

NUMBER OF COLORINGS OF  $R$ -PARTITE TURAN  
GRAPHS

by  
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## ABSTRACT

Let  $\mathcal{F}_{n,t_r(n)}$  consist of all simple graphs on  $n$  vertices and  $t_r(n)$  edges, where  $t_r(n)$  is the number of edges in the Turan's graph  $T_r(n)$  – the complete  $r$ -partite graph on  $n$  vertices with partition sizes as equal as possible. For a simple graph  $G$  and a positive integer  $\lambda$ , let  $P_G(\lambda)$  denote the number of proper vertex colorings of  $G$  with at most  $\lambda$  colors, and let  $f(n, t_r(n), \lambda) = \max\{P_G(\lambda) : G \in \mathcal{F}_{n,t_r(n)}\}$ . We prove that for all  $n \geq r \geq 2$ ,  $f(n, t_r(n), r + 1) = P_{T_r(n)}(r + 1)$  and that  $T_r(n)$  is the only extremal graph.

# Chapter 1

## INTRODUCTION

Most of the material in this chapter comes from a paper by Lazebnik, Pikhurko and Woldar [8], and a recent paper by Loh, Pikhurko and Sudakov [10].

All graphs in this paper are finite and undirected, and have neither loops nor multiple edges. For all missing definitions and facts which are mentioned but not proved, we refer the reader to Bollobás [5].

For a graph  $G$ , let  $V(G)$  and  $E(G)$  denote the vertex set of  $G$  and the edge set of  $G$ , respectively. Let  $|A|$  denote the cardinality of a set  $A$ . Let  $n = v(G) = |V(G)|$  and  $m = e(G) = |E(G)|$  denote the number of vertices the (*order*) of  $G$ , and number of edges the (*size*) of  $G$ , respectively. An edge  $\{x, y\}$  of  $G$  will also be denoted by  $xy$ , or  $yx$ . For sets  $X, Y$ , let  $X - Y = X \setminus Y$ . For  $A \subseteq V(G)$ , let  $G[A]$  denote the subgraph of  $G$  *induced* by  $A$ , which means that  $V(G[A]) = A$ , and  $E(G[A])$  consists of all edges  $xy$  of  $G$  with both  $x$  and  $y$  in  $A$ . For a vertex  $v$  of  $G$ , let  $N(v) = N_G(v) = \{u \in V(G) : uv \in E(G)\}$  denote the *neighborhood* of  $v$  in  $G$ , and  $d(v) = d_G(v) = |N_G(v)|$  denote the *degree* of  $v$  in  $G$ . For  $A \subseteq V(G)$ , let  $d_A(v) = |A \cap N_G(v)|$  denote the number of neighbors of a vertex  $v$  in  $G$  which are in  $A$ . For two disjoint subsets  $A, B \subseteq V(G)$ , by  $G[A, B]$  we denote the bipartite subgraph of  $G$  such that  $V(G[A, B]) = A \cup B$ , and  $E(G[A, B])$  consists of all edges

of  $G$  with one end-vertex in  $A$  and the other in  $B$ .

Also, if  $f, g : \mathbb{N} \rightarrow \mathbb{R}$  are functions, then we say  $g = \Omega(f)$  if there exists a positive constant  $c$  and  $x_0$  such that  $g(x) \geq cf(x)$  for all  $x > x_0$ . A partition of a set  $S$  is a collection of its disjoint nonempty subsets,  $A_1, A_2, \dots, A_n$ , such that  $S = A_1 \cup A_2 \cup \dots \cup A_n$ .

A graph  $G$  is  $r$ -partite with nonempty vertex classes  $V_1, V_2, \dots, V_r$  if  $V(G)$  is the disjoint union of  $V_1, V_2, \dots, V_r$  and every edge connects two vertices in different vertex classes. We say  $G$  is a complete  $r$ -partite graph if it is  $r$ -partite and every two vertices in different vertex classes are connected.

