AN INVESTIGATION INTO COLLEGE STUDENTS’ LEARNING ABOUT LOGARITHMIC FUNCTIONS: A THORNY PROBLEM

by

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ABSTRACT

There is little research investigating students’ understanding of logarithms. As a result, there are few data-based resources to help mathematics educators improve the quality of their instruction on logarithms. This study investigated students’ learning of, and difficulties with, logarithms the context of a college Pre-Calculus course (MATH 115) offered at the University of Delaware. Identifying particular ideas or skills that students find challenging may help inform mathematics educators about how to better promote students’ fluency with logarithms. All students enrolled in MATH 115 were invited to complete a Pre-test (before the logarithm lessons were taught) and a Post-test (after the logarithm lessons were taught) assessing their fluency with logarithms. A subset of 7 participants participated in a series of three, individual interviews during the semester. Results indicate that students are able to significantly improve their fluency with logarithms after receiving instruction. Findings also suggest that students’ fluency with logarithms is associated with their ability to read a logarithm, understand the notation, and use logarithm terminology. Future research should collect data from a larger and more diverse samples of students in order to provide more insight into what researchers and educators can do to better support students’ learning of logarithms.
Chapter 1

INTRODUCTION

1.1 Motivation for the study

From my work as a tutor and classroom assistant, I have noticed that students often struggle when learning logarithms. Moreover, although logarithms are applicable to various areas of potential interest for students, such as banking and psychology, students are usually not aware of these applications. Unfortunately, there is little research investigating students' understanding of logarithms and difficulties with logarithms. Therefore, there are few data-based resources to help mathematics educators better support students’ learning of logarithms. The purpose of investigating students’ perceptions and developing understanding of logarithms is to collect information that can be used to help high school and college instructors develop more effective lessons that better support students’ learning.

1.2 Background and problem statement

I became interested in investigating students’ difficulties with logarithms while working as a mathematics tutor in the Mathematical Sciences Learning Laboratory (MSLL) at the University of Delaware (UD) for the last year and a half. MSLL is a unique environment that serves as a “one-stop shop” for students taking introductory mathematics courses at UD. MSLL is home to three courses: Intermediate Algebra (MATH 010), Pre-Calculus (MATH 115), and Integrated Approach to Calculus I (MATH 267). Students in these courses have their classes meet in MSLL, take their
exams in the MSLL Testing Lab, and attend MSLL Open Lab. MSLL Open Lab is a place where students any introductory course up to and including Calculus I can work on their math homework, study for a math quiz or exam, and receive free assistance from a tutor. The MSLL instructors have worked diligently to develop a shared curriculum (detailed lesson plans including classroom activities and online homework assignments) in each course to ensure that all students taking these courses have the same learning opportunities, no matter which section they are in.

The majority of students who utilize MSLL are freshman; they start out with a lot of spirit, charisma, and a positive attitude. But with that comes a reality they will soon learn: college mathematics is not like what they were taught in high school. It is something more complicated, theoretical, and for some it will feel “impossible.” From my experience as a tutor in MSLL, one of the most challenging mathematical topics for students is logarithms, and this is why I decided to focus on this topic. Since students will revisit logarithms in later coursework, the study of logarithms is a core mathematical topic.

Some scholars have investigated why students have such a difficult time with logarithms. For example, Kenney (2005) conducted a study of students' interpretations of logarithmic functions and how it leads them to solve logarithmic expressions. She concluded that students had no valid explanation as to why they chose their answers other than the “[log] could be canceled out” (p. 4). Of course, this is not the ideal answer. Her interviews revealed that students typically memorize, rather than understand, the fundamental properties of logarithmic functions. This leads to students being able to solve one type of problem, but not all kinds.
Williams (2011) has also researched students’ understanding of logarithms. She developed a theory that sometimes students become too reliant on translating logarithmic equations to exponential equations. She recommends that students should “primarily [focus] upon the object and process definitions for logarithms” (p. 70) instead of changing forms. For example, most students do not understand that $\log_b(a)$ is a real number. To many students, the idea that a number can be written as an expression that shows very little numerical information is confusing. Williams concluded that logarithms should be treated as objects rather than numbers. This lets students “perform operations on logarithms rather than to use the rules by rote memorization” (p. 8).

From examining the work of Kenney (2005), Williams (2011), and other researchers, I have constructed the following general problem statement: The goal of this study is to investigate students’ fluency with logarithms and identify the difficulties that students face when learning and working with logarithms. Identifying specific components that students struggle with may help reveal what educators can do to better promote students’ fluency.

1.3 Overview of methodology

The goal of my study is investigating students’ fluency with logarithms and to trace how it develops during the semester. To accomplish this goal, I invited all students enrolled in MATH 115 in the Fall 2017 semester to complete a Pre-test on logarithms, before the logarithm lessons were taught. The goal of the Pre-test was to collect information about students’ incoming skills and fluency with logarithms before taking MATH 115. I then conducted a series of three, one-on-one interviews with a subset of 7 participants who took the Pre-test. The first interview was conducted
before the logarithm lessons were taught. The goals of the first interview were to follow up on students’ responses to the Pre-test, to probe what, if any, experiences students had with logarithms in high school, and to collect baseline information on students’ incoming abilities with logarithms. The second interview was conducted in late October after a few of the logarithm lessons had been taught. The purpose of this interview was to connect with students and assess how they were feeling about logarithms and what they were currently understanding or having trouble with regarding logarithms. The third interview was conducted in the middle of November. At this time, students had completed all of the logarithms lessons. The goal of this final interview was to assess students’ exit-level fluency with logarithms. Finally, after the logarithm lessons had been taught, I invited all students enrolled in MATH 115 to complete a Post-test on logarithms. The goal of the Post-test was to assess students’ fluency with logarithms and identify what, if any, misconceptions remained.

1.4 Organization of the thesis

My thesis is organized primarily in chronological order of the related tasks in which I engaged, beginning in the Spring 2017 academic semester. Chapter 2 presents my literature review. I conducted a literature review including the history of logarithms and research on teaching and learning logarithms. It is important to be able to understand the history of the logarithm and the previous research that has been conducted on teaching logarithms before discussing any of my findings. In Chapter 3, I describe the methodology of my study. I describe my research participants, and the instruments and procedures I used to collect data. I then describe how I analyzed my data. In Chapter 4, I present the results of my study. First, I discuss the results of the Pre-test and compare them to the results of the Post-test. My remaining results are
organized into case studies of individual participants who took part in all three of the individual interviews. I conclude this chapter with a cross-case analysis, highlighting similarities and differences across the cases.

Chapter 5, the final chapter of my thesis, presents the discussion section. I summarize my findings, and then discuss their implications. In this chapter I also identify limitations of my study. Finally, I make recommendations about further research and how my study may inform future studies.
Chapter 2

LITERATURE REVIEW

2.1 Introduction

This chapter discusses the research conducted to better understand students’ learning and misconceptions while working with logarithmic functions, expressions, and applications. Research investigating students’ understanding of logarithms is at a minimum; thus, there is little guidance to help educators improve the quality of instruction in this subject. This chapter is organized into the following sections: (1) the definition and history of logarithms, (2-10) a review of several research studies, and (11) a conclusion.

2.2 The definition and history of logarithms

The invention of the logarithm does not specifically relate to how it is used in today’s modern world. The original intent of the logarithm was to simplify tedious multiplication and division problems that mathematicians encountered in the 16th and 17th centuries. Today, we view a logarithmic function to be the inverse of an exponential function. Thus, the definition of a logarithm has changed with time. The story of how the logarithm was derived dates back prior to its discovery in the 17th century (O’Connell, 2015).

In the 15th century, mathematicians Nicolas Chuquet and Michael Stifel were investigating relationships between arithmetic and geometric sequences. They wanted to construct notation for these sequences and relate them in terms of an exponential relationship. Chuquet and Stifel promoted the use of common mathematical notation. At the turn of the century, the use for common notation was critical as
mathematicians continued to find ways to condense mathematical expressions and communicate their work to others in the field (O’Connell, 2015).

In the 16th century, the Age of Exploration was in full momentum and one of the only methods to compute large numbers relied heavily on trigonometry. There was a huge demand for astronomers, physicists, cartographers, and mathematicians who could carry out tedious calculations of multiplication and division. The field of trigonometry in the 16th century centered on using circles that created angles to form triangles. Mathematicians based the trigonometric functions using a “non-unity radii, such as R = 10,000,000” (Clark & Montelle, 2011, p. 3). However, determining large quantities, such as the mass of a planet, caused issues for mathematicians as they did not yet have instruments to precisely carry out the multiplication and division of large numbers. Further investigation needed to be done, before the idea of a logarithm could be developed (Clark & Montelle, 2011).

Mathematicians Johannes Werner, Christopher Clavius, and Nicolai Reymers Ursus all contributed to condensing long expressions. These three men examined Greek concepts related to addition and subtraction called prosthesis and aphaeresis, respectively. While studying these ideas, Clavius, Ursus, and Werner constructed a new form to simplify expressions called prosthaphaeresis which was an extensive combination of the Greek ideas of prosthesis and aphaeresis (Clark & Montelle, 2011). As a result, the commonly used trigonometric identity, $2\cos(A)\cos(B) = \cos(A + B) + \cos(A - B)$ was discovered. This was the beginning of a new movement in mathematics, and mathematicians were continuing to work to find shorter ways to simplify long, tedious expressions.
The desire to develop common notation and derive methods for simplifying long, tedious multiplication and division expressions pushed mathematicians to the next level of thinking. How could they combine both of these methods in order to create a universal system of notation and knowledge for anyone in the field be able to use and understand? How could mathematicians and scientists work together on the same material even though they were many miles apart? These questions drove the need for continuing the work of simplifying expressions. There was not yet enough information in the 16th century that could support all of the issues mathematicians were facing at the time. Their questions would be answered in the 17th century.

During the end of the Age of Exploration, John Napier greatly contributed to the development of universal notation and simplification of trigonometric expressions. Napier devoted his life to studying astronomy and physics, and he is credited to be the individual who discovered the logarithm. He wanted to create a way that helped him, and other mathematicians, shorten the time required for lengthy computations. His approach was based on the work of mathematicians who came before him who studied notation and prosthaphaeresis. He used what they discovered and applied it to his ideas.

Napier invented the logarithm in a way that would be considered odd today. He did so by examining the rate of change between an arithmetic and geometric sequence (Fogleman, 2015). Napier started off with two particles, \( p_1 \) and \( p_2 \), on two parallel lines. Particles \( p_1 \) and \( p_2 \) both start at the same position on their corresponding line. The first line, \( l \), was an infinite line with fixed distances between all points \( x_1, x_2, x_3, \ldots, x_i \), where \( I \) was a natural number, such that the velocity of the particle could be expressed as \( |x_1 - x_2| = |x_3 - x_2| = |x_4 - x_3| = \ldots = |x_{i+1} - x_i| \). The
second line, \( m \), was a line segment with points, \( y_1, y_2, y_3, \ldots, y_n \), where \( n \) was a natural number, such that the particle’s “velocity was proportional to the distance remaining from the particle to the fixed terminal point of the line segment” and \( |x_i - x_1| \geq |y_n - y_1| \) (Clarke & Montelle, 2011, p. 4). Napier defined the distance between points \( y_n \) and \( y_{n-j} \) for any \( j \) that belongs to \( \{1, 2, 3, \ldots, n-1\} \) to be \( x = \sin(\varphi) \) \(^1\) and the length of line segment \( m \) to have a radius length, \( R = 10,000,000 \) \(^2\). From here, Napier wanted to define the rate of change between the fixed distances. By doing so he was able to construct the word “logarithm” from the Greek words, \( \text{logo} \) meaning proportion, and \( \text{arithmus} \) meaning number resulting in the invention of the word “logarithm”, a proportion of numbers. Next, Napier set the distance traveled by \( p_1 \) on line \( l \) to be the log(\( x \)), where \( x = \sin(\varphi) \). Therefore, the distance traveled by \( p_1 \) on line \( l \) is log(\( \sin(\varphi) \)). Napier drew a segment that connected the corresponding points on both lines, such that \( x_1 \) connected to \( y_1 \), \( x_2 \) connected to \( y_2 \), \( x_3 \) connected to \( y_3 \), all the way up to \( x_i \) and \( y_i \) being connected. (See Figures 1 and 2.) He observed that as \( p_2 \) traveled along \( m \), the proportion to its corresponding particle, \( p_1 \), was a ratio that was from the angle created between the two points. Napier concluded that when \( p_2 \) increased, then \( x \) decreased, and when \( p_1 \) increased, then \( y \) increased. He then connected this general conclusion back to arithmetic and geometric sequences. Napier claimed that as \( x \) decreased in the geometric sequence, then \( y \) increased in the arithmetic sequence.

Napier reported his findings in his book, \( \text{Mirifici Logarithmorum Canonis Descriptio} \),

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\(^1\) In the 15\(^{th}\), 16\(^{th}\), and 17th centuries, mathematicians used \( \sin(\varphi) \) which is not the same as \( \sin(\varphi) \). It was later discovered that \( \sin(\varphi) = R \sin(\varphi) \) with \( R \) being the radius of a circle (Clark & Montelle, 2011).

\(^2\) Recall that mathematicians at this time used spherical trigonometry, rather than what we use today which is triangular trigonometry.
published in 1614. In this publication, Napier constructed a table of values that accurately depicted the relationship between the proportion of the distances that the two particles traveled. Mathematicians understand that Napier’s curiosity was driven by the research performed by Chuquet and Stifel; however, mathematicians do not understand why he decided to act upon this idea by studying the proportion of the distances the two particles traveled (O’Connell, 2015). This publication was the first time the world ever saw a logarithm and is why John Napier is considered by many to be the inventor of it. Now, how did Napier’s results impact the movement to condense trigonometric expressions in order to make them easier to compute (Clark & Montelle, 2011?).
Figure 1. An example of Napier’s diagram to describe the relationship between arithmetic and geometric sequences.
Figure 2. An example of Napier’s notation from his experiment that showed the relationship between the distances of the geometric to the arithmetic sequences.

Over the next twenty years, Napier constructed a table of values, consisting of more than ten million entries, that in turn allowed values for φ to be found easily and quickly. By completing such a large-scale table with the values for φ, it was now possible to perform multiplication and division of large numbers in a timely manner. This table would be referenced as “The Logarithm Table” or “The Table of Logarithms” (Fogleman, 2015). Napier’s invention of the logarithm allowed
mathematicians all over the world to see standardized notation and provided a reference to help reduce the likelihood of error while performing long calculations. Therefore, Napier had a monumental impact on the field of mathematics by introducing the logarithm, but also creating a universal notation for a (new) concept as well. This was what several mathematicians had been trying to do for the past few centuries (Clark & Montelle, 2011).

Since logarithms are not what they once were, it is important to understand exactly what Napier was trying to do. His results of constructing such a large-scale table allowed for extreme calculations involving multiplication and division to be completed with little room for error. To show this, I will provide a basic example of how Napier’s table of logarithms simplified tedious multiplications. (See Figure 3). First, consider using the standard multiplication algorithm for finding the product 1,234,567 × 4,321, which is 5,334,564,007. This method is very tedious and time-consuming for this problem. However, when Napier constructed his original table of logarithm values, the process of finding the product of 1,234,567 and 4,321 became much easier. Napier concluded that you can break both numbers, 1,234,567 and 4,321, into their scientific representation of $10^6(1.234567) \times 10^3(4.321)$. Next, simplify the expression to $10^9 (1.234567 \times 4.321000)$.

Next, use “The Logarithm Table” 3 and look for the numbers 1.234567 and 4.321000. Napier proved in his 1614 publication that once you found these two values

3 Napier’s first publication of “The Logarithm Table” had only values with three significant figures. In my example, I used an extended version of Napier’s table that he later published which contained numerical values rounded to four significant figures. There are various versions of “The Table of Logarithms” that contain different number of significant figures and numerical entries.
in the table you can numerically represent them using a base of 10 such that 1.234567 can be rewritten as $10^{0.0899}$ and 4.321000 can be rewritten as $10^{0.6355}$. Therefore, your expression can be expressed as $10^9 \times 10^{0.0899} \times 10^{0.6355}$. By simplifying, we get $10^{9.7254}$, or approximately 5,313,736,321. This was much easier to compute for mathematicians at this time. The difference between the exact value and Napier’s estimated value is only 20,827,686. This is only a difference of 0.3904% of the total value. Napier did not stop constructing his table after publishing his book in 1614; he continued to work to construct a more precise table. Thus, the same process could easily be carried out with the numbers changing such as finding the product of 123,456,789 and 567,891,234. Napier’s table made this product much easier to compute (Clark & Montelle, 2011).
This was the start of one of Napier’s biggest projects as a mathematician, physicist, and astronomer. The creation of the logarithm and the early entries in “The Table of Logarithms” had begun to solve over a two-century year-old issue of finding an easy way to simplify the multiplication and division of numbers.

After his publication of *Mirifici Logarithmorum Canonis Descriptio*, many mathematicians and scientists took note of the great achievement that was made (Fogleman, 2015). They took Napier’s discovery and applied it to simplifying trigonometric identities such as, \(2\cos(A)\cos(B) = \cos(A + B) + \cos(A - B)\). Now instead of computing the exact values of A and B, Napier’s use of logarithms explained this
identity in a more simplified fashion, because now the multiplication and division of numbers could be rewritten as bases of 10 raised to some power. This method was much more efficient to use than dealing with the longer computations required by this trigonometric identity. As Napier continued his research in logarithms and simplifying multiplication and division, he constructed what is known as “Napier’s Bones.” Essentially, these “bones” were an adjustable and portable way to quickly perform multiplication of large numbers using a table-like format. This method of computation was later given the name as the “Slide Rule.”

Napier was not the only one interested in the power of the logarithm; it was time for other mathematicians to make contributions to this field (Weisstein, 2018). Joost Bürgi was next in line to make progress with the logarithm. Bürgi was a clockmaker by day, but by night he was a mathematician driven to reduce computational issues. His ambitions to reduce these issues were largely influenced by Napier’s publication in 1614. Once Bürgi noticed how Napier reduced multiplication and division-based problems, he wanted to continue to investigate different ways mathematicians can obtain a universal system of computation tools, such as tables. Similar to Napier, Bürgi decided to approach logarithms in the same manner by examining the relationship between arithmetic and geometric progressions; his objective was to compose a table with more than just multiplication and division values of logarithms, but to create a table that had square root and cube root values of logarithms as well. In his table, he distinctly labeled what was referred to as “red numbers” and “black numbers.” The red numbers were defined to be the logarithmic value of a number; the black numbers were defined to be the antilogarithmic value of a number (Fogleman, 2015). Bürgi expanded upon Napier’s table and created a more
legible and universal notation for reading a large table of values. Although Bürgi developed his version of the “Table of Logarithms,” he did not publicly disseminate his work in the same way that Napier did. Also, Bürgi’s work was written in German, opposed to Napier’s work which was published in English (a more universal language). Thus, Napier’s work, notation, and thinking process was adapted as the general framework for logarithms, even though Bürgi expanded Napier’s table into something that was more compact and that could evaluate more numbers in one table, in comparison to Napier who had multiple tables of values for different cases (Clark & Montelle, 2011).

History changed the original definition to the modern-day definition of the logarithm. The definition of the logarithm was defined in 1614 by Napier as the “ratio of two distances in a geometric form” (Fogleman, 2005, p. 1). This definition changed over time as the investigation into the purpose and uses of the logarithm evolved. Mathematicians like Henry Briggs (1561-1630) worked with Napier and Bürgi in order to continue to understand the logarithm. Briggs proposed that the common base for a logarithm should be a base of 10 and that log(1) = 0. Briggs’ contribution then changed how logarithms were perceived. Instead of working in Napier’s base of 10,000,000, Brigg suggested to shorten this to a base 10, therefore continuing the drive to represent large quantities as smaller, more manageable expressions. Because of Briggs, we start to see the development of the modern-day definition of the logarithm.

The next contribution to the modern-day definition of the logarithm arose from the collaboration of Johann Bernoulli, Gottfried Leibniz, and Leonhard Euler. These three mathematicians worked together to conclude that “each positive number has only
one-real logarithm, and that all its infinitely many logarithms are imaginary” (Euler, 2015, p. 10). From here, Euler continued by stating a logarithm shared similar properties to exponential functions. By doing so, Euler is given the credit for identifying that logarithmic and exponential functions are inverses of each other, in addition to distinctly defining the mathematical constant, $e$ (also known as Euler’s Number). It was at the end of the 17th century when the logarithm was given its modern definition of “the logarithm $\log_b(x)$ for a base, $b$, and a number, $x$, is defined to be the inverse of the function taking $b$ to the power of $x$, i.e., $b^x$. Therefore, for any $x$ and $b$, $x = \log_b(b^x)$” (Weisstein, 2018, p. 1).

The invention of the logarithm arose from the work of many mathematicians toto find ways to simplify long, tedious products and quotients of large numbers. It was thanks to John Napier who first published and constructed a definition of the logarithm that inspired other mathematicians such as Bürgi, Briggs, Bernoulli, Leibniz, and Euler. Without Napier’s publication of his Table of Logarithms and the work that led him to the construction of that table, the world may not be the same as it is today. We know that the logarithmic function is a powerful tool in many fields. Learners can take away from this history of logarithms the lesson that such mathematical topics do not get invented over the course of one’s own lifetime, but over centuries.

In the sections that follow, I review several research studies investigating students’ learning of logarithms and their difficulties with logarithms.
2.3 Berezovski: “An inquiry into high school students’ understanding of logarithms”

This section discusses a research project conducted by Berezovski in a two British Columbia high school classroom. Her study focused on the cognitive obstacles students faced while learning logarithms. She had a sample size of 27 students enrolled in a twelfth grade mathematics course. The course required students to have some prerequisite knowledge on exponential properties, and how to evaluate exponential expressions. Her objective was to determine what difficulties students faced while learning about logarithms. Berezovski found that students had issues seeing that logarithms can be expressed as numbers, and that there is room for improvement when teaching students the applications of logarithms.

The setting of her study was at Holy Cross Secondary School, which had a total population of 750 students. She decided to focus on two twelfth grade classrooms, because logarithmic and exponential functions was the second largest unit covered in this course, accounting for 17% of the total course curriculum. The students who attended Holy Cross Secondary School were “very motivated and could be considered “the cream of the crop”” (Berezovski, 2004, p. 31).

Berezovski (2004) collected data on students’ understanding of logarithmic functions by administering written questionnaires (one quiz, one unit test, and 6 open-ended questions) and observing and analyzing in-class discussions about the topics being presented. The questionnaires and discussions were aligned with the curriculum used in the school. The quiz assessed the introductory properties of logarithms and exponents. Berezovski asked students to rewrite a logarithm in its exponential form, and vice versa. In addition, she asked students to simplify logarithmic expressions in two problems, solve for value(s) of \( x \) in four problems, and solve one bonus question
assessing students’ ability to solve a generic logarithmic equation for a particular value. The unit test assessed students’ overall learning of exponential and logarithmic functions. It was composed of two parts. The first part of the unit test was multiple choice and focused on the general properties of exponents and logarithms and graph transformations. The second part was structured as a free-response section. This part evaluated students’ understanding of how to graph a logarithmic function and solve three application problems involving logarithmic and exponential functions. The in-class discussions were important as they provided in-the-moment information about how students were thinking about and interpreting logarithms as the lessons were taught.

Berezovski (2004) learned from the questionnaires that students had trouble when simplifying exponential expressions, rather than logarithmic expressions. For example, students performed most poorly on the question \(54^3 \div 8^3 \times \times 4^3\). She suspected that this was the lowest scored question because of students’ misunderstanding of exponents. She received several answers involving logarithms, but only two students correctly solved this expression. In addition, she learned that students have trouble solving for \(x\) when the equations contained logarithmic expressions that require multiple skills to be used. Some students could condense the logarithms but did not correctly eliminate the value for \(x\) that made the argument(s) negative. That is, students would be able to complete part of the problem, but not be able to fully connect all of the ideas needed to correctly solve the problem. Finally, she learned that all, but four students were able to give their own examples of logarithmic equations, the definition of a logarithm, the relationship between logarithmic and exponential functions, and when to use the properties.
In addition to the questionnaires, Berezovski (2004) learned from data collected from conducting the interviews and probing classroom discussions. In the interviews, she found that students struggled with the notation of a logarithm and describing how each part influences the others. For example, a student named Ryan “mistakenly connected the logarithm with a base, not with an exponent” (p. 73). She elaborated on this by concluding that Ryan struggled with defining each part of the logarithm. He was able to do some of the easier problems, but when the problems became more difficult, he got confused about what to do with the logarithm. In the classroom discussions, Berezovski noted how students were perceiving logarithms in the moment. During this phase of her project, she concluded that when students were presented with a general history of the logarithm, only three students could give an accurate retelling. She suggested that this might be because they did not take interest in the topic. Berezovski noted that one of the teachers encouraged students to call out their answers in class to promote an open discussion within the classroom, instead of picking on students to lead the discussion. She learned that in this classroom, some students who were more reserved spoke up more often than the students who were reserved in the other classrooms where this approach was not used.

