SCATTERING AND INVERSE SCATTERING IN THE PRESENCE OF COMPLEX BACKGROUND MEDIA

by

Fan Yang

A dissertation submitted to the Faculty of the University of Delaware in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Applied Mathematics

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ABSTRACT

Scattering and inverse scattering theory plays a central role in mathematical physics. For example, through the use of acoustic or electromagnetic waves, one can detect and identify objects that are hard to or cannot directly be observed as well as obtain information about the material properties of objects of interest.

However, in practical applications, the presence of complex background media in which the problems are considered restricts us from applying directly the existing theoretical results and numerical methods. This constraint requires delicate modification of well-established theory and development of alternate computational approaches.

In this thesis, we investigate the applicability of qualitative methods in inverse scattering theory for obtaining material properties and recovering shapes of unknown objects by using time-harmonic electromagnetic waves under different geometrical configurations. In particular, we first consider a 2D model where a bounded dielectric scatterer sits on an infinite metallic substrate. This is a model problem for non-destructive testing of aircraft coatings. We validate the application of the Linear Sampling Method (LSM) for detecting special frequencies called transmission eigenvalues for both isotropic and anisotropic media. Then we move to a 3D model where a bounded perfectly electric conducting object is located inside an infinite long perfectly electric conducting waveguide and justify the application of the LSM for reconstructing the shape of the object.

For both cases, we show that additional work needs to be done in order to recast standard results from scattering and inverse scattering theory to the model problems we consider. Also, this work gives an idea of the effort needed to adapt academic research to industrial applications.

Chapter 1

INTRODUCTION

1.1 Historical Review

Scattering and inverse scattering problems arise in various areas of mathematics and physics and have a long history. Examples include the non-destructive testing of materials [31, 45], medical imaging [46, 50], remote sensing [28, 56] and seismic exploration [34, 47]. As an important and attractive field, the mathematical theory of shape identification and the determination of material properties has progressed rapidly over the last few decades.

This thesis focuses on the use of electromagnetic waves governed by Maxwell's equations to probe remote objects. In order to better understand the inverse problem, it is worth mentioning that the corresponding forward problem is to find the total field and scattered field due to a known incident field when we know the material parameter(s) describing the scattering object together with its shape. In our case this forward problem is linear and well posed. The inverse problems we shall study uses measurements of the scattered field resulting from the interaction of a known incident field with an unknown scatterer in order to probe this object. From this data we are to recover the shape of the scattering object as well, possibly, as parameters pertinent to the scatterer such as its relative permittivity and permeability. The inverse problem is generally ill-posed and non-linear.

In developing inversion algorithms for target identification, the first techniques to emerge were based on the weak scattering approximation and in particular using the popular Born approximation [5, 27]. This is computationally efficient and often very successful. However, if the weak scattering approximation does not hold, the resulting inversion schemes may be inaccurate and result in poor reconstructions [60]. As a common alternative, constrained optimization methods (see, e.g., [6, 30]) are well developed and usually require an iterative approach together with solution of the forward problem at least once during each iteration. This approach can deal with a variety of constraints and is applicable with reduced measurement data, such as only one incident wave. But it is computationally expensive and so may be very slow and may also stop at local minima. Moreover, this method requires a priori information that may not be available such as the number of connected components of the scattering object and the type of boundary condition on the object.

More recently, a new class of approaches which avoid possibly incorrect linearization but which seek only partial information about the scattering object (optimization approaches try to recover a complete description of the scatterer) have been developed. These are now referred to as "qualitative methods" and usually need very limited a priori data. They are relatively faster than optimization approaches but require substantially more input data (that is, more measured data). Examples of such methods include, for instance, the Linear Sampling Method (LSM) [20, 25], the Factorization Method [40, 41], and the Method of Singular Sources [53, 54]. This thesis will focus on the LSM, and we shall give more background on the LSM later in thesis introduction.

1.2 Time Harmonic Electromagnetic Waves

For the model problems we shall consider in the sequel, the fields (incident, scattering and total fields) are time-harmonic electromagnetic waves and thus governed by Maxwell's equations. Depending on the application, these fields will satisfy different versions of Maxwell's equations depending on the assumption on the application. To connect the various models, we start with the full time dependent Maxwell's system in $\mathbb{R}^3$. In the time domain, the space and time dependent quantities $\mathcal{D}, \mathcal{H}, \mathcal{B}, \mathcal{E}$ and $\mathcal{J}$ and $\tilde{\rho}$ satisfy

$$\frac{\partial \mathcal{D}}{\partial t} - \nabla \times \mathcal{H} = -\mathcal{J}, \tag{1.1}$$

$$
\begin{aligned}
\frac{\partial \mathcal{B}}{\partial t} + \nabla \times \mathcal{E} &= 0, && (1.2)\\
\nabla \cdot \mathcal{B} &= 0, && (1.3)\\
\nabla \cdot \mathcal{D} &= \tilde{\rho}, && (1.4)
\end{aligned}
$$

where $\mathcal{D}$ is the electric displacement, $\mathcal{H}$ is the magnetic field, $\mathcal{J}$ is the current density; $\mathcal{B}$ is the magnetic induction, $\mathcal{E}$ is the electric field and $\tilde{\rho}$ is the charge density. Here equations (1.3) and (1.4) are consequences of equations (1.1) and (1.2) provided initial and boundary conditions are consistent with them.

Furthermore, we assume the following linear constitutive relations:

$$
\begin{aligned}
\mathcal{D} &= \tilde{\varepsilon}\mathcal{E} \quad \text{where } \tilde{\varepsilon} \text{ is permittivity}, && (1.5)\\
\mathcal{B} &= \tilde{\mu}\mathcal{H} \quad \text{where } \tilde{\mu} \text{ is permeability}, && (1.6)\\
\mathcal{J} &= \tilde{\sigma}\mathcal{E} \quad \text{where } \tilde{\sigma} \text{ is conductivity (Ohm's law)}, && (1.7)
\end{aligned}
$$

and $\tilde{\varepsilon}, \tilde{\mu}$ and $\tilde{\sigma}$ are given functions of position describing the electromagnetic properties of the medium.

Plugging (1.5) and (1.7) into (1.1) and (1.6) into (1.2), we obtain

$$
\begin{aligned}
\tilde{\varepsilon}\frac{\partial \mathcal{E}}{\partial t} + \tilde{\sigma}\mathcal{E} - \nabla \times \mathcal{H} &= 0, && (1.8)\\
\tilde{\mu}\frac{\partial \mathcal{H}}{\partial t} + \nabla \times \mathcal{E} &= 0. && (1.9)
\end{aligned}
$$

Now let $\mathcal{F}[\,\cdot\,]$ denote the Fourier transform in *time*, that is,

$$
\mathcal{F}[f] = \int_{-\infty}^{\infty} \exp(i\omega t) f(t)\, dt.
$$

Applying the Fourier transform in time to (1.8) - (1.9), and recalling the assumption that $\tilde{\varepsilon}$, $\tilde{\sigma}$ and $\tilde{\mu}$ are independent of time, we obtain

$$
\begin{aligned}
-i\omega\tilde{\varepsilon}\mathcal{F}[\mathcal{E}] + \tilde{\sigma}\mathcal{F}[\mathcal{E}] - \nabla \times \mathcal{F}[\mathcal{H}] &= 0,\\
-i\omega\tilde{\mu}\mathcal{F}[\mathcal{H}] + \nabla \times \mathcal{F}[\mathcal{E}] &= 0.
\end{aligned}
$$

We denote by ϵ_0, μ_0 the positive and constant electromagnetic coefficients for free space. Setting $\mathbf{E} = \sqrt{\epsilon_0}\,\mathcal{F}[\mathcal{E}]$, $\mathbf{H} = \sqrt{\mu_0}\,\mathcal{F}[\mathcal{H}]$, then substituting for $\mathcal{F}[\mathcal{E}]$ and $\mathcal{F}[\mathcal{H}]$ above gives

$$\begin{aligned} -i\omega\left(\tilde{\varepsilon} + i\frac{\tilde{\sigma}}{\omega}\right)\sqrt{\epsilon_0\mu_0}\,\frac{1}{\epsilon_0}\mathbf{E} - \nabla\times\mathbf{H} &= 0, \\ -i\omega\tilde{\mu}\sqrt{\epsilon_0\mu_0}\,\frac{1}{\mu_0}\mathbf{H} + \nabla\times\mathbf{E} &= 0. \end{aligned}$$

Let k denote the wave number for the background medium given by $k = \omega\sqrt{\epsilon_0\mu_0}$. Then defining the relative quantities $\varepsilon = \left(\frac{\tilde{\varepsilon}}{\epsilon_0} + i\frac{\tilde{\sigma}}{\omega\epsilon_0}\right)$ and $\mu = \frac{\tilde{\mu}}{\mu_0}$ (so $\varepsilon = 1$ and $\mu = 1$ in the background), we end up with the time harmonic system of Maxwell equations as follows: the unknown spatially dependent complex valued vector fields $\mathbf{E}$ and $\mathbf{H}$ satisfy

$$-ik\varepsilon\mathbf{E} - \nabla\times\mathbf{H} = 0, \tag{1.10}$$

$$-ik\mu\mathbf{H} + \nabla\times\mathbf{E} = 0. \tag{1.11}$$

Generally, in $\mathbb{R}^3$, we have

$$\mathbf{E}(\mathbf{x}) = \begin{pmatrix} E_1(x,y,z) \\ E_2(x,y,z) \\ E_3(x,y,z) \end{pmatrix}, \mathbf{H}(\mathbf{x}) = \begin{pmatrix} H_1(x,y,z) \\ H_2(x,y,z) \\ H_3(x,y,z) \end{pmatrix},$$

and ε is a matrix function of position.

Using (1.11) to replace $\mathbf{H}$ in (1.10), we can eliminate $\mathbf{H}$ and the time harmonic Maxwell's system can be reduced to the following second order system of equations:

$$\nabla\times(\mu^{-1}\nabla\times\mathbf{E}) - k^2\varepsilon\mathbf{E} = 0. \tag{1.12}$$

In certain geometric settings, equation (1.12) can be further simplified. Suppose that the coefficients ε and μ are independent of z (that is, the medium is translation invariant) and that we only seek solutions $\mathbf{E}$ and $\mathbf{H}$ propagating on the (x,y) plane and individually independent of z. We also assume that $\mu = 1$ (no magnetic components are present) and the medium is orthotropic so that

$$\varepsilon = \begin{pmatrix} \varepsilon_{11}(x,y) & \varepsilon_{12}(x,y) & 0 \\ \varepsilon_{12}(x,y) & \varepsilon_{22}(x,y) & 0 \\ 0 & 0 & \varepsilon_{33}(x,y) \end{pmatrix}.$$

In this case the vector Maxwell's equation (1.12) can be decomposed into two scalar equations in $\mathbb{R}^2$. The two models are referred to as Transverse Electric (TE) mode scattering and Transverse Magnetic (TM) mode scattering respectively in electrical engineering.

Using all these assumptions, for the TE case, the magnetic field is given by

$$\mathbf{H} = \mathbf{H}(x,y) = \begin{pmatrix} H_1(x,y) \\ H_2(x,y) \\ 0 \end{pmatrix},$$

and the electric field is given by

$$\mathbf{E} = \mathbf{E}(x,y) = \begin{pmatrix} 0 \\ 0 \\ E_3(x,y) \end{pmatrix},$$

where $E_3(x,y) = -\dfrac{1}{ik\varepsilon_{33}}\left(\dfrac{\partial H_2(x,y)}{\partial x} - \dfrac{\partial H_1(x,y)}{\partial y}\right)$.

Direct calculation then shows that $E_3(x,y)$ satisfies

$$\Delta E_3 + k^2 \varepsilon_{33} E_3 = 0.$$

For the TM case, the electric field is given by

$$\mathbf{E} = \mathbf{E}(x,y) = \begin{pmatrix} E_1(x,y) \\ E_2(x,y) \\ 0 \end{pmatrix},$$

and the magnetic field is given by

$$\mathbf{H} = \mathbf{H}(x,y) = \begin{pmatrix} 0 \\ 0 \\ H_3(x,y) \end{pmatrix},$$

where $H_3(x,y) = \dfrac{1}{ik\mu}\left(\dfrac{\partial E_2(x,y)}{\partial x} - \dfrac{\partial E_1(x,y)}{\partial y}\right)$.

Again direct calculation shows that $H_3(x,y)$ satisfies

$$\nabla \cdot (A\nabla H_3) + k^2 H_3 = 0,$$

where A is a 2×2 matrix obtained from ε given by

$$A = \frac{1}{det(\epsilon)}\epsilon \quad \text{where} \quad \epsilon = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{12} & \varepsilon_{22} \end{pmatrix}.$$

1.3 Review of Qualitative Methods for Inverse Scattering Problems

Before presenting the details of our work, we shall first review the general historical background of qualitative methods using a simple model problem in standard settings.

As an illustration, consider the following forward scattering problem for a simply connected bounded domain D (for the TE case in 2D where, following Colton and Kress [19], we use the notation $n(\mathbf{x}) = \varepsilon_{33}$): find u and u^s such that

$$\begin{cases} \Delta u + k^2 n(\mathbf{x}) u = 0 & \text{in} \quad D, \\ \Delta u^s + k^2 u^s = 0 & \text{in} \quad \mathbb{R}^2 \backslash \bar{D}, \\ u = u^i + u^s & \text{on} \quad \partial D, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial}{\partial \nu}(u^i + u^s) & \text{on} \quad \partial D, \\ \sqrt{r}\left(\dfrac{\partial u^s}{\partial r} - iku^s\right) \to 0 & \text{as} \quad r := |\mathbf{x}| \to \infty, \end{cases} \tag{1.13}$$

where u^i is the incident field assumed to satisfy the 2D Helmholtz equation $\Delta u^i + k^2 u^i = 0$ except possibly for isolated point(s) outside D. A typical incoming wave is the plane wave

$$u^i = \exp(ik\mathbf{x} \cdot \mathbf{d}), \tag{1.14}$$

where $\mathbf{d}$ ($|\mathbf{d}| = 1$) is the direction of propagation of the wave.

If there exists a frequency k and incident field u^i (not necessarily a plane wave) such that the scattering field u^s is zero, then $w := u|_D$ and $v := u^i|_D$ satisfy the following homogeneous problem

$$\begin{cases} \Delta w + k^2 n(\mathbf{x}) w = 0 & \text{in} \quad D, \\ \Delta v + k^2 v = 0 & \text{in} \quad D, \\ w - v = 0 & \text{on} \quad \partial D, \\ \dfrac{\partial w}{\partial \nu} - \dfrac{\partial v}{\partial \nu} = 0 & \text{on} \quad \partial D. \end{cases} \tag{1.15}$$

This problem is referred to as the Interior Transmission Eigenvalue problem for the Helmholtz equation and any value k such that this homogeneous problem has nontrivial solutions is then referred to as a Transmission Eigenvalue.

The interior transmission problem was first discussed by Colton and Monk [21, 22] and Kirsch [39] in the mid 1980s in connection with inverse scattering for acoustic waves in an inhomogeneous medium. For almost two decades after this, most results on the interior transmission problem were concerned with well-posedness of the non-homogeneous interior transmission problem. Concerning transmission eigenvalues themselves, by an application of the analytic Fredholm theory, one can often show that the transmission eigenvalues form at most a discrete set with infinity as the only possible accumulation point [19, 24]. Relatively recently, Päivärinta and Sylvester [52] showed that, in the case of scalar isotropic media a finite number of transmission eigenvalues exist provided the index of refraction is large enough. Kirsch then extended this existence result to the case of anisotropic media for both the scalar case and Maxwell's equations [42]. Subsequently, Cakoni and Haddar presented a general proof for the existence of transmission eigenvalues for a wide class of scattering problems [17]. Meanwhile, Cakoni, Colton and Haddar investigated the difficult case of a medium with cavities, i.e. regions with zero contrast [12]. Soon, Cakoni and Gintides refined this proof by removing the assumption on the size of the index of refraction [15]. Together with Haddar, they proved the existence of an infinite discrete set of transmission eigenvalues and provided new results on monotonicity properties of the eigenvalues [16]. This latter paper opened the way to determine properties of the index of refraction from transmission eigenvalues.

To obtain partial information on the material properties of the scattering object, one can try to use measurements of the interior transmission eigenvalues. This problem arises in inverse scattering theory for inhomogeneous media [21]. Of particular interest is the spectrum associated with the interior transmission eigenvalue problem, more specifically the existence of eigenvalues and their dependence on the material properties of the scatterer. On the one hand, it is important to know that transmission eigenvalues form a discrete set because one needs to avoid those frequencies that correspond to transmission eigenvalues in, for example, standard sampling methods for reconstructing the support of the scatterer. On the other hand, it is important to know whether the eigenvalues exist and to understand their connection with the index of refraction because one can then try to use the transmission eigenvalues to obtain information about physical properties of the scatterer [11]. Either way, the spectral properties of the interior transmission problem have become an interesting and current question in inverse scattering theory [18].

To recover the support of the scattering object, the Linear Sampling Method (LSM) turns out to be an efficient approach. To describe the rationale behind this method, we recall that the last condition in (1.13) (also called the Sommerfeld radiation condition) is imposed on the scattered field. This implies an asymptotic expansion of the scattered field u^s (given u^i the plane wave (1.14)) as follows:

$$u^s(\mathbf{x}) = \frac{e^{ikr}}{r}\left\{u_\infty(\hat{\mathbf{x}}, \mathbf{d}) + \mathcal{O}\left(\frac{1}{r}\right)\right\} \text{ as } r := |\mathbf{x}| \to \infty \text{ and } \hat{\mathbf{x}} = \frac{\mathbf{x}}{|\mathbf{x}|}.$$

Here $u_\infty(\hat{\mathbf{x}}, \mathbf{d})$ is called the Far Field Pattern (FFP) of the scattered wave.

The classical LSM is then based on the following Far Field Equation (FFE): find a function $g_z \in L^2(\mathcal{S})$ such that

$$(Fg_{\mathbf{z}})(\hat{\mathbf{x}}) := \int_{\mathcal{S}} u_\infty(\hat{\mathbf{x}}, \mathbf{d}) g_z(\mathbf{d})\, ds(\mathbf{d}) = \Phi_\infty(\hat{\mathbf{x}}, \mathbf{z}) \quad \text{for all } \hat{\mathbf{x}} \in \mathcal{S}, \tag{1.16}$$

where $\mathcal{S}$ is the unit circle and $\Phi_\infty(\hat{\mathbf{x}}, \mathbf{z})$ is the Far Field Pattern (FFP) of the field due to a point source located at the auxiliary point (sampling point) $\mathbf{z}$.

In 2D, for instance, the field due to a point source in vacuum is given by the radiating fundamental solution to the Helmholtz equation defined by $\Phi(\mathbf{x},\mathbf{z}) := \dfrac{i}{4}H_0^{(1)}(k|\mathbf{x}-\mathbf{z}|), \mathbf{x} \neq \mathbf{z}$ where $H_0^{(1)}$ is the zeroth order Hankel function of the first kind. Denoting the direction of the incident plane wave by $\mathbf{d} = (\cos(\phi), \sin(\phi))$ and the observation direction $\hat{\mathbf{x}} = (\cos(\theta), \sin(\theta))$, the FFE becomes

$$\int_0^{2\pi} u_\infty(\theta,\phi) g_z(\phi)\, d\phi = \gamma \exp(-ikr_{\mathbf{z}} \cos(\theta - \theta_{\mathbf{z}})) \quad \text{for } \theta \in [0, 2\pi),$$

where $\gamma = \dfrac{\exp(i\pi/4)}{\sqrt{8\pi k}}$ and $(r_{\mathbf{z}}, \theta_{\mathbf{z}})$ are the polar coordinates of sampling point $\mathbf{z}$.

Using the Far Field Equation (FFE), the reconstruction of the shape of D and its transmission eigenvalues can be extracted from the function $g_{\mathbf{z}}$.

To present the theoretical underpinning for this statement, we first need some more assumptions on the function $n(\mathbf{x})$. Following [10], let $n(\mathbf{x}) \in C(\bar{D})$ (note this can be generalized to $n(\mathbf{x}) > c > 0$ a.e. in D such that $n(\mathbf{x}) \in L^\infty(D)$ and $1/|n(\mathbf{x}) - 1| \in L^\infty(D)$, see [16]). In addition, let

$$n_* := \inf_{\mathbf{x}\in D} n(\mathbf{x}) > 0, \qquad n^* := \sup_{\mathbf{x}\in D} n(\mathbf{x}) < \infty.$$

Further assume that

$$\mathfrak{Im}\,(n(\mathbf{x})) = 0 \quad \text{and} \quad \text{either} \quad 0 < n^* < 1 \quad \text{or} \quad n_* > 1. \tag{1.17}$$

Also define the Herglotz wave function corresponding to $g_{\mathbf{z}} \in L^2[0, 2\pi]$ by

$$w_{g_{\mathbf{z}}}(\mathbf{x}) = \int_0^{2\pi} \exp(ik\mathbf{x}\cdot\mathbf{d}) g_{\mathbf{z}}(\phi)\, d\phi \quad \text{where} \quad \mathbf{d} = (\cos(\phi), \sin(\phi)).$$

Then we have the following theorems:

Theorem 1.3.1 *(TE case of Theorem 6.50 in [10]) Assume that D is a bounded domain having a C^2-boundary ∂D such that $\mathbb{R}^2 \backslash \bar{D}$ is connected, and n satisfies assumption (1.17). Furthermore, assume that k is not a transmission eigenvalue corresponding to the homogeneous interior transmission problem (1.15). Then we have that*

- *For* $\mathbf{z} \in D$ *and a given* $\epsilon > 0$, *there exists a function* $g_{\mathbf{z}}^{\epsilon} \in L^2[0, 2\pi]$ *such that*

$$\|Fg_{\mathbf{z}}^{\epsilon} - \Phi_{\infty}(\cdot, \mathbf{z})\|_{L^2[0,2\pi]} < \epsilon,$$

  and the Herglotz wave function $w_{g_{\mathbf{z}}^{\epsilon}}(\mathbf{x})$ *with kernel* $g_{\mathbf{z}}^{\epsilon}$ *converges in* $H^1(D)$ *to a function* $v \in H^1(D)$ *as* $\epsilon \to 0$, *where* (w, v) *is the unique solution of the following interior transmission problem*

$$\begin{cases} \Delta w + k^2 n(\mathbf{x}) w = 0 & in \quad D, \\ \Delta v + k^2 v = 0 & in \quad D, \\ w - v = \Phi(\cdot, \mathbf{z}) & on \quad \partial D, \\ \dfrac{\partial w}{\partial \nu} - \dfrac{\partial v}{\partial \nu} = \dfrac{\partial \Phi(\cdot, \mathbf{z})}{\partial \nu} & on \quad \partial D. \end{cases}$$

- *For* $\mathbf{z} \notin D$ *and a given* $\epsilon > 0$, *every function* $g_{\mathbf{z}}^{\epsilon} \in L^2[0, 2\pi]$ *that satisfies*

$$\|Fg_{\mathbf{z}}^{\epsilon} - \Phi_{\infty}(\cdot, \mathbf{z})\|_{L^2[0,2\pi]} < \epsilon$$

  is such that

$$\lim_{\epsilon \to 0} \|w_{g_{\mathbf{z}}^{\epsilon}}\|_{H^1(D)} = \infty,$$

  where again $w_{g_{\mathbf{z}}^{\epsilon}}$ *is the Herglotz wave function with kernel* $g_{\mathbf{z}}^{\epsilon}$.

Theorem 1.3.2 *(TE case of Theorem 6.51 in [10]) Assume that* n *satisfies assumption* (1.17). *Let* k *be a transmission eigenvalue corresponding to the homogeneous interior transmission problem* (1.15) *and* $g_{\mathbf{z}}^{\epsilon}$ *satisfy*

$$\|Fg_{\mathbf{z}}^{\epsilon} - \Phi_{\infty}(\cdot, \mathbf{z})\|_{L^2[0,2\pi]} < \epsilon.$$

Then for every $\mathbf{z} \in D$, *except possibly for a nowhere dense set,* $\|w_{g_{\mathbf{z}}^{\epsilon}}\|_{H^1(D)}$ *cannot be bounded as* $\epsilon \to 0$. *Here* $w_{g_{\mathbf{z}}^{\epsilon}}$ *is the Herglotz wave function with kernel* $g_{\mathbf{z}}^{\epsilon}$.

From Theorem 1.3.1, we see that the shape of D can be determined by the norm of $g_{\mathbf{z}}^{\epsilon}$ as it will be unbounded if the sampling point $\mathbf{z} \notin D$ and bounded for $\mathbf{z} \in D$ as $\epsilon \to 0$. Together with Theorem 1.3.2, Theorem 1.3.1 also suggests that for almost every $\mathbf{z} \in D$, $w_{g_{\mathbf{z}}^{\epsilon}}$ behaves differently if k is a transmission eigenvalue where the norm of $g_{\mathbf{z}}^{\epsilon}$ will be unbounded and if k is not a transmission eigenvalue where the norm of $g_{\mathbf{z}}^{\epsilon}$ will be bounded.

The Linear Sampling Method (LSM) described above was first introduced by Colton and Kirsch [20]. Its origin traces back to the Dual Space Method developed by Colton and Monk during the late 80's. This qualitative method drew considerable attention as a novel approach to inverse scattering theory due to its many advantages such as the requirement of no priori assumptions about the material or the geometry of the scattering object. Also, the numerical implementation of the LSM is very simple and fast since as seen from (1.16), sampling is done by solving an ill-posed linear integral equation for each sampling point $\mathbf{z}$. Many books (see, e.g., [10, 13, 19, 43, 48]) have included this approach as a way of determining the shape of an unknown object. Later on, the LSM extended to a broad range of applications. For instance, using limited aperture data on a subset of $\mathcal{S}$ in (1.16) [10] or using near field data excited by point sources [55]. The employment of the LSM under different geometric settings is also investigated such as inside an acoustic waveguide [49] (for inverse source problem, see [8]). Particularly, the waveguide effect arises due to the presence of the boundary of the waveguide which separates the wave into propagating modes and evanescent modes. Since only a finite number of propagating modes can be captured at long distance while all the other evanescent modes decay exponentially, it increases the ill-posedness of the reconstruction of the scatterer.

1.4 Framework of the Thesis

In this thesis, we shall investigate two model inverse scattering problems. The first concerns the determination of transmission eigenvalues in non-destructive testing and the second seeks to justify the LSM for reconstructing an unknown scattering object in a waveguide. Both involve scattering in non-constant or "complex" background media.

In Chapter 2, we consider an interior transmission problem arising in a non-destructive testing application. Specifically, it corresponds to inverse scattering for a bounded isotropic dielectric medium lying on an infinite perfectly conducting surface. The novelty here is that a mixed boundary condition appears due to the presence of the perfect electric conducting surface. This configuration has also been considered for the modeling of near field optical microscopes [29] and the simulation of the radiation of an antenna situated on a large metallic structure [4]. In particular, we investigate the 2D scalar case of this problem where, in the corresponding scattering problem, the dielectric medium is illuminated by time harmonic TE or TM polarized electromagnetic waves respectively. In both cases we formulate the interior transmission problem for the appropriate Helmholtz equation and show that the transmission eigenvalues form an infinite discrete set. We also derive an analogue of the Faber-Krahn inequality by converting the problem to a fourth order elliptic equation. We also show the existence of these eigenvalues by adapting the proof in [16] with necessary modifications. Lastly, we conduct various numerical experiments related to finding the first real transmission eigenvalue for both TE and TM scattering. We show that real transmission eigenvalues can be found from near field data, although in some cases the accuracy requirements on the data is very stringent.

In Chapter 3, we are concerned with applying the LSM in a 3D electromagnetic waveguide with bounded cross-section. This is motivated by practical applications, for example, the detection of clogs or defects in petroleum pipes buried under the sea floor. Analogous to the analysis for a 2D acoustic waveguide by Bourgeois and Lunéville [9], the aim is to understand how the LSM must be modified in the case of a 3D electromagnetic waveguide. This generalization is far from trivial in a number of places. To name a few, in the direct problem, the Rellich's Lemma (see Lemma 2.12 in [19]) for proving the uniqueness of the solution to the forward problem does not hold in a waveguide due to the fact that the information carried by the exponentially decaying modes cannot be captured in the far field. Also the problem requires a good understanding of the background dyadic Green's function and an elaboration of procedures for factorizing the near field operator where the near field data is collected on a cross section of the waveguide far away from the scattering object.

We start by justifying the forward problem. Importantly, we show the well-posedness of the forward problem because standard results for free space don't apply here. Then we move to the corresponding inverse problem to show how to adapt the LSM to the inverse electromagnetic waveguide problem. In particular, we analyze the background dyadic Green's function of the waveguide, its decomposition in the vicinity of singularity (point source), the representation formula of the scattered field as well as the reciprocity property to prove a uniqueness theorem for determining the unknown scatterer. Then we employ a factorization of the near field operator to justify the LSM for the waveguide. Finally, we describe a numerical approach we use to produce synthetic scattering data and numerical results for the reconstruction of an unknown scatterer.

In the final chapter of the thesis, we end up with some open problems and potential future work as well as mentioning other inverse problems that may be of interest.

Chapter 2

DIELECTRIC SCATTERER ON A CONDUCTING PLANE

2.1 Interior Transmission Problem

2.1.1 Configuration of the Problem

We consider a Perfect Electric Conductor (PEC) backed dielectric scattering object illuminated by point source(s). Let $D \subset \mathbb{R}^2$ be a bounded open set having a piecewise smooth Lipschitz boundary $\Gamma = \Gamma_a \cup \Gamma_m$ such that Γ_a is the interface between the background dielectric medium D_a and the domain D and Γ_m the interface between the infinite perfect electric conducting substrate D_m and D (see Figure 2.1). We assume that $D_m = \{(x, y)|y < 0\}$. The unit normal vector to ∂D directed into the exterior of D is denoted by ν.

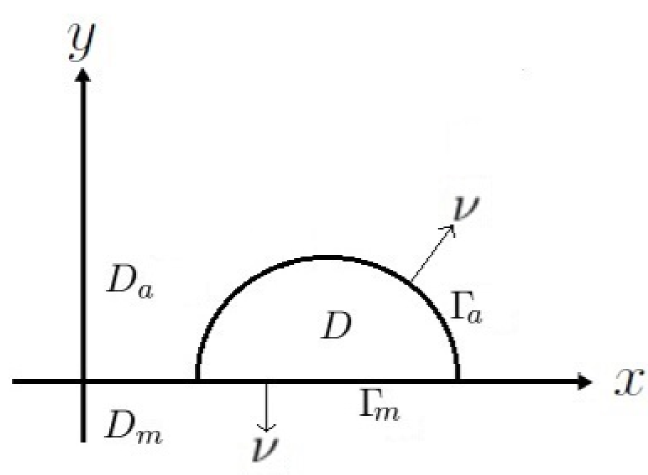

Figure 2.1: Configuration of plane supported domain.

If D is illuminated by time harmonic Transverse-Electric (TE) polarized electromagnetic waves, the corresponding scattering problem is for an isotropic inhomogeneous media. The scattering problem for inhomogeneous media illuminated by time harmonic Transverse-Magnetic (TM) polarized electromagnetic waves can give rise to an anisotropic problem. We will discuss these case by case in Section 2.1.4 and Section 2.1.5, respectively.

Remark 2.1.1 *Our proofs of the discreteness and existence of transmission eigenvalues in the following sections do not require Γ_m to be a segment of the x-axis. For example, Γ_m could be the finite union of smooth arcs. However, we impose the assumption that Γ_m is a segment of the x-axis because we use this fact in some later proofs (see Corollary 2.1.2 for TE case and Corollary 2.1.4 for TM case) and for the numerical results (see Section 2.2).*

2.1.2 Function Spaces and Preliminary Results

Concerning the spaces we will be using, first for $u, v \in L^2(D)$, let

$$(u,v)_D = \int_D u\,\bar{v}\,dx \quad \text{and} \quad \langle u,v\rangle_\Gamma = \int_\Gamma u\,\bar{v}\,ds$$

where the overbar denotes complex conjugate and denote $\Gamma = \Gamma_a \cup \Gamma_m$. Next, we introduce the usual energy spaces

$$\begin{aligned} H^1(D) &:= \{u \in L^2(D) \mid \nabla u \in (L^2(D))^2\}, \\ H^1_0(D) &:= \{u \in H^1(D) \mid u = 0 \text{ on } \Gamma\}. \end{aligned}$$

For the scalar isotropic case, we have the following Sobolev spaces

$$\begin{aligned} H(\mathrm{div}, D) &:= \{\mathbf{u} \in (L^2(D))^2 \mid \nabla\cdot\mathbf{u} \in L^2(D)\}, \\ H_0(\mathrm{div}, D) &:= \{\mathbf{u} \in H(\mathrm{div}, D) \mid \nu\cdot\mathbf{u} = 0 \text{ on } \Gamma\}, \\ H_{0a}(\mathrm{div}, D) &:= \{\mathbf{u} \in H(\mathrm{div}, D) \mid \nu\cdot\mathbf{u} = 0 \text{ on } \Gamma_a\}, \end{aligned}$$

and

$$\mathcal{H}(D) \;:=\; \{u \in H^1(D) \mid \nabla u \in H(\mathrm{div}, D)\}$$

$$
\begin{aligned}
&:= \{u \in L^2(D), \nabla u \in (L^2(D))^2, \nabla \cdot (\nabla u) = \Delta u \in L^2(D)\}, \\
\mathcal{H}_0(D) &:= \{u \in H_0^1(D) \mid \nabla u \in H_0(\mathrm{div}, D)\} \\
&:= \{u \in L^2(D), \nabla u \in (L^2(D))^2, \nabla \cdot (\nabla u) = \Delta u \in L^2(D), \\
&\quad\; u = 0 \text{ on } \Gamma, \nu \cdot \nabla u = 0 \text{ on } \Gamma\}, \\
\mathcal{H}_{0a}(D) &:= \{u \in H_0^1(D) \mid \nabla u \in H_{0a}(\mathrm{div}, D)\} \\
&:= \{u \in L^2(D), \nabla u \in (L^2(D))^2, \nabla \cdot (\nabla u) = \Delta u \in L^2(D), \\
&\quad\; u = 0 \text{ on } \Gamma, \nu \cdot \nabla u = 0 \text{ on } \Gamma_a\},
\end{aligned}
$$

equipped with the inner product

$$
(u, v)_{\mathcal{H}(D)} := (u, v)_D + (\nabla u, \nabla v)_D + (\Delta u, \Delta v)_D.
$$

Here $\mathcal{H}(D)$, $\mathcal{H}_0(D)$ and $\mathcal{H}_{0a}(D)$ are all Hilbert spaces, and $\mathcal{H}_0(D)$ is equivalent to the classical Sobolev space $H_0^2(D)$ (see, e.g., [16]).

For the scalar anisotropic case, for $\mathbf{u}, \mathbf{v} \in (L^2(D))^2$, let

$$
(\mathbf{u}, \mathbf{v})_D = \int_D \mathbf{u} \cdot \bar{\mathbf{v}} \, dx \quad \text{and} \quad \langle \mathbf{u}, \mathbf{v} \rangle_\Gamma = \int_\Gamma \mathbf{u} \cdot \bar{\mathbf{v}} \, ds.
$$

And we introduce the following Sobolev spaces

$$
H_{0a}^1(D) := \{u \in H^1(D) \mid u = 0 \text{ on } \Gamma_a\},
$$

and

$$
\begin{aligned}
\mathcal{G}(D) &:= \{\mathbf{u} \in H(\mathrm{div}, D) \mid \nabla \cdot \mathbf{u} \in H^1(D)\} \\
&:= \{\mathbf{u} \in (L^2(D))^2, \nabla \cdot \mathbf{u} \in L^2(D), \nabla(\nabla \cdot \mathbf{u}) \in (L^2(D))^2\}, \\
\mathcal{G}_{0a}(D) &:= \{\mathbf{u} \in H_0(\mathrm{div}, D) \mid \nabla \cdot \mathbf{u} \in H_{0a}^1(D)\} \\
&:= \{\mathbf{u} \in (L^2(D))^2, \nabla \cdot \mathbf{u} \in L^2(D), \nabla(\nabla \cdot \mathbf{u}) \in (L^2(D))^2, \\
&\quad\; \nu \cdot \mathbf{u} = 0 \text{ on } \Gamma, \nabla \cdot \mathbf{u} = 0 \text{ on } \Gamma_a\},
\end{aligned}
$$

equipped with the inner product

$$
(\mathbf{u}, \mathbf{v})_{\mathcal{H}(D)} := (\mathbf{u}, \mathbf{v})_D + (\nabla \cdot \mathbf{u}, \nabla \cdot \mathbf{v})_D + (\nabla\nabla \cdot \mathbf{u}, \nabla\nabla \cdot \mathbf{v})_D.
$$

With these definitions, we can further define the following continuous sesquilinear forms on $\mathcal{H}_{0a}(D) \times \mathcal{H}_{0a}(D)$ and $\mathcal{G}_{0a}(D) \times \mathcal{G}_{0a}(D)$, respectively:

$$\begin{aligned} \mathcal{C}(u, \xi) &:= (\nabla u, \nabla \xi)_D, && (2.1) \\ \mathcal{N}(\mathbf{u}, \eta) &:= (\nabla \cdot \mathbf{u}, \nabla \cdot \eta)_D . && (2.2) \end{aligned}$$

Let us denote by C and N the bounded linear operators from $\mathcal{H}_{0a}(D)$ to $\mathcal{H}_{0a}(D)$ and $\mathcal{G}_{0a}(D)$ to $\mathcal{G}_{0a}(D)$, respectively, defined using the Riesz representation theorem (see Theorem C.0.1) by

$$(Cu, \xi)_{\mathcal{H}_{0a}(D)} = \mathcal{C}(u, \xi) \qquad (2.3)$$

for all $\xi \in \mathcal{H}_{0a}(D)$ and

$$(N\mathbf{u}, \eta)_{\mathcal{G}_{0a}(D)} = \mathcal{N}(\mathbf{u}, \eta)$$

for all $\eta \in \mathcal{G}_{0a}(D)$.

Then, we have the compactness of the following operators:

Lemma 2.1.1 *$C : \mathcal{H}_{0a}(D) \longrightarrow \mathcal{H}_{0a}(D)$ is a compact operator.*

Proof: Let u_n be a bounded sequence in $\mathcal{H}_{0a}(D)$. Hence there exists a subsequence, denoted again by u_n, which converges weakly to u^0 in $\mathcal{H}_{0a}(D)$. Since ∇u_n is also bounded in $(H^1(D))^3$, from the Rellich compactness theorem we have that a suitable subsequence again denoted ∇u_n converges strongly to ∇u^0 in $(L^2(D))^3$. But

$$\begin{aligned} \|C(u_n - u^0)\|^2_{\mathcal{H}_{0a}(D)} &= (C(u_n - u^0), C(u_n - u^0))_{\mathcal{H}_{0a}(D)} \\ &= \mathcal{C}(u_n - u^0, C(u_n - u^0)) \\ &= (\nabla(u_n - u^0), \nabla[C(u_n - u^0)])_D \\ &\leq \|\nabla(u_n - u^0)\|_{L^2(D)} \|\nabla[C(u_n - u^0)]\|_{L^2(D)} \\ &\leq \|\nabla(u_n - u^0)\|_{L^2(D)} \|C(u_n - u^0)\|_{\mathcal{H}_{0a}(D)}, \end{aligned}$$

which implies

$$\|C(u_n - u^0)\|_{\mathcal{H}_{0a}(D)} \leq \|\nabla(u_n - u^0)\|_{L^2(D)}.$$

This proves that Cu_n converges strongly to Cu^0 and therefore C is compact. $\square$

Remark 2.1.2 *As an alternate definition, one can also define the continuous sesquilinear form on $H^1(D) \times H^1(D)$ such that*

$$\mathcal{C}(u, \xi) := (\nabla u, \nabla \xi)_D,$$

then the corresponding bounded linear operator C as in (2.3) will be from $H^1(D)$ to $H^1(D)$ and is also compact. However, in order to facilitate the analysis in the sequel, we shall use the definition given by (2.1).

Lemma 2.1.2 *$N : \mathcal{G}_{0a}(D) \longrightarrow \mathcal{G}_{0a}(D)$ is a compact operator.*

Proof: The proof is similar to the proof of Lemma 3.2 in [11]. □

2.1.3 Poincaré Type Inequality

Before discussing transmission eigenvalue problem for the geometric setting in Figure 2.1, we shall state the following Poincaré type inequalities which summarize the essential differences with the standard transmission eigenvalue problem:

Lemma 2.1.3 *For $u \in \mathcal{H}_{0a}(D)$, we have that*

$$\|\nabla u\|^2_{L^2(D)} \leq \frac{1}{\lambda(D)} \|\Delta u\|^2_{L^2(D)}, \tag{2.4}$$

where $\lambda(D)$ is the first eigenvalue of the buckled plate eigenvalue problem with Dirichlet boundary condition on Γ, Neumann boundary condition on Γ_a and Laplacian boundary condition on Γ_m. Specifically:

$$\begin{cases} -\Delta^2 v = \lambda \Delta v & \text{in} \quad D, \\ v = 0 & \text{on} \quad \Gamma, \\ \dfrac{\partial v}{\partial \nu} = 0 & \text{on} \quad \Gamma_a, \\ \Delta v = 0 & \text{on} \quad \Gamma_m. \end{cases}$$

Proof: First, from the Riesz representation theorem (see Theorem C.0.1), we define two operators as follows:

$$\begin{cases} A \quad : \mathcal{H}_{0a}(D) \longrightarrow \mathcal{H}_{0a}(D) & \text{bounded linear operator such that} \\ & (Au, v)_{\mathcal{H}(D)} = (\Delta u, \Delta v)_D \text{ for } u, v \in \mathcal{H}_{0a}(D), \\ B \quad : \mathcal{H}_{0a}(D) \longrightarrow \mathcal{H}_{0a}(D) & \text{bounded linear operator such that} \\ & (Bu, v)_{\mathcal{H}(D)} = (\nabla u, \nabla v)_D \text{ for } u, v \in \mathcal{H}_{0a}(D). \end{cases}$$

Then we have that

- The operator A is self-adjoint, positive definite (using the Poincaré type inequality (2.4) in $H^1_0(D)$).
- The operator B is self-adjoint, non-negative, compact (choosing B in place of C in Lemma 2.1.1).

Next, consider the following eigenvalue problem: find $\lambda \in \mathbb{R}$ and non-trivial $\phi \in \mathcal{H}_{0a}(D)$ such that

$$(\Delta\phi, \Delta\psi)_D = \lambda(\nabla\phi, \nabla\psi)_D \text{ for any } \psi \in \mathcal{H}_{0a}(D). \tag{2.5}$$

Then, by using the definition of operators A and B above, we have

$$(A\phi, \psi)_{\mathcal{H}} = \lambda(B\phi, \psi)_{\mathcal{H}} \text{ if and only if } A\phi = \lambda B\phi \text{ in } \mathcal{H}_{0a}(D).$$

Applying Theorem 2.1 and Theorem 2.2 in [17], we get

$$\frac{(A\phi, \phi)_{\mathcal{H}}}{(B\phi, \phi)_{\mathcal{H}}} = \frac{(\Delta\phi, \Delta\phi)_D}{(\nabla\phi, \nabla\phi)_D} \geq \lambda_1,$$

where $(\lambda_k)_{k\geq 1}$ (increasing sequence of positive real numbers) are the eigenvalues of $A\phi_k = \lambda_k B\phi_k$ and ϕ_k's are the corresponding eigenfunctions. Then

$$(\nabla\phi, \nabla\phi)_D \leq \frac{1}{\lambda_1}(\Delta\phi, \Delta\phi)_D.$$

Thus we have the desired Poincaré Inequality.

To justify that λ_k corresponds to the eigenvalues for the buckled plate eigenvalue problem stated above, given $\phi \in \mathcal{H}_{0a}(D)$ smooth enough (for example, $\phi \in H^4(D) \cap \mathcal{H}_{0a}(D)$) that satisfies the variational form in (2.5), integration by parts gives

$$\left(\Delta^2\phi + \lambda\Delta\phi, \psi\right)_D + \left\langle \Delta\phi, \frac{\partial\psi}{\partial\nu} \right\rangle_\Gamma = 0.$$

Choosing $\psi \in \mathcal{H}_0(D)$ so that $\nabla\psi \in H_0(\text{div}, D) \subset H_{0a}(\text{div}, D)$, we have

$$\left(\Delta^2\phi + \lambda\Delta\phi, \psi\right)_D = 0 \text{ in } \mathcal{H}_0(D).$$

Since $\mathcal{H}_0(D)$ is dense in $L^2(D)$, we have

$$\Delta^2\phi + \lambda\Delta\phi = 0 \text{ in the } L^2 \text{ sense.}$$

This leads to

$$\left\langle \Delta\phi, \frac{\partial\psi}{\partial\nu} \right\rangle_\Gamma = 0.$$

Now, since $\psi \in \mathcal{H}_{0a}(D)$ is arbitrary, we have $\Delta\phi|_{\Gamma_m} = 0$. The proof is done. $\square$

Note 2.1.1 *From Lemma 2.1.3 and the classical Poincaré inequality on $H_0^1(D)$, we can see that the norm $\|u\| = \|\Delta u\|_{L^2(D)}$ is equivalent to the one we defined for $\mathcal{H}_0(D)$ and $\mathcal{H}_{0a}(D)$.*

Lemma 2.1.4 *For $\nabla \cdot \mathbf{u} \in H_{0a}^1(D)$, we have that*

$$\|\nabla \cdot \mathbf{u}\|_{L^2(D)}^2 \leq \frac{1}{\mu(D)} \|\nabla\nabla \cdot \mathbf{u}\|_{L^2(D)}^2,$$

where $\mu(D)$ is the first eigenvalue of $-\Delta$ on D with Dirichlet boundary condition on Γ_a and Neumann boundary condition on Γ_m. Specifically:

$$\begin{cases} -\Delta v = \mu v & \textit{in} \quad D, \\ v = 0 & \textit{on} \quad \Gamma_a, \\ \dfrac{\partial v}{\partial \nu} = 0 & \textit{on} \quad \Gamma_m. \end{cases}$$

Proof: First, from the Riesz representation theorem (see Theorem C.0.1), we define two operators as follows:

$$\begin{cases} A \ : H_{0a}^1(D) \longrightarrow H_{0a}^1(D) & \text{bounded linear operator such that} \\ & (Au, v)_{H^1(D)} = (\nabla u, \nabla v)_D \text{ for } u, v \in H_{0a}^1(D), \\ B \ : H_{0a}^1(D) \longrightarrow H_{0a}^1(D) & \text{bounded linear operator such that} \\ & (Bu, v)_{H^1(D)} = (u, v)_D \text{ for } u, v \in H_{0a}^1(D). \end{cases}$$

Then we have that

- The operator A is self-adjoint, positive definite (using the standard Poincaré inequality in $H^1_{0a}(D)$ since Γ_a is assumed to have positive measure).
- The operator B is self-adjoint, non-negative, compact (choosing B in place of N in Lemma 2.1.2).

Next, consider the following eigenvalue problem: find $\mu \in \mathbb{R}$ and non-trivial $\phi \in H^1_{0a}(D)$ such that

$$(\nabla\phi, \nabla\psi)_D = \mu(\phi, \psi)_D \text{ for any } \psi \in H^1_{0a}(D). \tag{2.6}$$

Then, by using the definition of operators A and B above, we have

$$(A\phi, \psi)_{H^1} = \mu(B\phi, \psi)_{H^1} \text{ if and only if } A\phi = \mu B\phi \text{ in } H^1_{0a}(D).$$

Applying Theorem 2.1 and Theorem 2.2 in [17], we get

$$\frac{(A\phi, \phi)_{H^1}}{(B\phi, \phi)_{H^1}} = \frac{(\nabla\phi, \nabla\phi)_D}{(\phi, \phi)_D} \geq \mu_1,$$

where $(\mu_k)_{k\geq 1}$ (increasing sequence of positive real numbers) are the eigenvalues of $A\phi_k = \mu_k B\phi_k$ and ϕ_k's are the corresponding eigenfunctions. Then

$$(\phi, \phi)_D \leq \frac{1}{\mu_1}(\nabla\phi, \nabla\phi)_D.$$

Thus we have the desired Poincaré Inequality.

To justify that μ_k corresponds to the eigenvalues of $-\Delta$ stated above, given $\phi \in H^1_{0a}(D)$ smooth enough (for example, $\phi \in H^2(D) \cap H^1_{0a}(D)$) that satisfies the variational form in (2.6), integration by parts gives

$$(-\Delta\phi - \mu\phi, \psi)_D + \left\langle \frac{\partial\phi}{\partial\nu}, \psi \right\rangle_\Gamma = 0.$$

Choosing $\psi \in H^1_0(D) \subset H^1_{0a}(D)$, we have

$$(-\Delta\phi - \mu\phi, \psi)_D = 0 \text{ in } H^1_0(D).$$

Since $H^1_0(D)$ is dense in $L^2(D)$, we have

$$-\Delta\phi - \mu\phi = 0 \text{ in the } L^2 \text{ sense.}$$

This leads to

$$\left\langle \frac{\partial \phi}{\partial \nu}, \psi \right\rangle_{\Gamma} = 0.$$

Now, since $\psi \in H^1_{0a}(D)$ is arbitrary, we have $\left.\dfrac{\partial \phi}{\partial \nu}\right|_{\Gamma_m} = 0$. The proof is done. □

2.1.4 Scalar Isotropic Media

2.1.4.1 Formulation of the Problem

The interior transmission eigenvalue problem corresponding to the scattering problem for an isotropic inhomogeneous medium (TE mode electromagnetic scattering) with configuration as in Section 2.1.1 in $\mathbb{R}^2$ reads: find w and v in suitable function spaces such that (here $\mathbf{x} = (x, y)$):

$$\Delta w + k^2 n(\mathbf{x}) w = 0 \quad \text{in } D, \tag{2.7}$$

$$\Delta v + k^2 v = 0 \quad \text{in } D, \tag{2.8}$$

$$w - v = 0 \quad \text{on } \Gamma_a, \tag{2.9}$$

$$\frac{\partial w}{\partial \nu} - \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \Gamma_a, \tag{2.10}$$

$$w = 0, v = 0 \quad \text{on } \Gamma_m. \tag{2.11}$$

Note that (2.7) to (2.10) arise in the standard interior transmission eigenvalue problem when the domain D is immersed in a dielectric background. The boundary conditions (2.11) are due to the presence of the conducting surface. Here we assume that for some constant γ the positive real-valued function n is such that $n(\mathbf{x}) \geq \gamma > 0$ a.e. in D, $n \in L^\infty(D)$ and $1/|n(\mathbf{x}) - 1| \in L^\infty(D)$. Then, with the aid of function spaces defined in Section 2.1.2, the interior transmission eigenvalue problem becomes the following: Find $w \in L^2(D)$ and $v \in L^2(D)$ and $k \in \mathbb{R}$ such that $w - v \in \mathcal{H}_{0a}(D)$ satisfies (2.7), (2.8) and (2.11), that is,

$$\begin{aligned} \Delta w + k^2 n(\mathbf{x}) w &= 0 \quad \text{in} \quad D, \\ \Delta v + k^2 v &= 0 \quad \text{in} \quad D, \\ w = 0, v &= 0 \quad \text{on} \quad \Gamma_m, \end{aligned}$$

where we still need to give a suitable meaning to the last boundary condition.

The real transmission eigenvalues are the values of $k > 0$ for which this interior transmission problem has non-trivial solutions. The boundary conditions (2.9), (2.10) are incorporated in the fact that $w - v \in \mathcal{H}_{0a}(D)$.

2.1.4.2 Discreteness of Transmission Eigenvalues

Based on the analytic Fredholm theory, it is well known that, in the absence of a conducting surface, the set of transmission eigenvalues is at most discrete with $+\infty$ as the only possible accumulation point [19, 24, 58]. The goal here is to show that this is also true when the conducting surface is present.

It is worth mentioning that when Γ_m is a segment of the x-axis, we could prove the discreteness of the transmission eigenvalues using an image principle. Instead we use the analytic Fredholm theory because the results extend to more general Γ_m (see Remark 2.1.1). Yet an alternative approach is to use the standard Fredholm theory along the lines of [42].

First of all, following standard procedure [16], we write (2.7),(2.8) as an equivalent quadratic eigenvalue problem for $u = w-v \in \mathcal{H}_{0a}(D)$ for a fourth order differential equation in the following standard way: (2.7) $-$ (2.8) implies that

$$\Delta u + k^2 u = -k^2(n-1)w \text{ in } D. \tag{2.12}$$

Dividing both sides of (2.12) by $(n-1)$ and applying the operator $(\Delta + k^2 n)$ gives

$$(\Delta + k^2 n)\frac{1}{n-1}(\Delta u + k^2 u) = 0 \text{ in } D. \tag{2.13}$$

Also from (2.12) we have that

$$w = -\frac{1}{k^2}\frac{1}{n-1}(\Delta u + k^2 u).$$

Then the boundary condition $w = 0$ in (2.11) implies that

$$\frac{1}{n-1}(\Delta u + k^2 u) = 0 \text{ on } \Gamma_m,$$

which will turn out to be a natural boundary condition for u.

Note that in addition $u = w - v \in \mathcal{H}_{0a}(D)$ implies that

$$u = 0 \text{ on } \Gamma, \qquad \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma_a.$$

Thus if k is a transmission eigenvalue, then there is a non-trivial solution $u \in \mathcal{H}_{0a}(D)$ to the following problem:

$$\begin{cases} (\Delta + k^2 n)\dfrac{1}{n-1}(\Delta u + k^2 u) = 0 & \text{in} \quad D, \\ u = 0 & \text{on} \quad \Gamma, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on} \quad \Gamma_a, \\ \dfrac{1}{n-1}(\Delta u + k^2 u) = 0 & \text{on} \quad \Gamma_m. \end{cases}$$

To study this eigenvalue problem, we write it in variational form. To this end, we multiply (2.13) by the complex conjugate of a test function $\bar{\psi} \in \mathcal{H}_{0a}(D)$. Denote by $\beta = \dfrac{1}{n-1}(\Delta u + k^2 u)$, then integration by parts twice shows that

$$\left(\Delta\beta + k^2 n\beta, \psi\right)_D = (\beta, \Delta\psi)_D - \left\langle \beta, \frac{\partial \psi}{\partial \nu} \right\rangle_\Gamma + \left(k^2 n\beta, \psi\right)_D.$$

Using all the boundary conditions, we get the variational form of the interior transmission eigenvalue problem of finding a function $u \in \mathcal{H}_{0a}(D), u \neq 0$ and $k \in \mathbb{R}$ such that

$$\left(\frac{1}{n-1}(\Delta u + k^2 u), \Delta\psi + k^2 n\psi\right)_D = 0 \text{ for all } \psi \in \mathcal{H}_{0a}(D).$$

Notice that (2.13) can be rewritten as

$$\left(\frac{1}{n-1}(\Delta u + k^2 u), \Delta\psi + k^2\psi\right)_D + k^2\left(\Delta u + k^2 u, \psi\right)_D = 0.$$

By applying integration by parts again, we can finally reach the following equivalent form of finding a function $u \in \mathcal{H}_{0a}(D), u \neq 0$ and $k \in \mathbb{R}$ such that

$$\mathcal{B}_k(u, \psi) - k^2 \mathcal{C}(u, \psi) = 0 \text{ for all } \psi \in \mathcal{H}_{0a}(D), \tag{2.14}$$

where

$$\mathcal{B}_k(u,\psi) := \left(\frac{1}{n-1}(\Delta u + k^2 u), (\Delta\psi + k^2\psi)\right)_D + k^4(u,\psi)_D,$$

and $\mathcal{C}$ is defined as in (2.1).

In a similar fashion, the transmission eigenvalue problem can also be written as the problem of finding a non-trivial $u \in \mathcal{H}_{0a}(D)$ and k such that

$$\begin{cases} (\Delta + k^2)\dfrac{1}{1-n}(\Delta u + k^2 n u) = 0 & \text{in} \quad D, \\ u = 0 & \text{on} \quad \Gamma, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on} \quad \Gamma_a, \\ \dfrac{1}{1-n}(\Delta u + k^2 n u) = 0 & \text{on} \quad \Gamma_m, \end{cases}$$

where the natural boundary condition for u on Γ_m arises from the condition $v = 0$ in (2.11).

In this way, the corresponding variational form is to find a function $u \in \mathcal{H}_{0a}(D), u \neq 0$ and $k \in \mathbb{R}$ such that

$$\tilde{\mathcal{B}}_k(u,\psi) - k^2\mathcal{C}(u,\psi) = 0 \text{ for all } \psi \in \mathcal{H}_{0a}(D), \tag{2.15}$$

where

$$\begin{aligned} \tilde{\mathcal{B}}_k(u,\psi) &:= \left(\frac{1}{1-n}(\Delta u + k^2 n u), (\Delta\psi + k^2 n\psi)\right)_D + k^4(nu,\psi)_D \\ &= \left(\frac{n}{1-n}(\Delta u + k^2 u), (\Delta\psi + k^2\psi)\right)_D + (\Delta u, \Delta\psi)_D, \end{aligned}$$

and $\mathcal{C}$ is the same as before.

Clearly, $\mathcal{B}_k(\cdot,\cdot)$ and $\tilde{\mathcal{B}}_k(\cdot,\cdot)$ are continuous sesquilinear forms on $\mathcal{H}_{0a}(D) \times \mathcal{H}_{0a}(D)$. Let us denote by B_k and $\tilde{B}_k$ the bounded linear operators from $\mathcal{H}_{0a}(D)$ to $\mathcal{H}_{0a}(D)$ defined using the Riesz representation theorem (see Theorem C.0.1) by

$$(B_k u, \psi)_{\mathcal{H}_{0a}(D)} = \mathcal{B}_k(u,\psi), \tag{2.16}$$

$$(\tilde{B}_k u, \psi)_{\mathcal{H}_{0a}(D)} = \tilde{\mathcal{B}}_k(u,\psi), \tag{2.17}$$

for all $\psi \in \mathcal{H}_{0a}(D)$.

Now we can state and prove the following theorem:

Theorem 2.1.1 *(Discreteness) If* $\dfrac{1}{n(\mathbf{x})-1} > \alpha > 0$ *a.e. in* D *for some constant* $\alpha > 0$*, then*

1. *The set of transmission eigenvalues is at most discrete and does not accumulate at* 0*.*

2. *All real transmission eigenvalues, if they exist, are such that* $k^2 \geq \dfrac{\lambda(D)}{\sup_D(n)}$ *where* $\lambda(D)$ *is the first eigenvalue of the buckled plate eigenvalue problem stated in Lemma 2.1.3.*

Proof: To prove the first part of the theorem we consider the formulation (2.14). Indeed, following the proof in [17], we have

$$\begin{aligned}
\mathcal{B}_k(u,u) &= \left(\frac{1}{n(\mathbf{x})-1}(\Delta u + k^2 u), \Delta u + k^2 u\right)_D + k^4(u,u)_D \\
&\geq \alpha\|\Delta u + k^2 u\|^2_{L^2(D)} + k^4\|u\|^2_{L^2(D)} \\
&\geq \alpha\left(\|\Delta u\|_{L^2(D)} - \|k^2 u\|_{L^2(D)}\right)^2 + \left(k^2\|u\|_{L^2(D)}\right)^2 \\
&= \alpha X^2 - 2\alpha XY + (\alpha+1)Y^2,
\end{aligned}$$

where $X = \|\Delta u\|_{L^2(D)}$ and $Y = k^2\|u\|_{L^2(D)}$. Then we obtain

$$\mathcal{B}_k(u,u) \geq \epsilon\left(Y - \frac{\alpha}{\epsilon}X\right)^2 + \left(\alpha - \frac{\alpha^2}{\epsilon}\right)X^2 + (1+\alpha-\epsilon)Y^2, \tag{2.18}$$

for $\alpha < \epsilon < \alpha + 1$. By setting $\epsilon = \alpha + 1/2$, we have that

$$\mathcal{B}_k(u,u) \geq \frac{\alpha}{1+2\alpha}(X^2+Y^2).$$

Also notice that, using Lemma 2.1.3,

$$\begin{aligned}
\|u\|_{\mathcal{H}(D)} &= (u,u)_{\mathcal{H}(D)} = (u,u)_D + (\nabla u, \nabla u)_D + (\Delta u, \Delta u)_D \\
&= \left(\frac{Y}{k^2}\right)^2 + \|\nabla u\|^2_{L^2(D)} + X^2 \\
&\leq \left(\frac{Y}{k^2}\right)^2 + \frac{1}{\lambda(D)}\|\Delta u\|^2_{L^2(D)} + X^2 \\
&= \left(1 + \frac{1}{\lambda(D)}\right)X^2 + \left(\frac{1}{k^4}\right)Y^2.
\end{aligned}$$

Then the above estimate yields the existence of a constant $c_k > 0$ (independent of u and α) such that

$$\mathcal{B}_k(u,u) \geq c_k \frac{\alpha}{1+2\alpha}\|u\|^2_{\mathcal{H}(D)}.$$

Hence the sesquilinear form $\mathcal{B}_k(\cdot,\cdot)$ is coercive in $\mathcal{H}_{0a}(D) \times \mathcal{H}_{0a}(D)$ and consequently the operator $B_k : \mathcal{H}_{0a}(D) \longrightarrow \mathcal{H}_{0a}(D)$ is a bijection for fixed k.