The Turán graph  $T_r(n)$ ,  $r \geq 1$ , is the complete  $r$ -partite graph of order  $n$  with all parts of size either  $\lfloor n/r \rfloor$  or  $\lceil n/r \rceil$ . It is easy to argue that such a graph is unique. For example, if  $r = 1$ ,  $T_1(n) = \overline{K_n}$ , the graph on  $n$  vertices without any edges. If  $r = 2$ ,  $T_2(n)$  is  $K_{k,k}$  for  $n = 2k$ , and  $K_{k+1,k}$  for  $n = 2k + 1$ . If  $r = n$ ,  $T_n(n) = K_n$ . Let  $t_r(n) = e(T_r(n))$  denote the number of edges of  $T_r(n)$ .

Let  $\mathcal{F}_{n,m}$  consist of all  $(n, m)$ -graphs, that is, graphs of order  $n$  and size  $m$ . For a positive integer  $\lambda$ , let  $[\lambda] = \{1, 2, \dots, \lambda\}$ . A function  $c : V(G) \rightarrow [\lambda]$  such that  $c(x) \neq c(y)$  for every edge  $xy$  of  $G$  is called a *proper vertex coloring* of  $G$  in at most  $\lambda$  colors, or simply a  $\lambda$ -*coloring* of  $G$ . The set  $[\lambda]$  is often referred to as the *set of colors*.

The problem of counting the number  $P_G(\lambda)$  of  $\lambda$ -colorings of a given graph  $G$  has been the focus of much research over the past century. Although it is already NP-hard even to determine whether this number is nonzero, the function  $P_G(\lambda)$  itself

has very interesting properties. It was first introduced by Birkhoff [2], who proved that it is always a polynomial in  $\lambda$ . It is now called the *chromatic* polynomial of  $G$ . Although  $P_G(\lambda)$  has been studied for its own sake (e.g., Whitney [12] expressed its coefficients in terms of graph theoretic parameters), perhaps more interestingly there is a long history of diverse applications which has led researchers to minimize or maximize  $P_G(\lambda)$  over various families of graphs. In fact, Birkhoff's original motivation for investigating the chromatic polynomial was to use it to attack the famous four-color problem. Indeed, one way to show that every planar graph is 4-colorable is to minimize  $P_G(4)$  over all planar graphs, and show that the minimum is nonzero. In this direction Birkhoff [3] proved the tight lower bound  $P_G(\lambda) \geq \lambda(\lambda-1)(\lambda-2)(\lambda-3)^{n-3}$  for all  $n$ -vertex planar graphs  $G$  when  $\lambda \geq 5$ , later conjecturing with Lewis in [4] that it extends to  $\lambda = 4$  as well. An immediate corollary of Birkhoff's bound is that every planar graph is 5-colorable, a non-trivial fact.

Linial [9] arrived at the problem of minimizing the chromatic polynomial from a completely different motivation. The worst-case computational complexity of determining whether a particular function  $f : V(G) \rightarrow [\lambda]$  is a proper coloring is closely related to the number of acyclic orientations of a graph, which equals  $|P_G(-1)|$ , a famous result of Stanley [11], and is obtained by evaluating  $P_G(\lambda)$  at  $\lambda = -1$  and taking the absolute value of the result. Lower bounding the worst-case complexity therefore corresponds to minimizing  $|P_G(-1)|$  over the family  $\mathcal{F}_{n,m}$  of graphs with  $n$  vertices and  $m$  edges. Linial showed that surprisingly, for any  $n, m$ , there is a graph which simultaneously minimizes each  $|P_G(\lambda)|$  over  $\mathcal{F}_{n,m}$ , for every integer  $\lambda$ . This graph is simply a clique  $K_k$  with an additional vertex adjacent to  $l$  vertices of the  $K_k$ , plus  $n - k - 1$  isolated vertices, where  $k, l$  are the unique integers satisfying  $m = \binom{k}{2} + l$  with  $k > l \geq 0$ . At the end of his paper, Linial posed the problem of maximizing



$P_G(\lambda)$  over all graphs in  $\mathcal{F}_{n,m}$ .

Around the same time, Wilf arrived at exactly that maximization problem while analyzing the backtrack algorithm for finding a proper  $\lambda$ -coloring of a graph (see Bender and Wilf [1], and Wilf [13]). Although this generated much interest in the problem, it was only solved in sporadic cases. Let

$$f(n, m, \lambda) = \max\{P_G(\lambda) : G \in \mathcal{F}_{n,m}\}.$$

For known inequalities on  $f(n, m, \lambda)$  and related discussions, for optimizing the number of  $\lambda$ -colorings over other families of graphs, and for the connections between counting colorings and counting graph homomorphisms, see [6]–[8] and [10], and references therein. Here is a list of known “exact” results.

