m'(m'+3)-(m'+1)/2 = \frac{1}{6}((m')^2-3) = \frac{1}{6}((m+1)^2-3) = \frac{1}{6}(m^2+2m-2)$, so we are done in this case. □

See Figure 3.5 for an example of an independent set in $SR(m,3)$ for $m=3,4$.

3.3 Smallest Eigenvalue

Recall that the Hoffman ratio bound (Proposition 1.15) gives an upper bound on the independence number of a regular graph in terms of the smallest eigenvalue of the graph. Thus, when studying the independence number of a regular graph, it is sometimes useful to know the value of the smallest eigenvalue. Martin and Wagner [68] (see also Elkies [37]) found the smallest eigenvalue of $SR(m,n)$ when $n \geq \binom{m}{2}$.

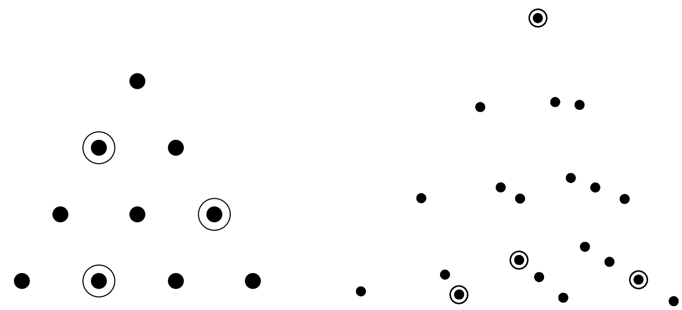


Figure 3.5: Maximal independent sets in $SR(3,3)$ and $SR(4,3)$.

Proposition 3.3. *If $n \geq \binom{m}{2}$, then the smallest eigenvalue of $SR(m,n)$ is $-\binom{m}{2}$.*

Proof. Martin and Wagner show directly that $-\binom{m}{2}$ is an eigenvalue of $SR(m,n)$ by constructing eigenvectors for that eigenvalue (see Proposition 3.19 in Section 3.10). Thus the smallest eigenvalue is at most $-\binom{m}{2}$. For each pair $1 \leq i < j \leq m$, let $SR(m,n)_{i,j}$ denote the vertex-spanning subgraph of $SR(m,n)$ containing only edges which are in the i,j direction. We see that $SR(m,n)_{i,j}$ is a disjoint union of complete graphs, so the smallest eigenvalue of $SR(m,n)_{i,j}$ is -1. Since $A(m,n)$ is the sum of the adjacency matrices of $SR(m,n)_{i,j}$, where the sum is taken over all $\binom{m}{2}$ pairs (i,j) with $1 \leq i < j \leq m$, the smallest eigenvalue of $A(m,n)$ is at least $-\binom{m}{2}$ (by Corollary 1.9 applied $\binom{m}{2}$ times). $\square$

Martin and Wagner show that when $n < \binom{m}{2}$, the graph $SR(m,n)$ has an eigenvalue $-n$ by constructing eigenvectors for that eigenvalue (see Proposition 3.20 in Section 3.10). Based on numerical evidence, they conjecture that $-n$ is the smallest eigenvalue in this case:

Conjecture 3.4. ([68, Conjecture 3.9]) *If $n < \binom{m}{2}$, then the smallest eigenvalue of $SR(m,n)$ is $-n$.*

We prove this conjecture and find the value of the smallest eigenvalue of $SR(m,n)$ for any m,n.

Proposition 3.5. *The smallest eigenvalue of $SR(m,n)$ is* $\max(-n, -\binom{m}{2})$.

Proof. By the results above we see that the smallest eigenvalue is at least $-\binom{m}{2}$. Furthermore, if $n < \binom{m}{2}$ then $-n$ is an eigenvalue, while if $n \geq \binom{m}{2}$ then $-\binom{m}{2}$ is an eigenvalue. It remains to show that the smallest eigenvalue is at least $-n$.

Consider the bipartite graph $\Delta(m,n)$ whose vertices are the m-tuples of nonnegative integers whose coordinates sum to at most n (that is, the union of $V(m,n)$ and the set $V'(m,n) = \{(x_1, x_2, \ldots, x_m) \mid 0 \leq x_i \leq n,\ \sum_{i=1}^{m} x_i < n\}$), where two vertices are adjacent when one has coordinate sum n, the other coordinate sum less than n, and they differ from each other in exactly one coordinate. Clearly $\Delta(m,n)$ is bipartite with partition $\{V(m,n), V'(m,n)\}$. We see that two vertices in $V(m,n)$ are adjacent in $SR(m,n)$ precisely when they have distance 2 in Δ. Indeed, if $x, y \in V(m,n)$ are (i,j)-neighbors in $SR(m,n)$, then they have all coordinates equal except i and j, and without loss of generality we have $x_i > y_i$ and $x_j < y_j$. Then the vertex $z \in V'(m,n)$ with $z_i = y_i$, $z_j = x_j$, and all other coordinates equal to those of x and y is adjacent to both x and y in $\Delta(m,n)$. In fact, we see that z is the only common neighbor of x and y in $\Delta(m,n)$. Conversely, we see that if $z \in V'(m,n)$ differs from $x \in V(m,n)$ only in the i coordinate and from $y \in V(m,n)$ only in the j coordinate, then x and y must be (i,j)-neighbors in $SR(m,n)$. If the adjacency matrix of $\Delta(m,n)$ is $\left(\begin{smallmatrix} 0 & N \\ N^\top & 0 \end{smallmatrix}\right)$, where the vertices are indexed by the partition $\{V(m,n), V'(m,n)\}$, then the x, y entry in $NN^\top$ is the number of common neighbors (in the graph $\Delta(m,n)$) in $V'(m,n)$ of the vertices $x, y \in V(m,n)$. We find that

$$NN^\top_{(x,y)} = \begin{cases} 1, & \text{if } x \sim y \text{ in } SR(m,n), \\ 0, & \text{if } x \neq y \text{ and } x \not\sim y \text{ in } SR(m,n), \\ n, & \text{if } x = y. \end{cases}$$

Indeed, we have already seen that if $x, y \in V(m,n)$ are adjacent in $SR(m,n)$, then they have exactly one common neighbor in $V'(m,n)$ in the graph $\Delta(m,n)$, while if $x, y \in V(m,n)$ are not adjacent in $SR(m,n)$ then they have no common neighbors in $V'(m,n)$ in the graph $\Delta(m,n)$. If $x = y$, the x, y entry of $NN^\top$ counts the number of neighbors in $V'(m,n)$ of the vertex $x \in V(m,n)$. Clearly every vertex in $V(m,n)$

has n neighbors in $V'(m,n)$. Indeed, for each coordinate i at which x has a nonzero entry, the vertex $x \in V(m,n)$ has x_i neighbors in $V'(m,n)$ that are equal to x in every coordinate except i (namely, the vertices which equal x everywhere except i, and in the coordinate i equal one of $0, 1, \ldots, x_i - 1$). Summing over all coordinates, we find that x has n neighbors in $V'(m,n)$. Thus $NN^\top$ has the form claimed above, so we find $A(m,n)+nI = NN^\top$. Since $MM^\top$ is positive semidefinite for any matrix M, this implies $A(m,n)+nI$ is positive semidefinite, so the smallest eigenvalue of $A(m,n)$ is at least $-n$. □

Combining Propositions 1.15 and 3.5 (and simplifying), we obtain the following bounds on $\alpha(m,n)$:

Corollary 3.6. *The independence number of $SR(m,n)$ satisfies*

$$\alpha(m,n) \leq \begin{cases} \binom{n+m-1}{m-1}/m & \text{if } n < \binom{m}{2}, \\ \binom{n+m-1}{m-1}\frac{m}{m+2n}, & \text{if } n \geq \binom{m}{2}. \end{cases}$$

However, we note that the bound given by Corollary 3.6 is not tight. For example, when $m = 3$ and $n = 4$ (so $n \geq \binom{m}{2}$) Corollary 3.6 implies $\alpha(3,4) \leq 4$, but by Proposition 3.1 we have $\alpha(3,4) = 3$. When $m = 4$ and $n = 3$ (so $n < \binom{m}{2}$) Corollary 3.6 implies $\alpha(4,3) \leq 5$, but by Proposition 3.2 we have $\alpha(4,3) = 4$.

3.4 Partial Spectrum

In this section, we construct an equitable partition of $SR(m,n)$ and calculate the eigenvalues of the corresponding quotient matrix. By Proposition 1.24, these eigenvalues are also eigenvalues of $SR(m,n)$. Recall that V_i is the set of vertices in $V(m,n)$ with exactly i nonzero coordinates.

Lemma 3.7. *Let $p = \min\{m,n\}$. Then the set $\{V_1, V_2, \ldots, V_p\}$ is an equitable partition of $V(m,n)$ with a tridiagonal quotient matrix. For $1 \leq i \leq p$, each vertex in V_i has $i(i-1)$ neighbors in V_{i-1}, $(n-i)(i-1)+i(m-i)$ neighbors in V_i, and $(n-i)(m-i)$ neighbors in V_{i+1}.*

Proof. Consider a vertex $v = (v_1, \ldots, v_m) \in V_i$, and let I be the set of coordinates of v which are nonzero (so $|I| = i$). Any vertex with more than $i+1$ or fewer than $i-1$ nonzero coordinates cannot be adjacent to v, since such a vertex would differ from v in more than two coordinates.

Suppose $u = (u_1, \ldots, u_m) \in V_{i-1}$ is adjacent to v. Then each of the $i-1$ nonzero coordinates of u must be in I, and there exists unique $j \in I$ such that $u_j = 0$ (so u and v differ in coordinate j). Since u must differ from v in exactly one other coordinate, there exists $k \in I \setminus \{j\}$ such that $u_k = v_j + v_k$ and $u_\ell = v_\ell$ for $\ell \neq j, k$. There are i possible coordinates j and $i-1$ possible coordinates k for each j, so v has $i(i-1)$ neighbors in V_{i-1}.

Suppose $u \in V_i$ is a neighbor of v. In this case there are two possibilities. First, it may be the case that u is also nonzero for each coordinate in I (and 0 elsewhere). In this case, there exist $j, k \in I$ such that $v_j + v_k = u_j + u_k$ and $v_\ell = u_\ell$ for $\ell \neq j, k$. There are $v_j + v_k - 2$ possible pairs u_j, u_k for each j, k. Thus we find v has

$$\sum_{j,k \in I} (v_j + v_k - 2) = (i-1) \sum_{j \in I} v_j - \sum_{j,k \in I} 2 = (i-1)n - 2\binom{i}{2} = (n-i)(i-1)$$

such neighbors in V_i. The other possibility is that u and v share only $i-1$ nonzero coordinates. Then there exist $j \in I$ and $k \in \{1, \ldots, m\} \setminus I$ such that $u_j = 0$, $u_k = v_j$, and $v_\ell = u_\ell$ for $\ell \neq j, k$. There are i possible coordinates j and $m - i$ possible coordinates k for each j, so v has $i(m-i)$ such neighbors.

Finally, suppose $u \in V_{i+1}$ is a neighbor of v. Then u is nonzero in each coordinate in I as well as in one other coordinate. Then there exist $j \in I$ and $k \in \{1, \ldots, m\} \setminus I$ such that $u_j + u_k = v_j$ and $v_\ell = u_\ell$ for $\ell \neq j, k$. For any j there are $m-i$ choices for k, so for fixed j there are $(v_j - 1)(m-i)$ such pairs u_j, u_k. Thus v has $\sum_{j \in I} (v_j - 1)(m-i) = (n-i)(m-i)$ neighbors in V_{i+1}.

Since $v \in V_i$ was chosen arbitrarily, this proves that $\{V_1, V_2, \ldots, V_p\}$ is an equitable partition of $V(m, n)$. $\square$

By Lemma 1.24, the eigenvalues of the quotient matrix of this partition are eigenvalues of $SR(m, n)$. To find the eigenvalues of the quotient matrix, we must first prove the following lemma. Let $(x)_k$ denote the falling factorial of x with k terms, that is, $(x)_k = x(x-1)\cdots(x-(k-1))$.

Lemma 3.8. *Let $F(\ell,k) = (-1)^\ell\binom{i-1}{\ell}(n-k)_{i-1-\ell}(m-k)_{i-1-\ell}(k-1)_\ell(k-2)_\ell$ and $S(j) = \sum_{\ell=0}^{j-2} F(\ell,j)$. If $j \geq 2$ is an integer then*

$$\begin{aligned} j(j-1)S(j) - \Big(j(j+1) + (n-j)(m-j) - i(n+m-i)\Big)S(j+1) \\ + (n-j-1)(m-j-1)S(j+2) = 0. \quad (3.1) \end{aligned}$$

Proof. We first note that we have

$$\begin{aligned} j(j-1)F(\ell,j) - \Big(j(j+1) + (n-j)(m-j) - i(n+m-i)\Big)F(\ell,j+1) \\ + (n-j-1)(m-j-1)F(\ell,j+2) \\ = F(\ell+1,j)R(\ell+1,j) - F(\ell,j)R(\ell,j), \quad (3.2) \end{aligned}$$

where

$$\begin{aligned} R(\ell,j) = \Bigg(j(j-1)\ell(m-(i-\ell-1)-j)(n-(i-\ell-1)-j) \\ \Big(j^2(m+n+1-3i) - j\big(2i^2 - i(2m+2n+4\ell-1) + \ell(m+n+1) + 2mn\big) \\ + i^2(\ell-1) - i(\ell-1)(m+n+\ell) + mn(\ell-1)\Big)\Bigg) \\ \Big/ \left((n-j)(m-j)\left((j-\ell)^2-1\right)(j-\ell)^2\right). \end{aligned}$$

Equation (3.2) is easily checked by dividing both sides by $F(\ell,j)$ and simplifying both sides. Then, since $F(j+1,j)R(j+1,j) = F(0,j)R(0,j) = 0$ and $F(j-1,j) = F(j,j) = F(j,j+1) = 0$, the lemma follows from summing both sides of (3.2) from $\ell = 0$ to j. □

The recurrence (3.1) and the function $R(\ell,j)$ were found using Zeilberger's algorithm (see [99]) as implemented in the `fastZeil` package for *Mathematica* by Paule and Schorn [77], which also provided an outline of the proof that (3.1) holds. For any fixed n and m we can now identify a subset of the eigenvalues of $SR(m,n)$.

Proposition 3.9. *For fixed m, n, let $p = \min\{m, n\}$. For each $i \in \{0, 1, \ldots, p-1\}$, $(m-i)(n-i) - n$ is an eigenvalue of $SR(m, n)$.*

Proof. By Lemma 3.7 we find that the $p \times p$ matrix

$$Q = \begin{pmatrix} a_1 & b_1 & 0 & \cdots & 0 \\ c_2 & a_2 & b_2 & \ddots & \vdots \\ 0 & c_3 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & a_{p-1} & b_{p-1} \\ 0 & \cdots & 0 & c_p & a_p \end{pmatrix}$$

with $a_i = (n-i)(i-1) + i(m-i)$, $b_i = (n-i)(m-i)$, and $c_i = i(i-1)$ is a quotient matrix of $A(m, n)$ with respect to the equitable partition $\{V_1, V_2, \ldots, V_p\}$. By Proposition 1.24, each eigenvalue of Q is also an eigenvalue of $A(m, n)$. We will see that the eigenvalues of Q are $\mu_i = (m-i)(n-i) - n$ with eigenvectors

$$v_i = (v_{i,j})_{j=1}^{p} = \left((-1)^i (n-1)_i (m-1)_i + (-1)^{i+1} i(n+m-i) \sum_{k=2}^{j} S(k) \right)_{j=1}^{p},$$

for $0 \le i \le p-1$. Indeed, by the structure of Q we have $Qv_i = \mu_i v_i$ if and only if $c_j v_{i,j-1} + a_j v_{i,j} + b_j v_{i,j+1} = \mu_i v_{i,j}$ for $1 \le j \le p$. We prove the latter by induction on j. Simplifying, we find that we must show that

$$i(n+m-i)\Bigg((n-1)_i (m-1)_i + j(j-1)S(j) \\ - (n-j)(m-j)S(j+1) - i(n+m-i) \sum_{k=2}^{j} S(k) \Bigg) = 0,$$

for $1 \le j \le p$. If $i = 0$, we are done. Otherwise, we must show that $G(j) = 0$ for $1 \le j \le p$, where

$$G(j) = (n-1)_i (m-1)_i + j(j-1)S(j) \\ - (n-j)(m-j)S(j+1) - i(n+m-i) \sum_{k=2}^{j} S(k).$$

It is straightforward to show that $G(1) = 0$ and $G(2) = 0$. For the induction step, we suppose that $G(j) = 0$ and show that $G(j+1) = 0$. We have $G(j+1) = G(j+1) - G(j)$ which, after simplifying, is equal to

$$-\Big(j(j-1)S(j) - \Big(j(j+1) + (n-j)(m-j) - i(n+m-i)\Big)S(j+1) \\ + (n-j-1)(m-j-1)S(j+2)\Big),$$

which is 0 by Lemma 3.8. This completes the proof. □

3.5 Spectrum of $SR(m,n)$ for $m \leq 3$ or $n \leq 3$

The graph $SR(1,n)$ is isomorphic to K_1 and thus has spectrum 0^1. The graph $SR(2,n)$ is isomorphic to K_{n+1} and has spectrum n^1, -1^n. The graph $SR(m,1)$ is isomorphic to K_m and has spectrum $(m-1)^1$, -1^{m-1}. The graph $SR(m,2)$ is isomorphic to the Johnson graph $J(m+1,2)$ and has spectrum $2(m-1)^1$, $(m-3)^m$, $-2^{(m+1)(m-2)/2}$ by Proposition 1.22.

Martin and Wagner [68] construct a complete set of eigenvectors for $SR(3,n)$ to prove:

Proposition 3.10. *The spectrum of $SR(3,n)$ is given by Table 3.1.*

If $n = 2k+1$:		If $n = 2k$:	
Eigenvalue	Multiplicity	Eigenvalue	Multiplicity
-3	$\binom{2k}{2}$	-3	$\binom{2k-1}{2}$
$-2, -1, \ldots, k-3$	3	$-2, -1, \ldots, k-4$	3
$k-1$	2	$k-3$	2
$k, \ldots, 2k-1$	3	$k-1, \ldots, 2k-2$	3
$2n$	1	$2n$	1

Table 3.1: Spectrum of $SR(3,n)$.

By refining the partition $\{V_1, V_2, V_3\}$ and noting a similarity between $SR(m,3)$ and the Johnson graph $J(m+2,3)$, we find the spectrum of $SR(m,3)$.

Proposition 3.11. *The spectrum of $SR(m,3)$ is given by Table 3.2.*

Eigenvalue	Multiplicity
$3(m-1)$	1
$2m-5$	m
$m-3$	$m-1$
$m-5$	$\binom{m}{2}$
-3	$m(m^2-7)/6$

Table 3.2: Spectrum of $SR(m, 3)$.

Proof. We consider the refinement $\mathcal{P}'$ of $\mathcal{P} = \{V_1, V_2, V_3\}$ obtained by considering also location of the nonzero elements. That is, let $\mathcal{P}'$ be the partition made up of the sets $V_1^i = \{3\mathbf{e}_i\}$ for $1 \leq i \leq m$, $V_2^{i,j} = \{\mathbf{e}_i + 2\mathbf{e}_j, 2\mathbf{e}_i + \mathbf{e}_j\}$ for $1 \leq i < j \leq m$, and $V_3^{i,j,k} = \{\mathbf{e}_i + \mathbf{e}_j + \mathbf{e}_k\}$ for $1 \leq i < j < k \leq m$. It is straightforward to show that $\mathcal{P}'$ is also equitable. Let R be the quotient matrix of the partition $\mathcal{P}'$. Then by Lemma 1.24 each eigenvalue of R is an eigenvalue of $SR(m, 3)$

Next, we consider the following correspondence between vertices in $SR(m, 3)$ and vertices in the Johnson Graph $J(m + 2, 3)$. For $1 \leq i \leq m$, the vertex $3\mathbf{e}_i \in V_1^i \subset V(m, 3)$ corresponds to the vertex $\{i, m + 1, m + 2\} \in V(J(m + 2, 3))$. For $1 \leq i < j \leq m$, the vertex $\mathbf{e}_i + 2\mathbf{e}_j \in V_2^{i,j} \subset V(m, 3)$ corresponds to the vertex $\{i, j, m + 1\} \in V(J(m + 2, 3))$ and the vertex $2\mathbf{e}_i + \mathbf{e}_j \in V_2^{i,j} \subset V(m, 3)$ corresponds to the vertex $\{i, j, m + 2\} \in V(J(m + 2, 3))$. For $1 \leq i < j < k \leq m$, the vertex $\mathbf{e}_i + \mathbf{e}_j + \mathbf{e}_k \in V_3^{i,j,k} \subset V(m, 3)$ corresponds to the vertex $\{i, j, k\} \in V(J(m + 2, 3))$. If we partition the vertices of $J(m+2, 3)$ according to the partition $\mathcal{P}'$ of the corresponding vertices of $SR(m, 3)$, it is straightforward to show that this partition $\mathcal{P}^*$ is also equitable with the same quotient matrix R as before. Thus by Lemma 1.24 the eigenvalues of R are common to $SR(m, 3)$ and $J(m + 2, 3)$.

Let E be the space orthogonal to the characteristic vectors of the partition $\mathcal{P}'$ (and thus, also of $\mathcal{P}^*$), or equivalently the space of vectors that sum to 0 on each set in $\mathcal{P}'$ (and thus, also in $\mathcal{P}^*$). By Proposition 1.24, to find the eigenvalues of R we can take the spectrum of $J(m + 2, 3)$ and remove those eigenvalues whose corresponding eigenvectors are in E (we call these eigenvectors E-eigenvectors and the corresponding eigenvalues E-eigenvalues). We note that the E-eigenvectors of $J(m+2,3)$ must be zero on all sets in the partition $\mathcal{P}^*$ containing only one vertex. Thus we can restrict ourselves to the subgraph induced by the sets in $\mathcal{P}^*$ containing two vertices (namely, $V_2^{i,j}$ for $1 \leq i < j \leq m$). That is, we restrict ourselves to the subgraph GJ whose vertices are those containing exactly one of $m+1$ and $m+2$. Those containing $m+1$ induce a copy of $J(m,2)$, as do those containing $m+2$. The only edges between a vertex containing $m+1$ and a vertex containing $m+2$ are those between vertices which have their other two elements equal (that is, those in the same set $V_2^{i,j}$ for some i,j. Thus GJ is isomorphic to $J(m,2)\square K_2$, and by Proposition 1.16 the adjacency matrix AJ of GJ is the matrix $T \otimes I_2 + I \otimes (J_2 - I_2)$, where T is the adjacency matrix of $J(m,2)$ (or equivalently of the triangular graph $T(m)$). That is, AJ is the matrix obtained by replacing the 1's in T by I_2, the diagonal 0's by $J_2 - I_2$, and the other 0's by O_2. With respect to the restriction of $\mathcal{P}^*$ to sets with 2 vertices, the adjacency matrix AJ of GJ has quotient matrix $T + I$. The E-eigenvalues of $J(m+2,3)$ are the eigenvalues of GJ with eigenvectors summing to 0 on each part of the partition $\mathcal{P}^*$, which (again by Lemma 1.24) are precisely the eigenvalues of AJ with the eigenvalues of the quotient matrix $T+I$ removed. The spectrum of $J(m,2)$ is $\{(2m-4)^1, (m-4)^{m-1}, -2^{m(m-3)/2}\}$ (by Proposition 1.22), so the spectrum of $T+I$ is $\{(2m-3)^1, (m-3)^{m-1}, -1^{m(m-3)/2}\}$ and by Proposition 1.16 the spectrum of AJ is $\{(2m-3)^1, (2m-5)^1, (m-3)^{m-1}, (m-5)^{m-1}, -1^{m(m-3)/2}, -3^{m(m-3)/2}\}$. Thus the E-eigenvalues of $J(m+2,3)$ are $\{(2m-5)^1, (m-5)^{m-1}, -3^{m(m-3)/2}\}$. Since the spectrum of $J(m+2,3)$ is $\{3(m-1)^1, (2m-5)^{m+1}, (m-5)^{(m+2)(m-1)/2}, -3^{(m+2)(m+1)(m-3)/6}\}$ by Proposition 1.22, the spectrum of R is $\{3(m-1)^1, (2m-5)^m, (m-5)^{\binom{m}{2}}, -3^{(m^2+2)(m-3)/6}\}$.

To complete the spectrum of $SR(m,3)$, we need only find the E-eigenvalues of $SR(m,3)$ and combine these with the eigenvalues of R. By the same argument as before, we can restrict ourselves to the subgraph GS induced by the sets in $\mathcal{P}'$ containing two vertices. We build the adjacency matrix AS of GS as follows. Let K be the directed complete graph K_m with vertices labeled $\{1,\ldots,m\}$ and edges oriented toward the vertex with larger label. Let M denote the signed vertex-edge incidence matrix of K. That is, M is the matrix with rows indexed by the vertices of K and columns indexed by the edges of K such that in the column of edge (i,j), $i<j$, there is a -1 in row i, a 1 in row j, and a 0 in every other row. Let $T' = M^\top M$. The rows and columns of T' are indexed by the edges of K, which are of the form (i,j), $i<j$. We see that we have

$$T'_{(i,j),(k,\ell)} = \begin{cases} 2, & \text{if } (i,j)=(k,\ell), \\ 1, & \text{if } i=k \text{ or } j=\ell \text{ (but not both)}, \\ -1, & \text{if } i=\ell \text{ or } j=k, \\ 0, & \text{else.} \end{cases}$$

We identify the oriented edge (i,j), $i<j$, with the vertex $\mathbf{e}_i + 2\mathbf{e}_j \in V(GS)$ and the opposite oriented edge (j,i), $i<j$, which is not in K, with the vertex $2\mathbf{e}_i + \mathbf{e}_j \in V(GS)$. We extend T' as follows: we replace each 1 by I_2, each -1 by $J_2 - I_2$, each 2 by $J_2 - I_2$, and each 0 by O_2. We will show that the resulting matrix is AS. There are $\binom{m}{2}$ blocks of size 2×2 in the extended matrix T^*. Let the first row and column of the $(i,j),(k,\ell)$ block correspond to the vertices (i,j) and (k,ℓ), and the second row and column of the same block correspond to the vertices (j,i) and (ℓ,k). By construction, if $(i,j)=(k,\ell)$ then the $(i,j),(k,\ell)$ block is

$$\begin{array}{r c} & \begin{array}{cc} (i,j) & (j,i) \end{array} \\ \begin{array}{r} (k,\ell)=(i,j) \\ (\ell,k)=(j,i) \end{array} & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{array}.$$

These are the correct entries in that block of AS, since $(i,j) \nsim (i,j)$ and $(j,i) \nsim (j,i)$, but $(i,j) \sim (j,i)$. If $i=k$ or $j=\ell$ (but not both), then the $(i,j),(k,\ell)$ block is

$$\begin{array}{r c} & \begin{array}{cc} (i,j) & (j,i) \end{array} \\ \begin{array}{r} (k,\ell)=(i,\ell) \\ (\ell,k)=(\ell,i) \end{array} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array} \quad \text{or} \quad \begin{array}{r c} & \begin{array}{cc} (i,j) & (j,i) \end{array} \\ \begin{array}{r} (k,\ell)=(k,j) \\ (\ell,k)=(j,k) \end{array} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array},$$

respectively. Again, we can easily verify that these are the correct entries in AS. If $i = \ell$ or $j = k$, then the $(i, j), (k, \ell)$ block is

$$\begin{array}{rc} & \begin{array}{cc}(i,j) & (j,i)\end{array} \\ \begin{array}{r}(k,\ell)=(k,i)\\ (\ell,k)=(i,k)\end{array} & \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\end{array} \quad \text{or} \quad \begin{array}{rc} & \begin{array}{cc}(i,j) & (j,i)\end{array} \\ \begin{array}{r}(k,\ell)=(j,\ell)\\ (\ell,k)=(\ell,j)\end{array} & \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\end{array},$$

respectively. As before, we can easily verify that these are the correct entries in AS. Finally, if $\{i, j\}$ and $\{k, \ell\}$ are disjoint, the $(i, j), (k, \ell)$ block is O_2, which is correct in AS. Thus the matrix T^* obtained by expanding T' as described is indeed AS. By Lemma 1.24, the E-eigenvalues of GS are the eigenvalues of AS whose eigenvectors sum to 0 on each block. Equivalently, if the blocks equal to $J_2 - I_2$ are replaced by $-(J_2-I_2)$, they are the eigenvalues (of the new matrix) whose eigenvectors are constant on each block. These are the eigenvalues of the quotient matrix T'' of the new matrix (with respect to the partition $\mathcal{P}'$ restricted to sets of two vertices). By construction, $T'' = M^\top M - 3I$. Since $MM^\top$ is the Laplacian matrix of the complete graph K_m, its eigenvalues are m^{m-1} and 0^1. Since $M^\top M$ and $MM^\top$ have the same nonzero eigenvalues and the same rank by Proposition 1.5, this implies that the spectrum of $M^\top M$ is $\{m^{m-1}, 0^{(m-1)(m-2)/2}\}$, so the E-eigenvalues of GS are $\{(m-3)^{m-1}, -3^{(m-1)(m-2)/2}\}$. Combining these with the spectrum of R, we find that the spectrum of $SR(m, 3)$ is $\{3(d-1)^1, (2m-5)^m, (m-3)^{m-1}, (m-5)^{\binom{m}{2}}, -3^{m(m^2-7)/6}\}$. □

There is a good reason for the similarity between the spectrum of $SR(m, 3)$ and that of $J(m+2, 3)$. Note the similarity between T and T': If we replace the diagonal entries of T by 2 and some (certain) 1's in T by -1, we obtain T'. Viewed another way, if B is the unsigned vertex-edge incidence matrix of K_m, then $B^\top B = T + 2I$. Then, replacing the diagonal 2's with $J - I_2$ and the 1's with I_2, we obtain AJ. Thus AS can be obtained from AJ by replacing certain I_2 blocks with $J - I_2$ blocks. In other words, $SR(m, 3)$ can be obtained from $J(m + 2, 3)$ by switching the adjacencies of a few vertices in V_2 according to which I_2 blocks were replaced by $J - I_2$ blocks.