To use the analytic Fredholm theory, we first have the following observations:

- The sesquilinear form $\mathcal{B}_k(\cdot,\cdot)$ is analytic in k.
- Denote by $\mathcal{L}(\cdot,\cdot)$ the set of all bounded linear operators from one Banach space to another. Define the operator valued function $f : k \in \mathbb{C} \to B_k \in \mathcal{L}(\mathcal{H}_{0a}(D), \mathcal{H}_{0a}(D))$ such that for each $u \in \mathcal{H}_{0a}(D)$, the function $f_u : k \in \mathbb{C} \mapsto B_k u \in \mathcal{H}_{0a}(D)$ is weakly analytic. This is true since for each $l \in [\mathcal{H}_{0a}(D)]^* := \mathcal{L}(\mathcal{H}_{0a}(D), \mathbb{C})$ where $*$ represents the dual space, we have that

  $$l(f_u(k)) = l(B_k u) = (B_k u, \psi)_{\mathcal{H}_{0a}(D)} = \mathcal{B}_k(u,\psi) \in \mathbb{C} \text{ for some } \psi \in \mathcal{H}_{0a}(D)$$

  is analytic in k. Then by Theorem C.0.2 and Theorem C.0.3, f is strongly analytic.
- By the Lax-Milgram Lemma (Theorem C.0.4), there exists a bounded linear inverse operator B_k^{-1} of B_k in a neighborhood of the positive real axis and in particular, this inverse B_k^{-1} is also strongly analytic in k.

Then we show that the operator $B_k - k^2 C : \mathcal{H}_{0a}(D) \longrightarrow \mathcal{H}_{0a}(D)$ is an isomorphism for $k > 0$ small enough. From (2.18) and Lemma 2.1.3, we have for $\alpha < \epsilon < \alpha+1$,

$$\begin{aligned}
\mathcal{B}_k(u,u) - k^2\mathcal{C}(u,u) &\geq \epsilon\left(Y - \frac{\alpha}{\epsilon}X\right)^2 + \left(\alpha - \frac{\alpha^2}{\epsilon}\right)X^2 + (1+\alpha-\epsilon)Y^2 - k^2\|\nabla u\|^2_{L^2(D)} \\
&\geq \left(\alpha - \frac{\alpha^2}{\epsilon}\right)X^2 + (1+\alpha-\epsilon)Y^2 - k^2\|\nabla u\|^2_{L^2(D)} \\
&\geq \left(\alpha - \frac{\alpha^2}{\epsilon} - \frac{k^2}{\lambda(D)}\right)\|\Delta u\|^2_{L^2(D)} + (1+\alpha-\epsilon)k^4\|u\|^2_{L^2(D)}.
\end{aligned}$$

Therefore, if $k^2 < (\alpha - \alpha^2/\epsilon)\lambda(D)$ for every $\alpha < \epsilon < \alpha+1$, then $B_k - k^2 C$ is invertible, whence the analytic Fredholm theory (see Theorem 8.26 in [19]) implies that the set of transmission eigenvalue is at most discrete.

In particular, by choosing $\alpha = \dfrac{1}{n^* - 1}$ where $n^* = \sup_D n(\mathbf{x})$ and taking ϵ arbitrarily close to $\alpha + 1$ we have that if $k^2 < \dfrac{\alpha}{\alpha+1}\lambda(D) = \dfrac{\lambda(D)}{\sup_D n(\mathbf{x})}$ then k is not a transmission eigenvalue.

The second part of the theorem is a consequence of the proof of part 1. $\square$

Alternatively, for $0 < n(\mathbf{x}) < 1$ we have,

Theorem 2.1.2 *(Discreteness) If* $\dfrac{n(\mathbf{x})}{1 - n(\mathbf{x})} > \alpha > 0$ *a.e. in* D *for some constant* $\alpha > 0$*, then*

1. *The set of transmission eigenvalues is at most discrete and does not accumulate at 0.*
2. *All transmission eigenvalues, if they exist, are such that* $k^2 \geq \lambda(D)$ *where* $\lambda(D)$ *is the first eigenvalue of the buckled plate eigenvalue problem stated in Lemma 2.1.3.*

Proof: The proof is similar to the proof of Theorem 2.1.1. Here we need to use the sesquilinear form (2.15). Again, following the proof in [17], similar to the derivation in Theorem 2.1.1, we have

$$\tilde{\mathcal{B}}_k(u,u) \;\; \geq \;\; \epsilon\left(X - \frac{\alpha}{\epsilon}Y\right)^2 + \left(\alpha - \frac{\alpha^2}{\epsilon}\right)Y^2 + (1 + \alpha - \epsilon)X^2, \qquad (2.19)$$

for $\alpha < \epsilon < \alpha + 1$, where $X = \|\Delta u\|_{L^2(D)}$ and $Y = k^2\|u\|_{L^2(D)}$.

Proceeding in the same way as in the first part of Theorem 2.1.1, we have again

$$\tilde{\mathcal{B}}_k(u,u) \geq \frac{\alpha}{1+2\alpha}(X^2 + Y^2).$$

Consequently, we can conclude again that

$$\tilde{\mathcal{B}}_k(u,u) \geq c_k \frac{\alpha}{1+2\alpha}\|u\|^2_{\mathcal{H}(D)},$$

where $c_k > 0$ is a constant independent of u and α, whence $\tilde{\mathcal{B}}_k(\cdot,\cdot)$ is a coercive sesquilinear form in $\mathcal{H}_{0a}(D) \times \mathcal{H}_{0a}(D)$.

Arguing exactly in the same way as in the first part of Theorem 2.1.1 we conclude from analytic Fredholm theory that $\mathcal{B}_k^{-1}$ is strongly analytical in k. Finally, to show that $\tilde{B}_k - k^2 C$ is invertible for k small enough, using (2.19) and Lemma 2.1.3 for $\alpha < \epsilon < \alpha + 1$ we have that

$$\tilde{\mathcal{B}}_k(u,u) - k^2\mathcal{C}(u,u) \geq \left(\alpha - \frac{\alpha^2}{\epsilon}\right) k^4 \|u\|^2_{L^2(D)} + \left(1 + \alpha - \epsilon - \frac{k^2}{\lambda(D)}\right) \|\Delta u\|^2_{L^2(D)}.$$

Therefore, if $k^2 < (\alpha + 1 - \epsilon)\lambda(D)$ for every $\alpha < \epsilon < \alpha + 1$, then $\tilde{B}_k - k^2 C$ is invertible, whence the analytic Fredholm theory implies that the set of transmission eigenvalue is at most discrete.

In particular, by taking $\epsilon > 0$ arbitrarily close to α we have that k such that $k^2 < \lambda(D)$ are not transmission eigenvalues.

The second part of the theorem is a consequence of the proof of part 1. □

Next we want to provide bounds on transmission eigenvalues. Notice that from the assumption of Theorem 2.1.1

$$\frac{1}{n(\mathbf{x}) - 1} > \alpha > 0 \text{ which means } n(\mathbf{x}) \geq \delta_* > 1$$

a.e. in D for some constant $\delta_* > 0$.

Similarly, from the assumption of Theorem 2.1.2

$$\frac{n(\mathbf{x})}{1 - n(\mathbf{x})} > \alpha > 0 \text{ which means } n(\mathbf{x}) \leq \delta^* < 1$$

a.e. in D for some constant $\delta^* > 0$.

Then, as a direct result of Theorem 2.1.1 and Theorem 2.1.2, we have the following consequence:

Corollary 2.1.1 *(Faber-Krahn inequality)*

1. *Assume that $n(\mathbf{x}) \geq \delta > 1$ for all $x \in D$ and some constant δ. Then, if k is a transmission eigenvalue,*
$$k^2 \geq \frac{\lambda(D)}{\sup_D n(\mathbf{x})}.$$
2. *Assume that $0 < \gamma \leq n(\mathbf{x}) \leq \delta < 1$ for all $x \in D$ and some constants γ and δ. Then, if k is a transmission eigenvalue,*
$$k^2 \geq \lambda(D).$$

Here $\lambda(D)$ is the first eigenvalue of the buckled plate eigenvalue problem stated in Lemma 2.1.3.

2.1.4.3 Existence of Transmission Eigenvalues

The next theorem confirms that transmission eigenvalue exists. It can be extended to more general Γ_m (see Remark 2.1.1). The key result we use for the existence is Lemma C.0.5.

Theorem 2.1.3 *Let $n \in L^\infty(D)$ satisfy either one of the following assumptions:*

(1). $1 + \alpha \le n_* \le n(\mathbf{x}) \le n^* \le \infty$,

(2). $0 < n_* \le n(\mathbf{x}) \le n^* < 1 - \beta$,

for some constants $\alpha > 0$ and $\beta > 0$. Then there exists an infinite set of transmission eigenvalues with $+\infty$ as the only accumulation point.

Proof: First of all, the discreteness of the transmission eigenvalues is proved in the last section by noticing that the first assumption satisfies the condition of Theorem 2.1.1 and the second assumption satisfies the condition of Theorem 2.1.2.

For existence, we adopt the proof of Theorem 2.5 in [16] with certain necessary modifications. Suppose assumption (1) holds which implies that

$$0 < \frac{1}{n^* - 1} \le \frac{1}{n - 1} \le \frac{1}{n_* - 1} < \infty.$$

Then B_k and C defined by (2.16) and (2.3) satisfy the requirements of Theorem C.0.5 with $X = \mathcal{H}_{0a}(D)$ and from the proof of Theorem 2.1.1 they also satisfy the assumption (1) of Theorem C.0.5 with $\tau_0 := k^2 < \lambda(D)/n^*$, that is, $B_{\tau_0} - \tau_0 C$ (equivalent to $B_k - k^2 C$) is positive on X.

Next, let k_{1,n_*} be the first transmission eigenvalue for the disk S of radius $R = 1$ and $n := n_*$. By a scaling argument, $k_{\epsilon,n_*} := k_{1,n_*}/\epsilon$ is the first transmission eigenvalue corresponding to the disk of radius $\epsilon > 0$ with $n := n_*$. Take $\epsilon > 0$ small enough such that D contains $m := m(\epsilon) \ge 1$ disjoint disks $S_\epsilon^1, S_\epsilon^2, \cdots, S_\epsilon^m$ of radius ϵ, then $k_{\epsilon,n_*} := k_{1,n_*}/\epsilon$ is the first transmission eigenvalue for each of these disks with $n := n_*$ and let $u^{S_\epsilon^j, n_*} \in H_0^2(S_\epsilon^j), j = 1, \cdots, m$ be the corresponding eigenfunction. The extension by zero $\tilde{u}^j$ of $u^{S_\epsilon^j, n_*}$ to the whole D is then in $H_0^2(D) \subset \mathcal{H}_{0a}(D)$. Furthermore, the vectors

$\{\tilde{u}^1, \tilde{u}^2, \ldots, \tilde{u}^m\}$ are linearly independent and orthogonal in $H_0^2(D) \subset \mathcal{H}_{0a}(D)$ and we have that

$$\begin{aligned} 0 &= \left(\frac{1}{n_* - 1} (\Delta \tilde{u}^j + k_{\epsilon,n_*}^2 \tilde{u}^j), \Delta \tilde{u}^j + k_{\epsilon,n_*}^2 \tilde{u}^j \right)_D \\ &= \frac{1}{n_* - 1} \|\Delta \tilde{u}^j + k_{\epsilon,n_*}^2 \tilde{u}^j\|_{L^2(D)}^2 + k_{\epsilon,n_*}^4 \|\tilde{u}^j\|_{L^2(D)}^2 - k_{\epsilon,n_*}^2 \|\nabla \tilde{u}^j\|_{L^2(D)}^2 \end{aligned}$$

for $j = 1, \ldots, m$.

Denote by $\mathcal{U}$ the m-dimensional subspace of $H_0^2(D) \subset \mathcal{H}_{0a}(D)$ spanned by $\{\tilde{u}^1, \tilde{u}^2, \ldots, \tilde{u}^m\}$. Then we have that for $\tau_1 := k_{\epsilon,n_*}^2$ and for every $\tilde{u} \in \mathcal{U}$

$$\begin{aligned} & (B_{\tau_1} \tilde{u} - \tau_1 C \tilde{u}, \tilde{u})_{\mathcal{H}_{0a}(D)} \\ = \;& (B_{k_{\epsilon,n_*}} \tilde{u} - k_{\epsilon,n_*}^2 C \tilde{u}, \tilde{u})_{\mathcal{H}_{0a}(D)} = \mathcal{B}_{k_{\epsilon,n_*}}(\tilde{u}, \tilde{u}) - k_{\epsilon,n_*}^2 \mathcal{C}_{k_{\epsilon,n_*}}(\tilde{u}, \tilde{u}) \\ = \;& \left(\frac{1}{n-1} \Delta \tilde{u} + k_{\epsilon,n_*}^2 \tilde{u}, \Delta \tilde{u} + k_{\epsilon,n_*}^2 \tilde{u} \right)_D + k_{\epsilon,n_*}^4 \|\tilde{u}\|_{L^2(D)}^2 - k_{\epsilon,n_*}^2 \|\nabla \tilde{u}\|_{L^2(D)}^2 \\ \leq \;& \frac{1}{n_* - 1} \|\Delta \tilde{u} + k_{\epsilon,n_*}^2 \tilde{u}\|_{L^2(D)}^2 + k_{\epsilon,n_*}^4 \|\tilde{u}\|^2 \, dx - k_{\epsilon,n_*}^2 \|\nabla \tilde{u}\|_{L^2(D)}^2 = 0. \end{aligned}$$

This means that assumption (2) of Theorem C.0.5 is also satisfied and therefore we can conclude that there are $m(\epsilon)$ transmission eigenvalues (counting multiplicity) inside $[\tau_0, \tau_1] = [\tau_0, k_{\epsilon,n_*}^2]$. Then by letting $\epsilon \to 0$, we can show that there exists an infinite countable set of transmission eigenvalues that accumulate at ∞.

If assumption (2) of the theorem holds, we have that

$$0 < \frac{n_*}{1 - n_*} \leq \frac{n}{1 - n} \leq \frac{n^*}{1 - n^*} < \infty.$$

Then use $\tilde{B}_k$ and C defined by (2.17) and (2.3), let $X = \mathcal{H}_{0a}(D)$, choose $\tau_0 := k^2 \leq \lambda(D)$, $\tau_1 := k_{\epsilon,n^*}^2$, proceed the similar argument, we can prove the same result. $\Box$

Note 2.1.2 *The first transmission eigenvalue for the disk S we mentioned in the above theorem is the first real non-zero eigenvalue of:*

$$\begin{cases} \Delta w + k^2 n_0 w = 0 & in \quad S, \\ \Delta v + k^2 v = 0 & in \quad S, \\ w - v = 0 & on \quad \partial S, \\ \dfrac{\partial w}{\partial \nu} - \dfrac{\partial v}{\partial \nu} = 0 & on \quad \partial S, \end{cases}$$

where $n_0 > 0$ is a constant and $n_0 \neq 1$. It can be computed by separation of variables and then finding the zeros of the appropriate determinant (see [16] for details).

Like Corollary 2.6 in [16], we can also obtain estimates for transmission eigenvalues by using Theorem 2.1.3. Our proof requires the flatness of boundary Γ_m as mentioned in Remark 2.1.1.

We denote by S_{r_1} the largest disk of radius r_1 such that $S_{r_1} \subset D$ and $\hat{S}_{r_2}$ the smallest half disk of radius r_2 such that $D \subset \hat{S}_{r_2}$. Denote k_{1,n_*} the first transmission eigenvalue corresponding to the disk S_1 of radius one with $n := n_*$ and $\hat{k}_{1,n^*}$ the first transmission eigenvalue corresponding to the half disk $\hat{S}_1$ of radius one with $n := n^*$.

We have seen that k_{1,n_*} can be computed since S_{r_1} is not touching the boundary of conducting surface. On the other hand, to find transmission eigenvalues for the half disk $\hat{S}_1$ with constant $n := \hat{n}_0$ ($\hat{n}_0 > 0, \hat{n}_0 \neq 1$), the corresponding interior transmission problem is to find eigenvalues of:

$$\begin{cases} \Delta w + k^2 \hat{n}_0 w = 0 & \text{in} \quad \hat{S}_1, \\ \Delta v + k^2 v = 0 & \text{in} \quad \hat{S}_1, \\ w - v = 0 & \text{on} \quad \partial\hat{S}_{1a}, \\ \dfrac{\partial w}{\partial \nu} - \dfrac{\partial v}{\partial \nu} = 0 & \text{on} \quad \partial\hat{S}_{1a}, \\ w = 0, v = 0 & \text{on} \quad \partial\hat{S}_{1m}, \end{cases}$$

where $\partial\hat{S}_{1a}$ is the interface of domain $\hat{S}_1$ and the background dielectric medium D_a and $\partial\hat{S}_{1m}$ is the interface of domain $\hat{S}_1$ and the infinite perfect electric conducting substrate D_m.

The idea here is to extend the half disk to a full disk S_1 so that the problem becomes one for the full disk considered in Note 2.1.2. Solutions to the Helmholtz equation for w and v are:

$$\begin{aligned} w \quad &: \quad J_m(k\sqrt{\hat{n}_0}r)\cos(m\theta) \quad \text{and} \quad J_m(k\sqrt{\hat{n}_0}r)\sin(m\theta), \quad m = 0, 1, \ldots \\ v \quad &: \quad J_m(kr)\cos(m\theta) \quad \text{and} \quad J_m(kr)\sin(m\theta), \quad m = 0, 1, \ldots. \end{aligned}$$

Then we drop the solutions that do not meet boundary condition on $\partial\hat{S}_{1m}$, that is, those depending on $\cos(m\theta)$ (even eigenfunctions) and make the remainder satisfy the boundary conditions on $\partial \hat{S}_{1a}$. Consequently, the transmission eigenvalues corresponding to the half disk can be computed by finding the zeros of the following determinant:

$$\det \begin{pmatrix} J_m(kr) & J_m(k\sqrt{\hat{n}_0}r) \\ -J_m'(kr) & -\sqrt{n}J_m'(k\sqrt{\hat{n}_0}r) \end{pmatrix}.$$

Note that $\sin(m\theta) = 0$ when $m = 0$, so $\hat{k}_{1,\hat{n}_0}$ is the first zero of

$$\det \begin{pmatrix} J_1(kR) & J_1(k\sqrt{\hat{n}_0}R) \\ -J_1'(kR) & -\sqrt{\hat{n}_0}J_1'(k\sqrt{\hat{n}_0}R) \end{pmatrix},$$

where $R = 1$. Correspondingly, when $n := n^*$, $\hat{k}_{1,n^*}$ is the first zero of

$$\det \begin{pmatrix} J_1(kR) & J_1(k\sqrt{n^*}R) \\ -J_1'(kR) & -\sqrt{n^*}J_1'(k\sqrt{n^*}R) \end{pmatrix}.$$

By a scaling argument, we have that $\hat{k}_{\epsilon,n^*} := \hat{k}_{1,n^*}/\epsilon$ is the first transmission eigenvalue corresponding to half disk of radius $\epsilon > 0$ with $n := n^*$.

For a given $0 < \epsilon \le r_1$ let $m(\epsilon) \in \mathbb{N}$ be the number of disjoint balls S_ϵ of radius ϵ that are contained in D, we have the following corollary:

Corollary 2.1.2 *Assume that $n(\mathbf{x}) \in L^\infty(D)$, then*

(1). If $1 + \alpha \le n_* \le n(\mathbf{x}) \le n^* < \infty$, then

$$0 < \frac{\hat{k}_{1,n^*}}{r_2} \le k_{1,D,n(\mathbf{x})} \le \frac{k_{1,n_*}}{r_1}.$$

There are at least $m(\epsilon)$ transmission eigenvalues in the interval $\left[\frac{\hat{k}_{1,n^*}}{r_2}, \frac{k_{1,n_*}}{\epsilon}\right]$.

(2). If $0 < n_* \le n(\mathbf{x}) \le n^* < 1 - \beta$, then

$$0 < \frac{\hat{k}_{1,n_*}}{r_2} \le k_{1,D,n(\mathbf{x})} \le \frac{k_{1,n^*}}{r_1}.$$

There are at least $m(\epsilon)$ transmission eigenvalues in the interval $\left[\frac{\hat{k}_{1,n_*}}{r_2}, \frac{k_{1,n^*}}{\epsilon}\right]$.

Proof: We first suppose that assumption (1) holds. Then, for any $u \in \mathcal{H}_{0a}(D)$, we have the following inequality for the Rayleigh quotient $(B_\tau u, u)_{\mathcal{H}_{0a}(D)}/(Cu, u)_{\mathcal{H}_{0a}(D)}$, that is,

$$\left.\frac{(B_\tau u, u)_{\mathcal{H}_{0a}(D)}}{(Cu, u)_{\mathcal{H}_{0a}(D)}}\right|_{n=n^*} \le \left.\frac{(B_\tau u, u)_{\mathcal{H}_{0a}(D)}}{(Cu, u)_{\mathcal{H}_{0a}(D)}}\right|_{n=n}.$$

Or equivalently,

$$\frac{\frac{1}{n^*-1}\|\Delta u+\tau u\|^2_{L^2(D)}+\tau^2\|u\|^2_{L^2(D)}}{\|\nabla u\|^2_{L^2(D)}} \quad \le \quad \frac{\int_D \frac{1}{n(\mathbf{x})-1}|\Delta u+\tau u|^2\,dx+\tau^2\|u\|^2_{L^2(D)}}{\|\nabla u\|^2_{L^2(D)}} \tag{2.20}$$

From formula (2.1) in [16], corresponding to our case,

$$\lambda_j = \min_{W\subset\mathcal{U}_j}\left(\max_{u\in\mathcal{W}\setminus\{0\}}\frac{(B_\tau u,u)_{\mathcal{H}_{0a}(D)}}{(Cu,u)_{\mathcal{H}_{0a}(D)}}\right), \tag{2.21}$$

where $\mathcal{U}_j$ denotes the set of all j dimensional subspaces $\mathcal{W}$ of $\mathcal{H}_{0a}(D)$ such that $\mathcal{W}\cap\ker(C)=\{0\}$. We now argue that for an arbitrary $\tau>0$,

$$\lambda_1(\tau,\hat{S}_{r_2},n^*)\overset{①}{\le}\lambda_1(\tau,D,n^*)\overset{②}{\le}\lambda_1(\tau,D,n(\mathbf{x})), \tag{2.22}$$

where $\lambda_1(\tau,D,n^*)$ and $\lambda_1(\tau,D,n(\mathbf{x}))$ are the first eigenvalue of the auxiliary problem for D and $n^*,n(\mathbf{x})$, respectively, whereas $\lambda_1(\tau,\hat{S}_{r_2},n^*)$ is the first eigenvalue of the auxiliary problem for $\hat{S}_{r_2}$ and n^*. The auxiliary eigenvalue problems are

$$B_\tau u-\lambda(\tau)Cu=0 \text{ where } u\in\mathcal{H}_{0a}(D) \text{ if } 1/(n-1)>\gamma>0,$$

and

$$\tilde{B}_\tau u-\lambda(\tau)Cu=0 \text{ where } u\in\mathcal{H}_{0a}(D) \text{ if } n/(1-n)>\gamma>0.$$

Clearly, from (2.21), inequality ② holds because of (2.20); inequality ① also holds by noting that the extension by zero $\hat{u}$ of $u\in\mathcal{U}_1\subset\mathcal{H}_{0a}(D)$ to the whole $\hat{S}_{r_2}$ is in $\hat{\mathcal{U}}_1\subset\mathcal{H}_{0a}(\hat{S}_{r_2})$.

Thus, we have

- For $\tau_1:=(k_{1,n_*}/r_1)^2, S_{r_1}\subset D$, from the proof Theorem 2.1.3 we have that $\lambda_1(\tau_1,D,n(\mathbf{x}))-\tau_1\le 0$.
- For $\tau_0:=(\hat{k}_{1,n^*}/r_2)^2, D\subset\hat{S}_{r_2}$, we have $\lambda_1(\tau_0,\hat{S}_{r_2},n^*)-\tau_0=0$ and hence from (2.22) we see that $\lambda_1(\tau_0,D,n(\mathbf{x}))-\tau_0\ge 0$.

Therefore, the first eigenvalue $k_{1,D,n(\mathbf{x})}$ corresponding to D and $n(\mathbf{x})$ is between $\hat{k}_{1,n^*}/r_2$ and $k_{1,n_*}/r_1$. Also there is no transmission eigenvalue for D and $n(\mathbf{x})$ that is less than $\hat{k}_{1,n^*}/r_2$ (see Corollary 2.6 in [16] for a similar argument). The case for $0<n_*\le n(\mathbf{x})\le n^*<1-\beta$ can be proven in a similar way. □

Remark 2.1.3 *From the proof of Theorem 2.1.1, Theorem 2.1.2 and Corollary 2.1.2 we have that*

1. *If* $1 + \alpha \leq n_* \leq n(\mathbf{x}) \leq n^* < \infty$, *then*

$$k_{1,D,n(\mathbf{x})} \geq \max\left(\frac{\hat{k}_{1,n^*}}{r_2}, \sqrt{\frac{\lambda(D)}{n^*}}\right).$$

2. *If* $0 < n_* \leq n(\mathbf{x}) \leq n^* < 1 - \beta$, *then*

$$k_{1,D,n(\mathbf{x})} \geq \max\left(\frac{\hat{k}_{1,n_*}}{r_2}, \sqrt{\lambda(D)}\right).$$

Here $\lambda(D)$ *is the first eigenvalue of the buckled plate eigenvalue problem stated in Lemma 2.1.3.*

2.1.5 Scalar Anisotropic Media

2.1.5.1 Formulation of the Problem

The interior transmission eigenvalue problem corresponding to the scattering problem for an anisotropic inhomogeneous medium (TM mode electromagnetic scattering) with configuration as in Figure 2.1 in $\mathbb{R}^2$ reads: find $\hat{w}$ and $\hat{v}$ in suitable function spaces such that

$$\nabla \cdot (A(x)\nabla \hat{w}) + k^2 \hat{w} = 0 \quad \text{in} \quad D, \tag{2.23}$$

$$\Delta \hat{v} + k^2 \hat{v} = 0 \quad \text{in} \quad D, \tag{2.24}$$

$$\hat{w} - \hat{v} = 0 \quad \text{on} \quad \Gamma_a, \tag{2.25}$$

$$\frac{\partial \hat{w}}{\partial \nu_A} - \frac{\partial \hat{v}}{\partial \nu} = 0 \quad \text{on} \quad \Gamma_a, \tag{2.26}$$

$$\frac{\partial \hat{w}}{\partial \nu_A} = 0, \frac{\partial \hat{v}}{\partial \nu} = 0 \quad \text{on} \quad \Gamma_m, \tag{2.27}$$

where $\dfrac{\partial \hat{w}}{\partial \nu_A} = \nu(x) \cdot A(x)\nabla \hat{w}$.

Note that (2.23) to (2.26) arise in the standard interior transmission eigenvalue problem when the domain D is immersed in a dielectric background. The boundary conditions (2.27) are due to the presence of the conducting surface. Here we assume that A is a real valued 2×2 matrix-valued function whose entries are piecewise continuously differentiable functions in $\bar{D}$ with (possible) jumps along piecewise smooth curves such that A is symmetric and $\bar{\xi} \cdot A\xi \geq \gamma|\xi|^2$, $\bar{\xi} \cdot A^{-1}\xi \geq \beta|\xi|^2$ for all $\xi \in \mathbb{C}^2$ and $x \in \bar{D}$ where γ, β are positive constants.

Let $\mathbf{w} = A(x)\nabla\hat{w}$ and $\mathbf{v} = \nabla\hat{v}$, then following the derivation in [11], the interior transmission eigenvalue problem for this case becomes the following: Find $\mathbf{w} \in (L^2(D))^2, \mathbf{v} \in (L^2(D))^2$ and $k \in \mathbb{R}$ such that $\mathbf{w} - \mathbf{v} \in \mathcal{G}_{0a}(D)$ and the functions satisfy

$$\nabla(\nabla \cdot \mathbf{w}) + k^2 A^{-1}\mathbf{w} = 0 \quad \text{in} \quad D, \tag{2.28}$$

$$\nabla(\nabla \cdot \mathbf{v}) + k^2\mathbf{v} = 0 \quad \text{in} \quad D, \tag{2.29}$$

$$\nu \cdot \mathbf{w} = 0, \nu \cdot \mathbf{v} = 0 \quad \text{on} \quad \Gamma_m. \tag{2.30}$$

The boundary conditions (2.25), (2.26) are incorporated in the fact that $\mathbf{w} - \mathbf{v} \in \mathcal{G}_{0a}(D)$. Thus we establish the follow result:

Lemma 2.1.5 *If k is a transmission eigenvalue, that is, if there exists non-trivial functions $\hat{w} \in H^1(D)$ and $\hat{v} \in H^1(D)$ that satisfie (2.23) to (2.27), then $\mathbf{w} = A(x)\nabla\hat{w} \in (L^2(D))^2$ and $\mathbf{v} = \nabla\hat{v} \in (L^2(D))^2$ satisfy $\mathbf{w} - \mathbf{v} \in \mathcal{G}_{0a}(D)$ and (2.28) to (2.30).*

2.1.5.2 Discreteness of Transmission Eigenvalues

Similar to Section 2.1.4, the goal here is to show that the set of transmission eigenvalues is at most discrete with $+\infty$ as the only possible accumulation point when the conducting surface is present.

We can write (2.28),(2.29) as an equivalent eigenvalue problem for $\mathbf{u} = \mathbf{w} - \mathbf{v} \in \mathcal{G}_{0a}(D)$ for a fourth order differential equation in the following way: (2.28) $-$ (2.29) implies that

$$\nabla(\nabla \cdot \mathbf{u}) + k^2\mathbf{u} = -k^2(A^{-1} - I)\mathbf{w} \text{ in } D. \tag{2.31}$$

Multiplying both sides of (2.31) by $(A^{-1}-I)^{-1}$ and applying the operator $(\nabla\nabla\cdot +k^2A^{-1})$ gives

$$(\nabla\nabla\cdot +k^2A^{-1})(A^{-1}-I)^{-1}(\nabla\nabla\cdot\mathbf{u}+k^2\mathbf{u})=0 \text{ in } D. \tag{2.32}$$

Also from (2.31) we have that

$$\mathbf{w}=-\frac{1}{k^2}(A^{-1}-I)^{-1}(\nabla\nabla\cdot\mathbf{u}+k^2\mathbf{u}).$$

Then the boundary condition $\nu\cdot\mathbf{w}=0$ in (2.30) implies that

$$\nu\cdot(A^{-1}-I)^{-1}(\nabla\nabla\cdot\mathbf{u}+k^2\mathbf{u})=0 \text{ on } \Gamma_m.$$

Note that in addition $\mathbf{u}=\mathbf{w}-\mathbf{v}\in\mathcal{G}_{0a}(D)$ implies that

$$\nu\cdot\mathbf{u}=0 \text{ on } \Gamma, \qquad \nabla\cdot\mathbf{u}=0 \text{ on } \Gamma_a.$$

Thus if k is a transmission eigenvalue, then there is a non-trivial solution $\mathbf{u}\in\mathcal{G}_{0a}(D)$ to the following problem:

$$\begin{cases} (\nabla\nabla\cdot +k^2A^{-1})(A^{-1}-I)^{-1}(\nabla\nabla\cdot\mathbf{u}+k^2\mathbf{u})=0, & \text{in} \quad D,\\ \nu\cdot\mathbf{u}=0 & \text{on} \quad \Gamma,\\ \nabla\cdot\mathbf{u}=0 & \text{on} \quad \Gamma_a,\\ \nu\cdot(A^{-1}-I)^{-1}(\nabla\nabla\cdot\mathbf{u}+k^2\mathbf{u})=0 & \text{on} \quad \Gamma_m. \end{cases}$$

To study this eigenvalue problem, we write it in variational form. To this end, we multiply (2.32) by the complex conjugate of a test function $\bar{\psi}\in\mathcal{G}_{0a}(D)$, denote by $\beta=(A^{-1}-I)^{-1}(\nabla\nabla\cdot\mathbf{u}+k^2\mathbf{u})$, then using the fact that A is symmetric, integration by parts twice shows that

$$\left(\nabla\nabla\cdot\beta+k^2A^{-1}\beta,\psi\right)_D=-\left\langle\beta\cdot\nu,\nabla\cdot\psi\right\rangle_\Gamma+\left(\beta,\nabla(\nabla\cdot\psi)\right)_D+\left(\beta,k^2A^{-1}\psi\right)_D.$$

Using all the boundary conditions, we get the variational problem of finding a function $\mathbf{u}\in\mathcal{G}_{0a}(D),\mathbf{u}\neq 0$ and $k\in\mathbb{R}$ such that

$$\left((A^{-1}-I)^{-1}(\nabla\nabla\cdot\mathbf{u}+k^2\mathbf{u}),\nabla\nabla\cdot\psi+k^2A^{-1}\psi\right)_D=0 \text{ for all } \psi\in\mathcal{G}_{0a}(D). \tag{2.33}$$

Using the fact that $A^{-1}(A^{-1}-I)^{-1} = [(A^{-1}-I)+I](A^{-1}-I)^{-1} = I + (A^{-1}-I)^{-1}$, (2.33) can be rewritten as

$$\left((A^{-1}-I)^{-1}(\nabla\nabla\cdot\mathbf{u}+k^2\mathbf{u}), \nabla\nabla\cdot\psi+k^2\psi\right)_D + \left(\nabla\nabla\cdot\mathbf{u}+k^2\mathbf{u}, k^2\psi\right)_D = 0.$$

By applying integration by parts again, we can finally reach the following equivalent form of finding a function $\mathbf{u}\in\mathcal{G}_{0a}(D), \mathbf{u}\neq 0$ and $k\in\mathbb{R}$ such that

$$\mathcal{M}_k(\mathbf{u},\psi) - k^2\mathcal{N}(\mathbf{u},\psi) = 0 \text{ for all } \psi\in\mathcal{G}_{0a}(D),$$

where

$$\mathcal{M}_k(\mathbf{u},\psi) := \left((A^{-1}-I)^{-1}(\nabla\nabla\cdot\mathbf{u}+k^2\mathbf{u}), \nabla\nabla\cdot\psi+k^2\psi\right)_D + k^4\left(\mathbf{u},\psi\right)_D,$$

and $\mathcal{N}$ is defined as in (2.2).

In a similar fashion, the transmission eigenvalue problem can also be written as the problem of finding a non-trivial $\mathbf{u}\in\mathcal{G}_{0a}(D)$ and k such that

$$\begin{cases} (\nabla\nabla\cdot+k^2)(I-A^{-1})^{-1}(\nabla\nabla\cdot\mathbf{u}+k^2A^{-1}\mathbf{u}) = 0 & \text{in} \quad D, \\ \nu\cdot\mathbf{u} = 0 & \text{on} \quad \Gamma, \\ \nabla\cdot\mathbf{u} = 0 & \text{on} \quad \Gamma_a, \\ \nu\cdot(I-A^{-1})^{-1}(\nabla\nabla\cdot\mathbf{u}+k^2A^{-1}\mathbf{u}) = 0 & \text{on} \quad \Gamma_m, \end{cases}$$

where the natural boundary condition for $\mathbf{u}$ on Γ_m arises from the condition $\nu\cdot\mathbf{v}=0$ in (2.30).

In this way, the corresponding variational form is to find a function $\mathbf{u}\in\mathcal{G}_{0a}(D), \mathbf{u}\neq 0$ and $k\in\mathbb{R}$ such that

$$\tilde{\mathcal{M}}_k(\mathbf{u},\psi) - k^2\mathcal{N}(\mathbf{u},\psi) = 0 \text{ for all } \psi\in\mathcal{G}_{0a}(D),$$

where

$$\tilde{\mathcal{M}}_k(\mathbf{u},\psi) := \left((I-A^{-1})^{-1}(\nabla\nabla\cdot\mathbf{u}+k^2A^{-1}\mathbf{u}), \nabla\nabla\cdot\psi+k^2A^{-1}\psi\right)_D + k^4\left(A^{-1}\mathbf{u},\psi\right)_D,$$

and $\mathcal{N}$ is the same as before.

The analysis now proceeds along similar lines to the analysis of the TE case. Clearly, $\mathcal{M}_k(\cdot,\cdot)$ and $\tilde{\mathcal{M}}_k(\cdot,\cdot)$ are continuous sesquilinear forms in $\mathcal{G}_{0a}(D) \times \mathcal{G}_{0a}(D)$. Let us denote by M_k and $\tilde{M}_k$ the bounded linear operators from $\mathcal{G}_{0a}(D)$ to $\mathcal{G}_{0a}(D)$ defined using the Riesz representation theorem by

$$\begin{aligned}(M_k \mathbf{u}, \psi)_{\mathcal{G}_{0a}(D)} &= \mathcal{M}_k(\mathbf{u}, \psi), \\ (\tilde{M}_k \mathbf{u}, \psi)_{\mathcal{G}_{0a}(D)} &= \tilde{\mathcal{M}}_k(\mathbf{u}, \psi),\end{aligned}$$

for all $\psi \in \mathcal{G}_{0a}(D)$.

Now we can state the following theorems:

Theorem 2.1.4 *(Discreteness) Assume that* $\bar{\xi} \cdot (A^{-1} - I)^{-1}\xi \geq \alpha|\xi|^2$ *in* D *and for all* $\xi \in \mathbb{C}^2$ *where* $\alpha > 0$ *is a constant. Then*

1. *The set of transmission eigenvalues is at most discrete and does not accumulate at* 0.
2. *All transmission eigenvalues, if they exist, are such that* $k^2 \geq \dfrac{\alpha}{1+\alpha}\mu(D)$ *where* $\mu(D)$ *is the first eigenvalue of* $-\Delta$ *on* D *with boundary conditions stated in Lemma 2.1.4.*

Theorem 2.1.5 *(Discreteness) Assume that* $\bar{\xi} \cdot A^{-1}(I - A^{-1})^{-1}\xi \geq \alpha|\xi|^2$ *in* D *and for all* $\xi \in \mathbb{C}^2$ *where* $\alpha > 0$ *is a constant. Then*

1. *The set of transmission eigenvalues is at most discrete and does not accumulate at 0.*
2. *All transmission eigenvalues, if they exist, are such that* $k^2 \geq \mu(D)$ *where* $\mu(D)$ *is the first eigenvalue of* $-\Delta$ *on* D *with boundary conditions stated in Lemma 2.1.4.*

The proof of these two theorems is similar to the proof of Theorem 3.1 and Theorem 3.2 in [11] and it is worth pointing out that whenever the Poincaré inequality is applied in the proof, it should be the one from Lemma 2.1.4 as using Lemma 2.1.3 in our proof of Theorem 2.1.1 and Theorem 2.1.2.

Next, we want to provide bounds on transmission eigenvalues and we have the following corollary:

Corollary 2.1.3 *(Faber-Krahn inequality)*

1. *Assume that* $\|A^{-1}\|_2 \geq \delta > 1$ *for all* $\mathbf{x} \in D$ *and some constant* δ*. Then, if* k *is a transmission eigenvalue,*
$$k^2 \geq \frac{\lambda(D)}{\sup_D \|A^{-1}\|_2}.$$

2. *Assume that* $0 < \beta \leq \|A^{-1}\|_2 \leq \delta < 1$ *for all* $\mathbf{x} \in D$ *and some constants* β *and* δ*. Then, if* k *is a transmission eigenvalue,*
$$k^2 \geq \lambda(D).$$

Here $\lambda(D)$ *is the first eigenvalue of* $-\Delta$ *on* D *with boundary conditions stated in Lemma 2.1.4.*

Proof: The proof is similar to the proof of Theorem 3.3 in [11]. □

2.1.5.3 Existence of Transmission Eigenvalues

The next result confirms that transmission eigenvalue exists:

Theorem 2.1.6 *Given* A *defined in Section 2.1.5.1, let* $A^{-1} \in L^{\infty}(D, \mathbb{R}^{2\times 2})$ *satisfy either one of the following assumptions:*

(1). $1 + \alpha \leq n_* \leq (\bar{\xi} \cdot A(x)^{-1}\xi) \leq n^* \leq \infty,$

(2). $0 < n_* \leq (\bar{\xi} \cdot A(x)^{-1}\xi) \leq n^* < 1 - \beta,$

for every $\xi \in \mathbb{C}^2$ *such that* $\|\xi\| = 1$ *and some constants* $\alpha > 0$ *and* $\beta > 0$*. Then there exists an infinite set of transmission eigenvalues with* $+\infty$ *as the only accumulation point.*

Proof: This theorem can be proven in the same way as Theorem 2.1.3. In particular, the proof is similar to the discussion of Problem 2 of Section 2.3 as well as Theorem 2.10 in [16] by noting that $\mathcal{H}_0(D)$ in that reference is a subspace of $\mathcal{G}_{0a}(D)$ here. □

In the same way as in our discussion in Note 2.1.2, we need the first transmission eigenvalue corresponding to the disk of radius one as well as the half disk of radius one, respectively.

For the full unit disk S_1, we need to consider the first real non-zero eigenvalue of

$$\begin{cases} \Delta w + k^2 n_0 w = 0 & \text{in} \quad S_1, \\ \Delta v + k^2 v = 0 & \text{in} \quad S_1, \\ w - v = 0 & \text{on} \quad \partial S_1, \\ \dfrac{1}{n_0}\dfrac{\partial w}{\partial \nu} - \dfrac{\partial v}{\partial \nu} = 0 & \text{on} \quad \partial S_1, \end{cases} \tag{2.34}$$

where $n_0 > 0$ is a constant and $n_0 \neq 1$. This is standard and can be computed by separation of variables and then finding the zeros of the appropriate determinant.

For the half unit disk $\hat{S}_1$, we need to consider the first real non-zero eigenvalue of

$$\begin{cases} \Delta w + k^2 \hat{n}_0 w = 0 & \text{in} \quad \hat{S}_1, \\ \Delta v + k^2 v = 0 & \text{in} \quad \hat{S}_1, \\ w - v = 0 & \text{on} \quad \partial \hat{S}_{1a}, \\ \dfrac{1}{\hat{n}_0}\dfrac{\partial w}{\partial \nu} - \dfrac{\partial v}{\partial \nu} = 0 & \text{on} \quad \partial \hat{S}_{1a}, \\ \dfrac{1}{\hat{n}_0}\dfrac{\partial w}{\partial \nu} = 0, \dfrac{\partial v}{\partial \nu} = 0 & \text{on} \quad \partial \hat{S}_{1m}, \end{cases} \tag{2.35}$$

where $\hat{n}_0 > 0$ is a constant and $\hat{n}_0 \neq 1$. Similar to the argument before Corollary 2.1.2, we can convert this to a standard problem by extending the half disk $\hat{S}_1$ to a full disk S_1 and drop the solutions that do not meet the boundary conditions on $\partial \hat{S}_{1m}$, that is, odd eigenfunctions. Then the eigenvalues can be computed by finding the zeros of the appropriate determinant.

Without confusion, we use the same notation as in Section 2.1.4.3 and denote by k_{1,n_*} and $\hat{k}_{1,n^*}$ the first transmission eigenvalue of problem (2.34) with $n_0 := n_*$ and problem (2.35) with $n_0 := n^*$, respectively. With no surprises, we have results similar to Corollary 2.1.2 and Remark 2.1.3:

Corollary 2.1.4 *Assume that $A^{-1} \in L^\infty(D, \mathbb{R}^{2\times 2})$, and let $k_{1,D,A^{-1}(x)}$ be the first transmission eigenvalue for (2.28) to (2.30) with $\mathbf{w} - \mathbf{v} \in \mathcal{G}_{0a}(D)$. Then*

(1). If $1+\alpha \le n_* \le (\bar{\xi}\cdot A(x)^{-1}\xi) \le n^* \le \infty$ for every $\xi \in \mathbb{C}^2$ such that $\|\xi\| = 1$, and some constant $\alpha > 0$, then

$$0 < \frac{\hat{k}_{1,n^*}}{r_2} \le k_{1,D,A^{-1}(x)} \le \frac{k_{1,n_*}}{r_1}.$$

There are at least $m(\epsilon)$ transmission eigenvalues in the interval $\left[\frac{\hat{k}_{1,n^*}}{r_2}, \frac{k_{1,n_*}}{\epsilon}\right]$.

(2). If $0 < n_* \le (\bar{\xi}\cdot A(x)^{-1}\xi) \le n^* < 1-\beta$ for every $\xi \in \mathbb{C}^2$ such that $\|\xi\| = 1$, and some constant $\beta > 0$, then

$$0 < \frac{\hat{k}_{1,n_*}}{r_2} \le k_{1,D,A^{-1}(x)} \le \frac{k_{1,n^*}}{r_1}.$$

There are at least $m(\epsilon)$ transmission eigenvalues in the interval $\left[\frac{\hat{k}_{1,n_*}}{r_2}, \frac{k_{1,n^*}}{\epsilon}\right]$.

Remark 2.1.4 *From the proof of Theorem 2.1.4, Theorem 2.1.5 and Corollary 2.1.4 we have that*

1. *If $1+\alpha \le n_* \le (\bar{\xi}\cdot A(x)^{-1}\xi) \le n^* \le \infty$, then*

$$k_{1,D,A^{-1}(x)} \ge \max\left(\frac{\hat{k}_{1,n^*}}{r_2}, \sqrt{\frac{\lambda(D)}{n^*}}\right).$$

2. *If $0 < n_* \le (\bar{\xi}\cdot A(x)^{-1}\xi) \le n^* < 1-\beta$, then*

$$k_{1,D,A^{-1}(x)} \ge \max\left(\frac{\hat{k}_{1,n_*}}{r_2}, \sqrt{\lambda(D)}\right).$$

Here $\lambda(D)$ is the first eigenvalue of $-\Delta$ on D with boundary conditions stated in Lemma 2.1.4.

2.2 Numerical examples

In this section we shall present some numerical investigations of the determination of transmission eigenvalues for both isotropic media ((2.7) to (2.11)) and orthotropic media ((2.23) to (2.27)) from scattering data. We shall restrict ourselves to simple 2D cases with flat Γ_m (see Remark 2.1.1) in order to make a comparison with the well-studied results for the cases where the conducting surface is absent. In particular we assume:

- Isotropic media: $n(\mathbf{x}) > 0$ $(n(\mathbf{x}) \neq 1)$,
- Orthotropic media: $A = \dfrac{1}{n(\mathbf{x})} I$ with $n(\mathbf{x}) > 0$ $(n(\mathbf{x}) \neq 1)$,

where $n(\mathbf{x})$ is a piecewise constant function and I is 2×2 identity matrix.

2.2.1 Configuration

We consider scattering by a domain D illuminated by TE / TM polarized waves due to point sources located along a line above D as shown in Figure 2.2 where D lies on an infinite perfectly electrically conducting half plane $\{(x, y) \mid y < 0\}$. In Figure 2.2

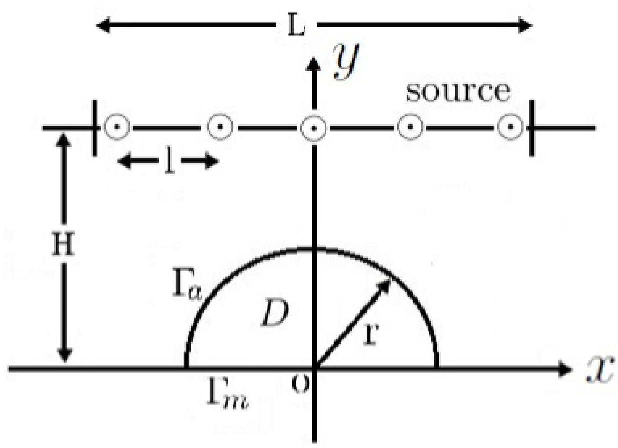

Figure 2.2: Geometric notation for plane supported scattering problem.

and later in this section we use the following notation

D	:	The dielectric scatterer (not necessarily a half disk),
r	:	Radius of the scatterer when D is a half disk,
Γ_a	:	Interface between scatterer and air,
Γ_m	:	Interface between scatterer and metallic substrate,
H	:	Height of the line where point sources are located,
L	:	Length of the line where point sources are located,

l : Distance between neighboring point sources.

Here the receivers are at the same locations as the point sources.

2.2.2 Formulae and Methods

2.2.2.1 Image Theory and Green's Function

The domain of the scattering problem is the upper half plane. In order to either generate synthetic data from a forward solver or compute the inverse problem, we shall need the Green's function due to a point source located in the upper half plane. To facilitate the computation, we use image theory to extend the domain to the entire plane by putting a point sink at the mirrored position of each point source with respect to the x-axis (see Figure 2.3). Then, without violating the boundary condition on Γ_m, the Green's function on the entire domain becomes the source-sink combination in pairs, that is,

- TE case:

$$G(\mathbf{x}, \mathbf{z}) = H_0^{(1)}(k|\mathbf{x} - \mathbf{z}^+|) - H_0^{(1)}(k|\mathbf{x} - \mathbf{z}^-|), \tag{2.36}$$

- TM case:

$$G(\mathbf{x}, \mathbf{z}) = H_0^{(1)}(k|\mathbf{x} - \mathbf{z}^+|) + H_0^{(1)}(k|\mathbf{x} - \mathbf{z}^-|), \tag{2.37}$$

where $H_0^{(1)}$ represents Hankel function of the first kind of order 0, and if $\mathbf{z}^+ = (z_1, z_2)$, $z_2 > 0$, then $\mathbf{z}^- = (z_1, -z_2)$.

Note 2.2.1 *For TE case, the boundary condition on Γ_m can be satisfied when the extension is an odd extension of the solution in the upper half plane and, for TM case, when the extension is an even extension of the solution in the upper half plan.*

2.2.2.2 Forward Problem

To generate the synthetic data for the forward problem, we compute the total field inside and outside the scatterer $(D \cup D')$ illuminated by one pair of source-sink combination given by (2.36) or (2.37). For example, in the TE case, we have

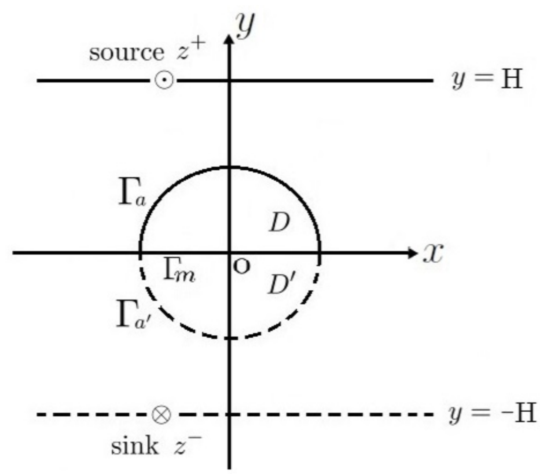

Figure 2.3: Computational domain for the plane supported transmission eigenvalue problem.

1. Series representation

   For a disk, we have explicit formulas for the total field in polar coordinate:

$$u = \begin{cases} u_{in} = \displaystyle\sum_{m=-\infty}^{\infty} \alpha_m J_m(k\sqrt{n(\mathbf{x})}r)\exp(im\theta) & \text{in} \quad D \cup D', \\ u_{out} = \displaystyle\sum_{m=-\infty}^{\infty} \beta_m H_m^{(1)}(kr)\exp(im\theta) + u^{inc} & \text{in} \quad (D \cup D')^c, \end{cases} \tag{2.38}$$

   where J_m represents Bessel function of order m, $H_m^{(1)}$ represents Hankel function of the first kind of order m, (r, θ) represents the polar coordinates of $\mathbf{x}$ and α_m and β_m are coefficients that can be determined by boundary conditions on the interface $|\mathbf{x}| = r$ as follows:

$$\begin{cases} u_{in} = u_{out} & \text{(Continuity Condition)}, \\ \dfrac{\partial u_{in}}{\partial r} = \dfrac{\partial u_{out}}{\partial r} & \text{(Flux Condition)}. \end{cases}$$

2. Coupling procedure

   For a general domain, we employ the finite element method (using quadratic finite elements) inside a circular artificial domain Ω (enclosing $D \cup D'$ and excluding the point sources and sinks) coupled with series representation outside Ω. In this way, the governing equations in each domain are:

   - In Ω:

$$\begin{cases} \Delta u + k^2 n(\mathbf{x}) u = 0, \\ n(\mathbf{x}) \neq 1 \text{ in } D \cup D', \\ n(\mathbf{x}) = 1 \text{ in } (D \cup D')^c. \end{cases}$$

- In Ω^c:
$$\begin{cases} \Delta u^s + k^2 u^s = 0, \\ u = u^s + u^{inc}, \\ u^s \text{ satisfies the Sommerfeld radiation condition.} \end{cases}$$

On $\partial\Omega$, a coupling procedure is carried out by using the Neumann to Dirichlet (NtD) mappings:

$$T : \lambda \longrightarrow u|_{\partial\Omega} \quad \text{where} \quad \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = \lambda,$$

where λ can be written as series expansion

$$\lambda = \sum_m \lambda_m \phi_m = \sum_m \lambda_m \exp(im\theta), \tag{2.39}$$

and T diagonalizes in the Fourier space (see [35]).

2.2.2.3 Inverse Problem

For solving the inverse problem of determining the shape of D or computing the transmission eigenvalues, we use the Linear Sampling Method (LSM) which is based on the Near Field Equation (NFE): we seek $g_{\mathbf{z}} \in L^2(\Sigma)$ such that

$$(Ng_{\mathbf{z}})(\mathbf{x}) := \int_\Sigma u^s(\mathbf{x},\mathbf{y}) g_{\mathbf{z}}(\mathbf{y})\, ds(\mathbf{y}) = G(\mathbf{x},\mathbf{z}), \qquad \forall \mathbf{x} \in \Upsilon \tag{2.40}$$

where

- $u^s(\cdot,\mathbf{y})$ is the scattered field (in the presence of the scatterer) due to the incident wave for the appropriate source at $\mathbf{y}$. It is measured on Σ and hence is referred to as "Near Field Data" (NFD).

- $g_{\mathbf{z}}(\mathbf{y})$ is the indicator function due to a source at $\mathbf{z}$.

- $G(\cdot,\mathbf{z})$ is the background Green's function (in the absence of the scatterer) due to a source at $\mathbf{z}$. This is just the fundamental solution for a point source at $\mathbf{z}$ given by (2.36) or (2.37) as appropriate.

- Σ is the curve on which the point sources lie. Integration is performed over this curve. Specifically, after discretization, point sources are uniformly distributed on a line of length L above the scatterer with height H (see Figure 2.2).

- Υ is the curve on which the receivers are located. For simplicity, as is usual with the LSM, they are on the same locations as the point sources. So $\Upsilon = \Sigma$.

Usually, the NFD is corrupted with random noise of size, for example, 1% in the relative L^2 norm. Specifically, for a finite number of sources and receivers, the NFD is represented by a matrix N, then for each entry $N_{i,j}$, the corresponding corrupted data is $\tilde{N}_{i,j} = N_{i,j}(1 + \epsilon\eta)$ where ϵ is the noise level (for example, 1%) and $\eta \in \mathbb{R}$ is a random number following uniform distribution on $[-1, 1]$. It is worth mentioning that for some scatterers with certain values of $n(\mathbf{x})$, the transmission eigenvalues which can be determined from the NFD are very sensitive to the noise level. Thus, we sometimes assume less than 1% random relative noise on the data as noted in the upcoming sections.

The integral equation (2.40) is discretized by using the Trapezoidal rule and collocation at the integration points and then the discrete problem is solved using techniques appropriate for ill-posed problems such as Tikhonov Regularization or the Truncated Singular Value Decomposition (TSVD) [33]. The choice of parameters for these methods are based on either an empirical value or approaches such as the Generalized Morozov Principle [13], the L-Curve Criterion or Generalized Cross-Validation (GCV) [33].

In practice, for the reconstruction of the shape of D, we choose several sampling points in the region we expect D to be found, compute the discrete l^2 norm of $g_{\mathbf{z}}$, and make contour plot of the reciprocal of these values. We expect the shape of D are indicated by contour lines of the reciprocal of the norm of $g_{\mathbf{z}}$. For the determination of transmission eigenvalues, we choose several sampling points inside the original scatterer D (assumed a priori known), average the discrete l^2 norm of $g_{\mathbf{z}}$ at these points for different wavenumbers, and plot these values against wavenumber. We expect that transmission eigenvalues are indicated by peaks in the graph.

Note that in applications both procedures would be applied to measured data directly with no need to solve any forward problem.

Remark 2.2.1 *It is worth pointing out that, in this thesis, we have focused on the novel problem of analyzing the interior transmission eigenvalue problem with mixed boundary* *data. With this well in hand and the discussion in Sections 4.5, 4.6, 6.5 and 6.6 of [10], it should be possible to prove the analogue of Theorem 1.3.1 and Theorem 1.3.2 using the near field equation with limited aperture data.*

2.2.3 Numerical Results

2.2.3.1 Forward Problem

To give some idea of the fields for the forward problem, we present results from our forward solvers for the TE case described in Section 2.2.2.2. Specifically, we use series representations (2.38) of the scattered field outside and the total field inside for scattering from a half disk on a perfect conducting plane. And we use the coupling of finite element solution and series representation for scattering from a half square on a perfect conducting plane. The parameters are:

- Wavenumber: $k = 4.5$
- Relative permittivity of scatterer: $n(\mathbf{x}) = 4$
- Location of point source in polar coordinates: $\mathbf{z}^+ = (\rho, \phi) = (3, \pi/2)$ and thus $\mathbf{z}^- = (\rho, \phi) = (3, -\pi/2)$
- For half disk (D_1)
  1. Radius of half disk: $r = 1/2$
  2. Terms kept in the series representation (2.38) of background Green's function: from $m = -14$ to $m = 14$ so that all the modes of major impact are included (see plot of coefficients in Figure 2.4).
- For half square (D_2)
  1. Size of half square: $[-0.5, 0.5] \times [0, 0.5]$
  2. Radius of circular artificial domain for finite element solver: 1.25
  3. Terms kept in the series representation (2.39) for NtD mapping in coupling procedure: from $m = -30$ to $m = 30$ so that all the coefficients λ_m of major impact are included (see plot of coefficients in Figure 2.4).

A plot of the total field (real part, imaginary part and absolute value) due to a half disk (D_1) and the coefficients α_m's and β_m's in the series expansions (see (2.38)) are given in Figure 2.4.

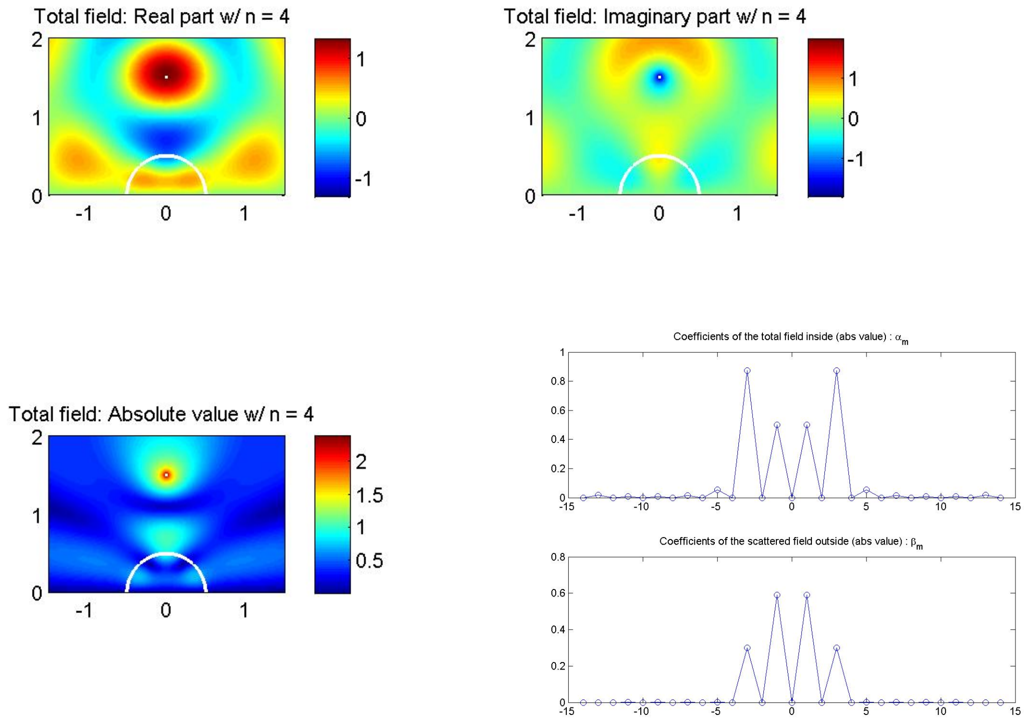

Figure 2.4: Plot of total field in the presence of a half disk D_1 of radius 1/2 (white curve) using image theory and coefficients for series expansions of the total field inside D_1 and the scattered field outside D_1 (see (2.38)).

