Domain decomposition methods via boundary integral equations

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Abstract

Domain decomposition methods are designed to deal with coupled or transmission problems for partial differential equations. Since the original boundary value problem is replaced by local problems in substructures, domain decomposition methods are well suited for both parallelization and coupling of different discretization schemes. In general, the coupled problem is reduced to the Schur complement equation on the skeleton of the domain decomposition. Boundary integral equations are used to describe the local Steklov–Poincaré operators which are basic for the local Dirichlet–Neumann maps. Using different representations of the Steklov–Poincaré operators we formulate and analyze various boundary element methods employed in local discretization schemes. We give sufficient conditions for the global stability and derive corresponding a priori error estimates. For the solution of the resulting linear systems we describe appropriate iterative solution strategies using both local and global preconditioning techniques.

Key words: domain decomposition, boundary integral equations, boundary element methods, preconditioning techniques.

1 Introduction

Domain decomposition methods were originally designed to solve boundary value problems in complicated domains. We mention here only the famous alternating Schwarz method [25]. Since modern parallel computers are available, these methods have become very useful in the numerical analysis of partial differential equations, in particular, with respect to the development of efficient

algorithms for the numerical solution of complicated problems, see e.g. [35]. Due to the decomposition into substructures, domain decomposition methods are well suited for the coupling of different discretization schemes such as finite and boundary element methods, see e.g. [5,7,10]. In finite element methods, the domain decomposition approach is often applied to construct efficient preconditioners for parallel computations. This is mainly based on a splitting of the global trial space into local ones arising from the domain decomposition. Applying these ideas to boundary integral equations leads to additive Schwarz methods based on a decomposition of the boundary into overlapping or non–overlapping parts, see e.g. [12,19,21,32].

Here we will concentrate our considerations to geometry–based domain decomposition methods where the original boundary value problem is reduced to local subproblems involving appropriate coupling conditions. When assuming boundary conditions of either Dirichlet or Neumann type on the local subdomain boundaries, the solution of the local subproblems defines local Dirichlet–Neumann or Neumann–Dirichlet maps. Hence, in domain decomposition methods we need to find the complete Cauchy data on the skeleton. This results in a variational formulation to find either the Dirichlet or Neumann data on the skeleton, and the remaining data are determined by the local problems and the coupling conditions. Using boundary integral equations we are able to describe the Dirichlet–Neumann map by the Steklov–Poincaré operator which admits different representations. Analyzing the mapping properties of local boundary integral operators [8,9,34], we get unique solvability of the resulting boundary integral variational problem. Moreover, applying a standard Galerkin scheme, we get stability and quasi–optimal a priori error estimates for the approximate solution. However, boundary integral representations of the Steklov–Poincaré operator involve inverse integral operators. Hence we are not able to compute the corresponding stiffness matrices exactly. Therefore we have to define suitable boundary element approximations and we need to derive related stability and error estimates, see e.g. [13,24,26]. Finally, we will discuss the efficient solution of the resulting linear systems by appropriate iterative methods in parallel. Here we need local and global preconditioning matrices.

2 Domain decomposition methods

As a model problem, we consider the Dirichlet boundary value problem

$$L(x)u(x) = f(x) \quad \text{for } x \in \Omega, \qquad u(x) = g(x) \quad \text{for } x \in \Gamma. \tag{2.1}$$

Here $\Omega \subset I\!R^n$, $n = 2$ or $n = 3$ is a bounded domain with Lipschitz boundary $\Gamma = \partial\Omega$ and $L(\cdot)$ is a formally positive elliptic partial differential operator of

second order. Applications of (2.1) are, for example, boundary value problems in potential theory and in elastostatics. In domain decomposition methods, we begin with the decomposition of Ω. Let

$$\overline{\Omega} = \bigcup_{i=1}^{p} \overline{\Omega}_i, \tag{2.2}$$

be a subdivision into p non–overlapping subdomains Ω_i. Note that this decomposition can be done either due to the geometrical form of Ω or due to some properties of the partial differential operator involved in (2.1). In particular, for $x \in \Omega_i$ we assume that $L(x) = L_i$ is a partial differential operator with constant coefficients which can be different in different subdomains. Without loss of generality we also assume that the local subdomain boundaries $\Gamma_i = \partial\Omega_i$ are strong Lipschitz. We denote by $\Gamma_{ij} = \Gamma_i \cap \Gamma_j$ for $i, j = 1, \ldots, p$ local coupling boundaries, and define the skeleton Γ_S of the domain decomposition (2.2) by

$$\Gamma_S = \bigcup_{i=1}^{p} \Gamma_i = \Gamma \cup \bigcup_{i,j=1}^{p} \Gamma_{ij}. \tag{2.3}$$

Defining $u_i(x) = u(x)$ for $x \in \Omega_i$, instead of (2.1) we need to consider local boundary value problems

$$L_i u_i(x) = f(x) \quad \text{for } x \in \Omega_i, \qquad u_i(x) = g(x) \quad \text{for } x \in \Gamma_i \cap \Gamma_D. \tag{2.4}$$

In addition to the boundary conditions in (2.4), we need also appropriate coupling conditions across all local coupling boundaries Γ_{ij}. More precisely, let $T_i u_i(x)$ denote the conormal derivative of u_i defined for $x \in \Gamma_i$ almost everywhere. Then, the natural coupling conditions, induced by (2.1), are

$$u_i(x) = u_j(x), \quad (T_i u_i)(x) + (T_j u_j)(x) = 0 \quad \text{for } x \in \Gamma_{ij}. \tag{2.5}$$

As will be seen, the essence of the domain decomposition methods amounts to reduce the solution of the original boundary value problem (2.1) to the solutions of local boundary value problems (2.4), (2.5). According to (2.5) we may formulate different domain decomposition methods, additive and multiplicative Schwarz methods, leading to different discretization techniques as well. In what follows we will restrict ourselves to the case that the first coupling condition in (2.5), $u_i(x) = u_j(x)$ for $x \in \Gamma_{ij}$ is required to be satisfied pointwise, while the second condition will be required in a week sense only.

We now need some function spaces. We denote by $H^{1/2}(\Gamma_S)$ the trace space of $H^1(\Omega)$ equipped with the norm

$$||u||_{H^{1/2}(\Gamma_S)} := \left\{ \sum_{i=1}^{p} ||u_{|\Gamma_i}||^2_{H^{1/2}(\Gamma_i)} \right\}^{1/2}. \tag{2.6}$$

Let $u \in H^{1/2}(\Gamma_S)$ with $u(x) = g(x)$ for $x \in \Gamma_D$. Then we define the restrictions $u_i(x) = u(x)$ for $x \in \Gamma_i$ which implies that $u_i(x) = u_j(x)$ for $x \in \Gamma_{ij}$. Now we consider local Dirichlet boundary value problems

$$L_i u_i(x) = f(x) \quad \text{for } x \in \Omega_i, \qquad u_i(x) = u(x) \quad \text{for } x \in \Gamma_i \tag{2.7}$$

and define the corresponding local Dirichlet–Neumann maps

$$\mathbf{T}_i u(x) := (T_i u_i)(x) \quad \text{for } x \in \Gamma_i. \tag{2.8}$$