3.6 Integrality of the Spectrum of $SR(m, n)$

A graph is called integral if all of its eigenvalues are integers. We have seen that $SR(m, n)$ is integral for $m \leq 3$ or $n \leq 3$. Martin and Wagner [68] checked by computer and found that if $m = 4$ and $n \leq 30$ or $m = 5$ and $n \leq 25$ then the spectrum of $SR(m, n)$ consists only of integers. This numerical evidence led them to make the following conjecture.

Conjecture 3.12. ([68, Conjecture 1.3]) *The graph $SR(m, n)$ is integral for any m, n.*

Integrality of a graph typically implies that the graph has some nice combinatorial structure. Integral graphs are apparently very rare. For example, for fixed valency k there are only finitely many connected, k-regular integral graphs (see [5] for a survey on integral graphs, including a proof of this fact). Also, for fixed n almost all graphs on n vertices are not integral (see [2]). Integral graphs have applications in quantum computing and quantum physics (see [91]).

Let $N = |V(m, n)| = \binom{n+m-1}{m-1}$. We prove Conjecture 3.12 by proving that each basis vector for the space $\mathbb{R}^N$ can be mapped to the zero vector by $\prod_i (A(m, n) - c_i I)$ for some sequence of integers $\{c_i\}$.

Associate with each vertex in $V(m, n)$ a distinct number in the set $\{1, 2, \ldots, N\}$. A basis for $\mathbb{R}^N$ is given by the vectors e_x ($x \in V(m, n)$) that equal 1 in the coordinate associated with x and 0 in all other coordinates.

For $S \subseteq V(m, n)$, let $e_S := \sum_{x \in S} e_x$. For partitions Π of the set of coordinate positions $\{1, \ldots, m\}$ and nonnegative integral vectors z indexed by Π that sum to n, let $S_{\Pi,z}$ be the set of all $u \in V(m, n)$ with $\sum_{i \in \pi} u_i = z_\pi$ for all $\pi \in \Pi$. The basis vector e_x can be realized as e_S where $S = S_{\Omega,z}$, Ω is the partition of $\{1, \ldots, m\}$ into singletons, and $z_{\{i\}} = x_i$ for each $i \in \{1, \ldots, m\}$.

For a vector y indexed by a partition Π, let $\tilde{y}$ be the sequence of pairs $(y_\pi, |\pi|)$ ($\pi \in \Pi$) sorted lexicographically: with the y_π in nondecreasing order, and for given y_π with the $|\pi|$ in nondecreasing order.

For partitions Π and Σ and vectors z and y indexed by Π and Σ, respectively, order pairs (Π, z) and (Σ, y) by $(\Sigma, y) < (\Pi, z)$ when $|\Sigma| < |\Pi|$, or when $|\Sigma| = |\Pi|$ and $\tilde{y} \neq \tilde{z}$, and in the first place j where $\tilde{y}$ and $\tilde{z}$ differ, the pair $\tilde{y}_j$ is lexicographically smaller than the pair $\tilde{z}_j$.

We are now ready to prove Conjecture 3.12.

Theorem 3.13. *All of the eigenvalues of $SR(m, n)$ are integers.*

Proof. We show that for each $x \in V(m, n)$ there is a sequence of integers c_{xi} (i in some indexing set I_x dependent on x) such that $\left(\prod_{i\in I_x}(A(m, n) - c_{xi}I)\right) e_x = 0$. Then $p(A(m, n)) := \prod_{x\in V(m,n)} \prod_{i\in I_x}(A(m, n) - c_{xi}I)v = 0$ for all $v \in \mathbb{R}^N$, so all eigenvalues of $A(m, n)$ are among the integers c_{xi}.

We begin with the basis vectors e_x for $x \in V(m, n)$ ($e_x = e_S$ for $S = S_{\Omega,z}$) and, for each x, find an integer c such that $(A(m, n) - cI)e_S \in U$, where U is a subspace spanned by a set of some e_T for $T = S_{\Sigma,y}$ with $(\Sigma, y) < (\Omega, z)$. We repeat the process for each of these e_T (showing that for some integer c, $(A(m, n) - cI)e_T$ is 0 or lies in a subspace spanned by some $e_{T'}$ with $T' = S_{\Sigma',y'}$ and $(\Sigma', y') < (\Sigma, y)$) until there is no smaller (Σ', y'), which occurs when Σ' has only one part, the whole set $\{1, \ldots, m\}$. Then the only possible y' has only one entry, n, and this pair (Σ', y') is smaller than every other pair, so the process must terminate there, when $U = U_1$ is the space spanned by $e_{V(m,n)}$ (since $S_{\Sigma',y'} = V(m, n)$) and we have $(A(m, n) - cI)e_{V(m,n)} = 0$, where $c = (m - 1)n$. Thus, if we can prove that for arbitrary $S = S_{\Pi,z}$ we have $(A(m, n) - cI)e_S$ lies in a subspace U spanned by a set of some e_T for $T = S_{\Sigma,y}$ with $(\Sigma, y) < (\Pi, z)$, then we are done.

For $j \neq k$, let A_{jk} be the matrix that describes only (i, j)-edges. In other words, the (x, y) entry of A_{jk} is 1 if x and y are (j, k)-neighbors and 0 otherwise. Then $A(m, n) = \sum A_{jk}$. If $(A_{jk} - c_{jk}I)u \in U$ for all j, k then $(A(m, n) - cI)u \in U$ for $c = \sum c_{jk}$. For fixed (Π, z), $S = S_{\Pi,z}$, and $j \neq k$, we will show there exists an integer c_{jk} such that the image $(A_{jk} - c_{jk}I)e_S$ lies in a subspace U spanned by a set of some e_T for $T = S_{\Sigma,y}$ with $(\Sigma, y) < (\Pi, z)$.

An integral vector in $\mathbb{R}^N$ can be viewed as a multiset where the $x \in V(m,n)$ occur with certain multiplicities. For any vertex u and any pair of coordinates j,k, the u entry of $A_{jk}e_S$ (or the number of times u occurs in the multiset $A_{jk}e_S$) is $(A_{jk}e_S)_u = |\{v \in S \mid u,v \text{ are } (j,k)\text{-neighbors}\}|$, the number of (j,k)-neighbors of u which are in S.

Fix (Π, z) and $S = S_{\Pi,z}$. Recall we wish to show that for each pair $j \neq k$ there exists an integer c_{jk} such that the image $(A_{jk} - c_{jk}I)e_S$ lies in a subspace U spanned by a set of some e_T for $T = S_{\Sigma,y}$ with $(\Sigma, y) < (\Pi, z)$.

We first handle all pairs j,k that belong to the same part of Π. Note that S induces a regular subgraph Γ_S of $SR(m,n)$, since Γ_S is a copy of the Cartesian product $\prod_{\pi\in\Pi} SR(|\pi|, z_\pi)$. Every edge in Γ_S is a (j,k)-edge where j,k are in the same part in Π. Further, if j,k are in the same part in π then there are no (j,k)-edges from S to $V(m,n) \setminus S$. Thus, if $A_\Pi = \sum_{\pi\in\Pi}\sum_{j,k\in\pi} A_{jk}$, then $A_\Pi e_S$ is constant on S (the value is the valency of Γ_S) and 0 on $V(m,n) \setminus S$. This implies $(A_\Pi - c_S I)e_S = 0$, where c_S is the valency of Γ_S. Thus we are done when j,k are in the same part in Π.

Suppose $j \in \pi$, $k \in \rho$, where $\pi, \rho \in \Pi$, $\pi \neq \rho$ (we label j and k such that $(z_\pi, |\pi|) \leq (z_\rho, |\rho|)$ in lexicographic order). Abbreviate $\pi \cup \{k\}$ by $\pi + k$ and $\pi \setminus \{j\}$ by $\pi - j$. We will show that the image $(A_{jk} + I)e_S$ equals $S_1 - S_2$, where S_1 is the sum of all e_T with $T = S_{\Sigma,y}$, $\Sigma = (\Pi \setminus \{\pi,\rho\}) \cup \{\pi - j, \rho + j\}$ (omitting $\pi - j$ if it is empty), and y agrees with z except that $y_{\pi-j} \leq z_\pi$ and $y_{\rho+j} \geq z_\rho$ (of course $y_{\pi-j} + y_{\rho+j} = z_\pi + z_\rho$, and $y_{\pi-j} = 0$ if $\pi - j$ is empty), and S_2 is the sum of all e_T with $T = S_{\Sigma,y}$, $\Sigma = (\Pi \setminus \{\pi,\rho\}) \cup \{\pi + k, \rho - k\}$, and y agrees with z except that $y_{\pi+k} < z_\pi$ and $y_{\rho-k} > z_\rho$ (of course $y_{\pi+k} + y_{\rho-k} = z_\pi + z_\rho$, and $\rho - k$ is not empty because $y_{\rho-k} > 0$). Because we label j and k such that $(z_\pi, |\pi|) \leq (z_\rho, |\rho|)$, we see that in S_1 and S_2, (Σ, y) occur only with $(\Sigma, y) < (\Pi, z)$, so that $S_1 - S_2$ is in a subspace U spanned by a set of some e_T for $T = S_{\Sigma,y}$ with $(\Sigma, y) < (\Pi, z)$. Thus, if we prove $(A_{jk} + I)e_S = S_1 - S_2$, then we are done.

First, note that the entries of S, S_1, and S_2 are either 0 or 1 (so we can view them as sets, rather than multisets). Indeed, for fixed Σ, a vertex cannot be in $S_{\Sigma,y}$ for two different vectors y. Viewed as sets, S, S_1, and S_2 satisfy $S_2 \subseteq S_1$ (so $S_1 - S_2$

can still be viewed as a set), $S \subseteq S_1$, and $S \cap S_2 = \emptyset$. Indeed, if $u \in S_2$ then $\sum_{i\in\pi-j} u_i \leq \sum_{i\in\pi+k} u_i < z_\pi$, so $u \in S_1$. If $s \in S$, then $\sum_{i\in\pi-j} s_i \leq z_\pi$ (so $s \in S_1$) and $\sum_{i\in\pi+k} s_i \geq z_\pi$ (so $S \notin S_2$). Thus, since $(A_{jk}e_S)_s = 0$ for $s \in S$ (because the (j,k)-neighbors in S of $s \in S$ must have j,k in the same part of Π), we have $((A_{jk}+I)e_S)_s = 1 = (S_1 - S_2)_s$ for every $s \in S$.

Each vertex $u \notin S$ has at most one (j,k)-neighbor in S, namely that s such that $s_i = u_i$ for $i \neq j,k$, $s_j = z_\pi - \sum_{i\in\pi-j} u_i$ (so $\sum_{i\in\pi} s_i = z_\pi$), and $s_k = z_\rho - \sum_{i\in\rho-k} u_i$ (so $\sum_{i\in\rho} s_i = z_\rho$), if such s exists (that is, if s_j and s_k given above are both nonnegative). Note that if u is counted in S_1 then $\sum_{i\in\pi-j} u_i = y_{\pi-j} \leq z_\pi$, so $s_j \geq 0$.

Suppose $u \notin S$ has a (j,k)-neighbor $s \in S$. Since $\sum_{i\in\pi} s_i = z_\pi$, it follows that $\sum_{i\in\pi-j} u_i = \sum_{i\in\pi-j} s_i \leq z_\pi$, so that $u \in S_1$, but $\sum_{i\in\rho-k} u_i = \sum_{i\in\rho-k} s_i \leq z_\rho$, so that $u \notin S_2$. Thus if $u \notin S$ has a (j,k)-neighbor in S, then $((A_{jk}+I)e_S)_u = 1 = (S_1 - S_2)_u$. If $u \in S_1$ ($u \notin S$) does not have a (j,k)-neighbor in S, then the candidate s above with $s_j = z_\pi - \sum_{i\in\pi-j} u_i$ and $s_k = z_\rho - \sum_{i\in\rho-k} u_i$ does not exist. As we saw above, this implies that $s_k < 0$, so $\sum_{i\in\rho-k} u_i > z_\rho$ and $u \in S_2$. Then $((A_{jk}+I)e_S)_u = 0 = (S_1 - S_2)_u$ and we have proved that $((A_{jk}+I)e_S)_u = (S_1 - S_2)_u$ for every $u \in V(m,n)$. This completes the proof. $\square$

3.7 Diameter

In this section we determine the diameter of $SR(m,n)$. The diameter of a graph is often related to the spectrum (see, for example, Proposition 1.14 and [17, Proposition 4.7.1]). To bound the diameter of $SR(m,n)$ from above we will need the following lemma.

Lemma 3.14. *Let $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_m)$ be vertices in $SR(m,n)$ in V_k with the same k coordinates nonzero. Then* $\operatorname{dist}(x,y) \leq k-1$.

Proof. Without loss of generality, we may assume that the first k coordinates of x and y are nonzero and the rest are zero. We may also assume that for some ℓ we have

$y_i > x_i$ for $1 \leq i \leq \ell$ and $y_i \leq x_i$ for $\ell < i \leq k$. Then we have a chain of adjacencies

$$
\begin{aligned}
y &= (y_1, y_2, y_3, y_4, \dots, y_\ell, y_{\ell+1}, \dots, y_k, 0, \dots, 0) \\
&\sim (y_1 + (y_2 - x_2), x_2, y_3, y_4, \dots, y_\ell, y_{\ell+1}, \dots, y_k, 0, \dots, 0) \\
&\sim (y_1 + (y_2 - x_2) + (y_3 - x_3), x_2, x_3, y_4, \dots, y_\ell, y_{\ell+1}, \dots, y_k, 0, \dots, 0) \\
&\vdots \\
&\sim (y_1 + \sum_{i=2}^{\ell} (y_i - x_i), x_2, \dots, x_\ell, y_{\ell+1}, \dots, y_k, 0, \dots, 0) \\
&\vdots \\
&\sim (y_1 + \sum_{i=2}^{k-1} (y_i - x_i), x_2, \dots, x_{k-1}, y_k, 0, \dots, 0) \\
&\sim (x_1, \dots, x_k, 0 \dots, 0) = x,
\end{aligned}
$$

which takes exactly $\ell - 1$ steps to get to the 4th line followed by at most $k - \ell$ steps to get to the last line, for a total of at most $k - 1$ steps. Note that $\sum_{i=2}^{j} (y_i - x_i)$ is positive for $2 \leq j \leq \ell$, so every m-tuple up to line 4 is indeed a vertex of $SR(m, n)$, and $\sum_{i=j+1}^{k} (x_i - y_i)$ is nonnegative for $\ell \leq j < k$, so

$$
\begin{aligned}
y_1 + \sum_{i=2}^{j} (y_i - x_j) &= \sum_{i=1}^{j} y_i - \sum_{i=2}^{j} x_i \\
&= \left(n - \sum_{i=j+1}^{k} y_i \right) - \left(n - x_1 - \sum_{i=j+1}^{k} x_i \right) \\
&= x_1 + \sum_{i=j+1}^{k} (x_i - y_i) \geq x_1 > 0
\end{aligned}
$$

for $\ell \leq j < k$ and every m-tuple after line 4 is also a vertex of $SR(m, n)$. □

Proposition 3.15. *For any fixed m, n, the diameter of $SR(m, n)$ is* $\min\{m - 1, n\}$.

Proof. We first show that $\operatorname{diam}(SR(m, n)) \leq n$. For $x, y \in V(m, n)$, each of x and y is a sum of n standard basis vectors for $\mathbb{R}^m$, so there exist sequences $(i_1, i_2, \dots, i_n)$ and

$(j_1, j_2, \dots, j_n)$ such that $x = \sum_{k=1}^n \mathbf{e}_{i_k}$ and $y = \sum_{k=1}^n \mathbf{e}_{j_k}$ Then we have a chain of at most n adjacencies

$$\begin{aligned}
y &= \sum_{k=1}^{n} \mathbf{e}_{j_k} \\
&\sim \sum_{k=1}^{n} \mathbf{e}_{j_k} + (\mathbf{e}_{i_1} - \mathbf{e}_{j_1}) \\
&\sim \sum_{k=1}^{n} \mathbf{e}_{j_k} + (\mathbf{e}_{i_1} - \mathbf{e}_{j_1}) + (\mathbf{e}_{i_2} - \mathbf{e}_{j_2}) \\
&\vdots \\
&\sim \sum_{k=1}^{n} \mathbf{e}_{j_k} + \sum_{k=1}^{\ell} (\mathbf{e}_{i_k} - \mathbf{e}_{j_k}) \\
&\vdots \\
&\sim \sum_{k=1}^{n} \mathbf{e}_{j_k} + \sum_{k=1}^{n} (\mathbf{e}_{i_k} - \mathbf{e}_{j_k}) \\
&= \sum_{k=1}^{n} \mathbf{e}_{i_k} = x,
\end{aligned}$$

so $\operatorname{dist}(x, y) \le n$.

Next we show that $\operatorname{diam}(SR(m, n)) \le m - 1$. For any $x \in V_i$ and $y \in V_j$, we must show $\operatorname{dist}(x, y) \le m - 1$. First, suppose that $i + j \le m$. There is a vertex z in V_1 such that $\operatorname{dist}(x, z) = i - 1$ and $\operatorname{dist}(y, z) \le j$ (indeed, one can simply choose z so that the nonzero coordinate in z is one of the nonzero coordinates in x), so in this case $\operatorname{dist}(x, y) \le i + j - 1 = m - 1$. If $i + j = m + k$ for some $k \ge 1$, then x and y share at least k nonzero coordinates and there are vertices z_x and z_y in V_k which have the same k nonzero coordinates (k of those coordinates in which both x and y are nonzero) such that $\operatorname{dist}(x, z_x) = i - k$ and $\operatorname{dist}(y, z_y) = j - k$. By Lemma 3.14, $\operatorname{dist}(z_x, z_y) \le k - 1$, so $\operatorname{dist}(x, y) \le i - k + j - k + k - 1 = i + j - k - 1 = m - 1$.

We have proved that $\operatorname{diam}(SR(m, n)) \le \min\{m - 1, n\}$. It is straightforward to show that this bound can be reached. Indeed, if $m - 1 \le n$ (so $m < n$), then for $x = n\mathbf{e}_1$ and $y = (n - m)\mathbf{e}_1 + \sum_{i=1}^m \mathbf{e}_i$ we have $\operatorname{dist}(x, y) = m - 1$, and if $n \le m - 1$ (so $n < m$) then for $x = n\mathbf{e}_1$ and $y = \sum_{i=2}^{n+1} \mathbf{e}_i$ we have $\operatorname{dist}(x, y) = n$. $\square$

In terms of simplicial rooks, Proposition 3.15 implies that one can place a pair of rooks on a simplicial chess board of dimension $m-1$ with $n+1$ tiles in each direction such that it takes $\min\{m-1, n\}$ moves for one rook to reach the other, but there is no way to place the pair so that it takes $\min\{m-1, n\}+1$ moves.

3.8 Clique Number

In this section we find the clique number $\omega(m,n) = \omega(SR(m,n))$ of $SR(m,n)$ and give examples of maximal cliques of that size.

Proposition 3.16. *For any fixed m, n, $\omega(m,n) = \max\{m, n+1\}$.*

Proof. If $m = 1$ then $SR(1,n) \cong K_1$ and $\omega(1,n) = 1$. Since $SR(2,n) \cong K_{n+1}$ and $SR(m,1) \cong K_m$, we have $\omega(2,n) = n+1$ and $\omega(m,1) = m$, so in those cases we are done. We also recall that $SR(m,2)$ is isomorphic to the Johnson graph $J(m+1,2) \cong T(m+1)$ with clique number $\omega(T(m+1)) = m$ (indeed, $T(m+1)$ is the line graph of K_{m+1}, and the m edges incident with a single vertex in K_{m+1} are pairwise adjacent in $T(m+1)$). Thus we may assume that $m \geq 3$ and $n \geq 3$.

Fix $m, n \geq 3$. The set V_1 is a clique of size m, while the set $\{x\mathbf{e}_1 + y\mathbf{e}_2 \mid x, y \geq 0,\ x+y = n\}$ is a clique of size $n+1$ (see Figure 3.6). Thus $\omega(m,n) \geq \max\{m, n+1\}$.

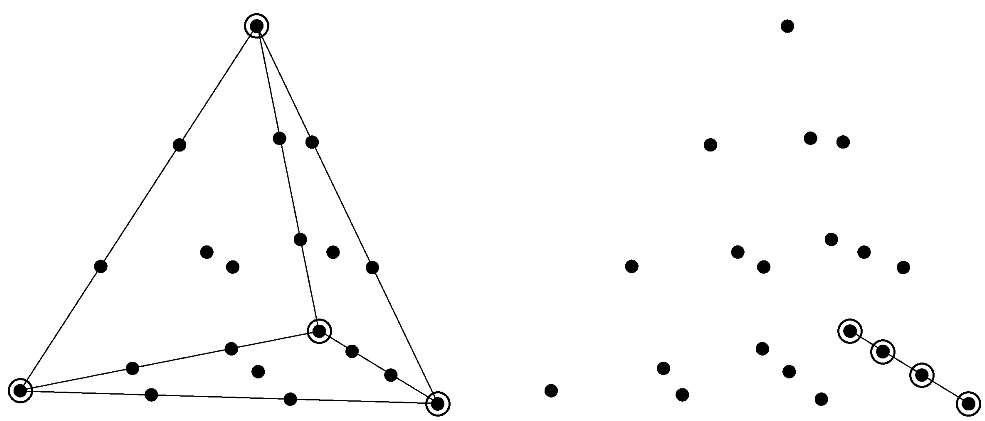


Figure 3.6: Maximal cliques of size $m = 4$ and $n+1 = 4$ in $SR(4,3)$.

Let C be a clique in $SR(m,n)$. We may assume $|C| \geq \max\{m, n+1\}$. For $u, v \in V(m,n)$, let $D(u,v)$ denote the set of coordinates where u and v differ. Thus, if u

and v are (i,j)-neighbors, then $D(u,v) = \{i,j\}$, $u_i + u_j = v_i + v_j$, and $u_k = v_k$ for all $k \neq i,j$. Since each pair of vertices in C is adjacent, each pair differs in exactly two coordinates. Suppose $u, v \in C$ and $D(u,v) = \{i,j\}$, and let $C' = C \setminus \{u,v\}$.

We first note that for all $x \in C'$, we must also have $i \in D(u,x)$ or $j \in D(u,x)$ (or both). To see that this is true, suppose there exists $x \in C'$ such that $D(u,x) \cap \{i,j\} = \emptyset$. Then there must be some $k, \ell \neq i,j$ such that $D(u,x) = \{k,\ell\}$. This implies that $D(v,x) = \{i,j,k,\ell\}$, so $v \not\sim x$, a contradiction.

Next, we see that if $D(u,w) = \{i,j\}$ for some $w \in C'$, then $D(u,x) = \{i,j\}$ for all $x \in C'$. Suppose not. That is, suppose $D(u,w) = \{i,j\}$ and there exists $z \in C'$ such that $D(u,z) = \{i,k\}$ for some $k \neq j$ (as noted above, we cannot have both $i \notin D(u,z)$ and $j \notin D(u,z)$). Note that $D(u,v) = \{i,j\}$ and $D(u,w) = \{i,j\}$ implies $D(v,w) = \{i,j\}$. Then must have $D(v,z) = \{j,k\}$ and $D(w,z) = \{j,k\}$, which implies $z_i = v_i$ and $z_i = w_i$, a contradiction (since $v_i \neq w_i$). Thus if any three vertices in C differ pairwise in the same two coordinates i,j, then every pair of vertices in C differs at i,j. In this case $|C| \leq u_i + u_j + 1 \leq n + 1$.

Next, suppose it is not the case that three vertices in C differ pairwise in the same two coordinates. That is, suppose for each $x \in C'$ we have either $i \in D(u,x)$ or $j \in D(u,x)$, but not both. Then either $i \in D(u,x)$ for all $x \in C'$ or $j \in D(u,x)$ for all $x \in C'$. Further, if $k \in D(u,x)$ for some $x \in C'$, $k \neq i,j$, then $k \notin D(u,y)$ for all $y \in C' \setminus \{x\}$. To prove the first assertion, suppose it is not true. That is, suppose there exist $w, z \in C'$ such that $D(u,w) = \{i,k\}$ and $D(u,z) = \{j,\ell\}$. If $k \neq \ell$ then $D(w,z) = \{i,j,k,\ell\}$, a contradiction. Thus $k = \ell$ and $D(u,z) = \{j,k\}$. Then we find that $D(v,w) = \{j,k\}$, $D(v,z) = \{i,k\}$, and $D(w,z) = \{i,j\}$. This implies $w_i = v_i$, $z_j = v_j$, and $w_k = z_k$. Combining these equalities with $w_i + w_k = u_i + u_k$ and $z_j + z_k = u_j + u_k$ we obtain $u_i + u_k - v_i = w_k = z_k = u_j + u_k - v_j$, from which we obtain $u_i - v_i = u_j - v_j$. Since $u_i + u_j = v_i + v_j$, this implies $u_i = v_i$, a contradiction. Thus the first assertion is true, and we assume without loss of generality that $i \in D(u,x)$ for all $x \in C'$. To prove the second assertion, note that if there exist $w, z \in C'$ such that $k \in D(u,w)$ and $k \in D(u,z)$, then $D(u,w) = D(u,z) = D(w,z) = \{i,k\}$. This contradicts that no three vertices in C differ pairwise in the same two coordinates. Thus both assertions are true. This implies $|C| \leq m$, since for each $x \in C \setminus \{u\}$ there is a distinct coordinate k such that $D(u, x) = \{i, k\}$. There are only $m - 1$ possible values for k (since $i \neq k$), so $|C \setminus \{u\}| \leq m - 1$.