A plot of the total field (real part, imaginary part and absolute value) due to a half square (D_2) and the coefficients λ_m's in the NtD mapping (see (2.39)) using the coupling procedure is given in Figure 2.5.

2.2.3.2 Inverse Problem for Shape Reconstruction

To illustrate the reconstruction of the half disk D_1 and half square D_2 described in Section 2.2.3.1, we put several point sources/receivers on a line above D_1 and D_2, respectively, as shown in Figure 2.2. For each point source, we compute the scattered field using the forward solvers described in Section 2.2.3.1 for D_1 and D_2, respectively, and collect the scattering data at all the receivers. Then we use Tikhonov regularization

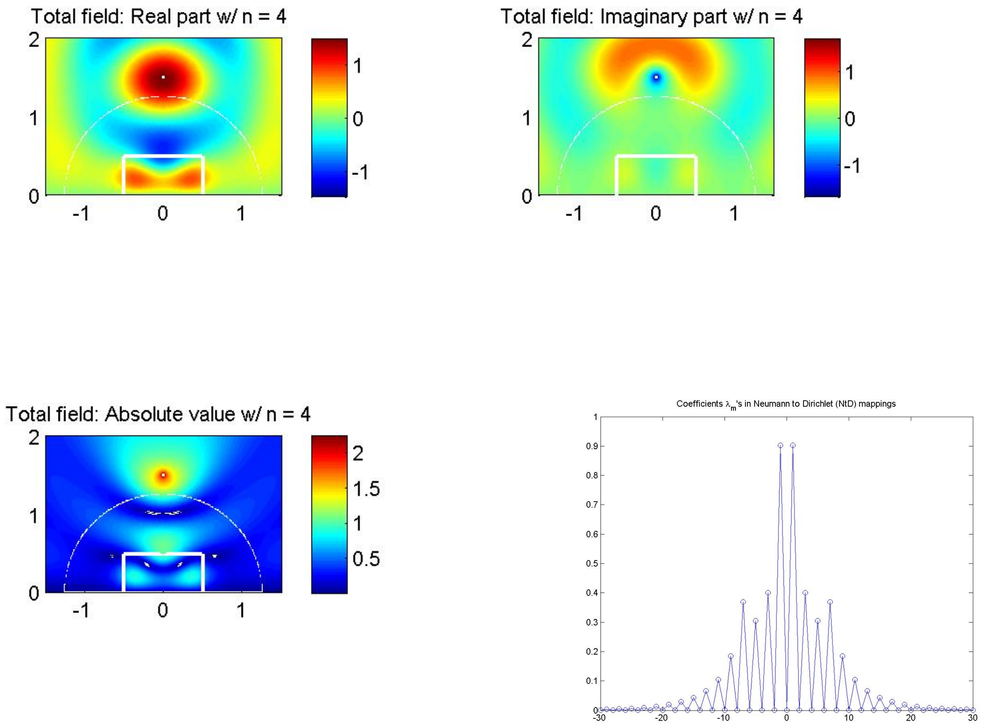

Figure 2.5: Plot of total field in the presence of a half square D_2 of size $[-0.5, 0.5] \times [0, 0.5]$ (white line) using image theory and coefficients λ_m's in series expansion of NtD mappings (see (2.39)).

combined with the Generalized Morozov Principle to solve the NFE (2.40).

Other parameters are:

- Location of point sources/receivers: uniformly distributed along a line of height $y = H = 2/3$ between $x = -L/2 = -3/2$ and $x = L/2 = 3/2$, distance between neighboring points is $\lambda/10$ where $\lambda = 2\pi/k$ is the wavelength.
- Level of random relative noise added on the scattering data: %1.
- Region of sampling points: a rectangle of size $[-1, 1] \times [0.05, 1]$.

The contour plot of the reconstruction of half disk D_1 is given in Figure 2.6 and the contour plot of the reconstruction of half square D_2 is given in Figure 2.7.

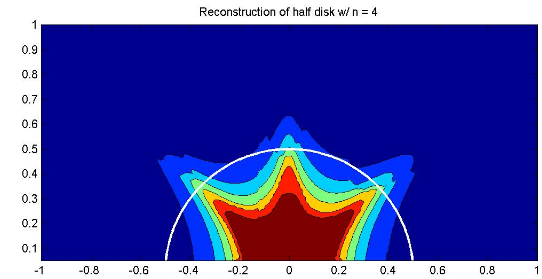

Figure 2.6: Contour plot of reconstruction of half disk D_1 of radius $1/2$ (white curve) with relative permittivity $n(\mathbf{x}) = 4$ by LSM under wavenumber $k = 4.5$.

2.2.3.3 Computation of Transmission Eigenvalues

To validate the computation of transmission eigenvalues from near field data, we shall present some numerical results for both TE and TM cases. Specifically, we test the method on a half disk and a half square with a constant $n(\mathbf{x})$, respectively. In particular, let the radius of half disk be $r = 1/2$, the size of half square be $[-0.5, 0.5] \times [0, 0.5]$ as in Section 2.2.3.1 and $n(\mathbf{x}) = 4$ or $n(\mathbf{x}) = 16$. The other parameters are:

- Wavenumber k

k	TE case	TM case
$n(\mathbf{x}) = 4$	from 4.5 to 8.5 with step size 0.02	from 4.0 to 8.0 with step size 0.02
$n(\mathbf{x}) = 16$	from 1.5 to 3.5 with step size 0.01	from 1.5 to 3.5 with step size 0.01

Table 2.1: Range of wavenumber for computation of transmission eigenvalue

- Location of point sources: uniformly distributed along $y = H = 3/2$ between $x = -L/2 = -3/2$ and $x = L/2 = 3/2$, distance between neighboring points is $\lambda/10$ where $\lambda = 2\pi/k$ is the wave length.
- Sample points $\mathbf{z} = (z_1, z_2)$ for computing average of $\|g_{\mathbf{z}}\|_2$: points inside the scatterer which are not close to the boundary. Specifically,

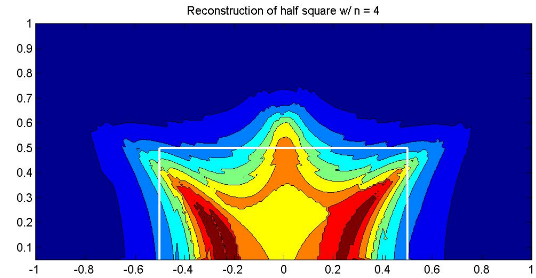

Figure 2.7: Contour plot of reconstruction of half square D_2 of size $[-0.5, 0.5] \times [0, 0.5]$ (white line) with relative permittivity $n(\mathbf{x}) = 4$ by LSM under wavenumber $k = 4.5$.

- For half disk: points satisfying $|\mathbf{z}| < 0.45$ and $z_2 > 0.1$.
- For half square: points satisfying $|z_1| < 0.45$ and $0.1 < z_2 < 0.45$.

We concentrate on detecting the first transmission eigenvalue which is important for non-destructive testing and then compare our results with well-studied transmission eigenvalues for the full disk or the full square in the absence of a conducting surface. In particular, for the half disk, we shall present results for both cases with $n(\mathbf{x}) = 4$ and $n(\mathbf{x}) = 16$. For the half square, since it is much harder to come up with an explicit formula for transmission eigenvalues of a full square, we just present results for TE case with $n(\mathbf{x}) = 16$ and TM case with $n(\mathbf{x}) = 4$ which can be immediately compared with the published data [14, 23].

We also note that we use the coupling procedure (for the TE case, it is described in Section 2.2.2.2) to generate the synthetic data both for the scattering of a half disk and a half square.

1. TE case
   First, we list the first few transmission eigenvalues for full disk and full square in the absence of the conducting background (see Table 2.2). In Table 2.2, the results for $n(\mathbf{x}) = 16$ are from Table 1 and Table 3 in [23], respectively. The

TE	Full Disk	Full Square
$n(\mathbf{x}) = 4$	5.8052, 6.7684, 6.8241, 7.9529	/
$n(\mathbf{x}) = 16$	1.9880, 2.6129, 3.2240	≈ 1.89, ≈ 2.46, ≈ 2.47, ≈ 2.89

Table 2.2: Standard transmission eigenvalue for disk of radius 1/2 and unit square for the TE case.

result for $n(\mathbf{x}) = 4$ for the full disk can be obtained by using the same argument as in Section 3.1 of [23].

(a) Half Disk

i. $n(\mathbf{x}) = 16$

We apply Tikhonov Regularization combined with the Generalized Morozov Principle [13]. The NFD is corrupted with 1% random relative noise. The result is shown in Figure 2.8. Vertical lines indicate trans-

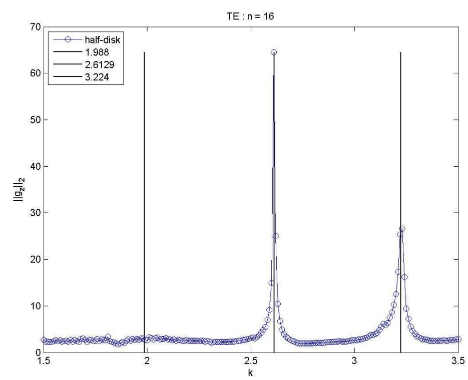

Figure 2.8: TE case, half disk, $n(\mathbf{x}) = 16$.

mission eigenvalues of the full disk and peaks in the graph should give eigenvalues of the half disk. The leftmost eigenvalue of the full disk is not an eigenvalue of the conducting surface backed half disk since to be an eigenfunction for the half disk, the eigenfunction of the full disk would need to be an odd function, which is not the case. Thus the first eigenvalue is different for the conducting surface backed scatterer and is also well discriminated.

Actually, from the discussion before Corollary 2.1.2, the first transmission eigenvalue due to the above set up can be computed explicitly (≈ 2.61). In comparison with Table 2.2 and the corresponding derivation in Section 3.1 of [23], the first real transmission eigenvalue in our case should be the second real transmission eigenvalue for a full disk in the absence of conducting surface. We indeed observe good agreement from Figure 2.8.

ii. $n(\mathbf{x}) = 4$

We use exactly the same approach with the same noise level as for $n(\mathbf{x}) = 16$. The result is shown in Figure 2.9. Note that in this plot we

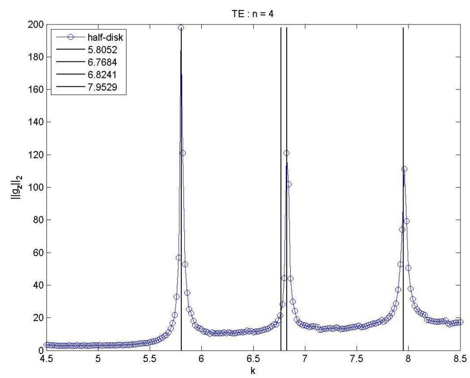

Figure 2.9: TE case, half disk, $n(\mathbf{x}) = 4$.

detect the first transmission eigenvalue of the full disk. This is consistent with the fact that this transmission eigenvalue actually corresponds to an odd eigenfunction for the full disk. Also, we do not detect the second transmission eigenvalue for a full disk because that one indeed corresponds to an even eigenfunction, and hence is not an eigenvalue of the half disk.

(b) Half Square with $n(\mathbf{x}) = 16$

It turns out that for this case, the result (see Figure 2.10) is not as promising as for the previous examples when using the same approach and noise level as before. The peaks of $\|g_{\mathbf{z}}\|_2$ do not align with any eigenvalues for the full square. Practically, the norm of $g_{\mathbf{z}}$ is sensitive to the choice of regularization parameter λ as well as the size of the random relative noise. However, we can still obtain the desired result with proper control of the random noise level and parameter λ. As an illustration, the result for Tikhonov Regularization with an empirical choice for $\lambda = 10^{-12}$ and $10^{-3}\%$ random relative noise on the NFD is shown in Figure 2.11. It is worth pointing out that the possible reason for issues with detecting the first transmission eigenvalue (for example, in Figure 2.10, there are two peaks fairly close to the desired location of transmission eigenvalues) is that the second and third transmission eigenvalues for the full square are very close to each other.

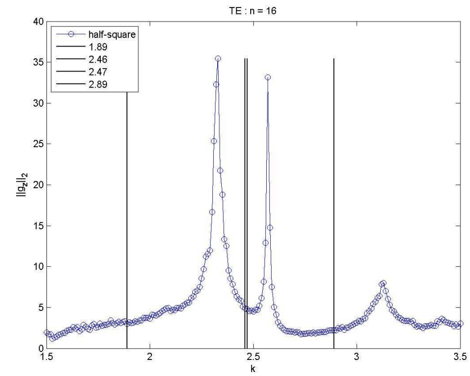

Figure 2.10: TE case, half square, $n(\mathbf{x}) = 16$.

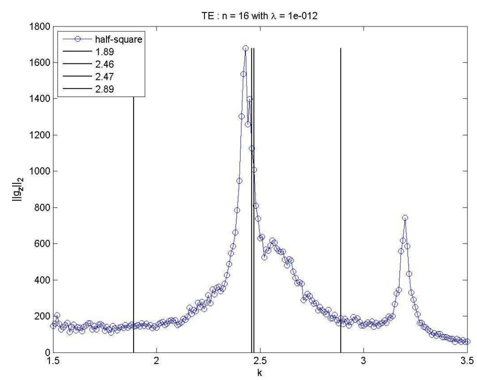

Figure 2.11: TE case, half square, $n(\mathbf{x}) = 16$, $10^{-3}\%$ random relative noise, regularization parameter $\lambda = 10^{-12}$.

2. TM Case
   First, we list the first few eigenvalues for full disk and full square (see Table 2.3). The result for the full square is from Table 1 in [14]. The results for the full disk can be obtained by mimicking the discussion of Section 3.1 in [23], the difference is that instead of having problem as in Note 2.1.2, we have problem (2.34).

   (a) Half Disk

      i. $n(\mathbf{x}) = 16$
         We use Tikhonov Regularization combined with the Generalized Morozov Principle. The NFD is corrupted with 1% random relative noise. The result is shown in Figure 2.12. Here the first transmission eigenvalue detected is the same one for the full disk since the corresponding eigenfunction is even.

TM	Full Disk	Full Square
$n(\mathbf{x}) = 4$	5.8052, 6.8008, 7.5660, 7.6066	≈ 5.3
$n(\mathbf{x}) = 16$	2.0840, 2.6129, 2.6633, 3.2656	/

Table 2.3: Standard transmission Eigenvalue for disk of radius 1/2 and unit square for the TM case.

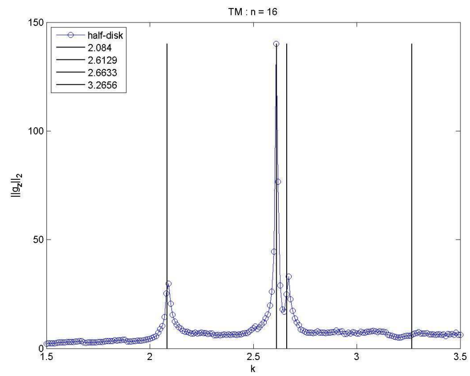

Figure 2.12: TM case, half disk, $n(\mathbf{x}) = 16$.

ii. $n(\mathbf{x}) = 4$

We first use the same approach with the same noise level as for the previous example. The result is shown in Figure 2.13. Clearly, this is not a successful determination of the eigenvalues. Indeed, similar to the half square with $n(\mathbf{x}) = 16$ in the TE case, we have sensitivity issues depending on the choice of the regularization parameter λ and the random relative noise level.

It turns out that, with proper control of the random relative noise level, the Tikhonov Regularization with an empirical choice for λ can detect the first eigenvalue. As an illustration, the results for Tikhonov Regularization with parameter $\lambda = 10^{-12}$ and $10^{-2}\%$ random relative noise on the NFD is shown in Figure 2.14.

Alternative Measurement Geometries

- If we change the location of points sources by letting Σ in the NFE (2.40) be an arc of radius 3/2 between angles $\pi/6$ and $5\pi/6$, we get better results (see Figure 2.15). As usual we use Tikhonov Regularization combined with the Generalized Morozov Principle and 1% random relative noise on the NFD.
- If we elongate Σ in the NFE (2.40) by letting the length of line be $L = 6$ so that the point sources are uniformly distributed between

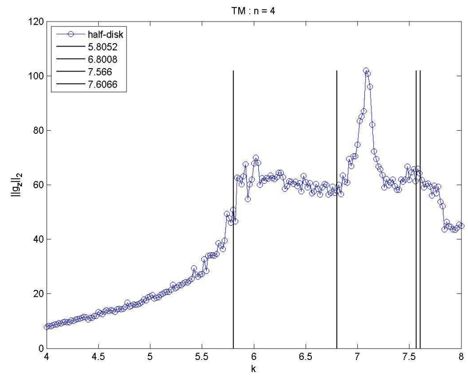

Figure 2.13: TM case, half disk, $n(\mathbf{x}) = 4$.

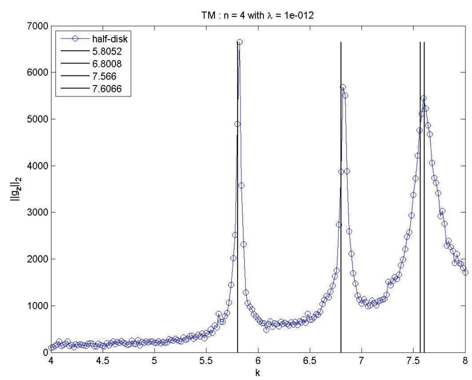

Figure 2.14: TM case, half disk, $n(\mathbf{x}) = 4$, $10^{-2}\%$ random relative noise, regularization parameter $\lambda = 10^{-12}$.

$x = -L/2 = -3$ and $x = L/2 = 3$, we can also get good results (see Figure 2.16).

(b) Half Square with $n(\mathbf{x}) = 4$
First we present the result for Tikhonov Regularization combined with Generalized Morozov Principle (see Figure 2.17). The NFD is corrupted with 1% random relative noise. In the same way as for the half disk with $n(\mathbf{x}) = 4$ above, the result is not accurate. However, with proper control of the regularization parameter and noise level, we can observe good agreement between the peaks and known eigenvalues. For instance, the result for Tikhonov Regularization with parameter $\lambda = 10^{-12}$ and $10^{-3}\%$ random relative noise on the NFD is shown in Figure 2.18. Here the first transmission eigenvalue detected should be the first one for the full square since the corresponding

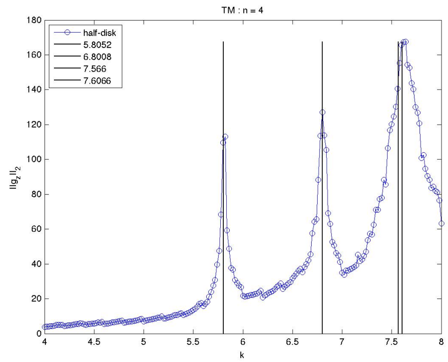

Figure 2.15: TM case, half disk, $n(\mathbf{x}) = 4$, using curved measurement geometry.

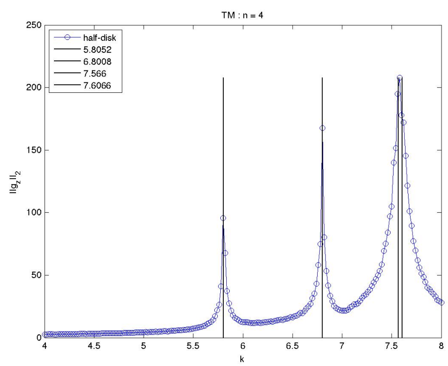

Figure 2.16: TM case, half disk, $n(\mathbf{x}) = 4$, using elongated measurement geometry.

eigenfunction is even.

Alternative Measurement Geometries. Similar to the case of a half disk with $n(\mathbf{x}) = 4$, we can have good results by letting Σ be an arc or elongating the length of Σ (see Figure 2.19 and Figure 2.20, respectively).

Remark 2.2.2 *The numerical results above show that the determination of transmission eigenvalues for the TM case is more delicate than for the TE case. This is not uncommon as, for instance, we note that numerical examples in Cossonniere's thesis [26] also suggest that determining TM mode transmission eigenvalues from far field data is more difficult than for the TE mode (see Figure 6.13 of [26] where the LSM predicts eigenvalues shifted from their true values).*

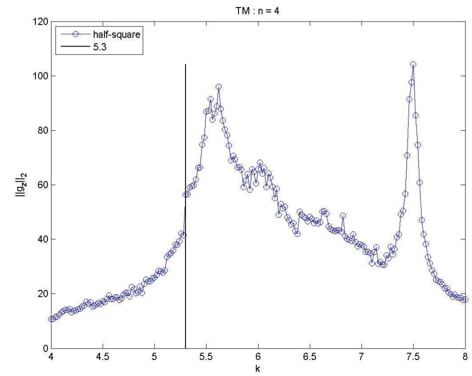

Figure 2.17: TM case, half square, $n(\mathbf{x}) = 4$.

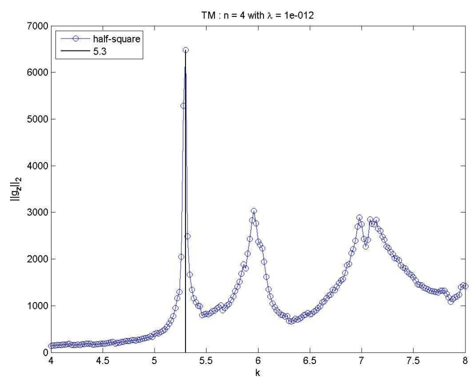

Figure 2.18: TM case, half square, $n(\mathbf{x}) = 4$, $10^{-3}\%$ random relative noise, regularization parameter $\lambda = 10^{-12}$.

In conclusion, we have shown that, for both TE and TM scattering, it is possible to identify the first transmission eigenvalue from near field data with a standard regularization approach (Tikhonov Regularization combined with the Generalized Morozov Principle). Some problems show great sensitivity to noise (for example, the TM case for half disk or half square with $n(\mathbf{x}) = 4$), but even in these cases a proper choice of regularization parameter and noise level will also give good results. Another option is to change the geometry of the measurement line. However, its effectiveness is restricted. For instance, it turns out that when two transmission eigenvalues are close to each

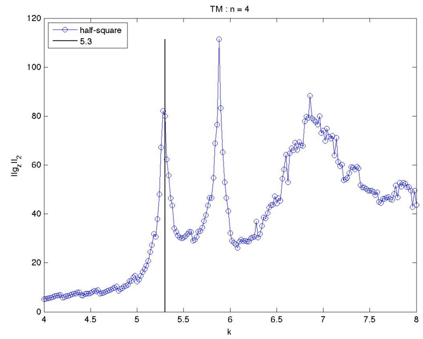

Figure 2.19: TM case, half square, $n(\mathbf{x}) = 4$, using curved measurement geometry.

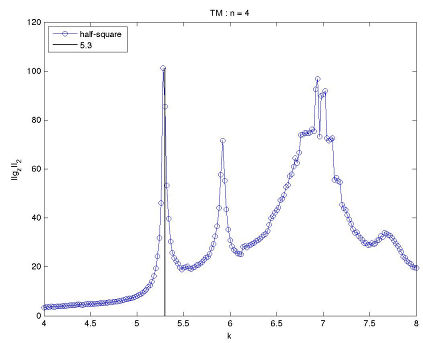

Figure 2.20: TM case, half square, $n(\mathbf{x}) = 4$, using elongated measurement geometry.

other (TE case for the half square with $n(\mathbf{x}) = 16$), the change to the measurement array (curve or elongation) will not improve the results.

As a final remark, this chapter appeared in [63] and more discussion on finding real transmission eigenvalues using other regularization techniques can also be found in that paper.

Chapter 3

PERFECTLY ELECTRIC CONDUCTING SCATTERER IN A WAVEGUIDE

3.1 The Forward Problem

Note that the notation used for this chapter on the waveguide problem is redefined compared to the previous chapter.

3.1.1 Configuration and Problem Description

We consider a waveguide occupying the domain $W = \Sigma \times \mathbb{R}$ in $\mathbb{R}^3$ where Σ is a simply connected bounded convex and open domain in $\mathbb{R}^2$. The boundary of W is piecewise smooth and denoted by Γ with outward normal $\mathbf{n}_\Gamma$. A scatterer D with smooth boundary is located inside the waveguide away from Γ with outward normal $\mathbf{n}_D$ (see Figure 3.1). We assume that the waveguide is filled with air (or vacuum) such that $\epsilon_0 = \mu_0 = 1$ where ϵ_0 and μ_0 represent the background electric permittivity and magnetic permeability, respectively. In the following, we will denote (x, y, z) a generic point of W.

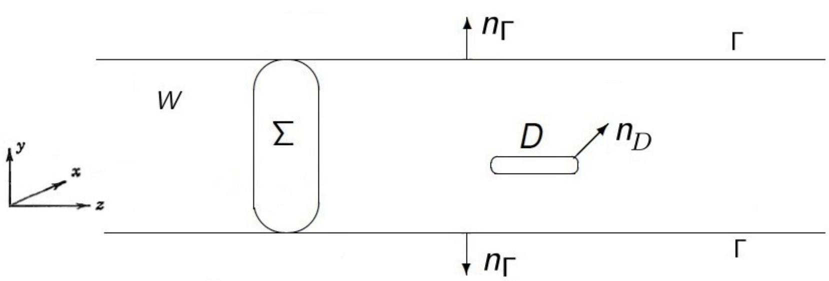

Figure 3.1: 2D view of generic configuration of waveguide in the presence of scatterer.

Note that the axis of the waveguide is parallel to the z-axis. We shall also denote by $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ the standard unit vectors in $\mathbb{R}^3$.

Let $\mathbf{u}^i$ denote an incident field excited by an electric point source at $\mathbf{x}_0 = (x_0, y_0, z_0)$ located on a cross section $\Sigma \times \{z_0\}$ with polarization vector $\mathbf{p}$ ($|\mathbf{p}| = 1$). The incident field is assumed to satisfy the Maxwell system in the absence of the scatterer D, that is

$$\nabla \times \nabla \times \mathbf{u}^i - k^2 \mathbf{u}^i = \mathbf{F} \quad \text{in} \quad W,$$
$$\mathbf{n}_\Gamma \times \mathbf{u}^i = 0 \quad \text{on} \quad \Gamma,$$

where $\mathbf{F} = \mathbf{p}\delta_{\mathbf{x}_0}$. Here $k = \omega\sqrt{\epsilon_0\mu_0}$ is the wavenumber where ω is the angular frequency.

Furthermore, we assume that the scatterer D and the boundary of the waveguide Γ are perfect electric conductors (PEC), for example, made of metal. Then the scattering problem we wish to solve is the model problem of finding the total field $\mathbf{u}$ such that

$$\nabla \times \nabla \times \mathbf{u} - k^2 \mathbf{u} = \mathbf{F} \quad \text{in} \quad W \backslash \bar{D}, \tag{3.1}$$
$$\mathbf{n}_D \times \mathbf{u} = 0 \quad \text{on} \quad \partial D, \tag{3.2}$$
$$\mathbf{u} = \mathbf{u}^i + \mathbf{u}^s \quad \text{in} \quad W \backslash \bar{D}, \tag{3.3}$$
$$\mathbf{n}_\Gamma \times \mathbf{u} = 0 \quad \text{on} \quad \Gamma, \tag{3.4}$$
$$\mathbf{u}^s \text{ satisfies an appropriate radiation condition} \quad \text{as} \quad z \to \pm\infty, \tag{3.5}$$

where the vector function $\mathbf{u}^s$ represents the unknown scattered field. The radiation condition here means that the scattered field $\mathbf{u}^s$ should be an outgoing wave that can be represented by modes, such that for each mode, it is either outward propagating or decaying exponentially away from the scatterer. It will be given explicitly in Definition 3.1.1 after we have developed some more notation.

Correspondingly, the forward problem for the scattered field $\mathbf{u}^s$ is:

$$\begin{cases} \nabla \times \nabla \times \mathbf{u}^s - k^2 \mathbf{u}^s = 0 \quad \text{in} \quad W \backslash \bar{D}, \\ \mathbf{n}_D \times \mathbf{u}^s = -\mathbf{n}_D \times \mathbf{u}^i \quad \text{on} \quad \partial D, \\ \mathbf{n}_\Gamma \times \mathbf{u}^s = 0 \quad \text{on} \quad \Gamma, \\ \mathbf{u}^s \text{ satisfies an appropriate radiation condition} \quad \text{as} \quad z \to \pm\infty. \end{cases} \tag{3.6}$$

3.1.2 Modal Solutions to Maxwell's Equation

To describe the radiation condition and also as a vital part of our inverse algorithms, we need an expansion of the dyadic Green's function (see Section 3.3.1) in terms of waveguide modes. We start by developing the well known modal solution in the waveguide.

3.1.2.1 Modal Solutions in the Waveguide

First, from Appendix A, we see that a mode $\mathbf{U}$ of the scattered field $\mathbf{u}^s$ can be represented by a combination of members of two families of solutions:

$$M = \nabla \times (\tilde{u}\mathbf{z}) \qquad \text{or} \qquad N = \frac{1}{k}\nabla \times \nabla \times (\tilde{v}\mathbf{z}),$$

where the generating function $\tilde{u}$ can be written in the form $\tilde{u}(x, y, z) = u(x, y)e^{ihz}$ and $\tilde{v}(x, y, z) = v(x, y)e^{ihz}$. The function u satisfies the 2D scalar Helmholtz equation on Σ:

$$\Delta_\Sigma u + \xi^2 u = 0 \qquad \text{where} \qquad k^2 = \xi^2 + h^2 \ (h \neq 0)$$

with an appropriate boundary condition on $\partial\Sigma = \Sigma \cap \Gamma$. Similarly, v also satisfies the 2D scalar Helmholtz equation on Σ.

Note 3.1.1 *The constant in the definition of N is convenient for the following relations*

$$N = \frac{1}{k}\nabla \times M \qquad \textit{and} \qquad M = \frac{1}{k}\nabla \times N. \tag{3.7}$$

Next we derive the boundary condition for u on $\partial\Sigma$ such that M and N satisfy the PEC boundary condition on Γ in (3.6).

- Consider the first family of solutions

  $$M = \nabla \times (u(x, y)e^{ihz}\hat{\mathbf{z}}),$$

  which satisfies the Maxwell's equation in (3.6) as long as u satisfies

  $$\Delta_\Sigma u + \lambda^2 u = 0$$

on Σ and suitable boundary conditions on $\partial\Sigma = \Sigma \cap \Gamma$, and h is chosen so that $h^2 = k^2 - \lambda^2$.

To investigate the PEC boundary conditions, we want

$$\mathbf{n}_\Gamma \times M = \mathbf{n}_\Gamma \times \nabla \times (u(x,y)e^{ihz}\hat{\mathbf{z}}) = 0.$$

Since

$$M = \nabla \times (u(x,y)e^{ihz}\hat{\mathbf{z}}) \quad = \quad \begin{pmatrix} \frac{\partial u}{\partial y} \\ -\frac{\partial u}{\partial x} \\ 0 \end{pmatrix} e^{ihz},$$

if $\mathbf{n}_\Gamma = (n_1, n_2, 0)^T$, we have

$$\mathbf{n}_\Gamma \times M \quad = \quad \begin{pmatrix} n_1 \\ n_2 \\ 0 \end{pmatrix} \times \begin{pmatrix} \frac{\partial u}{\partial y} \\ -\frac{\partial u}{\partial x} \\ 0 \end{pmatrix} e^{ihz} = - \begin{pmatrix} 0 \\ 0 \\ n_1\frac{\partial u}{\partial x} + n_2\frac{\partial u}{\partial y} \end{pmatrix} e^{ihz} = 0,$$

which implies that $\dfrac{\partial u}{\partial n} = 0$ on $\partial\Sigma$.

- Consider the second family of solutions

$$N = \frac{1}{k}\nabla \times \nabla \times (v(x,y)e^{ihz}\hat{\mathbf{z}}),$$

which satisfies the Maxwell's equation in (3.6) as long as v satisfies

$$\Delta_\Sigma v + \mu^2 v = 0$$

on Σ and appropriate boundary conditions on $\partial\Sigma = \Sigma \cap \Gamma$, and $h^2 = k^2 - \mu^2$.

To investigate the PEC boundary conditions, we want

$$\mathbf{n}_\Gamma \times N \quad = \quad \mathbf{n}_\Gamma \times \frac{1}{k}(\nabla \times \nabla \times (v(x,y)e^{ihz}\hat{\mathbf{z}})) = 0.$$

Using relation (3.7), straightforward calculation gives

$$\begin{aligned} \mathbf{n}_\Gamma \times N \quad &= \quad \mathbf{n}_\Gamma \times \left(\frac{1}{k}\nabla \times M\right) \\ &= \quad \begin{pmatrix} n_1 \\ n_2 \\ 0 \end{pmatrix} \times \left[\frac{1}{k}\begin{pmatrix} \frac{\partial v}{\partial x} ih \\ \frac{\partial v}{\partial y} ih \\ v\mu^2 \end{pmatrix} e^{ihz}\right] \\ &= \quad \frac{1}{k}\begin{pmatrix} n_2 v\mu^2 \\ -n_1 v\mu^2 \\ \left(n_1\frac{\partial v}{\partial y} - n_2\frac{\partial v}{\partial x}\right) ih \end{pmatrix} e^{ihz} = 0, \end{aligned}$$

which implies that $v = 0$ on Σ.

Therefore, in summary, we obtain

- The first family is given by

$$M = \nabla \times (u(x,y)e^{ihz}\hat{\mathbf{z}}) = \begin{pmatrix} \frac{\partial u}{\partial y} \\ -\frac{\partial u}{\partial x} \\ 0 \end{pmatrix} e^{ihz},$$

where λ is an eigenvalue and u is the corresponding non-trivial eigenfunction that satisfies

$$\begin{aligned} \Delta_\Sigma u + \lambda^2 u = 0 \quad &\text{in} \quad \Sigma, \\ \frac{\partial u}{\partial n} = 0 \quad &\text{on} \quad \partial\Sigma = \Sigma \cap \Gamma. \end{aligned}$$

- The second family is given by

$$N = \frac{1}{k}\nabla \times \nabla \times (v(x,y)e^{ihz}\hat{\mathbf{z}}) = \frac{1}{k}\begin{pmatrix} \frac{\partial v}{\partial x} ih \\ \frac{\partial v}{\partial y} ih \\ v\mu^2 \end{pmatrix} e^{ihz},$$

where μ is an eigenvalue and v is the corresponding non-trivial eigenfunction that satisfies

$$\begin{aligned} \Delta_\Sigma v + \mu^2 v = 0 \quad &\text{in} \quad \Sigma, \\ v = 0 \quad &\text{on} \quad \partial\Sigma = \Sigma \cap \Gamma. \end{aligned}$$

Note 3.1.2 *If alternatively the magnetic wall boundary condition* $\mathbf{n}_\Gamma \times (\nabla \times \mathbf{U}) = 0$ *is considered on* Γ*, then for the first family of solutions* M*, we have*

$$\mathbf{n}_\Gamma \times \nabla \times M \quad = \quad \begin{pmatrix} n_1 \\ n_2 \\ 0 \end{pmatrix} \times \begin{pmatrix} \frac{\partial u}{\partial x} ih \\ \frac{\partial u}{\partial y} ih \\ u\lambda^2 \end{pmatrix} e^{ihz} = \begin{pmatrix} n_2 u \lambda^2 \\ -n_1 u\lambda^2 \\ \left(n_1 \frac{\partial u}{\partial y} - n_2 \frac{\partial u}{\partial x}\right) ih \end{pmatrix} e^{ihz}.$$

From this we see that u *must satisfy the Dirichlet boundary condition* $u = 0$ *on* Σ.

On the other hand, for the second family of solutions N*, using relation* (3.7)*, we have*

$$\begin{aligned} &\mathbf{n}_\Gamma \times \nabla \times N = \mathbf{n}_\Gamma \times (kM) \\ = \quad &\begin{pmatrix} n_1 \\ n_2 \\ 0 \end{pmatrix} \times \left[k \begin{pmatrix} \frac{\partial v}{\partial y} \\ -\frac{\partial v}{\partial x} \\ 0 \end{pmatrix} e^{ihz} \right] = -k \begin{pmatrix} 0 \\ 0 \\ n_1 \frac{\partial u}{\partial x} + n_2 \frac{\partial u}{\partial y} \end{pmatrix} e^{ihz}. \end{aligned}$$

This means that v *should satisfy the Neumann boundary condition* $\frac{\partial v}{\partial n} = 0$ *on* Σ.

3.1.2.2 Eigenvalue Problem on the Cross Section of the Waveguide

To further investigate the modal solution, we are led to study the eigenvalue problems of finding non-trivial solutions u and v together with corresponding eigenfunctions u and v such that

$$(E1)\begin{cases} \Delta u + \lambda^2 u = 0 & \text{in} \quad \Sigma, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on} \quad \partial\Sigma. \end{cases} \qquad \text{and} \qquad (E2)\begin{cases} \Delta v + \mu^2 v = 0 & \text{in} \quad \Sigma, \\ v = 0 & \text{on} \quad \partial\Sigma. \end{cases}$$

Both of these problems are standard eigenvalue problems for the Laplacian and we have the following theorems (Theorems 8.5 and 8.6 in [59]) under the geometric assumptions on Σ in the introduction to this chapter (Section 3.1.1):

Theorem 3.1.1 *Problem $(E1)$ has a countable family of eigenpairs (u_m, λ_m) where $u_m \in H^1(\Sigma), m = 0, 1, \ldots$ with $u_m \neq 0$ and $\lambda_m \in \mathbb{R}$. We may choose $\lambda_0 = 0$ and $\lambda_m \geq \lambda_{m-1}, m \geq 1$. In addition, $\{u_m\}_{m=0}^{\infty}$ may be chosen as an orthonormal basis in $L^2(\Sigma)$. Moreover, the sequence $\{u_m/\sqrt{\lambda_m + 1}\}_{m\geq 0}$ constitutes an orthogonal basis in $H^1(\Sigma)$.*

Theorem 3.1.2 *Problem $(E2)$ has a countable family of eigenpairs (v_m, μ_m) where $v_m \in H_0^1(\Sigma), m = 1, 2, \ldots$ with $v_m \neq 0$ and $\mu_m \in \mathbb{R}, \mu_m > 0$. We may choose $\mu_m \geq \mu_{m-1}$ for all $m \geq 2$. In addition, $\{v_m\}_{m=0}^{\infty}$ may be chosen as an orthonormal basis in $L^2(\Sigma)$. Moreover, the sequence $\{v_m/\sqrt{\mu_m}\}_{m\geq 0}$ constitutes an orthogonal basis in $H_0^1(\Sigma)$.*

Remark 3.1.1 *Since Σ is convex, we know that both u_m and v_m are in $H^2(\Sigma)$ for each m, n.*

Now we can specify precisely the modal solutions corresponding to the PEC boundary conditions on the waveguide wall. The two families are:

1. First family:

$$M_m = \nabla \times (u_m e^{ih_m z}\hat{\mathbf{z}}), m = 1, 2, \ldots$$

   where

$$h_m = \begin{cases} \sqrt{k^2 - \lambda_m^2} & \text{if} \quad k^2 > \lambda_m^2, \\ i\sqrt{\lambda_m^2 - k^2} & \text{if} \quad k^2 < \lambda_m^2, \end{cases}$$

where we choose the positive square root. Note that the mode corresponding to $\lambda_0 = 0, h_0 = k$ has u_0 constant and so does not contribute to the solution.

2. Second family:

$$N_m = \frac{1}{k}\nabla \times \nabla \times (v_m e^{ig_m z}\hat{\mathbf{z}}), m = 1, 2, \ldots$$

where

$$g_m = \begin{cases} \sqrt{k^2 - \mu_m^2} & \text{if} \quad k^2 > \mu_m^2, \\ i\sqrt{\mu_m^2 - k^2} & \text{if} \quad k^2 < \mu_m^2, \end{cases}$$

where again we choose the positive square root.

Here we see that it is possible that all the h_m and g_m are imaginary so that no traveling waves occur.

For the reminder of the thesis we assume that we avoid the cut-off frequencies k such that $\lambda_m^2 = k^2$ and $\mu_m^2 = k^2$, that is, we assume $h_m = \sqrt{k^2 - \lambda_m^2} \neq 0$ and $g_m = \sqrt{k^2 - \mu_m^2} \neq 0$ for all $m = 1, 2, \ldots$. Then we will see in detail in the sequel that there exists other possible exceptional frequencies which will impact our analysis and should also be avoided.

3.1.3 Weak Formulation and Variational Formulation

3.1.3.1 Function Spaces

In order to further analyze the forward problem and its well-posedness, we shall introduce all the function spaces to be considered in the later sections.

1. Spaces for the unbounded domain

   For a general domain $\mathfrak{D} \subset \mathbb{R}^3$, we define the usual energy space for electromagnetic field as:

$$H(\text{curl}, \mathfrak{D}) = \{\mathbf{u} \in (L^2(\mathfrak{D}))^3 \mid \nabla \times \mathbf{u} \in (L^2(\mathfrak{D}))^3\}$$

   equipped with norm

$$\|\mathbf{u}\|_{H(\text{curl},\mathfrak{D})} = \left(\|\mathbf{u}\|^2_{L^2(\mathfrak{D})} + \|\nabla \times \mathbf{u}\|^2_{L^2(\mathfrak{D})}\right)^{1/2}.$$

   In order to investigate the scattering problem in a weak sense, we consider the waveguide in the presence of scatterer (see Figure 3.1) and define $H_{loc}(\text{curl}, W\backslash\bar{D})$ the space of functions $\mathbf{u} \in H(\text{curl}, W_{(-R,R)}\backslash\bar{D})$ for any R sufficiently large where

$W_{(-R,R)}$ is the segment of waveguide bounded by cross sections Σ_R and Σ_{-R} and containing D in its interior. We also denote by

$$H_0(\mathrm{curl}, W\backslash\bar{D}) = \{\mathbf{u} \in H(\mathrm{curl}, W\backslash\bar{D}) \mid \mathbf{n}_\Gamma\times\mathbf{u} = 0 \text{ on } \Gamma \text{ and } \mathbf{n}_D\times\mathbf{u} = 0 \text{ on } \partial D\},$$

and

$$H_{loc,0}(\mathrm{curl}, W\backslash\bar{D}) = \{\mathbf{u} \in H_{loc}(\mathrm{curl}, W\backslash\bar{D}) \mid \mathbf{n}_\Gamma\times\mathbf{u} = 0 \text{ on } \Gamma \text{ and } \mathbf{n}_D\times\mathbf{u} = 0 \text{ on } \partial D\}.$$

2. Spaces for bounded domains

   For the variational formulation, we shall reduce the unbounded domain to a bounded region containing D in its interior. A sketch of this is shown in Figure 3.2 where

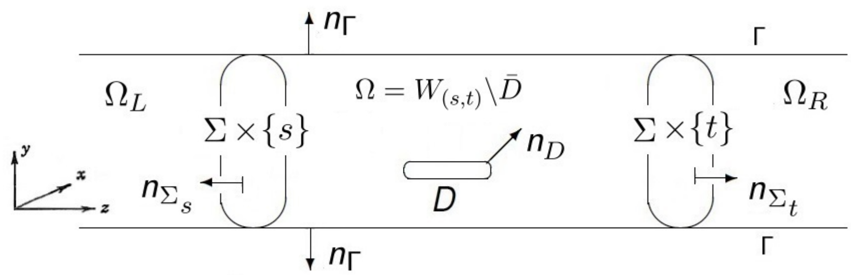

Figure 3.2: Bounded sub-domain of waveguide in the presence of scatterer.

$W_{(s,t)}$	:	Sub-domain of waveguide bounded by cross sections $\Sigma \times (s,t)$,
Ω	:	Domain inside $W_{(s,t)}$ excluding the scatterer D,
Ω_L	:	Unbounded domain on the left hand side of cross section $\Sigma_s = \Sigma \times \{s\}$,
Ω_R	:	Unbounded domain on the right hand side of cross section $\Sigma_t = \Sigma \times \{t\}$.

Taking into account the boundary conditions in the waveguide, we define

$$X = \{\mathbf{u} \in H(\mathrm{curl}, \Omega) \mid \mathbf{n}_\Gamma \times \mathbf{u} = 0 \text{ on } \Gamma \text{ and } \mathbf{n}_D \times \mathbf{u} = 0 \text{ on } \partial D\}.$$

To state the corresponding trace space, note that the standard trace space for $H(\mathrm{curl}, \Omega)$ is

$$\begin{aligned} H^{-1/2}(\mathrm{div}, \partial\Omega) &= \{\mathbf{f} \in (H^{-1/2}(\partial\Omega))^3 \mid \text{there exists } \mathbf{v} \in H(\mathrm{curl}, \Omega) \\ &\qquad \text{such that } \mathbf{n}_\Omega \times \mathbf{v}|_{\partial\Omega} = \mathbf{f}\}, \end{aligned}$$

where $\mathbf{n}_\Omega = -\mathbf{n}_D$ or $\mathbf{n}_{\Sigma s}$ or $\mathbf{n}_{\Sigma t}$ or $\mathbf{n}_\Gamma$, $\partial\Omega = \partial D \cup \Sigma_s \cup \Sigma_t \cup \Gamma_{(s,t)}$ and $\Gamma_{(s,t)} = \partial\Sigma \times (s,t)$ is the boundary of segment of the waveguide bounded by Σ_s and Σ_t.

Then, we define specifically the trace space on the cross sections:

$$\begin{aligned}\widetilde{H}^{-1/2}(\mathrm{div}, \Sigma_s) &= \{\mathbf{f} \in (H^{-1/2}(\Sigma_s))^3 \mid \text{there exists } \mathbf{v} \in X \\ &\qquad \text{such that } \mathbf{n}_\Sigma \times \mathbf{v}|_{\Sigma_s} = \mathbf{f}\}.\end{aligned}$$

A similar definition holds $\widetilde{H}^{-1/2}(\mathrm{div}, \Sigma_t)$. Moreover, the dual space of all the trace spaces above are denoted by

$$H^{-1/2}(\mathrm{curl}, \partial\Omega), \widetilde{H}^{-1/2}(\mathrm{curl}, \Sigma_s), \widetilde{H}^{-1/2}(\mathrm{curl}, \Sigma_t),$$

respectively.

For the ease of analyzing the inverse problem, we also define the trace space on ∂D:

$$\begin{aligned}H^{-1/2}(\mathrm{div}, \partial D) &= \{\mathbf{f} \in (H^{-1/2}(\partial D))^3 \mid \text{there exists } \mathbf{v} \in X \\ &\qquad \text{such that } \mathbf{n}_{\partial D} \times \mathbf{v}|_{\partial D} = \mathbf{f}\}.\end{aligned}$$

Also, the dual space of this trace space is denoted by $H^{-1/2}(\mathrm{curl}, \partial D)$.

3.1.3.2 Differential Operators on the Cross Section

With the eigenvalue problems $(E1)$ and $(E2)$ introduced in Section 3.1.2.2, we shall also investigate the space of surface tangential vector fields on cross sections Σ given by

$$L^2_T(\Sigma) = \{\mathbf{w} \in (L^2(\Sigma))^3 \mid \mathbf{n}_\Sigma \cdot \mathbf{w} = 0 \text{ a.e. on } \Sigma\}.$$

First of all, we recall several differential operators on a simply connected bounded smooth surface $S \subset \mathbb{R}^3$. Let u be a differentiable scalar function and $\mathbf{v}$ a differentiable tangent vector function defined on S, denote by $\nabla_S u$, $\nabla_S \cdot \mathbf{v}$, $\nabla_S \times \mathbf{v}$, $\vec{\nabla}_S \times u$ the surface gradient, surface divergence, surface curl and vectorial surface curl, respectively. The scalar Laplace-Beltrami operator

$$\Delta_S u = \nabla_S \cdot (\nabla_S u) = -\nabla_S \times (\vec{\nabla}_S \times u),$$

and the vector Laplace-Beltrami operator

$$\Delta_S \mathbf{v} = \nabla_S(\nabla_S \cdot \mathbf{v}) - \vec{\nabla}_S \times (\nabla_S \cdot \mathbf{v})$$

(see Section 6.3 in [19] or Section 3.4 in [48]). Relations among these operators are given in Appendix B.3. Also note that these operators can be extended to a Lipschitz domain (see Section 3.4 in [48]).

Note 3.1.3 *With operators defined above, for a differentiable scalar function* $\phi = \phi(x,y)$ *defined on* Σ*, we have, for example,*

$$\nabla_\Sigma \phi = \begin{pmatrix} \frac{\partial \phi}{\partial x} \\ \frac{\partial \phi}{\partial y} \end{pmatrix}, \vec{\nabla}_\Sigma \times \phi = \begin{pmatrix} \frac{\partial \phi}{\partial y} \\ -\frac{\partial \phi}{\partial x} \end{pmatrix}.$$

Then we have the following Stoke's identity:

Lemma 3.1.1 *Let* $\{u_m, \lambda_m\}_{m\geq 0}$ *and* $\{v_n, \mu_n\}_{n\geq 1}$ *be eigenpairs to problem* $(E1)$ *and* $(E2)$ *defined in Theorem 3.1.1 and Theorem 3.1.2, respectively. Then the following Stoke's identity holds:*

$$\int_\Sigma \lambda_m^2 u_m\, v_n\, dx = \int_\Sigma (\nabla_\Sigma u_m)\cdot(\nabla_\Sigma v_n)\, dx = \int_\Sigma (\vec{\nabla}_\Sigma \times u_m)\cdot(\vec{\nabla}_\Sigma \times v_n)\, dx. \tag{3.8}$$

This identity also holds when v_n *is replaced by* u_n *or* λ_m, u_m *are replaced by* μ_m, v_m*.*

Proof: Obviously,

$$\int_\Sigma \lambda_m^2 u_m\, v_n\, dx = -\int_\Sigma \Delta_\Sigma u_m\, v_n\, dx.$$

Using Stoke's identities (B.16) and (B.17), we have

$$\begin{aligned} \int_\Sigma \lambda_m^2 u_m\, v_n\, dx &= -\int_\Sigma \nabla_\Sigma \cdot (\nabla_\Sigma u_m) v_n\, dx \\ &= \int_\Sigma (\nabla_\Sigma u_m)\cdot(\nabla_\Sigma v_n)\, dx - \int_{\partial\Sigma} \nu_{\partial\Sigma}\cdot \nabla_\Sigma u_m\, v_n\, ds, \end{aligned}$$

and

$$\begin{aligned} \int_\Sigma \lambda_m^2 u_m\, v_n\, dx &= \int_\Sigma \nabla_\Sigma \times (\vec{\nabla}_\Sigma \times u_m) v_n\, dx \\ &= \int_\Sigma (\vec{\nabla}_\Sigma \times u_m)\cdot(\vec{\nabla}_\Sigma \times v_n)\, dx + \int_{\partial\Sigma} \tau_{\partial\Sigma}\cdot \vec{\nabla}_\Sigma \times u_m\, v_n\, ds. \end{aligned}$$

Note that $\nu_{\partial\Sigma}\cdot\nabla_\Sigma u_m = 0$ and the identity (B.21) (or because $v_n = 0$ on $\partial\Sigma$), we have the desired identity (3.8). In a similar fashion, it is easy to see that the identity also holds when v_n is replaced by u_n or λ_m, u_m are replaced by μ_m, v_m. This completes the proof. □

Also we have the following lemma:

Lemma 3.1.2 *For a simply connected bounded Lipschitz domain $\Sigma \subset \mathbb{R}^2$, $L^2_T(\Sigma)$ admits an orthogonal basis given by $\{\nabla_\Sigma u_m\}_{m\geq 1}$ and $\{\vec{\nabla}_\Sigma \times v_n\}_{n\geq 1}$ where u_m and v_n are defined from Theorem 3.1.1 and Theorem 3.1.2.*

Proof: For any $\mathbf{w} \in L^2_T(\Sigma)$, we have the following Helmholtz decomposition

$$\mathbf{w} = \nabla_\Sigma p + \vec{\nabla}_\Sigma \times q,$$

where $p \in H^1(\Sigma)$ and $q \in H^1_0(\Sigma)$ (see Theorem 3.3 in [32] and Theorem 3.8 in [62]).

From Theorem 3.1.1 and Theorem 3.1.2, p and q can be written as series expansion in term of u_m and v_n

$$p = \sum_{m=0}^{\infty} \alpha_m u_m \quad \text{and} \quad q = \sum_{n=1}^{\infty} \beta_n v_m,$$

where u_m and v_n can be chosen such that $\|u_m\|^2_{L^2(\Sigma)} = 1, \|v_n\|^2_{L^2(\Sigma)} = 1$.

Thus, since $\nabla_\Sigma u_0 = 0$, $\mathbf{w}$ can be written as

$$\mathbf{w} = \sum_{m=1}^{\infty} \alpha_m \nabla_\Sigma u_m + \sum_{n=1}^{\infty} \beta_n \vec{\nabla}_\Sigma \times v_n.$$

Moreover, by Stokes identities (B.17) and (B.19),

$$\begin{aligned}
\int_\Sigma (\nabla_\Sigma u_m) \cdot (\vec{\nabla}_\Sigma \times v_n)\, dx &= \int_\Sigma (\nabla_\Sigma \times (\nabla_\Sigma u_m)) v_n \, dx - \int_{\partial\Sigma} (\tau_{\partial\Sigma} \cdot \nabla_\Sigma u_m) v_n \, ds \\
&= -\int_{\partial\Sigma} (\tau_{\partial\Sigma} \cdot \nabla_\Sigma u_m) v_n \, ds.
\end{aligned}$$

Since $v_n = 0$ on $\partial\Sigma$, we get

$$\int_\Sigma (\nabla_\Sigma u_m) \cdot (\vec{\nabla}_\Sigma \times v_n)\, dx = 0.$$

Thus $\{\nabla_\Sigma u_m\}_{m\geq 1}$ and $\{\vec{\nabla}_\Sigma \times v_n\}_{n\geq 1}$ form an orthogonal basis. □

With the eigenpairs introduced in Section 3.1.2.2, we are able to redefine the spaces on cross sections Σ. From Theorem 3.1.1 and Theorem 3.1.2, we see that for any function $w \in L^2(\Sigma)$, it can be expanded using $\{u_m\}_{m\geq 0}$ or $\{v_m\}_{m\geq 0}$. Explicitly,

$$w = \sum_{m=0}^{\infty} w_m^{(1)} u_m, \text{ where } w_m^{(1)} = \int_\Sigma w\, u_m \, ds,$$

$$\text{or } w = \sum_{m=1}^{\infty} w_m^{(2)} v_m, \text{ where } w_m^{(2)} = \int_{\Sigma} w\, v_m\, ds.$$

Then we can equivalently define the spaces H^s and H_0^s such that

$$\begin{aligned} H^s(\Sigma) &= \left\{ w = \sum_{m=0}^{\infty} w_m^{(1)} u_m \;\middle|\; \sum_{m=0}^{\infty} (1+\lambda_m^2)^s |w_m^{(1)}|^2 < \infty \right\}, \\ H_0^s(\Sigma) &= \left\{ w = \sum_{m=1}^{\infty} w_m^{(2)} v_m \;\middle|\; \sum_{m=1}^{\infty} (1+\mu_m^2)^s |w_m^{(2)}|^2 < \infty \right\}, \end{aligned}$$

equipped with equivalent norms

$$\|w\|^2_{H^s(\Sigma)} = \sum_{m=0}^{\infty} (1+\lambda_m^2)^s |w_m^{(1)}|^2,$$
$$\|w\|^2_{H_0^s(\Sigma)} = \sum_{m=1}^{\infty} (1+\mu_m^2)^s |w_m^{(1)}|^2,$$

respectively.

By Lemma 3.1.2, for any tangential vector field $\mathbf{w}$ on Σ, it can be written as

$$\mathbf{w} = \sum_{m=1}^{\infty} \alpha_m \nabla_\Sigma u_m + \sum_{n=1}^{\infty} \beta_n \vec{\nabla}_\Sigma \times v_n, \tag{3.9}$$

then $L_T^2(\Sigma)$ can be redefined as:

$$L_T^2(\Sigma) = \left\{ \mathbf{w} = \sum_{m=1}^{\infty} \alpha_m \nabla_\Sigma u_m + \sum_{n=1}^{\infty} \beta_n \vec{\nabla}_\Sigma \times v_n \;\middle|\; \sum_{m=1}^{\infty} \lambda_m^2 |\alpha_m|^2 + \sum_{n=1}^{\infty} \mu_n^2 |\beta_n|^2 < \infty \right\}$$

equipped with norm

$$\|\mathbf{w}\|^2_{L_T^2(\Sigma)} = \sum_{m=1}^{\infty} \lambda_m^2 |\alpha_m|^2 + \sum_{n=1}^{\infty} \mu_n^2 |\beta_n|^2.$$

We can also define $H_T^s(\Sigma)$ as

$$H_T^s(\Sigma) = \left\{ \mathbf{w} \;\middle|\; \sum_{m=1}^{\infty} \lambda_m^{2(s+1)} |\alpha_m|^2 + \sum_{n=1}^{\infty} \mu_n^{2(s+1)} |\beta_n|^2 < \infty \right\}$$

equipped with norm

$$\|\mathbf{w}\|^2_{H_T^s(\Sigma)} = \sum_{m=1}^{\infty} \lambda_m^{2(s+1)} |\alpha_m|^2 + \sum_{n=1}^{\infty} \mu_n^{2(s+1)} |\beta_n|^2.$$

Also notice that, by Lemma 3.1.1 and the Stokes identities (B.18),(B.19), the expansion (3.9) yields

$$\begin{aligned}\nabla_\Sigma \cdot \mathbf{w} &= \sum_{m=1}^{\infty} \alpha_m \lambda_m^2 u_m, \\ \nabla_\Sigma \times \mathbf{w} &= \sum_{m=1}^{\infty} \beta_m \mu_m^2 v_m.\end{aligned}$$

Thus, the trace spaces $\widetilde{H}^{-1/2}(\mathrm{div}, \Sigma)$ and $\widetilde{H}^{-1/2}(\mathrm{curl}, \Sigma)$ introduced in Section 3.1.3.1 (see also Section 3.5.3 in [48] and Section 5.4.1 in [51]) can be redefined as follows:

$$\begin{aligned}\widetilde{H}^{-1/2}(\mathrm{div}, \Sigma) &= \left\{ \mathbf{w} \in H_T^{-1/2}(\Sigma) \;\middle|\; \nabla_\Sigma \cdot \mathbf{w} \in H^{-1/2}(\Sigma) \right\}, \\ \widetilde{H}^{-1/2}(\mathrm{curl}, \Sigma) &= \left\{ \mathbf{w} \in H_T^{-1/2}(\Sigma) \;\middle|\; \nabla_\Sigma \times \mathbf{w} \in H^{-1/2}(\Sigma) \right\},\end{aligned}$$

equipped with equivalent norms

$$\begin{aligned}\|\mathbf{w}\|^2_{H^{-1/2}(\mathrm{div},\Sigma)} &= \sum_{m=1}^{\infty} \lambda_m^3 |\alpha_m|^2 + \sum_{n=1}^{\infty} \mu_n |\beta_n|^2, \\ \|\mathbf{w}\|^2_{H^{-1/2}(\mathrm{curl},\Sigma)} &= \sum_{m=1}^{\infty} \lambda_m |\alpha_m|^2 + \sum_{n=1}^{\infty} \mu_n^3 |\beta_n|^2.\end{aligned}$$

3.1.3.3 Radiation Condition

Now, we first state what we mean by the radiation condition.

Definition 3.1.1 *For all $|z|$ sufficiently large, a solution $\mathbf{U}$ of Maxwell's equation satisfies the outgoing radiation condition if $\mathbf{U} \in H_{loc}(\mathrm{curl}, W\backslash\bar{D})$ can be written as*

$$\mathbf{U} = \sum_{m=1}^{\infty} A_m M_m + B_m N_m$$

with coefficients $\{A_m\}, \{B_m\}, m = 1, 2, \dots$ where

$$M_m = \nabla \times (u_m e^{ih_m|z|}) \qquad \text{and} \qquad N_m = \frac{1}{k} \nabla \times \nabla \times (v_m e^{ig_m|z|}).$$

- *Modes for which h_m (or g_m) are real are said to be traveling waves and traveling modes satisfy a Sommerfeld type outgoing radiation condition along the axis of the waveguide, for example, for $z > 0$,*

$$\frac{\partial M_m}{\partial z} - ih_m M_m = \mathbf{0}.$$

- *Modes for which h_m (or g_m) are imaginary are said to be evanescent and decay along the axis of the waveguide. For example, for $z > 0$ and m large enough,*

$$M_m = \nabla \times (u_m e^{ih_m|z|}) = \nabla \times (u_m e^{-|h_m|z}) \to \mathbf{0} \text{ as } z \to \infty.$$

3.1.3.4 Blocked Waveguide Problem

Before analyzing the well-posedness of the forward problem and other results in the subsequent sections, we first consider a blocked waveguide problem in the absence of scatterer (for example, domain Ω_R or Ω_L in Figure 3.2).