The latter implies that the Neumann coupling condition in (2.5) can be rewritten as

$$\mathbf{T}_i u(x) + \mathbf{T}_j u(x) = 0 \quad \text{for } x \in \Gamma_{ij}. \tag{2.9}$$

Let $\widetilde{g} \in H^{1/2}(\Gamma_S)$ be an arbitrary but fixed extension of the given Dirichlet data g satisfying $\widetilde{g}(x) = g(x)$ for $x \in \Gamma_D$. By defining the test function space

$$W := \left\{ v \in H^{1/2}(\Gamma_S) \,:\, v(x) = 0 \text{ for } x \in \Gamma_D \right\}, \tag{2.10}$$

we have the variational formulation of (2.9) to find $\widetilde{u} \in W$ such that $u = \widetilde{u} + \widetilde{g}$ and

$$\sum_{i=1}^{p} \int_{\Gamma_i} \mathbf{T}_i u(x) \cdot v(x) \, ds_x \;=\; 0 \quad \text{for all } v \in W. \tag{2.11}$$

In what follows we will describe a boundary integral approach to express the local Dirichlet–Neumann maps (2.8) by using boundary integral operators, see e.g. [15,16]. Based on mapping properties of local boundary integral operators we show unique solvability of (2.11). Note that the local Dirichlet–Neumann maps can be expressed in terms of local domain bilinear forms for which the unique solvability of (2.11) follows directly based on the corresponding result of (2.1). In fact, using domain bilinear forms in some subdomains Ω_i for $i = 1, \ldots, q < p$, leads to a coupled finite and boundary element formulation.

3 Boundary integral operators

We now assume that for each subdomain Ω_i there **exists a corresponding fundamental solution** $U^i(x, y)$, see [23, Section 2.3] for a general discussion. Then the solution of the local subproblems (2.7) is given by the representation formulae

$$u_i(x) = \int_{\Gamma_i} U^i(x,y)(T_i u)(y) ds_y - \int_{\Gamma_i} T_{i,y} U^i(x,y) u(y) ds_y \qquad (3.1)$$
$$+ \int_{\Omega_i} U^i(x,y) f(y) dy \qquad \text{for } x \in \Omega_i.$$

Now we define the standard boundary integral operators locally for $x \in \Gamma_i$, the single layer potential operator

$$(V_i t_i)(x) = \int_{\Gamma_i} U^i(x,y) t_i(y) ds_y, \qquad (3.2)$$

the double layer potential operator

$$(K_i u_i)(x) = \int_{\Gamma_i} T_i U^i(x,y) u_i(y) ds_y \qquad (3.3)$$

and the adjoint double layer potential

$$(K_i' t_i)(x) = \int_{\Gamma_i} T_{i,x} U^i(x,y) t_i(y) ds_y \qquad (3.4)$$

as well as the hypersingular integral operator

$$(D_i u_i)(x) = -T_{i,x} \int_{\Gamma_i} T_i U^i(x,y) u_i(y) ds_y. \qquad (3.5)$$

The mapping properties of all local boundary integral operators defined above are well known, see e.g. [8,9]. In particular, the boundary integral operators are bounded for $|s| \leq 1$:

$$\begin{aligned} V_i &: H^{-1/2+s}(\Gamma_i) \to H^{1/2+s}(\Gamma_i), \\ D_i &: H^{1/2+s}(\Gamma_i) \to H^{-1/2+s}(\Gamma_i), \\ K_i &: H^{1/2+s}(\Gamma_i) \to H^{1/2+s}(\Gamma_i), \\ K_i' &: H^{-1/2+s}(\Gamma_i) \to H^{-1/2+s}(\Gamma_i). \end{aligned}$$

Moreover, without loss of generality, we assume that the local single layer potentials V_i are $H^{-1/2}(\Gamma_i)$–elliptic satisfying

$$\langle V_i w_i, w_i \rangle_{L_2(\Gamma_i)} \geq c \, ||w_i||^2_{H^{-1/2}(\Gamma_i)} \quad \text{for all } w_i \in H^{-1/2}(\Gamma_i). \qquad (3.6)$$

The local hypersingular integral operators D_i are assumed to be $H^{1/2}(\Gamma_i)$ semi–elliptic,

$$\langle D_i u_i, u_i \rangle_{L_2(\Gamma_i)} \geq c \, ||u_i||^2_{H^{1/2}(\Gamma_i)} \quad \text{for all } u_i \in H^{1/2}(\Gamma_i)_{/\mathbf{R}_i}. \qquad (3.7)$$

Here, $\mathbf{R}_i$ is the solution space of the local homogeneous Neumann boundary value problems defined by $L_i u_i = 0$ in Ω and $T_i u_i = 0$ on Γ_i.

In addition to the boundary integral operators defined above we will use the local Newton potentials given by

$$(N_i f)(x) \;=\; \int\limits_{\Omega_i} U^i(x,y) f(y) dy \quad \text{for } x \in \Gamma_i. \tag{3.8}$$

Then, the standard boundary integral equation related to the local partial differential equation in (2.7) is

$$(V_i t_i)(x) \;=\; (\frac{1}{2} I + K_i) u_i(x) - (N_i f)(x) \quad \text{for } x \in \Gamma_i. \tag{3.9}$$

Since the local single layer potential operators V_i are assumed to be invertible we can describe the local Dirichlet–Neumann map by

$$t_i(x) \;=\; (S_i u_i)(x) - V_i^{-1}(N_i f)(x) \quad \text{for } x \in \Gamma_i \tag{3.10}$$

using the Steklov–Poincaré operator

$$\begin{aligned}
(S_i u_i)(x) &= V_i^{-1}(\frac{1}{2} I + K_i) u_i(x) && (3.11)\\
&= [D_i + (\frac{1}{2} I + K_i') V_i^{-1} (\frac{1}{2} I + K_i)] u_i(x) && (3.12)\\
&= (\frac{1}{2} I - K_i')^{-1} D_i u_i(x). && (3.13)
\end{aligned}$$

Hence, the Dirichlet–Neumann map (2.8) can be written as

$$\mathbf{T}_i u(x) \;=\; (S_i u)(x) - V_i^{-1}(N_i f)(x) \quad \text{for } x \in \Gamma_i. \tag{3.14}$$

Inserting (3.14) into the variational problem (2.11) we get the boundary integral variational formulation: to find $\widetilde{u} \in W$ such that

$$\sum_{i=1}^{p} \int\limits_{\Gamma_i} (S_i \widetilde{u})(x) \cdot v(x) \; ds_x \;=\; \sum_{i=1}^{p} \int\limits_{\Gamma_i} \left[V_i^{-1}(N_i f)(x) - (S_i \widetilde{g})(x) \right] \cdot v(x) \; ds_x \tag{3.15}$$

holds for all $v \in W$.