- The value of  $f(n, m, 2)$  was determined, and all extremal graphs were characterized for all  $m, n$  in Lazebnik [6].
- In Lazebnik [7] it was proved that  $f(n, t_r(n), \lambda) = P_{T_r(n)}(\lambda)$ , and that  $T_r(n)$  is the only extremal graph for all  $r \geq 1$  and all large  $\lambda = \Omega(n^6)$ , as  $n \rightarrow \infty$ .
- In Lazebnik, Pikhurko and Woldar [8], it was shown that for all  $k \geq 1$ ,  $f(2k, k^2, 3) = P_{K_{k,k}}(3)$ , and that  $T_2(2k) = K_{k,k}$  is the only extremal graph. It was also shown in [8] that

$$f(2k, k^2, 4) \sim P_{T_2(2k)}(4) \sim (6 + o(1))4^k$$

as  $k \rightarrow \infty$ . This can be stated in other words, as the graph  $T_2(2k)$  is asymptotically extremal for  $\lambda = 4$ . Thus it extended the result from [7] to a small  $\lambda$ , namely  $\lambda = 3$ , but to only special bipartite Turán graphs  $T_2(2k)$ .

- Most recently, Loh, Pikhurko and Sudakov [10] proved that for every  $r \geq 3$ , there exists  $n_0 = n_0(r)$ , such that for all  $n \geq n_0(r)$ ,

$$f(n, t_r(n), r + 1) = P_{T_r(n)}(r + 1),$$

and that  $T_r(n)$  is the only extremal graph. This result extends the one on  $f(2k, k^2, 3)$  from [8] to all  $r \geq 3$ , but it holds only for sufficiently large  $n$ . Though an explicit lower bound on  $n_0$  was not specified in [10], our very conservative estimate is  $n_0 = \Omega(r^3)$  as  $r \rightarrow \infty$ , and we believe that the best lower bound on  $n_0$  for which the result in [10] holds is much greater.

Among many other very interesting results in [10], though with less connections to this thesis, is a proof for large  $m$  of a conjecture of Lazebnik from 1989 concerning the value of  $f(n, m, 3)$  and the structure of extremal graphs in the case when  $m \leq n^2/4$ . It stated that the extremal graphs are complete bipartite graphs with certain ratio of partition sizes minus a star plus some isolated vertices if necessary. In addition, they extended this result for all  $\lambda \geq 4$ , by proving that for large  $m$ , and  $m \approx \frac{1}{\lambda \log \lambda} n^2$ , the structure of extremal graphs is similar to the case of 3-colorings. But the most important feature of [10], in our opinion, is the development of machinery which allows one, in principle, to determine extremal graphs in many nontrivial ranges of  $m$  and  $n$ .

We are now ready to state the main result of this thesis.

**Theorem 1** *For all integers  $n, r$ , with  $n \geq r \geq 1$ ,*

$$f(n, t_r(n), r + 1) = P_{T_r(n)}(r + 1),$$

*and  $T_r(n)$  is the only extremal graph.*

In relation to the aforementioned results, Theorem 1 represents the following improvements.

- It extends the result  $f(n, t_r(n), \lambda) = P_{T_r(n)}(\lambda)$  ([7]) from large  $\lambda = \Omega(n^6)$  to small  $\lambda = r + 1$ .
- It generalizes the result  $f(2k, k^2, 3) = P_{T_2(2k)}(3)$  ([8]) from  $r = 2$  to all  $r \geq 2$ , and it covers the missing special case for  $r = 2$  and  $n = 2k + 1$ .
- It extends the result  $f(n, t_r(n), r + 1) = P_{T_r(n)}(r + 1)$  ([10]) from sufficiently large  $n$  to all  $n \geq r$ .

The proof of Theorem 1 is presented in Chapter 2. This proof grew out of our attempts to extend the aforementioned result on  $f(2k, k^2, 3)$  from [8] to  $f(3k, 3k^2, 4)$ . After finally resolving this case and simplifying the method several times, we began seeing the light: a much simpler and more general argument. It is based on a generalization of the main idea behind Theorem 3 in [8].