For simplicity, denote $\int_\Sigma \mathbf{f} \cdot \bar{\mathbf{g}}\, ds = \langle \mathbf{f}, \mathbf{g} \rangle_\Sigma$. The following lemma shows the well-posedness of the blocked waveguide problem:

Lemma 3.1.3 *Given $\mathbf{Q} \in \widetilde{H}^{-1/2}(div, \Sigma_t)$, there exists a unique solution $\mathbf{U} \in H_{loc}(\text{curl}, W_{(t,\infty)})$ to the following blocked waveguide problem*

$$\begin{cases} \nabla \times \nabla \times \mathbf{U} - k^2 \mathbf{U} = 0 & in \quad W_{(t,\infty)}, \\ \mathbf{n}_\Gamma \times \mathbf{U} = 0 & on \quad \Gamma_{(t,\infty)}, \\ \mathbf{n}_\Sigma \times \mathbf{U} = \mathbf{Q} & on \quad \Sigma_t, \\ \mathbf{U} \text{ satisfies the radiation condition} & as \quad z \to +\infty, \end{cases} \tag{3.10}$$

where $\Gamma_{(t,\infty)} = \partial\Sigma \times (t, \infty)$ and $\mathbf{n}_\Sigma = \hat{\mathbf{z}}$.

Proof: From Section 3.1.2.1 and using the radiation condition in Definition 3.1.1, $\mathbf{U}$ in $W_{(t,\infty)}$ can be written as a superposition of two families of modal functions which satisfy both the Maxwell's equation and the boundary conditions on $\Gamma_{(t,\infty)}$, that is

$$\mathbf{U} = \sum_{m=1}^{\infty} A_m M_m + \sum_{n=1}^{\infty} B_n N_n.$$

Note that due to the radiation condition, we choose terms involving only e^{ihz} in (A.2) (drop terms with e^{-ihz}). Then, the explicit form of $\mathbf{U}$ for $z > t$ can be written as

$$\begin{aligned} \mathbf{U} &= \sum_m A_m M_m + \sum_n B_n N_n \\ &= \sum_m A_m \nabla \times (u_m e^{ih_m(z-t)} \hat{\mathbf{z}}) + \sum_n B_n \frac{1}{k} \nabla \times \nabla \times (v_n e^{ig_n(z-t)} \hat{\mathbf{z}}) \end{aligned}$$

$$
\begin{aligned}
&= \sum_m A_m \left[\begin{pmatrix} \frac{\partial u_m}{\partial y} \\ -\frac{\partial u_m}{\partial x} \\ 0 \end{pmatrix} e^{ih_m(z-t)} \right] + \sum_n B_n \frac{1}{k} \left[\begin{pmatrix} \frac{\partial v_n}{\partial x} ig_n \\ \frac{\partial v_n}{\partial y} ig_n \\ v_n \mu_n^2 \end{pmatrix} e^{ig_n(z-t)} \right] \\
&= \sum_m A_m \left[\begin{pmatrix} \vec{\nabla}_\Sigma \times u_m \\ 0 \end{pmatrix} e^{ih_m(z-t)} \right] \\
&\quad + \sum_n B_n \frac{1}{k} \left[\begin{pmatrix} \nabla_\Sigma v_n \\ 0 \end{pmatrix} ig_n e^{ig_n(z-t)} + \begin{pmatrix} 0 \\ 0 \\ v_n \end{pmatrix} \mu_n^2 e^{ig_n(z-t)} \right].
\end{aligned}
$$

On Σ_t, we have

$$
\begin{aligned}
\mathbf{n}_\Sigma \times \mathbf{U}|_{\Sigma_t} &= \sum_m A_m \left(\mathbf{n}_\Sigma \times \begin{pmatrix} \vec{\nabla}_\Sigma \times u_m \\ 0 \end{pmatrix} \right) \\
&\quad + \sum_n B_n \left(\mathbf{n}_\Sigma \times \frac{1}{k} \left[\begin{pmatrix} \nabla_\Sigma v_n \\ 0 \end{pmatrix} ig_n + \begin{pmatrix} 0 \\ 0 \\ v_n \end{pmatrix} \mu_n^2 \right] \right).
\end{aligned}
$$

Using identities (B.11), (B.12) and $\mathbf{n}_\Sigma \times \begin{pmatrix} 0 \\ 0 \\ v_n \end{pmatrix} = \hat{\mathbf{z}} \times \begin{pmatrix} 0 \\ 0 \\ v_n \end{pmatrix} = 0$,

$$
\mathbf{Q} = \mathbf{n}_\Sigma \times \mathbf{U}|_{\Sigma_t} = \sum_m A_m \begin{pmatrix} \nabla_\Sigma u_m \\ 0 \end{pmatrix} + \sum_n -B_n \frac{1}{k} \left[\begin{pmatrix} \vec{\nabla}_\Sigma \times v_n \\ 0 \end{pmatrix} ig_n \right]. \tag{3.11}
$$

To obtain the unknown coefficients A_m and B_n, by Lemma 3.1.2 and Theorem 3.1.1, we have

$$
\begin{aligned}
a_m &\triangleq \left\langle \mathbf{n}_\Sigma \times \mathbf{U}|_{\Sigma_t}, \begin{pmatrix} \nabla_\Sigma u_m \\ 0 \end{pmatrix} \right\rangle_{\Sigma_t} = \left\langle \mathbf{Q}, \begin{pmatrix} \nabla_\Sigma u_m \\ 0 \end{pmatrix} \right\rangle_{\Sigma_t} \\
&= A_m \lambda_m^2 \|u_m\|^2_{L^2(\Sigma_t)} = A_m \lambda_m^2, \\
b_n &\triangleq \left\langle \mathbf{n}_\Sigma \times \mathbf{U}|_{\Sigma_t}, \begin{pmatrix} \vec{\nabla}_\Sigma \times v_n \\ 0 \end{pmatrix} \right\rangle_{\Sigma_t} = \left\langle \mathbf{Q}, \begin{pmatrix} \vec{\nabla}_\Sigma \times v_n \\ 0 \end{pmatrix} \right\rangle_{\Sigma_t}
\end{aligned}
$$

$$= \quad -B_n\mu_n^2\|v_n\|^2_{L^2(\Sigma_t)}\frac{ig_n}{k} = -B_n\mu_n^2\frac{ig_n}{k}.$$

Then the coefficients in (3.11) are

$$A_m = \frac{a_m}{\lambda_m^2} \qquad \text{and} \qquad B_n = -\frac{b_n}{\mu_n^2}\frac{k}{ig_n}.$$

Thus the solution $\mathbf{U}$ in $W_{(t,\infty)}$is given by

$$\begin{aligned}
\mathbf{U} &= \sum_m \frac{a_m}{\lambda_m^2}\left[\begin{pmatrix}\vec{\nabla}_\Sigma \times u_m \\ 0\end{pmatrix} e^{ih_m(z-t)}\right] \\
&\quad + \sum_n -\frac{b_n}{\mu_n^2}\frac{k}{ig_n}\frac{1}{k}\left[\begin{pmatrix}\nabla_\Sigma v_n \\ 0\end{pmatrix} ig_n e^{ig_n(z-t)} + \begin{pmatrix}0\\0\\v_n\end{pmatrix}\mu_n^2 e^{ig_n(z-t)}\right] \\
&= \sum_m \frac{a_m}{\lambda_m^2}\left[\begin{pmatrix}\vec{\nabla}_\Sigma \times u_m \\ 0\end{pmatrix} e^{ih_m(z-t)}\right] \\
&\quad + \sum_n -\frac{b_n}{\mu_n^2}\left[\begin{pmatrix}\nabla_\Sigma v_n \\ 0\end{pmatrix} e^{ig_n(z-t)}\right] - \frac{b_n}{ig_n}\left[\begin{pmatrix}0\\0\\v_n\end{pmatrix} e^{ig_n(z-t)}\right].
\end{aligned} \tag{3.12}$$

To show $\mathbf{U} \in H_{loc}(\text{curl}, W_{(t,\infty)})$, consider a bounded segment of the waveguide $W_{(t,l)}$ where $t < l < \infty$. First we compute the L^2 norm of $\mathbf{U}$ over $W_{(t,l)}$ which is given by

$$\|\mathbf{U}\|^2_{L^2(W_{(t,l)})} = \sum_m |a_m|^2\left(\frac{1}{\lambda_m^2}\right) I_{t,l,m} + \sum_n |b_n|^2\left(\frac{1}{\mu_n^2} + \frac{1}{|g_n|^2}\right) J_{t,l,n},$$

where

$$I_{t,l,m} = \int_t^l \left|e^{ih_m(z-t)}\right|^2 dz = \begin{cases} l-t & \text{if} \quad \lambda_m^2 \le k^2, \\ \dfrac{1}{2|h_m|}\left(1 - \dfrac{1}{e^{2|h_m|(l-t)}}\right) & \text{if} \quad \lambda_m^2 > k^2, \end{cases}$$

and

$$J_{t,l,n} = \int_t^l \left|e^{ig_n(z-t)}\right|^2 dz = \begin{cases} l-t & \text{if} \quad \mu_n^2 \le k^2, \\ \dfrac{1}{2|g_n|}\left(1 - \dfrac{1}{e^{2|g_n|(l-t)}}\right) & \text{if} \quad \mu_n^2 > k^2. \end{cases}.$$

To compute the L^2 norm of $\nabla \times \mathbf{U}$ over $W_{(t,l)}$, using relation (3.7), straightforward calculation shows that

$$
\begin{aligned}
\nabla \times \mathbf{U} &= \sum_m A_m \begin{pmatrix} \frac{\partial u_m}{\partial x} ih_m \\ \frac{\partial u_m}{\partial y} ih_m \\ u_m \lambda_m^2 \end{pmatrix} e^{ih_m(z-t)} + \sum_n B_n k \begin{pmatrix} \frac{\partial v_n}{\partial y} \\ -\frac{\partial v_n}{\partial x} \\ 0 \end{pmatrix} e^{ig_n(z-t)} \\
&= \sum_m a_m \left[\frac{ih_m}{\lambda_m^2} \begin{pmatrix} \nabla_\Sigma u_m \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ u_m \end{pmatrix} \right] e^{ih_m(z-t)} \\
&\quad + \sum_n -\beta_n \frac{1}{\mu_n^2} \frac{k^2}{ig_n} \begin{pmatrix} \vec{\nabla}_\Sigma \times v_n \\ 0 \end{pmatrix} e^{ig_n(z-t)}.
\end{aligned} \tag{3.13}
$$

Thus, we get

$$
\|\nabla \times \mathbf{U}\|^2_{L^2(W_{(t,l)})} = \sum_m |a_m|^2 \left(\frac{|h_m|^2}{\lambda_m^2} + 1 \right) I_{t,l,m} + \sum_n |b_n|^2 \left(\frac{1}{\mu_n^2} \frac{k^4}{|g_n|^2} \right) J_{t,l,n},
$$

where $I_{t,l,m}$ and $J_{t,l,n}$ are defined the same as before.

So the $H(\text{curl}, W_{(t,l)})$ norm of $\mathbf{U}$ is

$$
\begin{aligned}
\|\mathbf{U}\|^2_{H(\text{curl}, W_{(t,l)})} &= \|\mathbf{U}\|^2_{L^2(W_{(t,l)})} + \|\nabla \times \mathbf{U}\|^2_{L^2(W_{(t,l)})} \\
&= \sum_m |a_m|^2 \left(\frac{1}{\lambda_m^2} + \frac{|h_m|^2}{\lambda_m^2} + 1 \right) I_{t,l,m} \\
&\quad + \sum_n |b_n|^2 \left(\frac{1}{\mu_n^2} + \frac{1}{|g_n|^2} + \frac{1}{\mu_n^2} \frac{k^4}{|g_n|^2} \right) J_{t,l,n} \\
&= \sum_m |a_m|^2 \left(\frac{1 + |h_m|^2 + \lambda_m^2}{\lambda_m^2} \right) I_{t,l,m} \\
&\quad + \sum_n |b_n|^2 \left(\frac{|g_n|^2 + \mu_n^2 + k^4}{\mu_n^2 |g_n|^2} \right) J_{t,l,n}.
\end{aligned}
$$

Breaking the terms into two parts where h_m and g_n are real or imaginary, respectively and noting that

$$
|h_m|^2 = \begin{cases} k^2 - \lambda_m^2 & \text{if} \quad \lambda_m^2 \le k^2 \\ \lambda_m^2 - k^2 & \text{if} \quad \lambda_m^2 > k^2 \end{cases}, \qquad |g_n|^2 = \begin{cases} k^2 - \mu_n^2 & \text{if} \quad \mu_n^2 \le k^2 \\ \mu_n^2 - k^2 & \text{if} \quad \mu_n^2 > k^2 \end{cases},
$$

we get

$$\begin{aligned}
\|\mathbf{U}\|^2_{H(\mathrm{curl},W_{(t,l)})} &= \left[\sum_{\lambda_m^2\le k^2}|a_m|^2\left(\frac{1+k^2}{\lambda_m^2}\right)(l-t)\right.\\
&\quad\left.+\sum_{\lambda_m^2>k^2}|a_m|^2\left(\frac{1+2\lambda_m^2-k^2}{\lambda_m^2}\right)\frac{1}{2\sqrt{\lambda_m^2-k^2}}\left(1-\frac{1}{e^{2\sqrt{\lambda_m^2-k^2}(l-t)}}\right)\right]\\
&+\left[\sum_{\mu_n^2\le k^2}|b_n|^2\left(\frac{k^2+k^4}{\mu_n^2(k^2-\mu_n^2)}\right)(l-t)\right.\\
&\quad\left.+\sum_{\mu_n^2>k^2}|b_n|^2\left(\frac{2\mu_n^2-k^2+k^4}{\mu_n^2(\mu_n^2-k^2)}\right)\frac{1}{2\sqrt{\mu_n^2-k^2}}\left(1-\frac{1}{e^{2\sqrt{\mu_n^2-k^2}(l-t)}}\right)\right].
\end{aligned}$$

Meanwhile, by Lemma 3.1.2, we have series expansion of $\mathbf{Q}$ on Σ_t written as

$$\begin{aligned}
\mathbf{Q} &= \sum_m \alpha_m(\nabla_\Sigma u_m)+\sum_n \beta_n(\vec{\nabla}_\Sigma\times v_n)\\
&= \sum_m \frac{a_m}{\lambda_m^2}(\nabla_\Sigma u_m)+\sum_n \frac{b_n}{\mu_n^2}(\vec{\nabla}_\Sigma\times v_n).
\end{aligned}$$

Then

$$\begin{aligned}
\|\mathbf{Q}\|^2_{H^{-1/2}(\mathrm{div},\Sigma_t)} &= \sum_m \lambda_m^3|\alpha_m|^2+\sum_n \mu_n|\beta_n|^2\\
&= \sum_m \lambda_m^3\left|\frac{a_m}{\lambda_m^2}\right|^2+\sum_n \mu_n\left|\frac{b_n}{\mu_n^2}\right|^2\\
&= \sum_m |a_m|^2\frac{1}{\lambda_m}+\sum_n |b_n|^2\frac{1}{\mu_n^3}.
\end{aligned}$$

As $m,n\to\infty$, we have

$$\begin{aligned}
\left(\frac{1+2\lambda_m^2-k^2}{\lambda_m^2}\right)\frac{1}{2\sqrt{\lambda_m^2-k^2}}\left(1-\frac{1}{e^{2\sqrt{\lambda_m^2-k^2}(l-t)}}\right) &= \mathcal{O}\left(\frac{1}{\lambda_m}\right),\\
\left(\frac{2\mu_n^2-k^2+k^4}{\mu_n^2(\mu_n^2-k^2)}\right)\frac{1}{2\sqrt{\mu_n^2-k^2}}\left(1-\frac{1}{e^{2\sqrt{\mu_n^2-k^2}(l-t)}}\right) &= \mathcal{O}\left(\frac{1}{\mu_n^3}\right).
\end{aligned}$$

Therefore, we can conclude that

$$\|\mathbf{U}\|^2_{H(\mathrm{curl},W_{(t,l)})}\le C\|\mathbf{Q}\|^2_{H^{-1/2}(\mathrm{div},\Sigma_t)}$$

for some constant $C > 0$ independent of m, n. Hence we can conclude $\mathbf{U} \in H(\text{curl}, W_{(t,l)})$ for any $t < l < \infty$. Since l is arbitrary, we can conclude that $\mathbf{U} \in H_{loc}(\text{curl}, W_{(t,\infty)})$.

Hence the proof for existence is done and it remains to show the uniqueness of the solution, i.e. the solution to the problem with homogeneous boundary conditions ($\mathbf{Q} = \mathbf{0}$) is zero.

Since on any bounded segment of waveguide $W_{(t,l)}$ we have $\|\mathbf{U}\|^2_{H(\text{curl},W_{(t,l)})} \leq C\|\mathbf{Q}\|_{H^{-1/2}(\text{div},\Sigma_t)}$. So $\mathbf{U} = \mathbf{0}$ on $W_{(t,l)}$ if $\mathbf{Q} = \mathbf{0}$. By unique continuation principle for the Maxwell's equations (Theorem D.0.8), we have $\mathbf{U} = \mathbf{0}$ in $W_{(t,\infty)}$. Therefore, the proof is done. □

3.1.3.5 The "Dirichlet to Neumann" Map

Now we shall define an important operator, the analogue of the Dirichlet to Neumann (DtN) map, denoted T, for our upcoming analysis. Specifically, given tangential field $\mathbf{n}_\Sigma \times \mathbf{U}|_\Sigma$ on Σ, T is defined as

$$T(\mathbf{n}_\Sigma \times \mathbf{U}|_\Sigma) = \mathbf{n}_\Sigma \times (\nabla \times \mathbf{U})|_\Sigma, \tag{3.14}$$

where $\mathbf{U}$ satisfies (3.10). A similar operator can be defined by considering the analogue of (3.10) on $\Sigma \times (-\infty, s)$ for some fixed s. The analysis of the two operators is identical, so we will only give details for T on Σ_t and will find it useful to identify the operator on specific cross section using subscript. For example, T_t^+ on Σ_t using $\Sigma \times (t, \infty)$, T_s^- on Σ_s using $\Sigma \times (-\infty, s)$ and so on. For now, we take $T = T_t^+$.

To derive a series representation of T using modal solutions derived in Section 3.1.2.1, consider Ω_R in Figure 3.2 and choose $\mathbf{n}_\Sigma = \hat{\mathbf{z}}$. From (3.12) and (3.13) in Lemma 3.1.3, the solution $\mathbf{U}$ to the Maxwell's equation in Ω_R and $\nabla \times \mathbf{U}$ can be written in explicit form for $z > t$ as

$$\begin{aligned}
\mathbf{U} &= \sum_m A_m M_m + \sum_n B_n N_n \\
&= \sum_m \frac{a_m}{\lambda_m^2} \left[\begin{pmatrix} \vec{\nabla}_\Sigma \times u_m \\ 0 \end{pmatrix} e^{ih_m(z-t)} \right]
\end{aligned}$$

$$+\sum_n -\frac{b_n}{\mu_n^2}\left[\begin{pmatrix}\nabla_\Sigma v_n\\ 0\end{pmatrix} e^{ig_n(z-t)}\right] - \frac{b_n}{ig_n}\left[\begin{pmatrix}0\\ 0\\ v_n\end{pmatrix} e^{ig_n(z-t)}\right],$$

so that

$$\begin{aligned}
\nabla\times\mathbf{U} &= \sum_m A_m(\nabla\times M_m)+\sum_n B_n(\nabla\times N_n)\\
&= \sum_m a_m\left[\frac{ih_m}{\lambda_m^2}\begin{pmatrix}\nabla_\Sigma u_m\\ 0\end{pmatrix}+\begin{pmatrix}0\\ 0\\ u_m\end{pmatrix}\right]e^{ih_m(z-t)}\\
&\quad+\sum_n -\beta_n\frac{1}{\mu_n^2}\frac{k^2}{ig_n}\begin{pmatrix}\vec{\nabla}_\Sigma\times v_n\\ 0\end{pmatrix}e^{ig_n(z-t)},
\end{aligned}$$

where as before

$$\begin{aligned}
a_m &\triangleq \left\langle \mathbf{n}_\Sigma\times\mathbf{U}|_\Sigma, \begin{pmatrix}\nabla_\Sigma u_m\\ 0\end{pmatrix}\right\rangle_\Sigma = A_m\lambda_m^2\|u_m\|^2_{L^2(\Sigma)} = A_m\lambda_m^2,\\
b_n &\triangleq \left\langle \mathbf{n}_\Sigma\times\mathbf{U}|_\Sigma, \begin{pmatrix}\vec{\nabla}_\Sigma\times v_n\\ 0\end{pmatrix}\right\rangle_\Sigma = -B_n\mu_n^2\|v_n\|^2_{L^2(\Sigma)}\frac{ig_n}{k} = -B_n\mu_n^2\frac{ig_n}{k}.
\end{aligned}$$

Thus, using identities (B.11), (B.12), we get

$$\mathbf{n}_\Sigma\times\mathbf{U}|_{\Sigma_t} = \sum_m a_m\frac{1}{\lambda_m^2}\begin{pmatrix}\nabla_\Sigma u_m\\ 0\end{pmatrix}+\sum_n b_n\frac{1}{\mu_n^2}\begin{pmatrix}\vec{\nabla}_\Sigma\times v_n\\ 0\end{pmatrix},$$

and

$$\mathbf{n}_\Sigma\times(\nabla\times\mathbf{U})|_{\Sigma_t} = \sum_m -a_m\frac{ih_m}{\lambda_m^2}\begin{pmatrix}\vec{\nabla}_\Sigma\times u_m\\ 0\end{pmatrix}+\sum_n -b_n\frac{1}{\mu_n^2}\frac{k^2}{ig_n}\begin{pmatrix}\nabla_\Sigma v_n\\ 0\end{pmatrix}.$$

In summary, we have that on Σ_t,

$$\mathbf{U} = \sum_m a_m\frac{1}{\lambda_m^2}\begin{pmatrix}\vec{\nabla}_\Sigma\times u_m\\ 0\end{pmatrix}$$

$$
+\sum_n -b_n \left[\frac{1}{\mu_n^2} \begin{pmatrix} \nabla_\Sigma v_n \\ 0 \end{pmatrix} + \frac{1}{ig_n} \begin{pmatrix} 0 \\ 0 \\ v_n \end{pmatrix} \right], \tag{3.15}
$$

$$
\begin{aligned}
\nabla \times \mathbf{U} &= \sum_m a_m \left[\frac{ih_m}{\lambda_m^2} \begin{pmatrix} \nabla_\Sigma u_m \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ u_m \end{pmatrix} \right] \\
&\quad + \sum_n -\beta_n \frac{1}{\mu_n^2} \frac{k^2}{ig_n} \begin{pmatrix} \vec{\nabla}_\Sigma \times v_n \\ 0 \end{pmatrix},
\end{aligned}
$$

$$
\mathbf{n}_\Sigma \times \mathbf{U} = \sum_m a_m \frac{1}{\lambda_m^2} \begin{pmatrix} \nabla_\Sigma u_m \\ 0 \end{pmatrix} + \sum_n b_n \frac{1}{\mu_n^2} \begin{pmatrix} \vec{\nabla}_\Sigma \times v_n \\ 0 \end{pmatrix}, \tag{3.16}
$$

$$
\mathbf{n}_\Sigma \times (\nabla \times \mathbf{U}) = \sum_m -a_m \frac{ih_m}{\lambda_m^2} \begin{pmatrix} \vec{\nabla}_\Sigma \times u_m \\ 0 \end{pmatrix} + \sum_n -b_n \frac{1}{\mu_n^2} \frac{k^2}{ig_n} \begin{pmatrix} \nabla_\Sigma v_n \\ 0 \end{pmatrix}. \tag{3.17}
$$

By making a comparison of (3.16) and (3.17), together with identity (3.8), we have the following series representation of operator T given by

$$
\begin{aligned}
T(\mathbf{n}_\Sigma \times \mathbf{U}|_\Sigma) &= \sum_m \left\langle \mathbf{n}_\Sigma \times \mathbf{U}|_\Sigma, \begin{pmatrix} \nabla_\Sigma u_m \\ 0 \end{pmatrix} \right\rangle_\Sigma \left(-\frac{ih_m}{\lambda_m^2} \right) \begin{pmatrix} \vec{\nabla}_\Sigma \times u_m \\ 0 \end{pmatrix} \\
&\quad + \sum_n \left\langle \mathbf{n}_\Sigma \times \mathbf{U}|_\Sigma, \begin{pmatrix} \vec{\nabla}_\Sigma \times v_n \\ 0 \end{pmatrix} \right\rangle_\Sigma \left(-\frac{1}{\mu_n^2} \frac{k^2}{ig_n} \right) \begin{pmatrix} \nabla_\Sigma v_n \\ 0 \end{pmatrix}
\end{aligned} \tag{3.18}
$$

To facilitate the analysis later on, we shall now derive some properties of the operator T.

Lemma 3.1.4 *T is a bounded operator from $\widetilde{H}^{-1/2}(\mathrm{div}, \Sigma)$ to $\widetilde{H}^{-1/2}(\mathrm{div}, \Sigma)$.*

Proof: Let $\mathbf{U} \in X$ with its trace $\mathbf{n}_\Sigma \times \mathbf{U}|_\Sigma \in \widetilde{H}^{-1/2}(\mathrm{div}, \Sigma)$. Using the form of T given in (3.18), we have

$$
T(\mathbf{n}_\Sigma \times \mathbf{U}|_\Sigma) = \sum_m \left\langle \mathbf{n}_\Sigma \times \mathbf{U}|_\Sigma, \begin{pmatrix} \nabla_\Sigma u_m \\ 0 \end{pmatrix} \right\rangle_\Sigma \left(-\frac{ih_m}{\lambda_m^2} \right) \begin{pmatrix} \vec{\nabla}_\Sigma \times u_m \\ 0 \end{pmatrix}
$$

$$+\sum_n \left\langle \mathbf{n}_\Sigma \times \mathbf{U}|_\Sigma, \begin{pmatrix} \vec{\nabla}_\Sigma \times v_n \\ 0 \end{pmatrix} \right\rangle_\Sigma \left(-\frac{1}{\mu_n^2}\frac{k^2}{ig_n}\right) \begin{pmatrix} \nabla_\Sigma v_n \\ 0 \end{pmatrix}$$

$$= \sum_m a_m \left(-\frac{ih_m}{\lambda_m^2}\right) \begin{pmatrix} \vec{\nabla}_\Sigma \times u_m \\ 0 \end{pmatrix} + \sum_n b_n \left(-\frac{1}{\mu_n^2}\frac{k^2}{ig_n}\right) \begin{pmatrix} \nabla_\Sigma v_n \\ 0 \end{pmatrix}.$$

Then

$$\begin{aligned} \|T(\mathbf{n}_\Sigma \times \mathbf{U}|_\Sigma)\|^2_{H^{-1/2}(\mathrm{div},\Sigma)} &= \sum_m |a_m|^2 \lambda_m \left| -\frac{ih_m}{\lambda_m^2} \right|^2 + \sum_n |b_n|^2 \mu_n^3 \left| -\frac{1}{\mu_n^2}\frac{k^2}{ig_n} \right|^2 \\ &= \sum_m |a_m|^2 \frac{|k^2-\lambda_m^2|}{\lambda_m^3} + \sum_n |b_n|^2 \frac{|k^2|^2}{\mu_n|(k^2-\mu_n^2)|}. \end{aligned}$$

Meanwhile, from (3.16),

$$\mathbf{n}_\Sigma \times \mathbf{U}|_\Sigma = \sum_m a_m \frac{1}{\lambda_m^2} \begin{pmatrix} \nabla_\Sigma u_m \\ 0 \end{pmatrix} + \sum_n b_n \frac{1}{\mu_n^2} \begin{pmatrix} \vec{\nabla}_\Sigma \times v_n \\ 0 \end{pmatrix},$$

with norm

$$\begin{aligned} \|\mathbf{n}_\Sigma \times \mathbf{U}|_\Sigma\|^2_{H^{-1/2}(\mathrm{div},\Sigma)} &= \sum_m |a_m|^2 \lambda_m^3 \left| \frac{1}{\lambda_m^2} \right|^2 + \sum_n |b_n|^2 \mu_n \left| \frac{1}{\mu_n^2} \right|^2 \\ &= \sum_m |a_m|^2 \frac{1}{\lambda_m} + \sum_n |b_n|^2 \frac{1}{\mu_n^3}. \end{aligned}$$

As $m, n \to \infty$, we have

$$\frac{|k^2-\lambda_m^2|}{\lambda_m^3} = \mathcal{O}\left(\frac{1}{\lambda_m}\right) \quad \text{and} \quad \frac{|k^2|^2}{\mu_n|(k^2-\mu_n^2)|} = \mathcal{O}\left(\frac{1}{\mu_n^3}\right).$$

Therefore,

$$\|T(\mathbf{n}_\Sigma \times \mathbf{U}|_\Sigma)\|^2_{H^{-1/2}(\mathrm{div},\Sigma)} \le C \|\mathbf{n}_\Sigma \times \mathbf{U}|_\Sigma\|^2_{H^{-1/2}(\mathrm{div},\Sigma)}$$

for some constant $C > 0$ independent of m, n and this completes the proof. □

Lemma 3.1.5 *Let $k = k_* > 0$ be any positive real wavenumber such that $h_m, g_n \neq 0$ for all m, n. Then there exists a neighborhood $\mathfrak{B}$ of k_* in which the operator T depends analytically on the wavenumber $k \in \mathfrak{B}$.*

Proof: The proof is inspired from the proof of Lemma 2.4 in [2]. The numbers $h_m = \sqrt{k^2-\lambda_m^2}$ and $g_n = \sqrt{k^2-\mu_n^2}$ can be defined using the analytic extension of the square root to the complex plane with branch cuts $\{b_m^\lambda\}_{m\geq 0}$ and $\{b_n^\mu\}_{n\geq 0}$ in a sub-domain of complex plane $\{z\in\mathbb{C}, \mathfrak{Re} z\geq 0, \mathfrak{Im} z\leq 0\}$ which end at the points λ_m and μ_n, respectively. Explicitly, they are

$$\begin{aligned}
b_m^\lambda &:= [\lambda_m, \lambda_m - i\infty) = \{z\in\mathbb{C} \mid \mathfrak{Re} z = \lambda_m, \mathfrak{Im} z\leq 0\} \quad \text{for } m = 0,1,\ldots, \\
b_n^\mu &:= [\mu_n, \mu_n - i\infty) = \{z\in\mathbb{C} \mid \mathfrak{Re} z = \mu_n, \mathfrak{Im} z\leq 0\} \quad \text{for } n = 0,1,\ldots.
\end{aligned}$$

Then, h_m and g_n depend analytically on k in region of the complex plane except for these branch cuts.

By definition of k_*, there exists an open ball $\mathfrak{B}$ of radius r_* centered at k_* in which h_m and g_n depend analytically on k. Moreover, since $h_m, g_n \neq 0$, $\dfrac{1}{h_m}$ and $\dfrac{1}{g_n}$ also depend analytically on k in $\mathfrak{B}$.

For $\mathbf{U},\mathbf{V}\in X$, using series expansions (3.16) on Σ, we have

$$\begin{aligned}
\mathbf{n}_\Sigma\times\mathbf{U}|_\Sigma &= \sum_m a_m \frac{1}{\lambda_m^2}\begin{pmatrix}\nabla_\Sigma u_m\\ 0\end{pmatrix} + \sum_n b_n\frac{1}{\mu_n^2}\begin{pmatrix}\vec{\nabla}_\Sigma\times v_n\\ 0\end{pmatrix}, \\
\mathbf{n}_\Sigma\times\mathbf{V}|_\Sigma &= \sum_m \tilde{a}_m \frac{1}{\lambda_m^2}\begin{pmatrix}\nabla_\Sigma u_m\\ 0\end{pmatrix} + \sum_n \tilde{b}_n\frac{1}{\mu_n^2}\begin{pmatrix}\vec{\nabla}_\Sigma\times v_n\\ 0\end{pmatrix}.
\end{aligned}$$

Denote by $\mathbf{V}_T = (\mathbf{n}_\Sigma\times\mathbf{V}|_\Sigma)\times\mathbf{n}_\Sigma$. We show next that the boundary integral $\langle T(\mathbf{n}_\Sigma\times\mathbf{U}|_\Sigma), \mathbf{V}_T\rangle_\Sigma$ converges absolutely and uniformly for k in $\mathfrak{B}$. Using series expansion (3.17), we have

$$\begin{aligned}
&\langle T(\mathbf{n}_\Sigma\times\mathbf{U}|_\Sigma), \mathbf{V}_T\rangle \\
&= \left\langle \sum_m -a_m\frac{ih_m}{\lambda_m^2}\begin{pmatrix}\vec{\nabla}_\Sigma\times u_m\\ 0\end{pmatrix} + \sum_n -b_n\frac{1}{\mu_n^2}\frac{k^2}{ig_n}\begin{pmatrix}\nabla_\Sigma v_n\\ 0\end{pmatrix}, (\mathbf{n}_\Sigma\times\mathbf{V}|_\Sigma)\times\mathbf{n}_\Sigma\right\rangle_\Sigma.
\end{aligned}$$

Using identities (B.2) and (B.11),(B.12), we have

$$\langle T(\mathbf{n}_\Sigma\times\mathbf{U}|_\Sigma), \mathbf{V}_T\rangle_\Sigma$$

$$
\begin{aligned}
&= \left\langle \sum_m -a_m \frac{ih_m}{\lambda_m^2} \left(\mathbf{n}_\Sigma \times \begin{pmatrix} \vec{\nabla}_\Sigma \times u_m \\ 0 \end{pmatrix} \right) + \sum_n -b_n \frac{1}{\mu_n^2} \frac{k^2}{ig_n} \left(\mathbf{n}_\Sigma \times \begin{pmatrix} \nabla_\Sigma v_n \\ 0 \end{pmatrix} \right), \mathbf{n}_\Sigma \times \mathbf{V}|_\Sigma \right\rangle_\Sigma \\
&= \left\langle \sum_m -a_m \frac{ih_m}{\lambda_m^2} \begin{pmatrix} \nabla_\Sigma u_m \\ 0 \end{pmatrix} + \sum_n -b_n \frac{1}{\mu_n^2} \frac{k^2}{ig_n} \left(- \begin{pmatrix} \vec{\nabla}_\Sigma \times v_n \\ 0 \end{pmatrix} \right), \right. \\
&\qquad \left. \sum_m \tilde{a}_m \frac{1}{\lambda_m^2} \begin{pmatrix} \nabla_\Sigma u_m \\ 0 \end{pmatrix} + \sum_n \tilde{b}_n \frac{1}{\mu_n^2} \begin{pmatrix} \vec{\nabla}_\Sigma \times v_n \\ 0 \end{pmatrix} \right\rangle_\Sigma \\
&= \sum_m \langle a_m, \tilde{a}_m \rangle_\Sigma \left(-\frac{ih_m}{\lambda_m^2} \right) + \sum_n \langle b_n, \tilde{b}_n \rangle_\Sigma \left(\frac{k^2}{\mu_n^2 ig_n} \right) \\
&= \sum_m \left\langle a_m \frac{1}{\lambda_m^{1/2}}, \tilde{a}_m \frac{1}{\lambda_m^{1/2}} \right\rangle_\Sigma \left(-\frac{i}{\lambda_m} \right) h_m + \sum_n \left\langle b_n \frac{1}{\mu_n^{3/2}}, \tilde{b}_n \frac{1}{\mu_n^{3/2}} \right\rangle_\Sigma \left(\frac{\mu_n}{i} \right) \frac{k^2}{g_n}.
\end{aligned} \tag{3.19}
$$

Notice that

$$
\begin{aligned}
\left| -\frac{ih_m}{\lambda_m} \right| &= \left| \frac{\sqrt{k^2 - \lambda_m^2}}{\lambda_m} \right| = \left| \sqrt{\frac{k^2 - \lambda_m^2}{\lambda_m^2}} \right|, \\
\left| \frac{k^2 \mu_n}{ig_n} \right| &= |k|^2 \left| \frac{\mu_n}{\sqrt{k^2 - \mu_n^2}} \right| = |k|^2 \left| \sqrt{\frac{\mu_n^2}{k^2 - \mu_n^2}} \right|.
\end{aligned}
$$

Since they are bounded for all m's and n's, we have

$$
\begin{aligned}
& |\langle T(\mathbf{n}_\Sigma \times \mathbf{U}|_\Sigma), \mathbf{V}_T \rangle| \\
&= \left| \sum_m \left\langle a_m \frac{1}{\lambda_m^{1/2}}, \tilde{a}_m \frac{1}{\lambda_m^{1/2}} \right\rangle_\Sigma \left(-\frac{ih_m}{\lambda_m} \right) + \sum_n \left\langle b_n \frac{1}{\mu_n^{3/2}}, \tilde{b}_n \frac{1}{\mu_n^{3/2}} \right\rangle_\Sigma \left(\frac{k^2 \mu_n}{ig_n} \right) \right| \\
&\leq \sum_m \left| \left\langle a_m \frac{1}{\lambda_m^{1/2}}, \tilde{a}_m \frac{1}{\lambda_m^{1/2}} \right\rangle_\Sigma \right| \left| -\frac{ih_m}{\lambda_m} \right| + \sum_n \left| \left\langle b_n \frac{1}{\mu_n^{3/2}}, \tilde{b}_n \frac{1}{\mu_n^{3/2}} \right\rangle_\Sigma \right| \left| \frac{k^2 \mu_n}{ig_n} \right| \\
&\leq C \left(\sum_m \left| \left\langle a_m \frac{1}{\lambda_m^{1/2}}, \tilde{a}_m \frac{1}{\lambda_m^{1/2}} \right\rangle_\Sigma \right| + \sum_n \left| \left\langle b_n \frac{1}{\mu_n^{3/2}}, \tilde{b}_n \frac{1}{\mu_n^{3/2}} \right\rangle_\Sigma \right| \right) \\
&\leq C \left(\sqrt{\sum_m |a_m|^2 \frac{1}{\lambda_m}} \sqrt{\sum_m |\tilde{a}_m|^2 \frac{1}{\lambda_m}} + \sqrt{\sum_n |b_n|^2 \frac{1}{\mu_n^3}} \sqrt{\sum_n |\tilde{b}_n|^2 \frac{1}{\mu_n^3}} \right) \\
&\leq C \left(\sqrt{\sum_m |a_m|^2 \frac{1}{\lambda_m} + \sum_n |b_n|^2 \frac{1}{\mu_n^3}} \sqrt{\sum_m |\tilde{a}_m|^2 \frac{1}{\lambda_m} + \sum_n |\tilde{b}_n|^2 \frac{1}{\mu_n^3}} \right).
\end{aligned}
$$

for some constant $C > 0$ depending on radius r_* of $\mathfrak{B}$ but independent of $k \in \mathfrak{B}$.

On the other hand,

$$\begin{aligned}
\|\mathbf{n}_\Sigma \times \mathbf{U}|_\Sigma\|^2_{H^{-1/2}(\mathrm{div},\Sigma)} &= \sum_m \left| a_m \frac{1}{\lambda_m^2} \right|^2 \lambda_m^3 + \sum_n \left| b_n \frac{1}{\mu_n^2} \right|^2 \mu_n \\
&= \sum_m |a_m|^2 \frac{1}{\lambda_m} + \sum_n |b_n|^2 \frac{1}{\mu_n^3}, \\
\|\mathbf{n}_\Sigma \times \mathbf{V}|_\Sigma\|^2_{H^{-1/2}(\mathrm{div},\Sigma)} &= \sum_m |\tilde{a}_m|^2 \frac{1}{\lambda_m} + \sum_n |\tilde{b}_n|^2 \frac{1}{\mu_n^3}.
\end{aligned}$$

Thus, we get

$$|\langle T(\mathbf{n}_\Sigma \times \mathbf{U}|_\Sigma), \mathbf{V}_T \rangle| \quad \leq \quad C \|\mathbf{n}_\Sigma \times \mathbf{U}|_\Sigma\|_{H^{-1/2}(\mathrm{div},\Sigma)} \|\mathbf{n}_\Sigma \times \mathbf{V}|_\Sigma\|_{H^{-1/2}(\mathrm{div},\Sigma)}.$$

So the series for $\langle T(\mathbf{n}_\Sigma \times \mathbf{U}|_\Sigma), \mathbf{V}_T \rangle_\Sigma$ in (3.19) converges absolutely and uniformly.

To show the analyticity of $\langle T(\mathbf{n}_\Sigma \times \mathbf{U}|_\Sigma), \mathbf{V}_T \rangle_\Sigma$ for k in $\mathfrak{B}$. Note that $-\frac{i}{\lambda_m}$ and $\frac{\mu_n}{i}$ in (3.19) are independent of k and $h_m, \frac{k^2}{g_n}$ are analytic for k in $\mathfrak{B}$. Thus, exploiting the uniform convergence of (3.19), we can conclude that the series in (3.19) is analytic for k in $\mathfrak{B}$ (see Theorem 1 in Chapter 5 of [1]).

Therefore, operator T is weakly analytic for k in $\mathfrak{B}$. By the equivalence of weak and strong analyticity of a bounded linear operator (see Theorem VI.4 in [57]), T is strongly analytic for k in $\mathfrak{B}$ and the statement of the lemma follows. $\square$

With the definition of operator T, as a corollary of Lemma 3.1.3, we have the well-posedness of the bounded segment of waveguide in the absence of scatterer:

Corollary 3.1.1 *For $s < t$ and $\mathbf{Q} \in \widetilde{H}^{-1/2}(\mathrm{div}, \Sigma_s)$, the following problem has a unique solution $\mathbf{U} \in H(\mathrm{curl}, W_{(s,t)})$ such that*

$$\begin{cases}
\nabla \times \nabla \times \mathbf{U} - k^2 \mathbf{U} = 0 & \text{in} \quad W_{(s,t)}, \\
\mathbf{n}_\Gamma \times \mathbf{U} = 0 & \text{on} \quad \Gamma_{(s,t)}, \\
\mathbf{n}_\Sigma \times \mathbf{U} = \mathbf{Q} & \text{on} \quad \Sigma_s, \\
\mathbf{n}_\Sigma \times (\nabla \times \mathbf{U}) = T_t(\mathbf{n}_\Sigma \times \mathbf{U}) & \text{on} \quad \Sigma_t.
\end{cases} \tag{3.20}$$

where $\Gamma_{(s,t)} = \partial\Sigma \times (s,t)$ and $\mathbf{n}_\Sigma = \hat{\mathbf{z}}$.

Proof: By Lemma 3.1.3, the blocked waveguide problem has a solution and its restriction to $W_{(s,t)}$ solves (3.20). But by the definition of the DtN map the solution of (3.20) can be extended to a solution of (3.10), so the problems are equivalent and this completes the proof. $\square$

3.1.3.6 Weak Formulation on an Unbounded Domain

To further analyze the problem, we shall investigate the problem on an unbounded waveguide where the scatterer D is illuminated by point sources (imagine a point source located far to the left away from D in the waveguide W as in Figure 3.1) and then reduce it to a bounded domain (the sub-domain Ω in Figure 3.2 where the point source is located outside Ω in Ω_L).

First we consider solutions to the scattering problem (3.1) - (3.5) in a weak sense. Formally multiplying (3.1) by the complex conjugate of a smooth test function $\mathbf{v}$ and applying Green's identity we have that

$$\int_{W\backslash\bar{D}} \left[(\nabla \times \mathbf{u}) \cdot (\nabla \times \bar{\mathbf{v}}) - k^2 \mathbf{u} \cdot \bar{\mathbf{v}}\right] \, dx = \int_{W\backslash\bar{D}} \mathbf{F} \cdot \bar{\mathbf{v}} \, dx, \tag{3.21}$$

for all $\mathbf{v} \in (C_0^\infty(W\backslash\bar{D}))^3$ where $(C_0^\infty(W\backslash\bar{D}))^3$ is the set of compactly supported smooth vector functions in $W\backslash\bar{D}$.

The difficulty in using (3.21) to state the weak formulation is the presence of the singular source term corresponding to the point source at $\mathbf{x}_0$. To avoid this difficulty, we introduce a cut-off scalar function $\chi \in C^\infty(W\backslash\bar{D})$ such that

- $\chi = 1$ in a neighborhood of ∂D.
- $\chi = 0$ in a neighborhood of Γ and for all x ($x \notin \bar{D}$) with $|x| > L$ where L is chosen so that $\chi = 0$ on Σ_s and Σ_t.

Then the (global) weak solution of the scattering problem reads as follows:

Definition 3.1.2 *(Forward scattering problem) Given an incident field $\mathbf{u}^i$ due to a point source at $\mathbf{x}_0$, then $\mathbf{u}^s$ is a weak solution to the waveguide problem if $\mathbf{u}^s = \mathbf{w} - \chi\mathbf{u}^i$ where*

- $\mathbf{w} \in H_{loc,0}(\text{curl}, W\backslash\bar{D})$ *and for any* $\mathbf{v} \in (C_0^\infty(W\backslash\bar{D}))^3$,

$$\int_{W\backslash\bar{D}} \left[(\nabla \times \mathbf{w}) \cdot (\nabla \times \bar{\mathbf{v}}) - k^2 \mathbf{w} \cdot \bar{\mathbf{v}}\right] dx = \tilde{F}(\mathbf{v}), \tag{3.22}$$

where

$$\tilde{F}(\mathbf{v}) = \int_{W\backslash\bar{D}} \left[(\nabla \times (\chi \mathbf{u}^i)) \cdot (\nabla \times \bar{\mathbf{v}}) - k^2 (\chi \mathbf{u}^i) \cdot \bar{\mathbf{v}}\right] dx.$$

- *The field* $\mathbf{w}$ *satisfies the radiation condition. This will guarantee the radiation condition in Definition 3.1.1 on* $\mathbf{u}^s$.

Note that $\tilde{F}$ *is a bounded antilinear functional on* $H_{loc}(\text{curl}, W\backslash\bar{D})$.

3.1.3.7 Variational Formulation on a Bounded Domain

Now we derive the variational formulation on $\Omega = W_{(s,t)}\backslash\bar{D}$ in Figure 3.2 where the point source $\mathbf{x}_0$ is excluded (for example, $\mathbf{x}_0 \in \Omega_L$). On cross sections Σ_s and Σ_t, we prescribe boundary conditions that enforce the radiation conditions in Definition 3.1.1 using DtN mappings T (see (3.14)). Then the full statement of the forward problem for the total field $\mathbf{u}$ in Ω reads:

$$\nabla \times \nabla \times \mathbf{u} - k^2 \mathbf{u} = 0 \quad \text{in} \quad \Omega, \tag{3.23}$$

$$\mathbf{n}_D \times \mathbf{u} = 0 \quad \text{on} \quad \partial D, \tag{3.24}$$

$$\mathbf{u} = \mathbf{u}^i + \mathbf{u}^s \quad \text{in} \quad \Omega, \tag{3.25}$$

$$\mathbf{n}_\Gamma \times \mathbf{u} = 0 \quad \text{on} \quad \Gamma_{(s,t)}, \tag{3.26}$$

$$\mathbf{n}_\Sigma \times (\nabla \times \mathbf{u}^s) = T_s(\mathbf{n}_\Sigma \times \mathbf{u}^s) \quad \text{on} \quad \Sigma_s, \tag{3.27}$$

$$\mathbf{n}_\Sigma \times (\nabla \times \mathbf{u}^s) = T_t(\mathbf{n}_\Sigma \times \mathbf{u}^s) \quad \text{on} \quad \Sigma_t, \tag{3.28}$$

where $\mathbf{u}^i$ is the incident field due to point source $\mathbf{x}_0$ outside Ω that satisfies (3.23) and boundary condition (3.26) in Ω.

Correspondingly, the forward problem for the scattered field $\mathbf{u}^s$ in Ω is

$$\begin{cases} \nabla \times \nabla \times \mathbf{u}^s - k^2 \mathbf{u}^s = 0 & \text{in} \quad \Omega, \\ \mathbf{n}_D \times \mathbf{u}^s = -\mathbf{n}_D \times \mathbf{u}^i & \text{on} \quad \partial D, \\ \mathbf{n}_\Gamma \times \mathbf{u}^s = 0 & \text{on} \quad \Gamma_{(s,t)}, \\ \mathbf{n}_\Sigma \times (\nabla \times \mathbf{u}^s) = T_s(\mathbf{n}_\Sigma \times \mathbf{u}^s) & \text{on} \quad \Sigma_s, \\ \mathbf{n}_\Sigma \times (\nabla \times \mathbf{u}^s) = T_t(\mathbf{n}_\Sigma \times \mathbf{u}^s) & \text{on} \quad \Sigma_t. \end{cases}$$

To construct the variational formulation of this problem, we take dot product of (3.23) with the complex conjugate of a test function $\mathbf{v} \in X$ and integrate over Ω. Using integration by parts and the vector identity (B.2), we have

$$\begin{aligned} 0 &= \int_\Omega (\nabla \times \nabla \times \mathbf{u} - k^2 \mathbf{u}) \cdot \bar{\mathbf{v}}\, dx \\ &= \int_\Omega (\nabla \times \mathbf{u}) \cdot (\nabla \times \bar{\mathbf{v}})\, dx - \int_{\partial\Omega} \mathbf{n}_\Omega \cdot (\bar{\mathbf{v}} \times \nabla \times \mathbf{u})\, ds - k^2 \int_\Omega \mathbf{u} \cdot \bar{\mathbf{v}}\, dx \\ &= \int_\Omega (\nabla \times \mathbf{u}) \cdot (\nabla \times \bar{\mathbf{v}})\, dx - k^2 \int_\Omega \mathbf{u} \cdot \bar{\mathbf{v}}\, dx - \int_{\partial D} (\mathbf{n}_D \times \nabla \times \mathbf{u}) \cdot \bar{\mathbf{v}}\, ds \\ &\quad + \int_{\Gamma_{(s,t)}} (\mathbf{n}_\Gamma \times \nabla \times \mathbf{u}) \cdot \bar{\mathbf{v}}\, ds + \sum_{j=s,t} \int_{\Sigma_j} (\mathbf{n}_\Sigma \times \nabla \times \mathbf{u}) \cdot \bar{\mathbf{v}}\, ds. \end{aligned}$$

Since $\mathbf{n} \times \mathbf{v} = 0$ on $\Gamma_{(s,t)} \cup \partial D$, we get

$$\int_\Omega (\nabla \times \mathbf{u}) \cdot (\nabla \times \bar{\mathbf{v}})\, dx - k^2 \int_\Omega \mathbf{u} \cdot \bar{\mathbf{v}}\, dx + \sum_{j=s,t} \int_{\Sigma_j} (\mathbf{n}_\Sigma \times \nabla \times \mathbf{u}) \cdot \bar{\mathbf{v}}\, ds = 0.$$

Furthermore, because $\mathbf{u}^s = \mathbf{u} - \mathbf{u}^i$, we see that

$$\begin{aligned} &\int_{\Sigma_s} (\mathbf{n}_\Sigma \times \nabla \times \mathbf{u}) \cdot \bar{\mathbf{v}}\, ds \\ =\ &\int_{\Sigma_s} (\mathbf{n}_\Sigma \times \nabla \times (\mathbf{u} - \mathbf{u}^i)) \cdot \bar{\mathbf{v}}\, ds + \int_{\Sigma_s} (\mathbf{n}_\Sigma \times \nabla \times \mathbf{u}^i) \cdot \bar{\mathbf{v}}\, ds \\ =\ &\int_{\Sigma_s} T_s(\mathbf{n}_\Sigma \times (\mathbf{u} - \mathbf{u}^i)) \cdot \bar{\mathbf{v}}\, ds + \int_{\Sigma_s} (\mathbf{n}_\Sigma \times \nabla \times \mathbf{u}^i) \cdot \bar{\mathbf{v}}\, ds \\ =\ &\int_{\Sigma_s} T_s(\mathbf{n}_\Sigma \times \mathbf{u}) \cdot \bar{\mathbf{v}}_T\, ds + \int_{\Sigma_s} \left[(\mathbf{n}_\Sigma \times \nabla \times \mathbf{u}^i) - T_s(\mathbf{n}_\Sigma \times \mathbf{u}^i) \right] \cdot \bar{\mathbf{v}}_T\, ds, \end{aligned}$$

where $\bar{\mathbf{v}}_T = (\mathbf{n}_\Sigma \times \bar{\mathbf{v}}) \times \mathbf{n}_\Sigma$. A similar formula holds on Σ_t with operator T_t. Thus we have

$$\int_\Omega (\nabla \times \mathbf{u}) \cdot (\nabla \times \bar{\mathbf{v}})\, dx - k^2 \int_\Omega \mathbf{u} \cdot \bar{\mathbf{v}}\, dx + \sum_{j=s,t} \int_{\Sigma_j} T_j(\mathbf{n}_\Sigma \times \mathbf{u}) \cdot \bar{\mathbf{v}}_T\, ds$$

$$= \sum_{j=s,t} \int_{\Sigma_j} \left[T_j(\mathbf{n}_\Sigma \times \mathbf{u}^i) - (\mathbf{n}_\Sigma \times \nabla \times \mathbf{u}^i) \right] \cdot \bar{\mathbf{v}}_T \, ds.$$

Therefore, the (local) variational formulation of the scattering problem reads as follows:

Definition 3.1.3 *(Truncated Forward scattering problem) Given an incident field* $\mathbf{u}^i$ *satisfying* (3.23) *and* (3.26), $\mathbf{u} \in X$ *is said to be a solution of the variational problem in* Ω *if it satisfies*

$$\int_\Omega (\nabla \times \mathbf{u}) \cdot (\nabla \times \bar{\mathbf{v}}) \, dx \quad - \quad k^2 \int_\Omega \mathbf{u} \cdot \bar{\mathbf{v}} \, dx \\ + \sum_{j=s,t} \int_{\Sigma_j} T_j(\mathbf{n}_\Sigma \times \mathbf{u}) \cdot \bar{\mathbf{v}}_T \, ds = \hat{F}(\mathbf{u}^i, \mathbf{v}), \qquad (3.29)$$

for all $\mathbf{v} \in X$ *where*

$$\hat{F}(\mathbf{u}^i, \mathbf{v}) = \sum_{j=s,t} \int_{\Sigma_j} \left[T_j(\mathbf{n}_\Sigma \times \mathbf{u}^i) - (\mathbf{n}_\Sigma \times \nabla \times \mathbf{u}^i) \right] \cdot \bar{\mathbf{v}}_T \, ds.$$

Before investigating the well-posedness of variational problem in Definition 3.1.3, we want to point out the equivalence of weak formulation on unbounded domain and variational formulation on bounded domain given by the following theorem:

Theorem 3.1.3 *Suppose* $\mathbf{u}^s$ *is a weak solution of the (global) scattering problem in the sense of Definition 3.1.2. Then* $\mathbf{u} = \mathbf{w} + (1 - \chi)\mathbf{u}^i$ *is a solution to the (local) variational problem in Definition 3.1.3. Conversely, if* $\mathbf{u}$ *is a solution to the (local) variational problem in Definition 3.1.3, setting* $\mathbf{w} = \mathbf{u} - (1 - \chi)\mathbf{u}^i$*, it can be extended in a unique way to* $W \backslash \Omega$ *and so that* $\mathbf{u}^s$ *as an extension of* $\mathbf{u} - \mathbf{u}^i$ *satisfies the (global) scattering problem in Definition 3.1.2.*

Proof: First we prove that the restriction of a solution of the global scattering problem is a solution of the local scattering problem. Suppose $\mathbf{w}$ is a solution to the problem described in Definition 3.1.2. From equation (3.22), for $\mathbf{v} \in (C_0^\infty(W \backslash \bar{D}))^3$, we have

$$\tilde{F}(\mathbf{v}) \quad = \quad \int_{W \backslash \bar{D}} \left[(\nabla \times \mathbf{w}) \cdot (\nabla \times \bar{\mathbf{v}}) - k^2 \mathbf{w} \cdot \bar{\mathbf{v}} \right] \, dx$$

$$
\begin{aligned}
&= \int_{\Omega} \left[(\nabla\times\mathbf{w})\cdot(\nabla\times\bar{\mathbf{v}}) - k^2\mathbf{w}\cdot\bar{\mathbf{v}}\right]\,dx \\
&\quad + \int_{\Omega_L\cup\Omega_R} \left[(\nabla\times\mathbf{w})\cdot(\nabla\times\bar{\mathbf{v}}) - k^2\mathbf{w}\cdot\bar{\mathbf{v}}\right]\,dx.
\end{aligned}
$$

Denote $W_{(-\tilde{L},s)}$ a bounded segment of Ω_L where $\tilde{L} > s$, using integration by parts and the vector identity (B.2), we have

$$
\begin{aligned}
& \int_{W_{(-\tilde{L},s)}} \left[(\nabla\times\mathbf{w})\cdot(\nabla\times\bar{\mathbf{v}}) - k^2\mathbf{w}\cdot\bar{\mathbf{v}}\right]\,dx \\
=\ & \int_{W_{(-\tilde{L},s)}} (\nabla\times\nabla\times\mathbf{w})\cdot\bar{\mathbf{v}}\,dx \\
& + \int_{\partial W_{(-\tilde{L},s)}} (\nabla\times\mathbf{w})\cdot(\mathbf{n}\times\bar{\mathbf{v}})\,ds - \int_{W_{(-\tilde{L},s)}} k^2\mathbf{w}\cdot\bar{\mathbf{v}}\,dx \\
=\ & -\int_{\Sigma_s} \bar{\mathbf{v}}\cdot(\mathbf{n}_\Sigma\times\nabla\times\mathbf{w})\,ds + \int_{\Sigma_{-\tilde{L}}} (\nabla\times\mathbf{w})\cdot(\mathbf{n}_\Sigma\times\bar{\mathbf{v}})\,ds \\
=\ & -\int_{\Sigma_s} T_s(\mathbf{n}_\Sigma\times\mathbf{w})\cdot\bar{\mathbf{v}}\,ds + \int_{\Sigma_{-\tilde{L}}} (\nabla\times\mathbf{w})\cdot(\mathbf{n}_\Sigma\times\bar{\mathbf{v}})\,ds.
\end{aligned}
$$

By Cauchy-Schwarz inequality,

$$
\left|\int_{\Sigma_{-\tilde{L}}} (\nabla\times\mathbf{w})\cdot(\mathbf{n}_\Sigma\times\bar{\mathbf{v}})\,ds\right| \le \|(\nabla\times\mathbf{w})_T|_{\Sigma_{-\tilde{L}}}\|_{H^{-1/2}(\mathrm{div},\Sigma)}\|\mathbf{n}_\Sigma\times\bar{\mathbf{v}}|_{\Sigma_{-\tilde{L}}}\|_{H^{-1/2}(\mathrm{div},\Sigma)}.
$$

Since $\mathbf{w}$ satisfies radiation condition and thus can be expanded as series of outgoing and evanescent waves (bounded and convergent) for $|x| > L$, the series are uniformly convergent for $|x| \ge \tilde{L} > L$. Together with $\mathbf{v} \in (C_0^\infty(W\backslash\bar{D}))^3$, as $\tilde{L} \to \infty$, we have that

$$
\int_{\Omega_L} \left[(\nabla\times\mathbf{w})\cdot(\nabla\times\bar{\mathbf{v}}) - k^2\mathbf{w}\cdot\bar{\mathbf{v}}\right]\,dx = -\int_{\Sigma_s} T_s(\mathbf{n}_\Sigma\times\mathbf{w})\cdot\bar{\mathbf{v}}\,ds.
$$

Notice that the outward normal $\mathbf{n}_\Sigma = \hat{\mathbf{z}}$ is with respect to Ω_L. In view of the outward normal to Ω, it should be $\mathbf{n}_\Sigma = -\hat{\mathbf{z}}$ and we get

$$
\int_{\Omega_L} \left[(\nabla\times\mathbf{w})\cdot(\nabla\times\bar{\mathbf{v}}) - k^2\mathbf{w}\cdot\bar{\mathbf{v}}\right]\,dx = \int_{\Sigma_s} T_s(\mathbf{n}_\Sigma\times\mathbf{w})\cdot\bar{\mathbf{v}}\,ds.
$$

Similar result holds for Ω_R.