Thus $\omega(m, n) = \max\{m, n + 1\}$. $\square$

In terms of simplicial rooks, Proposition 3.16 implies that there is a configuration of $\max\{m, n+1\}$ rooks on a simplicial chess board of dimension $m-1$ with $n+1$ tiles in each direction such that every rook can attack every other rook in one move, but there is no such configuration of $\max\{m, n + 1\} + 1$ rooks.

3.9 Cospectral Mates

Recall that a graph G is called DS if any graph with the same spectrum of G is isomorphic to G (see Definition 1.32). Martin and Wagner [68] ask for which values of the parameters m and n the graph $SR(m, n)$ is determined by its spectrum. Since $SR(1, n)$, $SR(2, n)$, and $SR(m, 1)$ are complete graphs, they are DS (see [31] or Proposition 1.33). Since $SR(m, 2)$ is isomorphic to the triangular graph $T(m + 1)$, it is DS unless $m = 7$, since $T(k)$ is DS unless $k = 8$ (there are exactly four graphs with the spectrum of $SR(7, 2) \cong T(8)$, namely $T(8)$ and the three Chang graphs, see [22] and [23]). Since $SR(3, 3)$ is a 6-regular graph on 10 vertices, its complement is a cubic graph on 10 vertices. It is known that the cubic graphs on at most 12 vertices are DS (see [79]), so by Proposition 1.32 $SR(3, 3)$ is DS. Alternately, since there are only 19 cubic graphs on 10 vertices (see [83]), one can simply check their spectra and observe that they are all DS, so by Proposition 1.32 $SR(3, 3)$ is DS. When $m = 4$ and $n \geq 3$ or $n = 3$ and $m \geq 4$, we use GM-switching (Theorem 1.34) to find nonisomorphic cospectral mates of $SR(m, n)$.

Proposition 3.17. *The graph $SR(4, n)$ is not determined by its spectrum for $n \geq 3$.*

Proof. When $m = 4$ we see that V_1 is a GM-switching set. Indeed, V_1 is a clique of 4 vertices, each vertex in V_2 has 2 neighbors in V_1, and each vertex in $V_3 \cup V_4$ has no neighbors in V_1 (see Figure 3.7). Furthermore, the resulting graph G after switching is

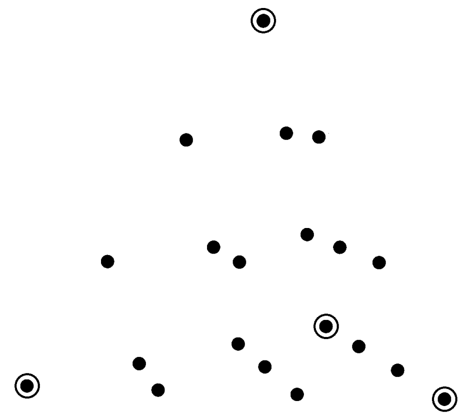


Figure 3.7: The set V_1 as a GM-switching set in $SR(4, 3)$.

not isomorphic to $SR(4, n)$. Indeed, it is not difficult to show that each vertex in V_1 or V_3 gains at least one additional distance-2 neighbor after switching, while each vertex in V_2 or V_4 retains the same number of distance-2 neighbors after switching, as we will see below.

Note that when switching with V_1 as the GM-switching set, the only possible changes in edges are between a vertex in V_1 and a vertex in V_2. Also, for $n \geq 3$ the diameter of $SR(4, n)$ is 3 by Proposition 3.15. First consider a vertex $v \in V_1$. Without loss of generality, let $v = (n, 0, 0, 0)$. The neighbors of v in $SR(4, n)$ are $V_1 \setminus \{v\}$ and $\bigcup_{x+y=n}\{(x, y, 0, 0), (x, 0, y, 0), (x, 0, 0, y)\}$. Thus the distance-2 neighbors of v in $SR(4, n)$ are $\bigcup_{x+y=n}\{(0, x, y, 0), (0, x, 0, y), (0, 0, x, y)\}$ and $\bigcup_{x+y+z=n}\{(x, y, z, 0),$ $(x, 0, y, z), (x, y, 0, z)\}$ (as well as v itself), for a total of $3(n-1) + 3\binom{n-1}{2} = 3\binom{n}{2}$ distance-2 neighbors (not counting v). After switching, the neighbors of v are $V_1 \setminus \{v\}$ and $\bigcup_{x+y=n}\{(0, x, y, 0), (0, x, 0, y), (0, 0, x, y)\}$, so the distance-2 neighbors if v in G are $\bigcup_{x+y=n}\{(x, y, 0, 0), (x, 0, y, 0), (x, 0, 0, y)\}$ and $\bigcup_{x+y+z=n}\{(x, y, z, 0), (x, 0, y, z),$ $(x, y, 0, z), (0, x, y, z)\}$ (as well as v itself), for a total of $3(n-1)+4\binom{n-1}{2} = 3\binom{n}{2} + \binom{n-1}{2}$ distance-2 neighbors (not counting v) in G. Note that if $n = 2$ then V_3 is empty (and $\binom{2-1}{2} = 0$), so there are no new distance-2 neighbors gained if $n = 2$. If $n \geq 3$, then $v \in V_1$ has more distance-2 neighbors in G than in $SR(4, n)$.

Next, consider $v \in V_2$. Without loss of generality, assume $v = (a, b, 0, 0)$ for some fixed a, b such that $a + b = n$. We see that the distance-2 neighbors of v in V_3 and V_4 are the same in $SR(4, n)$ and in G. Also, v has exactly two distance-2 neighbors in V_1 in both $SR(4, n)$ and G. If $u \in V_2$ is a distance-2 neighbor of v in $SR(4, n)$, then u and v have a common neighbor in V_1, V_2, or V_3. If they have a common neighbor in $V_2 \cup V_3$, then u and v are still distance-2 neighbors in G, since they are still not neighbors and they still have a common neighbor in $V_2 \cup V_3$ (recall that only edges between V_1 and V_2 are changed when switching). In $SR(4, n)$, the distance-2 neighbors of v in V_2 with a common neighbor in V_1 (that is, with common neighbor $(n, 0, 0, 0)$ or $(0, n, 0, 0)$) are $\bigcup_{x+y=n, x \neq a} \{(x, 0, y, 0), (x, 0, 0, y), (0, y, x, 0), (0, y, 0, x)\}$ (the requirement that $x \neq a$ ensures that the vertex is not a neighbor of v). We see immediately that these are exactly the distance-2 neighbors of v in V_2 with a common neighbor in V_1 in G, though the common neighbors in V_1 are instead $(0, 0, n, 0)$ or $(0, 0, 0, n)$. Thus $v \in V_2$ has the same number of distance-2 neighbors in $SR(4, n)$ and G.

Now we consider $v \in V_3$. Without loss of generality, let $v = (a, b, c, 0)$ for some fixed a, b, c such that $a + b + c = n$. The distance-2 neighbors of v in V_2, V_3, and V_4 are unchanged by switching. Before switching, vertices $(n, 0, 0, 0)$, $(0, n, 0, 0)$, and $(0, 0, n, 0)$ are the distance-2 neighbors of v in V_1. After switching, every vertex in V_1 is a distance-2 neighbor of v. Thus $v \in V_3$ has one more distance-2 neighbor in G than in $SR(4, n)$.

Finally, the distance-2 neighbors of a vertex in V_4 are unchanged by switching. Thus G is not isomorphic to $SR(4, n)$. □

Proposition 3.18. *The graph $SR(m, 3)$ is not determined by its spectrum for $m \geq 4$.*

Proof. For $m \geq 4$ the diameter of $SR(m, 3)$ is 3 by Proposition 3.15. Consider the set $C = \{3\mathbf{e}_1, 2\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + 2\mathbf{e}_2, 3\mathbf{e}_2\} \subset V(d, 3)$. Clearly C is a clique in $SR(m, 3)$. Any vertex with more than one nonzero entry in the last $m - 2$ coordinates is clearly not adjacent to any vertex of C. As we will see below, every other vertex in $V(m, 3) \setminus C$ is adjacent to exactly 2 vertices in C.

Each vertex in the set $\{\mathbf{e}_1 + 2\mathbf{e}_j \mid j \geq 3\}$ is adjacent to $\mathbf{e}_1 + 2\mathbf{e}_2$ and $3\mathbf{e}_1$. Each vertex in the set $\{\mathbf{e}_2 + 2\mathbf{e}_j \mid j \geq 3\}$ is adjacent to $2\mathbf{e}_1 + \mathbf{e}_2$ and $3\mathbf{e}_2$. Each vertex in the set $\{2\mathbf{e}_1 + \mathbf{e}_j \mid j \geq 3\}$ is adjacent to $2\mathbf{e}_1 + \mathbf{e}_2$ and $3\mathbf{e}_1$. Each vertex in the set $\{2\mathbf{e}_2 + \mathbf{e}_j \mid j \geq 3\}$ is adjacent to $\mathbf{e}_1 + 2\mathbf{e}_2$ and $3\mathbf{e}_2$. Each vertex in the set $\{\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_j \mid j \geq 3\}$ is adjacent to $\mathbf{e}_1 + 2\mathbf{e}_2$ and $2\mathbf{e}_1 + \mathbf{e}_2$. Finally, each vertex in the set $\{3\mathbf{e}_j \mid j \geq 3\}$ is adjacent to $3\mathbf{e}_1$ and $3\mathbf{e}_2$. Thus C is a GM-switching set (see Figure 3.8).

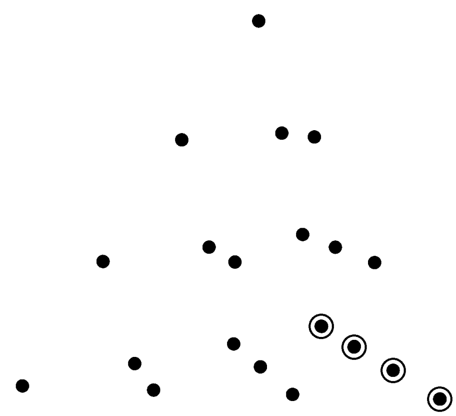


Figure 3.8: The set $\{3\mathbf{e}_1, 2\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + 2\mathbf{e}_2, 3\mathbf{e}_2\}$ as a GM-switching set in $SR(4,3)$.

Before switching, it is not difficult to show (using the same technique as in the proof of Proposition 3.17) that the m vertices in V_1 have distance-degree sequence $\left(3(m-1), 3\binom{m-1}{2}, \binom{m-1}{3}\right)$, the $2\binom{m}{2}$ vertices in V_2 have distance-degree sequence $\left(3(m-1), \binom{2m-3}{2}, \binom{m-2}{3}\right)$, and the $\binom{m}{3}$ vertices in V_3 have distance-degree sequence $\left(3(m-1), 3\binom{m-1}{2}, \binom{m-1}{3}\right)$. After switching, the $m-2$ vertices in $V_1 \backslash C$, the $2\binom{m}{2}$ vertices in V_2, and the $m-2+\binom{m-2}{2}$ vertices in the set $\{\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_j \mid j \geq 3\} \cup \{\mathbf{e}_j+\mathbf{e}_k+\mathbf{e}_\ell \mid j > k > \ell \geq 3\}$ have not changed their distance-degree sequence, but the 2 vertices $3\mathbf{e}_1$ and $3\mathbf{e}_2$ now have distance-degree sequence $\left(3(m-1), \binom{2m-3}{2}, \binom{m-2}{3}\right)$, and the $\binom{m-2}{2}$ vertices in the set $\{\mathbf{e}_j + \mathbf{e}_k + \mathbf{e}_\ell \mid j \in \{1,2\},\ k > \ell \geq 3\}$ now have distance-degree sequence $\left(3(m-1), 3\binom{m-1}{2} + 1, \binom{m-1}{3} - 1\right)$. It is not difficult to see that for $m \geq 4$ this results in a graph which is not isomorphic to $SR(m,3)$. $\square$

It is worth noting that the cospectral mates obtained for $SR(4,3)$ in Propositions

3.17 and 3.18 are not isomorphic to each other, so we have found 3 nonisomorphic graphs with spectrum $\{9^1, 3^4, 1^3, -1^6, -3^6\}$. In fact, we found that there are at least 336 pairwise nonisomorphic graphs cospectral to $SR(4,3)$, obtained by repeated GM-switching with respect to regular subgraphs of size 4.

3.10 Multiplicity of the Smallest Eigenvalue

As we mentioned in Section 3.3, Martin and Wagner [68] constructed eigenvectors for the eigenvalues $-\binom{m}{2}$ and $-n$ when $n \geq \binom{m}{2}$ or $n < \binom{m}{2}$, respectively. For details on these constructions, see Section 5.2. The number of linearly independent eigenvectors that they construct for each eigenvalue is a lower bound for the multiplicity of that eigenvalue. Thus Martin and Wagner proved the following Propositions.

Proposition 3.19. *The multiplicity of the eigenvalue $-\binom{m}{2}$ in $SR(m,n)$ is at least $\binom{n-\binom{m-1}{2}}{m-1}$.*

Note that this multiplicity is 0 when $n < \binom{m}{2}$. The Mahonian number $M(m,n)$ is the number of permutations on m letters that have precisely n inversions (see [88]).

Proposition 3.20. *The multiplicity of the eigenvalue $-n$ in $SR(m,n)$ is at least $M(m,n)$.*

Note that this multiplicity is 0 when $n > \binom{m}{2}$ As we have mentioned (see Conjecture 3.4), Martin and Wagner conjectured that the smallest eigenvalue of $SR(m,n)$ is $\max\{-n, -\binom{m}{2}\}$, which we proved in Section 3.3 (see Proposition 3.5). Martin and Wagner also conjectured that the multiplicity given in Proposition 3.20 is correct.

Conjecture 3.21. ([68, Conjecture 3.9]) *The multiplicity of the eigenvalue $-n$ in $SR(m,n)$ is exactly $M(m,n)$.*

We prove Conjecture 3.21 and also confirm that the multiplicity in Proposition 3.19 is correct.

Proposition 3.22. *The multiplicity of the eigenvalue $-\binom{m}{2}$ in $SR(m,n)$ is $\binom{n-\binom{m-1}{2}}{m-1}$. The multiplicity of the eigenvalue $-n$ in $SR(m,n)$ is $M(m,n)$.*

The proof below, primarily due to Brouwer, is substantially unchanged from [13].

Proof. We first consider the multiplicity of the eigenvalue $-\binom{m}{2}$.

For each vertex u, and $1 \le j < k \le m$, let $C_{jk}(u)$ be the (j,k)-clique on u, that is the set of all vertices v with $v_i = u_i$ for $i \ne j,k$. An eigenvector $a = (a_u)$ for the eigenvalue $-\binom{m}{2}$ must be a common eigenvector of all A_{jk} for the eigenvalue -1 (see the proof to Proposition 3.3). That means that $\sum_{v\in C} a_v = 0$ for each set $C = C_{jk}(u)$.

Order the vertices by $u > v$ when $u_d > v_d$ when $d = d_{uv}$ is the largest index where u, v differ. Suppose $u_i = s$ for some index i and $s \le m-i-1$. We can express a_u in terms of a_v for smaller v with $d_{uv} \ge m-s$ via $\sum_{v\in C} a_v = 0$, where $C = C_{i,m-s}(u)$. Indeed, this equation will express a_u in terms of a_v where $u_i + u_{m-s} = v_i + v_{m-s}$ and $v_j = u_j$ for $j \ne i, m-s$. If $v_i > s$ this is not a problem since $v_{m-s} < u_{m-s}$. If $t = v_i < s$, then by induction a_v in its turn can be expressed in terms of a_w where w is smaller and $d_{vw} \ge m-t > m-s$, so that w is smaller than u, and $d_{uw} > m-s$.

In this way we expressed a_u when $u_i \le m-i-1$ for some i. The free a_u have $u_i \ge m-i$ for all i, and the vector u' with $u'_i = u_i - (m-i)$ is nonnegative and sums to $n - \binom{m}{2}$. There are $\binom{n-\binom{m}{2}+m-1}{m-1} = \binom{n-\binom{m-1}{2}}{m-1}$ such vectors, so this is an upper bound for the multiplicity. We recall that by Proposition 3.19 this is also a lower bound. Thus the multiplicity of the eigenvalue $-\binom{m}{2}$ is exactly $\binom{n-\binom{m-1}{2}}{m-1}$.

By Proposition 3.20 the eigenvalue $-n$ has multiplicity at least as large as the Mahonian $M(m,n)$. We first note that the Mahonian number $M(m,n)$ is equal to the coefficient of t^n in the product $\prod_{i=2}^{m}(1+t+\cdots+t^{i-1})$ [88, Sequence #A008302].

Define N as in the proof of Proposition 3.5. Since $A+nI = NN^\top$, the multiplicity of the eigenvalue $-n$ is the nullity of N. We have already a lower bound, $M(m,n)$, so we need only to show that this is also an upper bound.

We first define a matrix P, and observe that N and P have the same column space, and hence the same rank. For $u, v \in \mathbb{N}^m$, write $u \preceq v$ when $u_i \le v_i$ for all i. Let P be the 0-1 matrix with the same row and column indices (elements of $\mathbb{N}^m$ with sum

m and sum smaller than m, respectively) where $P_{(x,y)} = 1$ when $y \preceq x$. Recall that N is the 0-1 matrix with $N_{(x,y)} = 1$ when x and y differ in precisely one coordinate position. Let $M(y)$ denote column y of the matrix M.

For $d = n - \sum y_i$ we find that

$$N(y) = \sum_{i=0}^{d-1} (-1)^i (d-i) \sum_{z \in W_i} P(y+z),$$

where W_i is the set of vectors in $\{0,1\}^m$ with sum i. Indeed, suppose that x and y differ in j positions. Then $j \le d$, and $N_{(x,y)} = \delta_{1j}$, while the x-entry of the right hand side is $\sum_{i=0}^{d} (-1)^i (d-i) \binom{j}{i} = j \sum_{i=1}^{j} (-1)^{i-1} \binom{j-1}{i-1} = j(1-1)^{j-1} = \delta_{1j}$. We see that $N(y)$ and $dP(y)$ differ by a linear combination of columns $P(y')$ where $\sum y'_i > \sum y_i$, and hence that N and P have the same column space.

Aart Blokhuis remarked that the coefficient of t^n in the product $\prod_{i=2}^{m} (1 + t + \cdots + t^{i-1})$ is precisely the number of vertices u satisfying $u_i < i$ for $1 \le i \le m$. Thus, it suffices to show that the rows of N (or P) indexed by the remaining vertices are linearly independent.

Consider a linear dependence between the rows of P indexed by the remaining vertices, and let P' be the submatrix of P containing the rows that occur in this dependence. Order vertices in reverse lexicographic order, so that u is earlier than u' when $u_h < u'_h$ and $u_i = u'_i$ for $i > h$. Let x be the last row index of P' (in this order). Let h be an index where the inequality $x_i < i$ is violated, so that $x_h \ge h$. Let e_i be the element of $\mathbb{N}^m$ that has all coordinates 0 except for the i-coordinate, which is 1. Let $z = x - he_h$. Let $H = \{1, \dots, h-1\}$. For $S \subseteq H$, let $\chi(S)$ be the element of $\mathbb{N}^m$ that has i-coordinate 1 if $i \in S$, and 0 otherwise.

Consider the linear combination $p = \sum_S (-1)^{|S|} P'(z + \chi(S))$ of the columns of P'. We shall see that p has x-entry 1 and all other entries equal to 0. But that contradicts the existence of a linear dependence.

If u is a row index of P', and not $z \preceq u$, then $p_u = 0$. If $z \preceq u$, and $z_i < u_i$ for some $i < h$, then the alternating sum vanishes, and $p_u = 0$. So, if $p_u \neq 0$, then u agrees with x in coordinates i, $1 \le i \le h-1$. For row x only $S = \emptyset$ contributes, and $p_x = 1$.

Finally, if $u \neq x$ then $u_i \geq x_i$ for $i \neq h$ and $\sum x_i = \sum u_i = n$ imply that $u_h < x_h$ and $u_i > x_i$ for some $i > h$, which is impossible, since x is the reverse lexicographically latest row index of P'. $\square$

3.11 Spectrum of $SR(m, n)$ for $m, n \geq 4$

In the proof to Proposition 3.11 we saw that $SR(m, 3)$ and the Johnson graph $J(m + 2, 3)$ are very similar graphs, and they have in common the eigenvalues of the quotient matrix of a common equitable partition that is a refinement of the partition $\{V_1, V_2, \ldots\}$, splitting each set further by location of the nonzero elements. This can be generalized to any m.

Both $SR(m, n)$ and $J(m + n - 1, n)$ are regular graphs of valency $n(m - 1)$ on $\binom{n+m-1}{m-1}$ vertices. We partition the vertices of each graph into sets V_i^S, where $S \subset \{1, 2 \ldots, m\}$ and $|S| = i$. In $SR(m, n)$ the set V_i^S consists of the vertices in $V(m, n)$ which are nonzero in the i coordinate if and only if $i \in S$. In $J(m + n - 1, n)$ the set V_i^S is the set of vertices containing every element of S and no other elements in $\{1, \ldots, m\}$. One can show that this partition is equitable for both graphs, and the corresponding quotient matrix is the same for both graphs (see Proposition 3.23 below). Proposition 1.24 implies that the eigenvalues of this quotient matrix are eigenvalues of both graphs. We find that these eigenvalues are precisely the eigenvalues of the unrefined partition, but with larger multiplicities.

Proposition 3.23. *The graphs $SR(m, n)$ and $J(m+n-1, n)$ have equitable partitions with the same quotient matrix Q, where Q has eigenvalues $(n - i)(m - i) - n$ with multiplicity $\binom{m}{i}$ for $0 \leq i \leq n - 1$, and multiplicity $\binom{m}{n} - 1$ for $i = n$. In particular, the spectrum of Q is a common part of the spectra of $SR(m, n)$ and $J(m + n - 1, n)$.*

We omit here the proof due to Brouwer; it can be found in [13].

Propositions 3.22 and 3.23 can be used to more easily find the spectrum of $SR(m, n)$ for fixed n. We first give an alternate proof of Proposition 3.11, which states that the spectrum of $SR(m, 3)$ is $(3m - 3)^1$, $(2m - 5)^m$, $(m - 3)^{m-1}$, $(m - 5)^{\binom{m}{2}}$, $(-3)^{m(m^2-7)/6}$.

Alternate proof of Proposition 3.11. Propositions 3.5 and 3.22 imply that -3 is the smallest eigenvalue of $SR(m,3)$ with multiplicity $M(m,3) = m(m^2-7)/6$. Proposition 3.23 implies that $(3m-3)^1$, $(2m-5)^m$, and $(m-5)^{\binom{m}{2}}$ are part of the spectrum of $SR(m,3)$. Thus we need only to show that $m-3$ is an eigenvalue of $SR(m,3)$ with multiplicity $m-1$.

Fix an index h, $1 \leq h \leq m$ and consider the vector p indexed by $V(m,3)$ that is 1 on vertices $2e_h + e_i$, -1 on vertices $e_h + 2e_i$, and 0 elsewhere. It is straightforward to verify that p is an eigenvector with eigenvalue $m-3$ and the m vectors defined in this way have only a single dependency (namely, they sum to 0). Thus $m-3$ is an eigenvalue of $SR(m,3)$ with multiplicity $m-1$, and we are done. $\square$

An even simpler proof is that, once we know all the eigenvalues except $(m-3)^{m-1}$, the fact that the sum of the eigenvalues is the trace of $A(m,3)$, which is 0, and the sum of the squares of the eigenvalues is the trace of $(A(m,3))^2$, which is the number of vertices times the valency, we immediately have $(m-3)^{m-1}$ as the remaining eigenvalues. This follows because the sum of the other eigenvalues is $-(m-1)(m-3)$ and the sum of their squares is $3(m-1)\binom{m+2}{m-1} - (m-1)(m-3)^2$, so the remaining $m-1$ eigenvalues must sum to $(m-1)(m-3)$ and their squares must sum to $(m-1)(m-3)^2$. Since the sum of numbers with fixed sum of squares is maximized when the numbers are all equal, this implies that the remaining $m-1$ eigenvalues are all $m-3$. This proof takes longer to write, but is conceptually simpler than the one above, which required us to find the eigenvectors of the missing eigenvalues.

Using the same technique, we can find the spectrum for larger fixed n:

Proposition 3.24. *The graph $SR(m,4)$ has spectrum given by Table 3.3.*

Proof. Propositions 3.5 and 3.22 imply that -4 is the smallest eigenvalue of $SR(m,4)$ with multiplicity $M(m,4) = m(m^3 + 2m^2 - 13m - 14)/24$. Proposition 3.23 implies that $(4m-4)^1$, $(3m-7)^m$, $(2m-8)^{\binom{m}{2}}$, and $(m-7)^{\binom{m}{3}}$ are part of the spectrum of $SR(m,4)$. Thus we need only to show that $(2m-5)^m$, $(m-4)^{\binom{m}{2}-1}$, and $(m-6)^{\binom{m}{2}}$ are eigenvalues of $SR(m,4)$.

Eigenvalue	Multiplicity
$4(m-1)$	1
$3m-7$	m
$2m-5$	m
$2m-8$	$\binom{m}{2}$
$m-4$	$\binom{m}{2}-1$
$m-6$	$\binom{m}{2}$
$m-7$	$\binom{m}{3}$
-4	$m(m^3+2m^2-13m-14)/24$

Table 3.3: Spectrum of $SR(m,4)$.

By Proposition 1.24, any eigenvector for one of the missing eigenvalues sums to zero on each part of the partition from Proposition 3.23, that is, on each set of vertices with given support S. Since there are unique vertices with support of sizes 1 or 4, these eigenvectors are 0 there, and we need only look at the vertices $3e_i + e_j$ and $2e_i + 2e_j$ and $2e_i + e_j + e_k$.

Fix an index h, $1 \le h \le m$ and consider the vector p (indexed by the vertices) that vanishes on each vertex where h is not in the support, is -1 on $2e_h + 2e_i$ and on $3e_h + e_i$, is 2 on $e_h + 3e_i$, is -2 on $2e_h + e_i + e_j$, and is 1 on $e_h + 2e_i + e_j$. It is straightforward to verify that this is an eigenvector with eigenvalue $2m-5$, and that the m vectors defined in this way are linearly independent. Thus $(2m-5)^m$ are eigenvalues of $SR(m,4)$.

Fix a pair of indices h, i, $1 \le h < i \le m$, and consider the vector p (indexed by the vertices) that is 1 on $e_h + 3e_j$, $2e_i + 2e_j$ and $2e_h + e_i + e_j$, is -1 on $e_i + 3e_j$, $2e_h + 2e_j$ and $e_h + 2e_i + e_j$, and is 0 elsewhere. It is straightforward to verify that this is an eigenvector with eigenvalue $m-6$, and that the $\binom{m}{2}$ vectors defined in this way are linearly independent. Thus $(m-6)^{\binom{m}{2}}$ are eigenvalues of $SR(m,4)$.