Theorem 1 *[5,7,14,16] The global boundary integral bilinear form*

$$a(v,w) \;=\; \sum_{i=1}^{p} \int\limits_{\Gamma_i} (S_i v)(x) \cdot w(x) \; ds_x \tag{3.16}$$

is bounded in $H^{1/2}(\Gamma_S)$ *and* W*–elliptic, i.e.,*

$$a(v,v) \;\geq\; c_1^S \cdot ||v||^2_{H^{1/2}(\Gamma_S)} \quad \textit{for all } v \in W. \tag{3.17}$$

PROOF. From the mapping properties of the local boundary integral operators we get

$$||S_i u_i||_{H^{-1/2}(\Gamma_i)} \;\leq\; c\;||u_i||_{H^{1/2}(\Gamma_i)} \quad \text{for all } u_i \in H^{1/2}(\Gamma_i).$$

Therefore,

$$\begin{aligned}
|a(u,v)| \leq & \sum_{i=1}^{p} |\langle S_i u_{|\Gamma_i}, v_{|\Gamma_i}\rangle_{L_2(\Gamma_i)}| \;\leq\; c\;\sum_{i=1}^{p} ||u_{|\Gamma_i}||_{H^{1/2}(\Gamma_i)} ||v_{|\Gamma_i}||_{H^{1/2}(\Gamma_i)} \\
\leq & c\;\left(\sum_{i=1}^{p} ||u_{|\Gamma_i}||^2_{H^{1/2}(\Gamma_i)}\right)^{1/2} \left(\sum_{i=1}^{p} ||v_{|\Gamma_i}||^2_{H^{1/2}(\Gamma_i)}\right)^{1/2} \\
= & c\;||u||_{H^{1/2}(\Gamma_S)} ||v||_{H^{1/2}(\Gamma_S)} \qquad \text{for all } u, v \in H^{1/2}(\Gamma_S).
\end{aligned}$$

For $u \in W$ we have $u(x) = 0$ for $x \in \Gamma_D$. Since there is at least one subdomain boundary Γ_{i^*} with $\Gamma_{i^*} \cap \Gamma_D \neq \emptyset$ we conclude $u_{|\Gamma_{i^*}} \in H^{1/2}(\Gamma_{i^*})_{/\mathbf{R}_i}$. We can repeat this argument recursively to get $u_{|\Gamma_i} \in H^{1/2}(\Gamma_i)_{/\mathbf{R}_i}$ for all $i = 1, \ldots, p$. Hence we have, using the symmetric representation (3.12),

$$\langle S_i u_{|\Gamma_i}, u_{\Gamma_i}\rangle_{L_2(\Gamma_i)} \;\geq\; \langle D_i u_{|\Gamma_i}, u_{|\Gamma_i}\rangle_{L_2(\Gamma_i)} \;\geq\; c\;||u_{|\Gamma_i}||^2_{H^{1/2}(\Gamma_i)}$$

Summation over $i = 1, \ldots, p$ gives (3.17). $\square$

With Theorem 1, all assumptions of the Lax–Milgram lemma are satisfied, hence there exists a unique solution $\widetilde{u} \in W$ satisfying the variational problem (3.15).

4 Boundary element methods

Let

$$W_h \;:=\; \text{span}\{\varphi_k\}_{k=1}^{M} \subset W \tag{4.1}$$

be a boundary element trial space with piecewise polynomial basis functions φ_k of polynomial degree μ. A suitable choice is the use of piecewise linear trial

functions with $\mu = 1$. For convenience, we define also local restrictions of W_h onto Γ_i, in particular,

$$W_{h,i} = \mathrm{span}\{\varphi_{k,i}\}_{k=1}^{M_i}. \tag{4.2}$$

Obviousely, for any $\varphi_{k,i} \in W_{h,i}$ there exists a unique basis function $\varphi_k \in W_h$ with $\varphi_{k,i} = \varphi_{k|\Gamma_i}$. By using the isomorphisms

$$\underline{u}_i \in I\!R^{M_i} \leftrightarrow u_{h,i} = \sum_{k=1}^{M_i} u_{i,k}\varphi_{k,i} \in W_{h,i}, \quad \underline{u} \in I\!R^{M} \leftrightarrow u_h = \sum_{k=1}^{M} u_k\varphi_k \in W_h,$$

there exist connectivity matrices $A_i \in I\!R^{M_i \times M}$ such that

$$\underline{u}_i = A_i \underline{u}\,. \tag{4.3}$$

We assume that there holds an approximation property of W_h in W,

$$\inf_{v_h \in W_h} ||v - v_h||_{H^{1/2}(\Gamma_S)} \;\leq\; \left(\sum_{i=1}^{p} h_i^{2s-1} \; ||v||^2_{H^s(\Gamma_i)} \right)^{1/2} \tag{4.4}$$

for all $v \in W \cap \prod_{i=1}^{p} H^s(\Gamma_i)$ and $s \leq \mu + 1$ where h_i is the local mesh size of the underlying boundary element mesh on Γ_i.

The Galerkin variational problem of (3.15) is to find a boundary element approximation $\widetilde{u}_h \in W_h$ satisfying

$$\sum_{i=1}^{p} \int_{\Gamma_i} (S_i \widetilde{u}_h)(x) v_h(x)\; ds_x \;=\; \sum_{i=1}^{p} \int_{\Gamma_i} \left[V_i^{-1}(N_i f)(x) - (S_i \widetilde{g})(x) \right] v_h(x)\; ds_x \tag{4.5}$$

for all test functions $v_h \in W_h$. This is equivalent to a system of linear equations, $S_h \underline{\widetilde{u}} = \underline{f}$, with a stiffness matrix S_h defined by

$$S_h[\ell, k] \;=\; \sum_{k=1}^{p} \langle S_i \varphi_k, \varphi_\ell \rangle_{L_2(\Gamma_i)} \;=\; \sum_{i=1}^{p} S_{h,i}[\ell, k] \quad \text{for } k, \ell = 1, \ldots, M. \tag{4.6}$$

Since the associated bilinear form is W–elliptic, Cea's lemma provides the quasi–optimal error estimate

$$||\widetilde{u} - \widetilde{u}_h||_{H^{1/2}(\Gamma_S)} \;\leq\; c \cdot \inf_{v_h \in W_h} ||\widetilde{u} - v_h||_{H^{1/2}(\Gamma_S)} \tag{4.7}$$

and, hence, convergence due to the approximation property of $W_h \subset W$. In fact, in order to assemble (4.6) we have to compute the local stiffness matrices defined by

$$S_{h,i}[\ell, k] \;=\; \langle S_i \varphi_{k|\Gamma_i}, \varphi_{\ell|\Gamma_i} \rangle_{L_2(\Gamma_i)} \tag{4.8}$$

using the definition of the local Steklov–Poincaré operators S_i. Note that all of these representations include a composition of different boundary integral operators including some inverse operators as well. Hence, the Galerkin scheme (4.5) cannot be realized exactly in general. Instead, we have to introduce some local approximations $\widetilde{S}_i$ leading to a computable scheme yielding almost optimal error estimates as in the exact Galerkin scheme. Therefore we may consider an approximated variational problem to find $\hat{u}_h \in W_h$ satisfying

$$\sum_{i=1}^{p} \int_{\Gamma_i} (\widetilde{S}_i \hat{u}_h)(x) v_h(x)\, ds_x = \sum_{i=1}^{p} \int_{\Gamma_i} \left[V_i^{-1}(N_i f)(x) - (S_i \widetilde{g})(x) \right] v_h(x)\, ds_x \quad (4.9)$$

for all test functions $v_h \in W_h$.