Hence, for any $\mathbf{v} \in (C_0^\infty(W\backslash\bar{D}))^3$, $\mathbf{w} \in H_{loc,0}(\text{curl}, W\backslash\bar{D})$ satisfies

$$\begin{aligned}\tilde{F}(\mathbf{v}) &= \int_\Omega \left[(\nabla\times\mathbf{w})\cdot(\nabla\times\bar{\mathbf{v}}) - k^2\mathbf{w}\cdot\bar{\mathbf{v}}\right]\,dx \\ &\quad + \sum_{j=s,t}\int_{\Sigma_j} T_j(\mathbf{n}_\Sigma\times\mathbf{w})\cdot\bar{\mathbf{v}}\,ds. \end{aligned} \tag{3.30}$$

On the other hand, since $\chi = 0$ in $\Omega_L \cup \Omega_R$, we have

$$\begin{aligned}\tilde{F}(\mathbf{v}) &= \int_{W\backslash\bar{D}} \left[(\nabla\times(\chi\mathbf{u}^i))\cdot(\nabla\times\bar{\mathbf{v}}) - k^2(\chi\mathbf{u}^i)\cdot\bar{\mathbf{v}}\right]\,dx \\ &= \int_\Omega \left[(\nabla\times(\chi\mathbf{u}^i))\cdot(\nabla\times\bar{\mathbf{v}}) - k^2(\chi\mathbf{u}^i)\cdot\bar{\mathbf{v}}\right]\,dx. \end{aligned} \tag{3.31}$$

Define $\mathbf{u} = \mathbf{w} + (1-\chi)\mathbf{u}^i$ in Ω, then $\mathbf{w} = \mathbf{u} - (1-\chi)\mathbf{u}^i = \mathbf{u} - \mathbf{u}^i + \chi\mathbf{u}^i$. Equating (3.30),(3.31) and substituting $\mathbf{w}$ gives

$$\begin{aligned} & \int_\Omega \left[(\nabla\times(\mathbf{u}-\mathbf{u}^i))\cdot(\nabla\times\bar{\mathbf{v}}) - k^2(\mathbf{u}-\mathbf{u}^i)\cdot\bar{\mathbf{v}}\right]\,dx \\ & + \sum_{j=s,t}\int_{\Sigma_j} T_j(\mathbf{n}_\Sigma\times(\mathbf{u}-\mathbf{u}^i))\cdot\bar{\mathbf{v}}\,ds \\ & + \int_\Omega \left[(\nabla\times(\chi\mathbf{u}^i))\cdot(\nabla\times\bar{\mathbf{v}}) - k^2(\chi\mathbf{u}^i)\cdot\bar{\mathbf{v}}\right]\,dx \\ & + \sum_{j=s,t}\int_{\Sigma_j} T_j(\mathbf{n}_\Sigma\times(\chi\mathbf{u}^i))\cdot\bar{\mathbf{v}}\,ds \\ = & \int_\Omega \left[(\nabla\times(\chi\mathbf{u}^i))\cdot(\nabla\times\bar{\mathbf{v}}) - k^2(\chi\mathbf{u}^i)\cdot\bar{\mathbf{v}}\right]\,dx. \end{aligned}$$

Since $\chi = 0$ on Σ_s and Σ_t, we get

$$\begin{aligned} & \int_\Omega \left[(\nabla\times(\mathbf{u}-\mathbf{u}^i))\cdot(\nabla\times\bar{\mathbf{v}}) - k^2(\mathbf{u}-\mathbf{u}^i)\cdot\bar{\mathbf{v}}\right]\,dx \\ & \qquad + \sum_{j=s,t}\int_{\Sigma_j} T_j(\mathbf{n}_\Sigma\times(\mathbf{u}-\mathbf{u}^i))\cdot\bar{\mathbf{v}}\,ds = 0. \end{aligned} \tag{3.32}$$

Thus, rearranging equation (3.32), using integration by parts and the vector identity (B.2) again, we have

$$\begin{aligned} & \int_\Omega \left[(\nabla\times\mathbf{u})\cdot(\nabla\times\bar{\mathbf{v}}) - k^2\mathbf{u}\cdot\bar{\mathbf{v}}\right]\,dx + \sum_{j=s,t}\int_{\Sigma_j} T_j(\mathbf{n}_\Sigma\times\mathbf{u})\cdot\bar{\mathbf{v}}\,ds \\ = & \sum_{j=s,t}\int_{\Sigma_j} T_j(\mathbf{n}_\Sigma\times\mathbf{u}^i)\cdot\bar{\mathbf{v}}\,ds + \int_\Omega \left[(\nabla\times\mathbf{u}^i)\cdot(\nabla\times\bar{\mathbf{v}}) - k^2\mathbf{u}^i\cdot\bar{\mathbf{v}}\right]\,dx \end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=s,t} \int_{\Sigma_j} T_j(\mathbf{n}_\Sigma \times \mathbf{u}^i) \cdot \bar{\mathbf{v}}\, ds \\
&\quad + \left[\int_\Omega (\nabla \times \nabla \times \mathbf{u}^i - k^2 \mathbf{u}^i) \cdot \bar{\mathbf{v}}\, dx + \int_{\partial\Omega} (\nabla \times \mathbf{u}^i) \cdot (\mathbf{n} \times \bar{\mathbf{v}})\, ds \right] \\
&= \sum_{j=s,t} \int_{\Sigma_j} \left[T_j(\mathbf{n}_\Sigma \times \mathbf{u}^i) - (\mathbf{n}_\Sigma \times (\nabla \times \mathbf{u}^i)) \right] \cdot \bar{\mathbf{v}}_T\, ds.
\end{aligned}
$$

Furthermore, note that

1. $\mathbf{u} \in X$ because
   - $\mathbf{u} \in H(\text{curl}, \Omega)$ since $\mathbf{w} \in H_{loc,0}(\text{curl}, W \backslash \bar{D})$ and $\mathbf{x}_0 \notin \Omega$.
   - $\mathbf{n} \times \mathbf{u} = 0$ on $\Gamma_{(s,t)} \cup \partial D$ since
     - $\chi = 1$ around ∂D, then $\mathbf{u} = \mathbf{w}$ and $\mathbf{n} \times \mathbf{u} = \mathbf{n} \times \mathbf{w} = 0$ on ∂D since $\mathbf{w} \in H_{loc,0}(\text{curl}, W \backslash \bar{D})$.
     - $\chi = 0$ around Γ, then $\mathbf{u} = \mathbf{w} + \mathbf{u}^i$ and $\mathbf{n} \times \mathbf{u} = \mathbf{n} \times \mathbf{w} + \mathbf{n} \times \mathbf{u}^i = 0$ on Γ since $\mathbf{w} \in H_{loc,0}(\text{curl}, W \backslash \bar{D})$ and boundary condition $\mathbf{n} \times \mathbf{u}^i = 0$ on Γ.
2. The space $\{\tilde{\mathbf{v}} \mid \tilde{\mathbf{v}} = \mathbf{v}|_\Omega \text{ for some } \mathbf{v} \in (C_0^\infty(W \backslash \bar{D}))^3\}$ is dense in X (see Theorem 4.1 in [48] for a similar proof).

Therefore, $\mathbf{u} = \mathbf{w} + (1 - \chi)\mathbf{u}^i$ is a solution to the (local) variational problem in Definition 3.1.3.

Now we prove that a solution to the (local) variational problem can be extended to a solution to the global scattering problem. Suppose $\mathbf{u} \in X$ is a solution of the variational problem in Definition 3.1.3, which is equivalent to (3.32). Letting $\mathbf{w} = \mathbf{u} - (1 - \chi)\mathbf{u}^i = \mathbf{u} - \mathbf{u}^i + \chi \mathbf{u}^i$ and reversing the steps starting from (3.32) we get the form in (3.30). That is, for any $\mathbf{v} \in X$, $\mathbf{w} \in X$ satisfies

$$
\int_\Omega \left[(\nabla \times \mathbf{w}) \cdot (\nabla \times \bar{\mathbf{v}}) - k^2 \mathbf{w} \cdot \bar{\mathbf{v}} \right] dx + \sum_{j=s,t} \int_{\Sigma_j} T_j(\mathbf{n}_\Sigma \times \mathbf{w}) \cdot \bar{\mathbf{v}} = \tilde{F}(\mathbf{v}),
$$

where

$$
\tilde{F}(\mathbf{v}) = \int_\Omega \left[(\nabla \times (\chi \mathbf{u}^i)) \cdot (\nabla \times \bar{\mathbf{v}}) - k^2 (\chi \mathbf{u}^i) \cdot \bar{\mathbf{v}} \right].
$$

Obviously this is also true for any $\mathbf{v} \in (C_0^\infty(W \backslash \bar{D}))^3$.

Extend $\mathbf{w}$ to Ω_R by solving

$$
\nabla \times \nabla \times \mathbf{w}_R - k^2 \mathbf{w}_R = 0 \quad \text{in} \quad \Omega_R,
$$

$$\begin{aligned} \mathbf{n}_\Sigma \times \mathbf{w}_R = \mathbf{n}_\Sigma \times \mathbf{w} \quad & \text{in} \quad \Sigma_t, \\ \mathbf{n}_\Sigma \times \mathbf{w}_R = 0 \quad & \text{on} \quad \Gamma_{(t,\infty)}, \\ \mathbf{w}_R \text{ satisfies radiation condition} \quad & \text{as} \quad z \to \infty, \end{aligned}$$

where $\mathbf{n}_\Sigma = \hat{\mathbf{z}}$ in view of the outward normal to Ω. By Lemma 3.1.3, this extension exists and is unique. A similar extension can be done for $\mathbf{w}_L$ in Ω_L.

Note that the extension

$$\tilde{\mathbf{w}} = \begin{cases} \mathbf{w}_L & \text{in} \quad \Omega_L, \\ \mathbf{w} & \text{in} \quad \Omega, \\ \mathbf{w}_R & \text{in} \quad \Omega_R. \end{cases}$$

belongs to space $H_{loc,0}(\text{curl}, W\backslash\bar{D})$ because

- $\mathbf{w} \in X$ in Ω.
- $\mathbf{w}_L \in H_{loc}(\text{curl}, \Omega_L)$ and $\mathbf{w}_R \in H_{loc}(\text{curl}, \Omega_R)$ as shown in Lemma 3.1.3.
- The tangential fields are continuous across Σ_s and Σ_t. Moreover, with definition of operator T, we know

  $$\mathbf{n}_\Sigma \times (\nabla \times \mathbf{w}) = T_t(\mathbf{n}_\Sigma \times \mathbf{w}) = \mathbf{n}_\Sigma \times (\nabla \times \mathbf{w}_R)$$

  (the same result holds for $\mathbf{w}_L$ on Σ_t) which makes the field $\tilde{\mathbf{w}}$ satisfy the Maxwell's equation and be in $H_{loc}(\text{curl}, W\backslash\bar{D})$ overall.

In addition, by definition of operator T_t on Σ_t such that

$$T_t(\mathbf{n}_\Sigma \times \mathbf{w}) = \mathbf{n}_\Sigma \times (\nabla \times \mathbf{w}_R) \text{ on } \Sigma_t$$

(same result holds for T_s with $\mathbf{w}_L$ on Σ_s) together with the compact support of test function $\mathbf{v} \in (C_0^\infty(W\backslash\bar{D}))^3$, we are able to reverse the derivation starting from (3.30) back to

$$\int_{W\backslash\bar{D}} \left[(\nabla \times \tilde{\mathbf{w}}) \cdot (\nabla \times \bar{\mathbf{v}}) - k^2 \tilde{\mathbf{w}} \cdot \bar{\mathbf{v}}\right] = \tilde{F}(\mathbf{v}),$$

where

$$\tilde{F}(\mathbf{v}) = \int_{W\backslash\bar{D}} \left[(\nabla \times (\chi \mathbf{u}^i)) \cdot (\nabla \times \bar{\mathbf{v}}) - k^2 (\chi \mathbf{u}^i) \cdot \bar{\mathbf{v}}\right],$$

which satisfies (3.22) in Definition 3.1.2.

Finally, from the construction of $\tilde{\mathbf{w}}$, we have shown that $\mathbf{w}$ can be extended uniquely to $W\backslash\Omega$ and satisfies radiation condition. So $\mathbf{u}^s = \tilde{\mathbf{w}} - \chi\mathbf{u}^i$ as an extension of $\mathbf{u} - \mathbf{u}^i$ also satisfies the radiation condition and this completes the proof. $\square$

3.2 Well-Posedness of the Forward Scattering Problem

In this section, we shall prove the well-posedness of the forward scattering problem on a bounded domain stated in Definition 3.1.3 by showing

1. The variational problem in Definition 3.1.3 may be reduced to an operator equation
$$(I - A)\mathbf{u} = f$$
on a suitable space X^+.

2. The operator A is compact and analytic for k in a suitable sub-domain of the complex plane.

3. For $k = ic$ with $c > 0$ small enough, the operator equation has at most one solution.

Then the well-posedness of the problem can be proved by using the Fredholm Alternative, except for an at most countable set of real wave numbers.

For simplicity, let $\Gamma = \Gamma_{(s,t)}$ and for $\mathbf{f}, \mathbf{g} \in X$, denote

$$\int_\Omega \mathbf{f} \cdot \bar{\mathbf{g}}\, dx = (f, g)_\Omega, \quad \int_{\Sigma_j} \mathbf{f} \cdot \bar{\mathbf{g}}\, ds = \langle \mathbf{f}, \mathbf{g} \rangle_{\Sigma_j} \text{ for } j = s, t.$$

Then equation (3.29) can be written as, $\mathbf{u} \in X$ satisfies

$$\begin{aligned}(\nabla \times \mathbf{u}, \nabla \times \mathbf{v})_\Omega - k^2(\mathbf{u}, \mathbf{v})_\Omega \quad &+ \quad \langle T_s(\mathbf{n}_\Sigma \times \mathbf{u}), \mathbf{v}_T \rangle_{\Sigma_s} \\ &+ \quad \langle T_t(\mathbf{n}_\Sigma \times \mathbf{u}), \mathbf{v}_T \rangle_{\Sigma_t} = \hat{F}(\mathbf{u}^i, \mathbf{v}), \qquad (3.33)\end{aligned}$$

for all $\mathbf{v} \in X$.

To make further progress, we need to use a Helmholtz decomposition for functions in X.

3.2.1 Helmholtz Decomposition of the Function Space X

The function space X for the weak solution given in (3.33) is too large for the direct analysis of (3.23) - (3.28). We need to factor out fields in the null space of the curl operator. To this end, we define the following potential space S:

$$S := \{p \in H^1(\Omega) \mid p = 0 \text{ on } \Gamma \cap \bar{\Omega}, p = \text{constant on } \partial D\}.$$

First, we have $\nabla S \subset X$ since $(\mathbf{n} \times \nabla p) \times \mathbf{n} = (\nabla p)_T = 0$ on each piece of boundary of Ω.

To understand the construction of the function space X^+, write $\mathbf{u} = \mathbf{u}^+ + \nabla p \in X$ for some $u^+ \in X$ and $p \in S$. Substituting into (3.33), we get

$$\begin{aligned}(\nabla \times (\mathbf{u}^+ + \nabla p), \nabla \times \mathbf{v})_\Omega \quad &- \quad k^2((\mathbf{u}^+ + \nabla p), \mathbf{v})_\Omega \\ &+ \quad \sum_{j=s,t} \langle T_j(\mathbf{n}_\Sigma \times (\mathbf{u}^+ + \nabla p)), \mathbf{v}_T \rangle_{\Sigma_j} = \hat{F}(\mathbf{u}^i, \mathbf{v}).\end{aligned}$$

Since $\nabla \times \nabla p = 0$, we have

$$\begin{aligned}(\nabla \times \mathbf{u}^+, \nabla \times \mathbf{v})_\Omega \quad &- \quad k^2((\mathbf{u}^+ + \nabla p), \mathbf{v})_\Omega \\ &+ \quad \sum_{j=s,t} \langle T_j(\mathbf{n}_\Sigma \times (\mathbf{u}^+ + \nabla p)), \mathbf{v}_T \rangle_{\Sigma_j} = \hat{F}(\mathbf{u}^i, \mathbf{v}).\end{aligned}$$

Now choose $\mathbf{v} = \nabla q \in \nabla S$ to obtain

$$-k^2((\mathbf{u}^+ + \nabla p), \nabla q)_\Omega + \sum_{j=s,t} \langle T_j(\mathbf{n}_\Sigma \times (\mathbf{u}^+ + \nabla p)), (\nabla q)_T \rangle_{\Sigma_j} = \hat{F}(\mathbf{u}^i, \nabla q).$$

After expansion, we get

$$\begin{aligned}-k^2(\mathbf{u}^+, \nabla q)_\Omega + \sum_{j=s,t} \langle T_j(\mathbf{n}_\Sigma \times \mathbf{u}^+), (\nabla q)_T \rangle_{\Sigma_j} & \\ -k^2(\nabla p, \nabla q)_\Omega + \sum_{j=s,t} \langle T_j(\mathbf{n}_\Sigma \times \nabla p)), (\nabla q)_T \rangle_{\Sigma_j} \quad &= \quad \hat{F}(\mathbf{u}^i, \nabla q).\end{aligned}$$

If we choose $\mathbf{u}^+$ such that $-k^2(\mathbf{u}^+, \nabla q)_\Omega + \sum_{j=s,t} \langle T_j(\mathbf{n}_\Sigma \times \mathbf{u}^+), (\nabla q)_T \rangle_{\Sigma_j} = 0$, then we must choose $p \in S$ such that

$$-\, k^2(\nabla p, \nabla q)_\Omega + \sum_{j=s,t} \langle T_j(\mathbf{n}_\Sigma \times \nabla p), (\nabla q)_T \rangle_{\Sigma_j} = \hat{F}(\mathbf{u}^i, \nabla q) \text{ for all } q \in S. \quad (3.34)$$

This motivates the following definition for $X^+ \subset X$:

$$X^+ := \left\{ \mathbf{w} \in X \;\middle|\; -k^2(\mathbf{w}, \nabla q)_\Omega + \sum_{j=s,t} \langle T_j(\mathbf{n}_\Sigma \times \mathbf{w}), (\nabla q)_T \rangle_{\Sigma_j} = 0 \text{ for all } q \in S \right\}.$$

Next, we shall prove that equation (3.34) has unique solution.

Lemma 3.2.1 *There exists a unique solution $p \in S$ such that*

$$-k^2(\nabla p, \nabla q)_\Omega + \sum_{j=s,t} \langle T_j(\mathbf{n}_\Sigma \times \nabla p), (\nabla q)_T \rangle_{\Sigma_j} = \hat{F}(\mathbf{u}^i, \nabla q) \textit{ for all } q \in \nabla S.$$

Proof: We shall prove the coercivity and boundedness of the following sesquilinear form

$$B(p, q) = k^2(\nabla p, \nabla q)_\Omega - \sum_{j=s,t} \langle T_j(\mathbf{n}_\Sigma \times \nabla p), (\nabla q)_T \rangle_{\Sigma_j}.$$

Then by the Lax-Milgram Lemma (Theorem C.0.4), the existence and uniqueness of $p \in S$ such that $B(p, q) = -\hat{F}(\mathbf{u}^i, \nabla q)$ for all $q \in S$ holds and the proof is done.

To analyze $\langle T_s(\mathbf{n}_\Sigma \times \nabla p), (\nabla p)_T \rangle_{\Sigma_s}$, using identity (B.11), we have $\mathbf{n}_\Sigma \times (\nabla p)|_\Sigma = -\vec{\nabla}_\Sigma \times p$. By Lemma 3.1.2, $-\vec{\nabla}_\Sigma \times p$ can be written as series representation using $\{\vec{\nabla}_\Sigma \times v_l\}_{l \geq 1}$, that is

$$-\vec{\nabla}_\Sigma \times p = \sum_l p_l^{(1)} \vec{\nabla}_\Sigma \times v_l.$$

Applying the operator T_s, we get, on Σ_s

$$\begin{aligned}
T_s(\mathbf{n}_\Sigma \times \nabla p) &= T_s(-\vec{\nabla}_\Sigma \times p) \\
&= \sum_m \left\langle \left(\sum_l p_l^{(1)} \vec{\nabla}_\Sigma \times v_l \right), \begin{pmatrix} \nabla_\Sigma u_m \\ 0 \end{pmatrix} \right\rangle_{\Sigma_s} \left(-\frac{ih_m}{\lambda_m^2} \right) \begin{pmatrix} \vec{\nabla}_\Sigma \times u_m \\ 0 \end{pmatrix} \\
&\quad + \sum_n \left\langle \left(\sum_l p_l^{(1)} \vec{\nabla}_\Sigma \times v_l \right), \begin{pmatrix} \vec{\nabla}_\Sigma \times v_n \\ 0 \end{pmatrix} \right\rangle_{\Sigma_s} \left(-\frac{1}{\mu_n^2} \frac{k^2}{ig_n} \right) \begin{pmatrix} \nabla_\Sigma v_n \\ 0 \end{pmatrix} \\
&= 0 + \sum_n \left(p_n^{(1)} \mu_n^2 \right) \left(-\frac{1}{\mu_n^2} \frac{k^2}{ig_n} \right) \begin{pmatrix} \nabla_\Sigma v_n \\ 0 \end{pmatrix} \\
&= \sum_n \left(-\frac{p_n^{(1)} k^2}{ig_n} \right) \begin{pmatrix} \nabla_\Sigma v_n \\ 0 \end{pmatrix}.
\end{aligned}$$

On the other hand, using identity (B.10), $(\nabla p)_T = \nabla_\Sigma p$. By Lemma 3.1.2 again, it can be written as series representation using $\{\nabla_\Sigma u_m\}_{m\geq 1}$, that is

$$(\nabla p)_T = \nabla_\Sigma p = \sum_m p_m^{(2)} \nabla_\Sigma u_m.$$

Since $p = 0$ on $\partial\Sigma$, by Lemma 3.1.1, we have

$$\begin{aligned}
&\langle T_s(\mathbf{n}_\Sigma \times \nabla p), (\nabla p)_T\rangle_{\Sigma_s} \\
= &\left\langle \sum_n \left(-\frac{p_n^{(1)}k^2}{ig_n}\right)\begin{pmatrix}\nabla_\Sigma v_n\\ 0\end{pmatrix}, \sum_m p_m^{(2)}\begin{pmatrix}\nabla_\Sigma u_m\\ 0\end{pmatrix}\right\rangle_{\Sigma_s} \\
= &\left\langle \sum_n -\left\langle -\vec{\nabla}_\Sigma \times p, \begin{pmatrix}\vec{\nabla}_\Sigma \times v_n\\ 0\end{pmatrix}\right\rangle_{\Sigma_s} \frac{1}{\mu_n^2}\frac{k^2}{ig_n}\begin{pmatrix}\nabla_\Sigma v_n\\ 0\end{pmatrix}, \right. \\
&\left. \sum_m \left\langle \nabla_\Sigma p, \begin{pmatrix}\nabla_\Sigma u_m\\ 0\end{pmatrix}\right\rangle_{\Sigma_s} \frac{1}{\lambda_m^2}\begin{pmatrix}\nabla_\Sigma u_m\\ 0\end{pmatrix}\right\rangle_{\Sigma_s} \\
= &\left\langle \sum_n -\langle -p, \mu_n^2 v_n\rangle_{\Sigma_s}\frac{1}{\mu_n^2}\frac{k^2}{ig_n}\begin{pmatrix}\nabla_\Sigma v_n\\ 0\end{pmatrix}, \sum_m \langle p, \lambda_m^2 u_m\rangle_{\Sigma_s}\frac{1}{\lambda_m^2}\begin{pmatrix}\nabla_\Sigma u_m\\ 0\end{pmatrix}\right\rangle_{\Sigma_s} \\
= &\left\langle \sum_n -\langle -p, v_n\rangle_{\Sigma_s}\frac{k^2}{ig_n}v_n, \sum_m \langle p, u_m\rangle_{\Sigma_s}\lambda_m^2 u_m\right\rangle_{\Sigma_s} \\
= &-\left\langle \sum_n \langle p, v_n\rangle_{\Sigma_s} v_n \frac{ik^2}{g_n}, \sum_m \langle p, u_m\rangle_{\Sigma_s} u_m\lambda_m^2\right\rangle_{\Sigma_s}.
\end{aligned}$$

The same equality holds for T_t on Σ_t. Therefore,

$$\begin{aligned}
B(p,p) &= k^2(\nabla p, \nabla p)_\Omega - \sum_{j=s,t}\langle T_j(\mathbf{n}_\Sigma \times \nabla p), (\nabla p)_T\rangle_{\Sigma_j} \\
&= k^2(\nabla p, \nabla p)_\Omega + \sum_{j=s,t}\left\langle \sum_n \langle p, v_n\rangle_{\Sigma_j} v_n\frac{ik^2}{g_n}, \sum_m \langle p, u_m\rangle_{\Sigma_j} u_m\lambda_m^2\right\rangle_{\Sigma_j}.
\end{aligned}$$

Note that $p|_\Sigma \in L^2(\Sigma)$ and thus by Theorem 3.1.1 and Theorem 3.1.2, for $j = s, t$, it has expansion

$$p|_\Sigma = \sum_n \langle p, v_n\rangle_\Sigma v_n \quad \text{or} \quad p|_\Sigma = \sum_m \langle p, u_m\rangle_\Sigma u_m.$$

Also there there exists a finite number n^* such that g_n is real for $n \leq n^*$ and imaginary for $n > n^*$. Thus, using the Poincaré inequality (notice $p = 0$ on $\Gamma \cap \bar{\Omega}$), we have

$$\mathfrak{Re}(B(p,p))$$
$$\begin{cases} = k^2(\nabla p, \nabla p)_\Omega \geq C\|p\|^2_{H^1(\Omega)} & \text{when} \quad n \leq n^*, \\ \geq k^2(\nabla p, \nabla p)_\Omega + \sum_{j=s,t} \|p\|^2_{L^2(\Sigma_j)} k^2 \lambda_1^2 \inf \left\{ \dfrac{1}{|g_n|} \right\}_{n > n*} \geq C\|p\|^2_{H^1(\Omega)} & \text{when} \quad n > n^*, \end{cases}$$

for some constant $C > 0$. So the sesquilinear form $B(p,q)$ is coercive.

For the boundedness, we have

$$\begin{aligned} |B(p,q)| &\leq |k^2(\nabla p, \nabla q)_\Omega| + \left| \sum_{j=s,t} \langle T_j(\mathbf{n}_\Sigma \times \nabla p), (\nabla q)_T \rangle_{\Sigma_j} \right| \\ &\leq k^2 \|\nabla p\|_{L^2} \|\nabla q\|_{L^2} + \sum_{j=s,t} \|T_j(\mathbf{n}_\Sigma \times \nabla p)\|_{H^{-1/2}(\mathrm{div},\Sigma)} \|(\nabla q)_T\|_{H^{-1/2}(\mathrm{curl},\Sigma)} \\ &\leq k^2 \|\nabla p\|_{H(\mathrm{curl})} \|\nabla q\|_{H(\mathrm{curl})} + \sum_{j=s,t} \|\mathbf{n}_\Sigma \times \nabla p|_{\Sigma_j}\|_{H^{-1/2}(\mathrm{div},\Sigma)} \|(\nabla q)_T\|_{H^{-1/2}(\mathrm{curl},\Sigma)} \\ &\leq k^2 \|\nabla p\|_{H(\mathrm{curl})} \|\nabla q\|_{H(\mathrm{curl})} + C\|\nabla p\|_{H(\mathrm{curl})} \|\nabla q\|_{H(\mathrm{curl})} \\ &\leq C\|\nabla p\|_{H(\mathrm{curl})} \|\nabla q\|_{H(\mathrm{curl})}, \end{aligned}$$

for some constant $C > 0$.

Thus, by the Lax-Milgram Lemma (Theorem C.0.4), there exists a unique solution $p \in S$ to $B(p,q) = -\hat{F}(\mathbf{u}^i, \nabla q)$ and this completes the proof. □

It is worth mentioning that this lemma asserts a unique solution $p \in S$ to $B(p,q) = l(\mathbf{v})$ where l is any continuous linear functional of $\mathbf{v} = \nabla q \in \nabla S$.

Now, similar to Lemma 10.3 in [48], we have the following Helmholtz decomposition:

Lemma 3.2.2 *(Helmholtz Decomposition) The space ∇S is a closed subspace of X, and we may write the direct sum*

$$X = \nabla S \oplus X^+.$$

Proof: The space ∇S in closed in X since S is closed in $H^1(\Omega)$. To show the subspace X^+ is closed, we have for fixed $q \in S$, the linear functionals $\mathbf{u} \mapsto (\mathbf{u}, \nabla q)_\Omega$ and $\mathbf{u} \mapsto$

$\langle T_j(\mathbf{n}_\Sigma \times \mathbf{u}), (\nabla q)_T\rangle_{\Sigma_j} (j = s, t)$ are bounded on $H(\text{curl}, \Omega)$. Since for $\mathbf{u} \in H(\text{curl}, \Omega)$, by Cauchy-Schwarz inequality, we have

$$|(\mathbf{u}, \nabla q)_\Omega| \quad \leq \quad \|\mathbf{u}\|_{H(\text{curl})} \|\nabla q\|_{H(\text{curl})} = \|\mathbf{u}\|_{H(\text{curl})} \|\nabla q\|_{L^2}, \tag{3.35}$$

and for $j = s, t$,

$$\begin{aligned} |\langle T_j(\mathbf{n}_\Sigma \times \mathbf{u}), (\nabla q)_T\rangle_{\Sigma_j}| &\leq \|T_j(\mathbf{n}_\Sigma \times \mathbf{u})\|_{H^{-1/2}(\text{div},\Sigma)} \|(\mathbf{n}_\Sigma \times \nabla q) \times \mathbf{n}_\Sigma\|_{H^{-1/2}(\text{curl},\Sigma)} \\ &\leq C\|\mathbf{n}_\Sigma \times \mathbf{u}\|_{H^{-1/2}(\text{div},\Sigma)} \|(\mathbf{n}_\Sigma \times \nabla q) \times \mathbf{n}_\Sigma\|_{H^{-1/2}(\text{curl},\Sigma)} \\ &\leq C\|\mathbf{u}\|_{H(\text{curl})} \|\nabla q\|_{H(\text{curl})} = C\|\mathbf{u}\|_{H(\text{curl})} \|\nabla q\|_{L^2}, \end{aligned} \tag{3.36}$$

for some constant $C > 0$. Here we have used the boundedness of operator T (Lemma 3.1.4), the boundedness of the trace operator from $H(\text{curl}, \Omega)$ to $\widetilde{H}^{-1/2}(\text{div}, \Sigma)$ and the boundedness of the trace operator from $H(\text{curl}, \Omega)$ to $\widetilde{H}^{-1/2}(\text{curl}, \Sigma)$.

Consider a Cauchy sequence $\{\mathbf{u}_n\} \in X^+$ that converges strongly to some function $\mathbf{u} \in X$, by definition of X^+, we see that

$$\begin{aligned} 0 &= -k^2(\mathbf{u}_n, \nabla q)_\Omega + \sum_{j=s,t} \langle T_j(\mathbf{n}_\Sigma \times \mathbf{u}_n), (\nabla q)_T\rangle_{\Sigma_j} \\ &= \left(-k^2(\mathbf{u}_n - \mathbf{u}, \nabla q)_\Omega + \sum_{j=s,t} \langle T_j(\mathbf{n}_\Sigma \times (\mathbf{u}_n - \mathbf{u})), (\nabla q)_T\rangle_{\Sigma_j}\right) \\ &\quad + \left(-k^2(\mathbf{u}, \nabla q)_\Omega + \sum_{j=s,t} \langle T_j(\mathbf{n}_\Sigma \times \mathbf{u}), (\nabla q)_T\rangle_{\Sigma_j}\right). \end{aligned}$$

Letting $n \to \infty$ and using the fact that $\mathbf{u}_n \to \mathbf{u}$ in X together with inequalities (3.35), (3.36), we have

$$-k^2(\mathbf{u}, \nabla q)_\Omega + \sum_{j=s,t} \langle T_j(\mathbf{n}_\Sigma \times \mathbf{u}), (\nabla q)_T\rangle_{\Sigma_j} = 0.$$

Thus, $\mathbf{u} \in X^+$ as well and the space X^+ is closed.

To show that $X = \nabla S \oplus X^+$, consider $\mathbf{u} \in X$. By Lemma 3.2.1, there exists a unique $p \in S$ such that

$$-k^2(\nabla p, \mathbf{v})_\Omega \quad + \quad \sum_{j=s,t} \langle T_j(\mathbf{n}_\Sigma \times \nabla p), \mathbf{v}_T\rangle_{\Sigma_j}$$

$$= \quad (\nabla \times \mathbf{u}, \nabla \times \mathbf{v})_\Omega - k^2(\mathbf{u}, \mathbf{v})_\Omega + \sum_{j=s,t} \langle T_j(\mathbf{n}_\Sigma \times \mathbf{u}), \mathbf{v}_T \rangle_{\Sigma_j},$$

for all $\mathbf{v} = \nabla q \in \nabla S$.

Define $\mathbf{u}^+ = \mathbf{u} - \nabla p$, then $\mathbf{u}^+ \in X^+$ is seen directly from the variational equation above.

Finally, we have to show that $\nabla S \cap X^+ = \mathbf{0}$. Suppose $\mathbf{u} = \nabla p \in \nabla S \cap X^+$, then for all $q \in S$, consider the variational form

$$\begin{aligned} 0 &= (\nabla \times \mathbf{u}, \nabla \times \nabla q)_\Omega - k^2(\mathbf{u}, \nabla q)_\Omega + \sum_{j=s,t} \langle T_j(\mathbf{n}_\Sigma \times \mathbf{u}), (\nabla q)_T \rangle_{\Sigma_j} \\ &= -k^2(\nabla p, \nabla q)_\Omega + \sum_{j=s,t} \langle T_j(\mathbf{n}_\Sigma \times \nabla p), (\nabla q)_T \rangle_{\Sigma_j}. \end{aligned}$$

Again, by Lemma 3.2.1, there exists a unique $p \in S$ that solves the above equation. Since $p = 0$ is obviously a solution, it is the only one. This completes the proof. □

3.2.2 Variational Analysis of the Forward Scattering Problem

Given the Helmholtz decomposition (Lemma 3.2.2), every solution $\mathbf{u} \in X$ of equation (3.33) can be written as $\mathbf{u} = \nabla p + \mathbf{u}^+$ where $\nabla p \in \nabla S$ and $\mathbf{u}^+ \in X^+$. Plugging the expansion for $\mathbf{u}$ in (3.33), we have

$$(\nabla \times \mathbf{u}^+, \nabla \times \mathbf{v})_\Omega - k^2((\mathbf{u}^+ + \nabla p), \mathbf{v})_\Omega + \sum_{j=s,t} \langle T_j(\mathbf{n}_\Sigma \times (\mathbf{u}^+ + \nabla p)), \mathbf{v}_T \rangle_{\Sigma_j} = \hat{F}(\mathbf{u}^i, \mathbf{v}).$$

Therefore, we get, for every $\mathbf{v} \in X$,

$$\begin{aligned} &(\nabla \times \mathbf{u}^+, \nabla \times \mathbf{v})_\Omega - k^2(\mathbf{u}^+, \mathbf{v})_\Omega + \sum_{j=s,t} \langle T_j(\mathbf{n}_\Sigma \times \mathbf{u}^+), \mathbf{v}_T \rangle_{\Sigma_j} \\ = \quad &\hat{F}(\mathbf{u}^i, \mathbf{v}) + k^2(\nabla p, \mathbf{v})_\Omega - \sum_{j=s,t} \langle T_j(\mathbf{n}_\Sigma \times \nabla p), \mathbf{v}_T \rangle_{\Sigma_j}. \end{aligned}$$

Note that p can be determined by solving the variational equation in Lemma 3.2.1. Thus we have derived a variational equation for $\mathbf{u}^+ \in X^+$. That is, the function $\mathbf{u}^+ \in X^+$ is such that for all $\mathbf{v} \in X^+$, it satisfies the variational problem

$$(\nabla \times \mathbf{u}^+, \nabla \times \mathbf{v})_\Omega - k^2(\mathbf{u}^+, \mathbf{v})_\Omega + \sum_{j=s,t} \langle T_j(\mathbf{n}_\Sigma \times \mathbf{u}^+), \mathbf{v}_T \rangle_{\Sigma_j} = G(\mathbf{u}^i, \mathbf{v}), \qquad (3.37)$$

where

$$G(\mathbf{u}^i, \mathbf{v}) = \hat{F}(\mathbf{u}^i, \mathbf{v}) + k^2(\nabla p, \mathbf{v})_\Omega - \sum_{j=s,t} \langle T_j(\mathbf{n}_\Sigma \times \nabla p), \mathbf{v}_T \rangle_{\Sigma_j}.$$

To continue, we shall now prove some useful lemmas:

Lemma 3.2.3 *The function space X^+ is compactly embedded in $(L^2(\Omega))^3$.*

Proof: Consider the definition of a compact embedding, we shall show that for any bounded sequence $\mathbf{u}_n \in X^+$, there exists a subsequence again denoted by $\mathbf{u}_n$ and $\mathbf{u}_0 \in X^+$ such that $\mathbf{u}_n$ strongly converges to $\mathbf{u}_0$ in $(L^2(\Omega))^3$.

Since $\mathbf{u}_n \in X^+$, we have, for all $q \in S$

$$-k^2(\mathbf{u}_n, \nabla q)_\Omega + \sum_{j=s,t} \langle T_j(\mathbf{n}_\Sigma \times \mathbf{u}_n), (\nabla q)_T \rangle_{\Sigma_j} = 0. \tag{3.38}$$

By choosing $q \in H_0^1(\Omega) \subset S$, then $(\nabla q)_T = \nabla_\Sigma q = 0$. Applying integration by parts, we get

$$-k^2(\mathbf{u}_n, \nabla q)_\Omega = k^2(\nabla \cdot \mathbf{u}_n, q)_\Omega = 0.$$

So $\nabla \cdot \mathbf{u}_n = 0$.

Now we extend $\mathbf{u}_n$ to Ω_R by solving

$$\begin{aligned}
\nabla \times \nabla \times \mathbf{u}_n^R - k^2 \mathbf{u}_n^R = 0 &\quad \text{in} \quad \Omega_R, \\
\mathbf{n}_\Sigma \times \mathbf{u}_n^R = \mathbf{n}_\Sigma \times \mathbf{u}_n &\quad \text{on} \quad \Sigma_t, \\
\mathbf{n}_\Gamma \times \mathbf{u}_n^R = 0 &\quad \text{on} \quad \Gamma, \\
\mathbf{u}_n^R \text{ satisfies radiation condition} &\quad \text{as} \quad z \to \infty,
\end{aligned}$$

where $\mathbf{n}_\Sigma = \hat{\mathbf{z}}$ in view of the outward normal to Ω. Of course $\nabla \cdot \mathbf{u}_n^R = 0$ by taking the divergence of the Maxwell's equation above.

In a similar fashion, we can do the same thing in Ω_R to obtain $\mathbf{u}_n^R$ and $\nabla \cdot \mathbf{u}_n^R = 0$ with matching tangential field on the interface Σ_t.

Define $\tilde{\mathbf{u}}_n$ as

$$\tilde{\mathbf{u}}_n = \begin{cases} \mathbf{u}_n^L & \text{in} \quad \Omega_L, \\ \mathbf{u}_n & \text{in} \quad \Omega, \\ \mathbf{u}_n^R & \text{in} \quad \Omega_R. \end{cases}$$

Similar to the reasoning in the proof of Theorem 3.1.3, we have $\tilde{\mathbf{u}}_n \in H_{loc}(\text{curl}, W\backslash\bar{D})$.

Next, we show that the normal component on the cross sections Σ_s and Σ_t also matches. By applying integration by parts to (3.38), we get

$$-k^2\left(-(\nabla\cdot\mathbf{u}_n, q)_\Omega + \sum_{j=s,t}\langle\mathbf{n}_\Sigma\cdot\mathbf{u}_n, q\rangle_{\Sigma_j}\right) + \sum_{j=s,t}\langle T_j(\mathbf{n}_\Sigma\times\mathbf{u}_n), \nabla_\Sigma q\rangle_{\Sigma_j} = 0.$$

Using Stokes identity (B.16), note that $q=0$ on $\partial\Sigma$, we have

$$-k^2\sum_{j=s,t}\langle\mathbf{n}_\Sigma\cdot\mathbf{u}_n, q\rangle_{\Sigma_j} - \sum_{j=s,t}\langle\nabla_\Sigma\cdot T(\mathbf{n}_\Sigma\times\mathbf{u}_n), q\rangle_{\Sigma_j} = 0.$$

This implies, on Σ_t

$$-k^2\mathbf{n}_\Sigma\cdot\mathbf{u}_n = \nabla_\Sigma\cdot T_t(\mathbf{n}_\Sigma\times\mathbf{u}_n).$$

Using definition of T obtained from (3.14), we get

$$-k^2\mathbf{n}_\Sigma\cdot\mathbf{u}_n = \nabla_\Sigma\cdot T_t(\mathbf{n}_\Sigma\times\mathbf{u}_n^R) = \nabla_\Sigma\cdot\left(\mathbf{n}_\Sigma\times(\nabla\times\mathbf{u}_n^R)\right).$$

Using series representation given by (3.17) on Σ_t, we have

$$\mathbf{n}_\Sigma\times(\nabla\times\mathbf{u}_n^R) = \sum_m -a_m\frac{ih_m}{\lambda_m^2}\begin{pmatrix}\vec{\nabla}_\Sigma\times u_m\\ 0\end{pmatrix} + \sum_n -b_n\frac{1}{\mu_n^2}\frac{k^2}{ig_n}\begin{pmatrix}\nabla_\Sigma v_n\\ 0\end{pmatrix}.$$

By Stokes identity (B.18), we get

$$\nabla_\Sigma\cdot\left(\mathbf{n}_\Sigma\times(\nabla\times\mathbf{u}_n^R)\right) = 0 + \sum_n -b_n\frac{1}{\mu_n^2}\frac{k^2}{ig_n}(-\mu^2 v_n) = \sum_n b_n\frac{k^2}{ig_n}v_n.$$

Meanwhile, using series representation given by (3.15) on Σ_t, we have

$$\begin{aligned}-k^2\mathbf{n}_\Sigma\cdot\mathbf{u}_n^R \quad &= \quad -k^2\mathbf{n}_\Sigma\cdot\left\{\sum_m a_m\frac{1}{\lambda_m^2}\begin{pmatrix}\vec{\nabla}_\Sigma\times u_m\\ 0\end{pmatrix}\right.\\ &\qquad\left. + \sum_n -b_n\left[\frac{1}{\mu_n^2}\begin{pmatrix}\nabla_\Sigma v_n\\ 0\end{pmatrix} + \frac{1}{ig_n}\begin{pmatrix}0\\0\\v_n\end{pmatrix}\right]\right\}\end{aligned}$$

$$
\begin{aligned}
&= -k^2 \sum_n -b_n \frac{1}{ig_n} \left[\mathbf{n}_\Sigma \cdot \begin{pmatrix} 0 \\ 0 \\ v_n \end{pmatrix} \right] \\
&= \sum_n \beta_n \frac{k^2}{ig_n} v_n.
\end{aligned}
$$

Thus, we obtain

$$
-k^2 \mathbf{n}_\Sigma \cdot \mathbf{u}_n = \nabla_\Sigma \cdot \left(\mathbf{n}_\Sigma \times (\nabla \times \mathbf{u}_n^R) \right) = -k^2 \mathbf{n}_\Sigma \cdot \mathbf{u}_n^R,
$$

that is,

$$
\mathbf{n}_\Sigma \cdot \mathbf{u}_n = \mathbf{n}_\Sigma \cdot \mathbf{u}_n^R.
$$

In a similar fashion, we can also conclude $\mathbf{n}_\Sigma \cdot \mathbf{u}_n = \mathbf{n}_\Sigma \cdot \mathbf{u}_n^L$ on Σ_t.

This means that the normal component also matches on the cross sections Σ_s, Σ_t and consequently $\tilde{\mathbf{u}}_n \in H_{loc}(\mathrm{div}, W \backslash \bar{D})$ and $\nabla \cdot \tilde{\mathbf{u}}_n = 0$. Hence, we can conclude that

$$
\begin{aligned}
\tilde{\mathbf{u}}_n \in Y_{W\backslash \bar{D}} \quad := \quad & \{ \mathbf{v} \in H_{loc}(\mathrm{curl}, W\backslash \bar{D}) \cap H_{loc}(\mathrm{div}, W \backslash \bar{D}) \mid \\
& \mathbf{n}_\Gamma \times \mathbf{v} = 0 \text{ on } \Gamma, \mathbf{n}_D \times \mathbf{v} = 0 \text{ on } \partial D, \nabla \cdot \mathbf{v} = 0 \text{ in } W \backslash \bar{D} \}.
\end{aligned}
$$

Now, choose a smooth cut-off scalar function χ such that

$$
\chi = \begin{cases} 1 & \text{in} \quad \Omega, \\ 0 & \text{in} \quad W \backslash \bar{D} \text{ when } |z| > l > \max\{|s|, |t|\}. \end{cases}
$$

Consider the sequence $\{\chi \tilde{\mathbf{u}}_n\} \subset X_{W_{(-l,l)} \backslash \bar{D}}$. We have the following facts:

$$
\begin{aligned}
\| \chi \tilde{\mathbf{u}}_n \|_{L^2(W_{(-l,l)} \backslash \bar{D})} &< \infty, \\
\| \nabla \times (\chi \tilde{\mathbf{u}}_n) \|_{L^2(W_{(-l,l)} \backslash \bar{D})} &< \infty, \\
\| \nabla \cdot (\chi \tilde{\mathbf{u}}_n) \|_{L^2(W_{(-l,l)} \backslash \bar{D})} &< \infty, \\
\mathbf{n} \times (\chi \tilde{\mathbf{u}}_n) = 0 \quad & \text{on} \quad \partial W_{(-l,l)} \backslash \bar{D}.
\end{aligned}
$$

Then by the standard compactness result of $H_0(\mathrm{curl}) \cap H(\mathrm{div}^0)$ in $(L^2)^3$ (see Corollary 3.49 in [48]), there exists a subsequence again denoted by $\{\chi \tilde{\mathbf{u}}_n\}$ that strongly convergence to a function $\tilde{\mathbf{u}} \in (L^2(W_{(-l,l)} \backslash \bar{D}))^3$. With restriction to Ω, we have $(\chi \tilde{\mathbf{u}}_n)|_\Omega = \tilde{\mathbf{u}}_n$

converges strongly to $\tilde{\mathbf{u}}|_\Omega$ in $(L^2(\Omega))^3$ and the proof of compactly embedding of X^+ to $(L^2(\Omega))^3$ is done. $\square$

The next lemma gives a decomposition of the operator T:

Lemma 3.2.4 *Operators T_j for $j = s, t$ in (3.37) can be written as $T_j = T_j^0 + T_j^c$ where T_j^0 is positive and T_j^c is a compact perturbation.*

Proof: Consider the operator T_t on Σ_t. We separate T_t into two parts, using series expansion of T from (3.18),

$$\begin{aligned} T_t(\mathbf{n}_\Sigma \times \mathbf{u}^+|_{\Sigma_t}) &= T_{1t}(\mathbf{n}_\Sigma \times \mathbf{u}^+|_{\Sigma_t}) + T_{2t}(\mathbf{n}_\Sigma \times \mathbf{u}^+|_{\Sigma_t}) \\ &= \sum_m \left\langle (\mathbf{n}_\Sigma \times \mathbf{u}^+|_{\Sigma_t}), \begin{pmatrix} \nabla_\Sigma u_m \\ 0 \end{pmatrix} \right\rangle_{\Sigma_t} \left(-\frac{ih_m}{\lambda_m^2}\right) \begin{pmatrix} \vec{\nabla}_\Sigma \times u_m \\ 0 \end{pmatrix} \\ &\quad + \sum_n \left\langle (\mathbf{n}_\Sigma \times \mathbf{u}^+|_{\Sigma_t}), \begin{pmatrix} \vec{\nabla}_\Sigma \times v_n \\ 0 \end{pmatrix} \right\rangle_{\Sigma_t} \left(-\frac{1}{\mu_n^2}\frac{k^2}{ig_n}\right) \begin{pmatrix} \nabla_\Sigma v_n \\ 0 \end{pmatrix}. \end{aligned}$$

We analyze each part individually. For operator T_{1t}, first notice that there exists a finite number m^* such that h_m is real for $m \le m^*$ and imaginary for $m > m^*$. Let

$$\tilde{h}_m = \begin{cases} 0 & \text{if} \quad m \le m^*, \\ |h_m| & \text{if} \quad m > m^*. \end{cases}$$

Define operator $\tilde{T}_{1t}$ as follows:

$$\tilde{T}_{1t}(\mathbf{n}_\Sigma \times \mathbf{u}^+|_{\Sigma_t}) = \sum_m \left\langle (\mathbf{n}_\Sigma \times \mathbf{u}^+|_{\Sigma_t}), \begin{pmatrix} \nabla_\Sigma u_m \\ 0 \end{pmatrix} \right\rangle_{\Sigma_t} \left(\frac{\tilde{h}_m}{\lambda_m^2}\right) \begin{pmatrix} \vec{\nabla}_\Sigma \times u_m \\ 0 \end{pmatrix}.$$

Then, we have

$$(T_{1t} - \tilde{T}_{1t})(\mathbf{n}_\Sigma \times \mathbf{u}^+|_{\Sigma_t}) = \sum_{m=0}^{m^*} \left\langle (\mathbf{n}_\Sigma \times \mathbf{u}^+|_{\Sigma_t}), \begin{pmatrix} \nabla_\Sigma u_m \\ 0 \end{pmatrix} \right\rangle_{\Sigma_t} \left(-\frac{ih_m}{\lambda_m^2}\right) \begin{pmatrix} \vec{\nabla}_\Sigma \times u_m \\ 0 \end{pmatrix}.$$

Since $T_t : \widetilde{H}^{-1/2}(\mathrm{div}, \Sigma_t) \mapsto \widetilde{H}^{-1/2}(\mathrm{div}, \Sigma_t)$ is bounded (Lemma 3.1.4), so is $T_{1t} - \tilde{T}_{1t}$. Also, $T_{1t} - \tilde{T}_{1t}$ is linear and of finite rank, so $T_{1t} - \tilde{T}_{1t}$ is a compact operator (see Theorem 8.1-4 in [44]).

Next we show that operator $\tilde{T}_{1t}$ is positive. Specifically, using identities (B.2) and (B.12), we have

$$
\begin{aligned}
&\left\langle \tilde{T}_{1t}(\mathbf{n}_\Sigma \times \mathbf{u}^+|_{\Sigma_t}), (\mathbf{u}^+)_T \right\rangle_{\Sigma_t} \\
= &\left\langle \sum_m \left\langle (\mathbf{n}_\Sigma \times \mathbf{u}^+|_{\Sigma_t}), \begin{pmatrix} \nabla_\Sigma u_m \\ 0 \end{pmatrix} \right\rangle_{\Sigma_t} \left(\frac{\tilde{h}_m}{\lambda_m^2}\right) \begin{pmatrix} \vec{\nabla}_\Sigma \times u_m \\ 0 \end{pmatrix}, (\mathbf{u}^+)_T \right\rangle_{\Sigma_t} \\
= &\sum_m \left\langle (\mathbf{n}_\Sigma \times \mathbf{u}^+|_{\Sigma_t}), \begin{pmatrix} \nabla_\Sigma u_m \\ 0 \end{pmatrix} \right\rangle_{\Sigma_t} \left(\frac{\tilde{h}_m}{\lambda_m^2}\right) \left\langle \begin{pmatrix} \vec{\nabla}_\Sigma \times u_m \\ 0 \end{pmatrix}, (\mathbf{n}_\Sigma \times \mathbf{u}^+|_{\Sigma_t}) \times \mathbf{n}_\Sigma \right\rangle_{\Sigma_t} \\
= &\sum_m \left\langle (\mathbf{n}_\Sigma \times \mathbf{u}^+|_{\Sigma_t}), \begin{pmatrix} \nabla_\Sigma u_m \\ 0 \end{pmatrix} \right\rangle_{\Sigma_t} \left(\frac{\tilde{h}_m}{\lambda_m^2}\right) \left\langle \mathbf{n}_\Sigma \times \begin{pmatrix} \vec{\nabla}_\Sigma \times u_m \\ 0 \end{pmatrix}, \mathbf{n}_\Sigma \times \mathbf{u}^+|_{\Sigma_t} \right\rangle_{\Sigma_t} \\
= &\sum_m \left\langle (\mathbf{n}_\Sigma \times \mathbf{u}^+|_{\Sigma_t}), \begin{pmatrix} \nabla_\Sigma u_m \\ 0 \end{pmatrix} \right\rangle_{\Sigma_t} \left(\frac{\tilde{h}_m}{\lambda_m^2}\right) \left\langle \begin{pmatrix} \nabla_\Sigma u_m \\ 0 \end{pmatrix}, \mathbf{n}_\Sigma \times \mathbf{u}^+|_{\Sigma_t} \right\rangle_{\Sigma_t} \\
= &\sum_m \left| \left\langle (\mathbf{n}_\Sigma \times \mathbf{u}^+|_{\Sigma_t}), \begin{pmatrix} \nabla_\Sigma u_m \\ 0 \end{pmatrix} \right\rangle_{\Sigma_t} \right|^2 \frac{\tilde{h}_m}{\lambda_m^2} \\
= &\sum_{m>m^*} \left| \left\langle (\mathbf{n}_\Sigma \times \mathbf{u}^+|_{\Sigma_t}), \begin{pmatrix} \nabla_\Sigma u_m \\ 0 \end{pmatrix} \right\rangle_{\Sigma_t} \right|^2 \frac{|h_m|}{\lambda_m^2} > 0.
\end{aligned}
$$

For operator T_{2t}, we shall show that it is already a compact operator. Consider a bounded sequence $\{\mathbf{u}_n^+\} \subset X^+$, then for each $\mathbf{u}_n^+$, we have

$$
T_{2t}(\mathbf{n}_\Sigma \times \mathbf{u}_n^+) = \sum_n \left\langle (\mathbf{n}_\Sigma \times \mathbf{u}_n^+|_{\Sigma_t}), \begin{pmatrix} \vec{\nabla}_\Sigma \times v_n \\ 0 \end{pmatrix} \right\rangle_{\Sigma_t} \left(-\frac{1}{\mu_n^2}\frac{k^2}{ig_n}\right) \begin{pmatrix} \nabla_\Sigma v_n \\ 0 \end{pmatrix},
$$

and then

$$
\begin{aligned}
\|T_{2t}(\mathbf{n}_\Sigma \times \mathbf{u}_n^+)\|^2_{H^{-1/2}(\mathrm{div},\Sigma)} &= \sum_n \mu_n^3 \left| \left\langle (\mathbf{n}_\Sigma \times \mathbf{u}_n^+|_{\Sigma_t}), \begin{pmatrix} \vec{\nabla}_\Sigma \times v_n \\ 0 \end{pmatrix} \right\rangle_{\Sigma_t} \left(-\frac{1}{\mu_n^2}\frac{k^2}{ig_n}\right) \right|^2 \\
&= \sum_n \frac{1}{\mu_n} \left| \left\langle (\mathbf{n}_\Sigma \times \mathbf{u}_n^+|_{\Sigma_t}), \begin{pmatrix} \vec{\nabla}_\Sigma \times v_n \\ 0 \end{pmatrix} \right\rangle_{\Sigma_t} \left(-\frac{k^2}{ig_n}\right) \right|^2.
\end{aligned}
$$

Meanwhile, since

$$\mathbf{u}_n^+|_{\Sigma_t} = \sum_m a_m \frac{1}{\lambda_m^2} \begin{pmatrix} \vec{\nabla}_\Sigma \times u_m \\ 0 \end{pmatrix} + \sum_n -b_n \left[\frac{1}{\mu_n^2} \begin{pmatrix} \nabla_\Sigma v_n \\ 0 \end{pmatrix} + \frac{1}{ig_n} \begin{pmatrix} 0 \\ 0 \\ v_n \end{pmatrix} \right],$$

we get

$$\begin{aligned}
\mathbf{n}_\Sigma \cdot \mathbf{u}_n^+|_{\Sigma_t} &= 0 + \sum_n -b_n \left[0 + \frac{1}{ig_n} \mathbf{n}_\Sigma \cdot \begin{pmatrix} 0 \\ 0 \\ v_n \end{pmatrix} \right] \\
&= \sum_n -b_n \frac{1}{ig_n} v_n \\
&= \sum_n \left\langle \mathbf{n}_\Sigma \times \mathbf{u}_n^+|_{\Sigma_t}, \begin{pmatrix} \vec{\nabla}_\Sigma \times v_n \\ 0 \end{pmatrix} \right\rangle_{\Sigma_t} \left(-\frac{1}{ig_n} \right) v_n,
\end{aligned}$$

and then

$$\|\mathbf{n}_\Sigma \cdot \mathbf{u}_n^+\|_{H^{-1/2}(\Sigma)}^2 = \sum_n \frac{1}{\sqrt{1+\mu_n^2}} \left| \left\langle \mathbf{n}_\Sigma \times \mathbf{u}_n^+|_{\Sigma_t}, \begin{pmatrix} \vec{\nabla}_\Sigma \times v_n \\ 0 \end{pmatrix} \right\rangle_{\Sigma_t} \left(-\frac{1}{ig_n} \right) \right|^2.$$

Because $\dfrac{1}{\sqrt{1+\mu_n^2}} = \mathcal{O}\left(\dfrac{1}{\mu_n}\right)$ as $n \to \infty$ and k is a constant, using the boundedness of trace operator from $H(\mathrm{div}, \Omega)$ to $H^{-1/2}(\Sigma)$, we have that

$$\begin{aligned}
\|T_{2t}(\mathbf{n}_\Sigma \times \mathbf{u}_n^+)\|_{H^{-1/2}(\mathrm{div},\Sigma)}^2 &\leq C \|\mathbf{n}_\Sigma \cdot \mathbf{u}_n^+\|_{H^{-1/2}(\Sigma)}^2 \\
&\leq C \|\mathbf{u}_n^+\|_{H(\mathrm{div},\Omega)} = C \left(\|\mathbf{u}_n^+\|_{L^2(\Omega)} + \|\nabla \cdot \mathbf{u}_n^+\|_{L^2(\Omega)} \right) \\
&= C \|\mathbf{u}_n^+\|_{L^2(\Omega)},
\end{aligned}$$

for some constant $C > 0$.

Therefore, there exists a strongly convergent sequence in $\widetilde{H}^{-1/2}(\mathrm{div}, \Sigma_t)$. Since X^+ is compactly embedded in $(L^2(\Omega))^3$ (Lemma 3.2.3), T_{2t} is compact.

Overall, we see that

$$T_t = T_{1t} + T_{2t} = \tilde{T}_{1t} + (T_{1t} - \tilde{T}_{1t}) + T_{2t},$$

where $T_t^0 = \tilde{T}_{1t} > 0$ and $T_t^c = (T_{1t} - \tilde{T}_{1t}) + T_{2t}$ is compact. Similar result holds for T_s on Σ_s and the proof is done. □

3.2.3 Existence and Uniqueness

With Lemma 3.2.4, we can rewrite (3.37) as finding $\mathbf{u}^+ \in X^+$ such that

$$\begin{aligned}
G(\mathbf{u}^i, \mathbf{v}) &= (\nabla \times \mathbf{u}^+, \nabla \times \mathbf{v})_\Omega - k^2(\mathbf{u}^+, \mathbf{v})_\Omega + \sum_{j=s,t} \langle T_j(\mathbf{n}_\Sigma \times \mathbf{u}^+), \mathbf{v}_T \rangle_{\Sigma_j} \\
&= (\nabla \times \mathbf{u}^+, \nabla \times \mathbf{v})_\Omega + (\mathbf{u}^+, \mathbf{v})_\Omega + \sum_{j=s,t} \langle \tilde{T}_{1j}(\mathbf{n}_\Sigma \times \mathbf{u}^+), \mathbf{v}_T \rangle_{\Sigma_j} \\
&\quad -(k^2+1)(\mathbf{u}^+, \mathbf{v})_\Omega \\
&\quad + \sum_{j=s,t} \langle (T_{1j} - \tilde{T}_{1j})(\mathbf{n}_\Sigma \times \mathbf{u}^+), \mathbf{v}_T \rangle_{\Sigma_j} + \sum_{j=s,t} \langle T_{2j}(\mathbf{n}_\Sigma \times \mathbf{u}^+), \mathbf{v}_T \rangle_{\Sigma_j},
\end{aligned}$$

for all $\mathbf{v} \in X^+$.