Having found all desired eigenvalues except one, it is not necessary to construct eigenvectors for the final one. By the same argument following the alternate proof to Proposition 3.11, and noting that the sum of the known eigenvalues is $-(\binom{m}{2}-1)(m-4)$ and the sum of their squares is $4(m-1)\binom{m+3}{m-1} - (\binom{m}{2}-1)(m-4)^2$, we find that the remaining eigenvalues are $(m-4)^{\binom{m}{2}-1}$ $\square$

For larger values of m and n, we have only conjectures and tables of spectra. For the first several values of m, the spectrum of $SR(m,5)$ is given by Table 3.4. The

Eigenvalue	Multiplicity
$5(m-1)$	1
$4m-9$	m
$3m-7$	m
$3m-11$	$\binom{m}{2}$
$2m-5$	$m-1$
$2m-7$	$\binom{m}{2}$
$2m-11$	$\binom{m}{3}$
$m-5$	$\binom{m}{3}-1$
$m-6$	$m(m-2)$
$m-8$	$2\binom{m}{3}$
$m-9$	$\binom{m}{4}$
-5	$M(m,5)$

Table 3.4: Spectrum of $SR(m,5)$.

spectra of $SR(4,n)$ and $SR(5,n)$ are given for the first several values of n in Tables 3.5 and 3.6. For $SR(4,n)$, we give the following description of the spectrum, which is correct for the values in the table.

Let $a^m \downarrow b$ denote sequence of eigenvalues and multiplicities found as follows: the eigenvalues are the integers c with $a \geq c \geq b$, where the first multiplicity is m, and each following multiplicity is 2 larger for even c, and 10 larger for odd c. For the first several values of n, the spectrum of $SR(4,n)$ consists of:

(i) $(3n)^1$,

(ii) b^4 for all odd integers b, where $2n-3 \geq b \geq n-1$,

(iii)

(iii)	$n=2r$	$(n-4)^{3n-1}$, $(n-6)^6$, $(n-7)^{16} \downarrow (n-8)/2$
	$n=2r+1$	$(n-2)^3$, $(n-4)^{3n-3}$, $(n-6)^9$, $(n-7)^{12} \downarrow (n-7)/2$

(iv) for $q = \lceil n/3 - 4 \rceil$:

$n = 4s$	$(2s-5)^{3n-12}$, $(2s-6)^{3n-26} \downarrow q$
$n = 4s+1$	$(2s-4)^{3n-7}$, $(2s-5)^{3n-21}$, $(2s-6)^{3n-23} \downarrow q$
$n = 4s+2$	$(2s-4)^{3n-16}$, $(2s-5)^{3n-22} \downarrow q$
$n = 4s+3$	$(2s-3)^{3n-3}$, $(2s-4)^{3n-25}$, $(2s-5)^{3n-19} \downarrow q$

(v) if $n \equiv 0 \pmod 3$ one additional eigenvalue $n/3 - 4$,

(vi)

$n = 6t$	$(2t-5)^{4n-12}$, $(2t-6)^{4n-16}$, $(2t-7)^{4n-16} \downarrow (-5)$
$n = 6t+1$	$(2t-4)^{4n-32}$, $(2t-5)^{4n-7}$, $(2t-6)^{4n-20}$, $(2t-7)^{4n-14} \downarrow (-5)$
$n = 6t+2$	$(2t-4)^{4n-24}$, $(2t-5)^{4n-8}$, $(2t-6)^{4n-21}$, $(2t-7)^{4n-12} \downarrow (-5)$
$n = 6t+3$	$(2t-4)^{4n-16}$, $(2t-5)^{4n-12}$, $(2t-6)^{4n-20} \downarrow (-5)$
$n = 6t+4$	$(2t-3)^{4n-28}$, $(2t-4)^{4n-11}$, $(2t-5)^{4n-16}$, $(2t-6)^{4n-18} \downarrow (-5)$
$n = 6t+5$	$(2t-3)^{4n-20}$, $(2t-4)^{4n-12}$, $(2t-5)^{4n-17}$, $(2t-6)^{4n-16} \downarrow (-5)$

(vii) $(-6)^{\binom{n-3}{3}}$.

n	Spectrum of $SR(4,n)$
0	0^1
1	$3^1\ -1^3$
2	$6^1\ 1^4\ -2^5$
3	$9^1\ 3^4\ 1^3\ -1^6\ -3^6$
4	$12^1\ 5^4\ 3^4\ 0^{11}\ -2^6\ -3^4\ -4^5$
5	$15^1\ 7^4\ 5^4\ 3^3\ 1^{12}\ -1^9\ -2^8\ -3^4\ -4^8\ -5^3$
6	$18^1\ 9^4\ 7^4\ 5^4\ 2^{17}\ 0^6\ -1^{16}\ -2^3\ -3^{12}\ -4^8\ -5^8\ -6^1$
7	$21^1\ 11^4\ 9^4\ 7^4\ 5^3\ 3^{18}\ 1^9\ 0^{12}\ -1^{18}\ -3^{21}\ -4^8\ -5^{14}\ -6^4$
8	$24^1\ 13^4\ 11^4\ 9^4\ 7^4\ 4^{23}\ 2^6\ 1^{16}\ 0^{18}\ -1^{12}\ -2^8\ -3^{24}\ -4^{11}\ -5^{20}\ -6^{10}$
9	$27^1\ 15^4\ 13^4\ 11^4\ 9^4\ 7^3\ 5^{24}\ 3^9\ 2^{12}\ 1^{22}\ 0^{20}\ -1^7\ -2^{20}\ -3^{24}\ -4^{16}\ -5^{26}\ -6^{20}$
10	$30^1\ 17^4\ 15^4\ 13^4\ 11^4\ 9^4\ 6^{29}\ 4^6\ 3^{16}\ 2^{18}\ 1^{28}\ 0^{14}\ -1^{12}\ -2^{29}\ -3^{24}\ -4^{22}\ -5^{32}\ -6^{35}$
11	$33^1\ 19^4\ 17^4\ 15^4\ 13^4\ 11^4\ 9^3\ 7^{30}\ 5^9\ 4^{12}\ 3^{22}\ 2^{24}\ 1^{30}\ 0^8\ -1^{24}\ -2^{32}\ -3^{27}\ -4^{28}\ -5^{38}$ -6^{56}
12	$36^1\ 21^4\ 19^4\ 17^4\ 15^4\ 13^4\ 11^4\ 8^{35}\ 6^6\ 5^{16}\ 4^{18}\ 3^{28}\ 2^{30}\ 1^{24}\ 0^{11}\ -1^{36}\ -2^{32}\ -3^{32}\ -4^{34}$ $-5^{44}\ -6^{84}$
13	$39^1\ 23^4\ 21^4\ 19^4\ 17^4\ 15^4\ 13^4\ 11^3\ 9^{36}\ 7^9\ 6^{12}\ 5^{22}\ 4^{24}\ 3^{34}\ 2^{32}\ 1^{18}\ 0^{20}\ -1^{45}\ -2^{32}$ $-3^{38}\ -4^{40}\ -5^{50}\ -6^{120}$
14	$42^1\ 25^4\ 23^4\ 21^4\ 19^4\ 17^4\ 15^4\ 13^4\ 10^{41}\ 8^6\ 7^{16}\ 6^{18}\ 5^{28}\ 4^{30}\ 3^{40}\ 2^{26}\ 1^{20}\ 0^{32}\ -1^{48}\ -2^{35}$ $-3^{44}\ -4^{46}\ -5^{56}\ -6^{165}$
15	$45^1\ 27^4\ 25^4\ 23^4\ 21^4\ 19^4\ 17^4\ 15^4\ 13^3\ 11^{42}\ 9^9\ 8^{12}\ 7^{22}\ 6^{24}\ 5^{34}\ 4^{36}\ 3^{42}\ 2^{20}\ 1^{27}\ 0^{44}$ $-1^{48}\ -2^{40}\ -3^{50}\ -4^{52}\ -5^{62}\ -6^{220}$

Table 3.5: Spectra of $SR(4,n)$.

n	Spectrum of $SR(5,n)$
0	0^{1}
1	4^{1} -1^{4}
2	8^{1} 2^{5} -2^{9}
3	12^{1} 5^{5} 2^{4} 0^{10} -3^{15}
4	16^{1} 8^{5} 5^{5} 2^{10} 1^{9} -1^{10} -2^{10} -4^{20}
5	20^{1} 11^{5} 8^{5} 5^{4} 4^{10} 3^{10} 1^{10} 0^{9} -1^{25} -3^{20} -4^{5} -5^{22}
6	24^{1} 14^{5} 11^{5} 8^{5} 6^{10} 5^{10} 4^{9} 3^{10} 1^{30} 0^{10} -1^{10} -2^{45} -4^{25} -5^{15} -6^{20}
7	28^{1} 17^{5} 14^{5} 11^{5} 8^{14} 7^{10} 6^{10} 5^{10} 3^{29} 2^{10} 1^{35} -1^{55} -2^{16} -3^{35} -4^{25} -5^{25} -6^{25} -7^{15}
8	32^{1} 20^{5} 17^{5} 14^{5} 11^{5} 10^{10} 9^{10} 8^{10} 7^{19} 5^{20} 4^{10} 3^{40} 2^{14} 1^{15} 0^{75} -1^{25} -2^{20} -3^{72} -4^{15} -5^{55} -6^{25} -7^{30} -8^{9}
9	36^{1} 23^{5} 20^{5} 17^{5} 14^{5} 12^{10} 11^{14} 10^{10} 9^{20} 7^{20} 6^{9} 5^{40} 4^{10} 3^{45} 2^{5} 1^{86} 0^{45} -1^{40} -2^{50} -3^{65} -4^{51} -5^{60} -6^{45} -7^{40} -8^{25} -9^{4}
10	40^{1} 26^{5} 23^{5} 20^{5} 17^{5} 14^{15} 13^{10} 12^{10} 11^{20} 10^{9} 9^{20} 7^{40} 6^{10} 5^{54} 4^{10} 3^{25} 2^{110} 1^{50} 0^{36} -1^{90} -2^{60} -3^{65} -4^{100} -5^{55} -6^{80} -7^{50} -8^{45} -9^{15} -10^{1}
11	44^{1} 29^{5} 26^{5} 23^{5} 20^{5} 17^{5} 16^{10} 15^{10} 14^{14} 13^{20} 12^{10} 11^{20} 9^{39} 8^{10} 7^{50} 6^{10} 5^{60} 3^{125} 2^{80} 1^{55} 0^{50} -1^{137} -2^{50} -3^{115} -4^{110} -5^{80} -6^{104} -7^{75} -8^{65} -9^{35} -10^{5}
12	48^{1} 32^{5} 29^{5} 26^{5} 23^{5} 20^{5} 18^{10} 17^{15} 16^{10} 15^{20} 14^{10} 13^{29} 11^{30} 10^{10} 9^{50} 8^{14} 7^{60} 6^{10} 5^{40} 4^{135} 3^{86} 2^{70} 1^{85} 0^{90} -1^{140} -2^{76} -3^{175} -4^{95} -5^{140} -6^{110} -7^{119} -8^{85} -9^{65} -10^{15}
13	52^{1} 35^{5} 32^{5} 29^{5} 26^{5} 23^{5} 20^{15} 19^{10} 18^{10} 17^{24} 16^{10} 15^{30} 13^{30} 12^{9} 11^{50} 10^{10} 9^{60} 8^{15} 7^{65} 5^{170} 4^{105} 3^{85} 2^{71} 1^{145} 0^{110} -1^{140} -2^{145} -3^{201} -4^{95} -5^{215} -6^{110} -7^{175} -8^{109} -9^{105} -10^{35}
14	56^{1} 38^{5} 35^{5} 32^{5} 29^{5} 26^{5} 23^{5} 22^{10} 21^{10} 20^{15} 19^{20} 18^{10} 17^{30} 16^{9} 15^{30} 13^{50} 12^{10} 11^{64} 10^{10} 9^{70} 8^{15} 7^{45} 6^{165} 5^{130} 4^{95} 3^{105} 2^{95} 1^{202} 0^{110} -1^{180} -2^{215} -3^{200} -4^{134} -5^{275} -6^{125} -7^{235} -8^{140} -9^{155} -10^{70}

Table 3.6: Spectra of $SR(5,n)$.

Chapter 4

LARGE REGULAR GRAPHS WITH FIXED VALENCY AND SECOND EIGENVALUE

4.1 The Second Eigenvalue of Regular Graphs

The second adjacency eigenvalue of a regular graph is a parameter of interest in the study of graph connectivity and expanders (see [3, 17, 54], for example). In this chapter, we investigate the maximum order $v(k, \lambda)$ of a connected k-regular graph whose second largest eigenvalue is at most some given parameter λ. As a consequence of work of Alon and Boppana, and of Serre [3, 25, 66, 71, 73, 74, 86], we know that $v(k, \lambda)$ is finite for $\lambda < 2\sqrt{k-1}$. The recent proof of Marcus, Spielman and Srivastava [67] of the existence of infinite families of Ramanujan graphs of any degree at least 3 implies that $v(k, \lambda)$ is infinite for $\lambda \geq 2\sqrt{k-1}$.

In this chapter, we will investigate $v(k, \lambda)$ for various values of k and λ. The graphs meeting this bound are called extremal expanders. The parameter $v(k, -1)$ can be determined using the fact that a graph with second eigenvalue at most -1 is a complete graph (see Lemma 1.29 and the following comments). Thus $v(k, -1) = k + 1$ and the unique graph meeting this bound is K_{k+1}. The parameter $v(k, 0)$ can be determined using the fact that a graph with exactly one positive eigenvalue must be a complete multipartite graph (see Lemma 1.30). Indeed, the largest k-regular multipartite graph is clearly the complete bipartite graph $K_{k,k}$, since a k-regular t-partite graph has $tk/(t-1)$ vertices. Thus $v(k, 0) = 2k$, and $K_{k,k}$ is the unique graph meeting this bound. The values of $v(k, -1)$ and $v(k, 0)$ also follow from Theorem 4.11 (see Section 4.4).

Results from Bussemaker, Cvetković, and Seidel [19] and Cameron, Goethals, Seidel, and Shult [21] give a characterization of the regular graphs with smallest eigenvalue $\lambda_{\min} \geq -2$. Since by Proposition 1.20 the second eigenvalue of the complement of a regular graph is $\lambda_2 = -1-\lambda_{\min}$, the regular graphs with second eigenvalue $\lambda_2 \leq 1$ are also characterized. This characterization can be used to find that $v(k, 1)$ (see Section 4.4).

The values remaining to be investigated are $v(k, \lambda)$ for $1 < \lambda < 2\sqrt{k-1}$. The parameter $v(k, \lambda)$ has also been studied by Teranishi and Yasuno [92] from a design theory perspective, Nozaki [76] from a linear programming point of view, Koledin and Staníc [58, 59, 89] from a spectral graph theory perspective, Høholdt and Justesen [53], and Richey, Shutty and Stover [81]. Høholdt and Justesen [53] focused on bipartite regular graphs and applied their results to the construction of graph codes while Richey, Shutty, and Stover [81] implemented Serre's quantitative version of the Alon-Boppana Theorem [86] to obtain upper bounds for $v(k, \lambda)$ for several values of k and λ. For certain values of k and λ, these authors made some conjectures about $v(k, \lambda)$.

In Section 4.3 we determine $v(k, \lambda)$ explicitly for several values of (k, λ), confirming or disproving several conjectures in [81], and we find the graphs (in many cases unique) which meet our bounds. A summary of our bounds is contained in the following table.

(k, λ)	$v(k, \lambda)$	Graph meeting bound	Unique?	Reference
$(3,\sqrt{2})$	14	Heawood graph	yes	Proposition 4.6
$(3,\sqrt{3})$	18	Pappus graph	yes	Proposition 4.9
$(3,\gamma \approx 1.8662)$	18	see Figure 4.7(b)	yes	Proposition 4.9
(3,2)	30	Tutte-Coxeter graph	yes	Proposition 4.4
$(4,\sqrt{5}-1)$	10	circulant graph $Ci_{10}(1,4)$	yes	Proposition 4.7
(4,2)	35	Odd graph O_4	yes	Proposition 4.5
(4,3)	≥ 728	incidence graph of $GH(3,3)$	?	Proposition 4.8

Portions of the remainder of this chapter represent joint work with Sebastian Cioabă, Jack Koolen, and Hiroshi Nozaki on the paper "Large regular graphs of given valency and second eigenvalue" [27]. In particular, Koolen gave the outline of a different proof of Proposition 4.4 that gave me the idea for Lemma 4.3. Koolen also gave the idea for Lemma 4.2 without proof. Theorem 4.11 and Proposition 4.12 are due to Koolen and Nozaki. The remaining results are mine.

4.2 Interlacing Bound

In this section we make use of Proposition 1.28 to find a bound on the number of vertices in a graph with induced subgraphs with certain properties. The idea is that if G satisfies $\lambda_2(G) \leq \lambda$ and G contains a subgraph H such that $\rho(H) > \lambda$, then the subgraph K induced by $\Gamma_{\geq 2}(H)$ must satisfy $\rho(K) \leq \lambda$. Indeed, if $\rho(K) > \lambda$, then the subgraph of G induced by $V(H) \cup \Gamma_{\geq 2}(H)$ is isomorphic to the disjoint union $H + K$, which satisfies $\lambda_2(H+K) \geq \min(\rho(H), \rho(K)) > \lambda$, which contradicts Proposition 1.28. From $\rho(K) \leq \lambda$ we can bound $|V(G)|$ in terms of $|V(H)|$ and $|E(H)|$. Further, using Lemma 4.2 we can make a similar argument even if $\rho(H) = \lambda$. The interlacing bound is given in Lemma 4.3.

The following lemma can be easily verified.

Lemma 4.1. *Each of the graphs in Figure 4.1 has spectral radius greater than 2.*

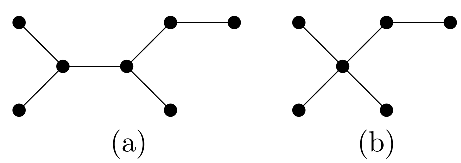


Figure 4.1: Two small graphs with spectral radius greater than 2.

Lemma 4.2. *Suppose G is a connected, k-regular graph with second largest eigenvalue $\lambda_2(G) \leq \lambda < k$, and H is an induced subgraph of G with $\overline{d}(H) \geq \lambda$. Then for the subgraph K induced by $\Gamma_{\geq 2}(H)$ we have $\overline{d}(K) \leq \lambda$, with equality only if $\overline{d}(H) = \lambda_2(G) = \lambda$.*

Proof. Consider the quotient matrix Q of the partition $\{V(H), \Gamma_1(H), \Gamma_{\geq 2}(H)\}$ of $V(G)$. We have

$$Q = \begin{pmatrix} \alpha & k-\alpha & 0 \\ \gamma & k-(\gamma+\epsilon) & \epsilon \\ 0 & k-\beta & \beta \end{pmatrix},$$

where $\alpha = \overline{d}(H)$, $\beta = \overline{d}(K)$, and γ and ϵ are the average numbers of neighbors in H and K, respectively, of the vertices in $\Gamma_1(H)$. By Proposition 1.27, the eigenvalues of Q interlace those of G, so we must have $\lambda_2(Q) \leq \lambda_2(G) \leq \lambda$. It is straightforward to verify that $\lambda_1(Q) = k$ and

$$\lambda_2(Q) = \frac{1}{2}\left(\alpha + \beta - (\gamma + \epsilon) + \sqrt{\Delta}\right), \tag{4.1}$$

where $\Delta = (\alpha + \beta - (\gamma + \epsilon))^2 - 4(\alpha\beta - \beta\gamma - \alpha\epsilon)$. By hypothesis we have $\alpha \geq \lambda$. If also $\beta \geq \lambda$, then we find that $\alpha = \beta = \lambda_2(Q) = \lambda$, as we will prove below.

Indeed, if both $\alpha > \lambda$ and $\beta > \lambda$, then by Proposition 1.28 we have $\lambda_2(G) \geq \lambda_2(H + K) > \lambda$, a contradiction. Suppose $\alpha \geq \lambda$ and $\beta \geq \lambda$. If $\alpha = \beta = \lambda$, then (4.1) becomes $\lambda_2(Q) = \lambda$. Otherwise we must have $\alpha > \beta = \lambda$ or $\beta > \alpha = \lambda$. If $\sqrt{\Delta} \geq \gamma + \epsilon$, then clearly $\lambda_2(Q) > \lambda$, a contradiction. If $\sqrt{\Delta} < \gamma + \epsilon$, then $\Delta < (\gamma + \epsilon)^2$, which implies $(\alpha - \beta)(\alpha - \beta + 2(\epsilon - \gamma)) < 0$. Thus we have either $\alpha > \beta$ and $\epsilon < \gamma - \frac{1}{2}(\alpha - \beta)$, or $\beta > \alpha$ and $\gamma < \epsilon - \frac{1}{2}(\beta - \alpha)$. Suppose the former is true. Then $\beta = \lambda$ and we can write $\alpha = \beta + s = \lambda + s$ and $\epsilon = \gamma - \frac{s}{2} - t$ for some $s, t > 0$. Then (4.1) becomes

$$\lambda_2(Q) = \frac{1}{4}\left(4\lambda - 4\gamma + 3s + 2t + \sqrt{\Delta'}\right),$$

where $\Delta' = 16\gamma^2 + (s - 2t)^2 - 8\gamma(s + 2t)$. If $\sqrt{\Delta'} > 4\gamma - 3s - 2t$, then clearly $\lambda_2(Q) > \lambda$, a contradiction. If $\sqrt{\Delta'} \leq 4\gamma - 3s - 2t$, then $\Delta' \leq (4\gamma - 3s - 2t)^2$, which implies $\gamma \leq \frac{s}{2} + t$. However, this implies $\epsilon = \gamma - \frac{s}{2} - t \leq 0$, a contradiction. If $\beta > \alpha$ and $\gamma < \epsilon - \frac{1}{2}(\beta - \alpha)$, the same argument holds (simply swap the roles of α and β and of γ and ϵ in the above argument). Thus we cannot have $\alpha \geq \lambda$ and $\beta \geq \lambda$ unless $\alpha = \beta = \lambda$, so we must have $\beta < \lambda$ or $\alpha = \beta = \lambda_2(Q) = \lambda$. $\square$

Lemma 4.3. *Suppose G is a connected, k-regular graph with second largest eigenvalue $\lambda_2(G) \leq \lambda < k$. If G contains an induced subgraph H on s vertices with t edges and either $\overline{d}(H) \geq \lambda$ or $\rho(H) > \lambda$, then*

$$|V(G)| \leq s + \frac{2k - \lambda - 1}{k - \lambda}(ks - 2t). \tag{4.2}$$

Proof. Since G is k-regular, we have $|E(H, \Gamma_1(H))| = ks - 2t$, which implies $|\Gamma_1(H)| \leq ks - 2t$. We will show that $|\Gamma_{\geq 2}(H)| \leq \frac{k-1}{k-\lambda} |\Gamma_1(H)|$, which completes the proof that (4.2) holds.

First, note that each vertex in $\Gamma_1(H)$ has a neighbor in H, so each such vertex has at most $k-1$ neighbors in $\Gamma_{\geq 2}(H)$. Then $|E(\Gamma_1(H), \Gamma_{\geq 2}(H))| \leq (k-1) |\Gamma_1(H)|$. If $\overline{d}(H) \geq \lambda$ then by Lemma 4.2 we have $\overline{d}(K) \leq \lambda$, where K is the subgraph induced by $\Gamma_{\geq 2}(H)$. If not, then $\rho(H) > \lambda$, so $\rho(K) \leq \lambda$ by Proposition 1.28 (and thus $\overline{d}(K) \leq \lambda$ also by Proposition 1.12). In either case we have $\overline{d}(K) \leq \lambda$. Since G is k-regular, this implies that the average number of neighbors in $\Gamma_1(H)$ of the vertices in $\Gamma_{\geq 2}(H)$ is at least $k - \lambda$, so $|E(\Gamma_1(H), \Gamma_{\geq 2}(H))| \geq (k - \lambda) |\Gamma_{\geq 2}(H)|$. Thus $(k - \lambda) |\Gamma_{\geq 2}(H)| \leq (k - 1) |\Gamma_1(H)|$, which completes the proof. $\square$

Lemma 4.3 is especially useful when $\lambda \leq 2$, since there are many graphs H with $\overline{d}(H) \geq 2$ or $\rho(H) > 2$.

For convenience we note that in the 3-regular case with $\lambda = 2$, (4.2) becomes $|V(G)| \leq 10s - 6t$ and in the 4-regular case with $\lambda = 2$, (4.2) becomes $|V(G)| \leq 11s - 5t$.

4.3 Results of the Interlacing Bound

Richey, Shutty, and Stover [81] show that $v(3, 2) \leq 105$, but they note that the largest 3-regular graph with $\lambda_2 \leq 2$ that they are aware of is the Tutte-Coxeter graph on 30 vertices. Richey, Shutty, and Stover [81] conjecture that $v(3, 2) = 30$. We confirm their conjecture and show that the Tutte-Coxeter graph is the unique graph which meets this bound.

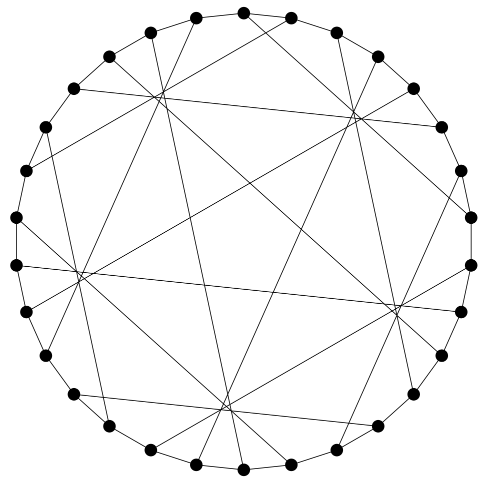


Figure 4.2: The Tutte–Coxeter graph.

Proposition 4.4. *If G is a connected, 3-regular graph with second largest eigenvalue $\lambda_2(G) \leq 2$, then G has at most 30 vertices, with equality if and only if G is the Tutte-Coxeter graph (see Figure 4.2).*

Proof. Note that G cannot be a tree, so G contains a cycle C_m for some $m \geq 3$. Note also that C_m has m vertices and edges and $\overline{d}(C_m) = 2$. Thus, if the girth of G is 3, 4, 5, 6, 7, or 8, then Lemma 4.3 implies that G has at most 12, 16, 20, 24, 28, or 32 vertices, respectively.

Also, if the girth of G is $m > 5$, then G induces a subgraph isomorphic to the graph in Figure 4.1(a). Indeed, G induces a C_m but does not induce C_i for $i < m$. Every vertex in the C_m must have a neighbor outside the C_m, and no two vertices in the C_m can share a neighbor outside the C_m (since the girth is m). In particular, there is a pair of vertices adjacent in the C_m each with a distinct neighbor outside the C_m. These neighbors are not adjacent, since the girth is $m > 5$. Thus as claimed G induces a subgraph isomorphic to the graph in Figure 4.1(a) (which has 7 vertices and 6 edges), so Lemmas 4.1 and 4.3 imply G has at most 34 vertices. By Lemma 1.1, if the girth of G is more than 8 then G must have at least 46 vertices, so the girth of G must be at most 8 and we have $|V(G)| \leq 32$.