Theorem 2 *Let*

$$\widetilde{a}(u_h, v_h) = \sum_{i=1}^{p} \int_{\Gamma_i} (\widetilde{S}_i u_h)(x) v_h(x)\, ds_x \quad (4.10)$$

be bounded in $H^{1/2}(\Gamma_S)$ and W_h–elliptic, i.e.

$$\widetilde{a}(v_h, v_h) \geq \widetilde{c} \cdot ||v_h||^2_{H^{1/2}(\Gamma_S)} \quad \textit{for all } v_h \in W_h.$$

Then there exists a unique solution of the approximate variational problem (4.9) *satisfying the error estimate*

$$||\widetilde{u} - \hat{u}_h||_{H^{1/2}(\Gamma_S)} \leq c \cdot \left\{ ||\widetilde{u} - \widetilde{u}_h||_{H^{1/2}(\Gamma_S)} + \sum_{i=1}^{p} ||(S_i - \widetilde{S}_i)u||_{H^{1/2}(\Gamma_S)} \right\}. \quad (4.11)$$

Note that Theorem 2 is a variant of the first Strang lemma for some perturbation of an elliptic bilinear form, see [6, Theorem 4.1.1].

To define suitable local approximations $\widetilde{S}_i$ of the Steklov–Poincaré operators S_i, we first define local trial spaces

$$Z_{h,i} = \operatorname{span}\{\psi_k^i\}_{k=1}^{N_i} \subset H^{-1/2}(\Gamma_i) \quad \text{for} \quad i = 1, \ldots, p. \quad (4.12)$$

Again we may use piecewise polynomial trial functions of polynomial degree ν, for example trial functions with piecewise constant basis functions where $\nu = 0$. We assume that for each $Z_{h,i}$ there holds an approximation property:

$$\inf_{\tau_{h,i} \in Z_{h,i}} ||w_i - \tau_{h,i}||_{H^{-1/2}(\Gamma_i)} \leq c\, h_i^{\sigma+\frac{1}{2}} ||w_i||_{H^{\sigma}(\Gamma_i)} \quad (4.13)$$

for all $w_i \in H^{\sigma}(\Gamma_i)$ with $\sigma \leq \nu + 1$.

For an arbitrarily given function $u_i \in H^{1/2}(\Gamma_i)$ the application of the Steklov–Poincaré operator can be written, using the symmetric representation (3.12), as

$$(S_i u_i)(x) \;=\; (D_i u_i)(x) + (\frac{1}{2} I + K_i')w_i(x) \quad \text{for } x \in \Gamma_i$$

where w_i satisfies the equation

$$\langle V w_i, \tau_i \rangle_{L_2(\Gamma_i)} \;=\; \langle (\frac{1}{2} I + K_i) u_i, \tau_i \rangle_{L_2(\Gamma_i)} \quad \text{for all } \tau_i \in H^{-1/2}(\Gamma_i). \tag{4.14}$$

This motivates us to define suitable approximations $\widetilde{S}_i$ of the local Steklov–Poincaré operators S_i as follows: The Galerkin discretization of (4.14) is to find $w_{h,i} \in Z_{h,i}$ satisfying

$$\langle V w_{h,i}, \tau_{h,i} \rangle_{L_2(\Gamma_i)} \;=\; \langle (\frac{1}{2} I + K_i) u_i, \tau_{h,i} \rangle_{L_2(\Gamma_i)} \quad \text{for all } \tau_{h,i} \in Z_{h,i}. \tag{4.15}$$

Applying standard arguments we get by Cea's lemma the quasi–optimal error estimate

$$||w_i - w_{h,i}||_{H^{-1/2}(\Gamma_i)} \;\leq\; c_i \cdot \inf_{\tau_{h,i}} ||w_i - \tau_{h,i}||_{H^{-1/2}(\Gamma_i)} \tag{4.16}$$

yielding convergence by the approximation property of the trial space $Z_{h,i}$. Now we can define an approximate Steklov–Poincaré operator as

$$(\widetilde{S}_i u_i)(x) := (D_i u_i)(x) + (\frac{1}{2} I + K_i')w_{h,i}(x) \quad \text{for } x \in \Gamma_i. \tag{4.17}$$

Note that from (4.17) with (4.16) we get

$$||(S_i - \widetilde{S}_i)u_i||_{H^{-1/2}(\Gamma_i)} \;\leq\; ||w_i - w_{i,h}||_{H^{-1/2}(\Gamma_i)}. \tag{4.18}$$

In case of the symmetric approximation (4.17) of the local Steklov–Poincaré operators S_i the following theorem is valid, see also [1,14,24].

Theorem 3 *Let the approximated bilinear form* (4.10) *be defined by the use of the symmetric approximation* (4.17) *of the local Steklov–Poincaré operators S_i. Then it follows that the assumptions of Theorem 2 are satisfied, and in particular, there holds the quasi–optimal error estimate*

$$\begin{aligned} &||\widetilde{u} - \hat{u}_h||_{H^{1/2}(\Gamma_S)} \\ &\leq c \left\{ \inf_{v_h \in W_h} ||\widetilde{u} - v_h||_{H^{1/2}(\Gamma_S)} + \sum_{i=1}^{p} \inf_{\tau_{h,i} \in Z_{h,i}} ||S_i \widetilde{u}_i - \tau_{h,i}||_{H^{-1/2}(\Gamma_i)} \right\}. \end{aligned} \tag{4.19}$$

PROOF. From (4.15) we conclude the stability estimate

$$||w_{h,i}||_{H^{-1/2}(\Gamma_i)} \leq c \cdot ||u_i||_{H^{1/2}(\Gamma_i)}.$$

and therefore

$$\begin{aligned}||\widetilde{S}_i u_i||_{H^{-1/2}(\Gamma_i)} &\leq ||D_i u_i||_{H^{-1/2}(\Gamma_i)} + ||(\frac{1}{2}I + K_i')w_{h,i}||_{H^{-1/2}(\Gamma_i)} \\ &\leq c \left\{||u_i||_{H^{1/2}(\Gamma_i)} + ||w_{h,i}||_{H^{-1/2}(\Gamma_i)}\right\} \leq c\, ||u_i||_{H^{1/2}(\Gamma_i)}.\end{aligned}$$

Hence, for $u, v \in W$ we have, with the help of the Cauchy–Schwarz inequality,

$$\begin{aligned}|\widetilde{a}(u,v)| &\leq \sum_{i=1}^{p} |\langle \widetilde{S}_i u, v\rangle_{L_2(\Gamma_i)}| \leq \sum_{i=1}^{p} ||\widetilde{S}_i u||_{H^{-1/2}(\Gamma_i)} ||v||_{H^{1/2}(\Gamma_i)} \\ &\leq c \sum_{i=1}^{p} ||u||_{H^{1/2}(\Gamma_i)} ||v||_{H^{1/2}(\Gamma_i)} \leq c\, ||u||_{H^{1/2}(\Gamma_S)} ||v||_{H^{1/2}(\Gamma_S)}\end{aligned}$$

and therefore the boundedness of $\widetilde{a}(\cdot,\cdot)$. Since the local single layer potentials V_i are $H^{-1/2}(\Gamma_i)$–elliptic, this gives with (4.15)

$$\begin{aligned}\langle \widetilde{S}_i v, v\rangle_{L_2(\Gamma_i)} &= \langle D_i v, v\rangle_{L_2(\Gamma_i)} + \langle (\frac{1}{2}I + K_i')w_{h,i}, v\rangle_{L_2(\Gamma_i)} \\ &= \langle D_i v, v\rangle_{L_2(\Gamma_i)} + \langle V w_{h,i}, w_{h,i}\rangle_{L_2(\Gamma_i)} \geq \langle D_i v, v\rangle_{L_2(\Gamma_i)}\end{aligned}$$

and therefore

$$\widetilde{a}(v,v) \geq \sum_{i=1}^{p} \langle D_i v, v\rangle_{L_2(\Gamma_i)}.$$