Overall, Berezovski’s (2004) research suggests that researchers and educators need to reexamine how logarithms are first introduced. She believed that since Napier was able to construct the logarithm, prior to the invention of the exponential function, then there is a way to present logarithms to students in a similar manner. She proposed a hands-on learning activity based on Napier’s “Slide Rule.” She thought that this could be a potential lesson to introduce logarithms for students as it promoted students’ critical thinking skills about the concepts being developed. I view her
research as something that can be explored more in-depth, because I believe that the way students perceive logarithms strongly impacts their learning of them. I will use similar assessment tools to collect my data.

2.4 Kastberg: “Understanding mathematical concepts: The case of the logarithmic function”

This section discusses research conducted by Kastberg in two college algebra courses at a rural southeastern college. Kastberg had three objectives. The first was to assess students’ understanding of logarithmic functions. The second goal was to assess what influences students’ understanding of logarithmic functions. The third goal was to assess students’ understanding about how to apply logarithms to solve problems.

Kastberg’s initial sample was a total of 29 students. The 29 participants were contacted by email and phone to conduct a first meeting where he gathered background information such as how they perceived themselves as a math student, if they felt comfortable being observed performing math problems, and if they were still interested in participating in his study. After the meeting, Kastberg chose 12 students, six from each section of college algebra, to participate in nine interviews.

Kastberg’s (2002) data analysis was composed of multiple elements. First, he transcribed each interview for each participant. He viewed this time as a way to personally get to know each student in a different way than just hearing them speak. Next, he made notes on each interview that he felt was relevant to his study. He later summarized each set of notes to help him create case summaries. Once Kastberg had a general outline of all of his interviews, he created five coding categories: conception, representation (written, oral, pictorial, and tabular), connection, applications, and ways
of knowing. Using these five categories, he was able to start breaking each interview into different pieces: evidence, students’ beliefs, and themes. Kastberg was able to take all of this information from each category to build a “story” for each participant.

Kastberg (2002) found that there is a strong connection between students’ understanding of any math concept, and the instruction they receive. He also claimed that instructors should be flexible in how they teach the material. He suggested that there “should be ways to alter [students’] beliefs about understanding… while providing opportunities for them” to expand their knowledge of mathematical conceptions (p. 185). Finally, he reported that students had trouble with explaining their work. He made several observations that suggested that students were working based off of memory, especially when using the logarithmic properties. So when he asked students to explain what they wrote down, they would have difficulties doing so. Kastberg suggested instead of giving students a series of properties to memorize, instructors should challenge students to think about where the properties come from. He claimed that this could improve students’ understanding of the properties.

Kastberg’s (2002) work influenced my research since we have similar research interests. We both want to better understand students’ obstacles and misconceptions about logarithmic functions. He conducted a total of nine interviews, using some questions that I will tie into my thesis project. Some questions that I would like to incorporate into my interviews are simplifying and expanding logarithmic expressions, allowing students to describe their math background (specifically their algebra background), identifying issues students faced in their previous math classes and what they did to fix them, and allowing students to express their understanding of logarithmic functions in different forms.
2.5 Kenney: “Students’ understanding of logarithmic function notation”

This section discusses research conducted by Kenney in two different college Pre-Calculus courses at North Carolina State University. Her goal was to investigate the idea that students must first learn the notation of the basic logarithmic function; understanding the notation is one of the most influential factors in whether students will master logarithms. Her framework was based on Gray and Tall’s (1994) work dealing with mathematical symbols and students’ perceptions. She wanted to look at the “understanding and conceptions that have been formed as a result of learning” in comparison to other studies that looked at the learning process (Kenney, 2005, p. 2).

Kenney (2005) collected data from two samples of participants. The first sample was 59 students who took MA-111. The second sample was two students enrolled in MA-107. Kenney’s project was broken up into two phases. In Phase I, she administered a five-problem written questionnaire to 59 students enrolled in MA-111. The questionnaire was designed to collect information about how “students interpret and manipulate logarithmic functions” (p. 2). Phase II involved a series of three interviews with two students enrolled in MA-107. (She had to exclude students who took MA-111 because of time constraints.) The interviews assessed how students represented logarithms and how they interpreted the notation. The two phases of her project were closely related because she did not develop her interview questions until after she determined the results of Phase I, in order to have results that closely supported one another.

The results of Kenney’s research indicated that students did not have a “perceptual understanding of logarithms” (Kenney, 2005, p. 3). She observed that if students were presented with a logarithmic expression, they would naturally want to convert to an exponential form. When students were presented with two or more
logarithmic expressions, they would not want to convert to exponential form but tried to get rid of the logarithms by cancellation. Kenney believed that these inclinations stem from students not understanding the notation of the logarithm, and she proposed that if they did understand the notation, then these issues may not arise.

In summary, I will use some ideas from Kenney’s paper. I find it interesting to investigate how students interpret the notation of a logarithm. I think that would be a good indicator that would explain other factors related to students’ fluency with logarithms. Kenney specifically showed different ways students interpret logarithmic notation and she suggested further investigation into how teachers are presenting this information to students. I want to obtain a larger sample size with my interview participants, because I think having more than two individuals would provide better data to explain students’ perceptions toward and fluency with logarithms.

2.6 Kenney & Kastberg: “Links in learning logarithms”

This paper is constructed differently than other papers reviewed in this section. In this paper, Kenney and Kastberg “discuss and share evidence of students’ difficulties collected from various courses over time. We share concepts related to logarithms that could help students build an understanding of these functions, and we present some ways that misconceptions related to these concepts are manifested to suggest what teachers can listen for as they explore logarithms with students” (p. 13).

The first section of the paper focuses on conceptual understanding of logarithmic functions. The authors argue that notation is essential to student learning about logarithmic functions because other researchers have concluded that if students do not understand the notation of logarithms, then they are more likely to struggle with how to use logarithms. Next, the authors discussed how logarithms need to be
introduced as functions and how interpreting the notation of the logarithmic function is critical. They concluded that if “logarithmic functions [could be] characterized by their own unique set of properties” then that will help students better understand the notation (Kenney & Kastberg, 2013, p. 6). They stress that when students encounter a problem that requires applying the properties of logarithms, they tend to forget the various methods of working with functions. Students focus more on the algebraic steps instead of connecting the properties to the function itself. They suggest students’ misconceptions about logarithmic notation prevent students from connecting the properties of logarithms to the concepts they already learned.

In the second section of their paper, Kenney and Kastberg (2013) discuss how teachers can help support students’ understanding of logarithmic functions. Their first idea is about “building connections to prior experiences” (p. 7). They suggest that teachers should “connect students’ experiences with functions” (p. 7) throughout teaching logarithms so they can improve students’ understanding that logarithms are functions and just because they involve properties, they can still be viewed as something that has an input and output. They suggest that discussing how the square root function has a restricted domain can then be applied to the logarithmic function. Kenney and Kastberg state that the square root function has notation that students find easier to interpret and understand. Thus, by incorporating ideas and functions that students are more comfortable with, they may see that the logarithmic function has similar properties. Therefore, the authors claim that this may help support students’ understanding of the different parts of the logarithm and improve their understanding of the notation.
Next, the authors discussed how teachers need to pay close attention to “both the words… and ideas that students share about their understanding” (Kenney & Kastberg, 2013, p. 7). They point out that sometimes teachers can get lost in explaining the concepts fully and forget to assess how students are understanding the processes. For example, they point out that students typically want to cancel out the logarithms, because teachers refer to this language during class time when solving logarithmic equations. So, from the students’ perspective, they interpret this to mean that they can physically get rid of the logarithms, which is not accurate. Furthermore, Kenney and Kastberg provide discussion questions for teachers to incorporate into their lessons, so that teachers can assess how students are interpreting logarithms. Some questions suggested are: “What does \( \log_3 \) mean in the expressions \( \log_3(x) \) and \( \log_3(x + 1) \)?”, “Give an example of an exponential equation whose solution is a negative number...”, and “Produce an argument that could convince a friend that \( \log_b(M + N) \neq \log_b(M) + \log_b(N) \)” (p. 8). By focusing on building connections to past topics and paying close attention to the language they are using, teachers can better support students in developing a deep understanding of logarithmic functions.

I appreciated this paper because I think I can apply the ideas of paying close attention to wording for my questions on the Pre-test, Interviews, and Post-test so that students have a clear understanding about how to perform the tasks at hand. I believe that applying this idea of paying close attention to syntax will help identify students’ misconceptions about notation and provide information regarding students’ interpretations of logarithmic expressions and equations. Kenney and Kastberg (2013) described how it is important to make sure to assess what students are knowing in the moment. Thus, I will schedule my interviews so that I can track students’ progress in
their attitudes and interpretations of logarithms at different stages along the learning process.

2.7 **Liang & Wood: “Working with logarithms: Students’ misconceptions and errors”**

Liang and Wood (2005) conducted research in three Singapore primary schools. The students’ ages ranged from 16-18 years old. Their goal was to examine students’ understanding of logarithms. All three schools followed the same standards because the goal was to help students pass the Primary School Leaving Examination. Schools A, B, and C consistently had the same scores on this exam. The researchers had a sample of 124 students: 43 were from School A, 42 were from School B, and 39 were from School C. School A served as the control group; Schools B and C served as the experimental groups. The control group was taught as if they were not involved in the experiment. Schools B and C participated in a series of lessons, written by Liang and Wood. This differed than the normal instruction used in School A because it presented logarithms through a more active learning style. The experimental groups represented male and female students which were randomly selected from eight classes, six from School B and two from School C, taught by different math teachers. The researchers selected students who had a wide range of math abilities in order to obtain a representative sample.

Liang and Wood (2005) developed an instrument to specifically assess “students’ learning difficulties, misconceptions and errors in logarithms” called the Test of Students’ Understanding of Logarithms, or ToSUL (p. 58). Their test was composed of 47 items. They collected information from various textbooks and a modern version of Bloom’s Taxonomy to help organize the test into three sections:
Computation or Knowledge, Understanding, and Application. They believed that assessing students’ understanding in these three areas would provide significant data to better understand students’ learning. The Computation or Knowledge part (23 items) assessed students’ abilities to define the “laws of logarithms, as well as simple manipulation or computation… with two or three steps” (p. 58). The Understanding part (14 items) assessed students’ ability to recall the properties of logarithms and understand properties. The Application part (10 items) assessed students’ ability to “develop their own techniques for solving problems that they probably have not met in a textbook” (p. 59).

Liang and Wood (2005) report that students knew how to solve logarithmic equations, but the students often struggled in the conceptual understanding. Students interpret algebra to be a series of steps to follow in order to get the right solution. So, when they encounter logarithms where their algebraic understanding gets tested, they fall short in their performance in comparison to other units on functions. The application of the properties of logarithms seems to cause students to forget their algebra skills since they cannot connect what they have previously learned and apply it to these properties. Students get confused as to which step to perform when, indicating that the notation of the logarithm occasionally hinders students’ understanding. Liang and Wood claim that logarithms could be looked at as “a study of relationships among quantities” (p. 54). This may promote students’ understanding of logarithmic functions because they would see a logarithm as a relationship instead of as a series of steps.

I can take away a lot of information from Liang and Woods’ (2005) research project. Even though they tested students in the Singapore education system, it seems
that similar students’ misunderstandings are present. I admire how they broke down the ToSUL into three sections, so that they are focusing on three specific aspects of students’ fluency. I think this would be a good way to construct my test as well. From their study, I realize that I will have to carefully construct my questions so that they are readable to the students in the way that I want them to be. Liang and Wood suggest that sometimes the instructions students are given can be confusing, therefore I will make my instructions as clear as possible with little room for misinterpretation.

2.8 Mulqueeney: “How do students acquire an understanding of logarithmic concepts?”

Mulqueeney (2012) conducted research in a mid-west United States The goal of her study was to “investigate how students at the collegiate level acquire[d] an understanding of logarithmic concepts and how the symbolic notation contributes to this cognitive understanding ... and pedagogical strategies to help students” move from one level of understanding to the next (p. 30). Mulqueeney aligned her research with what she found when investigating different methods of assessing students’ learning about logarithmic functions.

Mulqueeney (2012) collected data from 167 students enrolled in an introductory Pre-Calculus course. All 167 students took a pre-assessment, one week before the end of the semester. This test assessed students’ understanding of various logarithmic properties and their concepts. Out of the 167 students, Mulqueeney then carefully selected four participants to participate in the second part of her project. The four participants participated in a teaching experiment in which they had to complete an initial “mathematical beliefs survey” (p. 92), attend six teaching sessions, complete tasks assigned by the instructor, and answer questions about their work when indicated.
by the instructor. Completing her project in these stages allowed Mulqueeny to collect data about how students not only performed on the logarithm tasks, but also about how they verbally explained their solutions. She hypothesized that this would provide some give her insight into students’ misunderstanding of the notation as well as other misconceptions.

The four interview participants were first given a survey that assessed three different themes: “mathematics by intimidation”, “mathematics promotes deeper understandings”, and “mathematics is about right answers”” (p. #). From their responses, Mulqueeny (2012) was able to determine how students viewed mathematics generally. Part I of the teaching experiment assessed students’ understanding of exponential functions, following the Action-Process-Object-Schema (APOS) theory. Part II assessed students’ understanding of the relationship between logarithmic and exponential functions. Part III assessed students’ understanding of logarithmic properties. Part IV assessed students’ understanding of the relationship between logarithmic and exponential functions to “explore when it would be appropriate to use logarithmic notation to solve algebraic equations” (p. 169). Part V assessed students’ abilities to expand on their understanding of logarithms and eventually construct a “proof for the addition, subtraction, and multiplication properties for logarithms” (p. 118). Part VI was not conducted due to time constraints.

Mulqueeny’s (2012) findings indicated a few ideas for exploration. First, she suggests that when students first encounter logarithms, they struggle to make sense of the notation. They lose track of what they are doing, simply because they cannot read a logarithm. Students’ see the word “log” and interpret that as a word, instead of as a function that can be evaluated. Next, Mulqueeny found that students report they do
not care for the logarithm as they did not understand how someone could come up with such notation, when all of the other functions presented in the course are completely different. This suggests that students were “overwhelmed” by the notation and did not see the point in it. She suggests that teachers “force [students] to struggle for meaning; otherwise, the students will continue with their [misunderstandings of the] representations of logarithms” (p. 265).

Reviewing Mulqueeny’s study of students’ understanding of logarithmic functions, I can take away her close eye for detail. She collected a lot of information from just four participants, to the point where she was able to form detailed conclusions about them. She used the assessment of the 167 students to help support her general findings, however I learned that her interviews revealed more than the assessment. I have learned to pay close attention to my interviews, as they will form the majority of my evidence for my paper. I have to be able to take the time to really understand what students are saying and why they are saying it. By doing so, will be able to achieve my research goal, which is to assess students’ fluency with logarithms and identify their misconceptions.

2.9 Weber: “Developing students’ understanding of exponents and logarithms”

Weber (2002) conducted research at a university in the southern United States and had two research goals: (1) to have students “understand the act of ‘taking an exponent’ as a process” while using a computer program and (2) to have students “understand \( b^x \) as the number that is the product of \( x \) factors of \( b \) and \( \log_b(m) \) as the number of factors of \( b \) that are in the number of \( m \)” while using pen and pencil (p. 5). Weber examined two sections of College Algebra with fifteen students in each section. One was a control group and the other was the experimental group. The control group
ran as a traditional lecture-styled classroom. The experimental group was composed of many parts. The first part was related to Weber’s first goal, and it focused on students’ abilities to write a computer program, in groups of two or three, to show the relationship of how “‘taking an exponent’ [is] a process” (p. 4). He suggested that if students are able to develop a code to display the idea of what it means to be exponential, then students will “reflect on the steps of an operation and interiorize it into a process” (p. 3). This was done by creating a loop that “performed multiplication of integers as repeated addition” (p. 3). The instructor would help students conceptually but did not help them write the actual code. This forced the students to develop an understanding of what it means to be exponential.

Weber believed by constructing the knowledge of exponential functions using computer program they could see the iteration of the function. Students performed tasks such that they would be able to identify the relationship between exponential functions and the multiplication of a number a certain number of times. The second part of Weber’s (2002) experiment was related to his second research goal, and focused students on the relationship between $b^x$ and $\log_b(m)$. To do so, Weber carried out a series of worksheets in the experimental group on which students recorded their responses describing logarithmic and exponential functions as objects. Students worked in groups of two or three to complete the exercises. There were examples given on the worksheets, but again the teacher acted only as a guide for students. After the first three weeks of his project, Weber conducted interviews with all 30 participants. The interviews assessed three areas: basic computations, rules, and conceptual understanding. By doing so, Weber believed that he could identify the various difficulties students were having while learning logarithms.
After the interviews were over, Weber (2002) compared the two groups. The majority of his data was analyzed using quantitative tools. He scored each student’s response to each question from the interview as correct or incorrect so that he was able to quantify the students’ responses. Since he broke down the interview into three parts, it was easy for him to identify specific issues within each area that students had difficulties with. This enabled Weber to form his conclusions. He suggests that teachers should incorporate group work to help students develop a conceptual understanding of logarithms, as well as incorporate individual work to promote procedural fluency. Weber concluded students in the traditional classroom excelled in procedural knowledge, whereas the experimental classroom excelled in conceptual knowledge. The students in the experimental group had more difficulty recalling the properties of exponential and logarithmic expressions, while the traditional classroom excelled in this area.

I can take away the following from Weber’s study: First, it seems that his experimental group was taught in an active-learning environment similar to MSLL. This suggests that I could incorporate similar ideas on my assessments and in my interviews. Also, I learned that it is important to assess students’ understanding of the relationship between logarithmic and exponential functions because not incorporating this may change how I view my results. Weber suggests that assessing this concept is crucial to identifying students’ misconceptions with logarithms.

2.10 Williams: “A conceptual framework for student understanding of logarithms”

Williams (2011) conducted research in a mathematics education class at a western university in the United States, examining future secondary mathematics
teachers who already had received an introduction to logarithms. The class “explored the concepts behind mathematical topics” (p. 3). Williams’ goal was to create a “research-based framework of what it means to understand logarithms” (p. 3). Williams (2011) administered a short survey to the students assessing their “willingness to participate, knowledge of logarithms, and ability to explain their thinking” (p. 26). Using the survey responses, she eliminated students who did not want to be interviewed. Next, she analyzed students’ understanding of logarithms. If a participant showed little to no evidence of understanding logarithms, she eliminated them. Finally, Williams went through the remaining eligible participants to identify those so that a “variety of thinking was represented” (p. 26). From this analysis, she selected four participants to take part in four interviews for her study. Each of the four interviews followed the same protocol. The first interview assessed the “object and process conceptions of logarithms” which tied back to her framework (p. 26). The second interview assessed students’ understanding of the logarithmic function and application problems. The third interview assessed the same topics as interview one. The fourth interview assessed remaining topics not addressed in the previous three interviews.

Williams (2011) first transcribed each of her interviews for each participant. She noted that in her data analysis she adjusted her interview protocols “according to what [she] found as [she] collected data” from the surveys and classroom observations (p. 31). After going through each interview, she wrote a short memo on each interview for each participant. Next, she adjusted her framework and modified her categories based on each interview. She would continue this process until every interview was incorporated into her framework. As she went back and adjusted her
framework, she noticed that there were some categories that did not have a lot of evidence, and some that had more than enough evidence. After completing analyses of her interviews, she created one framework that represented all four participants.

Williams (2011) concluded that (1) there was a difference between “the practice of switching forms… and the process of meaning for logarithms,” and (2) her framework was successful in identifying students’ misunderstandings of logarithmic functions (p. 71). She suggested that more research should be done in creating a framework to help future secondary mathematics teachers teach logarithms effectively and be able to identify students’ misconceptions in the classroom.

Williams’ conclusions support that a stronger framework needs to be constructed so that students will develop fluency in logarithms. I will want to incorporate her keen eye for detail when analyzing her interviews. I concluded that if I can develop an understanding of what my interview participants said, then I may be able to form well-supported conclusions.

2.11 Conclusion

Over the course of reviewing the existing literature, I have gained some insight as to why students struggle with logarithms, and what approaches have positively impacted ‘students’ understanding. I have designed my methodology based on the research that I reviewed. Taking everything into consideration, the best way to understand what students do and do not understand about logarithms, and how their thinking changes over the course of instruction on logarithms, is to assess students’ fluency through both written tasks and through one-on-one interviews in which I can
probe student’s thinking. As you will see in Chapter 3, this is exactly how I designed my study.
Chapter 3

METHODOLOGY

This chapter describes the methodology of my thesis. First, I describe the study setting and participants. Next, I describe the data collected – a Pre-test, a set of three interviews, and a Post-test – and the procedures I used to collect this data. Finally, I describe how I analyzed the data. Throughout the rest of my thesis, I will refer to specific logarithmic properties in the following way:

- **Product Property:** When a student is able to rewrite the logarithm of a product as the sum of the logarithms of the individual factors, and vice versa: \( \log(AB) = \log(A) + \log(B) \). To know this property, students must be able to identify and correctly apply it.

- **Quotient Property:** When a student is able to rewrite the logarithm of a quotient as the difference of the logarithm of the numerator and the denominator, and vice versa: \( \log\left(\frac{A}{B}\right) = \log(A) - \log(B) \). To know this property, students must be able to identify and correctly apply it.

- **Power Property:** When a student is able to rewrite the logarithm of an argument raised to a power as the power times the logarithm of just the argument, and vice versa \( \log(B^A) = A \log(B) \). To know this property, students must be able to identify and correctly apply it.

### 3.1 Setting and Participants

#### 3.1.1 Setting

This study was conducted at the University of Delaware (UD) in the Fall 2017 semester. In particular, the study was situated in the context of MATH 115, a Pre-
calculus course offered at UD that involves the study of logarithms. While some students take MATH 115 as a terminal mathematics course, most students enroll in the course as a pre-requisite for MATH 221, a Calculus course required of many business majors and other social science majors at UD. I am quite familiar with this course because I work at UD as a Classroom Assistant for the Department of Mathematical Sciences. Over the course of four semesters, I have been a Classroom Assistant for six sections of MATH 115, working with approximately 350 students.

MATH 115 is a MSLL course, and so it employs active learning strategies in the classroom. As a result, students spend time in each class working together in groups to solve mathematics problems designed to help them learn the material. The instructors give a series of mini-lectures during each class meeting to help guide students to build skills and understand concepts. In the course, students work through a series of shared lessons that introduce students to several essential function families – linear, piecewise, exponential, logarithmic, polynomial, rational, and trigonometric functions. MATH 115 was designed to have a common, collaboratively developed curriculum (shared lesson plans) to ensure that students’ learning opportunities are similar across different sections of the same course. No matter which instructor a student has, the student will have similar learning opportunities in class. Students are assigned homework through an online, adaptive learning and assessment program called ALEKS. ALEKS works with each individual student and is designed to support their development of procedural fluency.

There are five lessons that develop skills and ideas about logarithms and logarithmic functions in MATH 115. These lessons span six, 75-minute class sessions. In the first lesson, logarithmic functions are introduced as the inverse to
exponential functions. Students learn how to relate the notation of a logarithm to the notation of an exponential expression, find the domain of a logarithmic function, describe the different parts of a logarithm, and graph the parent function. In the second lesson, the properties of logarithms are introduced. Students learn how to use the change of base formula, expand logarithms, condense logarithms, interpret the constant \( e \), and use logarithms in a simple application problem. In the third logarithm lesson, the Uniqueness Property of Logarithms is introduced. Students learn how to use the property to manipulate exponential expressions and solve for \( x \) in the exponent. In the fourth lesson, the two main formulas for compound interest are introduced to students, \( A = Pe^{rt} \) and \( A = P(1+\frac{r}{n})^{nt} \). Students learn how to use these formulas by applying them to solve word problems. In the fifth lesson, the final type of application problem is introduced, namely continuous exponential growth and decay. Students learn how logarithms can be used to solve problems that relate to a substance’s half-life. In the five lessons, students investigate the relationship between logarithmic and exponential functions, develop and apply the properties of logarithms, solve logarithmic and exponential equations, and study real world logarithmic applications. Each lesson is organized to introduce students to each topic so that they can develop proficiency with logarithms and logarithmic functions.

Students in MATH 115 are able to earn up to 20 points (which constitutes 1% of their overall course grade) engaging in “Course Analysis” activities. The purpose of these activities is to help the instructors collect data on students' performance, growth, knowledge, and perceptions that will help the instructors improve the course in future semesters. Each semester, students are presented with various opportunities to earn the 20 Course Analysis points throughout the semester (e.g., completion of a
survey, participation in a research study). These points could be offered to students participating in various aspects of my study.

### 3.1.2 Participants

In the Fall 2017 semester, there were 11 sections of MATH 115, taught by six different instructors. There were approximately 700 students enrolled in the course. In early September, all students enrolled in the course were invited by email to take the Pre-test (described in more detail in Section 3.1.3). A total of 249 students completed the Pre-test. A subset of 22 students who completed the Pre-test and who met particular criteria (described in Section 3.1.4) were invited to participate in a series of three interviews. Of these, 10 students completed all three interviews. A few weeks after the logarithm lessons had been taught, all students enrolled in the course were invited by email to take the Post-test (described in more detail in Section 3.1.5). A total of 306 students completed the Post-test. A total of 155 students completed both the Pre-test and the Post-test.