If G has 30 or 32 vertices, then the girth of G must be 8. Partitioning the vertices of G according to their distance from a vertex u in a cycle of length 8, we find that $|\Gamma_1(u)| = 3$, $|\Gamma_2(u)| = 6$, $|\Gamma_3(u)| = 12$, and $|\Gamma_{\geq 4}(u)| = m$, $m \in \{8, 10\}$, with quotient matrix

$$Q = \begin{pmatrix} 0 & 3 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & \beta & 3-\beta \end{pmatrix},$$

where β is the average number of neighbors in $\Gamma_3(u)$ of the vertices in $\Gamma_{\geq 4}(u)$. Note that this implies $m\beta = 24$ (just count $|E(\Gamma_3(u), \Gamma_{\geq 4}(u))|$ in two ways). Proposition 1.27 implies that $\lambda_2(Q) \leq \lambda_2(G) \leq 2$. If $|V(G)| = 32$, then $m = 10$, $\beta = 12/5$, and $\lambda_2(Q) = \gamma \approx 2.04 > 2$, where γ is the largest root of $f(x) = 5x^4 + 12x^3 - 23x^2 - 48x + 6$, a contradiction. If $|V(G)| = 30$, then $m = 8$, $\beta = 3$, and by Lemma 1.25 G must be a distance-regular graph with intersection array $\{3, 2, 2, 2; 1, 1, 1, 3\}$. The Tutte-Coxeter graph is known to be the unique distance-regular graph with this intersection array (see, for example, [14, Theorem 7.5.1]).

Thus G has at most 30 vertices, with equality if and only if G is the Tutte-Coxeter graph. □

Richey, Shutty, and Stover [81] show that $v(4, 2) \leq 77$ and conjecture that the largest 4-regular graph with $\lambda_2 \leq 2$ is the so-called rolling cube graph on 24 vertices (that is, the bipartite double of the cuboctohedral graph). We disprove their conjecture and show that $v(4, 2) = 35$ and the Odd graph O_4 is the unique graph which meets this bound.

Proposition 4.5. *If G is a connected, 4-regular graph with second largest eigenvalue $\lambda_2(G) \leq 2$, then $|V(G)| \leq 35$, with equality if and only if G is the Odd graph O_4 (see Figure 4.3).*

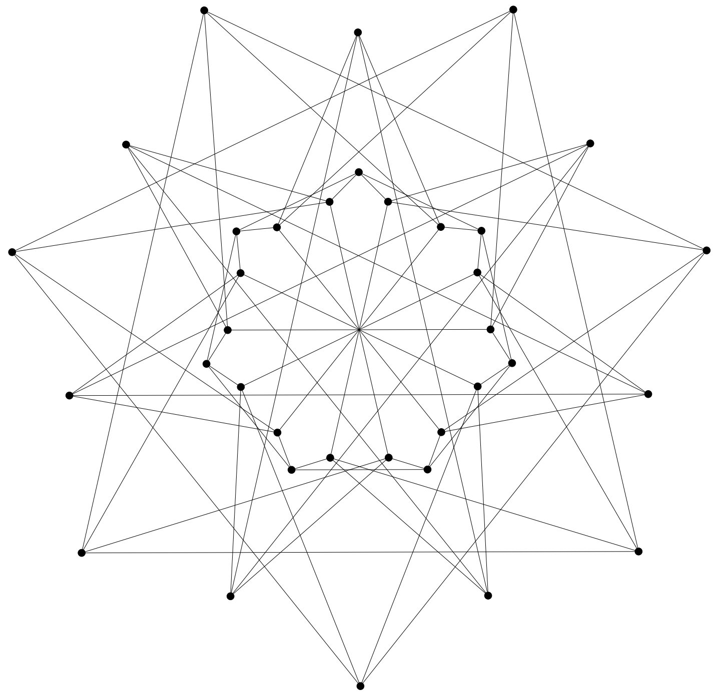


Figure 4.3: The Odd graph O_4.

Proof. A similar argument to the one above using Lemma 4.3 shows that if G has girth 3, 4, 5, or 6, then G has at most 18, 24, 30, or 36 vertices, respectively. A 4-regular graph with girth at least 5 must induce a subgraph isomorphic to the graph in Figure 4.1(b), which implies G has at most 41 vertices by Lemmas 4.1 and 4.3. By Lemma 1.1, if the girth of G is more than 6 then G must have at least 53 vertices, a contradiction. Thus G has girth at most 6 and at most 36 vertices.

If G has more than 30 vertices, then G must have girth 6. Partitioning the vertices according to their distance from a vertex u in a cycle of length 6, we find that $|\Gamma_1(u)| = 4$, $|\Gamma_2(u)| = 12$, $|\Gamma_{\geq 3}(u)| = m = |V(G)| - 17$, with quotient matrix

$$Q = \begin{pmatrix} 0 & 4 & 0 & 0 \\ 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & \beta & 4-\beta \end{pmatrix}, \tag{4.3}$$

where β is the average number of neighbors in $\Gamma_2(u)$ of the vertices in $\Gamma_{\geq 3}(u)$. Proposition 1.27 implies that $\lambda_2(Q) \leq \lambda_2(G) \leq 2$. Note that we have $m\beta = 36$ (by counting

$|E(\Gamma_2(u), \Gamma_{\geq 3}(u))|$ in two ways). Thus if $|V(G)| = 36$ then $m = 19$, $\beta = 36/19$ and $\lambda_2(Q) = \gamma \approx 2.02$, where γ is the largest root of $f(x) = 19x^3 + 36x^2 - 97x - 108$, a contradiction. Thus $|V(G)| \leq 35$.

If $|V(G)| = 35$, then as seen above G has girth 6 and thus has quotient matrix Q from (4.3) with $\beta = 2$. Then $\lambda_2(Q) = 2$ with eigenvector $y = (-6, -3, 0, 1)^\top$. Since we have $\lambda_2(G) \leq 2$, Proposition 1.27 implies that $\lambda_2(G) = 2$. Let S be the characteristic matrix of the partition with quotient matrix Q, so that $S^\top A(G)S = KQ$ and $S^\top S = K$, where $K = \mathrm{diag}(1, 4, 12, 18)$. Since G is 4-regular, we have $\lambda_1(G) = 4$ and $A\mathbf{1} = 4\mathbf{1}$, and Rayleigh's inequalities (Proposition 1.26) give us $\frac{x^\top A(G)x}{x^\top x} \leq \lambda_2(G) = 2$ for any $x \in \mathrm{span}\{\mathbf{1}\}^\perp$ (that is, for any x whose entries sum to 0), with equality if and only if x is an eigenvector of $A(G)$ with eigenvalue 2. For $v \in V(G)$ we have

$$(Sy)_v = \begin{cases} -6, & v = u, \\ -3, & v \in \Gamma_1(u), \\ 0, & v \in \Gamma_2(u), \\ 1, & v \in \Gamma_{\geq 3}(u), \end{cases}$$

and it is straightforward to verify that $Sy \in \mathrm{span}\{\mathbf{1}\}^\perp$ and $\frac{(Sy)^\top A(G)Sy}{(Sy)^\top Sy} = \frac{y^\top S^\top A(G)Sy}{y^\top S^\top Sy} = \frac{y^\top KQy}{y^\top Ky} = 2$, which implies that Sy is an eigenvector of $A(G)$ with eigenvalue 2. Then for any $v \in \Gamma_{\geq 3}(u)$ we have $2 = 2(Sy)_v = (A(G)Sy)_v = \sum_{w \sim v}(Sy)_w$, which, since vertices in $\Gamma_{\geq 3}(u)$ have neighbors only in $\Gamma_2(u)$ and $\Gamma_{\geq 3}(u)$, implies that v has exactly 2 neighbors in $\Gamma_{\geq 3}(u)$. This implies that the partition is equitable, so by Lemma 1.25 G is distance-regular with intersection array $\{4, 3, 3; 1, 1, 2\}$. The Odd graph O_4 is the unique distance-regular graph with that intersection array (see [72]), which implies G is O_4. □

In addition to finding $v(k, \lambda)$, it is also interesting to find the smallest possible second eigenvalue greater than 1 among k-regular graphs for fixed k (there is always a k-regular graph with second eigenvalue 1, for example the complement of the line graph of $K_{2,k+1}$). Below we find this eigenvalue for $k = 3$ and $k = 4$.

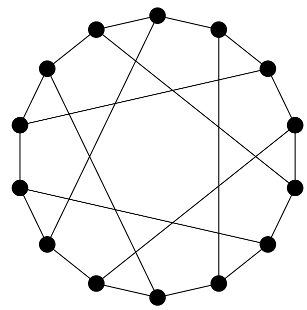


Figure 4.4: The Heawood graph.

Proposition 4.6. *If G is a connected, 3-regular graph with $\lambda_2(G) > 1$, then $\lambda_2(G) \geq \sqrt{2}$, with equality if and only if G is the Heawood graph (see Figure 4.4).*

Proof. We first note that the average degree of any cycle is $2 > \sqrt{2}$. The average degree of $K_{1,3}$ is $3/2 > \sqrt{2}$. If G has girth 3 or 4, then Lemma 4.3 implies $|V(G)| \leq \frac{6}{7}(\sqrt{2}+10) \approx 9.78$ or $\frac{8}{7}(\sqrt{2}+10) \approx 13.04$, respectively. Since G is 3-regular, this implies $|V(G)| \leq 8$ or 12, respectively. If G has girth more than 3, then G induces $K_{1,3}$, so Lemma 4.3 implies $|V(G)| \leq \frac{2}{7}(53+6\sqrt{2}) \approx 17.57$. Since G is 3-regular, this implies $|V(G)| \leq 16$. Lemma 1.1 implies that a graph with girth more than 6 has at least 22 vertices, so G has girth at most 6.

We partition the vertices of G by $P_1 = \{V(H), \Gamma_1(H), \Gamma_{\geq 2}(H)\}$, where H is a subgraph of G isomorphic to C_m, where m is the girth of G. This partition has quotient matrix Q given by

$$Q = \begin{pmatrix} 2 & 1 & 0 \\ \gamma & 3-(\alpha+\gamma) & \alpha \\ 0 & \beta & 3-\beta \end{pmatrix},$$

where $\gamma\,|\Gamma_1(H)| = m$ (by counting $|E(H, \Gamma_1(H))|$) and $\alpha\,|\Gamma_1(H)| = \beta\,|\Gamma_{\geq 2}(H)|$ (by counting $|E(\Gamma_1(H), \Gamma_{\geq 2}(H))|$). By Proposition 1.27 we must have $\lambda_2(Q) \leq \sqrt{2}$.

We first suppose G has girth 3. Lemma 1.1 implies $V(G) \geq 4$, so $4 \leq |V(G)| \leq 8$. If $|V(G)| = 4$, then $G \cong K_4$, and we have $\lambda_2(G) = -1$. If $|V(G)| = 6$, it is straightforward to show that $G \cong C_3 \square K_2$, and we have $\lambda_2(G) = 1$. Either case is a contradiction. If $|V(G)| = 8$ then $\Gamma_1(H)$ has 2 or 3 vertices. If $|\Gamma_1(H)| = 2$, then we have $|\Gamma_{\geq 2}(H)| = 3$, $\gamma = 3/2$, and depending on whether there is an edge in $\Gamma_1(H)$ or not we have $\alpha = 1/2$ or $3/2$, $\beta = 1/3$ or 1, and $\lambda_2(Q) = \frac{1}{3}(\sqrt{13}+4) \approx 2.54$ or 2, respectively. Either case is a contradiction. If $|\Gamma_1(H)| = 3$, then $|\Gamma_{\geq 2}(H)| = 2$, $\gamma = 1$, and depending on whether there is an edge in $\Gamma_{\geq 2}(H)$ or not we have $\beta = 2$ or 3, $\alpha = 4/3$ or 2, and $\lambda_2(Q) = 5/3$ or $\frac{1}{2}(\sqrt{17}-1) \approx 1.56$, respectively. Either case is a contradiction. Thus G cannot have girth 3.

Suppose G has girth 4. Then Lemma 1.1 implies $V(G) \geq 6$, so we have $6 \leq |V(G)| \leq 12$. If $|V(G)| = 6$, then $G \cong K_{3,3}$ and we have $\lambda_2(G) = 0$. If $|V(G)| = 8$, then it is straightforward to verify that G must either be the 3-cube Q_3 or the graph in Figure 4.5. In either case we have $\lambda_2(G) = 1$, a contradiction. If $|V(G)| = 10$,

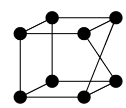


Figure 4.5: A 3-regular graph on 8 vertices with girth 4.

then $\Gamma_1(H)$ has 2, 3, or 4 vertices. If $|\Gamma_1(H)| = 2$, then $|\Gamma_{\geq 2}(H)| = 4$, $\gamma = 2$, $\alpha = 1$, $\beta = 1/2$, and $\lambda_2(Q) = \frac{1}{4}(\sqrt{41}+3) \approx 2.35$, a contradiction. If $|\Gamma_1(H)| = 3$, then $|\Gamma_{\geq 2}(H)| = 3$, $\gamma = 4/3$, and $\alpha = \beta$. Then $\alpha \leq 5/3$ (because the center entry of Q, $3-(\alpha+\gamma)$, must be nonnegative), so $\beta \leq 5/3$, which implies $\Gamma_{\geq 2}(H)$ has at least 2 edges. Since G has girth 4, $\Gamma_{\geq 2}(H)$ cannot have 3 edges, so $\Gamma_{\geq 2}(H)$ has exactly 2 edges, $\alpha = \beta = 5/3$, and $\lambda_2(Q) = \frac{1}{2}(\sqrt{241}+7) \approx 1.88$, a contradiction. If $|\Gamma_1(H)| = 4$, then $|\Gamma_{\geq 2}(H)| = 2$, $\gamma = 2$, and depending on whether there is an edge in $\Gamma_{\geq 2}(H)$ or not we have $\beta = 2$ or 3, $\alpha = 1$ or $3/2$, and $\lambda_2(Q) = \frac{1}{2}(\sqrt{5}+1) \approx 1.62$ or $3/2$, respectively. Either case is a contradiction. If $|V(G)| = 12$, then $\Gamma_1(H)$ must be a coclique on 4 vertices (otherwise $|E(\Gamma_1(H), \Gamma_{\geq 2}(H))| \leq 6$, and Lemma 4.2 implies $\overline{d}(\Gamma_{\geq 2}(H)) \leq \sqrt{2}$, so $|E(\Gamma_1(H), \Gamma_{\geq 2}(H))| \geq (3-\sqrt{2})\,|\Gamma_{\geq 2}(H)|$, so we have $|\Gamma_{\geq 2}(H)| < 6/(3-\sqrt{2}) \approx 3.78$, which implies $|V(G)| < 11.78$, a contradiction). Then we have $|\Gamma_1(H)| = |\Gamma_{\geq 2}(H)| = 4$, $\gamma = 1$, $\alpha = \beta = 2$, and $\lambda_2(Q) = \sqrt{3}$. This is a contradiction, so G cannot have girth 4.

Suppose G has girth 5. Then Lemma 1.1 implies $V(G) \geq 10$, so we have $10 \leq |V(G)| \leq 16$. The Petersen graph with 10 vertices and $\lambda_2 = 1$ is the unique

$(3,5)$-cage (see [52]), so G cannot have 10 vertices. Note we must have $|\Gamma_1(H)| = 5$ and $\gamma = 1$, since vertices in H cannot have common neighbors outside of H (otherwise the girth is at most 4, a contradiction). If $|V(G)| = 12$, then we have $|\Gamma_{\geq 2}(H)| = 2$, and depending on whether there is an edge in $\Gamma_{\geq 2}(H)$ or not we have $\beta = 2$ or 3, $\alpha = 4/5$ or $6/5$, and $\lambda_2(Q) = \frac{1}{5}(2\sqrt{6}+3) \approx 1.58$ or $\frac{1}{10}(\sqrt{241}-1) \approx 1.45$, respectively. Either case is a contradiction.

If $|V(G)| = 14$ or 16, partition the vertices of G according to their distance from a fixed vertex u. We have $|\Gamma_1(u)| = 3$, $|\Gamma_2(u)| = 6$, and $|\Gamma_{\geq 3}(u)| = k \in \{4, 6\}$. Then G has quotient matrix Q given by

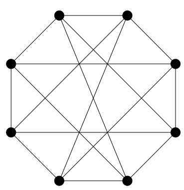
$$\begin{pmatrix} 0 & 3 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 2-\alpha & \alpha \\ 0 & 0 & \beta & 3-\beta \end{pmatrix}, \tag{4.4}$$

where $k\beta = 6\alpha$ (by counting $|E(\Gamma_2(u), \Gamma_{\geq 3}(u))|$). As before, Proposition 1.27 implies $\lambda_2(Q) \leq \sqrt{2}$. We may assume there is an edge in $\Gamma_2(u)$ since G has girth 5, so we have $0 < \alpha \leq 5/3$. The matrix Q has characteristic polynomial $\phi_Q(x) = (x-3)f(x)$, where $f(x) = x^3 + (\alpha+\beta-2)x^2 + (\beta-5)x - 3\alpha - 2\beta + 6$. Thus $\lambda_1(Q) = 3$. If $|V(G)| = 14$, we have $k = 4$, $6\alpha = 4\beta$ (so $\beta = \frac{3}{2}\alpha$), $f(\sqrt{2}) = (\alpha-2)(3\sqrt{2}-2) \leq (5/3-2)(3\sqrt{2}-2) < 0$, and $f(3) = 42\alpha > 0$, so Q has an eigenvalue between $\sqrt{2}$ and 3, a contradiction. If $|V(G)| = 16$, we have $k = 6$, $\alpha = \beta$, $f(\sqrt{2}) = 2 - 3\sqrt{2} + (\sqrt{2}-1)\alpha \leq 2 - 3\sqrt{2} + 5(\sqrt{2}-1)/3 < 0$, and $f(3) = 16\alpha > 0$, so Q has an eigenvalue between $\sqrt{2}$ and 3, a contradiction. Thus G cannot have girth 5.

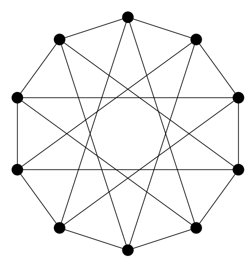
Finally, if G has girth 6, we again partition the vertices with respect to their distance from a vertex u. As with girth 5 we have $|\Gamma_1(u)| = 3$, $|\Gamma_2(u)| = 6$, $|\Gamma_{\geq 3}(u)| = k$, and quotient matrix Q given by (4.4). As before, Proposition 1.27 implies $\lambda_2(Q) \leq \sqrt{2}$. Since the girth is 6 we have $\alpha = 2$ and $k\beta = 12$ (again by counting $|E(\Gamma_2(u), \Gamma_{\geq 3}(u))|$). Then Q has characteristic polynomial $\phi_Q(x) = (x-3)f(x)$, where $f(x) = x^3 + \beta x^2 + (\beta-5)x - 2\beta$. We have $f(\sqrt{2}) = \sqrt{2}(\beta-3)$ and $f(3) = 2(5\beta+6)$, so if $\beta < 3$ then Q has an eigenvalue between $\sqrt{2}$ and 3, a contradiction. Thus $\beta = 3$, $k = 4$, $|V(G)| = 14$, and by Lemma 1.25 G is distance regular with intersection array $\{3, 2, 2; 1, 1, 3\}$. The Heawood graph is the unique distance-regular graph with that intersection array (see [9]), so G is the Heawood graph. $\square$

Note that this implies $v(3, \sqrt{2}) = 14$. We remark that Proposition 4.6 can also be proved by a computer search once we have bounded $V(G)$ depending on the girth as in the beginning of the proof. One can simply check the girth 3 graphs on at most 8 vertices, the girth 4 graphs on at most 12 vertices, and girth 5 and 6 graphs on at most 16 vertices. The number of graphs which must be checked is only 94 (see, for example, [83] for tables with the numbers of 3-regular graphs of given order and girth).

Proposition 4.7. *If G is a connected, 4-regular graph with $\lambda_2(G) > 1$, then $\lambda_2(G) \geq \sqrt{5}-1$, with equality if and only if G is either the graph in Figure 4.6(a) or the circulant graph $Ci_{10}(1, 4)$ (see Figure 4.6(b)).*



(a) The 4-regular graph G on 8 vertices with $\lambda_2(G) = \sqrt{5} - 1$.



(b) The Circulant graph $\mathrm{Ci}_{10}(1, 4)$.

Figure 4.6: The 4-regular graphs $\lambda_2 = \sqrt{5} - 1$.

Proof. It is straightforward to verify that the two graphs mentioned have second eigenvalue $\sqrt{5} - 1$. Indeed, the graph in Figure 4.6(a) has characteristic polynomial $f(x) = x^4(x-4)(x^2+2x-4)(x+2)$ and the circulant graph $\mathrm{Ci}_{10}(1, 4)$ has characteristic polynomial $g(x) = x^5(x-4)(x^2+2x-4)^2$. We also note that the spectral radius of $K_{1,2}$ is $\sqrt{2} > \sqrt{5}-1$. Suppose G is a connected, 4-regular graph with $\lambda_2(G) \leq \sqrt{5}-1$. If G has girth 3, then G induces a C_3 and Lemma 4.3 implies $|V(G)| \leq \frac{9}{10}(15+\sqrt{5}) \approx 15.51$. This implies $|V(G)| \leq 15$. If G has girth more than 3, then G induces $K_{1,2}$, so Lemma 4.3 implies $|V(G)| \leq \frac{1}{5}(17 + 6\sqrt{5}) \approx 19.68$. This implies $|V(G)| \leq 19$. A 4-regular, girth 6 graph must have at least 26 vertices by Lemma 1.1, so G must have girth at most 5.

We checked by computer all 4-regular graphs on at most 15 vertices and found that, in each case where $\lambda_2(G) > 1$, we have $\lambda_2(G) \geq \sqrt{5}-1$, with equality if and only if G is either the graph in Figure 4.6(a) or the circulant graph $\mathrm{Ci}_{10}(1,4)$. Thus, if we can show that $|V(G)| \leq 15$, we are done. Since we have already shown that if G has girth 3 then $|V(G)| \leq 15$, we may assume that G has girth 4 or 5.

If G has girth 5 and at most 19 vertices, then G must be the Robertson graph, the unique (4,5)-cage (see [82]). A direct computation shows that the Robertson graph has second eigenvalue $\frac{1}{2}(\sqrt{21}-1) > \sqrt{5}-1$, so G does not have girth 5.

Suppose that G has girth 4. Partitioning the vertices of G with respect to their distance from an arbitrary vertex u we find $|\Gamma_1(u)| = 4$ and $|\Gamma_{\geq 2}(u)| = m$. Then G has quotient matrix

$$Q = \begin{pmatrix} 0 & 4 & 0 \\ 1 & 0 & 3 \\ 0 & \beta & 4-\beta \end{pmatrix},$$

where β is the average number of neighbors in $\Gamma_1(u)$ of the vertices in $\Gamma_{\geq 2}(u)$. Note that this implies $m\beta = 12$ (by counting $|E(\Gamma_1(u), \Gamma_{\geq 2}(u))|$), so if $|V(G)| = 16$, 17, 18, or 19, then we have $m = 11$, 12, 13, or 14, $\beta = 12/11$, 1, 12/13, or 6/7, and $\lambda_2(Q) = \frac{2}{11}(\sqrt{97}-3)$, $\frac{1}{2}(\sqrt{13}-1)$, $\frac{2}{13}(\sqrt{139}-3)$, or $\frac{1}{7}(\sqrt{163}-3)$, respectively. Each of these is larger than $\sqrt{5}-1$, which contradicts Proposition 1.27. Thus if G has girth 4, we have $|V(G)| \leq 15$. □

Note that this implies $v(4, \sqrt{5}-1) = 10$. It would be interesting to find a proof of Proposition 4.7 which does not require a computer search. For the proof above the computer must check 906,331 graphs.

Richey, Shutty, and Stover conjectured that $v(4,3) = 27$ and the largest 4-regular graph with $\lambda_2 \leq 3$ is the Doyle graph on 27 vertices. We disprove this conjecture and show that $v(4,3) \geq 728$, because there is a distance-regular graph on 728 vertices which is 4-regular with $\lambda_2 = 3$: the incidence graph of a generalized hexagon $GH(3,3)$

(see, for example, [15]). It turns out this is the largest such graph (though perhaps not the only one), but we need more powerful bounds to prove it (see Section 4.4). Here, we can prove that if the girth of a 4-regular graph with second eigenvalue 3 is at least 12, then the graph must have the same intersection array as $GH(3,3)$, as seen below.

Proposition 4.8. *If G is a connected, 4-regular graph with girth at least 12 and $\lambda_2(G) \leq 3$, then G is a distance regular graph on 728 vertices and intersection array* $\{4,3,3,3,3,3;1,1,1,1,1,4\}$.

Proof. Since G is 4-regular with girth 12, by partitioning the vertices of G according to their distance from a vertex u we find that $|\Gamma_1(u)| = 4$, $|\Gamma_2(u)| = 12$, $|\Gamma_3(u)| = 36$, $|\Gamma_4(u)| = 108$, $|\Gamma_5(u)| = 324$, and $|\Gamma_{\geq 6}(u)| = m$, where $m = |V(G)| - 485$, with quotient matrix

$$Q = \begin{pmatrix} 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & \beta & 4-\beta \end{pmatrix},$$

where β is the average number of neighbors in $\Gamma_5(u)$ of the vertices in $\Gamma_{\geq 6}(u)$. Note that this implies $m\beta = 3 \cdot 324 = 972$ (by counting $|E(\Gamma_5(u), \Gamma_{\geq 6}(u))|$). The characteristic polynomial of Q is $\phi_Q(x) = (x-4)f(x)$, where $f(x) = x^6 + \beta x^5 + (\beta - 16)x^4 - 12\beta x^3 - (9\beta - 63)x^2 + 27\beta x + 9\beta - 36$. Clearly 4 is an eigenvalue of Q. We have $f(3) = 9(\beta - 4)$ and $f(4) = 485\beta + 972$, so for $\beta < 4$ we have $f(3) < 0$ and $f(4) > 0$, which implies Q has an eigenvalue between 3 and 4. This is a contradiction, since $\lambda_2(G) \leq 3$ by Proposition 1.27, so we must have $\beta = 4$ and $m = 243$. Then Lemma 1.25 implies that G is a distance-regular graph on 728 vertices with intersection array $\{4,3,3,3,3,3;1,1,1,1,1,4\}$. □

Richey, Shutty, and Stover conjectured that $v(3, 1.9) = 18$. We confirm this conjecture, and show that there are exactly two graphs meeting this bound.

Proposition 4.9. *If G is a connected, 3-regular graph with second largest eigenvalue $\lambda_2(G) \leq 1.9$, then $|V(G)| \leq 18$, with equality if and only if G is the Pappus graph (see Figure 4.7(a)) or the graph in Figure 4.7(b).*

(a) The Pappus graph.

(b) A graph with $\lambda_2 = \gamma \approx 1.8662$, the largest root of $f(x) = x^3 + 2x^2 - 4x - 6$.

Figure 4.7: The 3-regular graphs on 18 vertices with $\lambda_2 < 1.9$.

Proof. We note again that any cycle has spectral radius 2. It is straightforward to verify that the graph in Figure 4.8 also has spectral radius 2. Then, by Lemma 4.3,

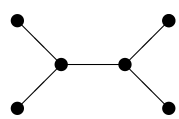


Figure 4.8: A graph with spectral radius 2.

if G has girth 3, 4, 5, 6, or 7, then G has at most 11.45, 15.27, 19.09, 22.91, or 26.73 vertices, respectively. Since G is 3-regular, this implies G has at most 10, 14, 18, 22, or 26 vertices, respectively. A 3-regular graph with girth at least 5 contains as an induced subgraph isomorphic to the graph in Figure 4.8, so by Lemma 4.3 such a graph has at most 28.54 vertices, hence at most 28 vertices. A 3-regular graph of girth 8 has at least 30 vertices by Lemma 1.1 (or note that the Tutte-Coxeter graph is the unique (3,8)-cage, see [93] and [94]), so we have shown that a 3-regular graph G with $\lambda_2(G) \leq 1.9$ and more than 18 vertices must have girth 6 or 7.