In Chapter 3 we mention several related open problems for future investigation.

## Chapter 2

### PROOF OF THEOREM 1

**Lemma 2** *Let  $n$  and  $r$  be positive integers, such that  $1 \leq r \leq n$ . Let  $k = \lfloor \frac{n}{r} \rfloor \geq 1$ , and let  $s = n - rk$ ,  $0 \leq s < r$ . Then*

$$(i) P_{T_r(n)}(r+1) = (r+1)!(s2^k + (r-s)2^{k-1} - (r-1)).$$

$$(ii) \text{ For } n \geq r+1, P_{T_r(n-1)}(r+1) < P_{T_r(n)}(r+1).$$

*Proof of (i):* Denote the maximal independent sets of  $T_r(n)$  by  $V_1, V_2, \dots, V_r$ , such that  $|V_1| = |V_2| = \dots = |V_s| = k+1$ , and  $|V_{s+1}| = |V_{s+2}| = \dots = |V_r| = k$ . Take a proper  $(r+1)$ -coloring of  $T_r(n)$ . It is clear that it must use at least  $r$  colors, and that if it uses all  $r+1$  colors that there exists exactly one  $V_i$  whose points are colored using two colors. Therefore, in order to compute  $P_{T_r(n)}(r+1)$ , we consider three cases.

**Case 1:** Exactly  $r$  colors are used.

Obviously, there are exactly  $\binom{r+1}{r} r! = (r+1)!$  colorings in this case.

**Case 2:** All  $r+1$  colors are used, and there exists exactly one  $i$ ,  $1 \leq i \leq s$ , such that  $V_i$ 's points are colored in two colors.

There are  $s$  ways to choose such  $V_i$ , and there are  $\binom{r+1}{2}$  ways to choose two colors for it. There are  $2^{|V_i|} - 2$  ordered partitions of  $V_i$  into 2 subsets, therefore there are  $2^{|V_i|} - 2$  ways of coloring it with the chosen two colors. Finally, there are  $(r-1)!$  ways to color the remaining  $(r-1)$   $V_j$ 's with the remaining  $(r-1)$  colors. So, there is a total of

$$s \cdot \binom{r+1}{2} \cdot (2^{|V_i|} - 2) \cdot (r-1)! = s(r+1)!(2^k - 1)$$

colorings in this case.

**Case 3:** All  $r+1$  colors are used, and there exists exactly one  $i$ ,  $s+1 \leq i \leq r$ , such that  $V_i$ 's points are colored in two colors.

There are  $r-s$  ways to choose such  $V_i$ , and there are  $\binom{r+1}{2}$  ways to choose two colors for it. There are  $2^{|V_i|} - 2$  ordered partitions of  $V_i$  into 2 subsets, therefore there are  $2^{|V_i|} - 2$  ways of coloring it with the chosen two colors. Finally, there are  $(r-1)!$  ways to color the remaining  $(r-1)$   $V_j$ 's with the remaining  $(r-1)$  colors. So, there is a total of

$$(r-s) \cdot \binom{r+1}{2} \cdot (2^{|V_i|} - 2) \cdot (r-1)! = (r-s)(r+1)!(2^{k-1} - 1)$$

colorings in this case.

Therefore, we have a total of

$$(r+1)! + s(r+1)!(2^k - 1) + (r-s)(r+1)!(2^{k-1} - 1) = (r+1)!(s2^k + (r-s)2^{k-1} - (r-1))$$

colorings, as desired.

*Proof of (ii):* Assume  $s = 0$ . Then,  $n = rk$ , since  $n \geq r + 1$ ,  $k \geq 2$ , and  $n - 1 = r(k - 1) + (r - 1)$ . Using part (i), we obtain

$$\begin{aligned}
P_{T_r(n)}(r + 1) - P_{T_r(n-1)}(r + 1) &= (r + 1)!(r 2^{k-1} - (r - 1)) \\
&\quad - (r + 1)!((r - 1)2^{k-1} + 2^{k-2} - (r - 1)) \\
&= (r + 1)!(2^{k-1} - 2^{k-2}) \\
&> 0
\end{aligned}$$

However, if  $s \geq 1$ , then by part (i), we find

$$\begin{aligned}
P_{T_r(n)}(r + 1) - P_{T_r(n-1)}(r + 1) &= (r + 1)!(s 2^k + (r - s) 2^{k-1} - (r - 1)) \\
&\quad - (r + 1)!((s - 1)2^k + (r - s + 1)2^{k-1} - (r - 1)) \\
&= (r + 1)!(2^k - 2^{k-1}) \\
&> 0
\end{aligned}$$

Therefore,  $P_{T_r(n)}(r + 1) > P_{T_r(n-1)}(r + 1)$ . ■

With this lemma, we are ready to prove Theorem 1.