### 3.1.3 Pre-test

In early September, I sent an email to all students enrolled in MATH 115 inviting them to participate in the first component of my study, which was completing a Pre-test on logarithms. Students were informed that they would earn five Course Analysis points by completing the Pre-test, but that otherwise their performance on the Pre-test would not affect their course grade in anyway. I offered multiple sessions during the last two weeks of September for students to take the Pre-test. Students were able to choose a time that worked for them.
The goal of the Pre-test was to assess students’ in-coming fluency with logarithms. (See Table 1.) It consisted of eight items. The first three items collected information about students’ background in mathematics. The fourth item assessed whether students could identify some characteristics and properties of logarithms. The fifth item assessed students’ ability to convert from logarithmic to exponential form. The sixth and seventh items assessed students’ ability to condense and expand logarithms using the Product, Quotient, and Power Properties, respectively. The eighth item assessed students’ ability to apply multiple logarithm skills to obtain.

The Pre-test administered in the MSLL Testing Laboratory and proctored by me. Students had 60 minutes to complete the Pre-test. Before students started, I went over consent form that outlined the protocol of the Pre-test. Students read the three pages, initialed the bottom right hand corner of each page, signed their name (print and official signature) and printed the date they took the exam on the last page. Calculators were not allowed. Students were given as much additional scrap paper as needed; students were strongly encouraged to show all work on the Pre-test. A copy of the consent form may be found in Appendix F. In the end, a total of 249 students enrolled in MATH 115 completed the Pre-test.
<table>
<thead>
<tr>
<th>Question Number</th>
<th>Product Property</th>
<th>Quotient Property</th>
<th>Power Property</th>
<th>Knowledge of the common and natural logarithm bases</th>
<th>Converting from logarithmic to exponential form</th>
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Table 1. A summary of the skills assessed in the Pre- and Post-test.

Students read the consent form, initialed the bottom right hand corner of each page, signed their name (print and official signature), and printed the date they took the exam on the last page. Students were given as much additional scrap paper as needed, but they were strongly encouraged to show all work on the Pre-test.

3.1.4 Interviews

A set of three individual interviews were conducted with 10 of the participants. In order to collect information from students with varied backgrounds and incoming knowledge of logarithms, I identified students to invite to the interview component of
my study based on their performance on the Pre-test. Students who completed the Pre-test were grouped into three categories – DAT, ADT, or TAD – based on their performance. These categories placed students into different percentiles based on their performance on the Pre-test. Students in category DAT scored in the lowest percentile, in exhibiting little to no understanding of logarithmic properties and concepts. Students in category ADT scored in the middle percentile; they exhibited some understanding of logarithms and got approximately 50% of the problems correct. Students in category TAD scored in the highest percentile; they exhibited considerable fluency in some or all logarithm concepts taught in MATH 115. Once participants were categorized in this way, I assigned a random three-digit number to each participant in each of the three groups. From here, I calculated the number and percentage of students in each category. The number of students in each category was as follows: 106 (47.32%) DAT students, 96 (42.86%) ADT students, and 22 (9.82%) TAD students. Most of the students placed into the low and medium fluency categories.

I selected 10 students each from groups DAT and ADT and two students from group TAD, for a total of 22 students, to participate in the interviews. I emailed each of these students in the last week of September and invited them to participate in Interview 1. Of the 22 students I contacted, 13 responded and participated in Interview 1. These 13 students were then invited by email in the second week of October to participate in Interview 2. Twelve of these 13 students participated in Interview 2. Finally, students who took part in Interview 2 were invited by email in the first week of November to participate in Interview 3; ten of these twelve students participated in Interview 3. Each student who participated in Interviews 1 and 2
earned five additional Course Analysis points per interview. Students who participated in Interview 3 received a $30 Barnes and Noble gift card.

The purpose of the interviews was to track a purposefully selected sample of students and to better understand their growth and development of their understanding of logarithms over the course of the semester. The interviews were designed to solicit students’ individual perspectives about logarithms and fluency with logarithms. I adopted a practice from Kastberg’s (2005) research to assess ‘students’ learning throughout the process. Kastberg suggested that asking students personal questions about their perceptions of logarithms might influence their fluency. Each interview was held in a private room on the ‘university’s campus. Each interview lasted about 30 minutes and was audio recorded and transcribed.

The goal of Interview 1 was to collect additional information about the ‘students’ incoming familiarity and skills with logarithms, as well as their general math background. I wanted to get an idea of what initial understandings students came in with because that could influence what they learned during the course. I interviewed participants individually, so that they were not influenced by other interviewee’s responses.

In Interview 1, each participant was asked the same set of nine questions. (See Appendix G for the Interview 1 protocol.) The initial questions in the first interview pertained to the ‘students’ backgrounds and their responses to the Pre-test. Questions 1 and 3 were Likert-scale items. Question 1 asked students to evaluate the difficulty level of the Pre-test. Question 3 assessed students’ familiarity with the material assessed on the Pre-test. Question 2 asked students how they felt while taking the Pre-test. I hypothesized that students’ incoming algebra skills might influence their ability
to master logarithm skills, and so Questions 4, 5, and 6 assessed students’ algebra backgrounds. Question 6 was a Likert-scale item designed to evaluate students’ confidence in their algebra skills. Questions 7 and 8 assessed ‘students’ familiarity with vocabulary related to logarithms. I asked students to give me their first reaction to words such as "logarithm," "exponential function," and to equations that relate to logarithms. The rest of the interview was designed to probe the students' responses to the items on the Pre-test. I asked each participant to revisit and explain their thinking to most, if not all, of their responses on the Pre-test in order to assess their incoming ideas and understandings related to logarithms.

Interview 2 followed the same structure as Interview 1 – it was audio recorded, transcribed, and held in a private room on the university's campus. The goal of this interview was to identify what knowledge and skills students were developing from the first few logarithm lessons. Thus, Interview 2 was purposefully scheduled in the middle of the logarithm lessons.
Question 1 in Interview 2 was designed to assess whether students could define what a logarithm is and why logarithms are important. (See Table 2.) Questions 2 and 4 were Likert-scale items designed to assess 'students' understanding of logarithms and their comfort in explaining and working with them. Question 3 focused on students' initial impressions of logarithms; also, it prompted students to think about what concepts or ideas were easy for them to initially understand. The final question, Question 5, was an open-ended set of 11 items that asked students to
expand or condense logarithmic expressions, and to solve logarithmic equations. Students were not expected to be able to complete all portions of the questions, but all questions could be answered if students were starting to develop mastery of logarithm properties and concepts. After students attempted to solve all 11 parts of Question 5, I asked them to explain their reasoning behind each of their solutions. The entire set of questions for Interview 2 can be found in Appendix D.

Like the first two interviews, Interview 3 was audio recorded, transcribed, and held in a one-to-one private setting. The goal of Interview 3 was to assess students’ fluency with logarithms after all of the logarithm lessons in the course had been taught. Questions 1, 4, 5, 6, and 7 evaluated students’ understanding of particular aspects of logarithms, such as the relationship between logarithmic and exponential functions and applications of logarithms to the real world. The interview questions also asked students to reflect on their journey of learning logarithms and what had been easy or difficult for them, and their overall perception of logarithms. Questions 2 and 3 were designed to assess students’ fluency in applying logarithm properties and evaluating logarithmic expressions. After the participants completed these items, I asked them to explain their work and their reasoning for each problem. The complete set of questions for Interview 3 can be found in Appendix E.

3.1.5 Post-test

After the Pre-test and all three interviews took place, I emailed all MATH 115 students and invited them to participate in the Post-test. MATH 115 students were sent a sign-up allowing them to schedule the Post-test for a time slot that was convenient for them. A total of 306 students completed the Post-test. Students who completed the Post-test earned five Course Analysis points.
The goal of the Post-test was to assess students’ exit understanding and skill with logarithms at the end of the logarithms unit. The Post-test was administered about 2-3 weeks after the six logarithm lessons were taught. It was important to administer the Post-test a few weeks after the logarithm lessons in order to assess what students had learned, as well as retained, about logarithms.

The Post-test was identical to the Pre-test, except for Questions 1-3. Recall that on the Pre-test, these questions asked students about their mathematics background and familiarity with logarithms. On the Post-test, these questions were replaced with questions about their current perceptions of logarithms, what challenged them, and what they thought the relationship between logarithmic and exponential functions was. The rest of the questions were the same. The procedure for signing up and administering the Post-test was identical to the Pre-test procedures and protocol. Students who completed the Post-test were compensated with five Course Analysis points. The Post-test can be found in Appendix B.

3.2 Data Analysis

In this section, I describe how I analyzed data collected from the Pre-test, the three interviews, and the Post-test.

3.2.1 Pre-test

Students’ responses to Questions 1 – 3 of the Pre-test provided information about their incoming mathematics background and familiarity with logarithms. This information was vital when I selected my interview participants, because I did not want the participants to have overlapping math backgrounds and perceptions. I wanted to choose students with different perspectives and different backgrounds for my
interviews. To choose a diverse subsample, I used a random number generator. First, I selected ten participants from group DAT. I programmed the random number generator to choose values from 100-999. I started the generator to give me one random number; if that number was assigned to a student I selected them, if not I started the generator again. I repeated this process until I obtained ten potential participants in group DAT. I repeated this same process for selecting ten students from group ADT and two students from group TAD. After I identified a total of 22 potential interview participants, I reviewed their responses to the first three questions on the Pre-test. Recall that these three questions asked students to state when they first saw logarithms (year, grade, class, or any information pertaining to these), to define what a logarithm was, and to explain the relationship between logarithmic and exponential functions. I verified that selected students from groups DAT and ADT had a diverse set of responses, so that each student was somewhat different than the other. For group TAD, I verified that their responses were similar, if not the same such that there was little variance in the responses. The purpose of this was to make sure I had two students who had substantial knowledge of logarithms. So, I had two groups of students with varying responses and two students who had a strong knowledge of logarithms entering MATH 115. This produced a subsample that would reflect the sample of the participants who took the Pre-test.

Students’ responses to Questions 4 – 8 of the Pre-test were scored using a strict scale of 0, 1, or 2 points for each question part. A score of 0 points was assigned if the student did not attempt the problem or did not exhibit any correct step or idea related to the problem. A response received a score of 1 point if the response exhibited partial, but incomplete, understanding. A response received a score of 2 points if the
response exhibited full and correct understanding and resulted in the correct answer. I then computed the total number of points earned (out of a total of 36 points) for each student. Since all MATH 115 students were invited to take the Pre-test and then invited again to take the Post-test, I found the intersection of the two sets of participants and scored tests only for those 155 students who took both the Pre-test and the Post-test. After scoring the 155 Pre-tests, I uploaded the scores to Google Sheets and computed the mean, median, mode, range, and standard deviation.

3.2.2 Interviews

The process of analyzing the interviews started with transcribing all 30 interviews with the ten participants. I transcribed only the interviews for the 10 participants who participated in all three interviews. In addition, I categorized what each open-ended question in all three interviews assessed. (See Table 2.) I excluded the three participants who did not take part in all three interviews because I was not able to successfully follow their entire journey of learning logarithms.

I started off by analyzing three different transcripts from the first interview. Instead of entering the analysis with predefined ideas and codes, I let the students’ own words guide me to identify key themes, patterns, and ideas. After reviewing the first three transcripts, I was able to start constructing general categories such as math background, attitudes towards logarithms, and general knowledge of logarithms to capture different information from the three different students. From here, I began to analyze each participant one at a time, reading and re-reading each of the three interviews from beginning to end. I describe this process as coming to understand and write each participant’s “story.” I first created the story for one of the original three
participants, Natalie. After creating the story of the three interviews with her. Starting with Natalie’s first interview, I then went back and repeated that interview, I went through the interviews from the other two participants, process for Emma and Jerry. Next, I looked for similar trends that supported or did not support Natalie’s explanations and reasoning. I then proceeded to analyze the interviews of the other nine participants, reading and re-reading them thoroughly. I treated each interview as an individual snapshot of a participant’s experience; the three snapshots formed from the three interviews over time contributed to the development of each participant’s “story.”

After reading through the 30 interviews, I identified specific trends and behaviors exhibited among the ten participants. From here, I started to refine my codes by explicitly looking at the details of their explanations (e.g., Are they thorough? Are they vague? Are their explanations confusing?), their use of specific terminology related to logarithms (e.g., “argument”), and any evidence of their attitude toward logarithms. I then reread the interviews and highlighted key phrases, responses, and words that fit each person into those categories and created a one- to three-page analytic memo on each participant, for each interview. The analytic memos contain summaries and essential sections of the interviews needed to create a full story of the participant’s learning of logarithms. I went back through the memos and reanalyzed them and established a second, more refined version of the analytic memos for each participant.

After refining the participants’ stories, I identified specific examples to support the categories I created. From here, I constructed a table that summarized the story of each participant from the analytic memos into shorter, more condensed statements that
connected the main ideas and codes from one interview to the next. After finalizing my codes, I was able to begin to start forming conclusions from the interviews.

The conclusions from the interviews were then used to conduct a cross-case analysis to identify similarities and differences across the cases. The cross-analysis section discusses the similarities and differences between participants. I was able to then identify practices that students used that coincided with their increasing fluency with logarithms. Comparing and contrasting each participant allowed me to form more general conclusions about students’ development of fluency with logarithms.

### 3.2.3 Post-test

I analyzed the first three questions to the Post-test to assess students’ abilities to define a logarithm, give an example of an application of a logarithm, and state the relationship between logarithmic and exponential functions. These questions also provided information about students’ perceptions of logarithms through the course of the five lessons. The process of analyzing the Post-tests was identical to the Pre-test. I used the same scale of 0-2 points to score the Post-tests, and the raw score for each was out of 36. Again, I calculated the mean, median, mode, range, and standard deviation for the total scores and for the scores on each individual problem. In the end, I produced two sets of analyses: one set includes data only the seven interview participants, and the other set includes data from all but the seven interview participants.

### 3.2.4 Cross Analyses

I performed two different cross-case analyses, (1) first, on the Pre-test and Post-test scores and (2) second, on the interview participants. The purpose of the Pre-
test and Post-test cross-case analysis was to observe if there was a statistical significance in the increase of scores for Questions 4-8. I was unable to perform a cross analysis on Questions 1-3 because of time constraints and because they assessed students’ math background, attitude towards learning logarithms, ability to define a logarithm, and ability to identify the relationship between logarithmic and exponential functions. I used SAS to conduct a t-test with 95% confidence to test for statistical significance. The purpose of the cross analysis of the interview participants was to identify similar themes across different students with different math backgrounds. This could help form conclusions about the overall sample, if they have overlapping difficulties or strengths from learning logarithms.
Chapter 4
RESULTS

This chapter discusses the results of my project. I begin by presenting findings from the Pre-test and Post-test, including an examination of changes in students’ performance. I then present individual case studies of my seven interview participants. Recall, 7 out of 10 of the participants that participated in all three interviews participated in the Post-test. Thus, I created 7 cases from the students who completed every aspect of my project. The cases were written to capture each participants’ journey and to describe the extent to which each developed fluency in logarithms over the course of the semester.

4.1 How does students’ performance change from the Pre-test to the Post-test?

This section describes results from the Pre-test and the Post-test. All of this information is provided in Figures 4, 5, and 6. I begin by discussing the results from the Pre-test, followed by the results from the Post-test and concluding with an analysis of changes in students’ performance from Pre-test to Post-test.

4.1.1 Students’ performance on the Pre-test

The mean scores for each of Questions 4 through 8 on the Pre-test can be found in Table 3. Recall the minimum score on each question was a 0, and the maximum score on each question was a 2. A student earned a score of 1 if the student had some work correct but did not obtain the final correct answer. The top five highest mean scores on the Pre-test, starting from fifth, were Questions 4a, 4e, 4c, 5b, 5a with scores of 1.30, 1.32, 1.34, 1.48, and 1.63 respectively. What is interesting is that three out of five of these items are True/False questions. A possible explanation
for this would be that since students could guess, most likely a student would guess if they had some idea of what the question was asking. So, it was not surprising to see these questions appear in the top five. In addition, Questions 5a and 5b assessed students’ ability to convert from logarithmic to exponential form. These two questions could have looked familiar to students since they were basic expressions to convert. Thus, the five highest mean scores were not surprising.

Another observation I made from looking at the mean scores of the Pre-test was that the bottom five scores, Questions 8c, 6b, 7c, 7b, and 5c, with scores 0.40, 0.50, 0.55, 0.59, and 0.61, respectively. These questions were more distributed towards the end of the Pre-test, which were the more challenging items. Question 6b assessed students’ ability to apply both the Product and Power Properties. Question 7c assessed students’ ability to apply the Product, Quotient, and Power Properties. Question 8c assessed students’ ability to apply the Quotient Property and to convert from logarithmic to exponential form. Thus, three of the five questions with the lowest mean scores involved applying multiple skills. Figure 4 provides the mean scores on each item of the Pre-test, ordered from least to greatest.

Overall, students’ performance on the Pre-test indicates that they entered MATH 115 with some knowledge of logarithms and their properties. However, their performance also indicates that they had much still to learn (or recall) about logarithms. The mean total score on the Pre-test was a 16.46 out of 36 points.
<table>
<thead>
<tr>
<th>Question</th>
<th>Mean Score (Out of 2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4a</td>
<td>1.3</td>
</tr>
<tr>
<td>4b</td>
<td>0.7</td>
</tr>
<tr>
<td>4c</td>
<td>1.34</td>
</tr>
<tr>
<td>4d</td>
<td>1.14</td>
</tr>
<tr>
<td>4e</td>
<td>1.32</td>
</tr>
<tr>
<td>4f</td>
<td>0.87</td>
</tr>
<tr>
<td>5a</td>
<td>1.63</td>
</tr>
<tr>
<td>5b</td>
<td>1.48</td>
</tr>
<tr>
<td>5c</td>
<td>0.61</td>
</tr>
<tr>
<td>6a</td>
<td>0.65</td>
</tr>
<tr>
<td>6b</td>
<td>0.5</td>
</tr>
<tr>
<td>6c</td>
<td>0.72</td>
</tr>
<tr>
<td>7a</td>
<td>0.94</td>
</tr>
<tr>
<td>7b</td>
<td>0.59</td>
</tr>
<tr>
<td>7c</td>
<td>0.55</td>
</tr>
<tr>
<td>8a</td>
<td>1.13</td>
</tr>
<tr>
<td>8b</td>
<td>0.9</td>
</tr>
<tr>
<td>8c</td>
<td>0.4</td>
</tr>
</tbody>
</table>

Table 3. Mean scores on the Pre-test.

4.1.2 Students’ performance on the Post-test

The mean scores for each of Questions 4 through 8 on the Post-test can be found in Table 4. The top five highest mean scores on the Post-test were Questions 4c, 7b, 7a, 5a, and 4a with scores of 1.83, 1.88, 1.89, 1.92, and 1.99, respectively. In both tests, Questions 4a, 4c, and 5a were in top five highest mean scores. These items consist of two True/False items, one converting forms item, and two expanding logarithms items. The top five scores were a mixture of problem types. However,
when referring to Table 1 in Section 3.1.3, the topics that these questions covered dealt with the Product, Quotient, and Power Properties, knowing the common base of 10, and converting from logarithmic to exponential form. Thus, the top five scores were from a mixture of problem types. Note that these questions assessed only one skill each, so students did not have to perform more than one operation or property to solve the problem.

Another observation that I made from reviewing the mean scores on the Post-test was that the lowest scored questions, starting with 18th, were 8c, 5c, 4d, 4f, and 7c. In both tests, Questions 5c, 7c, and 8c were in the bottom five lowest mean scores. Just like the top five scores, the bottom five are well distributed across the test. Question 8c assessed students’ ability to apply the Quotient Property and convert from logarithmic to exponential form. Question 4d assessed students’ knowledge of the base of the common logarithm and ability to convert from logarithmic to exponential form. Question 7c assessed students’ ability to apply the Product, Quotient, and Power Properties. Three out of the five questions required application of multiple skills. Figure 5 provides the mean scores on each item of the Post-test, ordered from least to greatest. Finally, students’ mean score on the Post-test was a 28.99 out of 36, which was higher than their mean score on the Pre-test.
Table 4. Mean scores on the Post-test.

### 4.1.3 Cross Analysis of the Pre-test and Post-test

A comparison of the mean scores between the Pre- and Post-test can be seen in Appendix H. The information when finding statistical significance for Questions 4-8 can be seen in Figures 7, 8, and 9.

To compare students’ performance on each item from the Pre-test to the Post-test, I found the difference of the mean scores (Post-test mean minus Pre-test mean) and then divided that by the score on the Pre-test to determine the percent increase.
from Pre- to Post-test. This analysis revealed that students’ mean scores at least doubled from Pre- to Post-test on nine of the items: Questions 4b (160%), 5c (150%), 6a (160%), 6b (170%), 6c (160%), 7a (150%), 7b (170%), 7c (160%), and 8c (160%).

Another similarity across both the Pre- and Post-test share was that three of the five lowest-scoring questions from each test assessed students’ ability to perform multiple logarithm skills for one problem. This indicates that students have a more difficult time solving logarithm problems that involve multiple skills. This aligns with research findings from Kenney (2005) and Liang and Wood (2005). Both concluded that students struggle with applying multiple logarithm properties at once. My data aligns with their findings.

To determine if students’ scores on the Post-test were significantly higher than the scores on the Pre-test, I performed a t-test using SAS. Results from this analysis indicate that the mean scores on the Post-test were significantly higher for seventeen of the eighteen items. The only item for which there was not a significant difference was Question 4d. (See Table 6.) These findings indicate that students significantly improved their fluency with logarithms. Thus, students left the course with increased fluency in what is known to be a troubling topic.

4.2 How does the subsample of interviewees compare to the larger sample of students who took the Pre-test and Post-test?

I conducted two evaluations: one on the full set of 155 scores that included my 7 interview participants and one on the subset of 148 scores that excluded my interview participants. I was then able to examine the performance of my interview participants and compare them to the general sample of 155 participants. Of the 10 participants who participated in all three interviews, seven completed the Post-test.
### Table 5. Mean and median scores on the Pre-test and Post-test for different samples of participants.

<table>
<thead>
<tr>
<th>Participants</th>
<th>Pre-test Mean</th>
<th>Post-test Mean</th>
<th>Pre-test Median</th>
<th>Post-test Median</th>
</tr>
</thead>
<tbody>
<tr>
<td>Full sample (n=155)</td>
<td>16.46</td>
<td>29.13</td>
<td>15</td>
<td>30</td>
</tr>
<tr>
<td>Sample excluding interviewees (n=148)</td>
<td>16.36</td>
<td>28.99</td>
<td>15</td>
<td>30</td>
</tr>
<tr>
<td>Subsample of interviewees (n=7)</td>
<td>18.71</td>
<td>32.17</td>
<td>20</td>
<td>35</td>
</tr>
</tbody>
</table>

Recall that the individuals who participated in both the Pre-test and the Post-test were evaluated using a 0, 1, or 2-point scaling system. Thus, the maximum score on each test was 36 points. The following information can be seen in Table 5. Including the interview participants, the mean and median score on the Pre-test was 16.46 and 15, respectively. Excluding the interview participants, the mean and median score on the Pre-test was 16.36 and 15, respectively. The Post-test, including my interview participants, had a mean and median score of 29.13 and 30, respectively. The Post-test, excluding my interview participants, had a mean and median score of 28.99 and 30, respectively. It is compelling to observe that my subsample of the seven interview participants who took both the Pre-test and the Post-test, performed slightly higher than the sample alone. The interviewees had Pre-test mean and median scores of 18.71 and 20, respectively. The interviewees had Post-test mean and median scores of 32.17 and 35, respectively. The difference between the mean scores on both assessments were slightly higher than the sample excluding the interview participants. The exact difference of the mean score on the Pre-test was 2.35. This was a difference of about one question completely correct or two questions with partially correct...
answers. The exact difference of the mean score on the Post-test was 3.18. This was a difference of about one question completely correct with one question as partially correct or three questions with partially correct answers. This suggests that my interview participants may have entered and exited with slightly more knowledge about logarithms than the rest of the sample. Due to time constraints, I was unable to test to see if the differences scores were statistically significant. My sample of interviewees scored higher, but I could have investigated if that difference in scores were statistically significant?

In the sections that follow, I present individual case studies or “stories” from each of the seven interview participants who completed both the Pre- and Post-test. These stories provide a more detailed picture of students’ challenges and successes with developing fluency with logarithms.
Figure 4. Mean scores on the Pre-test.
Figure 5. Mean scores on the Post-test.
Figure 6. Means procedure and t-test outputs for questions 4a-4f on both tests.
Figure 7. Means procedure and t-test outputs for questions 5a-6c on both tests.
Figure 8. Means procedure and t-test outputs for questions 7a-8c on both tests.
Table 6. The results of statistical significance testing on each item from Pre-test to Post test 4.