If G has 26 vertices, then G has girth 7. By partitioning the vertices according to their distance from an arbitrary vertex u, we find $|\Gamma_1(u)| = 3$, $|\Gamma_2(u)| = 6$, $|\Gamma_{\geq 3}(u)| = 16$. Then G has quotient matrix

$$Q = \begin{pmatrix} 0 & 3 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & \beta & 3-\beta \end{pmatrix},$$

with $16\beta = 12$ (by counting $|E(\Gamma_2(u), \Gamma_{\geq 3}(u))|$), so $\beta = 3/4$. By Proposition 1.27, we must have $\lambda_2(Q) \leq 1.9$, but we find that $\lambda_2(Q) = \gamma \approx 1.9009$, where γ is the largest root of the polynomial $f(x) = 4x^3 + 3x^2 - 17x - 6$, a contradiction. This implies that G has at most 24 vertices. In this case, the girth of G must still be 7. The McGee graph on 24 vertices is the unique (3,7)-cage ([69] and [94]). Since the McGee graph has second eigenvalue 2, we have proved that G does not have girth 7.

Now, if G has more than 18 vertices then G must have girth 6 and at most 22 vertices. Among 3-regular graphs, we checked by computer the 32 graphs with girth 6 on 20 vertices and the 385 graphs with girth 6 on 22 vertices and found that each has second eigenvalue more than 1.9. Thus G has at most 18 vertices. If G has 18 vertices, then G must have girth 5 or 6. Among 3-regular graphs, we checked by computer the 450 graphs with girth 5 on 18 vertices and found that each has second eigenvalue more than 1.9. We checked the 5 graphs with girth 6 on 18 vertices and found that all but two of them have second eigenvalue more than 1.9. The exceptions were the Pappus graph with second eigenvalue $\sqrt{3}$ and the graph in Figure 4.7(b) with second eigenvalue γ, where $\gamma \approx 1.8662$ is the largest root of $f(x) = x^3 + 2x^2 - 4x - 6$ (again, see [83] for tables with the numbers of 3-regular graphs of given order and girth). $\square$

Note that in addition to proving that $v(3, 1.9) = 18$, this result implies $v(3, \sqrt{3}) = 18$ and $v(3, \gamma \approx 1.8662) = 18$. It would be interesting to find a proof of Proposition 4.9 that does not require a computer search.

4.4 Linear Programming Bound

Recently, Nozaki [76] used linear programming to prove a bound on the number of vertices in a k-regular graph whose eigenvalues satisfy certain conditions:

Let $F_i^{(k)}$ be orthogonal polynomials defined by the recurrence formula

$$F_i^{(k)}(x) = xF_{i-1}^{(k)}(x) - (k-1)F_{i-2}^{(k)}(x), \tag{4.5}$$

for $i \geq 3$, where $F_0^{(k)}(x) = 1$, $F_1^{(k)}(x) = x$, and $F_2^{(k)}(x) = x^2 - k$.

Theorem 4.10. *Let G be a connected, k-regular graph with v vertices. Let $\tau_0 = k, \tau_1, \ldots, \tau_d$ be the distinct eigenvalues of G. Suppose there exists a polynomial $f(x) = \sum_{i \geq 0} f_i F_i^{(k)}(x)$ such that $f(k) > 0$, $f(\tau_i) \leq 0$ for any $i \geq 1$, $f_0 > 0$, and $f_i \geq 0$ for any $i \geq 1$. Then we have*

$$v \leq \frac{f(k)}{f_0}.$$

Let $T(k,t,c)$ be the $t \times t$ tridiagonal matrix with lower diagonal $(1, 1, \ldots, 1, c)$, upper diagonal $(k, k-1, \ldots, k-1)$, and with constant row sum k, where c is a positive real number. Let $M(k,t,c) = 1 + \sum_{i=0}^{t-3} k(k-1)^i + \frac{k(k-1)^{t-2}}{c}$. We have the following bound:

Theorem 4.11. *Let λ be the second largest eigenvalue of $T(k,t,c)$. Then we have $v(k,\lambda) \leq M(k,t,c)$, with equality if and only if any graph meeting the bound is a distance-regular graph with quotient matrix $T(k,t,c)$ with respect to the distance partition.*

The proof below, due to Koolen and Nozaki, is given in substantially the same form as in [27].

Proof. We first show that the eigenvalues of T coincide with the zeros of $\sum_{i=0}^{t-2} F_i(x) + F_{t-1}(x)/c$ (see [14, Section 4.1 B]). Indeed,

$$[F_0, F_1, \ldots, F_{t-1}/c]\, T = [xF_0, xF_1, \ldots, xF_{t-2}, (k-1)F_{t-2} + (k-c)F_{t-1}/c],$$

and

$$
\begin{aligned}
[F_0, F_1, \dots, F_{t-1}/c]\,(T - xI) &= [0, 0, \dots, 0, (k-1)F_{t-2} + (-x+k-c)F_{t-1}/c] \\
&= \left[0, 0, \dots, 0, (k-x)\left(\sum_{i=0}^{t-2} F_i + F_{t-1}/c\right)\right] \\
&= [0, 0, \dots, 0, (k-x)((c-1)G_{t-2} + G_{t-1})/c]
\end{aligned}
$$

by (4.5), where $G_i(x) = \sum_{j=0}^{i} F_j(x)$. From this equation, the zeros of $(k-x)((c-1)G_{t-2} + G_{t-1})$ are eigenvalues of T. The monic polynomials G_i form a sequence of orthogonal polynomials with respect to some positive weight on the interval $\left[-2\sqrt{k-1}, 2\sqrt{k-1}\right]$ [76]. Since the zeros of G_{t-2} and G_{t-1} interlace on $\left[-2\sqrt{k-1}, 2\sqrt{k-1}\right]$, the zeros of $(k-x)((c-1)G_{t-2} + G_{t-1})$ are simple. Therefore all eigenvalues of T coincide with the zeros of $(k-x)((c-1)G_{t-2} + G_{t-1})$, and are simple.

Let $\mu_1 = k > \mu_2 > \ldots > \mu_t$ be the eigenvalues of T. We show the polynomial

$$
f(x) = (x - \mu_2) \prod_{i=3}^{t} (x - \mu_i)^2 = \sum_{i=0}^{2t-3} f_i F_i(x)
$$

satisfies $f_i > 0$ for $i = 0, 1, \dots, 2t-3$. Note that it trivially holds that $f(k) > 0$, and $f(\mu) \le 0$ for any $\mu \le \mu_2$. The polynomial $f(x)$ can be expressed by

$$
f(x) = \frac{(c-1)G_{t-2} + G_{t-1}}{x - \mu_2} \left(\sum_{i=0}^{t-2} F_i + F_{t-1}/c\right).
$$

By [28, Theorem 3.1] (or by [76, Theorem 4]), $g(x) = ((c-1)G_{t-2} + G_{t-1})/(x - \mu_2)$ has positive coefficients in terms of $G_0, G_1, \dots, G_{t-1}$. This implies that $g(x)$ has positive coefficients in terms of $F_0, F_1, \dots, F_{t-1}$. Therefore $f_i > 0$ for $i = 0, 1, \dots, 2t-3$ by [76, Theorem 3].

The polynomial $g(x)$ can be expressed by $g(x) = \sum_{i=0}^{t-2} g_i F_i(x)$. By [76, Theorem 3], we have

$$
f_0 = \sum_{i=0}^{t-2} g_i F_i(k) = g(k).
$$

By Theorem 4.10 with $f(x)$, we have

$$\begin{aligned} v(k,\mu_2) \le \frac{f(k)}{f_0} &= \sum_{i=0}^{t-2} F_i(k) + F_{t-1}(k)/c \\ &= 1 + \sum_{i=0}^{t-3} k(k-1)^i + \frac{k(k-1)^{t-2}}{c}. \end{aligned}$$

By [76, Remark 2], the graph attaining the bound has girth at least $2t-2$, and at most t distinct eigenvalues. Therefore the graph is a distance-regular graph with quotient matrix $T(k,t,c)$ by [76, Theorem 6] and [34]. Conversely the distance-regular graph with quotient matrix $T(k,t,c)$ clearly attains the bound $M(k,t,c)$. □

Note that, by Lemma 1.25, $T(k,t,c)$ is the quotient matrix (with respect to the distance partition) of the distance-regular graph with diameter $t+1$ and intersection array $\{k, k-1, \dots, k-1; 1, \dots, 1, c\}$ (where $k-1$ and 1 each appear t times), if such a distance-regular graph exists. Further, $M(k,t,c)$ is clearly the number of vertices in such a graph. Thus, in addition to providing an upper bound on the number of vertices, Theorem 4.11 states that any distance-regular graph with such an intersection array must be maximal with respect to $v(k,\lambda)$.

Theorem 4.11 simplifies many of the proofs from Section 4.3, and in a few cases gives the extremal graph found in Section 4.3 immediately. We include the proofs in Section 4.3 for the sake of completeness and clarity, since the method is easier to understand than Theorem 4.11.

Table 4.1 gives some infinite families of graphs that meet the bound $M(k,t,c)$ for some values of k,t,c. Of course, by $PG(2,q)$, $GQ(q,q)$, and $GH(q,q)$, we mean the incidence graphs of those structures. The incidence graphs of $PG(2,q)$, $GQ(q,q)$, and $GH(q,q)$ are known to be unique for $q \le 8$, $q \le 4$, and $q \le 2$, respectively (see, for example, [14, Table 6.5 and the following comments]). The incidence graphs of $PG(2,2)$, $GQ(2,2)$, and $GH(2,2)$ are the Heawood graph, the Tutte-Coxeter graph (or Tutte 8-cage), and the Tutte 12-cage, respectively. Note that for $q=3$ in the last line of Table 4.1 we obtain $v(4,3) = 728$, which we were only able to partially prove in Section 4.3.

Table 4.1: Families of graphs meeting the bound $M(k,t,c)$.

(k,λ)	$v(k,\lambda)$	Graph meeting bound	Unique?	Ref.
$(2, 2\cos(2\pi/n))$	n	n-cycle C_n	yes	
$(k,-1)$	$k+1$	K_{k+1}	yes	
$(k,0)$	$2k$	$K_{k,k}$	yes	
$(q+1,\sqrt{q})$	$2(q^2+q+1)$	$PG(2,q)$	?	[14, 87]
$(q+1,\sqrt{2q})$	$2(q+1)(q^2+1)$	$GQ(q,q)$	?	[8, 14]
$(q+1,\sqrt{3q})$	$2(q+1)(q^4+q^2+1)$	$GH(q,q)$	?	[8, 14]

$PG(2,q)$: projective plane, $GQ(q,q)$: generalized quadrangle,
$GH(q,q)$: generalized hexagon, q: prime power

Table 4.2 gives some sporadic examples of graphs meeting the bound $M(k,t,c)$. For the values of t,c in Table 4.2, one needs only check the intersection array of the given graph.

Table 4.2: Sporadic of graphs meeting the bound $M(k,t,c)$.

(k,λ)	$v(k,\lambda)$	Graph meeting bound	Unique?	Ref.
$(3,1)$	10	Petersen graph	yes	[52]
$(4,2)$	35	Odd graph O_4	yes	[72]
$(7,2)$	50	Hoffman–Singleton graph	yes	[52]
$(5,1)$	16	Clebsch graph	yes	[45, 85]
$(10,2)$	56	Gewirtz graph	yes	[16, 41]
$(16,2)$	77	M_{22} graph	yes	[12, 51]
$(22,2)$	100	Higman–Sims graph	yes	[41, 51]

Theorem 4.11 and the characterization of graphs with least eigenvalue at least -2 can be used to find $v(k,1)$ for any k:

Proposition 4.12. *Let G be a connected k-regular graph with second largest eigenvalue at most 1, with $v(k,1)$ vertices. Then the following hold:*

(*i*) $v(2,1)=6$, *and G is the hexagon.*

(*ii*) $v(3,1)=10$, *and G is the Petersen graph.*

(*iii*) $v(4,1)=12$, *and G is the complement of graph no.* 186 *in Table 9.1 in [19].*

(*iv*) $v(5,1)=16$, *and G is the Clebsch graph.*

(*v*) $v(6,1) = 15$, *and* G *is the complement of the line graph of the complete graph with* 6 *vertices, or the complements of graphs nos.* 171–176 *in Table 9.1 in [19].*

(*vi*) $v(7,1) = 18$, *and* G *is the complements of graphs nos.* 177–180 *in Table 9.1 in [19].*

(*vii*) $v(8,1) = 21$, *and* G *is the complements of graphs nos.* 181, 182 *in Table 9.1 in [19].*

(*viii*) $v(9,1) = 24$, *and* G *is the complement of graph no.* 183 *in Table 9.1 in [19].*

(*ix*) $v(10,1) = 27$, *and* G *is the complement of the Schläfli graph.*

(*x*) $v(k,1) = 2k+2$ *for* $k \geq 11$, *and* G *is the complement of the line graph of* $K_{2,k+1}$.

In Table 4.3 we summarize the known values of $v(k,\lambda)$, $k \leq 22$, given in this section and Section 4.3.

Table 4.3: Summary of our results.

(k,λ)	$v(k,\lambda)$		(k,λ)	$v(k,\lambda)$		(k,λ)	$v(k,\lambda)$
$(2,-1)$	3		$(7,1)$	18		$(14,\sqrt{13})$	366
$(2,0)$	4		$(7,2)$	50		$(14,\sqrt{26})$	4760
$\left(2,\frac{1}{2}\left(\sqrt{5}-1\right)\right)$	5		$(8,-1)$	9		$(14,\sqrt{39})$	804468
$(2,1)$	6		$(8,0)$	16		$(15,-1)$	16
$(2,\sqrt{2})$	8		$(8,1)$	21		$(15,0)$	30
$\left(2,\frac{1}{2}\left(\sqrt{5}+1\right)\right)$	10		$(8,\sqrt{7})$	114		$(15,1)$	32
$(2,\sqrt{3})$	12		$(8,\sqrt{14})$	800		$(16,-1)$	17
$(3,-1)$	4		$(8,\sqrt{21})$	39216		$(16,0)$	32
$(3,0)$	6		$(9,-1)$	10		$(16,1)$	34
$(3,1)$	10		$(9,0)$	18		$(16,2)$	77
$(3,\sqrt{2})$	14		$(9,1)$	24		$(17,-1)$	18
$(3,\sqrt{3})$	18		$(9,2\sqrt{2})$	146		$(17,0)$	34
$(3,2)$	30		$(9,4)$	1170		$(17,1)$	36
$(3,\sqrt{6})$	126		$(9,2\sqrt{6})$	74898		$(18,-1)$	19
$(4,-1)$	5		$(10,-1)$	11		$(18,0)$	36
$(4,0)$	8		$(10,0)$	20		$(18,1)$	38
$(4,1)$	12		$(10,1)$	27		$(18,\sqrt{17})$	614
$(4,\sqrt{5}-1)$	10		$(10,2)$	56		$(18,\sqrt{34})$	10440
$(4,\sqrt{3})$	26		$(10,3)$	182		$(18,\sqrt{51})$	3017196
$(4,2)$	35		$(10,3\sqrt{2})$	1640		$(19,-1)$	20
$(4,\sqrt{6})$	80		$(10,3\sqrt{3})$	132860		$(19,0)$	38
$(4,3)$	728		$(11,-1)$	12		$(19,1)$	40
$(5,-1)$	6		$(11,0)$	22		$(20,-1)$	21
$(5,0)$	10		$(11,1)$	24		$(20,0)$	40
$(5,1)$	16		$(12,-1)$	13		$(20,1)$	42
$(5,2)$	42		$(12,0)$	24		$(20,\sqrt{19})$	762
$(5,2\sqrt{2})$	170		$(12,1)$	26		$(20,\sqrt{38})$	14480
$(5,2\sqrt{3})$	2730		$(12,\sqrt{11})$	266		$(20,\sqrt{57})$	5227320
$(6,-1)$	7		$(12,\sqrt{22})$	2928		$(21,-1)$	22
$(6,0)$	12		$(12,\sqrt{33})$	354312		$(21,0)$	42
$(6,1)$	15		$(13,-1)$	14		$(21,1)$	44
$(6,\sqrt{5})$	62		$(13,0)$	26		$(22,-1)$	23
$(6,\sqrt{10})$	312		$(13,1)$	28		$(22,0)$	44
$(6,\sqrt{15})$	7812		$(14,-1)$	15		$(22,1)$	46
$(7,-1)$	8		$(14,0)$	28		$(22,2)$	100
$(7,0)$	14		$(14,1)$	30			

Chapter 5

OPEN PROBLEMS AND FUTURE WORK

In this chapter we discuss open problems and future work in the areas of the problems discussed in Chapters 2–4.

5.1 The Graphs With All But Two Eigenvalues Equal to 0 or -2

In Section 2.3 we determined the set $\mathcal{G}$ of connected graphs with all but two eigenvalues equal to ± 1. We also showed that any graph G with all but two eigenvalues equal to ± 1 must be either a disjoint union of complete graphs with exactly two connected components different from K_2 or a disjoint union of a graph $H \in \mathcal{G}$ and some copies of K_2. A graph G in $\mathcal{G}$ has eigenvalues $r > 1$ and $s < -1$ with multiplicity 1, and ± 1 with any multiplicity. If a graph $G \in \mathcal{G}$ is a regular graph on n vertices, then by Proposition 1.20 the complement $\overline{G}$ of G has eigenvalues $n-1-r$ and $-1-s$ with multiplicity 1, and $0, -2$ with any multiplicity (the multiplicity of 0 and -2 in $\overline{G}$ is the multiplicity of -1 and 1, respectively, in G).

Thus it is natural to next classify the set $\overline{\mathcal{G}}$ of graphs with all but two eigenvalues equal to 0 or -2. Those that are regular are the complements of regular graphs in $\mathcal{G}$. Consider the complements of graphs in $\mathcal{G}$. It is possible that the complement of a nonregular graph in $\mathcal{G}$ is also in $\overline{\mathcal{G}}$, so we will check them all. The complements of $B_1(m)$ are $K_m \square K_2$ with spectrum $\{m, m-2, 0^{m-1}, -2^{m-1}\}$, and will be discussed in Case 3 below. The complements of $B_2(a,k)$ are disjoint unions of a coclique and a copy of $CP(k) \cong K_{2,2,\ldots,2}$ (which has only one eigenvalue not equal to 0 or -2, see Proposition 5.2). The complements of $B_3(\ell, m)$ are disjoint unions of two copies of $K_{2,2,\ldots,2}$ (one copy with ℓ parts and one with m parts), which will be found in Corollary 5.3. The complements of the graphs $B_4(4)$, $B_4(5)$, $B_6(3,5)$, and $B_6(4,4)$ have three eigenvalues different from $0, -2$. The complements of $B_5(6,5)$, $B_5(4,6)$, and $B_5(3,8)$ are disjoint unions of a coclique and a copy of $K_{1,m}$ for $m = 5, 6, 8$, respectively, and are found in Proposition 5.5.

Since the addition of an isolated vertex to a graph adds the eigenvalue 0 to the spectrum, if $G \in \overline{\mathcal{G}}$ then the disjoint union of G and an isolated vertex is also in $\overline{\mathcal{G}}$. For a graph G, we call the graph G^* obtained by removing all isolated vertices from G the main part of G.

Proposition 5.1. *If $G \in \overline{\mathcal{G}}$, then G^* has at most two connected components.*

Proof. Each connected component of G^* contributes a positive eigenvalue to the spectrum of G by Corollaries 1.11 and 1.19. Since G has at most two positive eigenvalues, this completes the proof. $\square$

Proposition 5.1 allows us to consider the problem in cases based on whether G^* has one or two connected components. We further divide the case where G^* is connected based on whether G has one or two positive eigenvalues. We characterize the graphs in $\overline{\mathcal{G}}$ in the first two cases below. For the last case (G^* is connected and G has two positive eigenvalues) we give a partial characterization. In future work we plan to finish the characterization of graphs in $\overline{\mathcal{G}}$.

Case 1: G^* Has Two Connected Components.

In this case, G has two positive eigenvalues (one coming from each connected component, see Corollary 1.19) and the rest of the eigenvalues of G are 0 or -2. The candidates for the connected components of G^* are given by the following proposition.

Proposition 5.2. *If H is a graph with exactly one eigenvalue not equal to 0 or -2, then H is either the complete bipartite graph $K_{1,4}$ or a cocktail party graph $CP(k)$ (see Definition 1.9) for some $k \geq 2$.*

Proof. We note that H has exactly one positive eigenvalue r, and the rest of the eigenvalues of H are 0 or -2. Chuang and Omidi [24] list all graphs that have exactly three distinct eigenvalues, each of which is at least -2 (see also their references [21] and [30]). The graphs $K_{1,4}$ and $CP(k)$, $k \geq 2$, are the only ones with exactly one eigenvalue not equal to 0 or -2. $\square$

With Proposition 5.2, the following result is immediate:

Corollary 5.3. *If $G \in \overline{\mathcal{G}}$ and G^* has two connected components, then G^* is isomorphic to the disjoint union $H + K$, where $H, K \in \{K_{1,4}\} \cup \{CP(k) \mid k \geq 2\}$.*

Case 2: G^* is Connected and G Has Exactly One Positive Eigenvalue

We will need the following theorem on the spectra of multipartite graphs [39].

Theorem 5.4. *Let $p_1 \leq p_2 \leq \cdots \leq p_k$ be positive integers and suppose G is the complete k-partite graph $K_{p_1,p_2,\ldots,p_k}$ on $n = \sum p_i$ vertices. Let $q_1 < q_2 < \cdots < q_t$ be the distinct part sizes of G (so for each i, $p_i = q_j$ for some j). Then G has exactly one positive eigenvalue, $k-1$ negative eigenvalues, and the rest of the eigenvalues are 0. The negative eigenvalues $\lambda_{n-k+2}, \ldots, \lambda_n$ satisfy $p_{i-1} \leq -\lambda_{n-k+i} \leq p_i$ for $i = 2, 3, \ldots, k$, and $t-1$ of the negative eigenvalues η_i ($i = 2, \ldots t$), satisfy $q_{i-1} < \eta_i < q_i$.*

Proposition 5.5. *If $G \in \overline{\mathcal{G}}$ has exactly one positive eigenvalue, then G^* is isomorphic to $K_{1,1,3}$, $K_{\ell,m}$ ($\ell, m \geq 1$ and not both equal to 2), or $K_{2,2,\ldots,2,m}$ ($m \neq 2$).*

Proof. Since G has one positive eigenvalue, G has a negative eigenvalue not equal to -2 with multiplicity 1, an eigenvalue -2 with any multiplicity (possibly 0), and no other negative eigenvalues. Lemma 1.30 implies G^* is a complete multipartite graph. If G^* is the complete bipartite graph $K_{\ell,m}$, then G has eigenvalues $\pm\sqrt{\ell m}$ each with multiplicity 1 and eigenvalue 0 with multiplicity $\ell + m - 2$ (possibly 0). If ℓ and m are not both equal to 2, this gives exactly two eigenvalues, $\pm\sqrt{\ell m}$, not equal to 0 or -2.

Suppose G^* is not bipartite. Then G^* is a complete multipartite graph $K_{p_1,p_2,\ldots,p_k}$, $k \geq 3$. Recall that G has exactly one negative eigenvalue (including multiplicity) not equal to -2. Theorem 5.4 implies that G^* has an eigenvalue strictly between the negative of any two distinct part sizes, and if G^* has a repeated part size m, then $-m$ is an eigenvalue. This implies that $K_{2,3,3}$ has two eigenvalues smaller than -2, so G^*

cannot induce $K_{2,3,3}$ as a subgraph. As a result, G^* cannot have two parts of size at least 3 and another part of size at least 2. Thus G cannot have four distinct part sizes, and if G^* has three distinct part sizes then one of them must be 1.

If G^* has only one part size, then it must be 2, since three parts of size m imply $-m$ is an eigenvalue with multiplicity 2. But $K_{2,\ldots,2}$ has only one eigenvalue different from 0 and -2, so G^* has at least two distinct part sizes.

Suppose G^* has exactly two distinct part sizes. We first consider the case that one of the sizes is 2. Let m be the other part size. Then there can be only one part of size m, since otherwise G^* has an eigenvalue between -2 and $-m$ as well as eigenvalue $-m$, a contradiction. In this case we find $G^* = K_{2,\ldots,2,m}$, with t parts of size 2 and $m \neq 2$, with spectrum $\lambda_1 > 0$, 0^{t+m-1}, -2^{t-1}, $\lambda_{2t+m} \in (-m,-2)$ (or $\lambda_{t+2} \in (-2,-1)$ if $m = 1$).

If G^* has two part sizes ℓ and m ($\ell, m \neq 2$), then G^* has at most three parts (thus exactly three parts). Indeed, if G^* has at least four parts, then G^* has at least two parts each of sizes ℓ and m or at least three parts of size ℓ. In the former case, G^* has eigenvalues $-\ell$ and $-m$, while in the latter case G^* has two eigenvalues $-\ell$. Either case is a contradiction, so G^* has exactly three parts and two part sizes, say two parts of size ℓ and one part of size m. We see that one of the part sizes must be 1, since otherwise G has an eigenvalue ℓ and one between ℓ and m, a contradiction (since neither ℓ nor any number between ℓ and m is 2). If ℓ is 1, then G^* is $K_{1,1,m}$ with characteristic polynomial $x^{m-1}\left(x^3 - (2m+1)x - 2m\right)$ (see, for example, [39], for computing the characteristic polynomial of complete multipartite graphs) and spectrum $\frac{1}{2}\left(1+\sqrt{1+8m}\right)$, 0^{m-1}, -1, $\frac{1}{2}\left(1-\sqrt{1+8m}\right)$. The last eigenvalue must be -2 if $G \in \overline{\mathcal{G}}$, so we must have $m = 3$ and G^* is isomorphic to $K_{1,1,3}$ with spectrum 3, 0^3, -1, -2. If $m = 1$, then G^* is $K_{1,\ell,\ell}$ with characteristic polynomial $x^{2\ell-2}\left(x^3 - \left(\ell^2+2\ell\right)x - 2\ell^2\right)$ and spectrum $\frac{1}{2}\left(\ell+\sqrt{\ell^2+8\ell}\right)$, $0^{2\ell-2}$, $\frac{1}{2}\left(\ell-\sqrt{\ell^2+8\ell}\right)$, $-\ell$. The second to last eigenvalue must be -2 if $G \in \overline{\mathcal{G}}$. However, we have $\sqrt{\ell^2+8\ell} < \sqrt{\ell^2+8\ell+16} = \sqrt{(\ell+4)^2} = \ell+4$, so $\frac{1}{2}\left(\ell-\sqrt{\ell^2+8\ell}\right) > \frac{1}{2}\left(\ell-(\ell+4)\right) > -2$, a contradiction.

Suppose G^* has three distinct part sizes $1 < \ell < m$ (recall one of the sizes must be 1). If $\ell = 2$ then G^* has an eigenvalue in each of the intervals $(-2, -1)$ and $(-m, -2)$, a contradiction. We have $2 < \ell < m$, and G^* has an eigenvalue in each of the intervals $(-\ell, -1)$ and $(-m, -\ell)$. Note that $K_{1,\ell,m}$ is induced by $K_{1,m,m}$, so if the eigenvalue in $(-\ell, -1)$ is -2, then $K_{1,\ell,m}$ has two eigenvalues at most -2, so by interlacing $K_{1,m,m}$ does as well. This is a contradiction, since we have seen above that the second smallest eigenvalue of $K_{1,m,m}$ is more than -2. This completes the proof. $\square$

Case 3: G^* is Connected and G Has Exactly Two Positive Eigenvalues

In this case, the smallest eigenvalue of G is -2. Such graphs have been characterized (see [21]) as line graphs, generalized line graphs, or one of finitely many graphs coming from exceptional root systems.