*Proof of Theorem 1.* We will use induction on  $n$ . The  $r = 1$  case is trivial, and the  $r = 2$ ,  $s = 0$  case was proved in Theorem 1 in [8], so we assume that  $r \geq 3$ , or  $r \geq 2$  and  $s \geq 1$ .

Now, if  $n = r$ , the result is obvious, as  $T_r(n) = T_n(n) = K_n$  and this is the only  $(n, t_r(n))$  graph. Therefore, suppose the theorem is true for all  $m$  such that

$2 \leq r \leq m < n = rk + s$ ,  $k = \lfloor \frac{n}{r} \rfloor \geq 1$ ,  $0 \leq s < r$ .

Let  $G$  be a  $(n, t_r(n))$  graph not isomorphic to  $T_r(n)$ . Then, by Turán's Theorem,  $G$  contains a subgraph isomorphic to  $K_{r+1}$ . Let the set of vertices of this complete subgraph be  $A = \{u_1, \dots, u_{r+1}\}$ . Our proof is divided into two cases, depending on whether the value of  $\sum_{i=1}^{r+1} d(u_i)$  is less than  $(r+1)((r-1)k + s)$ , or at least  $(r+1)((r-1)k + s)$ , and the arguments used in each case will differ.

**Case 1:** We assume that

$$d(u_1) + d(u_2) + \dots + d(u_{r+1}) \leq (r+1)((r-1)k + s) - 1.$$

We show that in this case at least one vertex  $u_i$  has degree small enough that its deletion from  $G$  results in a graph with more than  $t_r(n-1)$  edges, and the proof of the theorem will easily follow. Let  $u_i$  be a vertex in  $A$  with the lowest degree. Then

$$d(u_i) \leq \frac{1}{r+1} \sum_{j=1}^{r+1} d(u_j) < (r-1)k + s.$$

**Case 1.1:** Suppose that  $s \geq 1$ . As  $n = rk + s$ ,  $V(T_r(n-1))$  is partitioned into  $s-1$  parts each having  $k+1$  vertices and  $r-(s-1)$  parts each having  $k$  vertices. Therefore we have:

$$t_r(n) = t_r(rk + s) = \binom{s}{2} \cdot (k+1)^2 + s(r-s) \cdot (k+1)k + \binom{r-s}{2} \cdot k^2,$$

and

$$t_r(n-1) = t_r(rk + (s-1)) = \binom{s-1}{2} \cdot (k+1)^2 + (s-1)(r-s+1) \cdot (k+1)k + \binom{r-s+1}{2} \cdot k^2.$$

Therefore

$$t_r(n) - t_r(n-1) = (r-1)k + (s-1) \geq d(u_i).$$

**Case 1.2:** Suppose  $s = 0$ . In this case,  $V(T_r(n-1))$  is partitioned into  $r-1$  parts each having  $k$  vertices and one part having  $k-1$  vertices. So  $n-1 = rk-1 = r(k-1) + (r-1)$ , and we have:

$$t_r(n) = \binom{r}{2} \cdot k^2,$$

and

$$t_r(n-1) = \binom{r-1}{2} \cdot k^2 + \binom{r-1}{1} \cdot (k-1)k.$$

Therefore

$$t_r(n) - t_r(n-1) = (r-1)k > d(u_i).$$

Let  $G' = G[V(G) - \{u_i\}]$ . Then  $v(G') = n-1$  and  $e(G') > t_r(n-1)$ . Also,  $G'$  contains a copy of  $K_r$ , namely  $G'[A - \{u_i\}]$ . As  $u_i$  is adjacent to all its vertices, there exists at most one way to extend a proper  $(r+1)$ -coloring of  $G'$  to the one of  $G$ . Therefore,  $P_G(r+1) \leq P_{G'}(r+1)$ . Deleting edges from  $G'$ , we can obtain a graph  $G''$  such that  $v(G'') = n-1$  and  $e(G'') = t_r(n-1)$ . Then  $P_{G'}(r+1) \leq P_{G''}(r+1)$ , and, as  $n-1 \geq r$ , we have  $P_{G''}(r+1) \leq P_{T_r(n-1)}(r+1)$  by the induction hypothesis. Therefore, we have