### 4.3 Natalie’s Story

#### 4.3.1 Incoming attitudes about, and knowledge of, logarithms

Natalie came to MATH 115 with some exposure to and familiarity with logarithms. It seems that she had the ability to apply particular skills such as the Quotient Property, but she struggled with the Product and Power Properties. In addition, she seemed to have prior familiarity with the base for the common logarithm and the natural logarithm. Natalie sometimes used logarithm terminology and that

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4 *** p < 0.0001, ** **p < 0.01. * *p < 0.05.
seemed to help her describe her thinking process. However, she was not fluent in this vocabulary, and exhibited room for improvement.

Natalie completed several algebra-based courses prior to enrolling in MATH 115. In high school, she took Geometry, Algebra II, Pre-Calculus, and AP Calculus AB. She performed well in all of these courses, earning an A in each. She felt confident in her mathematical abilities because whenever “the teacher would go over something, [she] would get it right away” and she would only have to do a brief review prior to taking tests to refresh her skills (Interview 1). She seems to be a conceptual, rather than a procedural, learner who learns best by making sense of concepts. The two hardest topics Natalie recalled learning in high school were trigonometry and logarithms. She felt that both topics initially required a lot of memorization in order to develop a solid foundation.

The first time Natalie learned about logarithms was in her Pre-Calculus class during her junior year of high school. Her initial impressions were that they were difficult because of the memorization of the properties. Natalie learned best by learning concepts, not by memorizing. In order for Natalie to develop an understanding of logarithms she would “go to extra help to either understand it more or the teacher would tell [the students] little hints to memorize things” (Interview 1). Her determination to develop mathematical fluency in high school suggests that she had a positive attitude toward math and logarithms even though they were challenging at first. In the interview she discussed how the material on the Pre-test looked familiar and was comfortable with understanding what the questions were asking. However, Natalie struggled recalling all of the information.
Natalie’s familiarity with logarithms was evident in her responses on the Pre-test. For example, in response to Question 3 which dealt with identifying the relationship between logarithmic and exponential functions, her answer was “a logarithm function and an exponential function are inverses of each other.” In addition, on Question 2 she was able to give an example of what a logarithmic function looked like, \( y = \log_5(x) \).

Natalie earned a total score of 24 out of 36 on the Pre-test, a score that is eight points higher than the mean score. She earned full credit for questions 4b-4f, 5a-5c, 6c, 7a, 7c, and 8a. These problems involved the following knowledge and skills: knowing the base of the natural logarithm, knowing the Quotient Property and how to apply it, and converting from logarithmic to exponential form. The two questions that involved the Quotient Property for which she did not receive full credit were Questions 6a and 8c. Both of these questions involved polynomial expressions of more than one term.

The questions that Natalie did not perform well on dealt with the Product Property (Questions 4a and 6b) and the Power Property (Questions 6b and 7b). On Question 7b, she was not able to correctly apply the Power Property. Her thinking process for this question showed that she understood that \( x^3 \) could be rewritten as \((x)(x^2)\); however, she could not connect it to bring the power of 3 in front of the logarithmic expression as the coefficient. It seems Natalie exhibited strong skills applying the Quotient Property, but she seemed to need a refresher on the other properties.

The Pre-test included particular questions that required participants to know the base of the natural logarithm and common logarithm. Natalie successfully pointed
out on the Pre-test that the natural logarithm had a base of e and that a logarithm with no given base had a base of 10. Natalie’s knowledge of logarithmic notation was strong in Question 5, but weak in Questions 8b and 8c which required students to apply multiple skills and/or properties within the same problem. She could convert from logarithmic to exponential form when working with simple logarithmic expressions, however when the expressions became more complicated and required multiple steps, she could not complete the problem. Thus, Natalie came to MATH 115 with some strong skills, but she needed work in other areas.

In Interview 1, Natalie continued to exhibit some knowledge of logarithms and their relationship to exponential functions. She was also able to identify and distinguish quadratic functions from exponential functions.

Alex: I’m going to say a word or words and I want you to give me the first couple of things that come to mind.

Natalie: Okay.

Alex: Alright? When I say, exponent, what comes to mind?

Natalie: Exponent ... Like log, I guess, because they go hand in hand.

Alex: Exponential function.

Natalie: Logarithm function.

Alex: Logarithm.

Natalie: Exponent.

Alex: Logarithmic function.

Natalie: Exponential function.

Alex: Inverse functions.

Natalie: Log and exponential functions.
... 

*Alex:* $y = x^2 + 3x - 5$

*Natalie:* Quadratic.

...

*Alex:* $y = 3^{x+1}$

*Natalie:* Exponential.

Natalie was able to respond to this portion of the interview without exhibiting any common misconceptions. In particular, she was able to identify the inverse relationship between logarithmic and exponential functions.

Natalie had not yet fluent in applying the Quotient Property when at least one of the logarithms contained multiple terms in the argument. For Questions 6a and 8c, seen in Figure 9, her explanation for not dividing the arguments was that the logs “had the same base so I just assumed that I could get rid of the logs and just go down to the arguments that are inside” (Interview 1). Notice that she used the phrasing “could get rid of the logs,” indicating that she thought getting rid of the logarithms before condensing was how you solved for $x$. Thus, although she was able to correctly convert from logarithmic to exponential form in Question 5, here in Question 8c she did not apply the skill, because instead she “gets rid of the logs.” This prevented her from solving the problem correctly. However, her algebra skills seem to be strong. For example, she was able to correctly solve the quadratic equation she produced.
Natalie was able to partially describe the Product Property during the first interview. She stated that she knew there was a relationship between addition and multiplication but could not recall exactly what it was. She said that she just needed a refresher on what the property was, and then she could immediately do the problems with confidence. Furthermore, just as on the Pre-test, Natalie was able to identify in the interview that the natural logarithm has a base of $e$ and the common logarithm has a base of 10. Natalie also described how she was about to convert from logarithmic to exponential form. She described her thinking process to convert from logarithmic to exponential form as “it’s base to what is equal to the argument” (Interview 1).

In summary, Natalie came to MATH 115 with a solid algebra background and with experience studying logarithms in her junior year of high school, two years prior to taking MATH 115. She was able to apply the Quotient Property in simple cases but failed to apply the property in more complicated expressions. From the interviews, I was able to conclude that she had a positive outlook on learning mathematics. Natalie performed well on the Pre-test, scoring eight points higher than the mean, suggesting
that she has a somewhat stronger fluency with logarithms than the average MATH 115 student. The two areas that Natalie struggled with were applying the Product and Power Properties. One final observation was that in Natalie’s explanations, she sometimes referenced and used the word “argument” to describe the “inside” of the logarithm.

4.3.2 Midstream attitudes about, and knowledge of, logarithms

Recall that the goal of Interview 2 was to assess students’ knowledge of logarithms and ways of thinking about and working with logarithms now that they were midway through the logarithm lessons in the course. During Interview 2, Natalie exhibited an overall positive attitude towards logarithms but wondered about the purpose for learning them. She acknowledged that her initial difficulties with logarithms were “expanding and condensing them… It was difficult just trying to remember like all of the properties, but the more I [looked] at it, [the topics] got easier” (Interview 2).

Natalie took on the challenge of condensing and expanding logarithms. She did not appear to develop negative associations with the process nor did she indicate a negative attitude toward learning logarithms. Natalie reported that she spent approximately 0-2 hours of work outside of class in order to develop her knowledge of logarithms and she stated that time was used “doing homework, [but other than that she did not] really work on it outside of [class]” (Interview 2).

As in Interview 1, Natalie could identify that logarithmic and exponential functions were inverses, but she could not “really explain what it is” (Interview 2). Also, she was unable to give an example of a real-world application when she would
use logarithms. Her inability to give a real-life example may have prevented her from being able to clearly explain what a logarithm was.

During Interview 2, Natalie could easily state and apply the Product and Quotient Properties. To explain her work on the various exercises given in Interview 2, she noted “subtraction goes to dividing” and “adding goes to multiplication” (Interview 2). Natalie seems to have developed better fluency in applying these two properties, evidenced by her correctly solving all of the questions involving the application of these two properties.

The one concept that Natalie did initially struggle with was the Power Property. For example, when working on problem 2.5.a.c (evaluating $\frac{1}{2}\log(100)$), Natalie was unsure about what to do with the coefficient of one-half. She stated that “the one-half in the front is throwing me off” (Interview 2). At first, she was unable to identify the base of the logarithm given, but then she later determined that it was base 10. Once she identified that the common logarithm had a base of 10, she was able to explain her work, stating “So, 10 of what equals 100? That’s two, and so like I can just multiply two by a half” (Interview 2). Here, Natalie was able to solve the problem however she did not directly use the Power Property. The other question that involved the Power Property was Problem 2.5.b.c (expanding $\log_5(x^2)$). Here she was able to justify her answer of $2\log_5(x)$ by explaining that the 2 “would get brought out as like a factor or a coefficient” in front of the logarithm (Interview 2). Once she explained her work for this question, Natalie then went back and solved question 2.5.a.c correctly using this process. She obtained the same answer and recognized that her application of the Power Property verified her original answer.
During Interview 2, Natalie continued to identify that the common logarithm had a base of 10 and that the natural logarithm was “the same as log base e” (Interview 2). Looking at her work for Question 2.5.c.b, she represented the natural logarithm as a logarithm with a base e. (See Figure 10.) It seems that Natalie understands the notation for logarithms.

![Figure 10. Natalie’s work for question 2.5.c.b in interview 2.](image)

Natalie also seemed to have mastered converting between logarithmic and exponential form. She exhibited fluency in working between the two forms and understood the relationship. This was also indicated when she stated that the two functions were inverses of each other. When she converted from one form to another she stated, “the base to the outside term is equal to the argument” (Interview 2). Working through this logic she successfully completed all of the problems that required a conversion between forms. Natalie gave an interesting explanation for her work on Question 2.5.c.e. The problem was to solve \( \ln(x^2) = \ln(2x - 1) \). This problem assessed students’ ability to notice that since the logarithms have the same base, and since the natural logarithm function is a one-to-one function, the equation can be rewritten by setting the arguments equal to each other. Natalie explained that since the natural logarithm was “on both sides of the equal sign, you can just equal the
arguments to each other, and then solve for \( x \) that way” (Interview 2). She did not state the natural logarithms cancel out, suggesting that she recognizes that logarithms were “a type of expression” (Interview 2). Natalie’s treatment of a logarithm as an expression shows a small insight into how she perceived logarithms.

In summary, Natalie’s responses during Interview 2 indicate that she has developed fluency in many areas of logarithms after the first few Math 115 lessons. It is important to note that throughout Interview 2, Natalie did not use the logarithm terminology of “argument” or “base”, however she could work through the problems algebraically and provide detailed explanations. She also noted that the argument of a logarithm could not be zero or negative. This was important for Question 2.5.c.c because there were initially two solutions and by checking her work she was able to rule out one “solution”. Her responses in Interview 2 indicate that Natalie was developing increased fluency with logarithms.

4.3.3 Exiting attitudes about, and knowledge of, logarithms

After the five logarithm lessons were taught, Natalie remained motivated to understand logarithms, yet she could not explicitly define what a logarithm was or give a real-life application. She was able to state all of the different properties such as the Product, Quotient, and Power Properties, and she was able to note that the “argument can’t be less than zero, or equal to zero” (Interview 3). She frequently used the word “argument” at the very beginning of Interview 3 which suggests that she had become more comfortable using logarithm terminology. When I asked Natalie to reflect on her learning of logarithms and what her perception was from beginning to end, she stated that she thought “they’re just something for like a math class”
(Interview 3). She was not able to see the application of learning logarithms; however she was still able to strengthen her fluency with them over the course of the semester.

The rest of Interview 3 entailed asking Natalie to solve various problems involving logarithms. Natalie got every question correct. She was able to completely condense, expand, and evaluate logarithmic expressions with accuracy. Her verbal explanations aligned with what she wrote down on paper. For example, in Question 3.3.e she was able to explain her work with great detail. (See Figure 11 below.)

Figure 11. Natalie’s work for question 3.3.e in interview 3.

This problem was an example that showed that Natalie knew that the common logarithm had a base of 10, could apply the Quotient Property, could convert from
logarithmic to exponential form, and could verify that the argument of a logarithm had to be positive.

Natalie: “So, you make the 2 log(x) look the same. So, I brought the 2 to the x, with the x, and since the logs are being subtracted, you can make the arguments to division. And then I changed the log form to exponential form. And then, when I did that, I solved for x, I got zero and 1 over 100. But you can't have an argument of zero, so it's just 1 over 100.” (Interview 3).

It was evident that Natalie had developed fluency with the various skills for manipulating logarithms. Her incorporation of vocabulary such as the words “argument” and “exponential form” suggest that she better understood the various components of a logarithm and their relationship to their exponential counterpart. The work and explanations that Natalie provided in solving this problem were representative of the work and explanations she exhibited on all of the problems during Interview 2. Her explanations indicated that she has developed stronger fluency with logarithms.

Natalie also participated in the Post-test; she earned a perfect score of 36 out of 36 points. She was one of only twelve students who obtained a 100% on the Post-test. Natalie’s growth in fluency with logarithms was evident from beginning to end. Her solid algebra background from high school potentially enabled her to initially solve some logarithm problems without knowing or using all of the appropriate logarithmic properties. Natalie could identify that logarithmic and exponential functions were inverses, but she was never able to give an example of a real-life application. Natalie
started as an above average student with some fluency with logarithms and finished stronger than she started.

4.4 Fiona’s Story

4.4.1 Incoming attitudes about, and knowledge of, logarithms

Fiona was a first-semester freshman at the University of Delaware at the time of this study, so her math background only pertained to her high school experience. In high school, Fiona took Algebra I, Geometry, Algebra II and Trigonometry, and Pre-Calculus. She performed well in her high school math classes, earning “90’s and above,” (Interview 1) and so she felt comfortable with her algebra skills and could apply different skills when needed. Her biggest issue was that she did forget some concepts, but once she had a review she could easily recall all of the concepts and ideas from a particular topic.

Fiona came to MATH 115 with some familiarity of logarithms. She first saw logarithms in her senior year of high school while taking Pre-Calculus. Fiona’s biggest challenges in her high school math classes were related to logarithms because they “were really hard at first” (Interview 1). However, because she had a “really good teacher… after [the teacher] explained, [logarithms] became easier” (Interview 1). Logarithms were challenging for her because there was a lot of initial memorization and the notation was confusing “on where everything came from and trying to explain” it (Interview 1). Because of her teacher, she was able to become more comfortable with and less intimidated by logarithms. Fiona reported that the easier topics from high school mostly came from Algebra I and II, such as factoring. She thought these topics were easy and that was why she performed well in them. She
enjoyed Pre-Calculus because of her teacher and how the teacher would come “up with an easier way for [her] to understand [concepts]…it’s kind of a corny [way] to like remember how to explain [logarithms]” (Interview 1). Her positive attitude towards math meant that she would push herself and take time outside of class to continue to develop knowledge of math topics, if needed. The notation of logarithms was at first intimidating to Fiona, but with the help of her teacher, she was able to see logarithms as a topic that became easier.

Fiona earned 14 out of 36 points on the Pre-test, indicating that she had studied logarithms before, but entered MATH 115 with only partial knowledge. Fiona’s prior knowledge of the relationship between logarithmic and exponential functions was made clear both on the Pre-test and in Interview 1. On the Pre-test, Fiona was able to explicitly identify the different parts of a logarithm and convert it to exponential form. Her response to Question 3 about the relationship between logarithmic and exponential functions was thorough. (See Figure 12.) She could point out each component of the logarithm and relate it to the exponential representation. She was able to convert from logarithmic to exponential form accurately in this question.

![Image](image.png)

Figure 12. Fiona’s work for question 3 on the Pre-test.
In Interview 1, Fiona used an acronym she called B.A.E.: Base, Answer, Exponent. This was how she answered Question 3 correctly on the Pre-test and constantly referred back to it in the interview. She did not just state the acronym but gave a visual representation of what she meant that was identical to what she wrote for Question 3 on the Pre-test. In addition to using B.A.E., she knew that inverse functions were “the opposite, like switching x and y” (Interview 1).

Fiona could also identify that the natural logarithm was associated with e on the Pre-test and in Interview 1, but she reported that the questions involving e were “confuse[ing] with the e and everything” (Interview 1). Even though she knew that the natural logarithm had a base e, she was unsure what it represented as a whole. As a result, Fiona did not answer the questions on the Pre-test that dealt with the natural logarithm. On the test itself she stated that these problems looked “familiar, but I forgot how to solve it” (Pre-test, Question 5c). She also knew that the common logarithm had a base 10, evidenced in her response to Question 6c on the Pre-test where she wrote in a base of 10 in the appropriate spot. Even though she did not need to know the base specifically for this problem, she still wrote it down. She applied her “B.A.E.” idea on Question 8 on the Pre-test and was able to solve both 8a and 8b correctly, earning 4 of the 6 points for this question. Fiona came in with a strong knowledge of the relationship between logarithmic and exponential functions.

However, she did not know or could not apply the properties of logarithms. Fiona failed to apply the Product, Quotient, and Power Properties of logarithms when solving problems on the Pre-test. Instead of using these properties to condense logarithms, she dropped the “logs” and performed the indicated operation between the
logarithms on the arguments. For example, for Question 6c, she dropped the “log” notation and just subtracted the two arguments. (See Figure 13.)

![Image of logarithm expression](image)

**Figure 13.** Fiona’s work for question 6c on the Pre-test.

In summary, Fiona came into MATH 115 able to convert between logarithmic and exponential forms and knowing that the natural logarithm had a base of e and the common logarithm had a base of 10. She was able to explain her solutions to problems when they involved these skills. However, she was unable to correctly solve problems that required her to know and apply the Product, Quotient, and Power Properties of logarithms, and this led her to get Problems 6, 7, and 8a wrong on the Pre-test.

### 4.4.2 Midstream attitudes about, and knowledge of, logarithms

Fiona exhibited more fluency with logarithms in the second interview. She described logarithms incorporating logarithm vocabulary. Similar to Interview 1, Fiona’s attitude towards logarithms remained positive. She reported spending additional time outside of class, approximately 2-4 hours per week, working on developing her knowledge of logarithms. She said there were certain topics that were challenging such as expanding and condensing logarithmic expressions. These types
of problems took more time for her to work through and get correct. She defined that a logarithm was a function, but she could not give a real-life application. It was evident that Fiona had refreshed her knowledge of logarithmic notation and the terminology associated with each component of a logarithm.

Throughout the interview, Fiona consistently used the terminology “argument”, “base”, “exponential form” and “convert”. She incorporated these terms into her explanations of the problems. In Question 2.5.c.d she effectively used the vocabulary to solve a more complicated logarithmic expression. (See Figure 14.)

Fiona: Because first I took the arguments and divided them, so it was log base two, x plus one divided by four equals one. Then I changed that into exponential form, getting two to the first equals x plus one divided by four, which is two equals x plus one divided by four. Then I cross multiplied getting eight equals x plus one and then I subtracted the one on both sides and got seven. X equals seven.

Figure 14. Fiona’s work for question 2.5.c.d in Interview 2.

Fiona’s verbal and physical representations were identical. She clearly explained each step and described the steps using appropriate vocabulary. In contrast to Interview 1,
Fiona incorporated more vocabulary in her explanations, and correctly applied the properties in Interview 2.

In Interview 2, Fiona exhibited appropriate knowledge of the Quotient Property, stating that “when you subtract two logs with the same base you divide the numbers, like the arguments” (Interview 2). She explained that in order to combine logarithms that are subtracted, the logarithms had to have the same base. This indicates that she recognizes that if the logarithms did not have the same base, she could not have applied that property. In addition, she provided similar explanations to explain the Product and Power Properties. She consistently incorporated these explanations to answer all of the questions.

There was one question that Fiona struggled with during Interview 2. She had difficulties answering Question 2.5.c.b., which asked her to solve the equation, \( \ln(x) = 0 \). In order to work through the problem, she tried to give some form of explanation, but could not come to a final conclusion. She did identify that the natural logarithm had a base of \( e \), but knowledge of this did not seem to help her solve this problem. Fiona was able to convert from logarithmic to exponential form for this problem, yet could not figure out the value for \( x \). She never wrote anything down as a final answer; she just left the question unanswered.

Since this interview was conducted about halfway through the logarithm lessons, Fiona had recently learned about solving logarithmic equations with a logarithm on each side. She could apply this knowledge to Question 2.5.c.e. She explained that since “both of [the logarithms] have ln on both sides that means you could set \( x^2 \) equal to \( 2x - 1 \)” (Interview 2). However, she did not explain why you could set the two arguments equal to each other.
In summary, Fiona’s responses during Interview 2 indicated that she had developed stronger fluency with logarithms than she exhibited on the Pre-test and in Interview 1. She incorporated logarithm vocabulary into her explanations, properly applied the three properties of logarithms (Product, Quotient, and Power properties), and showed ability to solve logarithmic equations for \( x \). However, there was still room for improvement. For example, Fiona did not seem to recognize that the argument of a logarithm must be positive.

4.4.3 Exiting attitudes about, and knowledge of, logarithms

During Interview 3, Fiona’s attitude toward logarithms was reflected in her statement that “they are annoying”, but she was not intimidated by them (Interview 3). Beyond the use of the word “annoying”, Fiona never demonstrated a negative attitude toward logarithms.

Fiona stated that logarithms were functions and they were inverses of exponential functions. Moreover, she described what it meant for the two functions to be inverses of each other as “basically like flipping your \( x \) and your \( y \)” (Interview 3). Fiona revisited this idea throughout Interview 3, and later applied it on the Post-test.

The only application that Fiona could identify for logarithms would be when “solving exponential problems” (Post-test, Question 2). Even though she was not able to give an explicit example, she gave an example of a time one would apply logarithmic properties.

In Interview 3, Fiona demonstrated strong knowledge of logarithms. Fiona was able to correctly apply the Product, Quotient, and Power Properties. Recall that on the Pre-test and in Interview 1, Fiona had not been able to correctly condense or expand logarithmic expressions. She initially found condensing logarithms
challenging. However, in Interview 3 she demonstrated fluency in condensing and expanding logarithms, and she clearly identified when she would condense and why. In Question 2b, she simplified the logarithm more than what I had expected students to do. The task was to condense \( \log(x + 2) - \log(x + 2) \). She started off by noting that “subtracting means you're dividing” and this led her to get an answer of \( \log(1) \). Even though she did not evaluate this, she successfully concluded that the division of the two arguments simplified to one. The goal was to evaluate students’ use of applying the Quotient Property, but she took her answer to the next step by simplifying the argument further. Additionally, Fiona identified that the natural logarithm has a base of \( e \), that \( e \) is a number, and that the common logarithm has a base of 10.

![Figure 15. Fiona’s work for question 3.3.d in Interview 3.](image)

There was only one question that Fiona struggled on in Interview 3. Question 3.3.d assessed whether students remembered that they must check their work and verify that their solutions did not result in an argument that was zero or negative. Although Fiona correctly solved the problem, she did not remember to check her work. (See Figure 15.) She first simplified the expression using the Power Property, converted to exponential form, and obtained the solutions \( x = -1 \) and \( x = 1 \). She then stated that the argument could not be negative or zero, but her reasoning for why \( x = -1 \)
I was a solution was interesting. She explained that “you’re squaring \([x]\) and a negative number squared is positive” (Interview 3). Fiona was considering the right issue, but her mistake to not check her solutions in the *original* expression caused her to come to an incorrect conclusion. Other than this problem, Fiona performed well on all of the exercises in Interview 3.

Fiona received 36 out of 36 points on the Post-test. She was one of only twelve participants in my sample of 155 participants who earned a 100% on the Post-test. Fiona demonstrated strong fluency on the Post-test. Recall, she started with a Pre-test score of a 14. Thus, she increased her initial score by more than 61%. Her work on the Post-test was thorough and detailed, especially for Question 8. Question 8a evaluated students’ ability to convert from logarithmic to exponential form and then solve for \(x\). Question 8b required students to use the Power Property (or algebra) to manipulate an equation and then convert it into exponential form. Question 8c evaluated students’ ability to condense a logarithm using the Quotient Property, convert the equation into exponential form, solve for the \(x\)-values of a quadratic, and finally to verify that the solutions were valid. Fiona solved all three of these questions correctly. It seems Fiona’s hard work and dedication to learning logarithms – evidenced by the additional 2-4 hours per week she was spending outside of class – had paid off. It was evident that Fiona developed a stronger fluency with logarithms and that was supported by her accurate work and solid explanations during her interviews.
4.5 Emma’s Story

4.5.1 Incoming attitudes about, and knowledge of, logarithms

Like Fiona, Emma also did well in her high school math courses. She took Algebra I, Geometry, Algebra II, and Pre-Calculus; she earned A’s in all of these classes except for Algebra II, in which she earned a B+. Emma had not taken a college-level math course prior to taking MATH 115. Overall, Emma “remembers doing well in math” and has a positive attitude toward her ability to perform algebra techniques (Interview 1). Emma believes she learns math best by practicing problems and studying those problems repeatedly until she understands the process. One of the most challenging mathematical topics for Emma was when she learned how to transform graphs in Pre-Calculus. She thought it was “really confusing” and also “finding different things about [shapes]” in Geometry was challenging (Interview 1). Emma states that she does not “really retain that much over time” when it comes to her learning of mathematics (Interview 1). This could potentially be a downfall in Emma’s development of fluency with logarithms. However, on a scale of 1 to 10, with 1 being absolutely uncomfortable and 10 being full mastery of algebra techniques and skills, she did rate herself a 7 in terms of her confidence pertaining to her algebra skills.