Define the following sesquilinear form:

$$a(\mathbf{u}^+, \mathbf{v}) = (\nabla \times \mathbf{u}^+, \nabla \times \mathbf{v})_\Omega + (\mathbf{u}^+, \mathbf{v})_\Omega + \sum_{j=s,t} \langle \tilde{T}_{1j}(\mathbf{n}_\Sigma \times \mathbf{u}^+), \mathbf{v}_T \rangle_{\Sigma_j}.$$

Clearly $a(\mathbf{u}^+, \mathbf{v})$ is coercive because $\tilde{T}_1$ is positive and we have, for all $\mathbf{u}^+ \in X^+$,

$$a(\mathbf{u}^+, \mathbf{u}^+) \geq (\nabla \times \mathbf{u}^+, \nabla \times \mathbf{u}^+)_\Omega + (\mathbf{u}^+, \mathbf{u}^+)_\Omega = \|\mathbf{u}^+\|^2_{H(\mathrm{curl})}.$$

Also, $a(\mathbf{u}^+, \mathbf{v})$ is bounded because by Cauchy-Schwarz inequality

$$\begin{aligned}
|a(\mathbf{u}^+, \mathbf{v})| &\leq |(\nabla \times \mathbf{u}^+, \nabla \times \mathbf{v})_\Omega| + |(\mathbf{u}^+, \mathbf{v})_\Omega| + \sum_{j=s,t} |\langle \tilde{T}_{1j}(\mathbf{n}_\Sigma \times \mathbf{u}^+), \mathbf{v}_T \rangle_{\Sigma_j}| \\
&\leq C\|\mathbf{u}^+\|_{H(\mathrm{curl})}\|\mathbf{v}\|_{H(\mathrm{curl})} + \sum_{j=s,t} \|\tilde{T}_{1j}(\mathbf{n}_\Sigma \times \mathbf{u}^+)\|_{H^{-1/2}(\mathrm{div},\Sigma)} \|\mathbf{v}_T\|_{H^{-1/2}(\mathrm{curl},\Sigma)} \\
&\leq C\|\mathbf{u}^+\|_{H(\mathrm{curl})}\|\mathbf{v}\|_{H(\mathrm{curl})} + \sum_{j=s,t} \|T_{1j}(\mathbf{n}_\Sigma \times \mathbf{u}^+)\|_{H^{-1/2}(\mathrm{div},\Sigma)} \|\mathbf{v}_T\|_{H^{-1/2}(\mathrm{curl},\Sigma)} \\
&\leq C\|\mathbf{u}^+\|_{H(\mathrm{curl})}\|\mathbf{v}\|_{H(\mathrm{curl})} + \sum_{j=s,t} \|T_j(\mathbf{n}_\Sigma \times \mathbf{u}^+)\|_{H^{-1/2}(\mathrm{div},\Sigma)} \|\mathbf{v}_T\|_{H^{-1/2}(\mathrm{curl},\Sigma)} \\
&\leq C\|\mathbf{u}^+\|_{H(\mathrm{curl})}\|\mathbf{v}\|_{H(\mathrm{curl})} + \sum_{j=s,t} \|\mathbf{n}_\Sigma \times \mathbf{u}^+\|_{H^{-1/2}(\mathrm{div},\Sigma)} \|\mathbf{v}_T\|_{H^{-1/2}(\mathrm{curl},\Sigma)} \\
&= C\|\mathbf{u}^+\|_{H(\mathrm{curl})}\|\mathbf{v}\|_{H(\mathrm{curl})},
\end{aligned}$$

for some constant $C > 0$.

Next we shall show the equivalence of (3.37) to an operator equation. First define the operator $A : X^+ \mapsto X^+$ such that for all $\mathbf{f} \in X^+$, $A\mathbf{f} \in X^+$ satisfies

$$a(A\mathbf{f}, \mathbf{v}) = -(k^2+1)(\mathbf{f}, \mathbf{v})_\Omega + \sum_{j=s,t} \langle (T_{1j} - \tilde{T}_{1j})(\mathbf{n}_\Sigma \times \mathbf{f}), \mathbf{v}_T \rangle_{\Sigma_j}$$

$$+\sum_{j=s,t}\langle T_{2j}(\mathbf{n}_\Sigma\times\mathbf{f}),\mathbf{v}_T\rangle_{\Sigma_j} \text{ for all } \mathbf{v}\in X^+. \tag{3.39}$$

By the Lax-Milgram Lemma (Theorem C.0.4), this problem is well-posed and operator A is well defined and bounded.

Similarly, we can define $\mathbf{g}\in X^+$ such that

$$a(\mathbf{g},\mathbf{v}) = G(\mathbf{u}^i,\mathbf{v}).$$

Thus, the variational form (3.37) can be rewritten as a problem of finding $\mathbf{u}^+\in X^+$ such that

$$a(\mathbf{u}^+ + A\mathbf{u}^+ - \mathbf{g},\mathbf{v}) = 0 \text{ for all } \mathbf{v}\in X^+.$$

This implies that

$$(I+A)\mathbf{u}^+ = \mathbf{g} \text{ in } X^+. \tag{3.40}$$

To further analyze this operator equation, we have the following lemma

Lemma 3.2.5 *The map $A : X^+ \mapsto X^+$ is compact.*

Proof: Let $\{\mathbf{u}_n^+\}$ be a bounded sequence in X^+. Hence there exists a subsequence, denoted again by $\{\mathbf{u}_n^+\}$, which converges weakly to $\mathbf{u}_0\in X^+$. Then, by Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\|A(\mathbf{u}_n-\mathbf{u}_0)\|^2_{H(\text{curl})} &\le a\left(A(\mathbf{u}_n-\mathbf{u}_0),A(\mathbf{u}_n-\mathbf{u}_0)\right)\\
&= -(k^2+1)(\mathbf{u}_n-\mathbf{u}_0,A(\mathbf{u}_n-\mathbf{u}_0))_\Omega\\
&\quad+\sum_{j=s,t}\langle (T_{1j}-\tilde{T}_{1j})(\mathbf{n}_\Sigma\times(\mathbf{u}_n-\mathbf{u}_0)),(A(\mathbf{u}_n-\mathbf{u}_0))_T\rangle_{\Sigma_j}\\
&\quad+\sum_{j=s,t}\langle T_{2j}(\mathbf{n}_\Sigma\times(\mathbf{u}_n-\mathbf{u}_0)),(A(\mathbf{u}_n-\mathbf{u}_0))_T\rangle_{\Sigma_j}\\
&\le (k^2+1)\left|(\mathbf{u}_n-\mathbf{u}_0,A(\mathbf{u}_n-\mathbf{u}_0))\right|_\Omega\\
&\quad+\sum_{j=s,t}\left|\langle (T_{1j}-\tilde{T}_{1j})(\mathbf{n}_\Sigma\times(\mathbf{u}_n-\mathbf{u}_0)),(A(\mathbf{u}_n-\mathbf{u}_0))_T\rangle_{\Sigma_j}\right|\\
&\quad+\sum_{j=s,t}\left|\langle T_{2j}(\mathbf{n}_\Sigma\times(\mathbf{u}_n-\mathbf{u}_0)),(A(\mathbf{u}_n-\mathbf{u}_0))_T\rangle_{\Sigma_j}\right|
\end{aligned}$$

$$
\begin{aligned}
&\leq (k^2+1)\|\mathbf{u}_n-\mathbf{u}_0\|_{L^2}\|A(\mathbf{u}_n-\mathbf{u}_0)\|_{L^2} \\
&\quad +\sum_{j=s,t}\|(T_{1j}-\tilde{T}_{1j})(\mathbf{n}_\Sigma\times(\mathbf{u}_n-\mathbf{u}_0))\|_{H^{-1/2}(\mathrm{div},\Sigma)}\|A(\mathbf{u}_n-\mathbf{u}_0)\|_{H^{-1/2}(\mathrm{curl},\Sigma)} \\
&\quad +\sum_{j=s,t}\|T_{2j}(\mathbf{n}_\Sigma\times(\mathbf{u}_n-\mathbf{u}_0))\|_{H^{-1/2}(\mathrm{div},\Sigma)}\|A(\mathbf{u}_n-\mathbf{u}_0)\|_{H^{-1/2}(\mathrm{curl},\Sigma)} \\
&\leq C\|\mathbf{u}_n-\mathbf{u}_0\|_{L^2}\|A(\mathbf{u}_n-\mathbf{u}_0)\|_{H(\mathrm{curl})} \\
&\quad +C\sum_{j=s,t}\|(T_{1j}-\tilde{T}_{1j})(\mathbf{n}_\Sigma\times(\mathbf{u}_n-\mathbf{u}_0))\|_{H^{-1/2}(\mathrm{div},\Sigma)}\|A(\mathbf{u}_n-\mathbf{u}_0)\|_{H(\mathrm{curl})} \\
&\quad +C\sum_{j=s,t}\|T_{2j}(\mathbf{n}_\Sigma\times(\mathbf{u}_n-\mathbf{u}_0))\|_{H^{-1/2}(\mathrm{div},\Sigma)}\|A(\mathbf{u}_n-\mathbf{u}_0)\|_{H(\mathrm{curl})},
\end{aligned}
$$

for some constant $C>0$.

Thus, we have

$$
\begin{aligned}
\|A(\mathbf{u}_n-\mathbf{u}_0)\|_{H(\mathrm{curl})} &\leq C\|\mathbf{u}_n-\mathbf{u}_0\|_{L^2} \\
&\quad +C\sum_{j=s,t}\|(T_{1j}-\tilde{T}_{1j})(\mathbf{n}_\Sigma\times(\mathbf{u}_n-\mathbf{u}_0))\|_{H^{-1/2}(\mathrm{div},\Sigma)} \\
&\quad +C\sum_{j=s,t}\|T_{2j}(\mathbf{n}_\Sigma\times(\mathbf{u}_n-\mathbf{u}_0))\|_{H^{-1/2}(\mathrm{div},\Sigma)},
\end{aligned}
$$

for some constant $C>0$.

Since X^+ is compactly embedded in $(L^2(\Omega))^3$ (Lemma 3.2.3), and $T_{1j}-\tilde{T}_{1j}$ and T_{2j} are compact operators for $j=s,t$, we can conclude that $A\mathbf{u}_n$ converges strongly to $A\mathbf{u}_0$ and therefore A is compact. $\square$

Moreover, we need the following uniqueness result:

Theorem 3.2.1 *There exists at most one solution to the variational problem in Definition 3.1.3 for pure imaginary wavenumber k with $\mathfrak{Im}(k)>0$ being small.*

Proof: We shall show that solution to (3.33) with incident field $\mathbf{u}^i=\mathbf{0}$ is zero. By choosing test function $\mathbf{v}=\mathbf{u}$, we have

$$
(\nabla\times\mathbf{u},\nabla\times\mathbf{u})_\Omega-k^2(\mathbf{u},\mathbf{u})_\Omega+\sum_{j=s,t}\langle T_j(\mathbf{n}_\Sigma\times\mathbf{u}),u_T\rangle_{\Sigma_j}=\hat{F}(\mathbf{0},\mathbf{v})=0.
$$

Since k is purely imaginary,

$$
(\nabla\times\mathbf{u},\nabla\times\mathbf{u})_\Omega+|k|^2(\mathbf{u},\mathbf{u})_\Omega+\sum_{j=s,t}\langle T_j(\mathbf{n}_\Sigma\times\mathbf{u}),\mathbf{u}_T\rangle_{\Sigma_j}=0.
$$

For the term involving operator T, using identities (B.2) and (B.11),(B.12), we have

$$
\begin{aligned}
&\langle T(\mathbf{n}_\Sigma \times \mathbf{u}), \mathbf{u}_T\rangle_\Sigma \\
= \ &\langle T(\mathbf{n}_\Sigma \times \mathbf{u}), (\mathbf{n}_\Sigma \times \mathbf{u}) \times \mathbf{n}_\Sigma\rangle_\Sigma \\
= \ &\left\langle \sum_m \left\langle (\mathbf{n}_\Sigma \times \mathbf{u}|_\Sigma), \begin{pmatrix} \nabla_\Sigma u_m \\ 0 \end{pmatrix} \right\rangle_\Sigma \left(-\frac{ih_m}{\lambda_m^2}\right) \begin{pmatrix} \vec{\nabla}_\Sigma \times u_m \\ 0 \end{pmatrix}, (\mathbf{n}_\Sigma \times \mathbf{u}) \times \mathbf{n}_\Sigma \right\rangle_\Sigma \\
&+ \left\langle \sum_n \left\langle (\mathbf{n}_\Sigma \times \mathbf{u}|_\Sigma), \begin{pmatrix} \vec{\nabla}_\Sigma \times v_n \\ 0 \end{pmatrix} \right\rangle_\Sigma \left(-\frac{1}{\mu_n^2}\frac{k^2}{ig_n}\right) \begin{pmatrix} \nabla_\Sigma v_n \\ 0 \end{pmatrix}, (\mathbf{n}_\Sigma \times \mathbf{u}) \times \mathbf{n}_\Sigma \right\rangle_\Sigma \\
= \ &\left\langle \sum_m \left\langle (\mathbf{n}_\Sigma \times \mathbf{u}|_\Sigma), \begin{pmatrix} \nabla_\Sigma u_m \\ 0 \end{pmatrix} \right\rangle_\Sigma \left(-\frac{ih_m}{\lambda_m^2}\right) \left(\mathbf{n}_\Sigma \times \begin{pmatrix} \vec{\nabla}_\Sigma \times u_m \\ 0 \end{pmatrix}\right), \mathbf{n}_\Sigma \times \mathbf{u} \right\rangle_\Sigma \\
&+ \left\langle \sum_n \left\langle (\mathbf{n}_\Sigma \times u|_\Sigma), \begin{pmatrix} \vec{\nabla}_\Sigma \times v_n \\ 0 \end{pmatrix} \right\rangle_\Sigma \left(-\frac{1}{\mu_n^2}\frac{k^2}{ig_n}\right) \left(\mathbf{n}_\Sigma \times \begin{pmatrix} \nabla_\Sigma v_n \\ 0 \end{pmatrix}\right), \mathbf{n}_\Sigma \times \mathbf{u} \right\rangle_\Sigma \\
= \ &\left\langle \sum_m \left\langle (\mathbf{n}_\Sigma \times u|_\Sigma), \begin{pmatrix} \nabla_\Sigma u_m \\ 0 \end{pmatrix} \right\rangle_\Sigma \left(-\frac{ih_m}{\lambda_m^2}\right) \begin{pmatrix} \nabla_\Sigma u_m \\ 0 \end{pmatrix}, (\mathbf{n}_\Sigma \times \mathbf{u}) \right\rangle_\Sigma \\
&+ \left\langle \sum_n \left\langle (\mathbf{n}_\Sigma \times u|_\Sigma), \begin{pmatrix} \vec{\nabla}_\Sigma \times v_n \\ 0 \end{pmatrix} \right\rangle_\Sigma \left(-\frac{1}{\mu_n^2}\frac{k_{\mu_n}^2}{ih_n}\right) \left(-\begin{pmatrix} \vec{\nabla}_\Sigma \times v_n \\ 0 \end{pmatrix}\right), \mathbf{n}_\Sigma \times \mathbf{u} \right\rangle_\Sigma \\
= \ &\sum_m \left\langle (\mathbf{n}_\Sigma \times \mathbf{u}|_\Sigma), \begin{pmatrix} \nabla_\Sigma u_m \\ 0 \end{pmatrix} \right\rangle_\Sigma \left(-\frac{ih_m}{\lambda_m^2}\right) \left\langle \begin{pmatrix} \nabla_\Sigma u_m \\ 0 \end{pmatrix}, (\mathbf{n}_\Sigma \times \mathbf{u}) \right\rangle_\Sigma \\
&+ \sum_n \left\langle (\mathbf{n}_\Sigma \times \mathbf{u}|_\Sigma), \begin{pmatrix} \vec{\nabla}_\Sigma \times v_n \\ 0 \end{pmatrix} \right\rangle_\Sigma \left(\frac{1}{\mu_n^2}\frac{k^2}{ig_n}\right) \left\langle \begin{pmatrix} \vec{\nabla}_\Sigma \times v_n \\ 0 \end{pmatrix}, \mathbf{n}_\Sigma \times \mathbf{u} \right\rangle_\Sigma \\
= \ &\sum_m \left| \left\langle (\mathbf{n}_\Sigma \times \mathbf{u}|_\Sigma), \begin{pmatrix} \nabla_\Sigma u_m \\ 0 \end{pmatrix} \right\rangle_\Sigma \right|^2 \left(-\frac{ih_m}{\lambda_m^2}\right) \\
&+ \sum_n \left| \left\langle (\mathbf{n}_\Sigma \times \mathbf{u}|_\Sigma), \begin{pmatrix} \vec{\nabla}_\Sigma \times v_n \\ 0 \end{pmatrix} \right\rangle_\Sigma \right|^2 \left(\frac{1}{\mu_n^2}\frac{k^2}{ig_n}\right).
\end{aligned}
$$

Choosing k such that $0 < |k|^2 < \min\{\ \{\lambda_m^2\}_{m\geq 1}, \{\mu_n^2\}_{n\geq 1}\ \}$, we get

$$\begin{aligned}
\langle T(\mathbf{n}_\Sigma \times \mathbf{u}), \mathbf{u}_T\rangle_\Sigma &= \sum_m \left| \left\langle (\mathbf{n}_\Sigma \times \mathbf{u}|_\Sigma), \begin{pmatrix} \nabla_\Sigma u_m \\ 0 \end{pmatrix} \right\rangle_\Sigma \right|^2 \left(-\frac{i\, i\sqrt{|k|^2 + \lambda_m^2}}{\lambda_m^2} \right) \\
&\quad + \sum_n \left| \left\langle (\mathbf{n}_\Sigma \times \mathbf{u}|_\Sigma), \begin{pmatrix} \vec{\nabla}_\Sigma \times v_n \\ 0 \end{pmatrix} \right\rangle_\Sigma \right|^2 \left(\frac{1}{\mu_n^2} \frac{k^2}{i\, i\sqrt{|k|^2 + \mu_n^2}} \right) \\
&= \sum_m \left| \left\langle (\mathbf{n}_\Sigma \times \mathbf{u}|_\Sigma), \begin{pmatrix} \nabla_\Sigma u_m \\ 0 \end{pmatrix} \right\rangle_\Sigma \right|^2 \left(\frac{\sqrt{|k|^2 + \lambda_m^2}}{\lambda_m^2} \right) \\
&\quad + \sum_n \left| \left\langle (\mathbf{n}_\Sigma \times \mathbf{u}|_\Sigma), \begin{pmatrix} \vec{\nabla}_\Sigma \times v_n \\ 0 \end{pmatrix} \right\rangle_\Sigma \right|^2 \left(\frac{1}{\mu_n^2} \frac{|k|^2}{\sqrt{|k|^2 + \mu_n^2}} \right) > 0,
\end{aligned}$$

unless $\mathbf{u} = \mathbf{0}$. Therefore, there exists at most one solution. $\square$

Now, we shall prove the following existence result

Theorem 3.2.2 *Consider the variational problem: find* $\mathbf{u} \in X^+$ *such that equation* (3.37) *holds for all* $\mathbf{v} \in X^+$. *Then, for any real* k *except possibly for a discrete set of real wavenumber* $k^{(j)}$ *such that* $k^{(j)} \to \infty$ *as* $j \to \infty$, *this problem is uniquely solvable.*

Proof: First, we have shown that the variational form (3.37) is equivalent to the operator equation (3.40). Moreover, by Lemma 3.1.5 and variational form (3.39), the compact operator A depends analytically on the wavenumber k in an open connected sub-domain $\mathfrak{C}$ of complex plane $\mathbb{C}$ except a series of branch cuts as described in Lemma 3.1.5 for operator T. Then, by Theorem C.0.6, except possibly for a countable set of points, the operator equation

$$(I + A)\mathbf{u}^+ = \mathbf{0}$$

has the same number of linearly independent solutions in $\mathfrak{C}$.

Choose the wavenumber $k = ic$ for some constant $c > 0$ small enough such that $k \in \mathfrak{C}$. Note that the uniqueness result (Theorem 3.2.1) also holds for variational problem (3.37) in X, thus there exists at most one solution to equation $(I + A)\mathbf{u}^+ = \mathbf{0}$ in $\mathfrak{C}$. Because the trivial function $\mathbf{u}^+ = \mathbf{0}$ solves this equation, it is the only one.

Therefore, by Fredholm Alternative (Theorem C.0.7), the inhonmogeneous operator equation (3.40) is uniquely solvable and so is equation (3.37).

Finally, wavenumbers excluded for the well-posedness of this problem are isolated points from Theorem C.0.6 and λ_m's and μ_n's described in Lemma 3.1.5 such that $h_m = 0, g_n = 0$ which form a discrete set without accumulation points less than infinity. $\square$

Together with Lemma 3.2.1, we have the following result as a corollary of Theorem 3.2.2,

Corollary 3.2.1 *The variational problem problem described in Definition 3.1.3 is uniquely solvable for any real k except possibly for a discrete set of real wavenumbers $k^{(j)}$ such that $k^{(j)} \to \infty$ as $j \to \infty$.*

Also, as a direct result of Theorem 3.1.3, we have

Corollary 3.2.2 *The unique solution $\mathbf{u}$ given in Corollary 3.2.1, by setting $\mathbf{w} = \mathbf{u} - (1-\chi)\mathbf{u}^i$, can be extended in a unique way to $W \backslash \Omega$ and such that $\mathbf{u}^s$ as an extension of $\mathbf{u} - \mathbf{u}^i$ satisfies the (global) scattering problem in Definition 3.1.2.*

3.3 Inverse Problem

In this section, we shall provide a theoretical basis for the inverse problem in the waveguide geometry. The inverse problem we consider here is to identify the boundary ∂D of scatterer D using the scattering data (near field data) from D illuminated by point sources located far away from it. Specifically, there are two important results: the uniqueness of the scatterer and the justification of the Linear Sampling Method (LSM) to the reconstruction of the shape of scatterer.

3.3.1 Dyadic Green's Function

To initiate the analysis, we need to understand the background Green's function due to the waveguide. From Chapter 4 in [61], for electromagnetic waves, the Green's functions are dyadic functions with appropriate boundary conditions on Γ. In $\mathbb{R}^3$, a dyadic function is a second order tensor that can be written as a 3×3 matrix. Specifically, there are electric and magnetic Green's functions with each satisfying PEC condition and magnetic wall condition, respectively, on the boundary Γ of the waveguide. Denote by the electric type Green's function with subscript "e" and the magnetic type Green's function with subscript "m". For our analysis, we need the electric type dyadic Green's function with PEC condition on Γ, that is,

$$\nabla_{\mathbf{x}} \times \nabla_{\mathbf{x}} \times \mathbb{G}_e(\mathbf{x},\mathbf{y}) - k^2 \mathbb{G}_e(\mathbf{x},\mathbf{y}) = \mathbb{I}\delta(\mathbf{x}-\mathbf{y}),$$

with boundary condition

$$\mathbf{n}_\Gamma \times \mathbb{G}_e(\mathbf{x},\mathbf{y}) = \mathbf{0} \quad \text{on} \quad \Gamma,$$

where $\mathbf{x} = (x,y,z)$ represents an arbitrary point in $\mathbb{R}^3$ and $\mathbf{y} = (x',y',z')$ a point source; $\mathbb{I}$ is 3×3 identity matrix; $\delta(\mathbf{x}-\mathbf{y}) = \delta(x-x')\delta(y-y')\delta(z-z')$.

An explicit representation of the dyadic Green's functions can be written using the modal solutions defined in Section 3.1.2.1. To facilitate our analysis, we modify the notation for the modal solutions M and N with superscript "+" and "-" as follows:

$$\begin{aligned}
M^+ = M \quad &\text{and} \quad N^+ = N, \\
\nabla \times M^+ = \nabla \times M \quad &\text{and} \quad \nabla \times N^+ = \nabla \times N, \\
M^- = M\big|_{z=-z} \quad &\text{and} \quad N^- = N\big|_{z=-z}, \\
\nabla \times M^- = (\nabla \times M)\big|_{z=-z} \quad &\text{and} \quad \nabla \times N^- = (\nabla \times N)\big|_{z=-z}.
\end{aligned}$$

Here we see that superscript "-" means replacing z by $-z$. For example, for

$$\nabla \times M \quad = \quad \nabla \times \nabla \times (ue^{ihz}\hat{\mathbf{z}}) = \begin{pmatrix} \frac{\partial u}{\partial x} ih \\ \frac{\partial u}{\partial y} ih \\ u\lambda^2 \end{pmatrix} e^{ihz},$$

we have

$$\nabla \times M^- \quad = \quad \left(\nabla \times \nabla \times (ue^{ihz}\hat{\mathbf{z}})\right)\big|_{z=-z} = \begin{pmatrix} \frac{\partial u}{\partial x} ih \\ \frac{\partial u}{\partial y} ih \\ u\lambda^2 \end{pmatrix} e^{-ihz}.$$

With this notation, we have $\mathbb{G}_e$ in the following form:

- For $z > z'$,

$$\begin{aligned}\mathbb{G}_e(\mathbf{x},\mathbf{y}) &= -\frac{1}{k^2}\hat{\mathbf{z}}\hat{\mathbf{z}}^T\delta(\mathbf{x}-\mathbf{y}) \\ &\quad + \sum_{m=1}^{\infty}\left[c_m M_m^+(\mathbf{x})M_m^-(\mathbf{y})^T\right] + \sum_{n=1}^{\infty}\left[d_n N_n^+(\mathbf{x})N_n^-(\mathbf{y})^T\right], \quad (3.41)\end{aligned}$$

- For $z < z'$,

$$\begin{aligned}\mathbb{G}_e(\mathbf{x},\mathbf{y}) &= -\frac{1}{k^2}\hat{\mathbf{z}}\hat{\mathbf{z}}^T\delta(\mathbf{x}-\mathbf{y}) \\ &\quad + \sum_{m=1}^{\infty}\left[c_m M_m^-(\mathbf{x})M_m^+(\mathbf{y})^T\right] + \sum_{n=1}^{\infty}\left[d_n N_n^-(\mathbf{x})N_n^+(\mathbf{y})^T\right], \quad (3.42)\end{aligned}$$

where c_m, d_n $(m, n = 1, 2, \ldots)$ are coefficients depending on the eigenvalues of surface Laplacian on Σ and also the geometric shape of Σ. The terms $\hat{\mathbf{z}}\hat{\mathbf{z}}^T$, $M_m^+(\mathbf{x})M_m^-(\mathbf{y})^T$, $N_n^+(\mathbf{x})N_n^-(\mathbf{y})^T$, etc. are understood as column-row multiplication. For example, if $\mathbf{a} = (a_1, a_2, a_3)^T, \mathbf{b} = (b_1, b_2, b_3)^T$, then

$$\mathbf{a}\mathbf{b}^T = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}\begin{pmatrix} b_1 & b_2 & b_3 \end{pmatrix} = \begin{pmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & a_2b_2 & a_2b_3 \\ a_3b_1 & a_3b_2 & a_3b_3 \end{pmatrix}.$$

Note 3.3.1 *The electric dyadic Green's functions defined above is symmetric ($\mathbb{G}_e(\mathbf{x},\mathbf{y}) = \mathbb{G}_e(\mathbf{y},\mathbf{x})$) and can be separated into parts consisting of a singular matrix, non-zero sub-matrices and full matrices as follows:*

$$\mathbb{G}_e = -\frac{1}{k^2}\delta(\mathbf{x}-\mathbf{y})\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \sum_m c_m \begin{bmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 0 \end{bmatrix} + \sum_n d_n \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}.$$

Here the singular term $-\frac{1}{k^2}\delta(\mathbf{x}-\mathbf{y})$ contributes only when $\mathbf{x} = \mathbf{y}$. It arises from the discontinuity of the magnetic dyadic Green's function across a cross section Σ containing point source $\mathbf{y}$ (see Section 5.8 in [61]). However, this singular term will not affect our analysis.

Note 3.3.2 *It is worth mentioning that the governing equation for the magnetic type dyadic Green's function is*

$$\nabla_{\mathbf{x}} \times \nabla_{\mathbf{x}} \times \mathbb{G}_m(\mathbf{x}, \mathbf{y}) - k^2 \mathbb{G}_m(\mathbf{x}, \mathbf{y}) = \nabla_{\mathbf{x}} \times [\mathbb{I}\delta(\mathbf{x} - \mathbf{y})],$$

and the two dyadic Green's functions are related as follows (see Section 4.3 in [61])

$$\begin{aligned} \nabla_{\mathbf{x}} \times \mathbb{G}_e(\mathbf{x}, \mathbf{y}) &= \mathbb{G}_m(\mathbf{x}, \mathbf{y}), \\ \nabla_{\mathbf{x}} \times \mathbb{G}_m(\mathbf{x}, \mathbf{y}) &= \mathbb{I}\delta(\mathbf{x} - \mathbf{y}) + k^2 \mathbb{G}_e(\mathbf{x}, \mathbf{y}). \end{aligned}$$

In particular, the series representation for $\mathbb{G}_m$ *satisfying magnetic wall condition*

$$\mathbf{n}_\Gamma \times (\nabla_{\mathbf{x}} \times \mathbb{G}_m(\mathbf{x}, \mathbf{y})) = \mathbf{0} \quad on \quad \Gamma$$

is given by the following form:

- *For* $z > z'$,

$$\mathbb{G}_m(\mathbf{x}, \mathbf{y}) = \sum_{m=0}^{\infty} \left[c_m \left(\nabla \times M_m^+(\mathbf{x}) \right) M_m^-(\mathbf{y})^T \right] + \sum_{n=0}^{\infty} \left[d_n \left(\nabla \times N_n^+(\mathbf{x}) \right) N_n^-(\mathbf{y})^T \right],$$

- *For* $z < z'$,

$$\mathbb{G}_m(\mathbf{x}, \mathbf{y}) = \sum_{m=0}^{\infty} \left[c_m \left(\nabla \times M_m^-(\mathbf{x}) \right) M_m^+(\mathbf{y})^T \right] + \sum_{n=0}^{\infty} \left[d_n \left(\nabla \times N_n^-(\mathbf{x}) \right) N_n^+(\mathbf{y})^T \right].$$

3.3.1.1 Decomposition of Green's function

To facilitate the analysis of factorization of the near field operator in the sequel, denote $\mathbb{G} = \mathbb{G}_e$ unless stated otherwise and we shall show that $\mathbb{G} = \mathbb{G}_0 + \mathbb{J}$ where $\mathbb{J}$ is an infinitely differentiable remainder dyadic function so that the integral form with kernel $\mathbb{J}$ has well behaved properties which will allow us to employ the theory for integral form with kernel $\mathbb{G}_0$ to analyze that of $\mathbb{G}$.

Explicitly, $\mathbb{G}_0$ is the well-known free space dyadic Green's function given by

$$\mathbb{G}_0(\mathbf{x}, \mathbf{y}) = \Phi(\mathbf{x}, \mathbf{y})\mathbb{I} + \frac{1}{k^2} \nabla_{\mathbf{y}} \nabla_{\mathbf{y}} \Phi(\mathbf{x}, \mathbf{y}), \quad (\mathbf{x} \neq \mathbf{y}) \tag{3.43}$$

where $\mathbb{I}$ is a 3×3 identity matrix and $\nabla_{\mathbf{y}}\nabla_{\mathbf{y}}\Phi(\mathbf{x},\mathbf{y})$ is the Hessian matrix for Φ where $\Phi = \dfrac{\exp(ik|\mathbf{x}-\mathbf{y}|)}{4\pi|\mathbf{x}-\mathbf{y}|}$. Note that the ith column $\mathbb{G}_{0i}, i = 1,2,3$ of $\mathbb{G}_0$ satisfies the Maxwell's equation

$$\nabla_{\mathbf{y}} \times \nabla_{\mathbf{y}} \times \mathbb{G}_{0i} - k^2\mathbb{G}_{0i} = \mathbf{e}_i\delta_{\mathbf{x}} \quad \text{in} \quad \mathbb{R}^3,$$

where $\mathbf{e}_i$ is the unit vector along the ith coordinate axes, that is, $\hat{\mathbf{x}}$ or $\hat{\mathbf{y}}$ or $\hat{\mathbf{z}}$.

The major component of our analysis using dyadic Green's function is the continuity properties of layer potentials in the vicinity of the boundary of D. In turn, we consider the properties of layer potential with kernel $\mathbb{J}$ in a segment of the waveguide W enclosing D and bounded by cross sections Σ_l, Σ_{-l}, denote Ω_l. Then we have the following lemma:

Lemma 3.3.1 *The dyadic Green's function $\mathbb{G}$ for the waveguide W can be decomposed into two parts $\mathbb{G} = \mathbb{G}_0 + \mathbb{J}$ in a bounded segment $\Omega_l = W_{(-l,l)}$ of the waveguide including the scatterer D where $\mathbb{G}_0$ is the dyadic Green's function for the free space* (3.43) *and the remainder dyadic function $\mathbb{J}$ is infinitely differentiable in Ω_l, particularly, the neighborhood of ∂D.*

Proof: Since $\mathbb{J}(\mathbf{x},\mathbf{y}) = \mathbb{G}(\mathbf{x},\mathbf{y}) - \mathbb{G}_0(\mathbf{x},\mathbf{y})$, for a point source at $\mathbf{y} \in \Omega_l$, $\mathbb{J}$ satisfies the Maxwell's equation in Ω_l and $\mathbf{n}_\Gamma \times \mathbb{J} = \mathbf{n}_\Gamma \times (\mathbb{G} - \mathbb{G}_0)$ on $\Gamma_{(-l,l)}$. Moreover, we can impose the following impedance boundary condition on Σ_l and Σ_{-l}:

$$(\nabla \times \mathbb{J}) \times \mathbf{n}_\Sigma - ik\mathbb{J}_T = (\nabla \times (\mathbb{G} - \mathbb{G}_0)) \times \mathbf{n}_\Sigma - ik(\mathbb{G} - \mathbb{G}_0)_T \quad \text{on} \quad \Sigma_l \text{ and } \Sigma_{-l}.$$

With an analysis analogous to Section 12.2.1 in [48], the solution to Maxwell's equation in a bounded domain Ω_l with impedance boundary condition on part of the boundary $\partial\Omega_l = \Sigma_l \cup \Sigma_{-l} \cup \bar{\Gamma}_{(-l,l)}$ is uniquely solvable in $H(\text{curl}, \Omega_l)$, thus $\mathbb{J}$ exists and by uniqueness $\mathbb{J} = \mathbb{G} - \mathbb{G}_0$ on $\partial\Omega_l$. From the proof of Theorem 9.2 in [48], we have that components of $\mathbb{J}$ are smooth functions in a sub-domain of Ω_l away from $\partial\Omega_l$ but including D, moreover they are analytic by Theorem D.0.9. Therefore, we have the desired smoothness of $\mathbb{J}$ in the neighborhood of ∂D. $\square$

3.3.2 Uniqueness Result

In order to prove the uniqueness result, first we have the following two lemmas:

Lemma 3.3.2 *(Representation Formula) Let* $\mathbf{u}^s$ *be the solution of the following forward problem in* $W\backslash\bar{D}$*,*

$$\begin{cases} \nabla \times \nabla \times \mathbf{u}^s - k^2\mathbf{u}^s = 0 \quad \text{in} \quad W\backslash\bar{D}, \\ \mathbf{n}_\Gamma \times \mathbf{u}^s = \mathbf{0} \quad \text{on} \quad \Gamma, \\ \mathbf{n}_D \times \mathbf{u}^s = \mathbf{F} \quad \text{on} \quad \partial D, \\ \mathbf{n}_\Sigma \times (\nabla \times \mathbf{u}^s) = T_s(\mathbf{n}_\Sigma \times \mathbf{u}^s) \quad \text{on} \quad \Sigma_s, \\ \mathbf{n}_\Sigma \times (\nabla \times \mathbf{u}^s) = T_t(\mathbf{n}_\Sigma \times \mathbf{u}^s) \quad \text{on} \quad \Sigma_t, \end{cases} \tag{3.44}$$

where $\mathbf{F} \in H^{-1/2}(\mathrm{div}, \partial D)$ *and* $\mathbf{n}_D$ *is the inward normal to* ∂D*. Then for all* $\mathbf{x} \in W\backslash\bar{D}$*, we have the representation formula*

$$\mathbf{u}^s(\mathbf{x}) = \int_{\partial D} \mathbf{u}^s(\mathbf{y}) \cdot (\mathbf{n}_D \times [\nabla_{\mathbf{y}} \times \mathbb{G}(\mathbf{x}, \mathbf{y})]) - (\mathbf{n}_D \times [\nabla \times \mathbf{u}^s(\mathbf{y})]) \cdot \mathbb{G}(\mathbf{x}, \mathbf{y})\, ds(\mathbf{y})$$

where "$\cdot$" is understood as vector-matrix or matrix-vector multiplication depending on the position of dyadic and vector functions.

Note 3.3.3 *There exist a unique solution to the forward problem stated in Lemma 3.3.2 above. Because this forward problem is well-posed in* $\Omega = W_{(s,t)}\backslash\bar{D}$ *(Corollary 3.2.1), then by the definition of operators* T_s, T_t *and the uniqueness of solution in the blocked waveguide (Lemma 3.1.3), the solution can be extended uniquely to* $W\backslash\bar{D}$*.*

Proof: The proof is inspired from the proof of Lemma 2 in [9]. We analyze the problem in two regions:

First, we consider a point source at $\mathbf{x} \in W\backslash\bar{D}$. Let $r > 0$ be such that $B(\mathbf{x}, 2r) \in W\backslash\bar{D}$ where $B(\mathbf{x}, 2r)$ is the ball of radius $2r$ about $\mathbf{x}$. Define two new dyadic functions as follows

$$\tilde{\mathbb{G}}_{\mathbf{x}}(\mathbf{y}) = \begin{cases} 0 & \text{for} \quad \mathbf{y} \in B(\mathbf{x}, r) \quad \text{(close to the point source)}, \\ -\mathbb{G}(\mathbf{x}, \mathbf{y}) & \text{for} \quad \mathbf{y} \notin B(\mathbf{x}, r) \quad \text{(away from the point source)}, \end{cases}$$

and

$$
\begin{aligned}
\mathbb{G}_{\mathbf{x}}(\mathbf{y}) &= \tilde{\mathbb{G}}_{\mathbf{x}}(\mathbf{y}) + \mathbb{G}(\mathbf{x},\mathbf{y}) \\
&= \begin{cases} \mathbb{G}(\mathbf{x},\mathbf{y}) & \text{for} \quad \mathbf{y} \in B(\mathbf{x},r) \quad \text{(close to the point source),} \\ 0 & \text{for} \quad \mathbf{y} \notin B(\mathbf{x},r) \quad \text{(away from the point source).} \end{cases}
\end{aligned}
$$

In fact, $\mathbb{G}_{\mathbf{x}}(\mathbf{y})$ is the same as $\mathbb{G}(\mathbf{x},\mathbf{y})$ around the neighborhood of $\mathbf{x}$ and zero outside of $B(\mathbf{x},r)$, $\tilde{\mathbb{G}}_{\mathbf{x}}(\mathbf{y})$ is non-zero outside of $B(\mathbf{x},r)$.

Now take a test function $\mathbf{v} \in (C_0^\infty(W\backslash\bar{D}))^3$. In the sense of distribution on $W\backslash\bar{D}$, we have

$$
\begin{aligned}
&\int_{W\backslash\bar{D}} (\nabla\times\nabla\times\tilde{\mathbb{G}}_{\mathbf{x}}(\mathbf{y}) - k^2\tilde{\mathbb{G}}_{\mathbf{x}}(\mathbf{y}))\cdot\mathbf{v}(y)\,dy \\
&= \int_{W\backslash\bar{D}} \left(\tilde{\mathbb{G}}_{\mathbf{x}}(\mathbf{y})\right)^T \cdot (\nabla\times\nabla\times\mathbf{v}(\mathbf{y}) - k^2\mathbf{v}(\mathbf{y}))\,dy \\
&= -\int_{(W\backslash\bar{D})\backslash B(\mathbf{x},r)} \mathbb{G}(\mathbf{x},\mathbf{y})^T\cdot(\nabla\times\nabla\times\mathbf{v}(\mathbf{y}) - k^2\mathbf{v}(\mathbf{y}))\,dy \\
&= -\int_{(W\backslash\bar{D})\backslash B(\mathbf{x},r)} (\nabla\times\nabla\times\mathbf{v}(\mathbf{y}) - k^2\mathbf{v}(\mathbf{y}))\cdot\mathbb{G}(\mathbf{x},\mathbf{y})\,dy.
\end{aligned}
$$

Since $\nabla_{\mathbf{y}}\times\nabla_{\mathbf{y}}\times\mathbb{G}(\mathbf{x},\mathbf{y}) - k^2\mathbb{G}(\mathbf{x},\mathbf{y}) = 0$ for $\mathbf{y}\notin B(\mathbf{x},r)$, applying vector-dyadic identity (B.34), we have

$$
\begin{aligned}
&\int_{W\backslash\bar{D}} (\nabla\times\nabla\times\tilde{\mathbb{G}}_{\mathbf{x}}(\mathbf{y}) - k^2\tilde{\mathbb{G}}_{\mathbf{x}}(\mathbf{y}))\cdot\mathbf{v}(y)\,dy \\
&= \int_{(W\backslash\bar{D})\backslash B(\mathbf{x},r)} \mathbf{v}(\mathbf{y})\cdot(\nabla_{\mathbf{y}}\times\nabla_{\mathbf{y}}\times\mathbb{G}(\mathbf{x},\mathbf{y}) - k^2\mathbb{G}(\mathbf{x},\mathbf{y})) \\
&\quad -(\nabla\times\nabla\times\mathbf{v}(\mathbf{y}) - k^2\mathbf{v}(\mathbf{y}))\cdot\mathbb{G}(\mathbf{x},\mathbf{y})\,dy \\
&= \int_{(W\backslash\bar{D})\backslash B(\mathbf{x},r)} \mathbf{v}(\mathbf{y})\cdot(\nabla_{\mathbf{y}}\times\nabla_{\mathbf{y}}\times\mathbb{G}(\mathbf{x},\mathbf{y})) - (\nabla\times\nabla\times\mathbf{v}(\mathbf{y}))\cdot\mathbb{G}(\mathbf{x},\mathbf{y})\,dy \\
&= -\int_{\partial((W\backslash\bar{D})\backslash B(\mathbf{x},r))} \mathbf{n}_{\partial((W\backslash\bar{D})\backslash B(\mathbf{x},r))}\cdot(\mathbf{v}(\mathbf{y})\times[\nabla_{\mathbf{y}}\times\mathbb{G}(\mathbf{x},\mathbf{y})] + [\nabla\times\mathbf{v}(\mathbf{y})]\times\mathbb{G}(\mathbf{x},\mathbf{y}))\,ds(\mathbf{y}).
\end{aligned}
$$

Since $\mathbf{v} = \mathbf{0}$ in the neighborhood of Γ and ∂D, using dyadic identity (B.22), we have

$$
\begin{aligned}
&\int_{W\backslash\bar{D}} (\nabla\times\nabla\times\tilde{\mathbb{G}}_{\mathbf{x}}(\mathbf{y}) - k^2\tilde{\mathbb{G}}_{\mathbf{x}}(\mathbf{y}))\cdot\mathbf{v}(y)\,dy \\
&= -\int_{\partial B(\mathbf{x},r)} \mathbf{n}_{\partial B(\mathbf{x},r)}\cdot(\mathbf{v}(\mathbf{y})\times[\nabla_{\mathbf{y}}\times\mathbb{G}(\mathbf{x},\mathbf{y})]) + \mathbf{n}_{\partial B(\mathbf{x},r)}\cdot([\nabla\times\mathbf{v}(\mathbf{y})]\times\mathbb{G}(\mathbf{x},\mathbf{y}))\,ds(\mathbf{y})
\end{aligned}
$$

$$= \int_{\partial B(\mathbf{x},r)} \mathbf{v}(\mathbf{y}) \cdot \big(\mathbf{n}_{\partial B(\mathbf{x},r)} \times [\nabla_{\mathbf{y}} \times \mathbb{G}(\mathbf{x},\mathbf{y})]\big) - \big(\mathbf{n}_{\partial B(\mathbf{x},r)} \times [\nabla \times \mathbf{v}(\mathbf{y})]\big) \cdot \mathbb{G}(\mathbf{x},\mathbf{y})\, ds(\mathbf{y}),$$

where $\mathbf{n}_{\partial B(\mathbf{x},r)}$ is the unit inward normal to $B(\mathbf{x},r)$. Notice that

$$\begin{aligned} & \int_{W\backslash\bar{D}} (\nabla_{\mathbf{y}} \times \nabla_{\mathbf{y}} \times \mathbb{G}(\mathbf{x},\mathbf{y}) - k^2\mathbb{G}(\mathbf{x},\mathbf{y})) \cdot \mathbf{v}(y)\, dy \\ = & \int_{W\backslash\bar{D}} (\mathbb{I}\delta(\mathbf{x}-\mathbf{y})) \cdot \mathbf{v}(\mathbf{y})\, dy = \mathbf{v}(\mathbf{x}), \end{aligned}$$

we obtain, for any $\mathbf{v} \in (C_0^\infty(W\backslash\bar{D}))^3$,

$$\begin{aligned} & \int_{W\backslash\bar{D}} \big(\nabla \times \nabla \times \mathbb{G}_{\mathbf{x}}(\mathbf{y}) - k^2\mathbb{G}_{\mathbf{x}}(\mathbf{y})\big) \cdot \mathbf{v}(y)\, dy \\ = & \int_{W\backslash\bar{D}} \Big(\nabla \times \nabla \times \tilde{\mathbb{G}}_{\mathbf{x}}(\mathbf{y}) + \nabla_{\mathbf{y}} \times \nabla_{\mathbf{y}} \times \mathbb{G}(\mathbf{x},\mathbf{y}) - k^2(\tilde{\mathbb{G}}_{\mathbf{x}}(\mathbf{y}) + \mathbb{G}(\mathbf{x},\mathbf{y}))\Big) \cdot \mathbf{v}(\mathbf{y})\, dy \\ = & \int_{\partial B(\mathbf{x},r)} \mathbf{v}(\mathbf{y}) \cdot \big(\mathbf{n}_{\partial B(\mathbf{x},r)} \times [\nabla_{\mathbf{y}} \times \mathbb{G}(\mathbf{x},\mathbf{y})]\big) \\ & - \big(\mathbf{n}_{\partial B(\mathbf{x},r)} \times [\nabla \times \mathbf{v}(\mathbf{y})]\big) \cdot \mathbb{G}(\mathbf{x},\mathbf{y})\, ds(\mathbf{y}) + \mathbf{v}(\mathbf{x}). \end{aligned}$$

The above relation still holds when $\mathbf{v}$ is replaced by $\mathbf{u}^s$, the solution to Maxwell's equation. Because from the proof of Theorem 9.2 in [48], we have that, for $\mathbf{u}^s \in H_{loc}(\mathrm{curl}, W\backslash\bar{D})$ and a compact subset of $W\backslash\bar{D}$ including $B(\mathbf{x},2r)$ in its interior, its components are smooth functions when $\mathbf{x}$ is away from the boundary of that compact subset. By Theorem D.0.9, they are analytic and thus $\mathbf{u}^s \in (C^\infty(B(\mathbf{x},2r)))^3$.

Define a cut-off function $\chi \in C_0^\infty(B(\mathbf{x},2r))$ such that $\chi = 1$ on $\overline{B(\mathbf{x},r)}$, then $\chi\mathbf{u}^s \in (C_0^\infty(W\backslash\bar{D}))^3$ and since $\mathrm{supp}(\mathbb{G}_{\mathbf{x}}(\mathbf{y})) \subset \overline{B(\mathbf{x},r)}$,

$$\begin{aligned} & \int_{W\backslash\bar{D}} \big(\nabla \times \nabla \times \mathbb{G}_{\mathbf{x}}(\mathbf{y}) - k^2\mathbb{G}_{\mathbf{x}}(\mathbf{y})\big) \cdot \mathbf{u}^s(\mathbf{y})\, dy \\ = & \int_{W\backslash\bar{D}} \big(\nabla \times \nabla \times \mathbb{G}_{\mathbf{x}}(\mathbf{y}) - k^2\mathbb{G}_{\mathbf{x}}(\mathbf{y})\big) \cdot \chi\mathbf{u}^s(\mathbf{y})\, dy \\ = & \int_{W\backslash\bar{D}} \mathbb{G}_{\mathbf{x}}(\mathbf{y}) \cdot \big(\nabla \times \nabla \times (\chi\mathbf{u}^s(\mathbf{y})) - k^2(\chi\mathbf{u}^s(\mathbf{y}))\big)\, dy \\ = & \int_{B(\mathbf{x},r)} \mathbb{G}_{\mathbf{x}}(\mathbf{y}) \cdot \big(\nabla \times \nabla \times (\chi\mathbf{u}^s(\mathbf{y})) - k^2(\chi\mathbf{u}^s(\mathbf{y}))\big)\, dy \\ = & \ 0. \end{aligned}$$

Therefore, we obtain

$$0 = \int_{\partial B(\mathbf{x},r)} \mathbf{u}^s(\mathbf{y}) \cdot \big(\mathbf{n}_{\partial B(\mathbf{x},r)} \times [\nabla_{\mathbf{y}} \times \mathbb{G}(\mathbf{x},\mathbf{y})]\big)$$

$$- \left(\mathbf{n}_{\partial B(\mathbf{x},r)} \times [\nabla \times \mathbf{u}^s(\mathbf{y})]\right) \cdot \mathbb{G}(\mathbf{x},\mathbf{y})\, ds(\mathbf{y}) + \mathbf{u}^s(\mathbf{x}). \tag{3.45}$$

For the regions excluding the singularity, we consider a sub-domain $\Omega_B = \Omega\backslash\overline{B(\mathbf{x},r)} \subset \Omega$. Note that $\nabla_{\mathbf{y}} \times \nabla_{\mathbf{y}} \times \mathbb{G}(\mathbf{x},\mathbf{y}) - k^2\mathbb{G}(\mathbf{x},\mathbf{y}) = 0$ in Ω_B since $\overline{B(\mathbf{x},r)}$ is excluded and $\nabla \times \nabla \times \mathbf{u}^s(\mathbf{y}) - k^2\mathbf{u}^s(\mathbf{y}) = 0$ since $\mathbf{u}^s$ satisfies the Maxwell's equation.

By applying vector-dyadic identity (B.34) on Ω_B, we have

$$\begin{aligned}
0 &= \int_{\Omega_B} \mathbf{u}^s(\mathbf{y}) \cdot \left(\nabla_{\mathbf{y}} \times \nabla_{\mathbf{y}} \times \mathbb{G}(\mathbf{x},\mathbf{y}) - k^2\mathbb{G}(\mathbf{x},\mathbf{y})\right) \\
&\quad - \left(\nabla \times \nabla \times \mathbf{u}^s(\mathbf{y}) - k^2\mathbf{u}^s(\mathbf{y})\right) \cdot \mathbb{G}(\mathbf{x},\mathbf{y})\, dy \\
&= - \int_{\partial\Omega_B} \mathbf{n}_{\partial\Omega_B} \cdot [\mathbf{u}^s(\mathbf{y}) \times (\nabla_{\mathbf{y}} \times \mathbb{G}(\mathbf{x},\mathbf{y})) + (\nabla \times \mathbf{u}^s(\mathbf{y})) \times \mathbb{G}(\mathbf{x},\mathbf{y})]\; ds(\mathbf{y}) \\
&= - \Bigg[\int_{\Gamma} \mathbf{n}_{\Gamma} \cdot [\mathbf{u}^s(\mathbf{y}) \times (\nabla_{\mathbf{y}} \times \mathbb{G}(\mathbf{x},\mathbf{y})) + (\nabla \times \mathbf{u}^s(\mathbf{y})) \times \mathbb{G}(\mathbf{x},\mathbf{y})]\; ds(\mathbf{y}) \\
&\quad + \sum_{j=s,t} \int_{\Sigma_j} \mathbf{n}_{\Sigma_j} \cdot [\mathbf{u}^s(\mathbf{y}) \times (\nabla_{\mathbf{y}} \times \mathbb{G}(\mathbf{x},\mathbf{y})) + (\nabla \times \mathbf{u}^s(\mathbf{y})) \times \mathbb{G}(\mathbf{x},\mathbf{y})]\; ds(\mathbf{y}) \\
&\quad + \int_{\partial B(\mathbf{x},r)} \mathbf{n}_{\partial B(\mathbf{x},r)} \cdot [\mathbf{u}^s(\mathbf{y}) \times (\nabla_{\mathbf{y}} \times \mathbb{G}(\mathbf{x},\mathbf{y})) + (\nabla \times \mathbf{u}^s(\mathbf{y})) \times \mathbb{G}(\mathbf{x},\mathbf{y})]\; ds(\mathbf{y}) \\
&\quad + \int_{\partial D} \mathbf{n}_{D} \cdot [\mathbf{u}^s(\mathbf{y}) \times (\nabla_{\mathbf{y}} \times \mathbb{G}(\mathbf{x},\mathbf{y})) + (\nabla \times \mathbf{u}^s(\mathbf{y})) \times \mathbb{G}(\mathbf{x},\mathbf{y})]\; ds(\mathbf{y})\Bigg] \\
&\quad = -(\mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}).
\end{aligned}$$

Investigating term by term,

- Using vector-dyadic identity (B.22), we have

  $$\mathrm{I} = \int_{\Gamma} (\mathbf{n}_{\Gamma} \times \mathbf{u}^s(\mathbf{y})) \cdot (\nabla_{\mathbf{y}} \times \mathbb{G}(\mathbf{x},\mathbf{y})) - (\nabla \times \mathbf{u}^s(\mathbf{y})) \cdot (\mathbf{n}_{\Gamma} \times \mathbb{G}(\mathbf{x},\mathbf{y}))\; ds(\mathbf{y}).$$

  Since $\mathbf{n}_{\Gamma} \times \mathbf{u}^s = 0$ and $\mathbf{n}_{\Gamma} \times \mathbb{G}(\mathbf{x},\mathbf{y}) = 0$, we get $\mathrm{I} = 0$.

- For the integral on Σ_t, using vector-dyadic identity (B.22) again, we have

  $$\mathrm{II} \;=\; \int_{\Sigma_t} (\mathbf{n}_{\Sigma} \times \mathbf{u}^s(\mathbf{y})) \cdot (\nabla_{\mathbf{y}} \times \mathbb{G}(\mathbf{x},\mathbf{y})) + (\mathbf{n}_{\Sigma} \times (\nabla \times \mathbf{u}^s(\mathbf{y}))) \cdot \mathbb{G}(\mathbf{x},\mathbf{y})\, ds(\mathbf{y}).$$

  Without the loss of generality, assume $z > z' = t$. Using series expansion of functions (3.16) and (3.17) on cross section Σ_t and explicit forms of $\mathbb{G} = \mathbb{G}_e$ (3.41), we have, on Σ_t

  $$(\mathbf{n}_{\Sigma} \times \mathbf{u}^s(\mathbf{y})) \cdot (\nabla_{\mathbf{y}} \times \mathbb{G}(\mathbf{x},\mathbf{y}))$$

$$
\begin{aligned}
&= \left(\sum_m a_m \frac{1}{\lambda_m^2}\begin{pmatrix}\nabla_\Sigma u_m\\ 0\end{pmatrix}^T + \sum_n b_n \frac{1}{\mu_n^2}\begin{pmatrix}\vec{\nabla}_\Sigma \times v_n\\ 0\end{pmatrix}^T\right)(\mathbf{y})\\
&\quad \cdot\left(\sum_m \left[c_m M_m^+(\mathbf{x})(\nabla\times M_m^-(\mathbf{y}))^T\right] + \sum_n \left[d_n N_n^+(\mathbf{x})(\nabla\times N_n^-(\mathbf{y}))^T\right]\right)\\
&= \left(\sum_m a_m \frac{1}{\lambda_m^2}\begin{pmatrix}\nabla_\Sigma u_m\\ 0\end{pmatrix}^T + \sum_n b_n \frac{1}{\mu_n^2}\begin{pmatrix}\vec{\nabla}_\Sigma \times v_n\\ 0\end{pmatrix}^T\right)(\mathbf{y})\\
&\quad \cdot\left(\sum_m c_m \left[\begin{pmatrix}\vec{\nabla}_\Sigma \times u_m\\ 0\end{pmatrix} e^{ih_m z}\right](\mathbf{x})\left[\begin{pmatrix}\nabla_\Sigma u_m\\ 0\end{pmatrix}^T ih_m + \begin{pmatrix}0\\0\\u_m\lambda_m^2\end{pmatrix}^T\right](\mathbf{y})\right.\\
&\quad \left. + \sum_n d_n \left[\frac{1}{k}\left(\begin{pmatrix}\nabla_\Sigma v_n\\ 0\end{pmatrix} ig_n + \begin{pmatrix}0\\0\\v_n\mu_n^2\end{pmatrix}\right) e^{ig_n z}\right](\mathbf{x})\left[k\begin{pmatrix}\vec{\nabla}_\Sigma \times v_n\\ 0\end{pmatrix}^T\right](\mathbf{y})\right)\\
&= \sum_m c_m \left[\begin{pmatrix}\vec{\nabla}_\Sigma \times u_m\\ 0\end{pmatrix} e^{ih_m z}\right](\mathbf{x}) a_m ih_m + \sum_n d_n \left[\begin{pmatrix}\nabla_\Sigma v_n\\ 0\end{pmatrix} ig_n e^{ig_n z}\right](\mathbf{x}) b_n.
\end{aligned}
$$

On the other hand, on Σ_t

$$
\begin{aligned}
&(\mathbf{n}_\Sigma \times (\nabla\times \mathbf{u}^s(\mathbf{y})))\cdot \mathbb{G}(\mathbf{x},\mathbf{y})\\
&= \left(\sum_m -a_m \frac{ih_m}{\lambda_m^2}\begin{pmatrix}\vec{\nabla}_\Sigma \times u_m\\ 0\end{pmatrix}^T + \sum_n -b_n \frac{1}{\mu_n^2}\frac{k^2}{ig_n}\begin{pmatrix}\nabla_\Sigma v_n\\ 0\end{pmatrix}^T\right)(\mathbf{y})\\
&\quad \cdot\left(\sum_m \left[c_m M_m^+(\mathbf{x}) M_m^-(\mathbf{y})^T\right] + \sum_n \left[d_n N_n^+(\mathbf{x}) N_n^-(\mathbf{y})^T\right]\right)\\
&= \left(\sum_m -a_m \frac{ih_m}{\lambda_m^2}\begin{pmatrix}\vec{\nabla}_\Sigma \times u_m\\ 0\end{pmatrix}^T + \sum_n -b_n \frac{1}{\mu_n^2}\frac{k^2}{ig_n}\begin{pmatrix}\nabla_\Sigma v_n\\ 0\end{pmatrix}^T\right)(\mathbf{y})\\
&\quad \cdot\left(\sum_m c_m \left[\begin{pmatrix}\vec{\nabla}_\Sigma \times u_m\\ 0\end{pmatrix} e^{ih_m z}\right](\mathbf{x})\left[\begin{pmatrix}\vec{\nabla}_\Sigma \times u_m\\ 0\end{pmatrix}^T\right](\mathbf{y})\right.\\
&\quad + \sum_n d_n \left[\frac{1}{k}\left(\begin{pmatrix}\nabla_\Sigma v_n\\ 0\end{pmatrix} ig_n + \begin{pmatrix}0\\0\\v_n\mu_n^2\end{pmatrix}\right) e^{ig_n z}\right](\mathbf{x})\\
&\quad \left.\left[\frac{1}{k}\left(\begin{pmatrix}\nabla_\Sigma v_n\\ 0\end{pmatrix} ig_n + \begin{pmatrix}0\\0\\v_n\mu_n^2\end{pmatrix}\right)\right](\mathbf{y})\right)\\
&= \sum_m c_m \left[\begin{pmatrix}\vec{\nabla}_\Sigma \times u_m\\ 0\end{pmatrix} e^{ih_m z}\right](\mathbf{x})(-a_m ih_m) + \sum_n d_n \left[\begin{pmatrix}\nabla_\Sigma v_n\\ 0\end{pmatrix} ig_n e^{ig_n z}\right](\mathbf{x})(-b_n).
\end{aligned}
$$

Therefore, II $= 0$ on Σ_t for $z > z' = t$. Same results hold on Σ_t for $z < z' = t$ and on Σ_s by similar derivations.

- Using vector-dyadic identity (B.22) and equation (3.45), we have

$$\begin{aligned}\mathrm{III} &= \int_{\partial B(\mathbf{x},r)} -\mathbf{u}^s(\mathbf{y}) \cdot \left(\mathbf{n}_{\partial B(\mathbf{x},r)} \times [\nabla_{\mathbf{y}} \times \mathbb{G}(\mathbf{x},\mathbf{y})]\right) \\ &\quad + \left(\mathbf{n}_{\partial B(\mathbf{x},r)} \times [\nabla \times \mathbf{u}^s(\mathbf{y})]\right) \cdot \mathbb{G}(\mathbf{x},\mathbf{y})\, ds(\mathbf{y}) \\ &= \mathbf{u}^s(\mathbf{x}).\end{aligned}$$

Here $\mathbf{n}_{\partial B(\mathbf{x},r)}$ is the unit inward normal to $B(\mathbf{x},r)$.