$$P_G(r+1) \leq P_{G'}(r+1) \leq P_{G''}(r+1) \leq P_{T_r(n-1)}(r+1) < P_{T_r(n)}(r+1),$$

where the last inequality follows from Lemma 2(ii). This ends the proof of Case 1.

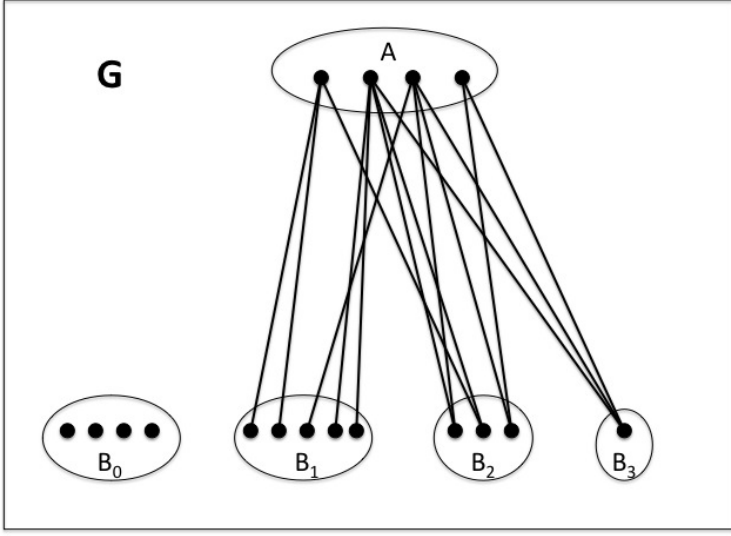
**Case 2:** We assume that

$$d(u_1) + d(u_2) + d(u_3) + \dots + d(u_{r+1}) \geq (r+1)((r-1)k + s).$$

For each  $i$ ,  $0 \leq i \leq r+1$ , let us define the following subsets of  $V(G) - A$ :

$$B_i = \{v \in V(G) - A \mid d_A(v) = i\}.$$





**Figure 2.1:** An example graph of  $G[V(G) - A, A]$  in the  $r = 3, \lambda = 4$  case.

If  $G$  contains a subgraph isomorphic to  $K_{r+2}$ , then  $P_G(r+1) = 0 < P_{T_r(n)}(r+1)$ , and the proof is finished. Therefore, we assume that  $G$  contains no  $(r+2)$ -clique. Then  $B_{r+1} = \emptyset$  and  $V(G)$  is the union of  $r+2$  pairwise disjoint subsets (with some possibly empty):

$$V(G) = A \cup B_0 \cup B_1 \cup \dots \cup B_r.$$

Let  $b_i = |B_i|$  for  $i = 0, \dots, r$ . Since  $G[A]$  is an  $(r+1)$ -clique,

$$e(G[A, V(G) - A]) = d(u_1) + d(u_2) + d(u_3) + \dots + d(u_{r+1}) - (r+1)r.$$

However, since every vertex in  $B_i$  is connected to exactly  $i$  vertices in  $A$ , therefore

$$e(G[A, B_i]) = ib_i.$$

As sets  $B_i$  are pairwise disjoint,

$$e(G[A, V(G) - A]) = e(G[A, B_0]) + e(G[A, B_1]) + \dots + e(G[A, B_r]),$$

and since

$$d(u_1) + d(u_2) + d(u_3) + \dots + d(u_{r+1}) \geq (r+1)((r-1)k + s),$$

we obtain that:

$$\sum_{i=0}^r ib_i \geq (r+1)((r-1)k + s) - (r+1)r. \quad (2.1)$$

In addition, since there are  $n - (r+1) = rk + s - (r+1)$  vertices in  $V(G) - A$ , and  $B_0, B_1, \dots, B_r$  are all disjoint sets, we have that

$$\sum_{i=0}^r b_i = rk + s - (r+1). \quad (2.2)$$

Now, by multiplying (2.2) by  $r$  and subtracting it from (2.1), we obtain

$$\sum_{i=0}^r (ib_i - rb_i) \geq (r+1)((r-1)k + s) - r(r+1) - (r^2k + rs - r(r+1)) = s - k.$$