Hence, the W–ellipticity of $\widetilde{a}(\cdot,\cdot)$ follows from the mapping properties of the assembled local hypersingular integral operators D_i. Now we can apply Theorem 2 to get the error estimate (4.11). Finally, (4.19) follows from (4.7), (4.18) and (4.16). $\square$

Note that in the symmetric approximation case the assumptions of Theorem 2 and therefore Theorem 3 hold without any restrictions on the definition of the trial spaces W_h and $Z_{h,i}$, only approximation properties have to be assumed. It turns out, to guarantee an optimal order of convergence, that the polynomial degree of the local trial spaces $Z_{h,i}$ should be choosen one degree less than the polynomial degree of the global trial space W_h. For example, one may use piecewise linear basis functions to define W_h while we can take piecewise constant trial functions for describing $Z_{h,i}$.

According to the symmetric approximation (4.17) of the local Steklov–Poincaré operators S_i we define local stiffness matrices as

$$D_{h,i}[\ell,k] = \langle D_i\varphi_{k,i},\varphi_{\ell,i}\rangle_{L_2(\Gamma_i)}, \qquad K_{h,i}[\widetilde{\ell},k] = \langle K_i\varphi_{k,i},\psi_{\widetilde{\ell},i}\rangle_{L_2(\Gamma_i)},$$
$$V_{h,i}[\widetilde{\ell},\widetilde{k}] = \langle V_i\psi_{\widetilde{k},i},\psi_{\widetilde{\ell},i}\rangle_{L_2(\Gamma_i)}, \qquad M_{h,i}[\widetilde{\ell},k] = \langle \varphi_{k,i},\psi_{\widetilde{\ell},i}\rangle_{L_2(\Gamma_i)}$$

for $k,\ell = 1,\ldots,M_i$ and $\widetilde{k},\widetilde{\ell} = 1,\ldots,N_i$. Then, the Galerkin discretization of the approximate Steklov–Poincaré operator $\widetilde{S}_i$ reads as

$$\widetilde{S}_{h,i} = D_{h,i} + (\frac{1}{2}M_{h,i}^\top + K_{h,i}^\top)V_{h,i}^{-1}(\frac{1}{2}M_{h,i} + K_{h,i}) \quad \text{for } i = 1,\ldots,p. \tag{4.20}$$

Hence, the approximated Galerkin formulation (4.9) is equivalent to the system of linear equations given by

$$\widetilde{S}_h\underline{u} := \sum_{i=1}^{p} A_i^\top \widetilde{S}_{h,i}A_i\hat{\underline{u}} = \sum_{i=1}^{p} A_i^\top \underline{f}_i =: \underline{f} \tag{4.21}$$

with the connectivity matrices A_i as introduced in (4.3) and with local vectors $\underline{f}_i$ defined by

$$f_{i,k} = \langle V_i^{-1}N_if - S_i\widetilde{g},\varphi_{k,i}\rangle_{L_2(\Gamma_i)} \quad \text{for } k = 1,\ldots,M_i; i = 1,\ldots,p.$$

The stiffness matrix $\widetilde{S}_h$ in (4.21) is symmetric and positive definite, hence we can use a standard preconditioned conjugate gradient scheme in parallel to solve (4.21) efficiently. The construction of appropriate preconditioning techniques will be discussed later in Section 5.

Defining

$$D_h = \sum_{i=1}^{p} A_i^\top D_{h,i}A_i, \qquad K_h = \sum_{i=1}^{p} K_{h,i}A_i,$$
$$V_h = \operatorname{diag}(V_{h,i})_{i=1}^{p}, \qquad M_h = \sum_{i=1}^{p} M_{h,i}A_i,$$

the linear system (4.21) can be written as a block system of the form

$$\begin{pmatrix} V_h & -\frac{1}{2}M_h - K_h \\ \frac{1}{2}M_h^\top + K_h^\top & D_h \end{pmatrix} \begin{pmatrix} \underline{w} \\ \hat{\underline{u}} \end{pmatrix} = \begin{pmatrix} \underline{0} \\ \underline{f} \end{pmatrix}. \tag{4.22}$$

Note that the stiffness matrix in (4.22) is either block skew–symmetric and positive definite, or by simple manipulations, symmetric but indefinite. Hence, for the iterative solution of (4.22) one may use any appropriate solver such as BiCGStab or GMRES applicable to nonsymmetric or indefinite systems. Instead, following [3,5] one can transform (4.22) into a symmetric and positive

definite system. Let $C_{V,i}$ be local preconditioning matrices for the discrete single layer potential operators satisfying the spectral equivalence inequalities

$$c_1^{V_i}\ (C_{V,i}\underline{w}_i,\underline{w}_i)\ \leq\ (V_{h,i}\underline{w}_i,\underline{w}_i)\ \leq\ c_2^{V_i}\ (C_{V,i}\underline{w}_i,\underline{w}_i)\quad \text{for all } \underline{w}_i\in I\!R^{N_i} \tag{4.23}$$

with positive constants $c_1^{V_i}$ and $c_2^{V_i}$. In addition, we assume $c_1^{V_i}>1$. This can be accomplished in general by some scaling of the preconditioning matrices $C_{V,i}$. Defining $C_V = \mathrm{diag}(C_{V_i})_{i=1}^p$ we then obtain the spectral equivalence inequalities

$$c_1^V\ (C_V\underline{w},\underline{w})\ \leq\ (V_h\underline{w},\underline{w})\ \leq\ c_2^V\ (C_V\underline{w},\underline{w})\quad \text{for all } \underline{w}\in I\!R^N \tag{4.24}$$

with $N=\sum\limits_{i=1}^{p} N_i$ and positive constants

$$1<c_1^V:=\min_{1\leq i\leq p} c_1^{V_i},\quad c_2^V:=\max_{1\leq i\leq p} c_2^{V_i}.$$

Due to the assumption $c_1^V>1$, instead of (4.22), we may solve the transformed linear system

$$\begin{aligned}\begin{pmatrix} V_hC_V^{-1}-I & 0\\ -(\frac{1}{2}M_h^\top+K_h^\top)C_V^{-1} & I\end{pmatrix}\begin{pmatrix} V_h & -\frac{1}{2}M_h-K_h\\ \frac{1}{2}M_h^\top+K_h^\top & D_h\end{pmatrix}\begin{pmatrix}\underline{w}\\ \hat{\underline{u}}\end{pmatrix}\\ =\begin{pmatrix} V_hC_V^{-1}-I & 0\\ -(\frac{1}{2}M_h^\top+K_h^\top)C_V^{-1} & I\end{pmatrix}\begin{pmatrix}\underline{0}\\ \underline{f}\end{pmatrix}.\end{aligned} \tag{4.25}$$

It turns out, see [3] for details, that the transformed stiffness matrix in (4.25) is now symmetric and positve definite. Hence we can use the preconditioned conjuate gradient scheme to solve (4.25) efficiently.