Thus, we get $\mathrm{III} = -\mathrm{IV}$, that is

$$\begin{aligned}\mathbf{u}^s(\mathbf{x}) &= -\int_{\partial D} \mathbf{n}_D \cdot (\mathbf{u}^s(\mathbf{y}) \times [\nabla_{\mathbf{y}} \times \mathbb{G}(\mathbf{x},\mathbf{y})] + [\nabla \times \mathbf{u}^s(\mathbf{y})] \times \mathbb{G}(\mathbf{x},\mathbf{y}))\, ds(\mathbf{y}) \\ &= \int_{\partial D} \mathbf{u}^s(\mathbf{y}) \cdot (\mathbf{n}_D \times [\nabla_{\mathbf{y}} \times \mathbb{G}(\mathbf{x},\mathbf{y})]) - (\mathbf{n}_D \times [\nabla \times \mathbf{u}^s(\mathbf{y})]) \cdot \mathbb{G}(\mathbf{x},\mathbf{y})\, ds(\mathbf{y}).\end{aligned}$$

Here $\mathbf{n}_D$ is the unit inward normal to D. This completes the proof. $\square$

Note 3.3.4 *If $\mathbf{n}_D$ represents the unit outward normal, the representation formula becomes*

$$\begin{aligned}\mathbf{u}^s(\mathbf{x}) &= \int_{\partial D} (-\mathbf{u}^s(\mathbf{y}) \times \mathbf{n}_D) \cdot [\nabla_{\mathbf{y}} \times \mathbb{G}(\mathbf{x},\mathbf{y})] + (\mathbf{n}_D \times [\nabla \times \mathbf{u}^s(\mathbf{y})]) \cdot \mathbb{G}(\mathbf{x},\mathbf{y})\, ds(\mathbf{y}) \\ &= \int_{\partial D} [\nabla_{\mathbf{y}} \times \mathbb{G}(\mathbf{x},\mathbf{y})]^T \cdot (\mathbf{n}_D \times \mathbf{u}^s(\mathbf{y})) + \mathbb{G}(\mathbf{x},\mathbf{y})^T \cdot (\mathbf{n}_D \times [\nabla \times \mathbf{u}^s(\mathbf{y})])\, ds(\mathbf{y}).\end{aligned}$$

Lemma 3.3.3 *(Reciprocity) Denote by $\mathbf{u}^s_{\mathbf{z}}(\mathbf{x})$ the solution of the forward problem in Lemma 3.3.2 with $\mathbf{F} = -\mathbf{n}_D \times (\mathbb{G}(\mathbf{x},\mathbf{z})\mathbf{p})$, i.e. the incident wave is due to a point source at $\mathbf{z} \in W\backslash\bar{D}$ with polarization $\mathbf{p}$ ($|\mathbf{p}| = 1$). Then for all $\mathbf{x},\mathbf{z} \in W\backslash\bar{D}$, we have*

$$\mathbf{u}^s_{\mathbf{z}}(\mathbf{x}) = \mathbf{u}^s_{\mathbf{x}}(\mathbf{z}).$$

Proof: First, by using the representation formula in Lemma 3.3.2, we have

$$\begin{aligned}\mathbf{u}^s_{\mathbf{z}}(\mathbf{x}) &= \int_{\partial D} \mathbf{u}^s_{\mathbf{z}}(\mathbf{y}) \cdot (\mathbf{n}_D \times \nabla_{\mathbf{y}} \times \mathbb{G}(\mathbf{x},\mathbf{y})\mathbf{p}) \\ &\quad - (\mathbf{n}_D \times \nabla \times \mathbf{u}^s_{\mathbf{z}}(\mathbf{y})) \cdot \mathbb{G}(\mathbf{x},\mathbf{y})\mathbf{p}\, ds(\mathbf{y}). \qquad (3.46)\end{aligned}$$

Interchanging the role of $\mathbf{x}$ and $\mathbf{z}$, i.e. $\mathbf{F} = -\mathbf{n}_D \times (\mathbb{G}(\mathbf{z},\mathbf{x})\mathbf{p})$ (with point source at $\mathbf{x} \in W\backslash\bar{D}$), gives

$$\mathbf{u}^s_{\mathbf{x}}(\mathbf{z}) = \int_{\partial D} \mathbf{U}_{\mathbf{x}}(\mathbf{y}) \cdot (\mathbf{n}_D \times \nabla_{\mathbf{y}} \times \mathbb{G}(\mathbf{z},\mathbf{y})\mathbf{p})$$

$$- (\mathbf{n}_D \times \nabla \times \mathbf{u}^s_{\mathbf{x}}(\mathbf{y})) \cdot \mathbb{G}(\mathbf{z}, \mathbf{y})\mathbf{p}\, ds(\mathbf{y}). \tag{3.47}$$

Besides, by Green's second identity, we have, for $\mathbf{x}, \mathbf{z} \notin \bar{D}$,

$$\begin{aligned}
&\int_{\partial D} \mathbb{G}(\mathbf{x}, \mathbf{y})\mathbf{p} \cdot (\mathbf{n}_D \times \nabla_{\mathbf{y}} \times \mathbb{G}(\mathbf{z}, \mathbf{y})\mathbf{p}) - (\mathbf{n}_D \times \nabla_{\mathbf{y}} \times \mathbb{G}(\mathbf{x}, \mathbf{y})\mathbf{p}) \cdot \mathbb{G}(\mathbf{z}, \mathbf{y})\mathbf{p}\, ds(\mathbf{y}) \\
= & - \int_{\partial D} \mathbf{n}_D \cdot [\mathbb{G}(\mathbf{x}, \mathbf{y})\mathbf{p} \times (\nabla_{\mathbf{y}} \times \mathbb{G}(\mathbf{z}, \mathbf{y})\mathbf{p}) - \mathbb{G}(\mathbf{z}, \mathbf{y})\mathbf{p} \times (\nabla_{\mathbf{y}} \times \mathbb{G}(\mathbf{x}, \mathbf{y})\mathbf{p})]\, ds(\mathbf{y}) \\
= & \int_D \mathbb{G}(\mathbf{x}, \mathbf{y})\mathbf{p} \cdot (\nabla_{\mathbf{y}} \times \nabla_{\mathbf{y}} \times \mathbb{G}(\mathbf{z}, \mathbf{y})\mathbf{p}) - \mathbb{G}(\mathbf{z}, \mathbf{y})\mathbf{p} \cdot (\nabla_{\mathbf{y}} \times \nabla_{\mathbf{y}} \times \mathbb{G}(\mathbf{x}, \mathbf{y})\mathbf{p})\, dy \\
= & \int_D \mathbb{G}(\mathbf{x}, \mathbf{y})\mathbf{p} \cdot \left(\nabla_{\mathbf{y}} \times \nabla_{\mathbf{y}} \times \mathbb{G}(\mathbf{z}, \mathbf{y})\mathbf{p} - k^2\mathbb{G}(\mathbf{z}, \mathbf{y})\mathbf{p} + k^2\mathbb{G}(\mathbf{z}, \mathbf{y})\mathbf{p}\right) \\
& -\mathbb{G}(\mathbf{z}, \mathbf{y})\mathbf{p} \cdot \left(\nabla_{\mathbf{y}} \times \nabla_{\mathbf{y}} \times \mathbb{G}(\mathbf{x}, \mathbf{y})\mathbf{p} - k^2\mathbb{G}(\mathbf{x}, \mathbf{y})\mathbf{p} + k^2\mathbb{G}(\mathbf{x}, \mathbf{y})\mathbf{p}\right)\, dy \\
= & \ 0
\end{aligned}$$

where the last equality follows from the fact that $\mathbf{y} \in D$ and $\mathbf{x}, \mathbf{z} \notin D$. Thus, we obtain

$$\begin{aligned}
0 = & \int_{\partial D} \mathbb{G}(\mathbf{x}, \mathbf{y})\mathbf{p} \cdot (\mathbf{n}_D \times \nabla_{\mathbf{y}} \times \mathbb{G}(\mathbf{z}, \mathbf{y})\mathbf{p}) \\
& - (\mathbf{n}_D \times \nabla_{\mathbf{y}} \times \mathbb{G}(\mathbf{x}, \mathbf{y})\mathbf{p}) \cdot \mathbb{G}(\mathbf{z}, \mathbf{y})\mathbf{p}\, ds(\mathbf{y}).
\end{aligned} \tag{3.48}$$

Now, integrating instead over $\Omega = W_{(s,t)} \backslash \bar{D}$, and using the similar argument as for evaluation of (II) in Lemma 3.3.2, we can also show that

$$0 = \int_{\partial D} \mathbf{u}^s_{\mathbf{x}}(\mathbf{y}) \cdot (\mathbf{n}_D \times \nabla \times \mathbf{u}^s_{\mathbf{z}}(\mathbf{y})) - (\mathbf{n}_D \times \nabla \times \mathbf{u}^s_{\mathbf{x}}(\mathbf{y})) \cdot \mathbf{u}^s_{\mathbf{z}}(\mathbf{y})\, ds(\mathbf{y}). \tag{3.49}$$

Note that instead of having $\nabla \times \nabla \times \mathbb{G}\mathbf{p} - k^2\mathbb{G}\mathbf{p} = \mathbf{p}\delta$, we use $\nabla \times \nabla \times \mathbf{u}^s - k^2\mathbf{u}^s = \mathbf{0}$ here.

Denote by $\tilde{\mathbf{u}}^s_\beta(\alpha) = \mathbf{u}^s_\beta(\alpha) + \mathbb{G}(\alpha, \beta)\mathbf{p}$. Using the symmetry of the dyadic Green's function $\mathbb{G}(\alpha, \beta)\mathbf{p} = \mathbb{G}(\beta, \alpha)\mathbf{p}$, we add (3.47) and (3.48) and obtain, for $\mathbf{x}, \mathbf{z} \notin \partial D$,

$$\begin{aligned}
\mathbf{u}^s_{\mathbf{x}}(\mathbf{z}) = & \int_{\partial D} (\mathbf{u}^s_{\mathbf{x}}(\mathbf{y}) + \mathbb{G}(\mathbf{x}, \mathbf{y})\mathbf{p}) \cdot (\mathbf{n}_D \times \nabla_{\mathbf{y}} \times \mathbb{G}(\mathbf{z}, \mathbf{y})\mathbf{p}) \\
& - [\mathbf{n}_D \times \nabla_{\mathbf{y}} \times (\mathbf{u}^s_{\mathbf{x}}(\mathbf{y}) + \mathbb{G}(\mathbf{x}, \mathbf{y})\mathbf{p})] \cdot \mathbb{G}(\mathbf{z}, \mathbf{y})\mathbf{p}\, ds(\mathbf{y}) \\
= & \int_{\partial D} (\mathbf{u}^s_{\mathbf{x}}(\mathbf{y}) + \mathbb{G}(\mathbf{y}, \mathbf{x})\mathbf{p}) \cdot (\mathbf{n}_D \times \nabla_{\mathbf{y}} \times \mathbb{G}(\mathbf{z}, \mathbf{y})\mathbf{p}) \\
& - [\mathbf{n}_D \times \nabla_{\mathbf{y}} \times (\mathbf{u}^s_{\mathbf{x}}(\mathbf{y}) + \mathbb{G}(\mathbf{y}, \mathbf{x})\mathbf{p})] \cdot \mathbb{G}(\mathbf{z}, \mathbf{y})\mathbf{p}\, ds(\mathbf{y}) \\
= & \int_{\partial D} \tilde{\mathbf{u}}^s_{\mathbf{x}}(\mathbf{y}) \cdot (\mathbf{n}_D \times \nabla_{\mathbf{y}} \times \mathbb{G}(\mathbf{z}, \mathbf{y})\mathbf{p})
\end{aligned}$$

$$-(\mathbf{n}_D \times \nabla \times \tilde{\mathbf{u}}^s_{\mathbf{x}}(\mathbf{y})) \cdot \mathbb{G}(\mathbf{z},\mathbf{y})\mathbf{p}\, ds(y). \tag{3.50}$$

Similarly, subtracting (3.49) from (3.46) will give us

$$\mathbf{u}^s_{\mathbf{z}}(\mathbf{x}) = \int_{\partial D} \mathbf{u}^s_{\mathbf{z}}(\mathbf{y}) \cdot (\mathbf{n}_D \times \nabla \times \tilde{\mathbf{u}}^s_{\mathbf{x}}(\mathbf{y})) - (\mathbf{n}_D \times \nabla \times \mathbf{u}^s_{\mathbf{z}}(\mathbf{y})) \cdot \tilde{\mathbf{u}}^s_{\mathbf{x}}(\mathbf{y})\, ds(\mathbf{y}). \tag{3.51}$$

Using these expressions, subtracting (3.51) from (3.50), we obtain

$$\begin{aligned}
& \mathbf{u}^s_{\mathbf{x}}(\mathbf{z}) - \mathbf{u}^s_{\mathbf{z}}(\mathbf{x}) \\
= & \int_{\partial D} \tilde{\mathbf{u}}^s_{\mathbf{x}}(\mathbf{y}) \cdot (\mathbf{n}_D \times \nabla \times \tilde{\mathbf{u}}^s_{\mathbf{z}}(\mathbf{y})) - (\mathbf{n}_D \times \nabla \times \tilde{\mathbf{u}}^s_{\mathbf{x}}(\mathbf{y})) \cdot \tilde{\mathbf{u}}^s_{\mathbf{z}}(\mathbf{y})\, ds(\mathbf{y}) \\
= & \int_{\partial D} (\tilde{\mathbf{u}}^s_{\mathbf{x}}(\mathbf{y}) \times \mathbf{n}_D) \cdot (\nabla \times \tilde{\mathbf{u}}^s_{\mathbf{z}}(\mathbf{y})) - (\nabla \times \tilde{\mathbf{u}}^s_{\mathbf{x}}(\mathbf{y})) \cdot (\tilde{\mathbf{u}}^s_{\mathbf{z}}(\mathbf{y}) \times \mathbf{n}_D)\; ds(\mathbf{y}).
\end{aligned}$$

Since by assumption,

$$\tilde{\mathbf{u}}^s_{\mathbf{x}}(\mathbf{y}) \times \mathbf{n}_D = [\mathbb{G}(\mathbf{y},\mathbf{x})\mathbf{p} + \mathbf{u}^s_{\mathbf{x}}(\mathbf{y})] \times \mathbf{n}_D = 0,$$

and

$$\tilde{\mathbf{u}}^s_{\mathbf{z}}(\mathbf{y}) \times \mathbf{n}_D = [\mathbb{G}(\mathbf{y},\mathbf{z})\mathbf{p} + \mathbf{u}^s_{\mathbf{z}}(\mathbf{y})] \times \mathbf{n}_D = 0,$$

we can conclude that $\mathbf{u}^s_{\mathbf{x}}(\mathbf{z}) - \mathbf{u}^s_{\mathbf{z}}(\mathbf{x}) = 0$. $\square$

Next, we shall prove the following uniqueness theorem:

Theorem 3.3.1 *Assume that D_1 and D_2 are two perfect electric conducting scatterers in the waveguide away from its boundary. For a fixed wave number k, if the tangential components of the scattered fields $\mathbf{u}^s_1(\cdot,\mathbf{y})$ and $\mathbf{u}^s_2(\cdot,\mathbf{y})$ for scatterers D_1 and D_2 respectively coincide on a cross section Σ for all incident fields $\mathbb{G}(\cdot,\mathbf{y})\mathbf{q}$ with $\mathbf{y} \in \Sigma$ and all polarizations $\mathbf{q}$, then $D_1 = D_2$.*

Note 3.3.5 *When the measurement is in the far field, this theorem is not true because the evanescent waves will die out before reaching infinity, leaving only a finite number of propagating modes (see [7] for a discussion in a 2D acoustic waveguide).*

Proof: The proof is a modification of the proof of Theorem 5.6 in [19]. Denote $\mathfrak{D} = W\backslash(D_1 \cup D_2)$, suppose $\mathbf{n}_\Sigma \times \mathbf{u}^s_1(\cdot,\mathbf{y}) = \mathbf{n}_\Sigma \times \mathbf{u}^s_2(\cdot,\mathbf{y})$ for all $\mathbf{y} \in \Sigma$ and polarization

$\mathbf{q}$. Let $\mathbf{w}(\mathbf{x},\mathbf{y}) = \mathbf{u}_1^s(\mathbf{x},\mathbf{y}) - \mathbf{u}_2^s(\mathbf{x},\mathbf{y})$, then $\mathbf{n}_\Sigma \times \mathbf{w} = 0$ on Σ, due to the uniqueness of solutions in the blocked waveguide problem (Lemma 3.1.3), we have $\mathbf{w} = 0$ on the left of Σ, by the unique continuation principle (Theorem D.0.8), $\mathbf{w}(\mathbf{x},\mathbf{y}) = 0$ for every $\mathbf{x} \in \mathfrak{D}, y \in \Sigma \subset \mathfrak{D}$ which means $\mathbf{u}_1^s(\mathbf{x},\mathbf{y}) = \mathbf{u}_2^s(\mathbf{x},\mathbf{y})$ and so is $\mathbf{u}_1^s(\mathbf{y},\mathbf{x}) = \mathbf{u}_2^s(\mathbf{y},\mathbf{x})$ by reciprocity relation (Lemma 3.3.3).

Now assume $D_1 \neq D_2$. Then, without the loss of generality, there exists $\mathbf{x}^* \in \mathfrak{D}$ such that $\mathbf{x}^* \in \partial D_1$ and $\mathbf{x}^* \notin \bar{D}_2$. Let

$$\mathbf{z}_n = \mathbf{x}^* + \frac{1}{n}\mathbf{n}(\mathbf{x}^*) \in \mathfrak{D}, \qquad n = 1, 2, \ldots$$

for n big enough. Here $\mathbf{n}(\mathbf{x}^*)$ is the unit outward normal to ∂D_1 at $\mathbf{x}^*$.

Imagine $\mathbf{z}_n$ as a point source on a cross section Σ_n, then the scattered field $\mathbf{u}_{1,n}^s(\mathbf{x},\mathbf{z}_n) = \mathbf{u}_{2,n}^s(\mathbf{x},\mathbf{z}_n)$ for every $\mathbf{x} \in \mathfrak{D}$. In particular, $\mathbf{u}_{1,n}^s(\mathbf{x}^*,\mathbf{z}_n) = \mathbf{u}_{2,n}^s(\mathbf{x}^*,\mathbf{z}_n)$.

Let $\mathbf{w}_n = \mathbf{u}_{2,n}^s(\mathbf{x},\mathbf{z}_n)$, replace $\mathbf{x}$ by $\mathbf{x}^*$, then

$$\lim_{n\to\infty} \mathbf{n}_\Sigma \times \mathbf{w}_n(\mathbf{x}^*,\mathbf{z}_n) = \lim_{n\to\infty} \mathbf{n}_\Sigma \times \mathbf{w}_n(\mathbf{z}_n,\mathbf{x}^*) = \mathbf{n}_\Sigma \times \mathbf{w}_n(\mathbf{x}^*,\mathbf{x}^*),$$

which is bounded due to the well-posedness of the forward problem for D_2 with point source at $\mathbf{x}^* \notin \bar{D}_2$.

On the other hand, we have that $\mathbf{w}_n = \mathbf{u}_{1,n}^s(\mathbf{x},\mathbf{z}_n)$, replace $\mathbf{x}$ by $\mathbf{x}^* \in \partial D_1$, then

$$\lim_{n\to\infty} \mathbf{n}_\Sigma \times \mathbf{w}_n(\mathbf{x}^*,\mathbf{z}_n) = \lim_{n\to\infty} -\mathbf{n}_\Sigma \times \mathbb{G}(\mathbf{x}^*,\mathbf{z}_n)\mathbf{q} = \infty.$$

Clearly, this contradicts with $\mathbf{u}_{1,n}^s(\mathbf{x}^*,\mathbf{z}_n) = \mathbf{u}_{2,n}^s(\mathbf{x}^*,\mathbf{z}_n)$ and therefore $D_1 = D_2$. $\square$

3.3.3 The Near Field Operator and its Factorization

First, assume that k^2 is not a Maxwell eigenvalue in D so that the well-posedness holds for the forward problem in D, that is $\nabla\times\nabla\times\mathbf{U} - k^2\mathbf{U} = \mathbf{0}$ in D with boundary data $\mathbf{n}_D \times \mathbf{U}|_{\partial D}$ on ∂D.

Next we introduce the near field operator:

Definition 3.3.1 *For $\mathbf{h} \in L_T^2(\Sigma)$, define the operator $N : L_T^2(\Sigma) \mapsto L_T^2(\Sigma)$ by*

$$N(\mathbf{h})(\mathbf{x}) := \int_\Sigma \mathbf{n}_\Sigma(\mathbf{x}) \times \mathbf{u}^s(\mathbf{x},\mathbf{y},\mathbf{h}(\mathbf{y}))\, ds(\mathbf{y}),$$

where $\mathbf{u}^s$ *represents the scattered field in the presence of* D*. It is called the Near Field Operator (NFO).*

With this definition, we can further introduce the Near Field Equation (NFE): Consider a point source $\mathbf{y}$ on a cross section Σ away from its boundary $\partial\Sigma$ with measurement at the same location. We seek a function $\mathbf{g} \in L^2_T(\Sigma)$ such that for $\forall \mathbf{x} \in \Upsilon$,

$$N(\mathbf{g})(\mathbf{x}) = \mathbf{n}_\Sigma(\mathbf{x}) \times \int_\Sigma \mathbf{u}^s(\mathbf{x}, \mathbf{y}, \mathbf{g}(\mathbf{y}))\, ds(\mathbf{y}) = \mathbf{n}_\Sigma(\mathbf{x}) \times \mathbb{G}(\mathbf{x}, \mathbf{z})\mathbf{q}|_\Sigma. \tag{3.52}$$

Here

- Σ represents a surface where incident fields are generated (location of point sources) and Υ represents a surface where the receivers are located. For simplicity, as is usual with the LSM, we choose $\Upsilon = \Sigma$.
- $\mathbf{u}^s(\mathbf{x}, \mathbf{y}, \mathbf{p})$ is the scattered field due to the incident field generated by a point source at $\mathbf{y}$ with polarization $\mathbf{p}$ in the presence of D. Moreover, it is a linear function of $\mathbf{p}$. So if $\mathbf{g} = g_1\hat{\mathbf{x}} + g_2\hat{\mathbf{y}}$, then $\mathbf{u}^s(\mathbf{x}, \mathbf{y}, \mathbf{g}) = \mathbf{u}^s(\mathbf{x}, \mathbf{y}, \hat{\mathbf{x}})g_1 + \mathbf{u}^s(\mathbf{x}, \mathbf{y}, \hat{\mathbf{y}})g_2$.
- $\mathbf{z}$ is a sampling source point inside and in the vicinity of D.
- $\mathbf{q}$ is an artificial polarization with $|\mathbf{q}| = 1$ associated with the sampling point $\mathbf{z}$.

The integral equation (3.52) is called the Near Field Equation (NFE).

Note that although the trace of an $H(\text{curl})$ function on a cross section Σ is in $\widetilde{H}^{-1/2}(\text{div}, \Sigma)$, we can define the near field operator from $L^2_T(\Sigma)$ to $L^2_T(\Sigma)$ on cross sections away from D due to the following lemma:

Lemma 3.3.4 *Given* $\mathbf{Q} \in \widetilde{H}^{-1/2}(\text{div}, \Sigma_t)$*, the tangential component of solution to the blocked waveguide problem* (3.10) *on any cross section* Σ_l *where* $t < l < \infty$ *is in* $L^2_T(\Sigma)$*.*

Proof: Since $\mathbf{Q} \in \widetilde{H}^{-1/2}(\text{div}, \Sigma_t)$, by Lemma 3.1.2, it has the following representation

$$\mathbf{Q} = \sum_m \alpha_m \nabla_\Sigma u_m + \sum_n \beta_n \vec{\nabla}_\Sigma \times v_n, \tag{3.53}$$

such that

$$\|\mathbf{Q}\|^2_{H^{-1/2}(\text{div},\Sigma)} = \sum_m \lambda_m^3 |\alpha_m|^2 + \sum_n \mu_n |\beta_n|^2 < \infty.$$

From Lemma 3.1.3, for $z > t$, the solution $\mathbf{U}$ to the blocked waveguide problem in $W_{(t,\infty)}$ is given by (3.12), that is,

$$\begin{aligned}
\mathbf{U} &= \sum_m \frac{a_m}{\lambda_m^2}\left[\begin{pmatrix} \vec{\nabla}_\Sigma \times u_m \\ 0 \end{pmatrix} e^{ih_m(z-t)}\right] \\
&\quad + \sum_n -\frac{b_n}{\mu_n^2}\left[\begin{pmatrix} \nabla_\Sigma v_n \\ 0 \end{pmatrix} e^{ig_n(z-t)}\right] - \frac{b_n}{ig_n}\left[\begin{pmatrix} 0 \\ 0 \\ v_n \end{pmatrix} e^{ig_n(z-t)}\right],
\end{aligned}$$

where, using (3.53),

$$\begin{aligned}
a_m &= \langle \mathbf{Q}, \nabla_\Sigma u_m \rangle_{\Sigma_t} = \alpha_m \lambda_m^2, \\
b_n &= \langle \mathbf{Q}, \vec{\nabla}_\Sigma \times v_n \rangle_{\Sigma_t} = \beta_n \mu_n^2.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\mathbf{U} &= \sum_m \alpha_m \left[\begin{pmatrix} \vec{\nabla}_\Sigma \times u_m \\ 0 \end{pmatrix} e^{ih_m(z-t)}\right] \\
&\quad + \sum_n -\beta_n \left[\begin{pmatrix} \nabla_\Sigma v_n \\ 0 \end{pmatrix} e^{ig_n(z-t)}\right] - \frac{\beta_n \mu_n^2}{ig_n}\left[\begin{pmatrix} 0 \\ 0 \\ v_n \end{pmatrix} e^{ig_n(z-t)}\right],
\end{aligned}$$

and on Σ_l, using identities (B.11),(B.12),

$$\begin{aligned}
\mathbf{n}_\Sigma \times \mathbf{U}|_{\Sigma_l} &= \sum_m \alpha_m \left[\begin{pmatrix} \nabla_\Sigma u_m \\ 0 \end{pmatrix} e^{ih_m(l-t)}\right] \\
&\quad + \sum_n \beta_n \left[\begin{pmatrix} \vec{\nabla}_\Sigma \times v_n \\ 0 \end{pmatrix} e^{ig_n(l-t)}\right].
\end{aligned}$$

So,

$$\begin{aligned}
\|\mathbf{n}_\Sigma \times \mathbf{U}|_{\Sigma_l}\|^2_{L^2_T(\Sigma)} &= \sum_m \lambda_m^2 \left|\alpha_m e^{ih_m(l-t)}\right|^2 + \sum_n \mu_n^2 \left|\beta_n e^{ig_n(l-t)}\right|^2 \\
&= \sum_m \lambda_m^3 |\alpha_m|^2 \left|\frac{e^{ih_m(l-t)}}{\sqrt{\lambda_m}}\right|^2 + \sum_n \mu_n |\beta_n|^2 \left|\sqrt{\mu_n} e^{ig_n(l-t)}\right|^2.
\end{aligned}$$

Since there exists finite numbers m^*, n^* such that h_m, g_n are real for $m \le m^*, n \le n^*$ and imaginary for $m > m^*, n > n^*$, respectively, we have

- For $m > m^*$

$$\left|\frac{e^{ih_m(l-t)}}{\sqrt{\lambda_m}}\right|^2 = \frac{e^{-2\sqrt{\lambda_m^2-k^2}(l-t)}}{\lambda_m} = \frac{1}{\lambda_m e^{2\sqrt{\lambda_m^2-k^2}(l-t)}} \to 0 \text{ as } m \to \infty.$$

- For $n > n^*$

$$\left|\sqrt{\mu_n}e^{ig_n(l-t)}\right|^2 = \mu_n e^{-2\sqrt{\mu_n^2-k^2}(l-t)} = \frac{\mu_n}{e^{2\sqrt{\mu_n^2-k^2}(l-t)}} \to 0 \text{ as } n \to \infty.$$

Therefore, by Abel's test, we have

$$\begin{aligned}
\|\mathbf{n}_\Sigma \times \mathbf{U}|_{\Sigma_l}\|^2_{L^2_T(\Sigma)} &= \sum_{m \le m^*} \lambda_m^3 |\alpha_m|^2 \left|\frac{e^{ih_m(l-t)}}{\sqrt{\lambda_m}}\right|^2 + \sum_{n \le n^*} \mu_n^2 |\beta_n|^2 \left|\sqrt{\mu_n}e^{ig_n(l-t)}\right|^2 \\
&\quad + \sum_{m > m^*} \lambda_m^3 |\alpha_m|^2 \left|\frac{e^{ih_m(l-t)}}{\sqrt{\lambda_m}}\right|^2 + \sum_{n > n^*} \mu_n^2 |\beta_n|^2 \left|\sqrt{\mu_n}e^{ig_n(l-t)}\right|^2 \\
&= \sum_{m \le m^*} \lambda_m^3 |\alpha_m|^2 \left|\frac{e^{ih_m(l-t)}}{\sqrt{\lambda_m}}\right|^2 + \sum_{n \le n^*} \mu_n^2 |\beta_n|^2 \left|\sqrt{\mu_n}e^{ig_n(l-t)}\right|^2 \\
&\quad + \sum_{m > m^*} \lambda_m^3 |\alpha_m|^2 \frac{1}{\lambda_m e^{2\sqrt{\lambda_m^2-k^2}(l-t)}} + \sum_{n > n^*} \mu_n^2 |\beta_n|^2 \frac{\mu_n}{e^{2\sqrt{\mu_n^2-k^2}(l-t)}} < \infty,
\end{aligned}$$

which implies $\mathbf{n}_\Sigma \times \mathbf{U}|_{\Sigma_l} \in L^2_T(\Sigma_l)$. $\square$

In order to facilitate the factorization of the operator N, we shall define several more operators:

Definition 3.3.2 *For* $\mathbf{g} \in H^{-1/2}(\mathrm{div}, \partial D)$, *define the operator* $B : H^{-1/2}(\mathrm{div}, \partial D) \mapsto L^2_T(\Sigma)$ *by*

$$B(\mathbf{g})(\mathbf{x}) := \mathbf{n}_\Sigma \times \mathbf{w}|_\Sigma,$$

where $\mathbf{w}$ *satisfies*

$$\begin{cases}
\nabla \times \nabla \times \mathbf{w} - k^2\mathbf{w} = 0 & \textit{in} \quad W \backslash \bar{D}, \\
\mathbf{n}_D \times \mathbf{w}|_{\partial D} = \mathbf{g} & \textit{on} \quad \partial D, \\
\mathbf{n}_\Sigma \times \mathbf{w}|_\Gamma = \mathbf{0} & \textit{on} \quad \Gamma, \\
\mathbf{w} \textit{ satisfies the radiation condition} & \textit{as} \quad z \to \pm\infty.
\end{cases}$$

Definition 3.3.3 *For* $\mathbf{h} \in L^2_T(\Sigma)$*, define the operator* $H : L^2_T(\Sigma) \mapsto H^{-1/2}(\mathrm{div}, \partial D)$ *by*

$$H(\mathbf{h})(\mathbf{x}) := \mathbf{n}_D(\mathbf{x}) \times \left[\int_\Sigma \mathbb{G}(\mathbf{x}, \mathbf{y}) \cdot (\mathbf{n}_\Sigma(\mathbf{y}) \times \mathbf{h}(\mathbf{y}))\, ds(\mathbf{y}) \right]_{\partial D},$$

where "$\cdot$" is understood as matrix-vector multiplication.

Choosing $\mathbf{g} = -H(\mathbf{h})$ when the scatterer D is a perfect electric conductor and by superposition, we have the following relation (also see Figure 3.3):

$$N = -BH.$$

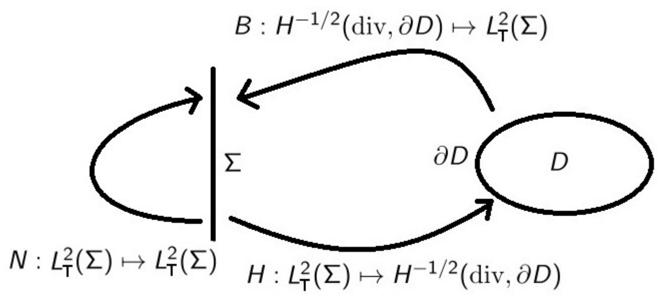

Figure 3.3: Illustration of superposition of the operators B and H.

Definition 3.3.4 *For* $\xi \in H^{-1/2}(\mathrm{curl}, \partial D)$*, let* $\mathbf{v}(\mathbf{x}) = \int_{\partial D} \mathbb{G}(\mathbf{x}, \mathbf{y}) \cdot (\mathbf{n}_D(\mathbf{y}) \times \xi(\mathbf{y}))\, ds(\mathbf{y})$*, define the operator* $F : H^{-1/2}(\mathrm{curl}, \partial D) \mapsto L^2_T(\Sigma)$ *by*

$$\begin{aligned} F(\xi)(\mathbf{x}) &:= \mathbf{n}_\Sigma(\mathbf{x}) \times \mathbf{v}|_\Sigma \\ &= \mathbf{n}_\Sigma(\mathbf{x}) \times \left[\int_{\partial D} \mathbb{G}(\mathbf{x}, \mathbf{y}) \cdot (\mathbf{n}_D(\mathbf{y}) \times \xi(\mathbf{y}))\, ds(\mathbf{y}) \right]_\Sigma, \end{aligned}$$

where we note that $\mathbf{v}$ *satisfies Maxwell's equations in* $W \backslash \bar{D}$ *and the radiation condition.*

Remark 3.3.1 *Note that* $\mathbb{G} = \mathbb{G}_0 + \mathbb{J}$ *(Lemma 3.3.1), then* $\mathbf{v}$ *is the electric field potential operator for the waveguide that consists of the usual electric field potential* *operator in $\mathbb{R}^3$ (see Section 6.3 in [19]) and a smooth perturbation. This can be seen using integration by part on the $\nabla_{\mathbf{y}}\nabla_{\mathbf{y}}$ term in definition of $\mathbb{G}_0$ (3.43).*

Definition 3.3.5 *For $\xi \in H^{-1/2}(\mathrm{curl}, \partial D)$, let $\mathbf{v}$ to be the same as in Definition 3.3.4, define the operator $S : H^{-1/2}(\mathrm{curl}, \partial D) \mapsto H^{-1/2}(\mathrm{div}, \partial D)$ by*

$$\begin{aligned} S(\xi)(\mathbf{x}) &:= \mathbf{n}_D \times \mathbf{v}|_{\partial D} \\ &= \mathbf{n}_D(\mathbf{x}) \times \left[\int_{\partial D} \mathbb{G}(\mathbf{x}, \mathbf{y}) \cdot (\mathbf{n}_D(\mathbf{y}) \times \xi(\mathbf{y}))\, ds(\mathbf{y}) \right]_{\partial D}. \end{aligned}$$

Remark 3.3.2 *Since $\mathbf{v}$ is the electric potential operator for the waveguide, S is just the electric field integral operator for the waveguide on ∂D.*

Again, choose $\mathbf{g} = \mathbf{n}_D \times \mathbf{v}|_{\partial D}$, by superposition, we have the following relation (also see Figure 3.4):

$$F = BS.$$

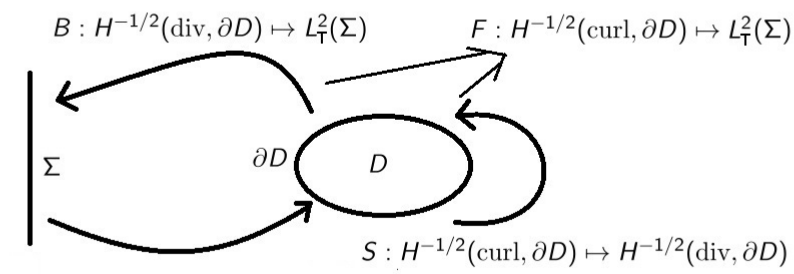

Figure 3.4: Illustration of the superposition of the operators B and S.

Thus, provided S is an isomorphism, we have $B = FS^{-1}$ and N has the following factorization (also see Figure 3.5):

$$N = -BH = -FS^{-1}H.$$

With all the operators defined above, we have the following lemmas regarding to their properties:

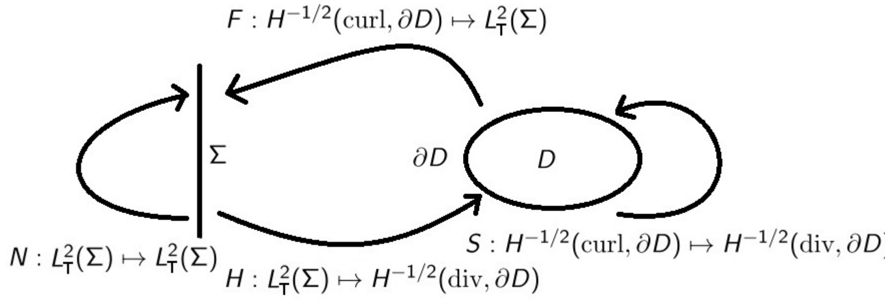

Figure 3.5: Illustration of the factorization of the operator N.

Lemma 3.3.5 *The operator S is an isomorphism if k^2 is not a Maxwell eigenvalue for D and k^2 is such that the forward problem in $W\backslash\bar{D}$ is well posed.*

Proof: First, from Lemma 3.3.1, the dyadic Green's function $\mathbb{G}$ has the decomposition $\mathbb{G} = \mathbb{G}_0 + \mathbb{J}$ where $\mathbb{J}$ is smooth in the neighborhood of D.

Define operator S_0 as operator S with kernel replaced by $\mathbb{G}_0$, that is

$$S_0(\xi)(\mathbf{x}) = \mathbf{n}_D(\mathbf{x}) \times \left[\int_{\partial D} \mathbb{G}_0(\mathbf{x},\mathbf{y}) \cdot (\mathbf{n}_D(\mathbf{y}) \times \xi(\mathbf{y}))\, ds(\mathbf{y})\right]_{\partial D}.$$

This is the standard electric field boundary operator. Then the operator $S - S_0$ given by

$$(S - S_0)(\xi)(\mathbf{x}) = \mathbf{n}_D(\mathbf{x}) \times \left[\int_{\partial D} \mathbb{J}(\mathbf{x},\mathbf{y}) \cdot (\mathbf{n}_D(\mathbf{y}) \times \xi(\mathbf{y}))\, ds(\mathbf{y})\right]_{\partial D}$$

has smooth kernel and is thus continuous (see Theorem 8.7-5 in [44]).

Thus we can use the properties of S_0 to prove the desired properties of S (injectivity and surjectivity) as follows:

First we prove the injectivity of S. Suppose $S(\xi) = 0$, denote A and A_0 the vector potentials with kernel $\mathbb{G}$ and $\mathbb{G}_0$, respectively, that is

$$\begin{aligned} A(\xi)(\mathbf{x}) &= \int_{\partial D} \mathbb{G}(\mathbf{x},\mathbf{y}) \cdot (\mathbf{n}_D(\mathbf{y}) \times \xi(\mathbf{y}))\, ds(\mathbf{y}), \qquad \mathbf{x} \notin \partial D, \\ A_0(\xi)(\mathbf{x}) &= \int_{\partial D} \mathbb{G}_0(\mathbf{x},\mathbf{y}) \cdot (\mathbf{n}_D(\mathbf{y}) \times \xi(\mathbf{y}))\, ds(\mathbf{y}), \qquad \mathbf{x} \notin \partial D. \end{aligned}$$

From the discussion at the end of Section 6.3 in [19], $A_0(\xi)$ defines two functions from $H^{-1/2}(\text{curl}, \partial D)$ to $\mathbf{u}_0^- \in H(\text{curl}, D)$ and $\mathbf{u}_0^+ \in H_{loc}(\text{curl}, \mathbb{R}^3 \backslash \bar{D})$. By the continuity of A_0 (see Theorem 6.12 in [19]), as $\mathbf{x}$ approaches ∂D, we have

$$\mathbf{n}_D \times \mathbf{u}_0^-|_{\partial D} = \mathbf{n}_D \times \mathbf{u}_0^+|_{\partial D} = \mathbf{n}_D \times A_0(\xi)|_{\partial D}.$$

This implies that $A(\xi)$ defines two functions $\mathbf{u}^- \in H(\text{curl}, D)$ and $\mathbf{u}^+ \in H_{loc}(\text{curl}, W\backslash\bar{D})$ with

$$\mathbf{n}_D \times \mathbf{u}^-|_{\partial D} = \mathbf{n}_D \times \mathbf{u}^+|_{\partial D} = \mathbf{n}_D \times A(\xi)|_{\partial D} = S(\xi) = \mathbf{0}.$$

From the uniqueness of the forward problem in D (since k^2 is not a Maxwell eigenvalue) and the assumption of the well-posedness of the forward problem in $W\backslash\bar{D}$, we obtain the lifting $\mathbf{u}^- = \mathbf{u}^+ = \mathbf{0}$ and then

$$\mathbf{n}_D \times (\nabla \times \mathbf{u}^-)|_{\partial D} = \mathbf{n}_D \times (\nabla \times \mathbf{u}^+)|_{\partial D} = \mathbf{0}.$$

To complete the proof of injectivity, we use the jump relation of $\mathbf{n}_D \times (\nabla \times A_0)$ on ∂D (see Theorem 6.12 of [19]) which implies that

$$\begin{aligned} \mathbf{n}_D \times \xi &= \mathbf{n}_D \times (\nabla \times \mathbf{u}_0^-)|_{\partial D} - \mathbf{n}_D \times (\nabla \times \mathbf{u}_0^+)|_{\partial D} \\ &= \mathbf{n}_D \times (\nabla \times \mathbf{u}^-)|_{\partial D} - \mathbf{n}_D \times (\nabla \times \mathbf{u}^+)|_{\partial D} \\ &= \mathbf{0}. \end{aligned}$$

Thus, $\xi = \mathbf{0}$ since $\mathbf{n}_D \cdot \xi = \mathbf{0}$ (because $\xi \in H^{-1/2}(\text{curl}, \partial D)$).

To show the surjectivity of S, let $\mathbf{g} \in H^{-1/2}(\text{div}, \partial D)$, then there are liftings $\mathbf{u}^-, \mathbf{u}^+$ that satisfy the Maxwell's equation in D and $\Omega = W_{(s,t)}\backslash\bar{D}$ (using the DtN maps on Σ_s, Σ_t as in (3.44)) with boundary data

$$\mathbf{n}_D \times \mathbf{u}^-|_{\partial D} = \mathbf{n}_D \times \mathbf{u}^+|_{\partial D} = \mathbf{g}.$$

Denote by $\xi \in H^{-1/2}(\text{curl}, \partial D)$ such that

$$\mathbf{n}_D \times \xi = \mathbf{n}_D \times (\nabla \times \mathbf{u}^+)|_{\partial D} - \mathbf{n}_D \times (\nabla \times \mathbf{u}^-)|_{\partial D},$$

and define

$$\mathbf{a}(\mathbf{x}) = \int_{\partial D} \mathbb{G}(\mathbf{x}, \mathbf{y}) \cdot (\mathbf{n}_D(\mathbf{y}) \times \xi(\mathbf{y}))\, ds(\mathbf{y}), \qquad \mathbf{x} \notin \partial D,$$

with the corresponding functions $\mathbf{a}^-$ and $\mathbf{a}^+$ in D and $W\backslash\bar{D}$. Obviously, $\mathbf{a}^+$ satisfies the radiation condition.

Using the continuity of the vector potential $\mathbf{a}$ (see Theorem 6.12 of [19]), we have

$$\mathbf{n}_D \times \mathbf{a}^+|_{\partial D} - \mathbf{n}_D \times \mathbf{a}^-|_{\partial D} = \mathbf{0}. \tag{3.54}$$

Using the jump relation of $\mathbf{n}_D \times (\nabla \times \mathbf{a})$ on ∂D (see Theorem 6.12 of [19]), we have

$$\mathbf{n}_D \times (\nabla \times \mathbf{a}^+)|_{\partial D} - \mathbf{n}_D \times (\nabla \times \mathbf{a}^-)|_{\partial D} = \mathbf{n}_D \times \xi. \tag{3.55}$$

Notice that the functions $\mathbf{u}^-$ and $\mathbf{u}^+$ satisfy the same relationship (3.54),(3.55) as $\mathbf{a}^-$ and $\mathbf{a}^+$ on ∂D. Define

$$\mathbf{u} = \begin{cases} \mathbf{u}^- & \text{in} \quad D, \\ \mathbf{u}^+ & \text{in} \quad \Omega. \end{cases} \qquad \text{and} \quad \mathbf{a} = \begin{cases} \mathbf{a}^- & \text{in} \quad D, \\ \mathbf{a}^+ & \text{in} \quad W\backslash\bar{D}. \end{cases}$$

By the definition of DtN maps on Σ_s, Σ_t and the well-posedness of blocked waveguide problem (Lemma 3.1.3), $\mathbf{u}^+$ can be extended uniquely to $W\backslash\bar{D}$. Since $\mathbf{a}$ also solves the problems in D and $W\backslash\bar{D}$, we can conclude that $\mathbf{u} = \mathbf{a}$ in the entire waveguide. Hence, we obtain

$$\mathbf{n}_D \times \mathbf{u}|_{\partial D} = \mathbf{n}_D \times \mathbf{a}|_{\partial D} = \mathbf{n}_D \times \left[\int_{\partial D} \mathbb{G}(\mathbf{x}, \mathbf{y}) \cdot (\mathbf{n}_D(\mathbf{y}) \times \xi(\mathbf{y}))\, ds(\mathbf{y})\right]_{\partial D}.$$

Therefore, we get $\mathbf{g} = S(\xi)$ and this completes the proof. □

Lemma 3.3.6 *The operators H, F and N defined in Definition 3.3.3, 3.3.4 and 3.3.1 are compact operators, they are also injective with dense range if k^2 is not a Maxwell eigenvalue for D and k^2 is such that the forward problem in $W\backslash\bar{D}$ is well posed.*

Proof: Recall the operator $H : L^2_T(\Sigma) \mapsto H^{-1/2}(\text{div}, \partial D)$ is such that, for $\mathbf{h} \in L^2_T(\Sigma)$,

$$H(\mathbf{h})(\mathbf{x}) = \mathbf{n}_D(\mathbf{x}) \times \left[\int_{\Sigma} \mathbb{G}(\mathbf{x}, \mathbf{y}) \cdot (\mathbf{n}_\Sigma(\mathbf{y}) \times \mathbf{h}(\mathbf{y}))\, ds(\mathbf{y})\right]_{\partial D}.$$

To prove compactness, since $\mathbb{G}$ is smooth for $(\mathbf{x},\mathbf{y}) \in \partial D \times \Sigma$, we see that H is compact because integral operator with smooth kernel is compact (see Theorem 8.7-5 in [44]).

For injectivity, for $\mathbf{h} \in L^2_T(\Sigma)$, consider

$$\mathbf{v_h}(\mathbf{x}) = \int_\Sigma \mathbb{G}(\mathbf{x},\mathbf{y}) \cdot (\mathbf{n}_\Sigma(\mathbf{y}) \times \mathbf{h}(\mathbf{y}))\, ds(\mathbf{y}).$$

Suppose that on ∂D, $\mathbf{n}_D \times \mathbf{v_h}|_{\partial D} = \mathbf{0} \in H^{-1/2}(\mathrm{div}, \partial D)$, then $\mathbf{v_h}$ solves Maxwell's equation in D with vanishing boundary data on ∂D. Because k^2 is not a Maxwell eigenvalue, $\mathbf{v_h}$ vanishes in D and then in the waveguide W with the help of the unique continuation theorem (Theorem D.0.8). In particular, $\mathbf{n}_\Sigma \times \mathbf{v_h}|_\Sigma = \mathbf{0}$. With the series representation of $\mathbf{h}$ on Σ,

$$\mathbf{0} = \mathbf{n}_\Sigma \times \mathbf{h} = \sum_m a_m \frac{1}{\lambda_m^2} \begin{pmatrix} \nabla_\Sigma u_m \\ 0 \end{pmatrix} + \sum_n b_n \frac{1}{\mu_n^2} \begin{pmatrix} \vec{\nabla}_\Sigma \times v_n \\ 0 \end{pmatrix},$$

and the orthogonality of $\nabla_\Sigma u_m$ and $\vec{\nabla}_\Sigma \times v_n$, we see that all the coefficients α_m, β_n are zeros which proves the injectivity of H.

To prove the denseness of the range, we shall prove that the adjoint operator $H^* : H^{-1/2}(\mathrm{curl}, \partial D) \mapsto L^2_T(\Sigma)$ is injective. For $\xi \in H^{-1/2}(\mathrm{curl}, \partial D)$ and $\mathbf{h} \in L^2_T(\Sigma)$, using identity (B.2), we have

$$\begin{aligned} \langle \xi, H(\mathbf{h}) \rangle_{\partial D} &= \left\langle \xi(\mathbf{x}), \mathbf{n}_D(\mathbf{x}) \times \left[\int_\Sigma \mathbb{G}(\mathbf{x},\mathbf{y}) \cdot (\mathbf{n}_\Sigma(\mathbf{y}) \times \mathbf{h}(\mathbf{y}))\, ds(\mathbf{y}) \right]_{\partial D} \right\rangle_{\partial D} \\ &= \left\langle \xi(\mathbf{x}) \times \mathbf{n}_D(\mathbf{x}), \left[\int_\Sigma \mathbb{G}(\mathbf{x},\mathbf{y}) \cdot (\mathbf{n}_\Sigma(\mathbf{y}) \times \mathbf{h}(\mathbf{y}))\, ds(\mathbf{y}) \right]_{\partial D} \right\rangle_{\partial D}. \end{aligned}$$

Using Fubini's theorem to interchange the integral over Σ and ∂D, we obtain

$$\langle \xi, H(\mathbf{h}) \rangle_{\partial D} = \left\langle \left[\int_{\partial D} (\xi(\mathbf{x}) \times \mathbf{n}_D(\mathbf{x}))^T \cdot \overline{\mathbb{G}(\mathbf{x},\mathbf{y})}\, ds(\mathbf{x}) \right]_\Sigma, \mathbf{n}_\Sigma(\mathbf{y}) \times \mathbf{h}(\mathbf{y}) \right\rangle_\Sigma.$$

Note that the Fubini's theorem is applicable here since

$$\xi \times \mathbf{n}_D = \nabla_{\partial D} \alpha + \vec{\nabla}_{\partial D} \times \beta,$$

with $\alpha \in H^{3/2}(\partial D)$ and $\beta \in H^{1/2}(\partial D)$ (see Theorem 3.8 in [62]). Of course $\nabla_{\partial D} \alpha \in L^2_T(\partial D)$ and so Fubini's theorem applies directly. For the other term, because $\vec{\nabla}_{\partial D} \times \beta \in$

$\{\mathbf{v} \in (H^{-1/2}(\partial D))^3 \mid \mathbf{n}_D \cdot \mathbf{v} = 0 \text{ a.e. on } \partial D\}$ and $L^2_T(\partial D)$ is dense in this space, Fubini's theorem also applies.

Using the identity (B.2) again, we have

$$\begin{aligned}
\langle \xi, H(\mathbf{h})\rangle_{\partial D} &= \left\langle \left[\int_{\partial D} (\xi(\mathbf{x}) \times \mathbf{n}_D(\mathbf{x}))^T \cdot \overline{\mathbb{G}(\mathbf{x},\mathbf{y})}\, ds(\mathbf{x})\right]_\Sigma \times \mathbf{n}_\Sigma(\mathbf{y}), \mathbf{h}(\mathbf{y}) \right\rangle_\Sigma \\
&= \left\langle \mathbf{n}_\Sigma(\mathbf{y}) \times \left[\int_{\partial D} (\mathbf{n}_D(\mathbf{x}) \times \xi(\mathbf{x}))^T \cdot \overline{\mathbb{G}(\mathbf{x},\mathbf{y})}\, ds(\mathbf{x})\right]_\Sigma, \mathbf{h}(\mathbf{y}) \right\rangle_\Sigma .
\end{aligned}$$

Using identity (B.27) and symmetry of $\mathbb{G}(\mathbf{x},\mathbf{y})$, we obtain

$$\begin{aligned}
\langle \xi, H(\mathbf{h})\rangle_{\partial D} &= \left\langle \mathbf{n}_\Sigma(\mathbf{y}) \times \left[\int_{\partial D} \overline{\mathbb{G}(\mathbf{x},\mathbf{y})}^T \cdot (\mathbf{n}_D(\mathbf{x}) \times \xi(\mathbf{x}))\, ds(\mathbf{x})\right]_\Sigma, \mathbf{h}(\mathbf{y}) \right\rangle_\Sigma \\
&= \left\langle \mathbf{n}_\Sigma(\mathbf{y}) \times \left[\int_{\partial D} \overline{\mathbb{G}(\mathbf{x},\mathbf{y})} \cdot (\mathbf{n}_D(\mathbf{x}) \times \xi(\mathbf{x}))\, ds(\mathbf{x})\right]_\Sigma, \mathbf{h}(\mathbf{y}) \right\rangle_\Sigma \\
&= \left\langle \mathbf{n}_\Sigma(\mathbf{y}) \times \left[\int_{\partial D} \overline{\mathbb{G}(\mathbf{y},\mathbf{x})} \cdot (\mathbf{n}_D(\mathbf{x}) \times \xi(\mathbf{x}))\, ds(\mathbf{x})\right]_\Sigma, \mathbf{h}(\mathbf{y}) \right\rangle_\Sigma .
\end{aligned}$$

Thus, we see that H^* is defined by

$$\begin{aligned}
H^*(\xi)(\mathbf{y}) &= \mathbf{n}_\Sigma(\mathbf{y}) \times \left[\int_{\partial D} \overline{\mathbb{G}(\mathbf{y},\mathbf{x})} \cdot (\mathbf{n}_D(\mathbf{x}) \times \xi(\mathbf{x}))\, ds(\mathbf{x})\right]_\Sigma \\
&= \overline{F}(\xi)(\mathbf{y}).
\end{aligned}$$

Interchanging $\mathbf{x}$ and $\mathbf{y}$ gives

$$\begin{aligned}
H^*(\xi)(\mathbf{x}) &= \mathbf{n}_\Sigma(\mathbf{x}) \times \left[\int_{\partial D} \overline{\mathbb{G}(\mathbf{x},\mathbf{y})} \cdot (\mathbf{n}_D(\mathbf{y}) \times \xi(\mathbf{y}))\, ds(\mathbf{y})\right]_\Sigma \\
&= \mathbf{n}_\Sigma(\mathbf{x}) \times \overline{\left[\int_{\partial D} \mathbb{G}(\mathbf{x},\mathbf{y}) \cdot \overline{(\mathbf{n}_D(\mathbf{y}) \times \xi(\mathbf{y}))}\, ds(\mathbf{y})\right]_\Sigma} \\
&= \overline{F(\overline{\xi})}(\mathbf{x}).
\end{aligned}$$

If $H^*(\xi) = 0$, using the series representation of the dyadic Green's function for $z > z'$ (3.41), we get

$$\mathbb{G}(\mathbf{x},\mathbf{y}) = \sum_m \left[c_m M_m^+(\mathbf{x}) M_m^-(\mathbf{y})^T\right] + \sum_n \left[d_n N_n^+(\mathbf{x}) N_n^-(\mathbf{y})^T\right].$$

Here the singular term vanishes since $(\mathbf{x},\mathbf{y}) \in \Sigma \times \partial D$. Then we have

$$\langle M_m^-(\mathbf{y}), \mathbf{n}_D(\mathbf{y}) \times \xi(\mathbf{y})\rangle_{\partial D} = 0,$$

$$\langle N_n^-(\mathbf{y}), \mathbf{n}_D(\mathbf{y}) \times \xi(\mathbf{y})\rangle_{\partial D} \quad = \quad 0,$$

which implies that the integrand of matrix vector multiplication is zero for $\mathbf{x}$ with $z > z'$ and then for all $\mathbf{x} \in W\backslash\bar{D}$ by using the unique continuation theorem (Theorem D.0.8). By applying the trace theorem from $H_{loc}(\text{curl}, W\backslash\bar{D})$ onto $H^{-1/2}(\text{div}, \partial D)$, we have that

$$\mathbf{n}_D(\mathbf{x}) \times \left[\int_{\partial D} \mathbb{G}(\mathbf{x}, \mathbf{y}) \cdot \overline{(\mathbf{n}_D(\mathbf{y}) \times \xi(\mathbf{y}))}\, ds(\mathbf{y})\right]_{\partial D} = \mathbf{0}\ (= S(\bar{\xi})).$$

Because operator S is an isomorphism, we have $\bar{\xi} = \mathbf{0}$ and thus $\xi = \mathbf{0}$ which completes the proof of injectivity of H^*.

For the operator F, as we have shown that $F(\xi) = \overline{H^*}(\xi)$, this proves that it is compact, injective with dense range since the operator H has the same property.

For the operator N, using the factorization $N = -FS^{-1}H$, we see that it is compact, injective with dense range as well by noticing that F, H are compact, injective with dense range and S is an isomorphism. This completes the proof. $\square$

3.3.4 Justification of the Linear Sampling Method

To provide a justification of the LSM for the waveguide, first we have the following lemma:

Lemma 3.3.7 $\mathbf{n}_\Sigma(\cdot) \times \mathbb{G}(\cdot, \mathbf{z})\mathbf{q}|_\Sigma \in B(H^{-1/2}(\text{div}, \partial D))$ *if and only if* $\mathbf{z} \in D$.

Proof: The proof follows the lines of Lemma 7.20 in [19]. First note that $B = FS^{-1}$ where S is an isomorphism and F is compact, injective with dense range.

If $\mathbf{z} \in D$, then $B\left(-\mathbf{n}_D \times \mathbb{G}(\cdot, \mathbf{z})\mathbf{q}|_{\partial D}\right) = \mathbf{n}_\Sigma \times \mathbb{G}(\cdot, \mathbf{z})\mathbf{q}|_\Sigma$.

If $\mathbf{z} \in W\backslash\bar{D}$ and assume that there exists $\mathbf{c} \in H^{-1/2}(\text{div}, \partial D)$ such that $B(\mathbf{c}) = \mathbf{n}_\Sigma \times \mathbb{G}(\cdot, \mathbf{z})\mathbf{q}|_\Sigma$. Then by the uniqueness of forward problem in $W\backslash\bar{D}$, the scattered field $\mathbf{w} \in H_{loc}(\text{curl}, W\backslash\bar{D})$ corresponding to the boundary data $\mathbf{c}$ and the incident field due to $\mathbb{G}(\cdot, \mathbf{z})\mathbf{q}$ coincide in $(W\backslash\bar{D})\backslash\{\mathbf{z}\}$. However, since $\mathbb{G} = \mathbb{G}_0 + \mathbb{J}$ away from Γ, and $\mathbb{G}_0\mathbf{q}$ is not locally integrable in $H(\text{curl})$, this leads to a contradiction. $\square$

Now, the main theorem we shall prove is the following:

Theorem 3.3.2 *Assume that k^2 is not a Maxwell eigenvalue for D and k^2 is such that the forward problem is well-posed. Let N be the Near Field Operator defined in Definition 3.3.1 for scattering from a perfect electric conductor, then the following holds:*

- *For $\mathbf{z} \in D$ and a given $\epsilon > 0$ there exists a function $\mathbf{g}_z^\epsilon \in L^2_T(\Sigma)$ such that*

$$\|N(\mathbf{g}_\mathbf{z}^\epsilon) - \mathbf{n}_\Sigma(\mathbf{x}) \times \mathbb{G}(\mathbf{x},\mathbf{z})\mathbf{q}|_\Sigma\|_{L^2_T(\Sigma)} < \epsilon \tag{3.56}$$

  and the vector potential field $\mathbf{U}_{\mathbf{g}_\mathbf{z}^\epsilon} = \displaystyle\int_\Sigma \mathbb{G}(\mathbf{x},\mathbf{y}) \cdot \mathbf{g}_\mathbf{z}^\epsilon(\mathbf{y})\, ds(\mathbf{y})$ with density function $\mathbf{g}_\mathbf{z}^\epsilon$ converges to the solution of Maxwell's equation with boundary condition $\mathbf{n}_D \times [\mathbf{U} + \mathbb{G}(\mathbf{x},\mathbf{z})\mathbf{q}] = 0$ in $H(\mathrm{curl}, D)$ as $\epsilon \to 0$.