In [11], Borovićanin showed that a line graph has exactly two positive eigenvalues if and only if it is an induced subgraph of one of the graphs in Figure 5.1 and contains either P_4 or $K_1 \nabla (K_1 + K_2)$ as an induced subgraph. Most such graphs will still have

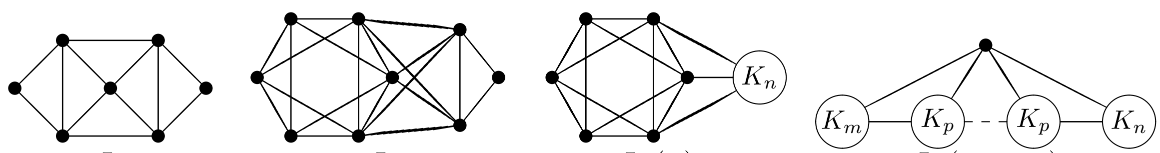


Figure 5.1: Graphs inducing line graphs with two positive eigenvalues.

more than two eigenvalues not equal to 0 or -2, but every line graph in $\overline{\mathcal{G}}$ with two positive eigenvalues must be induced by one of the graphs in Figure 5.1 (and must induce P_4 or $K_1 \nabla (K_1 + K_2)$). We checked by computer all subgraphs induced by the graphs L_1 and L_2 and found that none are in $\overline{\mathcal{G}}$. We found also that graph $L_3(n)$ does not induce any graph in $\overline{\mathcal{G}}$ for $n \leq 10$. The graph $K_t \square K_2$, which is the complement of $B_1(t)$, is an induced subgraph of $L_4(m, n, p)$ when $p \geq t$.

We do not know whether $L_3(n)$ induces any graphs in $\overline{\mathcal{G}}$ for larger n, or whether $L_4(m, n, p)$ induces any other graphs in $\overline{\mathcal{G}}$. We also do not know whether there are any generalized line graphs or exceptional root systems graphs in $\overline{\mathcal{G}}$ that have not already been found in Case 2. Haemers [50] conjectured that the graphs in Case 3 must be the complements of bipartite graphs, so they must be able to be partitioned into two cliques. For future work, we plan to prove this conjecture and use it to finish the characterization of the graphs in $\overline{\mathcal{G}}$.

5.2 Partial Permutohedra

In Section 3.10 we mention that Martin and Wagner [68] construct some linearly independent eigenvectors for the eigenvalues $-\binom{m}{2}$ and $-n$, giving a lower bound for the multiplicity of those eigenvalues Those eigenvectors are constructed as follows.

When $n \geq \binom{m}{2}$, Martin and Wagner construct *permutohedra* (so named because they are shapes defined by permutations) whose signed characteristic vector is an eigenvector of $SR(m, n)$ with eigenvalue equal to $-\binom{m}{2}$. The permutohedra are defined as follows. Let $p, w \in \mathbb{R}^m$ be vectors such that $P(p, w) = \{p + \sigma(w) \mid \sigma \in \mathcal{S}_m\}$ are distinct vertices in $SR(m, n)$. Then we say $P(p, w)$ is the permutohedron centered at p with offset w. Let $\mathbf{w}$ be the standard offset vector in $\mathbb{R}^m$, that is $\mathbf{w} = \mathbf{w}_m = ((1-m)/2, (3-m)/2, \ldots, (m-3)/2, (m-1)/2) = (i - (m+1)/2)_{i=1}^m$. Martin and Wagner show that there are

$$\binom{n - \binom{m-1}{2}}{m-1}$$

distinct $p \in \mathbb{R}^m$ such $P(p, \mathbf{w})$ is a permutohedron in $SR(m, n)$ whose signed characteristic vector is an eigenvector of $SR(m, n)$ with eigenvalue equal to $-\binom{m}{2}$, and in fact these vectors are linearly independent (see [68]). That is, they prove the following Proposition, which immediately implies Proposition 3.19.

Proposition 5.6. *For $p, w \in \mathbb{R}^m$ such that $P(p, w)$ are distinct vertices in $SR(m, n)$, define*

$$F_{p,w} = \sum_{\sigma \in \mathcal{S}_m} \operatorname{sgn}(\sigma) e_{p+\sigma(w)}.$$

Then each $F_{p,w}$ is an eigenvector of $SR(m, n)$ with eigenvalue $-\binom{m}{2}$, and for fixed w, the collection of all such $F_{p,w}$ is linearly independent.

(The vectors e_x for $x \in V(m,n)$ are defined in Section 3.6.) Choosing $w = \mathbf{w}$ yields the indicated multiplicity.

When $n < \binom{m}{2}$, Martin and Wagner [68] find so-called *partial permutohedra* in $SR(m,n)$ which lead to the eigenvalue $-n$ with multiplicity at least the Mahonian number $M(m,n)$. The partial permutohedra are obtained as follows. For a sequence of positive integers $c = (c_1, \ldots, c_m)$, the *skyline board* $\mathrm{Sky}(c)$ consists of a sequence of m columns, such that the i-th column contains c_i tiles. A *rook placement* on $\mathrm{Sky}(c)$ is a choice of one tile from each column. A rook placement is *proper* if no two chosen tiles are in the same row.

An inversion of a permutation $\pi = (\pi_1, \ldots, \pi_m) \in \mathcal{S}_m$ is a pair i, j such that $i < j$ and $\pi_i > \pi_j$. Let $\mathcal{S}_{m,n}$ denote the set of permutations of $\{1, 2, \ldots, m\}$ with exactly n inversions. Note that $|\mathcal{S}_{m,n}| = M(m,n)$. For each $\pi \in \mathcal{S}_{m,n}$, the *inversion word* of π is $a = a(\pi) = (a_1, \ldots, a_m)$, where a_i is the number of pairs i, j that are inversions of π (this is sometimes called the *Lehmer code* of π, see [60, 62]). Since π has n inversions, $a(\pi)$ is a vertex of $SR(m,n)$. We denote by $s(\pi)$ the skyline board $\mathrm{Sky}(a(\pi) + \mathrm{id})$, where $\mathrm{id} = (1, 2, \ldots, m)$. Note that $s(\pi)$ always has n tiles above the main diagonal $\mathrm{Sky}(1, \ldots, m)$, since for each i $s(\pi)$ has a_i tiles above the main diagonal in column i. A permutation $\sigma \in \mathcal{S}_m$ is π-admissible if it is a proper rook placement on $s(\pi)$ (note that if σ is a rook placement on a given skyline board, then it is automatically proper; the only way for σ to fail to be a proper rook placement on $s(\pi)$ is for some σ_i to be larger than $a_i + i$ for some i, so that the rook is placed above column i in $s(\pi)$). Clearly σ is π-admissible exactly when $x(\sigma) = a(\pi) + \mathrm{id} - \sigma$ is also a vertex in $SR(m,n)$. We denote by $\mathrm{Adm}(\pi)$ the set of all $\sigma \in \mathcal{S}_m$ that are π-admissible. The corresponding partial permutohedron is $\mathrm{Parp}(\pi) = \{x(\sigma) \mid \sigma \in \mathrm{Adm}(\pi)\}$. One can show that $\mathrm{Parp}(\pi)$ is the intersection of $V(m,n)$ with the permutohedron centered at $a(\pi) + \mathbf{w}$ and with offset $\mathbf{w}$, which is why they are called partial permutohedra. Martin and Wagner [68] prove that for each $\pi \in \mathcal{S}_{m,n}$, the signed characteristic vector of $\mathrm{Parp}(\pi)$ is an eigenvector for $SR(m,n)$ with eigenvalue $-n$, and moreover that these eigenvectors are linearly independent. That is, they prove the following Proposition, which immediately implies

Proposition 3.20.

Proposition 5.7. *For each $\pi \in \mathcal{S}_{m,n}$, let*

$$F_\pi = \sum_{\sigma \in \mathrm{Adm}(\pi)} \mathrm{sgn}(\sigma) e_{x(\sigma)}.$$

Then each F_π is an eigenvector of $SR(m,n)$ with eigenvalue $-n$, and the F_π are linearly independent.

(The vectors e_x for $x \in V(m,n)$ are defined in Section 3.6.)

Here we give an example of a permutation $\pi \in \mathcal{S}_{4,3}$ and explain how to find the corresponding subgraph $SR(4,3)|_{\mathrm{Parp}(\pi)}$. Let $\pi = (3,2,1,4)$. Then π has three inversions, because $\pi_1 > \pi_2$, $\pi_1 > \pi_3$, and $\pi_2 > \pi_3$. This also implies that the inversion word of π is $a = a(\pi) = (2,1,0,0)$. Then $s(\pi)$ is a skyline board consisting of 4 columns of tiles such that the first three columns have three tiles and the fourth column has four tiles (because $a(\pi) + \mathrm{id} = (3,3,3,4)$). We find that the π-admissible permutations in $\mathcal{S}_4$ are $\mathrm{Adm}(\pi) = \{(1,2,3,4), (1,3,2,4), (2,1,3,4), (2,3,1,4), (3,1,2,4), (3,2,1,4)\}$ (see Figure 5.2). Then the corresponding partial per-

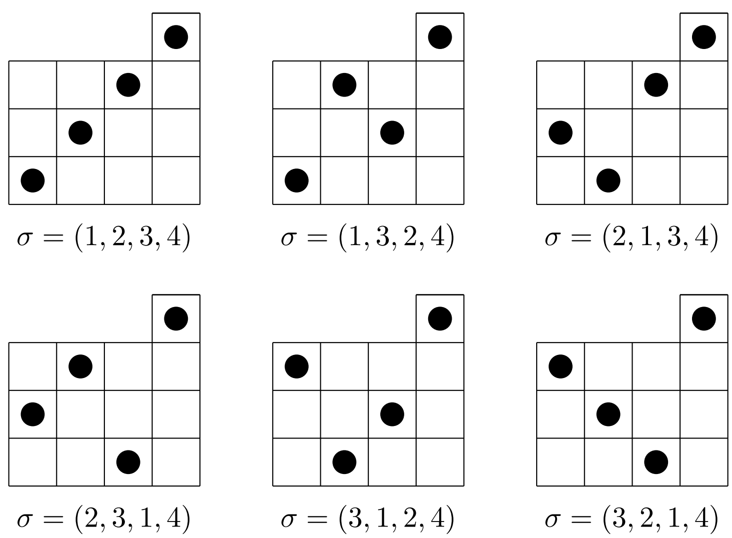


Figure 5.2: The π-admissible permutations for $\pi = (3,2,1,4)$.

mutohedron is $\mathrm{Parp}(\pi) = \{x(\sigma) \mid \sigma \in \mathrm{Adm}(\pi)\}$, where $x(\sigma) = a(\pi) + \mathrm{id} - \sigma$. Thus $\mathrm{Parp}(\pi) = \{(2,1,0,0), (2,0,1,0), (1,2,0,0), (1,0,2,0), (0,2,1,0), (0,1,2,0)\}$. We see

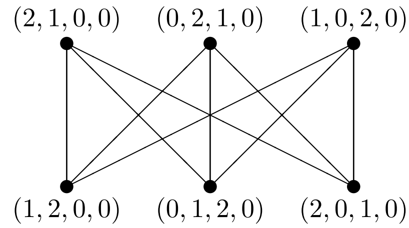


Figure 5.3: The subgraph $SR(4,3)|_{\mathrm{Parp}(\pi)}$ for $\pi = (3,2,1,4)$.

that $SR(4,3)|_{\mathrm{Parp}(\pi)} \cong K_{3,3}$ (see Figure 5.3), so $SR(4,3)|_{\mathrm{Parp}(\pi)}$ has integral spectrum $\{3^1, 0^4, -3^1\}$ by Proposition 1.10. The subgraph $SR(4,3)|_{\mathrm{Parp}(\pi)}$ can be seen within the context of the whole graph $SR(4,3)$ in Figure 5.4. It is straightforward to check that

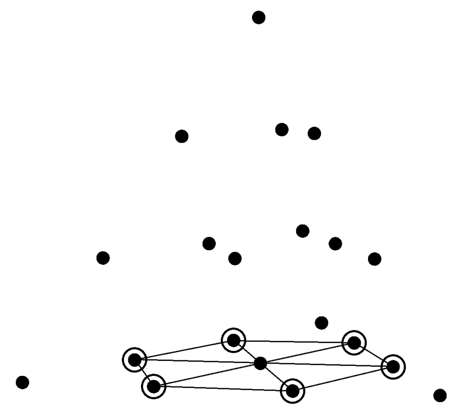


Figure 5.4: The subgraph $SR(4,3)|_{\mathrm{Parp}(\pi)}$ for $\pi = (3,2,1,4)$ within $SR(4,3)$.

F_π as defined in Proposition 5.7 is indeed an eigenvector of $SR(4,3)$ for the eigenvalue -3. We note also that the vertices $x(\sigma)$ and $x(\rho)$, $\sigma, \rho \in \mathrm{Adm}(\pi)$, are adjacent if and only if the corresponding permutations σ and ρ differ by a transposition. For example, the vertices $x((1,3,2,4)) = (2,0,1,0)$ and $x((3,1,2,4)) = (0,2,1,0)$ are adjacent vertices in $SR(4,3)|_{\mathrm{Parp}(\pi)}$, while $(1,3,2,4)$ and $(3,2,1,4)$ differ by the transposition $(1\ 3)$. This is true for any partial permutohedra for simplicial rook graphs. Finally, note that $SR(4,3)|_{\mathrm{Parp}(\pi)}$ is 3-regular and bipartite. As we will see, the subgraphs induced by partial permutohedra are always n-regular and bipartite.

Martin and Wagner also made the following conjecture regarding the subgraphs of $SR(m,n)$ induced by the partial permutohedra, which they verified for $m \leq 5$ (and $n \leq \binom{m}{2}$).

Conjecture 5.8. ([68, Conjecture 3.10]) *For each $\pi \in \mathcal{S}_{m,n}$, the subgraph induced by* $\mathrm{Parp}(\pi)$ *has integral spectrum.*

For the remainder of this section we present evidence in support of this conjecture. The conjecture is still unsolved.

Given $\sigma \in \mathrm{Adm}(\pi)$, the corresponding vertex $x(\sigma)$ in $SR(m,n)$ is $x(\sigma) = (x_1, \dots, x_m)$, where x_i is the number of tiles above the chosen rook placement (σ_i) in column i of $s(\pi)$. We can use this to prove the following lemma.

Lemma 5.9. *For any $\pi \in \mathcal{S}_{m,n}$, the induced subgraph $SR(m,n)|_{\mathrm{Parp}(\pi)}$ is bipartite and n-regular.*

Proof. Let $\sigma \in \mathrm{Adm}(\pi)$, so $x(\sigma) \in \mathrm{Parp}(\pi)$. The neighbors of $x(\sigma)$ in $SR(m,n)|_{\mathrm{Parp}(\pi)}$ are of the form $x(\rho)$, where $\rho \in \mathrm{Adm}(\pi)$. If $x(\rho)$ is adjacent to $x(\sigma)$, then ρ is a permutation corresponding to a rook placement in $s(\pi)$ identical to that of σ in all but two columns, say i and j. That is, $\rho_i = \sigma_j$, $\rho_j = \sigma_i$, and $\rho_\ell = \sigma_\ell$ for $\ell \neq i, j$, or in more compact form $\rho = (i\ j)\sigma$. This proves that $SR(m,n)|_{\mathrm{Parp}(\pi)}$ is bipartite, since adjacent vertices correspond to permutations with opposite signature. For each $i \in \{1, 2, \dots, m\}$, there are $a_i + i - \sigma_i$ empty tiles above the rook on tile σ_i, so there must be $a_i + i - \sigma_i$ columns j in $s(\pi)$ with $\sigma_j > \sigma_i$. Thus there are $a_i + i - \sigma_i$ permutations $\rho = (i\ j)\sigma$ that are proper tilings of $s(\pi)$. Summing over all $i \in \{1, 2, \dots, m\}$, we find that there are $\sum_{i=1}^{m}(a_i + i - \sigma_i) = n$ permutations ρ that differ from σ in exactly two entries and are proper tilings of $s(\pi)$, so $x(\sigma)$ has n neighbors in $SR(m,n)|_{\mathrm{Parp}(\pi)}$, which proves that the subgraph is n-regular. $\square$

The proof of Lemma 5.9 is contained in the proof in [68] that the signed characteristic vector of a partial permutohedron is an eigenvector of $SR(m,n)$. In addition to this general property of $SR(m,n)|_{\mathrm{Parp}(\pi)}$, we can determine more about the possible graphs $SR(m,n)|_{\mathrm{Parp}(\pi)}$ may be and how they arise.

Lemma 5.9 allows us to characterize the graphs $SR(m,3)|_{\mathrm{Parp}(\pi)}$.

Corollary 5.10. *For any $\pi \in \mathcal{S}_{m,3}$, the induced subgraph $SR(m,3)|_{\mathrm{Parp}(\pi)}$ is isomorphic to either the complete bipartite graph $K_{3,3}$ or the 3-cube Q_3.*

Proof. Because $K_{3,3}$ and Q_3 are the only 3-regular bipartite graphs on 6 and 8 vertices (it is straightforward to verify this, as there are only 2 cubic graphs on 6 vertices and 5 cubic graphs on 8 vertices [83]), respectively, it remains only to show that $|\mathrm{Parp}(\pi)| \in \{6,8\}$ for any $\pi \in \mathcal{S}_{m,3}$. Note that $|\mathrm{Parp}(\pi)| = |\mathrm{Adm}(\pi)|$. Fix $\pi \in \mathcal{S}_{m,3}$. Since $n = 3$, we know that $s(\pi)$ has exactly 3 tiles above the main diagonal. We consider cases based on the columns in which these tiles occur.

Case 1: The 3 tiles above the diagonal in $s(\pi)$ occur in 3 columns, $i < j < k$.

A proper rook placement σ on $s(\pi)$ must have $\sigma_\ell = \ell$ for all ℓ except possibly $\ell \in \{i, i+1, j, j+1, k, k+1\}$.

Case 1a: $j > i+1$ and $k > j+1$.

We have $\sigma_i, \sigma_{i+1} \in \{i, i+1\}$, $\sigma_j, \sigma_{j+1} \in \{j, j+1\}$, and $\sigma_k, \sigma_{k+1} \in \{k, k+1\}$. There are $2^3 = 8$ possibilities, so in this case $|\mathrm{Parp}(\pi)| = 8$.

Case 1b: $j = i+1$ and $k > j+1$.

We have $\sigma_i \in \{i, i+1\}$, $\sigma_j, \sigma_{j+1} \in \{i, i+1, i+2\} \setminus \{\sigma_i\}$, and $\sigma_k, \sigma_{k+1} \in \{k, k+1\}$. There are $2^3 = 8$ possibilities, so in this case $|\mathrm{Parp}(\pi)| = 8$.

Case 1c: $j > i+1$ and $k = j+1$.

We have $\sigma_i, \sigma_{i+1} \in \{i, i+1\}$, $\sigma_j \in \{j, j+1\}$, and $\sigma_k, \sigma_{k+1} \in \{j, j+1, j+2\} \setminus \{\sigma_j\}$. There are $2^3 = 8$ possibilities, so in this case $|\mathrm{Parp}(\pi)| = 8$.

Case 1d: $j = i+1$ and $k = j+1$.

We have $\sigma_i \in \{i, i+1\}$, $\sigma_j \in \{i, i+1, i+2\} \setminus \{\sigma_i\}$, and $\sigma_k, \sigma_{k+1} \in \{i, i+1, i+2, i+3\} \setminus \{\sigma_i, \sigma_j\}$. There are $2^3 = 8$ possibilities, so in this case $|\mathrm{Parp}(\pi)| = 8$.

Case 2: The 3 tiles above the diagonal in $s(\pi)$ occur in 2 columns: 1 in column i, 2 in column j.

A proper rook placement σ on $s(\pi)$ must have $\sigma_\ell = \ell$ for all ℓ except possibly $\ell \in \{i, i+1, j, j+1, j+2\}$.

Case 2a: $i+1<j$ or $i>j+2$.

We have $\sigma_i, \sigma_{i+1} \in \{i, i+1\}$, $\sigma_{j+1} \in \{j, j+1\}$, and $\sigma_j, \sigma_{j+2} \in \{j, j+1, j+2\} \setminus \{\sigma_{j+1}\}$. There are $2^3 = 8$ possibilities, so in this case $|\mathrm{Parp}(\pi)| = 8$.

Case 2b: $i+1=j$.

We have $\sigma_i \in \{i, i+1\}$, $\sigma_{j+1} \in \{i, i+1, i+2\} \setminus \{\sigma_i\}$, and $\sigma_j, \sigma_{j+2} \in \{i, i+1, i+2, i+3\} \setminus \{\sigma_i, \sigma_{j+1}\}$. There are $2^3 = 8$ possibilities, so in this case $|\mathrm{Parp}(\pi)| = 8$.

Case 2c: $i=j+1$.

We have $\sigma_j, \sigma_i, \sigma_{i+1} \in \{j, j+1, j+2\}$. There are $3! = 6$ possibilities, so in this case $|\mathrm{Parp}(\pi)| = 6$.

Case 2d: $i=j+2$.

We have $\sigma_{j+1} \in \{j, j+1\}$, $\sigma_j \in \{j, j+1, j+2\} \setminus \{\sigma_{j+1}\}$ and $\sigma_i, \sigma_{i+2} \in \{j, j+1, j+2, j+3\} \setminus \{\sigma_j, \sigma_{j+1}\}$. There are $2^3 = 8$ possibilities, so in this case $|\mathrm{Parp}(\pi)| = 8$.

Case 3: All 3 tiles above the diagonal in $s(\pi)$ occur in column i.

A proper rook placement σ on $s(\pi)$ must have $\sigma_\ell = \ell$ for all ℓ except possibly $\ell \in \{i, i+1, i+2, i+3\}$. We have $\sigma_{i+1} \in \{i, i+1\}$, $\sigma_{i+2} \in \{i, i+1, i+2\} \setminus \{\sigma_{i+1}\}$ and $\sigma_i, \sigma_{i+3} \in \{i, i+1, i+2, i+3\} \setminus \{\sigma_{i+1}, \sigma_{i+2}\}$. There are $2^3 = 8$ possibilities, so in this case $|\mathrm{Parp}(\pi)| = 8$.

This completes the proof. $\square$

We note that when $n = 3$, in almost all cases we find $|\mathrm{Parp}(\pi)| = 8$. We can show that $|\mathrm{Parp}(\pi)| = 6$ only in the case where, for some j, $s(\pi)$ has exactly $j+2$ tiles in columns j, $j+1$, and $j+2$, and exactly i tiles in column i for $i \neq j, j+1, j+2$. This corresponds to the case where π has 2 inversions at j, one inversion at $j+1$, and no other inversions, which occurs precisely when π is the transposition permutation $(j\ j+2)$. This idea generalizes to larger n in Proposition 5.13.

Proposition 5.11. *For any $\pi \in \mathcal{S}_{m,n}$, there is a permutation $\pi' \in \mathcal{S}_{m+1,n}$ such that $SR(m+1,n)|_{\mathrm{Parp}(\pi')} \cong SR(m,n)|_{\mathrm{Parp}(\pi)}$.*

Proof. Let $\pi \in \mathcal{S}_{m,n}$ and let π' be the permutation in $\mathcal{S}_{m+1}$ that acts on $\{1,\ldots,m\}$ in the same way as π and fixes $m+1$. Then $\pi' \in \mathcal{S}_{m+1,n}$, and we see that $s(\pi)$ and $s(\pi')$ are identical in the first m columns. In the $m+1$ column of $s(\pi')$ there are $m+1$ tiles, but any $\sigma' \in \mathrm{Adm}(\pi')$ must have $\sigma'_{m+1} = m+1$. There is a bijection between $\mathrm{Adm}(\pi)$ and $\mathrm{Adm}(\pi')$: for any $\sigma \in \mathrm{Adm}(\pi)$, there is a unique permutation $\sigma' \in \mathrm{Adm}(\pi')$ which acts on $\{1,\ldots,m\}$ in the same way as σ and fixes $m+1$. Clearly, for any $\rho, \sigma \in \mathrm{Adm}(\pi)$, the vertices $x(\rho')$ and $x(\sigma')$ of the corresponding $\rho', \sigma' \in \mathrm{Adm}(\pi')$ are adjacent if and only if $x(\rho)$ and $x(\sigma)$ are adjacent. Thus $SR(m+1,n)|_{\mathrm{Parp}(\pi')} \cong SR(m,n)|_{\mathrm{Parp}(\pi)}$. $\square$

Proposition 5.12. *For any $\pi \in \mathcal{S}_{m,n}$, there is a permutation $\pi' \in \mathcal{S}_{m+2,n+1}$ such that $SR(m+2,n+1)|_{\mathrm{Parp}(\pi')} \cong SR(m,n)|_{\mathrm{Parp}(\pi)} \Box K_2$.*

Proof. Let $\pi \in \mathcal{S}_{m,n}$ and let π' be the permutation in $\mathcal{S}_{m+2}$ that acts on $\{1,\ldots,m\}$ in the same way as π and inverts $m+1$ and $m+2$. Then $\pi' \in \mathcal{S}_{m+2,n+1}$, and we see that $s(\pi)$ and $s(\pi')$ are identical in the first m columns. In the $m+1$ and $m+2$ columns of $s(\pi')$ there are $m+2$ tiles, so any $\sigma' \in \mathrm{Adm}(\pi')$ may have either $\sigma'_{m+1} = m+1$ and $\sigma'_{m+2} = m+2$, or $\sigma'_{m+1} = m+2$ and $\sigma'_{m+2} = m+1$. We partition the set $\mathrm{Adm}(\pi')$ into two subsets. Let $\mathrm{Adm}(\pi')_1$ be the set of permutations in $\mathrm{Adm}(\pi')$ such that $\sigma'_{m+1} = m+1$ and $\sigma'_{m+2} = m+2$, and let $\mathrm{Adm}(\pi')_2$ be the set of permutations in $\mathrm{Adm}(\pi')$ such that $\sigma'_{m+1} = m+2$ and $\sigma'_{m+2} = m+1$. There is a bijection between $\mathrm{Adm}(\pi)$ and each of $\mathrm{Adm}(\pi')_1$ and $\mathrm{Adm}(\pi')_2$: for any $\sigma \in \mathrm{Adm}(\pi)$, there is a unique permutation $\sigma' \in \mathrm{Adm}(\pi')_1$ which acts on $\{1,\ldots,m\}$ in the same way as σ and fixes $m+1$ and $m+2$, and a unique permutation $\sigma'' \in \mathrm{Adm}(\pi')_2$ which acts on $\{1,\ldots,m\}$ in the same way as σ and inverts $m+1$ and $m+2$. Clearly, for any $\rho, \sigma \in \mathrm{Adm}(\pi)$, the vertices $x(\rho')$ and $x(\sigma')$ of the corresponding $\rho', \sigma' \in \mathrm{Adm}(\pi')_1$ are adjacent if and only if $x(\rho)$ and $x(\sigma)$ are adjacent, and similarly for ρ'' and $\sigma' \in \mathrm{Adm}(\pi')_2$. Further, the only neighbor of $x(\sigma')$ which arises from a permutation in $\mathrm{Adm}(\pi')_2$ is $x(\sigma'')$. Thus $SR(m+1,n)|_{\mathrm{Parp}(\pi')}$ is isomorphic to two copies of $SR(m,n)|_{\mathrm{Parp}(\pi)}$ connected by a perfect matching between corresponding vertices. That is, $SR(m+2,n+1)|_{\mathrm{Parp}(\pi')} \cong SR(d,n)|_{\mathrm{Parp}(\pi)} \Box K_2$. $\square$

Let T_k denote the k-th triangular number $T_k = \sum_{i=1}^{k} i$, and let CT_p denote the complete transposition graph on p letters, that is the graph whose vertices are the permutations in $\mathcal{S}_p$, where two vertices are adjacent if and only if they differ by a transposition. Seen another way, CT_p is the Cayley graph $\mathrm{Cay}(\mathcal{S}_p, \mathcal{T}_p)$, where $\mathcal{T}_p$ is the set of all transpositions in $\mathcal{S}_p$.