4.2 Hybrid approximation techniques

Instead of the symmetric approximation (4.17) based on the symmetric representation (3.12) one may use any other boundary element approximation of the local Steklov–Poincaré operators S_i as for example, the local representations (3.11) or (3.13). Following [27,33] we will describe a non–symmetric and a so–called "hybrid" boundary element scheme by discretizing the Steklov–Poincaré operator representation (3.11) (see also [11]).

For an arbitrarily given function $u_i\in H^{1/2}(\Gamma_i)$, the application of the Steklov–Poincaré operator S_i in view of (3.11) reads as

$$(S_iu_i)(x)\ =\ w_i(x)\quad \text{for } x\in\Gamma_i$$

where w_i is, as in the symmetric approximation, the unique solution of the variational problem (4.14). As in (4.15) we can define a corresponding Galerkin solution $w_{h,i} \in Z_{h,i}$. Therefore, an approximate Steklov–Poincaré operator is here given by

$$(\widetilde{S}u_i)(x) \;=\; w_{h,i}(x) \quad \text{for } x \in \Gamma_i, i = 1, \ldots, p. \tag{4.26}$$

Obviously, the error estimate (4.18) for $||(S - \widetilde{S})u_i||_{H^{-1/2}(\Gamma_i)}$ remains valid. As in the proof of Theorem 3 we can conclude that the bilinear form $\widetilde{a}(\cdot,\cdot)$ defined by the local approximations (4.26) is bounded in $H^{1/2}(\Gamma_S)$.

Theorem 4 *Let H_i be the local mesh size of the trial space W_h while h_i is the local mesh size of $Z_{h,i}$ respectively. Let the inverse inequality in W_h be valid locally,*

$$||w_{h|\Gamma_i}||_{H^s(\Gamma_i)} \;\leq\; c\; H_i^{\frac{1}{2}-s}\; ||w_{h,i}||_{H^{1/2}(\Gamma_i)}. \tag{4.27}$$

If $h_i \leq c_{0,i} H_i$ is satisfied with positive, sufficiently small constants $c_{0,i} \leq 1$, then the bilinear form $\widetilde{a}(\cdot,\cdot)$ defined by the approximation (4.26) *is W_h–elliptic.*

PROOF. For $v_h \in W_h$ we have by (3.17), (4.18), (4.16) and the inverse inequality, for $s \leq \nu + 1$,

$$\begin{aligned}
c_i^S ||v_h||^2_{H^{1/2}(\Gamma_S)} \leq & \sum_{i=1}^{p} \langle S_i v_h, v_h\rangle_{L_2(\Gamma_i)} \\
\leq & \sum_{i=1}^{p} \langle \widetilde{S}_i v_h, v_h\rangle_{L_2(\Gamma_i)} \;+\; \sum_{i=1}^{p} \langle (S_i - \widetilde{S}_i) v_h, v_h\rangle_{L_2(\Gamma_i)} \\
\leq & \sum_{i=1}^{p} \langle \widetilde{S}_i v_h, v_h\rangle_{L_2(\Gamma_i)} \;+\; \sum_{i=1}^{p} ||(S_i - \widetilde{S}_i) v_h||_{H^{-1/2}(\Gamma_i)} ||v_h||_{H^{1/2}(\Gamma_i)} \\
\leq & \sum_{i=1}^{p} \langle \widetilde{S}_i v_h, v_h\rangle_{L_2(\Gamma_i)} \;+\; \sum_{i=1}^{p} c_i h_i^{s+\frac{1}{2}} ||v_h||_{H^{s+1}(\Gamma_i)} ||v_h||_{H^{1/2}(\Gamma_i)} \\
\leq & \sum_{i=1}^{p} \langle \widetilde{S}_i v_h, v_h\rangle_{L_2(\Gamma_i)} \;+\; \sum_{i=1}^{p} \widetilde{c}_i (h_i/H_i)_i^{s+\frac{1}{2}} ||v_h||^2_{H^{1/2}(\Gamma_i)}
\end{aligned}$$

Hence, if $\widetilde{c}_i (h_i/H_i)^{s+\frac{1}{2}} \leq c_1^S/2$ is satisfied the theorem is proved. $\square$

When using the approximation $\widetilde{S}_i$ as defined in (4.26) then the local Galerkin discretization is given by

$$\widetilde{S}_{h,i} \;=\; M_{h,i}^{\top} V_{h,i}^{-1} (\frac{1}{2} M_{h,i} + K_{h,i}) \tag{4.28}$$

while the global system is given as in (4.21) by

$$\widetilde{S}_h \underline{\hat{u}} \;=\; \sum_{i=1}^{p} A_i^{\top} \widetilde{S}_{h,i} A_i \underline{\hat{u}} \;=\; \underline{f}. \tag{4.29}$$

The assembled stiffness matrix $\widetilde{S}_h$ is still positive definite but, in general, not symmetric. Therefore, we recommend a suitable preconditioned BiCGStab or GMRES algorithm for an efficient solution strategy. Moreover, the local stiffness matrices $\widetilde{S}_{h,i}$ as given in (4.28) are nonsymmetric perturbations of an originally symmetric stiffness matrix $S_{h,i}$. To keep the symmetry in the approximation of local Steklov–Poincaré operators, which is important when coupling boundary elements with a symmetric finite element scheme, one can introduce a modified hybrid discretization scheme [11,26]. That is again based on the representation (3.11) but on the formulation of the local Steklov–Poincaré operator S_i as

$$S_i = V_i^{-1}(\frac{1}{2}I + K_i)V_i V_i^{-1} \;=\; V_i^{-1} F_i V_i^{-1} \tag{4.30}$$

with the self–adjoint and computable operator

$$F_i \;=\; (\frac{1}{2}I + K_i)V_i \,. \tag{4.31}$$

As before, we can introduce an appropriate approximation of S_i, now based on the representation (4.30). Then, the local Galerkin discretization is given by

$$\widetilde{S}_{h,i} = M_{h,i}^{\top} V_{h,i}^{-1} F_{h,i} V_{h,i}^{-1} M_{h,i} \tag{4.32}$$

which is now a symmetric and positive definite matrix provided $F_{h,i}$ can be computed accurately. We remark that the computation of

$$F_{h,i}[\widetilde{\ell}, \widetilde{k}] \;=\; \langle F_i \psi_{\widetilde{k},i}, \psi_{\widetilde{\ell},i} \rangle_{L_2(\Gamma_i)}$$

for $\widetilde{k}, \widetilde{\ell} = 1, \ldots, N_i$ requires the evaluation of two boundary integral operators per matrix element. To ensure stability of the hybrid discretization scheme (4.32) we have to assume the stability assumption,

$$c \;||v_{h,i}||_{H^{1/2}(\Gamma_i)} \;\leq\; \sup_{w_{h,i} \in Z_{h,i}} \frac{|\langle v_{h,i}, w_{h,i} \rangle_{L_2(\Gamma_i)}|}{||w_{h,i}||_{H^{-1/2}(\Gamma_i)}} \tag{4.33}$$

for all $v_{h,i} \in W_{h,i}$, see [26] for details. In fact, for a local trial space $W_{h,i}$ we have to define trial spaces $Z_{h,i}$ in such a way that (4.33) is satisfied. Note that for a given $W_{h,i}$, the construction of $Z_{h,i}$ is not unique. We will describe three possible choices of $Z_{h,i}$ for the case that $W_{h,i}$ is spanned by piecewise linear continuous basis functions, see also [27].