Emma came to MATH 115 with some familiarity with logarithms. She had previously seen logarithms during her senior year of high school in Pre-Calculus. Her initial impression of logarithms was that they “were not difficult at first, however they got much harder as the year went on” (Pre-test). Her approach to developing an understanding of logarithms in high school was to do different problems “until [she] fully understood it, and then [she] would try it again with different numbers, so then
[she] would fully understand it” (Interview 1). Even though logarithms got more difficult, she believed that with her process of repeatedly practicing problems, she was able to eventually understand the material.

When I asked Emma to give me her first thoughts on the word “logarithm” she stated, “the word log and ln.” In addition, when I stated the phrase “logarithmic function” she replied by saying “numbers with functions, the word log” (Interview 1). Emma struggled here when giving her responses, suggesting that she did not have full understanding of logarithms. She did state that inverse functions mean “switching x and the y” (Interview 1). The Pre-test was not especially difficult for Emma; it was only challenging because she “didn’t remember exactly everything” (Interview 1) which may be why she scored a 21 out of a 36 on the Pre-test, which was well above the mean score on the Pre-test. Her attitude towards the Pre-test and logarithms in general was related to her ability to “remember” how to do the problems. She consistently used the word “remember” when discussing the Pre-test, and this aligns with her studying habits of working through the same problems repeatedly until she knew the process. Emma has had experience with logarithms before, and she would have felt more comfortable taking the Pre-test if she had been able to remember everything she had previously learned.

Emma was able to apply some prior knowledge about logarithms to the Pre-test. For example, Question 2 asked her to define a logarithm and describe a real-world application. Emma responded by giving a correct example of how to convert from logarithmic to exponential form. She gave the expression, \(\log_2(8) = x\) and rewrote this as \(2^x = 8\) and then replaced the \(x\) with a 3. This response indicates that her initial thought when asked to define a logarithm is to show how it relates to
exponential functions / equations. In addition, she stated that the relationship between logarithmic and exponential functions was that they are inverses of each other. Her work was consistent in Question 5a and 5b on the Pre-test where she converted from logarithmic to exponential form without any difficulty.

Emma also used the technique of converting from logarithmic to exponential form in her responses to Questions 8a and 8b. In 8a, she found the correct value for $x$ by immediately applying this technique and then solving for $x$. Again, in 8b she immediately performed the conversion, but she did not recognize that she needed to take care of the coefficient first. Emma was unable to apply the Power Property in this question. Emma was able to expand logarithms but was not able to condense them. Emma said “the subtraction sign means division” at many points during the interview, but she did not apply this relationship to condense logarithms (Interview 1). In addition, when asked to explain her work on the Pre-test, she frequently used the words “confused” and “remember.” In addition, she stated that she was slightly intimidated by Question 8c. (See Figure 16.) She knew that she had to divide, however she “didn’t know what to do with the $x$’s” (Interview 1). Emma said this intimidated her because she knew the first step but could not continue the problem. She could not figure out how to convert to exponential form because she put logarithmic expressions in both the numerator and the denominator instead of just dividing the arguments. On other questions on the Pre-test she was able to apply the Quotient Property correctly and could convert from logarithmic to exponential form, however she failed to apply them together in Question 8c. Overall, Emma was able to apply the Quotient and Power Properties in some situations like expanding logarithms,
but she struggled when condensing and manipulating more complicated logarithmic expressions.

Figure 16. Emma’s work for question 8c on the Pre-test.

Emma knew the base for the common logarithm. Emma answered “false” for Question 4d, \( \log(1) = 1 \); when explaining her response, she indicated that since the logarithm did not have a specified base, she thought “of a log base 10” (Interview 1). She knew that the common base was 10; however, she thought the base of the natural logarithm was 10 as well. This was also evident in her incorrect response to Question 4f, \( \ln(x) \) has a base of 10; Emma said this was “true.” Thus, Emma knew only one of the two common bases.

Emma’s previous knowledge of logarithms was evident on the Pre-test and in Interview 1. She could not recall every property from high school, however she knew the common logarithm had a base of 10, she expanded logarithms using the Quotient and Power Properties, and she showed some knowledge of converting from logarithmic to exponential form.
4.5.2 Midstream attitudes about, and knowledge of, logarithms

In Interview 2, Emma’s attitude towards logarithms was positive. She described a logarithm as “a mathematical function [and] … you have to use logs or ln’s to find out what is the exponent [value]” (Interview 2). She strongly believed that she was able to do well on logarithm problems in class and on ALEKS. When asked how confident she was being able to explain logarithms to someone else, with 1 being feeling uncomfortable and 10 being a good understanding and can explain a lot about them to another student, Emma quickly responded with an 8 because “I understand it” (Interview 2). Her attitude towards logarithms is positive and she believes in herself.

Emma reported spending some time outside of class working on logarithms. She said that at times the material was “more difficult than I expected” but stated that she “performed well on my assessment [the Pre-test]” (Interview 2). When I specifically asked her how much time she was spending outside of class, Emma responded by saying “about four hours or more” because she wanted to be able to understand the material rather than just memorize it (Interview 2). She did not have an issue devoting that much time out of class because it helped her strengthen her confidence and fluency in logarithms.

During Interview 2, Emma identified that the common logarithm has a base of 10 and the base of the natural logarithm is e. She also understood that “anything to the zero [power] is one” (Interview 2). Emma knew the purpose of condensing and expanding logarithms was to help evaluate logarithmic expressions. She understood when to apply this general knowledge to solving the exercises presented in this interview.

Furthermore, Emma was able to apply the Quotient Property in the exercises. When she was working on Question 2.5.c.d, \( \log_2(x + 1) - \log_2(4) = 1 \), Emma
clearly identified that when logarithms have the same base and are being subtracted, then she could combine the arguments by dividing them. She then converted from logarithmic to exponential form. When Emma noticed that the problem became easier when she converted from one form to the other, she identified that the next step was to get “rid of the four on the bottom by multiplying the equation by four” (Interview 2). She solved this problem algebraically and got the correct answer.

Emma had difficulty with the Power and Product Properties during Interview 2. For Question 2.5.b.a, when asked to expand the expression \( \ln(3x) \), she got confused. She initially thought the answer was \( \ln(x^3) \) but was unsure of her answer because “the three does not encounter the \( \ln \), so then I thought it would have been an exponent” (Interview 2). She identified the Power Property, but this problem does not use that property. She struggled with this problem because of her failure to recognize that the argument is a product. She concluded that because the argument was \( 3x \) that is equivalent to \( x + x + x \).

In contrast, when dealing with the problem \( \log_5(x) + \log_5(4x - 1) = 1 \), she immediately applied the Product Property. She stated that since the logarithms were being added then you could multiply the arguments. Emma then converted the equation into exponential form, which eventually resulted in the quadratic equation, \( 4x^2 - x - 5 = 0 \). At this point in the problem, she got stuck. She was unsure about how to factor the quadratic because the leading coefficient was 4. After talking with her a bit, I discovered the difficulty was not in solving a quadratic equation, but in solving a quadratic equation with a leading coefficient not equal to 1.

Emma did not apply the Power Property in the expression \( \ln(x^3) \) on the Pre-test and furthermore confused it with the Product Property in this problem. Later, she
correctly applied the Product Property in a different problem. This may indicate that she was able to work with the two properties but sometimes got confused about whether the argument was a product, or an expression raised to a power. Thus, it seems that at this stage Emma had partial knowledge of some of the properties, but not all of them.

In summary, during Interview 2 Emma exhibited a positive attitude towards logarithms and confidence in her ability to develop fluency with logarithms. She exhibited some knowledge of logarithmic properties including knowledge of the bases for the common logarithm and natural logarithm, the use of the Quotient Property, and being able to convert from logarithmic to exponential form. However, Emma was not fully fluent with the Product and Power Properties.

4.5.3 Exiting attitudes about, and knowledge of, logarithms

During the final interview, Emma showed growth in her knowledge of logarithms, and in attitude. Her attitude towards logarithms continued to remain positive; she felt confident in knowing what logarithms were and how to manipulate them. The easiest way for her to develop fluency in logarithms was to do “lots of practice… keep doing practice problems… and [review] my study guide” (Interview 3). She was unsure about when you would use logarithms in a real-world situation, however if she needed to deal with one she felt confident she could work through the problem. Her attitude throughout the whole process was positive.

Emma easily identified that logarithms and exponential functions were inverses each other. She described functions that are inverses of each other as when “the range and domain are like opposite of one another” (Interview 3) and they switch. She could state that the natural logarithm had a base of e and the common logarithm
had a base of 10, and she was able to apply this knowledge to solve problems in the interview. Her knowledge of how to convert from logarithmic to exponential form was solid; she could identify and state the different parts of a logarithm and relate it to the corresponding parts of an exponential expression. By Interview 3, Emma has developed strong fluency with logarithms.

In Interview 3, Emma was able to apply the Product, Quotient, and Power Properties correctly to almost all of the problems. In contrast to her responses during Interview 2, Emma was able to describe each property. She explained that the Product Property dealt with multiplication and addition, the Quotient Property dealt with division and subtraction, and the Power Property dealt with the coefficient and the power. However, Emma did exhibit one misconception with the Power and Product Properties, in her solution to Problem 3.2.f. (See Figure 17.) In this problem, Emma moved the power of 3 in front of the natural logarithm.

Figure 17. Emma’s work for question 3.2.f in Interview 3.

In addition, she then did not expand the resulting logarithm using the Product Property. Interestingly, she was able to apply these two properties separately in Problem 3.2.a, 3.2.d, and 3.2.e. Thus, the issue for Emma might be applying these two properties within the same problem. Or, the issue might be algebraic, in that she does not realize that only the y is being raised to the third power in the argument.
One new misconception arose for Emma in Interview 3. Recall that the last set of problems assessed students’ abilities to apply various logarithmic properties to solve for $x$. These problems were difficult for Emma to solve. For example, in solving Problem 3.3.c, Emma initially applied the Product Property, but from here she did not convert from logarithmic to exponential form. (See Figure 18.) Emma explained that she “got stuck because I couldn’t figure out the correct factoring” (Interview 3). Emma ran into this issue because she inappropriately took the log of the right side of her equation, instead of just converting to an exponential equation. Interestingly, in Interview 2, converting from logarithmic to exponential form was not an issue for her. In addition, she could not factor the resulting quadratic. The quadratic she derived was impossible; she needed to use the Quadratic Formula. So, instead of figuring out how to finish the problem, she continued and chose to leave her answer as the quadratic. Emma also had issues with similar problems and continued inserting a logarithm on the side without a logarithm, such as Question 3.3.d.

![Figure 18](image.png)

Figure 18. Emma’s work for questions 3.3.c and 3.3.d in Interview 3.
On the Post-test, Emma earned a score of 29 out of 36 points, which was an increase of 8 points from her score on the Pre-test. She stated that her biggest challenge learning logarithms was “with ln and solving for \( x \)... the different rules confused me... I resolved this through practice” (Post-test). She was again able to identify that logarithmic and exponential functions were inverses of each other and gave a clear example of what that would look like. She was not able to give an example of the application of a logarithm in real life. Emma’s general attitude toward logarithms remained positive.

Emma was able to successfully answer Questions 4a-4d but incorrectly marked the statement “\( \ln(x) \) has a base of 10” in Question 4e as “true” (Post-test). This also appeared on Question 5c, \( \ln(x) = 2 \), where she answered \( x = 2 \). In the problem, Emma said \( \ln(x) = 2 \) was equivalent to \( 10^2 = 100 \) and that was how she got \( x = 2 \). Recall that in Interview 2 she stated that the natural logarithm had a base of \( e \) and not 10. Thus, Emma’s knowledge of the base for the natural logarithm seems to be tenuous.

On the Post-test, Emma was able to successfully condense and expand logarithms using the appropriate properties to the fullest extent. For example, for Question 7b she put two answers, both of which were correct. She was able to identify that \( \log(x^3) = 3\log(x) = \log(x) + \log(x) + \log(x) \). However, she struggled with Question 8c, which combined several items. In this problem, she began by appropriately applying the Quotient Property of logarithms. However, instead of then converting to exponential form, her work suggests that she multiplied both sides by \( x - 1 \). Then she ignored the logarithm to get the quadratic equation. Her work led her to an equation that had no solution. Emma’s work on this problem is similar to her work during Interview 3, in that she seems to struggle with problems that require using
multiple skills and properties within the same problem. Her work suggests that her knowledge of and fluency with logarithms is more fragile than it appeared to be during Interview 2.

4.6 Jerry’s Story

4.6.1 Incoming attitudes about, and knowledge of, logarithms

Jerry came into MATH 115 with an algebra-based background and a positive attitude. He took Algebra, Geometry, Algebra II, and Pre-Calculus. He had no experience taking a college-level math course prior to MATH 115. He had a positive attitude toward mathematics, stating that he believed that he was pretty good because he would “always [receive] like a B+ or A” in his math classes. Jerry rated his confidence level with his algebra skills as a 7, on a scale of 1-10, with 1 being absolutely uncomfortable to 10 being complete mastery. He believed that he could “do a lot of the problems… but there’s always a chance I’d forget how to do something” (Interview 1).

In terms of prior learning of logarithms, Jerry reported he had studied logarithms in his senior year Pre-Calculus course, and “it might have actually been Algebra II,” but most definitely in Pre-Calculus. The topics on the Pre-test were not new for Jerry; in fact, the “concepts seemed familiar. There were just a few problems I don’t remember” (Interview 1). He had a neutral attitude towards logarithms because he viewed logarithms as “challenging…. [but] the repetition and seeing what I got wrong, and [why] I got it wrong really helped” (Interview 1). Jerry believed that logarithms got easier the more he practiced them, because he was then able to develop knowledge of the concepts involved.
Jerry entered MATH 115 with some knowledge of logarithms, as seen by his performance on the Pre-test. Jerry earned a score of 20 out of 36 points, which was above the mean score of 16 points. It seems that Jerry had knowledge about the principles of logarithms. He identified that logarithmic and exponential functions were inverses of each other, and he gave an example of this relationship. (See Figure 19.) It seems that Jerry already had some knowledge of how to convert from logarithmic to exponential functions. This was reflected in his work for Question 5 and for Questions 8a and 8b. He received 6 out of 6 points for Question 5 by correctly converting from logarithmic to exponential form in order to solve for the indicated variable, \( x \). (See Figure 19.) Jerry also knew that the base of the natural logarithm was \( e \), and that the base of the common logarithm was 10. For Problems 8a and 8b, Jerry correctly converted from logarithmic to exponential form. (See Figure 20.) His work in Question 8b also reveals that he came to the course knowing how to apply the Product Property.

Figure 19. Jerry’s work for questions 3 and 5 on the Pre-test.
Jerry did not exhibit any knowledge of how to condense or expand logarithms. On the Pre-test, for Questions 6 and 7 (condensing and expanding logarithms) he wrote “I don’t know how to do this” or “No idea.” The only part of Question six that he even attempted, Question 6a, asked students to condense $\log_5(2x^3 - x^2) - \log_5(x)$. Jerry combined the logarithms into one logarithm by incorrectly subtracting the arguments, instead of dividing them. It seems that Jerry did not know how to condense or simplify logarithms because of his minimal effort to do the problems.

In Interview 1, Jerry explained his work for the Pre-test. On the Pre-test, when asked to explain his work for solving a simple logarithmic equation in Question 5 he stated that he applied the “circle method” he learned in high school (Interview 1). The circle method for Jerry was a strategy that he used in order to correctly convert from logarithmic to exponential form. He recalled and applied this method throughout the Pre-test and in Interview 1. Furthermore, for Questions 6 and 7 which assessed Jerry’s ability to condense and expand logarithms, respectively, he stated that for Question 6a his answer was “it looked simpler… what also threw me off on 6b was the eight in front of the first log” (Interview 1). Jerry had an issue with the notation of Questions 6
and 7. In this context, notation was defined by the physical representation of the expression. He was unsure about what to do with the problems. Jerry explicitly stated that he “didn’t remember how to condense” the expressions in Questions 6, 7, and 8c (Interview 1). He did try “to remember different ways to solve it, but I think it just came down to I couldn’t remember… even though I can do them” (Interview 1). His acknowledgement that he could not recall the properties explained why he did not attempt the problems. However, he did believe that he would have been able to do them, if he had remembered the process, which reflects a positive attitude. Jerry’s explanations in Interview 1 aligned with his work on the Pre-test.

In summary, Jerry came to MATH 115 with some entry-level skills and knowledge of logarithms. He knew how to convert between logarithmic and exponential forms, identified the relationship between logarithmic and exponential functions and gave an example to support his claim, and had some ability to apply the Product Property. However, he did not know how to condense or expand logarithms. It seems that Jerry had a relatively neutral attitude towards logarithms in high school and had a positive one when looking forward to reviewing the material in class.

4.6.2 Midstream attitudes about, and knowledge of, logarithms

During Interview 2, Jerry seemed to have a positive outlook on learning logarithms thus far and reported that he felt “pretty good grasp[ing] all of the problems” (Interview 2). Jerry’s confidence and ability to explain logarithms was evident that he was strengthening his logarithm skills. He stated that it “takes [him] no time at all” doing the classwork and homework on ALEKS (Interview 2). He reported spending about 0-2 hours per week outside of class learning logarithms, which was only spent doing the homework. He believed that even though he could
not “explain [logarithms] too well” he was “confident in solving operations” (Interview 2).

During Interview 2 Jerry exhibited knowledge of logarithm terminology when condensing logarithms. Throughout the interview, Jerry referred back to the properties of logarithms to explain his thinking process. When I asked Jerry to condense \( \log_2(3) - \log_2(5) \), Jerry provided a detailed explanation referring to both the Product and Quotient Properties of logarithms:

\[
\text{Because one of the operations that you use when working with logs is, if you have two logs that have the same base then if it's addition then you would multiply but you would multiply the arguments. But if it's subtraction you'd divide the arguments.}
\]

He incorporated different words and phrases to support his explanation such as “same base”, “addition then you would multiply, “arguments” and “subtraction you’d divide the arguments.” This indicated that Jerry could use the vocabulary that was presented in the first few lessons and integrate it into his explanations.

In addition, Jerry exhibited knowledge of logarithm terminology when expanding logarithms. When Jerry was presented Question 2.5.b.a, \( \ln(3x) \), he explained his answer, \( \ln(3) + \ln(x) \), by “because the 3 and the x were multiplied together, so during the reverse of the operation they would be used in addition” (Interview 2). In this explanation, he used the word “reverse” which was an interesting word to describe the process of the Product Property. He provided similar explanations with the other problems involving how to expand logarithms.

As Jerry went through the last portion of the interview, he correctly answered all of the questions except 2.5.c.c, 2.5.c.d, and 2.5.c.e which asked students to solve logarithmic equations involving at least three different logarithmic expressions. His
solutions to these three problems revealed that he had a misunderstanding of the Quotient Property. (See Figure 21.) In all three cases that included a subtraction sign, Jerry separated the argument into two logarithms. Earlier, he had clearly defined what the Quotient Property was and when working on simple logarithmic expressions like $\log_2(3) - \log_2(5)$, he justified and condensed the logarithms correctly. However, in this situation Jerry did not perform well. In his explanations for these three questions he did not explain why he separated the arguments in this manner. Even though he was able to describe the Product and Quotient Properties, Jerry did not use those properties correctly when condensing complicated polynomials with multiple terms.

Finally, as in Interview 1, Jerry knew the base of the common and natural logarithms were 10 and e, respectively. Jerry was confident in his abilities to solve and simplify logarithm problems correctly. His explanations included logarithm terminology. He could condense and expand basic logarithmic expressions, however could not apply the properties consistently when working with more complicated expressions and equations.
4.6.3 Exiting attitudes about, and knowledge of, logarithms

Jerry’s attitude towards logarithms changed in the third interview as he reflected on his journey of relearning them. At first Jerry thought logarithms were “relatively easy…it didn’t get harder. It just got more tedious” (Interview 3). Jerry felt unsure about when he would apply logarithms in real life because “it’s not something they really teach you in class… I doubt you would actually have to [use] logarithms” (Interview 3). His once positive attitude towards logarithms became more negative. This shift seemed to be influenced by the fact he did not see when he would use logarithms in the real world.

Jerry developed knowledge of the fundamentals of logarithms including that the common logarithm has a base of 10, the natural logarithm has a base of e, e is a number, and the notion that logarithmic and exponential functions are inverses. In addition, he identified each part of a logarithm and related it to its exponential counterpart.

In Interview 3, Jerry answered every question correctly, except Question 3.2.b and 3.3.c. In Problem 3.2.b, seen in Figure 22, Jerry performed the same mistake that he exhibited in Interview 2. He separated the arguments by distributing the “log.” It is possible that Jerry performed this process out of an old habit because throughout the rest of the interview, he solved the other problems correctly and did not exhibit this mistake. This was not the only problem with multiple logarithm terms, so this one-time error in the final interview could have been just a careless mistake. Even though he began the problem incorrectly, reading from left to right starting at the second line of his work, Jerry correctly applied the Product and Quotient Properties. When I asked him to explain his work, he stated that he was “going to divide…. I’m first going to expand and then condense” the problem and then “distribute the log to both
of them” (Interview 3). He worked from “left to right [to] condense” the logarithmic expression using the Product and Quotient Properties. It seems that Jerry was not able to discern exactly when to use each one and therefore incorrectly “expanded” when there was nothing to expand.

In Question 3.3.c, Jerry correctly carried out the process with use of vocabulary to explain his work. (See Figure 23.) In this question, Jerry did not break up the argument in the second term; he automatically condensed the logarithm by using the Product Property. He said, “I have to first condense it, before I can solve it… because the bases are the same, I can multiply [the arguments] together” (Interview 3). He then converted to exponential form correctly and solved the resulting quadratic. From there, Jerry did not check his two solutions, \( x = 3 \) and \( x = -1 \) to verify that the argument would be positive. Instead, he moved onto the next question immediately. Jerry’s knowledge of logarithms grew from Interview 1 to
Interview 2 to Interview 3, however there were minor mistakes that persisted from interview to interview.

![Image]

Figure 23. Jerry’s work for question 3.3.c in Interview 3.

Jerry earned 35 out of 36 points on the Post-test. He lost only one point on Question 6a. He made the same mistake he made in Interviews 2 and 3 – separating the arguments into two different logarithmic expressions. (See Figure 24.) He wrote it as a quotient of logarithms, instead of as a quotient of the arguments. The reasoning behind why he performed this action was unclear in Interview 2, 3, and on the Post-test. He just had the idea to expand and then condense. However, in Question 6b, $8\log_2(x - 2) + \log_2(x - 5)$, Jerry correctly condensed the logarithm into its simplest form. He applied the Power Property followed by the Product Property without separating the arguments.
Jerry’s work in Interview 3 and the Post-test showed that he had developed near-complete fluency with logarithms. From the very beginning, Jerry knew that logarithmic and exponential functions were inverses, he knew the common bases of 10 and e, and naturally integrated logarithm vocabulary into his explanations and described them in detail. He did not put a lot of outside time learning logarithms because it came natural to him through practicing problems in class and on the 0-2 hours he spent on homework. He thought logarithms got easier in time but became more “tedious.” He originally had a positive attitude with logarithms, but his attitude changed to something more negative because he could not see the application of them. Logarithms might have been “tedious” for Jerry, however his growth from an original score of a 20 on the Pre-test to a score of a 35 (out of 36) on the Post-test score showed that Jerry’s knowledge of logarithms improved.

4.7 Ellen’s Story

4.7.1 Incoming attitudes about, and knowledge of, logarithms

Ellen did well in her high school math classes. She came into MATH 115 having taken Geometry, Algebra II, Pre-Calculus, and Calculus in high school. She never took a college-level math course prior to taking MATH 115. Ellen came in with
a strong algebra background covering many topics. She received “low B’s to high B’s” in her math classes and in her opinion, she “didn’t do as well” as she would have liked (Interview 1). Ellen wanted to take Pre-Calculus because she “wanted to review… before jumping into [Calculus]” at the university (Interview 1). Ellen was not too confident in her Calculus background, so she felt that taking MATH 115 would be her best option.

Ellen had exposure to logarithms prior to taking MATH 115. She saw logarithms for the first time in Pre-Calculus her junior year and “recalling the information was hard” while taking the Pre-test (Interview 1). The concepts on the Pre-test looked familiar to her, but she forgot the material since it had been awhile since she had seen them. Ellen reported that when she took Pre-Calculus in high school, trigonometry was difficult for her to understand, but not logarithms. She thought that trigonometry was harder than logarithms because of the “different functions that go with [the unit circle]” (Interview 1). Learning logarithms was easier for her to learn because it was not as much memorization and she could apply the different concepts. On the Pre-test, Ellen’s attitude towards logarithms was they were “fairly easy once [they] were taught” (Question 1).