Proposition 5.13. *For every pair of integers $k \geq 1$ and $t \geq 0$, if $n = T_k + t$ then $SR(m, n)$ contains partial permutohedra isomorphic to $CT_{k+1}\square_t K_2$ for every $m \geq k + 1 + 2t$.*

Proof. If $n = T_k$ for some k, let $m = k + 1$ and let π be the permutation $(k + 1 = m, k = m - 1, \ldots, 2, 1) \in \mathcal{S}_{m,n}$. Then each of the m columns in $s(\pi)$ has m tiles. The resulting $\mathrm{Parp}(\pi)$ has as vertices the $m!$ permutations of $(0, 1, \ldots, m - 1)$ with two vertices adjacent if and only if they differ by a single transposition. Thus $\mathrm{Parp}(\pi) \cong CT_m = CT_{k+1}$. Now, if $n = T_k + t$ for any pair of integers $k \geq 1$ and $t \geq 0$, let $n' = T_k$, $m' = k + 1$, and $m = k + 1 + 2t$. By the same argument as before, $SR(m', n')$ contains partial permutohedra isomorphic to CT_{k+1}. By Proposition 5.12 applied t times, $SR(m, n)$ contains partial permutohedra isomorphic to $CT_{k+1}\square_t K_2$. By Proposition 5.11 this is also true for $m > k + 1 + 2t$. $\square$

For example, when $n = 1 = T_1$, we obtain partial permutohedra with $2! = 2$ vertices, the graph $CT_2 \cong Q_1 \cong K_2$. When $n = 3 = T_2$, we obtain partial permutohedra with $3! = 6$ vertices ,the graph $CT_3 \cong K_{3,3}$. When $n = 6 = T_3$, we obtain partial permutohedra with $4! = 24$ vertices.

Proposition 5.14. *For any fixed $n \in \mathbb{N}$, for any $m > 2n$ and $\pi \in \mathcal{S}_{m,n}$, the induced subgraph $SR(m, n)|_{\mathrm{Parp}(\pi)}$ is isomorphic to $SR(2n, n)|_{\mathrm{Parp}(\pi')}$ for some $\pi' \in \mathcal{S}_{2n,n}$.*

Proof. As noted above, $s(\pi)$ has exactly n tiles above the main diagonal. Suppose columns $u_1, u_2, \ldots, u_k$ have exactly $t_1, t_2, \ldots, t_k$ tiles above the diagonal, respectively, where $\sum_{i=1}^{k} t_i = n$, so no other columns have tiles above the diagonal. It is clear that any proper rook placement σ on $s(\pi)$ must satisfy $\sigma_\ell = \ell$ for any $\ell \notin U = \bigcup_{i=1}^{k}\{u_i, u_i +$

$1, \ldots, u_i + t_i\}$. The corresponding vertex $x(\sigma)$ in $\mathrm{Parp}(\pi)$ must satisfy $x_\ell = 0$ for $\ell \notin U$. This is true for every $\sigma \in \mathrm{Adm}(\pi)$, so the corresponding vertices $x(\sigma) \in \mathrm{Parp}(\pi)$ are all identical in coordinates not in U. Thus, if a coordinate not in U is removed from every vertex of $\mathrm{Parp}(\pi)$, the resulting induced subgraph graph is isomorphic to $SR(m,n)|_{\mathrm{Parp}}(\pi)$. We see that $|U| \leq 2n$. To finish the proof, we note by the above argument that for any $\ell \notin U$ we can remove column ℓ (and the corresponding row) from $s(\pi)$ without affecting the graph induced by $\mathrm{Parp}(\pi)$. We can remove columns and rows from $s(\pi)$ in this way until exactly $2n$ remain. The resulting skyline board has $2n$ columns and still has n tiles above the main diagonal, so it can be obtained as $s(\pi')$ for some $\pi' \in \mathcal{S}_{2n,n}$, and we have $SR(m,n)|_{\mathrm{Parp}(\pi)} \cong SR(2n,n)|_{\mathrm{Parp}(\pi')}$. □

Proposition 5.14 allows us to characterize the graphs $SR(m,n)|_{\mathrm{Parp}(\pi)}$ by checking only the graphs $SR(2n,n)|_{\mathrm{Parp}(\pi)}$. For example, we find a much faster proof of Corollary 5.10 (although it requires a computer search).

Alternate proof of Corollary 5.10. Using Sage we find that $SR(6,3)|_{\mathrm{Parp}(\pi)}$ is always isomorphic to one of the indicated graphs. □

Corollary 5.15. *For any $\pi \in \mathcal{S}_{m,4}$, the induced subgraph $SR(m,4)|_{\mathrm{Parp}(\pi)}$ is isomorphic to either $K_{3,3} \square K_2$ or Q_4.*

Proof. Using Sage we find that $SR(8,4)|_{\mathrm{Parp}(\pi)}$ is always isomorphic to one of the indicated graphs. □

Propositions 5.11, 5.12, 5.13, and 5.14 allow us to get an idea of the way in which many subgraphs induced by partial permutohedra arise in $SR(m,n)$. We see that for fixed n, the set of subgraphs induced by partial permutohedra in $SR(m,n)$ contains the set of subgraphs induced by partial permutohedra in $SR(k,n)$ for every $k < m$. Fixing n and increasing m results in a graph including the same subgraphs induced by partial permutohedra and may only add new (nonisomorphic to those already obtained) partial permutahedra until $m = 2n$. For every $m > 2n$, $SR(m,n)$ contains exactly the same partial permutohedra as $SR(2n,n)$. Incrementing n gives a graph with partial permutohedra isomorphic to $G\Box K_2$ for partial permutohedra G in the smaller graph. This implies, for example, that $SR(m,n)$ always contains partial permutohedra isomorphic to Q_n, since $SR(2,1)$ contains partial permutohedron Q_1.

$SR(2,1)$ contains only $CT_2 \cong Q_1$ as a partial permutohedron, and similarly $SR(4,2)$ contains only $CT_2\Box K_2 \cong Q_2$. As we have seen, $SR(6,3)$ contains $CT_2\Box_2 K_2 \cong Q_3$, $CT_3 \cong K_{3,3}$, and no others. $SR(8,4)$ contains only $CT_2\Box_3 K_2 \cong Q_4$ and $CT_3\Box K_2 \cong K_{3,3}\Box K_2$. Here it is tempting to hope that every partial permutohedra arises as in Proposition 5.13. However, while $SR(10,5)$ contains $CT_2\Box_4 K_2 \cong Q_5$ and $CT_3\Box_2 K_2 \cong K_{3,3}\Box_2 K_2$, it also contains a third type of partial permutohedron. Similarly, $SR(12,6)$ contains CT_4 and $G\Box K_2$ for each partial permutohedron G of $SR(10,5)$, but it also contains a fifth type of partial permutohedron.

While CT_2, CT_3, and CT_4 are integral, and $G\Box K_2$ is integral for any integral graph G, it is unclear whether CT_k is integral for general k or whether the partial permutohedra not arising as in Proposition 5.13 are integral. We checked using Sage and found that for $n \leq 8$, every partial permutohedron induced in $SR(2n,n)$ (and thus in $SR(m,n)$ for any m) is integral. Thus we confirmed Conjecture 5.8 for $n \leq 8$ for all m.

5.3 Problems and Questions on the Second Eigenvalue of Regular Graphs

We conclude with some open questions and problems related to the work on second eigenvalues of regular graphs.

Question 5.1. *What is the value of $v(k,\sqrt{k})$ for any value of k?*

We have $T(k,4,k-\sqrt{k}) = \sqrt{k}$ and $M(k,4,k-\sqrt{k}) = 2k^2 + k^{3/2} - k - \sqrt{k} + 1$, which yields

$$v(k,\sqrt{k}) \leq 2k^2 + k^{3/2} - k - \sqrt{k} + 1.$$

The Odd graph O_4 meets this bound (see Proposition 4.5 and Table 4.1). We do not know what other graphs, if any, meet this bound. Odd graphs, in general, do not have $T(k,t,c)$ as a quotient matrix.

Problem 5.2. *Classify the k-regular graphs with second eigenvalue more than 1 but less than $\sqrt{2}$ for each $k \geq 3$.*

Recall that for $k = 3$ no such graph exists. For $k > 3$ we note that Lemma 4.3 with $H = K_3$ implies that a graph G with $\lambda_2(G) < \sqrt{2}$ and girth 3 satisfies $|V(G)| \leq 3(k-1)\left(1 + \frac{k-2}{k-\sqrt{2}}\right)$, and Lemma 4.3 with $H = K_{1,2}$ implies that such a graph with girth more than 3 satisfies $|V(G)| \leq 3 + (3k-4)\left(1 + \frac{k-1}{k-\sqrt{2}}\right)$ (note that in both cases we have $\rho(H) > \lambda_2(G)$). Combining this with Lemma 1.1 allows one to restrict the search to graphs with certain girth. For $k \geq 6$, $n_l(k,g)$ is larger than these bounds unless the girth is at most 4, and for $k = 4$ or 5 $n_l(k,g)$ is larger than these bounds unless the girth is at most 5. Thus the graphs sought in Problem 5.2 must have girth at most 5 for $k = 4, 5$ and girth at most 4 for $k \geq 6$.

Question 5.3. *Is there a k-regular graph with second eigenvalue $\sqrt{2}$ for every $k \geq 3$?*

Recall that for $k = 3$ the Heawood suffices. Using similar argument to the one above, one finds that Lemma 4.3 with $H = K_3$ or $H = K_{1,3}$ (so that $\deg(H) > \sqrt{2}$) implies that the number of vertices in a graph G with $\lambda_2(G) \leq \sqrt{2}$ and girth 3 satisfies $|V(G)| \leq 3(k-1)\left(1 + \frac{k-2}{k-\sqrt{2}}\right)$, and with girth more than 3 satisfies $|V(G)| \leq 4 + 2(2k-3)\left(1 + \frac{k-1}{k-\sqrt{2}}\right)$. Then, combining with Lemma 1.1 and arguing as before we find that the graphs sought in Question 5.3 must have girth at most 5 for $k = 4, 5, 6$ and girth at most 4 for $k \geq 7$.

Question 5.4. *Among regular graphs, what is the smallest second eigenvalue larger than 1?*

Yu [98] found a 3-regular graph G on 16 vertices (see Figure 5.5) with smallest

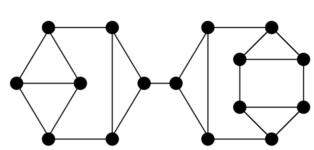


Figure 5.5: The unique 3-regular graph with largest least eigenvalue less than -2.

eigenvalue $\lambda_{\min} = \gamma \approx -2.0391$, where γ is the smallest root of $f(x) = x^6 - 3x^5 -$

$7x^4+21x^3+13x^2-35x-4$, and moreover proved that there is no connected, 3-regular graph with smallest eigenvalue in the interval $(\gamma,-2)$ (that is, among all connected, 3-regular graphs G has the largest least eigenvalue less than -2). Since the second eigenvalue of the complement of a regular graph is $\lambda_2=-1-\lambda_{\min}$ by Lemma 1.20, the complement $\overline{G}$ of G, a 12-regular graph on 16 vertices, has second eigenvalue $\lambda_2(\overline{G})=-1-\gamma\approx 1.0391$. We do not know if $\overline{G}$ has smallest second eigenvalue larger than 1 among regular graphs, but it is not unique. Indeed, the complement of the disjoint union $G+kK_4$ of G and k copies of K_4 is a connected, $(12+4k)$-regular graph on $16+4k$ vertices and second eigenvalue $\lambda_2(\overline{G+kK_4})=-1-\gamma$, so we have found an infinite family of regular graphs with second eigenvalue $-1-\gamma$.

BIBLIOGRAPHY

[1] A. Abdollahi, S. Janbaz, and M.R. Oboudi, Graphs cospectral with a friendship graph or its complement, *Transactions on Combinatorics* **2** (2013), 37–52.

[2] O. Ahmadi, N. Alon, I.F. Blake, and I.E. Shparlinski, Graphs with integral spectrum, *Lin. Alg. Appl.* **430** (2009), 544–546.

[3] N. Alon, Eigenvalues and expanders, *Combinatorica* **6** (1986), 83–96.

[4] S. Axler, *Linear Algebra Done Right, 2nd Ed.*, Springer, 2004.

[5] K. Balińska, D. Cvetković, Z. Radosavljević, S. Simić, and D. Stevanović, A survey on integral graphs, *Univ. Beograd. Publ. Elektrotehn. Fak.* **12** (2002), 42–65.

[6] M. Bataineh, An alternative proof of the Friendship Theorem, *Jordan Journal of Mathematics and Statistics* **2** (2009), 25–32.

[7] F. Belardo, B. Borovicanin, Q.X. Huang and J.F. Wang, On the two largest Q-eigenvalues of graphs, *Discrete Math.* **310** (2010), 2858-2866.

[8] C.T. Benson, Minimal regular graphs of girths eight and twelve, *Canad. J. Math.* **18** (1966), 1091–1094.

[9] N.L. Biggs, A.G. Boshier, and J. Shawe-Taylor, Cubic distance-regular graphs, *J. London Math. Soc. (2)* **33** (1986), 385-394.

[10] S.R. Blackburn, M.B. Paterson and D.R. Stinson, Putting dots in triangles, *J. Combin. Math. Combin. Comput.* **78** (2011), 23–32.

[11] B. Borovićanin, Line graphs with exactly two positive eigenvalues, *Publ. Inst. Math. (Beograd) (N.S.)* **72(86)** (2002), 39–47.

[12] A.E. Brouwer, The uniqueness of the strongly regular graph on 77 points, *J. Graph Theory* **7** (1983), no. 4, 455–461.

[13] A.E. Brouwer, S.M. Cioabă, W.H. Haemers, and J.R. Vermette, Notes on simplicial rook graphs, preprint at `http://arxiv.org/abs/1408.5615`.

[14] A.E. Brouwer, A.M. Cohen, and A. Neumaier, *Distance-regular graphs*, Springer, 1989.

[15] A.E. Brouwer and J.H. Koolen, The distance-regular graphs of valency four, *J. Alg. Combin.* **10** (1999), 5–24.

[16] A.E. Brouwer and W.H. Haemers, The Gewirtz graph: an exercise in the theory of graph spectra, *European J. Combin.* **14** (1993), 397–407.

[17] A.E. Brouwer and W.H. Haemers, Spectra of Graphs, Springer, 2012.

[18] J.M. Brunat, A proof of the friendship theorem by elementary methods. (Catalan) *Butll. Soc. Catalana Mat.* **7** (1992), 75–80.

[19] F.C. Bussemaker, D.H. Cvetković, and J.J. Seidel, Graphs related to exceptional root systems, *Report TH Eindhoven* 76-WSK-05, 1976.

[20] P.J. Cameron, *Combinatorics: Topics, Techniques, Algorithms.* Cambridge, 1994.

[21] P.J. Cameron, J.M. Goethals, J.J. Seidel, and E.E. Shult, Line graphs, root systems, and elliptic geometry, *J. Algebra* **43** (1976), 305–327.

[22] L.C. Chang, The uniqueness and nonuniqueness of triangular association schemes, *Sci. Record* **3** (1959) 604–613.

[23] L.C. Chang, Association schemes of partially balanced block designs with parameters $v = 28$, $n_1 = 12$, $n_2 = 15$ and $p^2_{11} = 4$, *Sci. Record* **4** (1960) 12–18.

[24] H. Chuang and G.R. Omidi, Graphs with three distinct eigenvalues and largest eigenvalue less than 8, *Lin. Alg. Appl.* **430** (2009) 2053–2062.

[25] S.M. Cioabă, On the extreme eigenvalues of regular graphs. *J. Combin. Theory Ser. B* **96** (2006), 367–373.

[26] S.M. Cioabă, W.H. Haemers, J.R. Vermette, and W. Wong, The graphs with all but two eigenvalues equal to ± 1, *J. Alg. Combin.* **41** (2015), 887–897.

[27] S.M. Cioabă, J.H. Koolen, H. Nozaki, and J.R. Vermette, Large regular graphs of given valency and second eigenvalue, preprint at `http://arxiv.org/abs/1503.06286`.

[28] H. Cohn and A. Kumar, Universally optimal distribution of points on spheres, *J. Amer. Math. Soc.* **20** (2007), 99–184.

[29] L. Collatz and U. Sinogowitz, Spektren endlicher Grafen. *Abh. Math. Sem. Univ. Hamburg* **21** (1957), 63–77.

[30] E.R. Van Dam, Nonregular graphs with three eigenvalues, *J. Combin. Theory Ser. B* **73** (1998) 101–118.

[31] E.R. van Dam and W.H. Haemers, Which graphs are determined by their spectrum? *Linear Algebra Appl.* **373** (2003), 241-272.

[32] E.R. van Dam and W.H. Haemers, Developments on spectral characterizations of graphs, *Discrete Math.* **309** (2009), 576-586.

[33] E.R. van Dam and E. Spence, Combinatorial designs with two singular values – I: uniform multiplicative designs, *J. Combinatorial Theory A* **107** (2004), 127–142.

[34] R.M. Damerell and M.A. Georgiacodis, On the maximum diameter of a class of distance-regular graphs, *Bull. London Math. Soc.* **13** (1981), 316–322.

[35] K.C. Das, Proof of conjectures on adjacency eigenvalues of graphs, *Discrete Math.* **313** (2013), 19–25.

[36] N. Elkies, *Graph theory glossary,* `http://www.math.harvard.edu/~elkies/FS23j.05/glossary_graph.html`

[37] N. Elkies, Answer to *Hexagonal Rooks*, Mathoverflow, August 1, 2012, `http://mathoverflow.net/questions/103540/`

[38] P. Erdős, A. Rényi and V. Sós, On a problem of graph theory, *Studia Sci. Math. Hungar.* **1** (1966), 215–235.

[39] F. Esser and F. Harary, On the spectrum of a complete multipartite graph, *Europ. J. Combinatorics* **1** (1980) 211–218.

[40] K. Fan, On a theorem of Weyl concerning eigenvalues of linear transformations I, *Proc. Nat. Acad. Sci.* **35** (1949), 652–655.

[41] A. Gewirtz, Graphs with maximal even girth, *Canad. J. Math.* **21** (1969), 915–934.

[42] C.D. Godsil, *Algebraic Combinatorics,* Chapman & Hall, 1993.

[43] C.D. Godsil, Answer to *What is the independence number of hamming graph?*, Mathoverflow, September 11, 2013, `http://mathoverflow.net/questions/141842/`

[44] C.D. Godsil and B.D. McKay, Constructing cospectral graphs, *Aequationes Math.* **25** (1982), 257–268.

[45] C.D. Godsil and G. Royle, *Algebraic graph theory*, Graduate Texts in Mathematics 207, Springer-Verlag, New York, 2001.

[46] H.H. Günthard and H. Primas, Zusammenhang von Graphentheorie und MO-Theorie von Molekeln mit Systemen konjugierter Bindungen, *Helv. Chim. Acta.* **39** (1956), 1645–1653.

[47] X. Gui, X. Liu, and Y. Zhang, The multi-fan graphs are determined by their Laplacian spectra, *Discrete Math.* **308** (2008), 4267–4271.

[48] W.H. Haemers, *Eigenvalue Techniques in Design and Graph Theory*, Math. Centre Tract 121, Mathematical Centre, Amsterdam, 1980.

[49] W.H. Haemers, Interlacing eigenvalues and graphs, *Lin. Alg. Appl.* **226–228** (1995), 593–616.

[50] W.H. Haemers, private communication.

[51] D.G. Higman and C.C. Sims, A simple group of order 44,352,000, *Math. Z.* **105** (1968), 110–113.

[52] A.J. Hoffman and R.R. Singleton, On Moore graphs with diameters 2 and 3, *IBM J. Res. Develop.* **4** (1960), 497–504.

[53] T. Høholdt and J. Justesen, On the sizes of expander graphs and minimum distances of graph codes, *Discrete Math.* **325** (2014), 38–46.

[54] S. Hoory, N. Linial and A. Wigderson, Expander graphs and their applications, *Bull. Amer. Math. Soc.* (2006) **46**, 439–561.

[55] J. E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge, 1990.

[56] C. Huneke, The friendship theorem, *The American Mathematical Monthly* **109** (2002) 192–194.

[57] T.P. Kirkman, On a problem in combinatorics. *Cambridge Dublin Math. J.* **2** (1847), 191–204.

[58] T. Koledin and Z. Staníc, Regular graphs whose second largest eigenvalue is at most 1, *Novi Sad J. Math.* **43** (2013), 145–153.

[59] T. Koledin and Z. Staníc, Regular graphs with small second largest eigenvalue, *Appl. Anal. Discrete Math.* **7** (2013), 235–249.

[60] C. Laisant, Sur la numération factorielle, application aux permutations (french), *Bulletin de la Société Mathématique de France* **16** (1888), 176–183.

[61] P. Lax, *Linear Algebra and its Applications, 2nd Ed.*, Wiley, 2007.

[62] D.H. Lehmer, Teaching combinatorial tricks to a computer, *Proc. Sympos. Appl. Math. Combinatorial Analysis, Amer. Math. Soc.* **10** (1960), 179–193.

[63] J.H. van Lint and R.M. Wilson, *A Course in Combinatorics, 2nd ed.*, Cambridge, 2006.

[64] J.Q. Longyear and T.D. Parsons, The friendship Theorem, *Indagationes Mathematicae (Proceedings)* **75** (1972) 257–262.

[65] L. Lovász, Spectra of graphs with transitive groups, *Periodica Mathematica Hungarica.* **6** (1975), 191–195.

[66] A. Lubotzky, R. Phillips, P. Sarnak, Ramanujan graphs, *Combinatorica* **8** (1988), 261–277.

[67] A. Marcus, D.A. Spielman, N. Srivastava, Interlacing Families I: Bipartite Ramanujan Graphs of All Degrees, *Annals of Math.*, to appear, also at `arXiv:1304.4132`

[68] J.L. Martin and J.D. Wagner, On the spectral of simplicial rook graphs, *25th International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2013)*, 373–386, *Discrete Math. Theor. Comput. Sci. Proc.*

[69] W.F. McGee, A minimal cubic graph of girth seven, *Canad. Math. Bull.* **3** (1960), 149–152.

[70] G.B. Mertzios and W. Unger, The friendship problem on graphs, *Proceedings of the International Conference on Relations, Orders and Graphs: Interaction with Computer Science*, Mahdia, Tunisia, 2008.

[71] B. Mohar, A strengthening and a multipartite generalization of the Alon–Boppana–Serre theorem. *Proc. Amer. Math. Soc.* **138** (2010), 3899–3909.

[72] A. Moon, Characterization of the odd graphs O_k by parameters, *Discrete Math.* **42** (1982), 91–97.

[73] A. Nilli, On the second eigenvalue of a graph, *Discrete Math.* **91** (1991), 207–210.

[74] A. Nilli, Tight estimates for eigenvalues of regular graphs. *Electron. J. Combin.* **11** (2004), no. 1, Note 9.

[75] G. Nivasch and E. Lev, Non-attacking queens on a triangle, *Mathematics Magazine* **78** (2005), 399–403.

[76] H. Nozaki, Linear programming bounds for regular graphs, available at `arXiv:1407.4562`

[77] P. Paule and M. Schorn, A Mathematica Version of Zeilbergers Algorithm for Proving Binomial Coefficient Identities, *J. Symbolic Comput.* **20** (1995), 673–698.

[78] M.M. Petrović, The Spectrum of Infinite Complete Multipartite Graphs, *Publ. Inst. Math. (Belgrade) (N.S.)* **31** (1982) 169–176.

[79] F. Ramazani and B. Tayfeh-Rezaie, Spectral characterization of some cubic graphs, *Graphs and Combin.* **28** (2012), 869–876.

[80] D.K. Ray-Chaudhuri and R.M. Wilson, Solution of Kirkman's schoolgirl problem. *Combinatorics, Proc. Sympos. Pure Math., Univ. California, Los Angeles, Calif., 1968* **19** (1971), 187–203.

[81] J. Richey, N. Shutty, and M. Stover, Finiteness theorems in spectral graph theory, available at `arXiv:1306.6548`.

[82] N. Robertson, The smallest graph of girth 5 and valency 4, *Bull. Amer. Math. Soc.* **70** (1964), 824–825.

[83] G. Royle, *Cubic Graphs*, available at:
`http://staffhome.ecm.uwa.edu.au/~00013890/remote/cubics/`

[84] A.J. Schwenk, Almost all trees are cospectral, in: F. Harary (Ed.), New Directions in the Theory of Graphs, Academic Press, New York, 1973, pp. 275–307.

[85] J.J. Seidel, Strongly regular graphs with $(-1, 1, 0)$ adjacency matrix having eigenvalue 3, *Linear Algebra and Appl.* **1** (1968), 281–298.

[86] J. Serre, Répartition asymptotique des valeurs propres de l'opérateur de Hecke T_p (French) [Asymptotic distribution of the eigenvalues of the Hecke operator T_p], *J. Amer. Math. Soc.* **10** (1997), 75–102.

[87] R. Singleton, On minimal graphs of maximum even girth, *J. Combin. Theory* **1** (1966), 306–332.

[88] N.J.A. Sloane, The On-Line Encyclopedia of Integer Sequences, published electronically at `http://oeis.org`, 2014.

[89] Z. Staníc, On regular graphs and coronas whose second largest eigenvalue does not exceed 1, *Linear and Multilinear Algebra* **58** (2010), 545–554.

[90] R.P. Stanley, *Enumerative Combinatorics*, Cambridge, 1999.

[91] D. Stevanović, Applications of graph spectra in quantum physics, *Selected topics on applications of graph spectra* (D. Cvetković, I. Gutman, eds.), Collection of Papers Vol. 14 (22), Mathematical Institute of the Serbian Academy of Sciences and Arts, Belgrade (2011), 85–112.

[92] Y. Teranishi and F. Yasuno, The second largest eigenvalues of regular bipartite graphs, *Kyushu J. Math.* **54** (2000), 39–54.

[93] W.T. Tutte, A family of cubical graphs, *Proc. Cambridge Phil. Soc.* **43** (1947), 459–474.

[94] W.T. Tutte, *Connectivity in Graphs*, University of Toronto Press, 1966.

[95] P. Vaderlind, R. Guy and L. Larson, *The Inquisitive Problem Solver*, MAA Problem Books Series 2002.

[96] J.J. Watkins, *Across the Board: The Mathematics of Chessboard Problems*, Princeton, 2012.

[97] H. Wilf, The friendship theorem, 1971 Combinatorial Mathematics and its Applications (Proc. Conf., Oxford, 1969) pp. 307–309, Academic Press, London.

[98] H. Yu, On the limit points of the smallest eigenvalues of regular graphs, *Des. Codes Cryptogr.* **65** (2012), no. 1–2, 77–88.

[99] D. Zeilberger, A Fast Algorithm for Proving Terminating Hypergeometric Series Identities. *Discrete Math.* **80**, (1990), 207-211.