i. Mesh refinement: As in Theorem 4 we can define $Z_{h,i}$ by using piecewise constant basis functions with respect to a sufficiently refined boundary element mesh compared with the underlying mesh of $W_{h,i}$. In this case we have to assume an inverse inequality, see (4.27). Therefore, this approach is applicable for quasi–uniform boundary element meshes only. For more details, see e.g. [14,26,33].

ii. Iso–parametric trial functions: We first consider the case $\widetilde{Z}_{h,i} = W_{h,i}$. Then the stability property (4.33) is strongly related to the stability of the corresponding L_2 projection Q_h onto $W_{h,i}$ in $H^{1/2}(\Gamma)$. The latter holds for a rather large class of non–uniform refinements based on adaptive strategies provided that certain local conditions are satisfied. We refer to [28] for a detailed discussion. Now we define $Z_{h,i}$ to be the trial space of piecewise linear but discontinuous basis functions. Obviously, $\widetilde{Z}_{h,i} \subset Z_{h,i}$ and the stability condition (4.33) remains valid.

iii. Non–matching boundary meshes: In both cases described above, the definition of $Z_{h,i}$ requires the use of appropriate trial functions satisfying (4.33) which implies a significant growth of the dimension N_i of the trial space $Z_{h,i}$. In view of (4.19), the optimal choice seems to be, to define $Z_{h,i}$ by piecewise constant basis functions where the mesh size of $W_{h,i}$ and $Z_{h,i}$ is almost equal. However, it is not possible to define $Z_{h,i}$ with respect to the same boundary element mesh as $W_{h,i}$, since then the corresponding mass matrix $M_{h,i}$ would become singular. Instead we can define $Z_{h,i}$ with respect to the mesh dual to that of $W_{h,i}$. In this case, (4.33) is satisfied again, also for non–uniform boundary element meshes; for a further discussion see [27,29].

For comparison we consider a simple numerical example. Let Ω be an L shaped domain with boundary Γ. We solve a mixed boundary value problem in one subdomain by using the approximation (4.26). In Table 4.1 we give the approximation errors for the boundary element solution according to Theorem 2 while in Table 4.2 we give the errors of the approximations of the Steklov–Poincaré operator. In both tables, M is the number of all boundary nodes while N is the degree of freedom needed for the definition of $\widetilde{S}$. Note that with respect to both, computational work as well as accuracy, the approach based on the dual mesh is favourable.

5 Preconditioning techniques

For the iterative solution of the linear systems (4.21) or (4.22) resulting from the symmetric approximation or (4.29) in case of the nonsymmetric approximation, we need to use some appropriate preconditioning techniques to reduce the number of iterations needed. In particular, we assume that there are given local preconditioning matrices $C_{V,i}$ satisfying the spectral equivalence inequalities

Table 1
Errors for the Boundary Element Solution

	Case *i.*		Case *ii.*		Case *iii.*	
M	N	$\|\|u-u_h\|\|_{L^2}$	N	$\|\|u-u_h\|\|_{L^2}$	N	$\|\|u-u_h\|\|_{L^2}$
32	64	2.04 –2	64	2.19 –2	38	1.74 –2
64	128	5.10 –3	128	5.41 –3	70	4.37 –3
128	256	1.28 –3	256	1.35 –3	134	1.11 –3
256	512	3.20 –4	512	3.36 –4	262	2.79 –4
512	1024	8.02 –5	1024	8.40 –5	518	7.02 –5

Table 2
Errors for the Approximation of the Steklov–Poincaré Operator

	Case *i.*		Case *ii.*		Case *iii.*	
M	N	$\|\|(S-\widetilde{S})u\|\|_{L^2}$	N	$\|\|(S-\widetilde{S})u\|\|_{L^2}$	N	$\|\|(S-\widetilde{S})u\|\|_{L^2}$
32	64	4.24 –1	64	3.20 –1	38	6.41 –1
64	128	1.84 –1	128	9.87 –2	70	3.30 –1
128	256	8.70 –2	256	2.74 –2	134	1.67 –1
256	512	4.27 –2	512	7.33 –3	262	8.42 –2
512	1024	2.13 –2	1024	1.99 –3	518	4.23 –2

$$\gamma_1^{V_i}\ (C_{V,i}\underline{w}_i, \underline{w}_i) \ \leq\ (V_{h,i}\underline{w}_i, \underline{w}_i) \ \leq\ \gamma_2^{V_i}\ (C_{V,i}\underline{w}_i, \underline{w}) \quad \text{for all}\, \underline{w}_i \in I\!R^{N_i} \qquad (5.1)$$

and $i = 1, \ldots, p$, as well as a global preconditioning matrix C_S satisfying

$$\gamma_1^S\ (C_S\underline{u}, \underline{u}) \ \leq\ (\widetilde{S}_h\underline{u}, \underline{u}) \ \leq\ \gamma_2^S\ (C_S\underline{u}, \underline{u}) \quad \text{for all}\, \underline{u} \in I\!R^M. \qquad (5.2)$$

5.1 Local preconditioners

To define local preconditioners $C_{V,i}$ for the local single layer potential operators V_i satisfying (5.1), one can apply different strategies. One approach is based on the use of geometrically similar and rotational symmetric domains which leads to block circulant matrices which can be used as local preconditioners [20]. Here, a proper ordering of the degrees of freedom has to be assumed. A classical approach, as in finite element methods, is the use of multigrid preconditioners for the local single layer potentials, which are operators of order -1 [2]. Another strategy is the use of multilevel methods such as additive or multiplicative Schwarz methods [19]. However, in both multigrid and multilevel

approaches a suitable mesh hierarchy has to be assumed. Here we will describe an approach [18,31] which does not require neither a proper ordering of the degrees of freedom nor a given mesh hierarchy. From the mapping properties of the local single layer potential operators V_i we get the spectral equivalence inequalities

$$c_1^{V_i} \, ||w_i||^2_{H^{-1/2}(\Gamma_i)} \, \leq \, \langle V w_i, w_i \rangle_{L_2(\Gamma_i)} \, \leq \, c_2^{V_i} \, ||w_i||^2_{H^{-1/2}(\Gamma_i)} \tag{5.3}$$

for all $w_i \in H^{-1/2}(\Gamma_i)$. On the other hand there hold the spectral equivalence inequalities