On the Pre-test, Ellen scored a 20 out of 36 points. She performed higher than the average score of 16 points. She was able to identify that logarithmic and exponential functions were inverses of each other and gave a clear example of how to convert from logarithmic to exponential form. She also defined that a logarithm was “an easier way to find an answer to an exponential function” (Question 3). She came in with some prior knowledge on the relationship between logarithmic and exponential functions.
Ellen’s prior knowledge about the skills that pertain to logarithms centered around two ideas, converting from logarithmic to exponential form and a loosely understood version of the Quotient Property. In Question 5, she was able to successfully switch from logarithmic to exponential form. (See Figure 25.) Her only issue was that she thought the natural logarithm had a base of 10 and which she incorrectly answered Questions 4f and 5c.

Figure 25. Ellen’s work for Question 5 on the Pre-test.

Ellen’s work for Question 5 on the Pre-test showed that she knew how to switch between logarithmic and exponential form. She missed Question 5c, only because she incorrectly identified the base of the natural logarithm to be 10. Ellen continued to show some knowledge of logarithms in Question 6c, 7a, and 7c. She could condense and expand logarithms using the Quotient property but incorrectly denoted her answers, which can be seen in Figure 26. In the interview, Ellen was able to identify that when you are subtracting logarithms you can condense into a single logarithm with a quotient of the arguments, and vice versa. In Question 7a, she broke up the numerator of the argument. She separated the logarithm into two expressions; in addition, Ellen she subtracted that quantity from the numerator. In Question 7c she
separated the numerator by expanding it into two logarithmic expressions. She then applied the Quotient Property. In Question 6c, she rewrote a difference of logarithms as a quotient of two logarithms (instead of a single logarithm of a quotient). It seems that Ellen has some prior knowledge of the properties of logarithms but could not consistently apply them correctly.

Figure 26. Ellen’s work for Questions 7a (left), 7c (middle), and 6c (right) on the Pre-test.

In Interview 1, I examined more about what Ellen knew about logarithms and asked her to explain some of the problems on the Pre-test. During the interview, I performed a brief exercise where I stated a word or phrase and asked students to give their initial reaction. Below is an excerpt from Ellen’s interview:

Alex: When I say the word exponent, what comes to mind?

Ellen: Something raised to a power.

... 

Alex: Logarithm?

Ellen: Log base answer equals exponent.

Alex: Logarithmic function?
Ellen: The inverse of an exponential function.

It seems that Ellen associated with the exponential function. She seems to be able to read the notation of a logarithm in this exercise as well by stating the word “base.” Ellen’s response to the equation, $y = 3^{x+1}$ was that it was an exponential function. It was clear during the interview that Ellen had some incoming knowledge of logarithms.

During the interview, Ellen explained her work on the Pre-test. She was able to correct herself on the Pre-test on Question 4c, $\log(100) = 2$, because “I forgot that if there’s no base, then it’s just 10, the base is 10. Then I remembered that, so then 10 squared would be 100” (Interview 1). Once Ellen recalled this information that the common logarithm had a base of 10, she was able to reason herself out of selecting “false” and changing her response to “true.” about

Ellen answered Question 6c incorrectly, but she had exhibited a general idea about the Quotient Property, so I asked her about her thinking process for that problem. Her explanation as to how she got her answer was that she “remembered when you’re subtracting logs, that’s division. I think I forgot that, when you add them, you multiply” (Interview 1). She knew something about the properties but from her statement, it is unclear if she knew what was being subtracted, divided, added or multiplied – logarithms or their arguments. She was “not confident” when working with the Product Property (Interview 1). This was also her thinking process when she expanded Questions 7a and 7c. (See Figure 26.) Ellen’s explanations throughout the Pre-test revealed that she did not have complete knowledge of the other properties and skills that are associated with logarithms.

In summary, Ellen worked hard in her high school math courses and was a B student. She thought that trigonometry, not logarithms, was the hardest topic that she learned in high school. Her teacher required the students to memorize the unit circle
and she had issues knowing all of the different functions that are involved in trigonometry. During the Pre-test, Ellen seemed to have some knowledge about converting from logarithmic to exponential form and recognized that the common logarithm has a base of 10. In addition, Ellen inconsistently applied the Quotient Property by associating subtraction with division and vice versa, however the issue was with her notation. Ellen had room for improvement in her explanations and knowledge of logarithms.

4.7.2 Midstream attitudes about, and knowledge of, logarithms

During Interview 2, Ellen’s attitude towards logarithms was somewhat negative. She was cautious in her explanations because she was not confident in the material. When I asked Ellen to rate her confidence with logarithms from a scale of 1 to 10, with 1 being absolutely uncomfortable and 10 being complete mastery, she categorized herself as a “4 because I can, like, explain how to solve them but…. The meaning behind them is… I’m not sure” (Interview 2). She stated that she was unsure about the various properties and how to solve logarithmic equations for the indicated variable. It seems that her attitude toward logarithms was more negative than positive because of her lack of confidence in solving problems involving logarithms.

Ellen reported she spent about 0-2 hours per week outside of class working on logarithms. That time was devoted to working on the ALEKS homework assignments. She got “most of them right” and only “a few here and there wrong” because she would make a careless error (Interview 2).

Ellen could identify the Product Property and referred to it throughout the interview. She associated addition with multiplication. Her use of the property allowed her to complete the problems in 2.5.c. Ellen ran into a conceptual issue in
Problem 2.5.c.c, \( \log_5(x) + \log_5(4x - 1) = 1 \). Her work suggests that she was not aware to check her final solutions to make sure the argument was positive. Her work throughout the problem is correct, until she got two answers, \( x = -1 \) and \( x = \frac{5}{4} \). The only solution to this problem is \( \frac{5}{4} \) because substituting in \( x = -1 \) produces a negative argument. Ellen failed to observe this, and just went on to Question 2.5.c.d. However, her work up to that point is correct. She applied the Product Property correctly and correctly rewrote the resulting equation in exponential form, so she could solve for \( x \).

Ellen’s knowledge of the Quotient Property was exhibited in Question 2.5.c.d. (See Figure 27.) The problem was to solve the equation \( \log_2(x + 1) - \log_2(4) = 1 \) for \( x \). Ellen was able to apply the Quotient Property and then “convert into exponential form” (Interview 2) for this problem. Her work indicates again that she was able to fluently convert from logarithmic to exponential form. This is supported as well from Interview 1 and her work on the Pre-test.

Figure 27. Ellen’s work for Question 8c on the Pre-test.

In addition to applying the Product and Quotient Properties, Ellen effectively applied the Power Property for both Questions 2.5.a.c (evaluate \( \frac{1}{2} \log(100) \)) and
2.5.b.c (evaluate $\log_{5}(x^2)$). She got both answers correct and stated that the “coefficient goes to the power” and “you bring the power to the front of the log” (Interview 2).

In summary, Ellen’s fluency with logarithms had improved since the Pre-test. She was now able to identify that the natural logarithm had a base of $e$, instead of 10. She exhibited knowledge of how to apply the Product, Quotient, and Power properties of logarithms. She was also able to give better explanations in the interview. Ellen frequently incorporated logarithm terminology such as the words “argument”, “change to exponential for”, “log base”, and identifying that “you can’t take the natural log of zero” (Interview 2). But, Ellen still had more to learn.

4.7.3 Exiting attitudes about, and knowledge of, logarithms

During Interview 3, Ellen’s attitude towards logarithms was more optimistic. She thought “logarithms become easier” throughout the lessons and if she “come across logs… I’ll know how to do the problem” (Interview 3). She believed in her abilities to solve the various problems that were presented in class and in the homework. Ellen exited the course with a positive attitude toward logarithms and confidence in her abilities to apply the different concepts and skills.

Ellen exhibited fluency with logarithms. She explained that “logarithmic functions and exponential functions, well they’re inverses” of each other and “if you can’t solve an exponential equation, you can find the inverse of that and solve it logarithmically” (Interview 3). Ellen was able to describe each component of the logarithm and its relationship to its exponential representation. For example, she stated that the “argument inside the logarithm [represented] the answer that you set the log equal to when you’re solving for it. The base of the logarithm represents what
you’re raising the power to and setting a logarithm equal to something represents what
the value is when you’re raising that base to a power” (Interview 3). This statement
indicates that Ellen had a strong knowledge of the notation of the logarithm. Ellen
also noted, “You can’t take the log of a negative number” (Interview 3). In addition,
she described the Product, Quotient, and Power Properties. Ellen understood the
relationship between logarithmic and exponential functions, explained the notation of
the logarithm, identified that the argument had to be positive, and was able to use the
three properties to condense and expand logarithms.

Ellen performed well on the exercises in Interview 3. Her application of the
Product, Quotient, and Power Properties showed that she mastered using them for
simple as well as more complicated logarithmic expressions. Ellen made one mistake
on the exercises, which contradicted what she had said at the beginning of the
interview about the argument of a logarithm needing to be positive. For Question
3.3.c, Ellen first condensed the logarithm by using the Product Property and then
converted into exponential form. (See Figure 28.) However, when she factored her
quadratic equation and obtained the two solutions $x = -1$ and $x = 3$, she did not go back
and check if her solutions would produce a positive argument. were valid. Even
though Ellen knew the argument must be positive, she forgot to check for that in the
end. However, in Problem 3.3.d, she did check both solutions to make sure they
worked in the argument. (See Figure 28.) For this problem28, Ellen removed her
solution of $x = -1$ because it made the argument negative. It seems that because the
condensed logarithm in 3.3.3.c allowed $x = -1$ to work, she included it in her final
answer. So, she was just a bit careless in not checking her solutions for the previous
Overall, throughout Interview 3, Ellen had a strong understanding of when to apply the different properties of logarithms and she thoroughly explained her work.

In Interview 3, I noticed that Ellen used more logarithm terminology such as the words “the properties of multiplication and division”, “base”, “the inside”, “argument”, “multiplied in a log, you can expand it to addition”, “division means you can subtract”, and “four to the zero [power] is one” (Interview 3). As she incorporated these words and phrases in her responses, she was able to self-correct her mistakes that she had made when working independently and silently.

Ellen did well on the Post-test, earning a score of 34 out of 36 points. Recall that Ellen received 15 out of 36 points on the Pre-test; thus, she more than doubled her initial score and increased her score by 18 points. Ellen only got one question entirely incorrect on the test. This was Question 4e, which was the true/false question “We can find the logarithm of a negative number.” She answered “True.” This contradicts
what she said in Interviews 2 and 3 where she stated that the argument had to be positive. It seems that while she knew that the argument must be positive, she could not always transfer this to a new situation or wording. It may be that her knowledge of this is tenuous.

Ellen also lost a point on Question 7c, which asked her to expand the expression $\ln\left(\frac{xy}{z^2}\right)$. Ellen was able to apply the Quotient and Power Properties; however she did not perform the Product Property. (See Figure 29.) Perhaps she did not recognize the product within the remaining argument, and so she did not realize she could take the problem further.

![Figure 29. Ellen’s work for Question 7c on the Post-test.](image)

In summary, Ellen’s progress from the Pre-test to the Post-test showed that she learned many skills and concepts associated with logarithms. She learned to incorporate terminology when explaining her answers. More than doubling her score from the Pre-test to the Post-test indicates that Ellen developed much stronger fluency with logarithms. She was able to state and apply the Product, Quotient, and Power Properties, identify that $e$ was a number, identify the common bases for logarithms were either 10 or $e$, and recognize that the argument had to be positive. Ellen’s journey from beginning to end showed considerable growth and fluency in logarithms.
4.8 Nancy’s Story

4.8.1 Incoming attitudes about, and knowledge of, logarithms

Of all the interview participants, Nancy had the most unique math background from high school. Instead of taking the standard Algebra, Geometry, Pre-Calculus, and Calculus course sequence, Nancy participated in an integrated math program; she took Integrated Math I, II, III and in her senior year she took AP Statistics. In the integrated math courses, content from Algebra I, Geometry, Algebra II, and Pre-Calculus is all woven together. Nancy did well in the integrated math courses, but she decided to break away from the integrated curriculum by taking AP Statistics senior year of high school because “I wanted to step away from it. I was too tired of it” (Interview 1). She also wanted to take AP Statistics because she wanted to have a college-level class before entering college. She performed well in her AP Statistics course and on the AP exam where she received a qualifying score.

Nancy’s initial impression of logarithms was negative. She first saw logarithms at the beginning of her sophomore year of high school in Integrated Math II. She did not like logarithms because she “never got to properly learn it… we didn’t spend that much time on it” (Interview 3). So, she was unsure of many of the problems on the Pre-test. Nancy also did not enjoy learning logarithms because it was “not like just solve an equation with using different” tools and skills it dealt with “complex things” (Interview 1). Throughout the interview Nancy used negative descriptions to describe learning logarithms such as “that’s a little scary,” there were “so many things you have to know to begin the problems,” and “knowing the rules was very difficult” (Interview 1). Since logarithms were a challenge for Nancy, her initial impression of them was negative.
On the Pre-test, Nancy received a score of 10 out of 36 points. She reported being “overwhelmed” by how much she had forgotten about logarithms (Interview 1). She was able to give an example of what a logarithmic function looks like and stated that logarithmic and exponential functions were inverses. For the true or false section, Nancy answered “I’m not sure” for Questions 4c (\(\log(100) = 2\)) and 4f (\(\ln(x)\) has a base of 10). Her justification for this was because those two questions “were super fuzzy. I know I have seen [them] before but I wasn’t sure how to do them” (Interview 1). It seems that Nancy did not know the base of the natural logarithm, which was then supported in her blank response to Question 5c which assessed whether students knew the base for the natural logarithm. The natural logarithm “freaked out” Nancy (Interview 1).

Nancy was able to correctly answer Question 4a which involved identifying the Product Property. She even explained that when you “multiply you are adding” (Interview 1). This is further evidenced by her work in Question 6b, where she multiplied the two arguments together. (See Figure 30.) Even though she did not correctly multiply the arguments, she knew to apply the Product Property because “if you had two logs added…. you multiply” the arguments (Interview 1). Nancy was “unsure of the rules” when dealing with the other two problems in Question 6 so she just combined the arguments by subtraction (Interview 1). “If I can remember the rules” that would have helped her condense the logarithms but “connecting the rules and they are fuzzy” (Interview 1).
Figure 30. Nancy’s work for Question 6b on the Pre-test.

Nancy continued to struggle on the second half of the Pre-test. For example, she incorrectly answered all 6 parts for Questions 7 and 8, only giving minimal responses to these questions involving expanding logarithms and solving logarithmic equations. For Question 7, Nancy was unable to explain her work because she was “confused” and did “not even know anymore” of the material being tested on the Pre-test (Interview 1). She could only provide a partial solution for Question 8b (Solve $3\log_3(x) = 3$) in which she just divided the equation by three on both sides and then “didn’t know what to do anymore” (Interview 1).

In summary, it seems that Nancy came to MATH 115 with little knowledge of logarithms. She could state and apply the Product Property, knew that logarithmic and exponential functions were inverses, and knew the physical representation of a logarithmic function. But she was not able to apply the Quotient or Power properties of logarithms. She had little experience from high school with logarithms because they were not sufficiently taught, and the lack of instruction led Nancy to develop a negative attitude towards them. Nancy struggled on the Pre-test because she could not remember any of the properties or ideas that it was assessing. Thus, Nancy had a lot of room for growth.
4.8.2 Midstream attitudes about, and knowledge of, logarithms

During Interview 2, Nancy still exhibited a negative attitude toward logarithms. She said that she would be able to perform “some basic operations” but whenever the rules and properties get “thrown together, I get confused” (Interview 2). Nancy said that she would feel more comfortable with logarithms if she would “sit down and memorize them” because if she did that she could apply them (Interview 2).

Nancy’s most consistent word throughout the interview was “memorize.” She displayed the mindset that if she could take the time to memorize the properties then she could focus on doing the problems and not get “intimidated” by them.

Nancy’s fluency logarithms seemed to be related to her negative attitude. Nancy reported spending about 0-2 hours per week outside of class on the homework. She acknowledged that “wasn’t the right amount of time. I needed to study more” (Interview 2). Nancy displayed some recognition that she would use logarithms in studying finance or growth and decay, but she could not define what a logarithm was. Nancy could not identify that the base of the natural logarithm was e, but she could identify that the base of the common logarithm was 10. This was shown in her solution to Question 2.5.c.a \((\log(100) = x)\). Nancy said that whenever there was not a base, it was “just a log base 10” (Interview 2). In Question 2.5.c.b \((\ln(0) = x)\), she could not identify that the argument of a logarithm must be positive.

In the exercises for Interview 2, Nancy exhibited knowledge of the logarithmic properties. She was able to define the Product and Quotient Properties and use them in the problems. She associated addition with multiplication and subtraction with division. Since she already knew the Product Property coming into this course, her explanations to those problems were identical to her explanations from the first interview. For Question 2.5.c.d, Nancy successfully applied the Quotient Property.
However, her work falls short after that initial first step. She performs algebraic moves on the argument instead of directly converting to exponential form. In fact, she never converts to exponential form. Instead, she divides both sides of the equation by $\log_2$. She was confident that you can divide by log base $n$, where $n$ is a natural number. She called this process getting the “exact form” of a logarithm (Interview 2). This suggests that Nancy did not really understand what the logarithm was or how the notation worked. In Question 2.5.c.e, Nancy showed that she knew that if two logarithms with the same base are equal to each other, then their arguments must be equal. (See Figure 31.) She correctly “focuses on the arguments” and factors the quadratic by “moving everything on one side and set it equal to zero” (Interview 2). the Uniqueness Property of Logarithms.

![Equation](image)

Figure 31. Nancy’s work for questions 2.5.c.d (top) and 2.5.c.e (bottom) in Interview 2.
Nancy left Interview 2 with some knowledge of logarithms. She could describe and apply the Product and Quotient Properties, and she could identify that the common base was a base of 10. However, she could not successfully switch from logarithmic to exponential form, and she could not identify that the base for the natural logarithm was e. Nancy said she was “confused” and “intimidated” throughout the interview when explaining her work. She pointed out that she should have spent more than two hours working on logarithms outside of class. Thus, Nancy has room to expand her knowledge of logarithms.

4.8.3 Exiting attitudes about, and knowledge of, logarithms

In the end, Nancy’s final attitude toward found logarithms was that they were to be “daunting overall” (Interview 3). She reflected how at first, she did not understand the steps that connected each one, but now the issue grew into the “methodology behind trying to solve” them (Interview 3). Nancy often hesitated when trying to determine what to do next in a problem. The physical presentation of logarithms had always intimidated her and still did. She was intimidated by the presence of the natural logarithm. She believed that “the ln problems are like just more confusing to look at” (Interview 3).

Despite her negative attitude, Nancy experienced significant growth in her knowledge of logarithms since the first and second interviews. In general, she viewed logarithms as problems that could be solved using “different properties that are similar to exponential properties” (Interview 3). It seems that Nancy has made the connection that because logarithmic and exponential functions are inverses, then the properties of the two are linked as well. This suggests that perhaps she recognized that the logarithm properties were related to the rules for exponents. She identified
that the base of the natural logarithm was $e$ and the base of the common logarithm was 10. Nancy consistently used the word “argument” in her explanations. Thus, Nancy’s knowledge of the basic properties of logarithms increased from the first and second interviews.

Nancy could define and apply the Product, Quotient, and Power Properties in the exercises in the third interview. She could state that addition went with multiplication, subtraction went with division, and the coefficient was what the argument was raised to. Her development of these skills was evident especially in Question 3.2.a (Condense $\log(x) + \log(y) – \log(z)$) and Question 3.2.f (Expand $\ln(xy^3)$). (See Figure 32.) Her solutions to these problems indicate that she was able to apply multiple properties within one problem. In Question 3.2.a, Nancy first wanted to condense the logarithm using the Product Property because when “you see adding two logs together... you’re multiplying the arguments together” (Interview 3). Once she observed there was subtraction then “you’re dividing the arguments” (Interview 3). Nancy incorporated the term “argument” in her reasoning for this problem as well as for the other problems. In Problem 3.2.f, Nancy knew that when “the arguments are being multiplied” then you add them together (Interview 3). Then she identified she had to use the Product Property because when expanding logarithms, “you want it to be as simple as possible” (Interview 3). Nancy developed new skills that she did not previously applying exhibit.
Figure 32. Nancy’s work for Questions 3.2.a (left) and 3.2.f (right) in Interview 3.

At times, Nancy could convert from logarithmic to exponential form. Recall that she did not exhibit this skill in the previous interview. One example of when she employed this skill was in her solution to Question 3.3.b. (See Figure 33.) The left side of Figure 33 shows Nancy’s first attempt at the problem; the right side of the figure shows Nancy’s second attempt. While working through this problem, Nancy hesitated, immediately stating, “Oh my goodness… I’m just going to jump around” (Interview 3). She then clarified that she was intimidated by this problem which was why she had some difficulty solving it initially. She knew that because she could not expand or condense then the next option was to “change the form of the equation” to find $x$ (Interview 3). She then solved for $x$ and obtained found the correct answer. Nancy did not always correctly convert from logarithmic to exponential form, but in this problem, she did.
However, Nancy had difficulty converting from logarithmic and exponential form in two other problems. In Problem 3.3.d (Solve $2 \log_4(x) = 0$), Nancy started off strong by dividing both sides by 2 but thought that $4^0 = 0$. (See Figure 34.) Therefore, she got an answer of $x = 0$. She did not go back and check her work, so she did not have an opportunity to catch her mistake. In addition, Nancy had difficulty with Problem 3.3.e. (See Figure 34.) In this problem, she knew that she first had to condense the logarithm using the Quotient Property. She correctly did that, but then “lost confidence” in solving the problem (Interview 3). Nancy explained that this problem did not have a step-by-step procedure compared to the other problems in this interview. She believed that this problem was different than the other ones presented. It was a problem where all of her skills could not solve the problem. This suggests that Nancy does not have confidence in solving more complicated logarithmic expressions. Thus, Nancy was able to convert from logarithmic to exponential form in simpler problems, but she struggled to apply this to more complicated expressions.
On the Post-test, Nancy earned a score of 24 out of 36 points, which was an increase of 14 points from her score on the Pre-test. Thus, Nancy improved her performance by more than doubling her original score. Her responses to the Post-test provide some insight into why she did not score higher on it. Question 1 on the Post-test asked what students struggle(d) with currently with logarithms and how they learned or did not learn. In response to this question, Nancy said that the more “complex looking logarithms [are when] I [get] overwhelmed” (Post-test). She continued her thoughts that she still struggles with this issue and noted that she is “confused about tackling these [types of] problems” (Post-test). In Question 2, which asked students to define or show an example of a logarithm and if they can give a real-world application, Nancy referred to the expression \( \log(x) = 2 \) and that logarithms can be used to model bacterial growth. In Question 3, which assessed if students can identify the relationship between logarithmic and exponential functions, she stated that logarithmic and exponential functions were inverses and “they undo each other” (Post-test). Even though Nancy seems to still be struggling with logarithms, she did have some knowledge about what a logarithm is.
Nancy missed several of the parts on Question 4 on the Post-test, True/False questions. (See Figure 35.) Although she received full credit for Questions 4a, 4c, 4f, she was unable to answer Question 4e and received no credit for Questions 4b and 4d. It seems that Nancy could apply the Product Property in Interview 3, however she did not recognize it on Question 4b. Her response to Question 4e was not surprising; Nancy had never said, in any of the interviews, that the argument of a logarithm had to be positive. She got those questions wrong in the interviews because she would not check her solution to verify that it was a real solution to the logarithmic equation. This suggests that Nancy may not understand that concept. Nancy’s responses in Question 4 suggests that she had not acquired full fluency with logarithms.

Figure 35. Nancy’s work for Question 4 on the Post-test.

Nancy received all points on Question 5, which meant that Nancy was able to convert from logarithmic to exponential form correctly for simple logarithmic expressions. Also, Nancy could identify that the natural logarithm had a base of e. In Question 6, Nancy struggled with the notation of the logarithm. Recall, that Question 6 assessed students’ abilities to condense logarithms using the Product,
Quotient, and Power Properties. Nancy struggled with the notation of the Quotient Property. (See Figure 36.) Nancy did recognize that she had to divide, however she divided the logarithms and not the arguments. This contradicts what she did in Interview 2 and Interview 3 where she correctly applied the Quotient Property. This suggests that Nancy could have forgotten how to correctly simplify logarithms using the Quotient Property, or that her knowledge of this property was tenuous. This claim can be further supported by her work in Question 6a. (See Figure 37.) In this question, Nancy applied the Quotient Property, but again incorrectly used the notation. She then simplified the numerator’s argument as well. So, Nancy may not be able to correctly apply the Quotient Property because of her misunderstanding of logarithmic notation. Nancy’s work in Question 6 suggests that she vaguely knows the Quotient Property, but not all of the properties.

Figure 36. Nancy’s work for Question 6c on the Post-test.
In Question 7, which assessed students’ ability to expand logarithms, Nancy was able to correctly expand the given logarithms. The only issue was she did not receive full credit for Question 7c. The problem was $\ln(\frac{x^2 y}{z^2})$ and Nancy answered the partially expanded version of the problem, $\ln(x y) - \ln(z^2)$. This shows that she could apply the Quotient Property, but she did not notice that she had to continue expanding the first terms. This was the only question that Nancy had an issue with in Question 7. She also did not expand the second term by using the Power Property on the exponent.