Hence,

$$\sum_{i=0}^r (i-r)b_i = \sum_{i=0}^{r-1} (i-r)b_i \geq s - k,$$

which gives

$$b_{r-1} \leq k - s - \sum_{i=0}^{r-2} (r-i)b_i. \quad (2.3)$$

Consider an  $(r+1)$ -coloring of  $G$ . Since all vertices of  $A$  are assigned distinct colors, and since every vertex in  $B_i$  is adjacent to  $i$  vertices of  $A$ , there are at most

$(r + 1) - i$  ways to color each vertex in  $B_i$ . As there are  $(r + 1)!$  ways to color  $A$ , using (2.3), we have that

$$\begin{aligned}
P_G(r + 1) &\leq (r + 1)! \prod_{i=0}^r (r + 1 - i)^{b_i} \\
&\leq (r + 1)! 2^{b_{r-1}} \prod_{i=0}^{r-2} (r + 1 - i)^{b_i} \\
&\leq (r + 1)! 2^{k-s} \prod_{i=0}^{r-2} \frac{(r + 1 - i)^{b_i}}{2^{(r-i)b_i}} \\
&\leq (r + 1)! 2^{k-s} \prod_{i=0}^{r-2} \left( \frac{r + 1 - i}{2^{r-i}} \right)^{b_i} \\
&\leq (r + 1)! 2^{k-s}.
\end{aligned}$$

As  $n = rk + s > r$ ,  $k = \lfloor \frac{n}{k} \rfloor \geq 1$ , and  $0 \leq s < r$ , we have either  $s \geq 1$ , or  $k \geq 2$ .

Suppose that  $s \geq 1$ . Then, as  $r \geq 2$ , we have

$$\begin{aligned}
P_{T_r(n)}(r + 1) - (r + 1)! 2^{k-s} &= (r + 1)! (s 2^k + (r - s) 2^{k-1} - (r - 1) - 2^{k-s}) \\
&\geq (r + 1)! (2^k + (r - 1) 2^{k-1} - (r - 1) - 2^{k-1}) \\
&= (r + 1)! (2^k + (r - 2) 2^{k-1} - (r - 1)) \\
&> 0.
\end{aligned}$$

Note that this extends the result in [8] for  $\lambda = 3$  from  $(2k, k^2)$ -graphs to  $(2k + 1, k(k + 1))$ -graphs, and, hence, proves Theorem 1 for  $r = 2$ ,  $\lambda = r + 1 = 3$  case.

Finally we assume that  $r \geq 3$  and  $s = 0$ . If  $k \geq 2$ , we obtain

$$\begin{aligned}
P_{T_r(n)}(r+1) - (r+1)!2^{k-s} &= (r+1)!(s2^k + (r-s)2^{k-1} - (r-1) - 2^{k-s}) \\
&= (r+1)!(r2^{k-1} - (r-1) - 2^k) \\
&= (r+1)!((r-2)2^{k-1} - (r-1)) \\
&\geq (r+1)!(2(r-2) - (r-1)) \\
&= (r+1)!(r-3) \\
&\geq 0,
\end{aligned}$$

with equality if and only if  $k = 2$ ,  $r = 3$ , and  $s = 0$ . This implies  $n = 2 \cdot 3 + 0 = 6$ .