- *For $\mathbf{z} \notin D$, every $\mathbf{g}_\mathbf{z}^\epsilon \in L^2_T(\Sigma)$ that satisfies (3.56) for a given $\epsilon > 0$ is such that*

$$\lim_{\epsilon\to 0} \|\mathbf{g}_\mathbf{z}^\epsilon\|_{L^2_T(\Sigma)} = \infty.$$

Proof: The proof follows the lines of Theorem 7.21 in [19]. Under the assumption of k we have the well-posedness of the interior Maxwell problem in $H(\mathrm{curl}, D)$. Given $\epsilon > 0$, since $H : L^2_T(\Sigma) \mapsto H^{-1/2}(\mathrm{div}, \partial D)$ is compact, injective with dense range, we can choose $\mathbf{g}_\mathbf{z}^\epsilon = \mathbf{n}_\Sigma \times \mathbf{h}_\mathbf{z}^\epsilon \in L^2_T(\Sigma)$ such that

$$\|H(\mathbf{h}_\mathbf{z}^\epsilon) + \mathbf{n}_D \times \mathbb{G}(\cdot,\mathbf{z})\mathbf{q}\|_{H^{-1/2}(\mathrm{div},\partial D)} < \frac{\epsilon}{\|B\|},$$

where $\|B\|$ is the standard induced norm of B. Then, recalling that $N = -BH$, we have

$$\begin{aligned}
\epsilon &> \|H(\mathbf{h}_\mathbf{z}^\epsilon) + \mathbf{n}_D \times \mathbb{G}(\cdot,\mathbf{z})\mathbf{q}|_{\partial D}\|_{H^{-1/2}(\mathrm{div},\partial D)} \|B\| \\
&\geq \|B(H(\mathbf{h}_\mathbf{z}^\epsilon) + \mathbf{n}_D \times \mathbb{G}(\cdot,\mathbf{z})\mathbf{q}|_{\partial D})\|_{L^2_T(\Sigma)} \\
&= \|N(\mathbf{g}_\mathbf{z}^\epsilon) - B\left(\mathbf{n}_D \times \mathbb{G}(\cdot,\mathbf{z})\mathbf{q}|_{\partial D}\right)\|_{L^2_T(\Sigma)} \\
&= \|N(\mathbf{g}_\mathbf{z}^\epsilon) - \mathbf{n}_\Sigma \times \mathbb{G}(\cdot,\mathbf{z})\mathbf{q}|_\Sigma\|_{L^2_T(\Sigma)}.
\end{aligned}$$

Now if $\mathbf{z} \in D$, then by the well-posedness of interior Maxwell problem, the convergence of $H(\mathbf{h}_\mathbf{z}^\epsilon) + \mathbf{n}_D \times \mathbb{G}(\cdot,\mathbf{z})\mathbf{q}|_{\partial D} \to 0$ as $\epsilon \to 0$ in $H^{-1/2}(\mathrm{div}, \partial D)$ implies the convergence $\mathbf{U}_{\mathbf{g}_\mathbf{z}^\epsilon} \to \mathbf{U}$ as $\epsilon \to 0$ in $H(\mathrm{curl}, D)$ where $\mathbf{U}$ solves the interior Maxwell equation with boundary condition $\mathbf{n}_D \times [\mathbf{U} + \mathbb{G}(\mathbf{x}, \mathbf{z})\mathbf{q}] = 0$. Thus, we are done with the proof of the first statement.

For the second statement, for $\mathbf{z} \notin D$, assume that there exists a sequence $\{\epsilon_n\}$ with $\epsilon_n \to 0$ and corresponding vector potentials $\mathbf{U}_n$ with kernel $\mathbf{g}_n := \mathbf{g}_{\mathbf{z}}^{\epsilon_n}$ such that $\|\mathbf{U}_n\|_{H(\mathrm{curl},D)}$ is bounded. Further we assume weak convergence $\mathbf{U}_n \rightharpoonup \mathbf{U} \in H(\mathrm{curl}, D)$ as $n \to \infty$.

Denote by $\mathbf{U}^s \in H_{loc}(\mathrm{curl}, W \backslash \bar{D})$ the solution to the exterior Maxwell problem with $\mathbf{n}_D \times \mathbf{U}^s|_{\partial D} = \mathbf{n}_D \times \mathbf{U}|_{\partial D}$ on ∂D with boundary data $\mathbf{n}_\Sigma \times \mathbf{U}^s|_\Sigma$ on Σ. Since $N(\mathbf{g}_n)$ gives the boundary data of exterior problem on Σ due to the incident field $-\mathbf{n}_D \times \mathbf{U}_n|_{\partial D}$ on ∂D, then from (3.56) we can conclude that there exists a function $\mathbf{g}_{\mathbf{z}} \in L^2_T(\Sigma)$ corresponding to the density function of $\mathbf{U}$ such that $N(\mathbf{g}) = -\mathbf{n}_\Sigma \times \mathbb{G}(\mathbf{x}, \mathbf{z})\mathbf{q}|_\Sigma$ and therefore $\mathbf{n}_\Sigma \times \mathbb{G}(\mathbf{x}, \mathbf{z})\mathbf{q}|_\Sigma \in B(H^{-1/2}(\mathrm{div}, \partial D))$. But this contradicts with Lemma 3.3.7, and the proof of the second statement is done. $\square$

3.4 Numerical Simulation

In this section, we shall describe some numerical simulations of the reconstruction of scattering objects in order to investigate the practical use of the Linear Sampling Method (LSM) inside the waveguide. Specifically, we use the Method of Fundamental Solutions (MFS) to generate synthetic scattering data to be collected at the receivers located on a cross section of the waveguide away from the scatterer. Because Theorem 3.3.2 shows that the near field equation is ill-posed, we shall use a regularization approach to solve a discrete version of the near field equation.

3.4.1 The Method of Fundamental Solutions

The basic idea of the Method of Fundamental Solutions (MFS) is that, by using the fields due to a finite number of point sources located inside the scatterer D (see, e.g., $\{\mathbf{o}_1, \mathbf{o}_2, \ldots, \mathbf{o}_M\}$ in Figure 3.6), we aim to simulate the scattered field $\mathbf{u}^s$ due to an incident field $\mathbf{u}^i$ outside of D. In particular, we match the field due to points sources inside D to the incident field on the boundary ∂D.

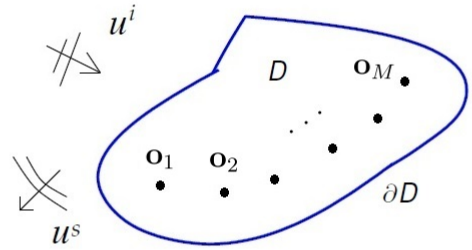

Figure 3.6: Illustration of the basic idea of the Method of Fundamental Solutions.

To further explain the rationale of this method, consider a closed surface S inside the scatterer D and a tangential vector field $\mathbf{g} \in L^2_T(S)$, then from Definition 3.3.3 and an extension of Lemma 3.3.6, we know that $H(\mathbf{g})$ is injective with dense range in $H^{-1/2}(\mathrm{div}, \partial D)$. Thus, given a tangential field on ∂D representing the perfect conducting data generated by an incident field, it can be approximated by

$$H(\mathbf{g})(\mathbf{x}) = \mathbf{n}_D(\mathbf{x}) \times \left[\int_S \mathbb{G}(\mathbf{x}, \mathbf{y}) \cdot \mathbf{g}(\mathbf{x}) \, ds(\mathbf{y}) \right]_{\partial D}. \tag{3.57}$$

If the incident field is denoted as usual by $\mathbf{u}^i$, then there exists $\mathbf{g} \in L^2_T(S)$ such that $H(\mathbf{g}) \approx -\mathbf{n}_D \times \mathbf{u}^i|_{\partial D}$ to any tolerance. Then we are able to use $\int_S \mathbb{G}(\mathbf{x}, \mathbf{y}) \cdot \mathbf{g}(\mathbf{y}) \, ds(\mathbf{y})$ to approximate scattering data $\mathbf{u}^s$ away from ∂D.

In numerical simulations, we can discretize the integral in (3.57) and still approximate $-\mathbf{n}_D \times \mathbf{u}^i|_{\partial D}$. To describe more in detail the implementation of the MFS for the waveguide, as shown in Figure 3.6, let $\{\mathbf{o}_1, \mathbf{o}_2, \ldots, \mathbf{o}_M\}$ be a set of M grid points inside D where a sequence of point sources are located. We also choose the polarization for each source. Then the field generated by these point sources in the waveguide is given using sums of Green's functions as follows:

$$\mathbf{u}_{\mathrm{MFS}}(\mathbf{x}) = \sum_{m=1}^{M} \sum_{j=1}^{3} \alpha_{mj} \mathbb{G}(\mathbf{x}, \mathbf{o}_m) \mathbf{p}_j, \tag{3.58}$$

where

$\mathbf{o}_m$: $Them^{\mathrm{th}}$ point source (grid point) inside D where $m = 1, 2, \ldots, M$,

$\mathbb{G}(\mathbf{x}, \mathbf{o}_m)$	:	Dyadic Green's function $\mathbb{G}_e$ due to point source at $\mathbf{o}_m$,
$\mathbf{p}_j$	:	Polarization of the point source at $\mathbf{y}_m$ where $j = 1, 2, 3$ and $\lvert\mathbf{p}_j\rvert = 1$,
α_{mj}	:	Undetermined scalar coefficient corresponding to the m_{th} grid point and j^{th} polarization,
$\mathbf{u}_{\mathrm{MFS}}(\mathbf{x})$	:	Electric field at an arbitrary point $\mathbf{x} \in W \backslash \bar{D}$.

Here $\mathbf{p}_j$ can be chosen as an orthonormal basis of $\mathbb{R}^3$, for example, $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$. Note that the representation (3.58) can be considered as a discretized version of the surface integral in (3.57).

Of course the MFS solution exactly satisfies Maxwell's equation in the waveguide, the boundary condition in the waveguide, and the radiation condition. Here only the boundary condition on ∂D needs to be approximated.

In order to correctly simulate the scattered field, we consider the following minimization problem:

$$\{\alpha_{mj}\} = \mathrm{argmin}\left\{\|\mathbf{n}_D \times \mathbf{u}_{\mathrm{MFS}} + \mathbf{n}_D \times \mathbf{u}^i\|^2_{L^2(\partial D)} + \|\nabla_{\partial D} \cdot (\mathbf{n}_D \times \mathbf{u}_{\mathrm{MFS}} + \mathbf{n}_D \times \mathbf{u}^i)\|^2_{L^2(\partial D)}\right\}.$$

Here we would prefer to use the $H^{-1/2}(\mathrm{div}, \partial D)$ norm but use the $H(\mathrm{div}, \partial D)$ norm instead because it is very challenging to use the $H^{-1/2}$ norm.

Now, using the identity (see, e.g. (6.43) in [19])

$$\nabla_{\partial D} \cdot (\mathbf{n}_D \times \mathbf{U}) = -\mathbf{n}_D \cdot (\nabla \times \mathbf{U}),$$

we are equivalently minimizing the following quantity

$$\{\alpha_{mj}\} = \mathrm{argmin}\left\{\|\mathbf{n}_D \times \mathbf{u}_{\mathrm{MFS}} + \mathbf{n}_D \times \mathbf{u}^i\|^2_{L^2(\partial D)} + \|\mathbf{n}_D \cdot (\nabla \times \mathbf{u}_{\mathrm{MFS}} + \nabla \times \mathbf{u}^i)\|^2_{L^2(\partial D)}\right\}. \tag{3.59}$$

In practice, we also approximate the surface of D using a triangular gird and approximate the norms on each triangular element by using a quadrature at a single quadrature point at the centroid of each element to approximate the integral. Since the number of triangular elements is much greater than the number of undetermined coefficients α_{mj}, we actually minimize the norms in the Least-Square sense as show in (3.59). For numerical implementation, we use the truncated singular value decomposition (SVD) to solve for the linear system.

Specifically, if the number of triangular element for the surface of D is N_T, then the resulting linear system is given by $B\alpha = \mathbf{c}$ and if $p = 2N_T, q = 3M, p \gg q$, then α is a $q \times 1$ vector consisting of all the unknown α_{mj}'s, $\mathbf{c}$ is a $p \times 1$ column vector due to the incident field $\mathbf{u}^i$ (that is, $-\mathbf{n}_D \times \mathbf{u}^i|_{\partial D}$ and $-\mathbf{n}_D \times (\nabla \times \mathbf{u}^i)|_{\partial D}$), and B is a $p \times q$ matrix. Given the Singular Value Decomposition (SVD) of $B = U\Lambda V^*$ where "$*$" represents the conjugate transpose, the normal equation $BB^*\alpha = B^*\mathbf{c}$ can be written as

$$V\Lambda^*\Lambda V^*\alpha = B^*\mathbf{c} = V\Lambda^* U^*\mathbf{c}.$$

Therefore, the optimal α is given by $\alpha = V\Lambda^\dagger U^*\mathbf{c}$ where "$\dagger$" represents the pseudoinverse.

In choosing the parameters, we shall take into the consideration the following:

- Number of terms truncated in the series expansion of the dyadic Green's function in $\mathbf{u}^i$ and $\mathbf{u}_{\text{MFS}}$:

  Since the problem is to collect the near field but far away from the scatterer, we shall include at least all the propagating modes, that is, all the m, n such that $h_m = \sqrt{k^2 - \lambda_m} > 0$ and $g_m = \sqrt{k^2 - \mu_m^2} > 0$. Also, we shall include more evanescent modes in order that (3.58) represents a better approximation. But the number of terms should be controlled since otherwise the whole problem will be very computationally expensive.

- Truncated SVD for solving Least-Square problem for obtaining coefficients $\{\alpha_{mj}\}$:

  To solve for the coefficients $\alpha = V\Lambda^\dagger U^*\mathbf{c}$, we shall arrange the singular values in Λ, denote $\{\theta_l\}_{l\geq 1}$, in decreasing order and truncate at $l = l_*$ when $\theta_1/\theta_{l_*} > b \gg 0$, for example $b = 10^{14}$.

- Number of grid points $\{\mathbf{o}_m\}_{m=1}^M$ and their locations inside D:

  Choosing the grid points inside the scatterer D requires additional work. Although it seems plausible to increase the number of grid points M inside D in (3.58) for a more accurate solution $\{\alpha_{mj}\}$ in the Least-Square sense, this is not for free in a practical implementation since, for instance, (3.58) is used to approximate a compact operator using finite dimensional approximation and thus the condition number of the problem will grow rapidly as M increases (see, e.g., Section 4.1 in [19], [3]). For the choice of locations of grid points, denote by $d(S, \partial D) = \inf\{|\mathbf{x} - \mathbf{y}|\}$ where $\mathbf{x} \in S, \mathbf{y} \in \partial D$. On one hand, if $d(S, \partial D)$ is small, the kernel $\mathbb{G}$ becomes highly peaked which needs fine discretization. On the other hand, if $d(S, \partial D)$ is big, $\mathbb{G}$ becomes smoother and this amplifies the the condition number for the problem. We shall use some heuristic ways to choose the number and location of grid points $\{\mathbf{o}_m\}_{m=1}^M$. For example, using a uniform distribution of points on a smaller sphere if D is a sphere or using a cube for the location of grid points if D is a cube.

As a further remark, although the drawback of MSF concerning the choice of grid points and the expense of including more terms in the series representation of dyadic Green's function hinders us from improving the results easily, this idea can serve for the generation of synthetic scattering data for the forward problem as it can be computed by using series representations which greatly reduces the computational complexity compared with traditional Galerkin-based methods such as the finite element method.

3.4.2 The Near Field Equation

With the synthetic data generated by using MFS in hand, we next consider the near field equation (3.52) in order to solve for the indicator function $\mathbf{g}$ for each sampling point $\mathbf{z}$.

For numerical simulations, instead of considering only the tangential field on cross section Σ, we incorporate more data by including all components of scattering data at measurements by solving the following integral equation: find $\mathbf{g} \in (L^2(\Sigma))^3$ such that

$$\tilde{N}(\mathbf{g})(\mathbf{x}) := \int_\Sigma \mathbf{u}^s(\mathbf{x}, \mathbf{y}, \mathbf{g}(\mathbf{y}))\, ds(\mathbf{y}) = \mathbb{G}(\mathbf{x}, \mathbf{z})\mathbf{q}|_\Sigma. \tag{3.60}$$

Since the scattered field $\mathbf{u}^s$ is a linear function of the vector function $\mathbf{g} = g_1\mathbf{e}_1 + g_2\mathbf{e}_2 + g_3\mathbf{e}_3$ where $\mathbf{e}_1 = \hat{\mathbf{x}}, \mathbf{e}_2 = \hat{\mathbf{y}}, \mathbf{e}_3 = \hat{\mathbf{z}}$, the left hand side of the integral euqation (3.60) may be equivalently written as

$$\begin{aligned}\int_\Sigma \mathbf{u}^s(\mathbf{x}, \mathbf{y}, \mathbf{g}(\mathbf{y}))\, ds(\mathbf{y}) &= \int_\Sigma \mathbf{u}^s(\mathbf{x}, \mathbf{y}, g_1\mathbf{e}_1 + g_2\mathbf{e}_2 + g_3\mathbf{e}_3)\, ds(\mathbf{y}) \\ &= \int_\Sigma \left[\sum_{k=1}^3 \mathbf{u}^s(\mathbf{x}, \mathbf{y}, \mathbf{e}_k) g_k\right] ds(\mathbf{y}).\end{aligned}$$

Componentwise, we have

$$\int_{\Sigma}\left[\sum_{k=1}^{3}\begin{pmatrix}\mathbf{u}_1^s(\mathbf{x},\mathbf{y},\mathbf{e}_k)\\ \mathbf{u}_2^s(\mathbf{x},\mathbf{y},\mathbf{e}_k)\\ \mathbf{u}_3^s(\mathbf{x},\mathbf{y},\mathbf{e}_k)\end{pmatrix} g_k\right] ds(y) = \left.\begin{pmatrix}(\mathbb{G}(\mathbf{x},\mathbf{z})\mathbf{q})_1\\ (\mathbb{G}(\mathbf{x},\mathbf{z})\mathbf{q})_2\\ (\mathbb{G}(\mathbf{x},\mathbf{z})\mathbf{q})_3\end{pmatrix}\right|_{\Sigma}. \tag{3.61}$$

To evaluate the integral numerically, we discretize the integral by applying a quadrature rule the choice of which depends on the cross section Σ of the waveguide. For example, we can use the Gauss-Jacobi quadrature rule for a cylindrical waveguide and the tensor product composite Midpoint rule for a rectangular waveguide.

Let $\mathbf{y}_j, j = 1, 2, \ldots, n$ be n point sources at quadrature points on Σ and $\mathbf{x}_i, i = 1, 2, \ldots, m$ be m receivers also on Σ. In fact, we take $m = n$ and $\mathbf{y}_j = \mathbf{x}_j, 1 \leq j \leq n$. We loop through point sources ($\mathbf{y}_j$'s) at all the quadrature points with all three polarizations $\mathbf{p} = \mathbf{e}_1, \mathbf{e}_2$ or $\mathbf{e}_3$ and collect measurements at all the receivers ($\mathbf{x}_i$'s). Then by collecting measurement at all the receivers ($\mathbf{x}_i$'s) due to a sampling point $\mathbf{z}$ with polarization $\mathbf{p} = \mathbf{e}_1, \mathbf{e}_2$ or $\mathbf{e}_3$, the equation (3.61) can be reduced to a linear system

$$A_{3m\times 3n}(\mathbf{g}_{\mathbf{z}})_{3n\times 1} = (\mathbf{b}_{\mathbf{z}})_{3m\times 1},$$

where the matrix $A_{3m\times 3n}$ consists of mn 3×3 blocks and each block records the measurement at the i^{th} receiver due to the j^{th} point source associated with weight of the quadrature point w_{ij}. For instance, the structure of $(ij)^{\text{th}}$ block can be written as follows:

$$w_{ij}\begin{pmatrix}\mathbf{u}_1^s(\mathbf{x}_i,\mathbf{y}_j,\mathbf{e}_1) & \mathbf{u}_1^s(\mathbf{x}_i,\mathbf{y}_j,\mathbf{e}_2) & \mathbf{u}_1^s(\mathbf{x}_i,\mathbf{y}_j,\mathbf{e}_3)\\ \mathbf{u}_2^s(\mathbf{x}_i,\mathbf{y}_j,\mathbf{e}_1) & \mathbf{u}_2^s(\mathbf{x}_i,\mathbf{y}_j,\mathbf{e}_2) & \mathbf{u}_2^s(\mathbf{x}_i,\mathbf{y}_j,\mathbf{e}_3)\\ \mathbf{u}_3^s(\mathbf{x}_i,\mathbf{y}_j,\mathbf{e}_1) & \mathbf{u}_3^s(\mathbf{x}_i,\mathbf{y}_j,\mathbf{e}_2) & \mathbf{u}_3^s(\mathbf{x}_i,\mathbf{y}_j,\mathbf{e}_3)\end{pmatrix}.$$

The column vector $(\mathbf{g}_{\mathbf{z}})_{3n\times 1}$ consists of a number of n 3×1 blocks with each block corresponding to the indicator function for the j^{th} point source. Lastly, the column vector $(\mathbf{b}_{\mathbf{z}})_{3m\times 1}$ consists of a number of m 3×1 blocks with each block corresponding to the measurements of the dyadic Green's function $\mathbb{G}(\mathbf{x},\mathbf{z})\mathbf{q}$ at the i^{th} receiver due to sampling point $\mathbf{z}$ with polarization $\mathbf{q} = \mathbf{e}_1, \mathbf{e}_2$ or $\mathbf{e}_3$.

Since there are three polarizations for each sampling point, we use the average of the discrete l^2 norms of indicator function $\mathbf{g}$ for each of these three polarizations for identification of the shape of the scatterer.

3.4.3 Numerical Results: Cylindrical Waveguide

For numerical results, we use a cylindrical waveguide. In particular, consider a cylindrical waveguide with cross section Σ a disk of radius a. From the discussion in Section 6.1 and 6.2 in [61], using cylindrical coordinates $(x, y, z) \to (r, \phi, z)$, two families of modal solutions M and N exist having double indices:

$$
\begin{aligned}
M_{mn} &= \nabla \times (u_{mn}(r,\phi)e^{ih_{mn}z}\hat{\mathbf{z}}), \quad \text{for } m \geq 1, n \geq 0, \\
N_{mn} &= \frac{1}{k}\nabla \times \nabla \times (v_{mn}(r,\phi)e^{ig_{mn}z}\hat{\mathbf{z}}), \quad \text{for } m \geq 1, n \geq 0,
\end{aligned}
$$

where

$$
\begin{aligned}
u_{mn}(r,\phi) &= J_n(\mu_{mn}r)e^{in\phi}, \ \mu_{mn} = \frac{q_{mn}}{a}, \\
v_{mn}(r,\phi) &= J_n(\lambda_{mn}r)e^{in\phi}, \ \lambda_{mn} = \frac{p_{mn}}{a}.
\end{aligned}
$$

Here q_{mn} represents the m^{th} root of the derivative of the n^{th} order of Bessel function $J_n(x)$ and p_{mn} represents the m^{th} root of the n^{th} order of Bessel function $J_n(x)$.

Correspondingly, the coefficients c_{mn} and d_{mn} in the dyadic Green's function $\mathbb{G}_e$ in (3.41) and (3.42) are:

$$
\begin{aligned}
c_{mn} &= \int_0^a J_n^2(\mu_{mn}r)r\,dr = \frac{a^2}{2\mu_{mn}^2}\left(\mu_{mn}^2 - \frac{n^2}{a^2}\right)J_n^2(\mu_{mn}a), \\
d_{mn} &= \int_0^a J_n^2(\lambda_{mn}r)r\,dr = \frac{a^2}{2\lambda_{mn}^2}\left[\frac{\partial J_n(\lambda_{mn}r)}{\partial r}\right]_{r=a}^2.
\end{aligned}
$$

The parameters related to the waveguide are listed as follows:

- Wave number: $k = 2\pi$ so the wavelength is 1.
- Radius of the circular pipe: $a = 1$
- Location of cross section for point source/receiver: $\Sigma \times \{z = 0\}$

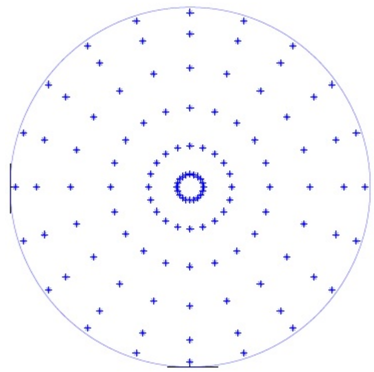

Figure 3.7: Illustration of generic distribution of point sources/receivers on a cross section of typical circular waveguide.

- Mesh on the cross section for point source/receiver: 10×10 mesh in polar coordinates with uniform distribution on $[0, 2\pi]$ in the angular direction and located at Gauss-Jacobi quadrature points along the radial direction.
- Number of terms kept in dyadic Green's function: $N = 15$ in order to include all propagating modes.
  - Number of propagating mode due to $k = 2\pi$: 11
  - Number of evanescent mode: $N - 11 = 4$
- Region of sampling points: a box of size $0.5 \times 0.5 \times 2$ centered at $(0, 0, 20)$.

Parameters for the scattering object D are listed below:

- Location and shape of D: a sphere of radius 0.2 centered at $(0, 0, 20)$.
- Location of grid points for MFS: on surface of a concentric sphere of radius 0.16.
- Distribution of grid points for MFS: 30 points distributed on six latitude circles with degrees $\pm 75^o$ (3 points on each cicle), $\pm 45^o$ (5 points on each circle) and $\pm 15^o$ (7 points on each circle) (see Figure 3.8 for a generic illustration).
- Noise on scattering data: no noise

The reconstruction of D by using Tikhonov regularization combined with Morozov discrepancy principle is given in Figure 3.9 (note that the positive z-axis in the plot is pointing upward).

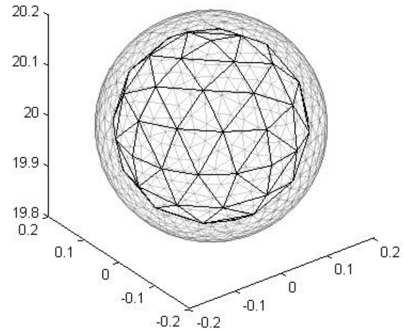

Figure 3.8: Illustration of generic distribution of grid points on a sphere.

If we keep more terms in dyadic Green's function ($N = 25$) and use more grid points for MFS (66 points uniformly distributed on a sphere of radius 0.17), the reconstruction can be improved (see Figure 3.10).

From the plot, we are able to reconstruct the scatterer in a reasonable sense by using LSM. Note that only one sided data is used.

As a further test, we also show the results for the reconstruction of a cube (with non-smooth boundary) in this waveguide for reference. Parameters for the scattering object D are listed below:

- Location and shape of D: a cube of side length 0.4 centered at $(0, 0, 20)$.
- Location of grid points for MFS: on surface of a concentric cube of side length 0.32.
- Distribution of grid points for MFS: 56 points uniformly distributed on six faces, four edges and eight vertices (see Figure 3.11 for a generic illustration)
- Noise on scattering data: no noise

Using the same inverse technique, the reconstruction of D is given in Figure 3.12.

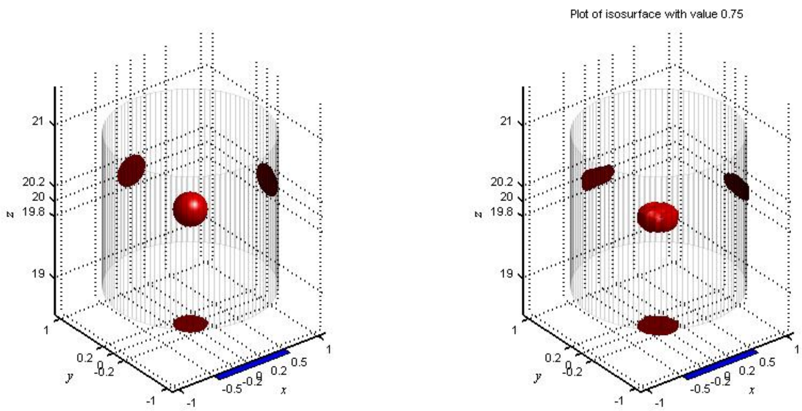

Figure 3.9: Left: Plot of original scattering object (sphere of radius 0.2). Right: Reconstruction of object with isosurface value 0.75.

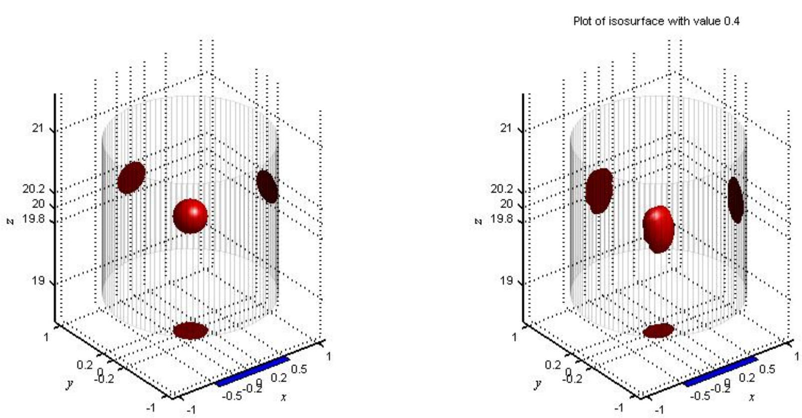

Figure 3.10: Left: Plot of original scattering object (sphere of radius 0.2). Right: Reconstruction of object with isosurface value 0.4.

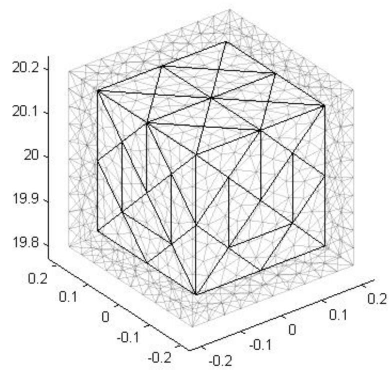

Figure 3.11: Illustration of generic distribution of grid points on a cube.

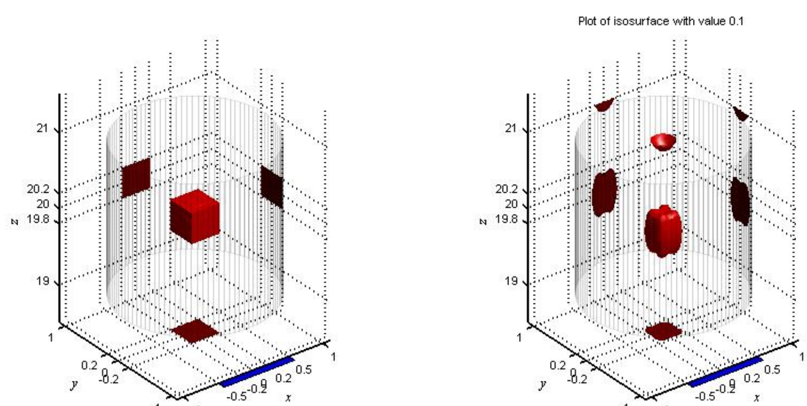

Figure 3.12: Left: Plot of original scattering object (cube of side length 0.4). Right: Reconstruction of object with isosurface value 0.1.

Chapter 4

CONCLUSION AND FUTURE WORK

In this thesis we have investigated problems arisen from scattering and inverse scattering theory with special characteristics in the background media, in particular, the presence of geometrical settings with prescribed material properties. We have shown the discreteness and existence of transmission eigenvalues and identified the first few real transmission eigenvalues for PEC backed scattering objects which may be useful for obtaining their material properties. We also justified the Linear Sampling Method (LSM) applied for the reconstruction of PEC objects inside a PEC waveguide. For each problem, we have developed methods that cater to the presence of structure in the background and have proved the standard scattering theory can be applied with proper modifications.

Meanwhile, there are still many interesting questions and future research opportunities based on the work in this thesis. To name a few, we have

1. Enlightened from the investigation of transmission eigenvalues, we can also consider the reconstruction of dielectric scattering objects in a 3D waveguide and look for the corresponding transmission eigenvalues. Furthermore, we can investigate the scattering and inverse scattering in a 3D waveguide where a dielectric scattering object sits on the wall of the waveguide.

2. There have been studies on transmission eigenvalues for objects with defects inside. The study can be then extended to the investigation of transmission eigenvalues for PEC backed scattering objects with defects. This will create mathematical difficulties if the defects are located on the interface between the object and conducting substrate. Also, one can study the problem where non-perfect conducting regions are present in the PEC substrate.

3. In Section 3.4, we discussed the advantages and drawbacks of implementation of ideas from Method of Fundamental Solutions. As an alternate, it will be very

beneficial to develop other forward solvers using Galerkin-based methods. We are currently studying on the theory and numerical approach on Ultra Weak Variational Formulation (UWVF) [37] for the waveguide and this will also serve for fields generation for other inverse scattering problems in electromagnetic theory.

4. The specialty of this thesis is to study the scattering and inverse scattering problem pertaining to non-standard backgrounds. As a problem of the same type, we have investigated an inverse scattering problem involving the design and use of a device called a “Hyperlens”. This device has been suggested by physicists [38]. It consists of a sequence of concentric thin layers of dielectric and metallic material with a hole inside used to enclose the unknown scatterer. The expectation is that this “lens” will enhance remote measurements of the scattered field in order to help identify the shape of objects placed in the lens better than in its absence. The goal is to provide a microscope with enhanced resolution. We have conducted various numerical simulations for imaging scatterers of different shapes in the cavity of the lens illuminated by incoming plane waves from multiple directions as well as new configurations by using multiple point sources close to the lens and collect data outside the lens. Future work could be the study of influence of material property of this “lens” on reconstruction of the scatterer and the justification of qualitative methods applied to this problem (e.g. LSM).

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Appendix A

DERIVATION OF MODAL SOLUTIONS FOR THE WAVEGUIDE

Here we present some details of modal solutions to the governing Maxwell's equation in (3.6), that is,

$$\nabla \times \nabla \times \mathbf{u}^s - k^2 \mathbf{u}^s = 0. \tag{A.1}$$

In particular, we use separation of variable to decompose solution to (A.1) into the two families of modes.

By taking the divergence of (A.1) we have $\nabla \cdot \mathbf{u}^s = 0$. Using the vector identity (B.8), (A.1) can be written as the vector Helmholtz equation

$$\Delta \mathbf{u}^s + k^2 \mathbf{u}^s = 0.$$

Let $\mathbf{u}^s_j = A_j(x,y)\theta_j(z), j = 1,2,3$. Then for each component,

$$(A_j)_{xx}\theta_j + (A_j)_{yy}\theta_j + (A_j)\theta_j'' + k^2(A_j)\theta_j = 0.$$

If $\Delta_{xy} = (\frac{\partial}{\partial x})^2 + (\frac{\partial}{\partial y})^2$, a standard separation of variables argument shows that

$$\Delta_{xy}(A_j) + \lambda^2 (A_j) = 0 \quad \text{and} \quad \theta_j'' + (k^2 - \lambda^2)\theta_j = 0.$$

For $\lambda \geq 0$, let

$$h = \begin{cases} \sqrt{k^2 - \lambda^2} & \text{if} \quad \lambda \leq k, \\ i\sqrt{\lambda^2 - k^2} & \text{if} \quad \lambda > k. \end{cases}$$

Then we have $\theta(z) = \alpha e^{ihz} + \beta e^{-ihz}$ for some constants α, β. The choice of α, β depends on whether we are to the left or the right of the scatterer D. Consider $\theta_j(z) = e^{ih_j z}$, then one mode of $\mathbf{u}^s$ is given by

$$\mathbf{U} = \begin{pmatrix} A_1(x,y)e^{ih_1 z} \\ A_2(x,y)e^{ih_2 z} \\ A_3(x,y)e^{ih_3 z} \end{pmatrix} = \begin{pmatrix} A_1(x,y) \\ 0 \\ 0 \end{pmatrix} e^{ih_1 z} + \begin{pmatrix} 0 \\ A_2(x,y) \\ 0 \end{pmatrix} e^{ih_2 z} + \begin{pmatrix} 0 \\ 0 \\ A_3(x,y) \end{pmatrix} e^{ih_3 z}.$$

We now show that we can assume that $h_1 = h_2 = h_3 = h$ so that $\mathbf{U}$ has the form

$$\mathbf{U} = \begin{pmatrix} A_1(x,y) \\ A_2(x,y) \\ A_3(x,y) \end{pmatrix} e^{ihz}. \tag{A.2}$$

Of course the same conclusion holds for e^{-ihz}.

Note A.0.1 *For the solvability results of our analysis, we will assume $h \neq 0$ (or $k^2 \neq \lambda^2$).*

Let $c_1 = ih_1, c_2 = ih_2, c_3 = ih_3$, then $\nabla \cdot \mathbf{U} = 0$ implies

$$A_{1,x}e^{c_1 z} + A_{2,y}e^{c_2 z} + A_3 c_3 e^{c_3 z} = 0.$$

Fix x, y, let $a_1 = A_{1,x}, a_2 = A_{2,y}, a_3 = A_3 c_3 = A_3 ih_3$, then for any z, we have

$$a_1 e^{c_1 z} + a_2 e^{c_2 z} + a_3 e^{c_3 z} = 0.$$

Set $z = 0$, so that $a_1 + a_2 + a_3 = 0$. Hence $a_3 = -(a_1 + a_2)$ and so

$$a_1 e^{c_1 z} + a_2 e^{c_2 z} - (a_1 + a_2)e^{c_3 z} = 0. \tag{A.3}$$

Taking the first derivative of (A.3) above and setting $z = 0$ gives

$$a_1 c_1 + a_2 c_2 - (a_1 + a_2)c_3 = 0,$$

so that

$$a_1(c_1 - c_3) + a_2(c_2 - c_3) = 0.$$

Taking the second derivative of (A.3) and setting $z = 0$ gives

$$a_1 c_1^2 + a_2 c_2^2 - (a_1 + a_2)c_3^2 = 0,$$

or

$$a_1(c_1 - c_3)^2 + a_2(c_2 - c_3)^2 = 0.$$

Combining these equations, we have linear system

$$\begin{pmatrix} 1 & 1 \\ c_1 + c_3 & c_2 + c_3 \end{pmatrix} \begin{pmatrix} a_1(c_1 - c_3) \\ a_2(c_2 - c_3) \end{pmatrix} = 0.$$

The determinant of the matrix is $c_2 - c_1$. We discuss the solutions case by case:

1. Suppose $c_2 \neq c_1$, then $a_1(c_1 - c_3) = 0$ and $a_2(c_2 - c_3) = 0$.

   - Suppose $a_1, a_2 \neq 0$, then $c_1 = c_3 = c_2$ which is a contradiction.
   - Suppose $a_1 = 0$ and $a_2 \neq 0$, then $a_3 = -a_2$ so that $A_3 c_3 = -A_{2,y}$. Also we have $c_2 = c_3$ but $c_3 \neq c_1$. This implies $h_2 = h_3$ and $h_3 \neq h_1$. But $a_1 = A_{1,x} = 0$ for all x, y. Hence $A_1 = A_1(y)$. This gives

     $$\mathbf{U} = \begin{pmatrix} A_1(y)e^{ih_1 z} \\ A_2(x,y)e^{ih_2 z} \\ A_3(x,y)e^{ih_2 z} \end{pmatrix} = \begin{pmatrix} A_1(y) \\ 0 \\ 0 \end{pmatrix} e^{ih_1 z} + \begin{pmatrix} 0 \\ A_2(x,y) \\ A_3(x,y) \end{pmatrix} e^{ih_2 z}.$$

     A short calculation shows that both $\begin{pmatrix} A_1(y) \\ 0 \\ 0 \end{pmatrix} e^{ih_1 z}$ and $\begin{pmatrix} 0 \\ A_2(x,y) \\ A_3(x,y) \end{pmatrix} e^{ih_2 z}$ satisfy the Maxwell's equations. Hence $\mathbf{U}$ can be separated into two modes of the required form (A.2).
   - In the same way, if $a_1 \neq 0$ and $a_2 = 0$, we have

     $$\mathbf{U} = \begin{pmatrix} A_1(x,y)e^{ih_1 z} \\ A_2(x)e^{ih_2 z} \\ A_3(x,y)e^{ih_1 z} \end{pmatrix} = \begin{pmatrix} 0 \\ A_2(x) \\ 0 \end{pmatrix} e^{ih_2 z} + \begin{pmatrix} A_1(x,y) \\ 0 \\ A_3(x,y) \end{pmatrix} e^{ih_1 z}.$$

     Again, both $\begin{pmatrix} 0 \\ A_2(x) \\ 0 \end{pmatrix} e^{ih_2 z}$ and $\begin{pmatrix} A_1(x,y) \\ 0 \\ A_3(x,y) \end{pmatrix} e^{ih_1 z}$ satisfy the Maxwell's equation and then $\mathbf{U}$ can be separated into two modes of the required form (A.2).
   - Suppose $a_1, a_2 = 0$ so that $a_3 = 0$. Hence $A_3 c_3 = 0$ and $c_1 \neq c_2$. If $c_1, c_2, c_3 \neq 0$, then $a_1 = A_{1,x} = 0, a_2 = A_{2,y} = 0, A_3 = 0$ for all x, y, or $A_1 = A_1(y), A_2 = A_2(x), A_3 = 0$. This gives

     $$\mathbf{U} = \begin{pmatrix} A_1(y)e^{ih_1 z} \\ A_2(x)e^{ih_2 z} \\ 0 \end{pmatrix}.$$

     This is ruled out since $\nabla \times \nabla \times \mathbf{U} - k^2 \mathbf{U} = 0$ implies $A_1'' - c_1^2 A_1 = 0$. But since $A_1'' - (c_1^2 + k^2)A_1 = 0$, this gives $A_1 = 0$, and in the same way $A_2 = 0$.

2. Suppose $c_2 = c_1$, then $(a_1 + a_2)(c_1 - c_3) = 0$.

   - Either $a_1 + a_2 \neq 0$ so that $c_2 = c_1 = c_3$ and then $h_2 = h_1 = h_3$. This gives, as desired,
   $$\mathbf{U} = \begin{pmatrix} A_1(x,y) \\ A_2(x,y) \\ A_3(x,y) \end{pmatrix} e^{ih_1 z}.$$
   - Alternatively, $a_1 + a_2 = 0$ which implies $a_3 = 0$, so $A_3 c_3 = 0$. If $c_3 \neq 0$, then $A_3 = 0$ for all x, y. This gives
   $$\mathbf{U} = \begin{pmatrix} A_1(x,y) \\ A_2(x,y) \\ 0 \end{pmatrix} e^{ih_1 z}.$$

In summary, we have shown the claimed form (A.2) for $\mathbf{U}$.

To further analyze the decomposition of $\mathbf{U}$, since $\mathbf{U}$ satisfies Maxwell's equation (A.1), we have

$$k^2\mathbf{U} = \nabla \times \nabla \times \begin{pmatrix} A_T e^{ihz} \\ 0 \end{pmatrix} + \nabla \times \nabla \times (A_3 e^{ihz}\hat{\mathbf{z}}),$$

where $A_T = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$. Then

$$\begin{aligned} \nabla \times \nabla \times \begin{pmatrix} A_T e^{ihz} \\ 0 \end{pmatrix} &= \nabla \times \left(\begin{pmatrix} -A_2 ih \\ A_1 ih \\ A_{2,x} - A_{1,y} \end{pmatrix} e^{ihz} \right) \\ &= \begin{pmatrix} A_2 h^2 \\ A_1 h^2 \\ (A_{1,x} + A_{2,y}) ih \end{pmatrix} e^{ihz} + \nabla \times ((A_{2,x} - A_{1,y}) e^{ihz}\hat{\mathbf{z}}). \end{aligned}$$

By the divergence free condition, $\nabla \cdot \mathbf{U} = A_{1,x} + A_{2,y} + ihA_3 = 0$, we have

$$\nabla \times \nabla \times \begin{pmatrix} A_T e^{ihz} \\ 0 \end{pmatrix} = \begin{pmatrix} A_2 h^2 \\ A_1 h^2 \\ A_3 h^2 \end{pmatrix} e^{ihz} + \nabla \times ((A_{2,x} - A_{1,y}) e^{ihz}\hat{\mathbf{z}})$$

$$= \quad h^2\mathbf{U} + \nabla \times ((A_{2,x} - A_{1,y})e^{ihz}\hat{\mathbf{z}}).$$

Thus,

$$k^2\mathbf{U} \quad = \quad h^2\mathbf{U} + \nabla \times ((A_{2,x} - A_{1,y})e^{ihz}\hat{\mathbf{z}}) + \nabla \times \nabla \times (A_3 e^{ihz}\hat{\mathbf{z}}).$$

So under our assumption that $k^2 - h^2 \neq 0$,

$$\begin{aligned} \mathbf{U} &= \frac{1}{k^2 - h^2}\nabla \times ((A_{2,x} - A_{1,y})e^{ihz}\hat{\mathbf{z}}) + \frac{1}{k^2 - h^2}\nabla \times \nabla \times (A_3 e^{ihz}\hat{\mathbf{z}}) \\ &= \frac{1}{\lambda^2}\nabla \times ((A_{2,x} - A_{1,y})e^{ihz}\hat{\mathbf{z}}) + \frac{1}{\lambda^2}\nabla \times \nabla \times (A_3 e^{ihz}\hat{\mathbf{z}}). \end{aligned}$$

The Maxwell's equation (A.1) is derived from the Maxwell's system

$$\begin{cases} \nabla \times \mathbf{E} - ik\mathbf{H} &= 0, \\ \nabla \times \mathbf{H} + ik\mathbf{E} &= 0, \end{cases}$$

where $\mathbf{E}$ and $\mathbf{H}$ are electric field and magnetic field, respectively.

If $\mathbf{E} = \mathbf{U} = \begin{pmatrix} A_1(x,y) \\ A_2(x,y) \\ A_3(x,y) \end{pmatrix} e^{ihz}$, then $\mathbf{H} = \begin{pmatrix} B_1(x,y) \\ B_2(x,y) \\ B_3(x,y) \end{pmatrix} e^{ihz}$, where $A_{2,x} - A_{1,y} = ikB_3(x,y)$ and therefore, $\mathbf{U}$ can be written as

$$\mathbf{U} = \frac{ik}{\lambda^2}\nabla \times (B_3 e^{ihz}\hat{\mathbf{z}}) + \frac{1}{\lambda^2}\nabla \times \nabla \times (A_3 e^{ihz}\hat{\mathbf{z}}),$$

where $\lambda^2 = k^2 - h^2$. Let $u = B_3\frac{ik}{\lambda^2}$ and $v = A_3\frac{k}{\lambda^2}$, then we have derived the following representation for $\mathbf{U}$:

$$\mathbf{U} = \nabla \times (ue^{ihz}\hat{\mathbf{z}}) + \frac{1}{k}\nabla \times \nabla \times (ve^{ihz}\hat{\mathbf{z}}).$$

Here u and v are called generating functions for $\mathbf{U}$ and $\hat{\mathbf{z}}$ is called the pilot vector.

Appendix B

IDENTITIES IN VECTOR CALCULUS

B.1 Vector Identities

Assume $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are vectors,

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}, \tag{B.1}$$

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}). \tag{B.2}$$

B.2 Differential Identities

Assume $\mathbf{u}, \mathbf{v}$ are vector functions and p, ϕ are scalar functions,

$$\nabla \times (\nabla p) = 0, \tag{B.3}$$

$$\nabla \cdot (\nabla \times \mathbf{v}) = 0, \tag{B.4}$$

$$\nabla \cdot (\phi \mathbf{v}) = \nabla \phi \cdot \mathbf{v} + \phi \nabla \cdot \mathbf{v}, \tag{B.5}$$

$$\nabla \times (\phi \mathbf{v}) = \phi \nabla \times \mathbf{v} + (\nabla \phi) \times \mathbf{v}, \tag{B.6}$$

$$\nabla \times (\mathbf{u} \times \mathbf{v}) = \mathbf{u}(\nabla \cdot \mathbf{v}) - (\mathbf{u} \cdot \nabla)\mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{u} - \mathbf{v}(\nabla \cdot \mathbf{u}), \tag{B.7}$$

$$\nabla \times (\nabla \times \mathbf{u}) = \nabla(\nabla \cdot \mathbf{u}) - \Delta \mathbf{u}, \tag{B.8}$$

$$\nabla \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot \nabla \times \mathbf{u} - \mathbf{u} \cdot \nabla \times \mathbf{v}. \tag{B.9}$$

In the (B.8), $\Delta \mathbf{u} = (\Delta u_1, \Delta u_2, \Delta u_3)$ in Cartesian coordinates only.

B.3 Differential Identities and Integral Theorems on a Surface

Let $S \subset \mathbb{R}^3$ be a smooth surface with unit normal $\mathbf{n}$ and let $\mathbf{v}$ and p be smooth functions defined in a neighborhood of S. The following identities hold:

$$\nabla_S p = (\mathbf{n} \times \nabla p|_S) \times \mathbf{n}, \tag{B.10}$$

$$\vec{\nabla}_S \times p = -\mathbf{n} \times \nabla_S p, \tag{B.11}$$

$$\mathbf{n} \times (\vec{\nabla}_S \times p) = \nabla_S p, \tag{B.12}$$

$$\nabla_S \times \mathbf{v} = -\nabla_S \cdot (\mathbf{n} \times \mathbf{v}), \tag{B.13}$$

$$\nabla_S \cdot \mathbf{v} = \nabla_S \times (\mathbf{n} \times \mathbf{v}), \tag{B.14}$$

$$\nabla_S \cdot (\mathbf{n} \times \mathbf{v}) = -\mathbf{n} \cdot (\nabla \times \mathbf{v})|_S. \tag{B.15}$$

Theorem B.3.1 *[Theorem 2.5.19 in [51] and Corollary 3.21 in [48]] Let $S \subset \mathbb{R}^2$ be a bounded simply connected Lipschitz domain with unit outward normal ν and unit tangent τ to ∂S. For $u \in C^1(\bar{S})$ and $\mathbf{v} \in (C^1(\bar{S}))^2$, the following Stokes identities hold:*

$$\int_S u \nabla_S \cdot \mathbf{v}\, dx = -\int_S (\nabla_S u \cdot \mathbf{v})\, dx + \int_{\partial S} \nu \cdot \mathbf{v}\, u\, ds, \tag{B.16}$$

$$\int_S u \nabla_S \times \mathbf{v}\, dx = \int_S (\vec{\nabla}_S \times u \cdot \mathbf{v})\, dx + \int_{\partial S} \tau \cdot \mathbf{v}\, u\, ds, \tag{B.17}$$

and

$$\nabla_S \cdot (\vec{\nabla}_S \times u) = 0, \tag{B.18}$$

$$\nabla_S \times (\nabla_S u) = 0. \tag{B.19}$$

Moreover, on ∂S, we have that

$$\nu \cdot \vec{\nabla}_\Sigma \times u = \tau \cdot \nabla_\Sigma u, \tag{B.20}$$

$$\tau \cdot \vec{\nabla}_\Sigma \times u = -\nu \cdot \nabla_\Sigma u. \tag{B.21}$$

B.4 Dyadic Identities

For dyadic function $\mathbb{G}$ written as 3×3 matrix, denote by $\mathbf{g}_l$ the lth column of $\mathbb{G}$ and define $\nabla \cdot \mathbb{G}$ to be the matrix with lth column $\nabla \cdot \mathbf{g}_l$ and $\nabla \times \mathbb{G}$ to be the matrix with lth column $\nabla \times \mathbf{g}_l$.

Assume $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are vector functions and $\mathbb{A}, \mathbb{B}, \mathbb{C}$ are dyadic functions,

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbb{C}) = -\mathbf{b} \cdot (\mathbf{a} \times \mathbb{C}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbb{C}, \tag{B.22}$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbb{C}) = \mathbf{b} \cdot (\mathbf{a} \times \mathbb{C}) - (\mathbf{a} \cdot \mathbf{b})\mathbb{C}, \tag{B.23}$$

$$\nabla \cdot (\nabla \times \mathbb{A}) = \mathbf{0}, \tag{B.24}$$

$$\nabla \times (\nabla \times \mathbb{A}) = \nabla(\nabla \cdot \mathbb{A}) - \Delta \mathbb{A}, \tag{B.25}$$

$$\mathbf{a} \times \mathbb{B} = -\left[\mathbb{B}^T \cdot \mathbf{a}\right]^T, \tag{B.26}$$

$$\mathbf{a} \cdot \mathbb{B} = \mathbb{B}^T \cdot \mathbf{a}, \tag{B.27}$$

$$\mathbb{C}^T \cdot (\mathbf{a} \times \mathbb{B}) = -(\mathbf{a} \times \mathbb{C})^T \cdot \mathbb{B}, \tag{B.28}$$

(also see Section 1-3 in [61]).

B.5 Integral Theorems

Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain with boundary $\partial\Omega$ and unit outward normal $\mathbf{n}_{\partial\Omega}$.

- If $\xi \in C^1(\bar{\Omega})$ and $\mathbf{u} \in (C^1(\bar{\Omega}))^3$, then

$$\int_\Omega \nabla \cdot \mathbf{u}\, \xi \, dx = -\int_\Omega \mathbf{u} \cdot \nabla \xi \, dx + \int_{\partial\Omega} \mathbf{n}_{\partial\Omega} \cdot \mathbf{u}\, \xi \, ds. \tag{B.29}$$

- (First Green's identity) If $\xi \in C^1(\bar{\Omega})$ and $\eta \in C^2(\bar{\Omega})$, then

$$\int_\Omega \Delta \eta \, \xi \, dx = -\int_\Omega \nabla \eta \cdot \nabla \xi \, dx + \int_{\partial\Omega} \frac{\partial \eta}{\partial \mathbf{n}_{\partial\Omega}} \xi \, ds. \tag{B.30}$$

- (Second Green's identity) If $\xi \in C^2(\bar{\Omega})$ and $\eta \in C^2(\bar{\Omega})$, then

$$\int_\Omega (\Delta \eta \, \xi - \eta \Delta \, \xi) \, dx = \int_{\partial\Omega} \left(\frac{\partial \eta}{\partial \mathbf{n}_{\partial\Omega}} \xi - \frac{\partial \xi}{\partial \mathbf{n}_{\partial\Omega}} \eta \right) ds. \tag{B.31}$$

- Suppose $\mathbf{u}$ and ϕ are in $(C^1(\bar{\Omega}))^3$, then

$$\int_\Omega \nabla \times \mathbf{u} \cdot \phi \, dx = \int_\Omega \mathbf{u} \cdot \nabla \times \phi \, dx + \int_{\partial\Omega} \mathbf{n}_{\partial\Omega} \times \mathbf{u} \cdot \phi \, ds. \tag{B.32}$$

Assume $\mathbf{p}, \mathbf{q}$ are vector functions and $\mathbb{P}, \mathbb{Q}$ are dyadic functions, then

- First vector-dyadic Green's identity

$$\int_\Omega \left[(\nabla \times \mathbf{p}) \cdot (\nabla \times \mathbb{Q}) - \mathbf{p} \cdot (\nabla \times \nabla \times \mathbb{Q}) \right] dx = \int_{\partial\Omega} \mathbf{n}_{\partial\Omega} \cdot [\mathbf{p} \times \nabla \times \mathbb{Q}] \, ds, \tag{B.33}$$

- Second vector-dyadic Green's identity

$$\begin{aligned}&\int_\Omega \left[(\nabla \times \nabla \times \mathbf{p}) \cdot \mathbb{Q} - \mathbf{p} \cdot (\nabla \times \nabla \times \mathbb{Q})\right] dx \\ &= \int_{\partial\Omega} \mathbf{n}_{\partial\Omega} \cdot \left[\mathbf{p} \times \nabla \times \mathbb{Q} + (\nabla \times \mathbf{p}) \times \mathbb{Q}\right] ds,\end{aligned} \tag{B.34}$$

- First dyadic-dyadic Green's identity

$$\begin{aligned}&\int_\Omega \left[(\nabla \times \mathbb{Q})^T \cdot (\nabla \times \mathbb{P}) - (\nabla \times \nabla \times \mathbb{Q})^T \cdot \mathbb{P}\right] dx \\ &= \int_{\partial\Omega} (\nabla \times \mathbb{Q})^T \cdot (\mathbf{n}_{\partial\Omega} \times \mathbb{P})\, ds,\end{aligned} \tag{B.35}$$

- Second dyadic-dyadic Green's identity

$$\begin{aligned}&\int_\Omega \left[(\mathbb{Q})^T \cdot (\nabla \times \nabla \times \mathbb{P}) - (\nabla \times \nabla \times \mathbb{Q})^T \cdot \mathbb{P}\right] dx \\ &= \int_{\partial\Omega} \left[(\nabla \times \mathbb{Q})^T \cdot (\mathbf{n}_{\partial\Omega} \times \mathbb{P}) + (\mathbb{Q})^T \cdot (\mathbf{n}_{\partial\Omega} \times \nabla \times \mathbb{P})\right] ds.\end{aligned} \tag{B.36}$$

Appendix C

THEOREMS IN FUNCTIONAL ANALYSIS

Theorem C.0.1 *[Theorem 3.8-4 in [44]]* ***(Riesz Representation)*** *Let H_1, H_2 be Hilbert spaces, $K = \mathbb{R}$ or $\mathbb{C}$ and*

$$h : H_1 \times H_2 \longrightarrow K$$

a bounded sesquilinear form. Then h has a representation

$$h(x, y) = \langle Sx, y \rangle$$

where $S : H_1 \longrightarrow H_2$ is a bounded linear operator. S is uniquely determined by h and has norm

$$\|S\| = \|h\|.$$

Theorem C.0.2 *[Corollary 8.23 in [19]] Let H_1 and H_2 be two Banach spaces and denote by $\mathcal{L}(H_1, H_2)$ the Banach space of bounded linear operators mapping H_1 into H_2. Let D be a domain in $\mathbb{C}$ and let $A : D \to \mathcal{L}(H_1, H_2)$ be an operator valued function such that for each $\varphi \in H_1$ the function $A\varphi : D \to H_2$ is weakly holomorphic. Then A is strongly holomorphic.*

Theorem C.0.3 *[Theorem 8.25 in [19]] Every analytic function is holomorphic and vice versa.*

Theorem C.0.4 *[Theorem 5.2.3 in [36]]* ***(Lax-Milgram Lemma)*** *Let a continuous sesquilinear form $a(u, v)$ be coercive on the Hilbert space H, that is, $a(u, v)$ satisfies*

$$|a(v, v)| \geq \alpha_0 \|v\|_H^2 \text{ for all } v \in H$$

with $\alpha_0 > 0$. Then to every bounded continuous linear functional $l(v)$ on H, there exists a unique solution $u \in H$ of the following variational equation

$$a(u, v) = l(v) \text{ for all } v \in H.$$

Furthermore,

$$\|u\|_H \leq \frac{1}{\alpha_0}\|l\|_{H^*}.$$

where "$*$" represents the dual space.

Theorem C.0.5 *(Theorem 2.1 in [16]) Let $\tau \mapsto A_\tau$ be a continuous mapping from $(0, \infty)$ to the set of self-adjoint and positive definite bounded linear operators on X and let B be a self-adjoint and non-negative compact bounded linear operator on X. We assume that there exists two positive constant $\tau_0 > 0$ and $\tau_1 > 0$ such that*

(1). $A_{\tau_0} - \tau_0 B$ is positive on X,

(2). $A_{\tau_1} - \tau_1 B$ is non-positive on an m dimensional subspace of X.

Then each of the equations $\lambda_j(\tau) = \tau$ for $j = 1, \ldots, m$, has at least one solution in $[\tau_0, \tau_1]$ where $\lambda_j(\tau)$ is the jth eigenvalue (counting multiplicity) of A_τ with respect to B, i.e. $ker(A_\tau - \lambda_j(\tau)B) \neq \{0\}$.

Theorem C.0.6 *[Theorem 3.6 in [2]] Let X denote a Hilbert space. Assume that $C \subset \mathbb{C}$ is an open connected set and that $A_k : X \mapsto X$ is a compact linear operator for all $k \in C$ that depends analytically on k. Then, for all $k \in C$ except possibly for some isolated points, the equation*

$$(I - A_k)u = 0$$

has the same number of linearly independent solutions.

Theorem C.0.7 *[Theorem 2.33 in [48]]* ***(Fredholm Alternative)*** *Let $B : X \to X$ be a bounded linear operator where X is a Hilbert space. Suppose $B = I + A$, where A is a compact operator and I is the identity operator. Then either*

1. *The homogeneous equation $Bu = 0$ has only the trivial solution $u = 0$ in X. In this case, for every $f \in X$, the inhomogeneous equation $Bu = f$ has a unique solution depending continuously on f; or*

2. *The homogeneous equation $Bu = 0$ has exactly p linearly independent solutions for some finite integer $p > 0$.*

Appendix D

THEOREMS IN ELECTROMAGNETIC THEORY

Theorem D.0.8 *[simplified version of Theorem 9.3 in [19]]* ***(Unique Continuation Principle)*** *Let G be a domain in $\mathbb{R}^3$ and let $\mathbf{E}, \mathbf{H} \in C^1(G)$ be a solution of*

$$\nabla \times \mathbf{E} - ik\mathbf{H} = 0, \qquad \nabla \times \mathbf{H} + ik\mathbf{E} = 0$$

Suppose $\mathbf{E}, \mathbf{H}$ vanishes in a neighborhood of some $\mathbf{x}_0 \in G$. Then $\mathbf{E}, \mathbf{H}$ is identically zero in G.

Theorem D.0.9 *[Theorem 6.3 in [19]] Any continuously differentiable solution to the Maxwell equations has analytic cartesian components.*

Appendix E

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