Nancy received partial credit for Question 8. She successfully answered Question 8a, $\log_2(x - 2) = 3$, but not Questions 8b and 8c. (See Figure 38.) Nancy’s mistake in Question 8b arises from a computation error rather than not knowing logarithmic properties. (See Figure 38.) In Question 8c, Nancy had a notation error, again. She divided the two logarithms instead of dividing the two arguments. She then proceeded with the problem from this incorrect representation. After multiplying both sides by the denominator, she obtained two logarithms equal to each other, and she was able to use a property of logarithms to produce a quadratic equation to solve. From here she was unable to proceed since she could not factor the quadratic equation. It seems that Nancy still struggled with the more complicated
logarithmic expressions, which she acknowledged herself in Question 1 of the Post-test.

Figure 38. Nancy’s work for Questions 8b and 8c on the Post-test.

In summary, Nancy’s growth over the weeks was noticeable, however she still misunderstood fundamental properties of logarithms including the notation of the logarithm, the idea that the argument of a logarithm must be positive, the application of the Quotient Property, and being able to accurately compute values of a logarithmic expression. It seems as if her “intimidation” and “confusion” of logarithms hindered her from developing full fluency in the subject. Nancy stated in the Post-test that with “more practice and knowledge” she could solve more logarithm problems.
4.9 Renee’s Story

4.9.1 Incoming attitudes about, and knowledge of, logarithms

Renee did well in her high school math classes. She took Algebra I, Geometry, Algebra II, and Pre-Calculus. She “got A’s in all” of these classes except Geometry, in which she earned a B (Interview 1). She was “more of an algebra person” so Geometry was more difficult for her (Interview 1). She felt comfortable with the algebra skills that she learned in high school and stated that she just needs to be “refreshed with [topics]” to do well in them. Since Renee is a visual learner, she found math difficult when she couldn’t represent a problem with a picture. She felt confident that when she would set up the problem correctly she could carry out the rest of it; her issue was to correctly set up the problem. When approaching a math problem, Renee would read it, and then go back to underline key information in the problem including numbers, the question, and vocabulary that looked familiar. This helped her determine what the question was asking.

Renee first saw logarithms during her senior year of high school. Initially, she felt that logarithms were not hard; it was just about making sure she understood the properties. She thought the Pre-test was difficult because she had to “think back on what my previous knowledge… and use that to… figure it out” (Interview 1). Even though Renee thought the Pre-test was hard, she scored a 27 out of 36 possible points, which is 11 points above the average. On the Pre-test, she explained and identified what a logarithm was and its relationship to the exponential function. (See Figure 39.) Her example of what a logarithm was and her explanation of how it related to an exponential function was more detailed than other students on the Pre-test. Both of her answers could suggest that she learned logarithms well in high
school. Instead of simply stating that exponential and logarithmic functions were inverses of each other, she gave an example showing how an equation in one form could be converted into the other form. Renee also knew that the base of the common logarithm is 10 and the base of the natural logarithm is e.

Figure 39. Renee’s responses to Questions 2 and 3 on the Pre-test.

Renee was able to identify a majority of the properties of logarithms. She performed well on Questions 6 and 7 which assessed condensing and expanding logarithms, respectively. In this set of six problems, Renee only missed two questions – Questions 6b and 7a – but she received partial credit for those two questions. In Question 6b, Renee could apply the Product Property, but she did not apply the Power Property. (See Figure 40.) In Question 7a, she applied the Quotient Property, but she also expanded one of the arguments incorrectly. (See Figure 40.) Renee exhibited knowledge of the properties but did not consistently apply them.
Throughout the interview, Renee explained her work for the problems. For Question 4, the true and false section, she got all of the questions correct except 4e, “We can find the logarithm of a negative number.” She was unsure what to put for this answer, so she circled “I’m not sure.” She stated that she thought “it really [was] false” but was not “confident enough to… give a true or false answer” (Interview 1). This suggests that Renee knew the correct answer, but her lack of confidence ultimately made her circle “I don’t know.” Her lack of confidence also caused her to miss Question 8c. Renee correctly carried out the process but made an algebra error. (See Figure 41.) In the interview, I asked her if she questions her answer of $x = -4$ and she responded that she was not sure if the argument could be negative, so she did not know what else to put.
Renee was able to correctly convert from logarithmic to exponential form on Question 5. She was able to successfully figure out Questions 5a and 5b but wrote that she did not know how to solve Question 5c, \( \ln(x) = 2 \). She missed Question 5c because she was unsure about the natural logarithm.

In summary, Renee overall had considerable incoming knowledge of logarithm concepts and properties. She scored significantly higher than the average on the Pre-test. She made a few algebraic errors on the Pre-test, which resulted in her losing some points. Her work and explanations throughout the interviews suggest that her main issue on the Pre-test could have been related to her lack of confidence on answering particular problems such as Questions 4e and her answer for 8c.

### 4.9.2 Midstream attitudes about, and knowledge of, logarithms

Renee’s attitude toward about logarithms in the second interview was positive. She believed that logarithms were hard at first because of the initial memorization of the properties, however they quickly became easier. She viewed logarithms as an
equation that would be applied to problems “when a [variable] is in the exponent” and you were trying to find that variable (Interview 2). She “[thought] they were fun” once she knew the properties (Interview 2). Renee had attitude.

Renee reported that she spent 2-4 hours per week outside of class working on logarithms. This time was spent on homework in the course pack and ALEKS. She believed that this was “an appropriate amount of time” to spend on the homework but “a little bit more time wouldn’t have hurt” (Interview 2). She felt confident in her abilities to apply the logarithm properties and solve logarithmic equations because of her practice in ALEKS.

Renee knew the three essential properties of logarithms and could state them. She clearly identified that she could only apply the properties if “the logs have the same base” (Interview 2). She described the Quotient Property as if you see two logarithms that are being “subtracted you would divide the two arguments” (Interview 2). When working with the Product Property she said that when “the logs have the same base… [and if] it’s addition…. you can condense them to be multiplied” (Interview 2). Renee defined the Power Property as “the front [number] of the log that means it’s an exponent” to the argument (Interview 2). She identified that the common logarithm has a base of 10 and the natural logarithm has a base of $e$.

Overall, Renee did well on the exercises in Interview 2, which assessed students’ learning of logarithms midway through the logarithm lessons. However, she did have a little trouble with Questions 2.5.c.b and 2.5.c.c. Question 2.5.c.b was $\ln(0) = x$ and assessed students’ knowledge about the argument of the logarithm. When we got to this question in the interview, Renee started to become unsure of her answers. She viewed this problem as one of the more difficult problems of the interview. She
first stated that “ln of zero isn’t a thing” but because it was “ln… [she] didn’t know” (Interview 2). The entire time Renee worked on this question, she reasoned herself through the problem but kept second-guessing herself. She knew that the argument “can’t be negative, but I don’t remember if it can be zero” (Interview 2). She stated that “$e^0 = 1$” but because the zero was in the argument it did not make sense that the answer would be 1. Finally, Renee came to the conclusion that “the argument has to be positive. It can’t be negative…the argument can’t be zero either” (Interview 2). Thus, in the end she was able to work through the problem. Renee also had trouble with Question 2.5.c.c, but it was not a logarithm issue, it was an algebra mistake. (See Figure 42.) She was able to correctly apply the Product Property, but she did not multiply or factor correctly. Thus, she got the wrong answer. She also knew that her answer of $-\frac{1}{4}$ was not a solution because it would have made the arguments negative.

![Figure 42. Renee’s work for Question 2.5.c.c in Interview 2.](image)

For the rest of Interview 2, Renee worked through the problems without any issues. She explained her work using logarithm terminology like “argument” and “exponential form.” Renee’s incorporation of the terminology may have helped her solve the problems because when she was not confident, her explanations helped her
think through the problem. She reported spending 2-4 hours outside of class working on homework. Overall, Renee performed quite well in Interview 2.

4.9.3 Exiting attitudes about, and knowledge of, logarithms

At the end, Renee’s had a positive attitude about logarithms. She still enjoyed doing logarithms. Renee could define a logarithm, knew that a logarithm was a function and identified it as the inverse to the exponential function. She knew that you would “need logarithms when you’re solving an exponential question… [when] you actually solve for \(x\)” in the power (Interview 2). Renee’s positive attitude also supported her learning of logarithms because it made it more enjoyable for her to do the homework and participate in class, as she stated in Interview 3.

Renee’s fluency with logarithms was evident throughout the interview, as she consistently used appropriate logarithm vocabulary and definitions. She identified that the two common bases were 10 and e; in addition, she knew the three properties of logarithms and defined them just like she did in Interview 2. While working through the problems in the interview, her explanations were thorough and detailed, using phrases like “logs with the same base” and switching “from logarithmic to exponential form” (Interview 2). Renee knew that the argument of a logarithm could only be positive.

In summary, it seems that Renee has developed considerable fluency was fluent with logarithms. In Interview 3, she correctly solved all of the questions correctly and explained her work in thorough detail. Her consistent use of terminology seemed to have help her work through the problems because she could identify what properties and procedures to perform. In addition, Renee’s fluency with logarithms was evident by her performance on the Post-test; she scored 36 out of 36
points possible on the Post-test. She clearly defined logarithms as functions and gave examples of the conversion of logarithm to exponential form. Her perfect scores on the third interview and Post-test indicate that Renee developed full fluency with logarithms.

4.10 Cross-Case Analysis

This section presents a cross-case analysis between all interview participants described in the previous sections. It will compare and contrast the different characteristics that each participant exhibited during their journey to develop fluency with logarithms. I will identify the skills and concepts that students learned easily as well as those that students struggled with, as well as possible factors that played a role in their struggles to learn logarithms.

4.10.1 Relationship between when students last studied logarithms and their math background affect Pre-test Post-test scores

One issue that I wanted to investigate was if any relationship or pattern could be detected between students’ prior knowledge, including when they last studied logarithms, and experiences affected their performance on the Pre-test and Post-test. Because all students received the same lessons in their Math 115 classes, students worked with problems similar had very similar learning opportunities. environments. Table 7 organizes my participants in terms of how long ago they last studied logarithms and then provides the participants’ scores on the Pre-test and the Post-test. Note that the Pre-test mean for the four students who had not studied logarithms for a year was 20.5, and the Pre-test mean for the two students who had not seen logarithms in
two years was 20. In contrast, the Pre-test mean for the one student who had not
studied logarithms in three years was only a 10. The In other words, the student who
had been away from logarithms for three years scored about half as many points on the
Pre-test. Although there are too few participants to draw firm conclusions, these data
suggest that there is not much difference in knowledge retention until 3 or more years
have passed. These data This situation only dealt with one student, however the
scores were still low in comparison to the other participants. This data also show that
each group of students increased their scores, on average, by about 14 points from the
Pre-test to the Post-test.

<table>
<thead>
<tr>
<th>Participant</th>
<th>Years Since Last Studied Logarithms</th>
<th>Pre-test Score</th>
<th>Post-test Score</th>
<th>Mean on the Pre-test</th>
<th>Mean on the Post-test</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fiona</td>
<td>1</td>
<td>14</td>
<td>36</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Emma</td>
<td>1</td>
<td>21</td>
<td>29</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Jerry</td>
<td>1</td>
<td>20</td>
<td>35</td>
<td>20.5</td>
<td>34</td>
</tr>
<tr>
<td>Renee</td>
<td>1</td>
<td>27</td>
<td>36</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Natalie</td>
<td>2</td>
<td>25</td>
<td>36</td>
<td>20</td>
<td>34.5</td>
</tr>
<tr>
<td>Ellen</td>
<td>2</td>
<td>15</td>
<td>33</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Nancy</td>
<td>3</td>
<td>10</td>
<td>24</td>
<td>10</td>
<td>24</td>
</tr>
</tbody>
</table>

Table 7. A comparison of the timed students learned logarithms to when they took
MATH 115. Comparing the means from the Pre-test and the Post-test.
4.10.2 Students’ ideas about the definition of logarithm students’ perspectives and about its applications

My interview participants were able to define a logarithm as a function and identify that it is the inverse of an exponential function. The participants were not able to formulate any other definition of a logarithm. The other typical response they gave was to start listing off the properties associated with logarithms. Participants were not able to provide a general definition of a logarithm as a function without immediately associating it with the exponential function.

Only 2 of the 7 participants were able to give some insight as to an example of when a logarithm would be used. Both Fiona and Emma said that logarithms would be applied when solving for a variable that was in the exponent position. Fiona was the only student who was able to give a real-world application of logarithms, namely bacteria growth and exponential decay.

The participants did not demonstrate that they fully understood that a logarithm could be a number. In Interviews 2 and 3, five questions (2.5.a.c, 2.5.c.a, 2.5.c.b, 3.2.b, and 3.2.c) involved logarithms that simplified to a numerical solution. Table 8 presents these five problems along with which students got each problem correct. Not a single participant was able to get all five questions correct, nor did any student correctly answer Question 3.2.b, \( \log(x + 2) - \log(x + 2) \). Collectively, students could simplify the logarithm, but not evaluate the expression to the point where they would get the correct numerical answer. This may suggest that students may not always have recognized a logarithm as a numerical value, unless given a simple problem. For example, Questions 2.5.c.a \( \log(100) = x \) and 2.5.c.b \( \ln(0) = x \) were simple logarithmic equations. These two problems were straightforward if you knew the base and how to convert to exponential form. In contrast, for Question 3.2.b
(log(x + 2) − log(x + 2)), students needed to apply a property before evaluating; only one student could reduce it to log(1) and that one student did not evaluate the expression. Thus, participants may not have always viewed logarithms as a numerical value, but only as a function. Even though the interview participants were slightly stronger than the full sample of 155 Pre-test students, the participants share a likely similarity: they understood logarithms to be functions and they may not know a logarithm could also be a number. It is possible that this is also due in part to the curriculum (lesson plans) that students received in the MATH 115 course. The focus in the course is on viewing logarithms as functions rather than as numerical values. Although students do spend a bit of time evaluating logarithmic expressions, the bulk of the curriculum focuses on studying the family of logarithmic functions.
<table>
<thead>
<tr>
<th>Participant</th>
<th>2.5.a.c</th>
<th>2.5.c.a</th>
<th>2.5.c.b</th>
<th>3.2.b</th>
<th>3.2.c</th>
<th>Total Number Correct (max = 5)</th>
</tr>
</thead>
<tbody>
<tr>
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<td>Yes</td>
<td>Yes</td>
<td>No</td>
<td>Yes</td>
<td>4</td>
</tr>
<tr>
<td>Fiona</td>
<td>No</td>
<td>Yes</td>
<td>No</td>
<td>No</td>
<td>Yes</td>
<td>2</td>
</tr>
<tr>
<td>Emma</td>
<td>No</td>
<td>Yes</td>
<td>Yes</td>
<td>No</td>
<td>No</td>
<td>2</td>
</tr>
<tr>
<td>Jerry</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>No</td>
<td>Yes</td>
<td>4</td>
</tr>
<tr>
<td>Ellen</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>No</td>
<td>Yes</td>
<td>4</td>
</tr>
<tr>
<td>Nancy</td>
<td>No</td>
<td>Yes</td>
<td>Yes</td>
<td>No</td>
<td>No</td>
<td>2</td>
</tr>
<tr>
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<td>Yes</td>
<td>Yes</td>
<td>No</td>
<td>Yes</td>
<td>3</td>
</tr>
<tr>
<td>Total Correct (max = 7)</td>
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<td>7</td>
<td>6</td>
<td>0</td>
<td>5</td>
<td></td>
</tr>
</tbody>
</table>

Table 8. Interviewee’s correct answers for questions that dealt with finding a value for expressions in interviews 2 and 3.

4.10.3 Correlation between Students’ Use of logarithmic terminology students’ responses and impact their Developing Fluency with Logarithms

Students used logarithm terminology differently in their explanations; this use may have affected, or been a reflection of, their knowledge of logarithms. In this section I will describe a possible correlation between logarithm terminology use and test performance.

Students’ use of terminology may be correlated with their performance on the Post-test, with students who used more terminology scoring higher. I rated the students’ use of logarithmic vocabulary on a two-point scale. A student who received a zero did not use any logarithmic terminology in their responses. A student who
received a score of one sometimes used logarithmic terminology in their explanations, but not consistently use. A student who received a score of two consistently used logarithmic terminology in their explanations. The results of students’ terminology frequency are presented in Table 9.

<table>
<thead>
<tr>
<th>Participant</th>
<th>Interview #</th>
<th>Pre-test Score</th>
<th>Post-test Score</th>
<th>Percent Difference in Students’ Scores</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fiona</td>
<td>2</td>
<td>2</td>
<td>14</td>
<td>36</td>
</tr>
<tr>
<td>Jerry</td>
<td>2</td>
<td>2</td>
<td>20</td>
<td>35</td>
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<tr>
<td>Renee</td>
<td>2</td>
<td>2</td>
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<td>36</td>
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<tr>
<td>Natalie</td>
<td>1</td>
<td>2</td>
<td>25</td>
<td>36</td>
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<tr>
<td>Ellen</td>
<td>0</td>
<td>1</td>
<td>15</td>
<td>33</td>
</tr>
<tr>
<td>Nancy</td>
<td>0</td>
<td>1</td>
<td>10</td>
<td>24</td>
</tr>
<tr>
<td>Emma</td>
<td>1</td>
<td>1</td>
<td>21</td>
<td>29</td>
</tr>
</tbody>
</table>

Table 9. Students’ use of logarithmic terminology and relationship to test performance.

Students like Fiona, Jerry and Renee quickly learned logarithmic terminology and their performance increased from the Pre-test to the Post-test. Renee, who scored high on the Pre-test, increased her score as high as it could still go, earning a 36 out of 36 on the Post-test. Fiona also earned a 36 out of 36 on the Post-test, increasing her score by 157%. Jerry increased his score by 75%, earning 35 out of 36 points. Natalie, who occasionally used logarithm terminology in her responses, also earned a 36 out of 36 on the Post-test, increasing her score by 50%. In contrast, Ellen and
Nancy had some growth in incorporating logarithm terminology into their responses, increasing from 0 points to 1 point between the second and third interview. Ellen’s score on the Post-test increased by 120% and Nancy’s score increased by 140%. Even though both more than doubled their scores, neither had a Post-test score higher than one of the 4 students who used terminology frequently. Nancy’s score on the Post-test was still below the mean score of 28.99 on the Post-test. Ellen’s score was only slightly higher than the mean score on the Post-test. Finally, Emma only occasionally incorporated logarithm vocabulary in her responses in both interviews. She only increased her score on the Post-test by 38%, the second lowest increase after Renee, who maxed out her score on the Post-test. Also, Emma’s score on the Post-test was lower than all of the other interview participants’ scores except for Nancy, who increased her score by an impressive 140%. Thus, these data hint at the possibility of a correlation between logarithmic terminology use and fluency with manipulating logarithms and solving logarithmic equations.

### 4.10.4 Students’ attitudes relationship toward Logarithms over Time

Throughout the interviews, I asked students about their attitude toward logarithms in order to assess if their attitudes had any effect on their performance. I categorized each student as having either a positive, neutral, or negative attitude toward logarithms. I determined this by examining their word choice and personal comments about the problems. Their attitude assessment can be found in Table 10. Natalie and Emma had a positive attitude towards logarithms throughout the interviews. Fiona was the only student who had only neutral and positive attitudes towards logarithms. Three students – Jerry, Ellen and Renee – exhibited all three kinds of attitudes toward logarithms. Nancy was the only student who had a
consistently negative attitude toward logarithms for all three interviews. There were no participants who had both neutral and negative attitudes towards logarithms.

<table>
<thead>
<tr>
<th>Name</th>
<th>Interview 1 Attitude</th>
<th>Interview 2 Attitude</th>
<th>Interview 3 Attitude</th>
</tr>
</thead>
<tbody>
<tr>
<td>Natalie</td>
<td>Positive</td>
<td>Positive</td>
<td>Positive</td>
</tr>
<tr>
<td>Fiona</td>
<td>Positive</td>
<td>Positive</td>
<td>Neutral</td>
</tr>
<tr>
<td>Emma</td>
<td>Positive</td>
<td>Positive</td>
<td>Positive</td>
</tr>
<tr>
<td>Jerry</td>
<td>Neutral</td>
<td>Positive</td>
<td>Negative</td>
</tr>
<tr>
<td>Ellen</td>
<td>Neutral</td>
<td>Negative</td>
<td>Positive</td>
</tr>
<tr>
<td>Nancy</td>
<td>Negative</td>
<td>Negative</td>
<td>Negative</td>
</tr>
<tr>
<td>Renee</td>
<td>Negative</td>
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</tbody>
</table>

Table 10. Students’ attitudes toward logarithms throughout the interviews.

4.10.5 Interview participants’ strengths and weaknesses on the Pre-test, interviews, and Post-test

This section identifies which questions and concepts all of the students got right, and which questions no student earned full credit on. This section will identify what topics students learned best, and what topics students are still struggling with after the logarithm lessons had been taught. It will go over how the seven stories come together and suggest information that may lead to identify students’ learning of logarithmic properties.
Table 11 shows all eleven questions that every interview participant answered correctly on the Pre-test, the second and third interviews, and the Post-test. Question 5a was the only question out of 18 on the Pre-test that all participants received full credit for. Question 5a assessed students’ ability to convert from logarithmic to exponential form in order to solve for $x$ on the right-hand side of a logarithmic equation. In Interview 2, Questions 2.5.a.a, 2.5.a.b, and 2.5.b.b were the only three questions, out of 11 total, for which all students got full credit. In Interview 3, Question 3.3.b was the only question, out of 11 total, for which all interview participants received full credit. Finally, on the Post-test, Questions 5a, 5b, 6b, 7a, 7b, and 8a were the only six questions, out of 18 total, for which all interview participants received full credit. (See Table 11.)

Interestingly, five out of the eleven questions dealt with knowing how to convert from logarithmic to exponential form. This may suggest that the participants were able to learn how to convert from logarithmic to exponential form more easily than learning how to use the other properties. Also, participants’ inclination to view logarithms as functions may also help explain why they were adept at converting from logarithmic to exponential form. Question 5a was both correctly answered by all participants on both the Pre-test and the Post-test. This again suggests that participants developed knowledge of the conversion process and were able to apply this skill to various problems.
### Table 11. The questions on the Pre-test, Interview 2, Interview 3, and the Post-test for which that all interviewees received full credit.

Comparing the number of questions that all students got right on the Pre-test to the Post-test shows a unique detail. On the Pre-test, there was only one question for which all students received full credit. On the Post-test, there were eight questions for which all students received full credit. Recall from Table 9, the students’ scores from Pre-test to Post-test increased by approximately fourteen points in each group. This may suggest there is a connection between the development of students’ knowledge of logarithms and applying the new knowledge to the 14-point increase. This could be from students learning new skills that are easier to enact, such as those involved in

<table>
<thead>
<tr>
<th>Question</th>
<th>Problem</th>
<th>Properties Tested</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pre-test: 5a</td>
<td>$\log_2(8) = x$</td>
<td>Converting from logarithmic to exponential form</td>
</tr>
<tr>
<td>Interview 2: 2.5.a.a</td>
<td>$\log_2(3) - \log_2(5)$</td>
<td>Quotient Property</td>
</tr>
<tr>
<td>Interview 2: 2.5.a.b</td>
<td>$\log_5(x) + \log_5(y)$</td>
<td>Product Property</td>
</tr>
<tr>
<td>Interview 2: 2.5.b.b</td>
<td>$\frac{x}{y}$</td>
<td>Quotient Property</td>
</tr>
<tr>
<td>Interview 3: 3.3.b</td>
<td>$\log_2(2x - 3) = 2$</td>
<td>Converting from logarithmic to exponential form</td>
</tr>
<tr>
<td>Post-test: 5a</td>
<td>$\log_2(8) = x$</td>
<td>Converting from logarithmic to exponential form</td>
</tr>
<tr>
<td>Post-test: 5b</td>
<td>$\log_x(32) = 5$</td>
<td>Converting from logarithmic to exponential form</td>
</tr>
<tr>
<td>Post-test: 6b</td>
<td>$8\log_2(x - 2) + \log_2(x - 5)$</td>
<td>Product and Power Properties</td>
</tr>
<tr>
<td>Post-test: 7a</td>
<td>$\ln \frac{2x - 5}{y}$</td>
<td>Quotient Properties</td>
</tr>
<tr>
<td>Post-test: 7b</td>
<td>$\log(x^3)$</td>
<td>Power Properties</td>
</tr>
<tr>
<td>Post-test: 8a</td>
<td>$\log_2(x - 2) = 3$</td>
<td>Converting from logarithmic to exponential form</td>
</tr>
</tbody>
</table>
Question 5c. Question 5c, $\ln(x) = 2$, is an easier problem to evaluate if the students knew what the base of the natural logarithm was. Solving this simple problem correctly would increase their score by 2 points.