Therefore we assume that  $r = 3$  and  $n = 6$ . In this case,  $A = \{u_1, \dots, u_4\}$ , and  $P_{T_r(n)}(r+1) = P_{T_3(6)}(4) = 4!(0 \cdot 2^2 + 3 \cdot 2^1 - 2) = 96$ . Now, since  $G$  has only 6 vertices, then  $|V(G) - A| = 2$ , and  $e(G[V(G) - A]) \leq e(K_2) = 1$ . In addition,

$$e(G) = t_3(6) = 12 = e(G[A]) + e(G[V(G) - A]) + e(G[A, V(G) - A]).$$

Therefore we have

$$12 \leq 6 + 1 + (d(u_1) + d(u_2) + d(u_3) + d(u_4) - 12),$$

which leads to

$$d(u_1) + d(u_2) + d(u_3) + d(u_4) \geq 17.$$

So,  $e(G[A, V(G) - A]) \geq 5$ . Let  $V(G) - A = \{x, y\}$ ,  $d_A(x) \geq d_A(y)$ . Then  $d_A(x) + d_A(y) \geq 5$ , and so  $d_A(x) \geq 3$ . If  $d_A(x) \geq 4$ , then  $G$  contains a copy of  $K_5$ . This implies  $P_G(4) = 0 < P_{T_3(6)}(4)$ .

If  $d_A(x) = 3$ , then  $d_A(y) = 2$ . Now, there exist  $4!$  ways to color properly the vertices in  $A$ . Each such coloring can be extended to a proper coloring of  $G$  in at

most two ways, as  $x$  can be colored uniquely, and  $y$  can be colored in at most 2 ways. This shows that  $P_G(4) \leq 4! \cdot 2 = 48$ , which is less than  $P_{T_r(n)}(r+1) = P_{T_3(6)}(4) = 96$ , and so this ends the proof of Case 2, and of Theorem 1. ■

## Chapter 3

### EXTENDING TOWARDS $R = 2, \lambda = 4$

Now, after we proved that  $T_r(n)$  is extremal for  $\lambda = r + 1$ , the obvious next step is  $\lambda = r + 2$ , and the simplest case of that would be  $r = 2, \lambda = 4$ . Unfortunately, we found that although our method for Case 1 was easily extended, the method for Case 2 was not. Still though, that is enough to generate the following result.

**Theorem 3** *Let  $G$  be a  $(n, t_2(n))$  graph. Suppose that  $P_G(4) > 6 \cdot 2^n$ , that  $G$  is not isomorphic to  $T_2(n)$ , and if  $H$  is a  $(k, t_2(k))$  graph with  $k < n$ , then  $P_H(4) \leq 6 \cdot 2^k$ . Then, for every vertex  $v$  in a triangle in  $G$ ,*

$$d(v) \geq t_2(n) - t_2(n - 1).$$

*Sketch of Proof.* Suppose that  $G$  satisfies the hypotheses listed above, that  $\{u, v, w\}$  is the vertex set of a triangle in  $G$ , and that  $d(w) < t_2(n) - t_2(n - 1)$ . Let  $G'$  be the induced graph with vertex set  $V(G) - \{w\}$ . Then,  $v(G') = n - 1$ , and  $e(G') > t_2(n - 1)$ . Therefore,  $G'$  has a  $(n - 1, t_2(n - 1))$ -subgraph  $G''$  containing a triangle. As  $P_{G'}(4) \leq P_{G''}(4)$ , and  $P_{G''}(4) \leq 6 \cdot 2^{n-1}$  by the theorem hypothesis,  $P_{G'}(4) \leq 6 \cdot 2^{n-1}$ . However, since  $w$  is adjacent to both  $u$  and  $v$ , which are also connected, every 4-coloring of  $G'$  is extendable to a 4-coloring of  $G$  in at most two ways. This implies that  $P_G(4) \leq 2 \cdot P_{G'}(4) \leq 6 \cdot 2^n$ , which contradicts the hypothesis. This finishes the proof. ■

Another possible way of extending our result is to examine graphs with  $n$  vertices, and a slightly different number of edges compared to  $t_r(n)$ , namely  $t_r(n) \pm a$ , where  $a$  is independent of  $n$  and small. This leads to the following conjectures for all  $n \geq r \geq 2$ .

**Conjecture 4** *Let  $G$  be a  $(n, t_r(n) - 1)$ -graph with the maximum number of  $r + 1$  colorings. Then,  $G$  is isomorphic to a subgraph of  $T_r(n)$ .*

**Conjecture 5** *Let  $G$  be a  $(n, t_r(n) + 1)$ -graph with the maximum number of  $r + 1$  colorings. Then,  $G$  contains a subgraph isomorphic to  $T_r(n)$ .*

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