$$c_1^{D_i} \, ||u_i||^2_{H^{1/2}(\Gamma_i)} \, \leq \, \langle D u_i, u_i \rangle_{L_2(\Gamma_i)} \, \leq \, c_2^{D_i} \, ||u_i||^2_{H^{1/2}(\Gamma_i)} \tag{5.4}$$

for all $u_i \in H^{1/2}(\Gamma_i)_{/\mathbf{R}_i}$. Hence, it follows that

$$I + D_i \, : \, H^{1/2}(\Gamma_i) \to H^{-1/2}(\Gamma_i)$$

is bounded and $H^{1/2}(\Gamma_i)$–elliptic. Therefore, with (5.3), the spectral equivalence inequalities

$$\gamma_1^i \, \langle (I + D_i)^{-1} w_i, w_i \rangle_{L_2(\Gamma_i)} \, \leq \, \langle V_i w_i, w_i \rangle_{L_2(\Gamma_i)} \, \leq \, \gamma_2^i \, \langle (I + D_i)^{-1} w_i, w_i \rangle_{L_2(\Gamma_i)} \tag{5.5}$$

hold for all $w_i \in H^{-1/2}(\Gamma_i)$. For the preconditioning matrix C_{V_i} defined by

$$C_{V_i}[\widetilde{\ell}, \widetilde{k}] \, = \, \langle (I + D_i)^{-1} \psi_{\widetilde{k},i}, \psi_{\widetilde{\ell},i} \rangle_{L_2(\Gamma_i)} \tag{5.6}$$

for $\widetilde{k}, \widetilde{\ell} \, = \, 1, \ldots, N_i$, the spectral equivalence inequalities (5.1) then follow from (5.5) with the positive constants $c_1^{V_i} = \gamma_1^i, c_2^{V_i} = \gamma_2^i$. Similar as for the Steklov–Poincaré operators S_i, in general one is not able to compute the matrix elements (5.6) directly. Instead we use an approximation

$$\widetilde{C}_{V_i} \, = \, \bar{M}_{h,i}^{\top} (\widetilde{M}_{h,i} + \widetilde{D}_{h,i})^{-1} \bar{M}_{h,i} \tag{5.7}$$

in terms of the local matrices

$$\begin{aligned}
\widetilde{D}_{h,i}[\widetilde{\ell}, \widetilde{k}] &= \langle D_i \widetilde{\varphi}_{\widetilde{k},i}, \widetilde{\varphi}_{\widetilde{\ell},i} \rangle_{L_2(\Gamma_i)}, \\
\widetilde{M}_{h,i}[\widetilde{\ell}, \widetilde{k}] &= \langle \widetilde{\varphi}_{\widetilde{k},i}, \widetilde{\varphi}_{\widetilde{\ell},i} \rangle_{L_2(\Gamma_i)}, \\
\bar{M}_{h,i}[\widetilde{\ell}, \widetilde{k}] &= \langle \widetilde{\varphi}_{\widetilde{k},i}, \psi_{\widetilde{\ell},i} \rangle_{L_2(\Gamma_i)}
\end{aligned}$$

where $\widetilde{W}_{h,i} := \operatorname{span} \left\{ \widetilde{\varphi}_{\widetilde{k},i} \right\}_{\widetilde{k}=1}^{N_i} \subset H^{1/2}(\Gamma_i)$ is an appropriate trial space to be used for the discretization of the local hypersingular integral operators D_i. As it was shown in [31], there holds the upper estimate

$$(\widetilde{C}_{V_i} \underline{w}_i, \underline{w}_i) \, \leq \, (C_{V_i} \underline{w}_i, \underline{w}_i) \quad \text{for all } \underline{w}_i \in I\!R^{N_i}. \tag{5.8}$$

Theorem 5 ([31]) *Assume the stability condition*

$$c_0 \; ||u_{h,i}||_{H^{1/2}(\Gamma)_i} \; \leq \; \sup_{w_{h,i} \in Z_{h,i}} \frac{|\langle w_{h,i}, u_{h,i} \rangle_{L_2(\Gamma_i)}|}{||w_{h,i}||_{H^{-1/2}(\Gamma_i)}} \quad \textit{for all } u_{h,i} \in \widetilde{W}_{h,i}. \tag{5.9}$$

Then,

$$\gamma_0 \; (C_{V_i} \underline{w}_i, \underline{w}_i) \; \leq \; (\widetilde{C}_{V_i} \underline{w}_i, \underline{w}_i) \quad \textit{for all } \underline{w}_i \in I\!R^{N_i}. \tag{5.10}$$

Note that the stability condition (5.9) is similar to the stability condition (4.33) needed in hybrid discretizations of the Steklov–Poincaré operators locally. Since (5.9) ensures the invertibility of the mass matrix $\bar{M}_{h,i}$, as a consequence we have from (5.7)

$$\widetilde{C}_{V_i}^{-1} \; = \; \bar{M}_{h,i}^{-1} (\widetilde{M}_{h,i} + \widetilde{D}_{h,i}) \bar{M}_{h,i}^{-\top}. \tag{5.11}$$

5.2 Parallel preconditioners

To construct a global preconditioning matrix C_S satisfying the spectral equivalence inequalities (5.2) we first note that there hold the spectral equivalence inequalities

$$c_1^{S_i} \; (S_{h,i} \underline{u}_i, \underline{u}_i) \; \leq \; (\widetilde{S}_{h,i} \underline{u}_i, \underline{u}_i) \; \leq \; c_2^{S_i} \; (S_{h,i} \underline{u}_i, \underline{u}_i) \tag{5.12}$$

for all $\underline{u}_i \in I\!R^{M_i}$. In case of the symmetric approximation given by (4.20), (5.12) follows from Theorem 3, since the assumptions of Theorem 2 are satisfied. When using either the non–symmetric approximation (4.28) or the hybrid approximation (4.32) we need to assume (4.33) to ensure (5.12). Hence, instead of (5.2) it is sufficient to construct a global preconditioning matrix C_S which is spectrally equivalent to the global bilinear form (3.16). Moreover, since the local Steklov–Poincaré operators S_i are spectrally equivalent to the local hypersingular integral operators D_i, we need only to find a preconditioning matrix for the modified bilinear form

$$\widetilde{a}(u, v) \; := \; \sum_{i=1}^{p} \langle D_i u_{|\Gamma_i}, v_{|\Gamma_i} \rangle_{L_2(\Gamma_i)} \quad \text{for } u, v \in W. \tag{5.13}$$

When using the symmetric approximation (4.20), the local Galerkin discretization of the hypersingular integral operators is already computed. Hence, the action of the preconditioner can be defined by the solution $\underline{v}$ of

$$\sum_{i=1}^{p} A_i^{\top} D_{h,i} A_i \underline{v} \; = \; \underline{r} \tag{5.14}$$

by any available efficient method, this defines an optimal preconditioning strategy. For example, we can use a standard multigird scheme as in [22] for the hypersingular integral operator to solve (5.14) in parallel, see for example [5]. Alternatively, we may solve (5.14) approximately by some suitable iterative scheme using some appropriate preconditioning strategy for the assembled Galerkin matrix D_h. Again we can use multigrid or multilevel preconditioners, or some additive Schwarz methods as described in [4] (for an application of the latter case, see [30]).

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