In Interview 2 and the Post-test, at least one student received full credit for each problem. However, there were questions for which no student earned full credit on the Pre-test and in Interview 3. There were two questions, Questions 6b and 8a, on the Pre-test for which no student earned full credit. (See Appendix A.) Question 6b assessed students’ knowledge of the Product and Power Properties. Question 8a assessed students’ skill of the Quotient Property and converting from logarithmic to exponential form. However, students corrected both of these mistakes on the Post-test; all seven participants received the full 2 points for each problem. For Interview 3, no student got full credit on Question 3.2.b (Condense: condense $\log(x+2) - \log(x+2)$). Students were able to apply the Quotient property, but they did not reduce the logarithm to its numerical value of 0. This again suggests that students may not be inclined to see logarithms as numerical value.
Chapter 5

DISCUSSION

In this chapter, I provide a summary of my results. I then identify some limitations of my study. Next, I identify what I see as possible implications of my work to the field of mathematics education and to instruction on logarithms in Pre-calculus courses like MATH 115. I conclude by identifying some next steps for researchers and for myself.

5.1 Summary of results chapter

By examining data from the Pre-tests, Post-tests, and interviews with seven interview participants, I was able to form some general conclusions about students’ learning of logarithms. In particular, I was able to identify some possible factors that may affect students’ learning of logarithmic functions. First, students struggled to understand the purpose or reasons for learning logarithms. Six out of seven of the interview participants were not able to provide an example of an application of logarithms in the real world. Not understanding the purpose of learning about logarithms and not knowing any real-world applications seem to go hand-in-hand. That is, the same students who did not know any real-world applications also tended to be the same students who reported not understanding why they needed to learn about logarithms. Second, my findings suggest that if students are able to read a logarithm and understand the notation, then they are more likely to identify and apply the skills necessary to solve logarithmic problems. Students who are able to interpret a logarithm developed a stronger fluency. Third, were more fluent. Thirdly my findings suggest that if students are able to employ logarithm terminology, then they are more likely to have better fluency with logarithm manipulations. Finally, my data
suggests that if students maintain a positive attitude toward learning logarithms, they are more likely to perform better than those with negative attitudes toward logarithms.

5.2 Limitations

There are a few limitations of my study that may have affected my results. First, my sample of 155 students who took both the Pre-test and the Post-test was only about 22% of the total MATH 115 population for the Fall 2017 semester. This sample was a convenience sample, in that students who participated were volunteers. As a result, I do not know if the 155 students who participated are a representative sample of the approximately 700 students taking Pre-Calculus that semester.

Second, only seven out of about 700 MATH 115 students participated in all five parts on my project. Thus, my study provides insight into only a small number of students’ experiences with and attitudes toward logarithms. These seven students were not necessarily representative of the entire population of students enrolled in MATH 115.

A third limitation is that MATH 115 at the University of Delaware is structured using a department-wide curriculum where every student, no matter the teacher or section, receives the same material in the same format. MATH 115 employs an active learning approach that engages students to work in small groups to learn mathematics. This is different than other colleges and institutions where Pre-Calculus is more typically offered in a traditional, lecture-style format. Therefore, my results may not be as applicable to institutions where Pre-calculus is taught using a different approach, such as a traditional lecture that does engage students in not always utilize active learning.
5.3 Field Implications for Research in Mathematics Education mathematics education

The amount of existing research on students’ learning of logarithms is minimal. I believe that more research should be done in this field because there is a large number of students that do not develop an understanding of logarithms; many students go on to take Calculus and need to understand and work with logarithms. Without this knowledge and fluency, students struggle and may fall behind in their field of study. Furthermore, this thesis provides a small insight into students’ attitudes towards logarithms and follows their journeys through five lessons. Investigating students’ interpretations and their thinking process may contribute to learning more about what educators can do better to increase students’ fluency. By interviewing the participants, I obtained an insider perspective on what students were thinking and how they were processing the material. I was able to form conclusions that could not have been derived from a test on which students simply write answers to answer questions. Further research investigating students’ thinking processes may help researchers gather information that can be used to identify how to best support students’ learning of logarithms.

5.4 Improve curriculum optimize students’ learning Implications for MATH 115 logarithmic functions?

First, my research suggests that having students read aloud the logarithm expression as an exercise may improve students’ abilities to understand the notation of the logarithm. There are limited exercises that deals with this concept, so I think adding additional ones would benefit the student. An example problem would be:

“Discuss with your groups about how you would read each expression:

a) $\log(x) = y$
b) \( \log_2(2) = 6 \)

c) \( \ln(e) = 1 \)

d) \( \log_4(2x - 1) = x^\prime \)

Just a few short exercises for students to verbally describe how to read a logarithm may help students identify the different components. This may then increase students’ understanding of what a logarithm means and how it can be evaluated. From my results, it seems that if students are able to read a logarithm, then they are able to apply more properties than students who cannot.

Another suggested revision would be to capitalize on the inverse relationship between exponential and logarithmic functions to better connect the properties of logarithms with their corresponding properties for exponents. For example, consider Jerry who discussed how the properties of logarithms and exponents relate but he could not explain exactly how they work together. Since MATH 115 focuses on family of functions and their relationships, creating a lesson that highlights the relationship between the different properties may help students understand where the properties came from and feel less of a need to simply memorize them. Also, by starting the lesson by discussing how previous functions work together to create inverses may help students see that the properties of inverse functions are similar. Since students may be intimidated by the logarithm notation, showing them something they have seen before may lessen the intimidation and promote discovery. By allowing students to actively learn through a problem-solving lesson, instead of automatically giving them the properties, may help them better understand where the properties come from and why it is more than just memorizing. My study suggests that when students just memorize the properties they do not learn as much as the
students who identify the relationships between logarithmic and exponential properties.

In summary, I recommend that the MATH 115 instructors do more to emphasize the notation of logarithms and to promote understanding the properties of logarithms instead of simply giving students a table of the properties. Allow the students time to attempt to discover the properties of logarithms, because they will remember that time when they struggled to figure this out.

5.5 Implications for Pre-Calculus results impact students taking pre-calculus at other Educational Institutions educational institutes

Even though my results are limited to an active learning environment and the University of Delaware’s unique curriculum, mathematics educators at other institutions can learn from my study. Findings from my study suggest that students may develop fluency in logarithms more easily if they are able to read a logarithm and understand the purpose of each part of the expression and how they relate to each other. This can be emphasized during instruction in various ways. For example, in a traditional lecture-style class, the instructor may discuss each component of the logarithm and ask essential questions like “What role does the base play in terms of the expression?”, “What happens if I manipulate the argument? How does that impact my expression?”, and “What role does the expression on the other side of the equals sign play in terms of the logarithm?” Discussing the answers to these questions explicitly may help students develop better understanding of the notation and how it works.

Findings from my study also suggest that if students continue to have a positive attitude toward learning mathematics and toward logarithms specifically, then
they may be more successful at developing fluency with logarithms. From my 
interview participants, I found that the student who had a negative attitude towards 
logarithms just memorized the properties and did not see any reason for understanding 
the properties. My study suggests that if students have positive attitudes toward 
learning logarithms, then they may be more likely to work harder and ultimately 
become more fluent.

5.6 Next Steps: Where do researchers and educators go from here?

Further investigation is important for researchers and mathematics educators 
working to improve students’ learning of logarithms. My study suggests that 
continuing to interview students and asking them about their thinking process may 
provide additional insights into what students are misunderstanding. Collecting 
information from a larger sample of students from a wider range of institutions would 
also give even more insight into what we as researchers and educators can do to 
improve student learning.

The next step is to continue our investigation of student learning of logarithms 
because there is little existing research to provide insight into why students are not 
developing fluency with logarithms. There are many different strategies to assess 
students’ learning and I believe the best way to learn about how to improve is by using 
both written assessments and individual interviews.
REFERENCES


Mulqueeny, E. S. (2012). *How do students acquire an understanding of logarithmic concepts?* (Doctoral dissertation, Kent State University) (pp. 1-343). Kent, OH.


Appendix A

PRE-TEST

Logarithms Pre-test

Thank you for taking the time to help support my Senior Thesis Research Project! The purpose of my project is to better understand students’ learning of logarithms. The more information you can write down, the more helpful it will be to my project. This information will be used to help future students better learn and master logarithmic expressions and equations.

The goal of this Pre-test is to collect information about what you already know about logarithms. Please try your best by writing down anything you know about the problem. Even an answer of “This looks familiar, but I do not recall what to do” is helpful!

You have 60 minutes to complete this Pre-test. Calculators are not allowed. There should be enough space to answer all of the questions, however if you need additional space, scrap paper will be provided. Please box, circle, star, etc. your final answer so it shows that you are done with the problem.

**Name (Please print neatly):**

________________________________________________________________________

**Which section of Math 115 are you in?**

- TR 8am, Tammy Rossi
- TR 8am, Anthony Mak
- TR 9:30am, Anthony Mak
- TR 9:30am, Giovanna Lisey
- TR 9:30am, Gail Headley
- TR 11am, Tammy Rossi
- TR 11am, Donna McQuillin
- TR 12:30pm, Donna McQuillin
- TR 12:30pm, Anthony Mak
- TR 2pm, Eirini Kilikian
- MW 8:40am, Giovanna Lisey

1) Have you ever been introduced to or studied logarithms before? If so, when did you first see them – which year / grade / class? Did you find them easy or difficult?
2) In your own words, define what a logarithm is (or show an example) and how it can be used in a real-world application. (You can just give an example problem here if that is easier.)

3) What is the relationship, if any, between a logarithmic and exponential function? For this question, you can either show an example or you can describe the relationship in words.

4) Circle True, False, or I’m not sure.

a) $\log_a(xy) = \log_a(x) + \log_a(y)$
   True       False       I’m not sure

b) $2\ln(3x - 4) = \ln(6x - 8)$
   True       False       I’m not sure

c) $\log(100) = 2$
   True       False       I’m not sure

d) $\log(1) = 1$
   True       False       I’m not sure

e) We can find the logarithm of a negative number.
   True       False       I’m not sure

f) $\ln(x)$ has a base of 10
   True       False       I’m not sure
5) Find the value of $x$ in each logarithmic equation below:

a) $\log_2(8) = x$

b) $\log_x(32) = 5$

c) $\ln(x) = 2$

6) Condense each logarithm:

a) $\log_5(2x^3 - x^2) - \log_5(x)$

b) $8 \log_2(x - 2) + \log_2(x - 5)$

c) $\log(xy^2) - \log(z)$
7) Expand each logarithm expression
   a) \( \ln\left(\frac{2x-5}{y}\right) \)

   b) \( \log(x)^3 \)

   c) \( \ln\left(\frac{xy}{z^2}\right) \)

8) Solve for \( x \). If there is no solution, write “No solution.”
   a) \( \log_2(x - 2) = 3 \)     b) \( 3\log_3(x) = 3 \)     c) \( \log_4(x^2) - \log_4(x - 1) = 1 \)
Appendix B

POST-TEST

Logarithms Post-test

Thank you for taking the time to help support my Senior Thesis Research Project! The purpose of my project is to better understand students’ learning of logarithms. The more information you can write down, the more helpful it will be to my project. This information will be used to help future students better learn and master logarithmic expressions and equations.

The goal of this Post-test is to collect information about what you now know about logarithms. Please try your best by writing down anything you know about the problem. Even an answer of “This looks familiar, but I do not recall what to do” is helpful!

You have 60 minutes to complete this Post-test. Calculators are not allowed. There should be enough space to answer all of the questions, however if you need additional space, scrap paper will be provided. Please box, circle, star, etc. your final answer so it shows that you are done with the problem.

Name (Please print neatly):

Which section of Math 115 are you in?

- TR 8am, Tammy Rossi
- TR 8am, Anthony Mak
- TR 9:30am, Anthony Mak
- TR 9:30am, Giovanna Lisey
- TR 9:30am, Gail Headley
- TR 11am, Tammy Rossi
- TR 11am, Donna McQuillin
- TR 12:30pm, Donna McQuillin
- TR 12:30pm, Anthony Mak
- TR 2pm, Eirini Kilikian
- MW 8:40am, Giovanna Lisey
1) Think of a specific issue or problem you struggled with involving logarithms. Are you still confused about it? Why do you think you initially had trouble with that issue/problem? How did you resolve it?

2) In your own words, define what a logarithm is (or show an example) and describe how it can be used in a real-world application. (You can just give an example problem here if that is easier.)

3) What is the relationship, if any, between a logarithmic and exponential function? For this question, you can either show an example or describe the relationship in words.

4) Circle True, False, or I’m not sure.
   a) $\log_a(xy) = \log_a(x) + \log_a(y)$
      True  False  I’m not sure
b) \(2 \ln(3x - 4) = \ln(6x - 8)\)  
   True  False  
   I’m not sure

c) \(\log(100) = 2\)  
   True  False  
   I’m not sure

d) \(\log(1) = 1\)  
   True  False  
   I’m not sure

e) We can find the logarithm of a negative number.  
   True  False  
   I’m not sure

f) \(\ln(x)\) has a base of 10  
   True  False  
   I’m not sure

5) Find the value of \(x\) in each logarithmic equation below:

   a) \(\log_2(8) = x\)  
   b) \(\log_x(32) = 5\)  
   c) \(\ln(x) = 2\)

6) Condense each logarithm:

   a) \(\log_5(2x^3 - x^2) - \log_5(x)\)

   b) \(8 \log_2(x - 2) + \log_2(x - 5)\)

   c) \(\log(xy^2) - \log(z)\)
7) Expand each logarithm expression
   a) \( \ln \left( \frac{2x-5}{y} \right) \)
   
   b) \( \log(x^3) \)
   
   c) \( \ln \left( \frac{xy}{z^2} \right) \)

8) Solve for \( x \). If there is no solution, write “No solution.”
   
   a) \( \log_2(x - 2) = 3 \)
   
   b) \( 3 \log_3(x) = 3 \)
   
   c) \( \log_4(x^2) - \log_4(x - 1) = 1 \)
Appendix C

INTERVIEW 1 QUESTIONS

1. Please rate the level of difficulty of the Pre-test on a scale of 1 to 10, with 1 being extremely easy, 5 being moderately difficult, and 10 being extremely difficult?

2. Describe how you were thinking/feeling while working on the Pre-test.
   a. Did you have difficulty understanding what the problems were asking for?
   b. What did you do to try to attempt the problems?

3. On a scale of 1 to 4, how familiar did the concepts on the exam look to you?
   • 1 = I have never seen or heard of a logarithm before the Pre-test.
   • 2 = I have seen logarithms before, but I don’t remember much. / It looks familiar and I think I have seen it. / The word “log” looks familiar but I am not confident about it.
   • 3 = I have seen logarithms before. / I could do some problems.
   • 4 = Everything looked familiar and I felt pretty confident that I could answer the questions or at least be able to attempt the problems with some prior knowledge.

4. What math courses did you take in high school? Have you taken any college-level math classes? How have you done in math classes, in general?

5. Let’s talk a bit about your algebra background.
   a. How did you do in your algebra-based courses?
   b. What topics did you find most challenging?
   c. What topics were easier to understand?
   d. Tell me about a time when you were struggling in an algebra class and what you did to improve your performance.

6. On a scale of 1 to 10, how would you describe your algebra skills right now, with 1 being absolutely uncomfortable and 10 being I have mastery in it.

7. What comes to mind when I say the word…
   a. Exponent
   b. Exponential function
   c. Logarithm
   d. Logarithmic function
e. Inverse functions

8. Describe what comes to mind when you see the following expressions and functions.
   a. $f(x) = x + 5$
   b. $h(x) = x^2 + 3x - 5$
   c. $y = mx + b$
   d. $g(x) = 3^{x+1}$

9. Do you have any concerns / questions about the research project, interviews or Post-test?

   Let’s look at your work on the problems on the Pre-test and talk about some of them…
Appendix D

INTERVIEW 2 QUESTIONS

1. How would you define what a logarithm is? Can you give an example of how logarithms may be used or why they are important?

2. On a scale of 1 to 10, how comfortable do you feel about explaining logarithms to someone else?
   - 1 = I feel absolutely uncomfortable and would not be able to explain them to another student
   - 5 = I feel confident performing some basic operations, but sometimes I get lost or stuck.
   - 10 = I have a good understanding of logarithms and could explain a lot about them to another student.

3. Let’s talk about what you might be having difficulty with when it comes to logarithms. Can you identify specific issues you’re struggling with?

4. On a scale of 1 to 5, how much time have you spent outside of class so far working on / studying logarithms (e.g., doing homework on logarithms, getting help from a tutor on logarithms, getting help on logarithms during your instructor’s office hours)?
   - 1 = No time
   - 2 = 0 - 2 hours
   - 3 = 2 - 4 hours
   - 4 = 4 - 6 hours
   - 5 = More than 6 hours

5. Let’s work on some logarithm problems. As you work on the problems, please describe each step you are taking and explain why you are doing it.
   a. Simplify / condense the following logarithmic expressions:
      a. \( \log_2 (3) - \log_2 (5) = ?? \)
b. \( \log_5(x) + \log_5(y) = ?? \)
c. \( \frac{1}{2} \log(100) = ?? \)

b. Expand the following logarithmic expressions:
   a. \( \ln(3x) \)
   b. \( \log\left(\frac{xy}{z}\right) \)
   c. \( \log_5(x^2) \)

c. Solve for \( x \):
   a. \( \log(100) = x \)
   b. \( \ln(0) = x \)
   c. \( \log_5(x) + \log_5(4x-1) = 1 \)
   d. \( \log_2(x+1) - \log_2(4) = 1 \)
   e. \( \ln(x^2) = \ln(2x-1) \)
Appendix E

INTERVIEW 3 QUESTIONS

1. Now that your class has completed the logarithms lessons, let’s talk some about your current understanding of logarithms.
   a. How would you define a logarithm?
   b. When are logarithms useful / why do we need them?
   c. What properties do you know about logs?

2. Simplify or expand the following expressions and explain each step / your thinking aloud.
   a. \(\log(x) + \log(y) - \log(z)\)
   b. \(\log(x+2) - (x+2)\)
   c. \(\ln(e)\)
   d. \(\log_2(xy)\)
   e. \(3\log_5(x)\)
   f. \(\ln(xy^3)\)

3. Solve for \(x\) and explain each step / your thinking aloud.
   a. \(\ln(x-1)=0\)
   b. \(\log_2(2x-3) = 2\)
   c. \(\log_3(x) + \log_3(x-2) = 1\)
   d. \(2\log_4(x) = 0\)
   e. \(\log(x) - 2\log(x) = 2\)

4. Think of a specific issue or problem that you struggled with logarithms. Do you still have issues or confusions about it? Why do you think you initially had trouble with that piece? How did you solve those problems?

5. How has your perception of or opinions about logarithms changed over the course of the semester? Do you feel like you have an understanding of the material? Do you feel like you could apply logarithms in real-life situations?

6. How would you describe the relationship between logarithmic functions and exponential functions?
7. Describe what each part of a logarithm represents.
   a. What does the argument (the “inside”) of the logarithm represent?
   b. What does the base of a logarithm represent?
   c. What does setting a logarithm equal to something represent? What is that value?
Appendix F

PRE-TEST AND POST-TEST CONSENT FORM

INFORMED CONSENT TO PARTICIPATE IN RESEARCH

Title of Project: An Investigation into College Students’ Learning about Logarithmic Functions: A Thorny Problem

Principal Investigator(s): Alexander Frketic, Dawn Berk, and Tammy Rossi

You are being invited to participate in a research study. This consent form tells you about the study including its purpose, what you will be asked to do if you decide to take part, and the risks and benefits of being in the study. Please read the information below and ask us any questions you may have before you decide whether or not you agree to participate. Your participation is voluntary, and you can refuse to participate or withdraw at any time without penalty or loss of benefits to which you are otherwise entitled. If you decide to participate, you will be asked to sign this form and a copy will be given to you to keep for your reference.

WHAT IS THE PURPOSE OF THIS STUDY?
This is a Senior Thesis project conducted by Alex Frketic. The purpose of this study is to better understand students’ obstacles in developing fluency with logarithmic expressions.
We estimate that approximately 400 MATH 115 students will participate in the pre/Post-test components of the study and 20 participants in the interview component. You are being invited to participate because you are enrolled in MATH 115, Pre-Calculus, at the University of Delaware. The only reason why you would be excluded from volunteering in this study is if you drop MATH 115.

WHAT WILL YOU BE ASKED TO DO?
This study consists of two components: a pre/Post-test component and an interview component. This consent form describes the pre/Post-test component. If you qualify for the interview component, you will receive an additional consent form describing that component of the study.

All MATH115 students will receive an email invitation to take a 30-60 minute Pre-test in early to mid-September. Several testing times will be offered to accommodate students’ schedules. The Pre-test will consist of about eight items assessing your current understanding of logarithms.

At the end of the semester, all MATH115 students will receive an email invitation to sign up for the Post-test. Again, several testing times will be offered to accommodate
students’ schedules. The Post-test will also take 30-60 minutes and will be very similar to the Pre-test.

WHAT ARE THE POSSIBLE RISKS AND DISCOMFORTS?
This research involves no risks to participants.

WHAT ARE THE POTENTIAL BENEFITS?
You will not directly benefit from taking part in this research. However, the knowledge gained from this study may contribute to our understanding of students’ perceptions of logarithms and what obstacles and misconceptions they have when learning them.

NEW INFORMATION THAT COULD AFFECT YOUR PARTICIPATION:
None

HOW WILL CONFIDENTIALITY BE MAINTAINED? WHO MAY KNOW THAT YOU PARTICIPATED IN THIS RESEARCH?
Only the project staff and your MATH 115 instructor will know that you participated in the research. All student names will be replaced by code numbers. An electronic file linking names and code numbers will be encrypted and stored on a secure, university-maintained server. Paper copies of the consent forms and data will be stored in a locked file cabinet in an office on campus. Electronic copies of all data will be stored in password-protected files on a secure university-maintained server. Only the project staff will have access to the secured data. All data will be kept indefinitely.

The research team will make every effort to keep all research records that identify you confidential. The findings of this research may be presented or published. If this happens, no information that reveals your identity will be shared.

The confidentiality of your records will be protected to the extent permitted by law. Your research records may be viewed by the University of Delaware Institutional Review Board, which is a committee formally designated to approve, monitor, and review biomedical and behavioral research involving humans. Records relating to this research will be kept for at least three years after the research study has been completed.

WILL THERE BE ANY COSTS TO YOU FOR PARTICIPATING IN THIS RESEARCH?
There are no costs to you for participating in this research study.

WILL YOU RECEIVE ANY COMPENSATION FOR PARTICIPATION?
If you participate in this component of the research, you will receive compensation for your time and effort. Students who complete the Pre-test will earn 5 course points; students who complete the Post-test will earn an additional 5 course points. Thus,
students that complete both the pre/Post-test will earn a total of 10 course points. Students who do not consent to participate will be able to earn these 10 points by completing other assignments from their MATH115 instructor at the end of the semester. The other assignments will be of comparable demand on your time and effort.

DO YOU HAVE TO TAKE PART IN THIS STUDY?
Taking part in this research study is entirely voluntary. You do not have to participate in this research. If you choose to take part, you have the right to stop at any time. If you decide not to participate or if you decide to stop taking part in the research at a later date, there will be no penalty or loss of benefits to which you are otherwise entitled. Your decision to stop participation, or not to participate, will not influence current or future relationships with the University of Delaware. As a student, if you decide not to take part in this research, your choice will have no effect on your academic status or your grade in the class.

WHO SHOULD YOU CALL IF YOU HAVE QUESTIONS OR CONCERNS?
If you have any questions about this study, please contact the Principal Investigator, Alex Frketic, at (302) 258-8879 or alexfrk@udel.edu. In addition, you may contact my Senior Thesis advisor, Dr. Dawn Berk, at (302) 831-6813 or berk@udel.edu. If you have any questions or concerns about your rights as a research participant, you may contact the University of Delaware Institutional Review Board at hsrb-research@udel.edu or (302) 831-2137.

Your signature on this form means that: 1) you are at least 18 years old; 2) you have read and understand the information given in this form; 3) you have asked any questions you have about the research and the questions have been answered to your satisfaction; and 4) you accept the terms in the form and volunteer to participate in the study. You will be given a copy of this form to keep.

____________________________  ______________________  ____________
Printed Name of Participant  Signature of Participant  Date

____________________________  ______________________  ____________
Person Obtaining Consent  Person Obtaining Consent  Date
(PRINTED NAME)  (SIGNATURE)
Appendix G

INTERVIEW CONSENT FORM

INFORMED CONSENT TO PARTICIPATE IN RESEARCH

Title of Project: An Investigation into College Students’ Learning about Logarithmic Functions: A Thorny Problem

Principal Investigator(s): Alexander Frketic, Dawn Berk, and Tammy Rossi

You are being invited to participate in a research study. This consent form tells you about the study including its purpose, what you will be asked to do if you decide to take part, and the risks and benefits of being in the study. Please read the information below and ask us any questions you may have before you decide whether or not you agree to participate. Your participation is voluntary, and you can refuse to participate or withdraw at any time without penalty or loss of benefits to which you are otherwise entitled. If you decide to participate, you will be asked to sign this form and a copy will be given to you to keep for your reference.

WHAT IS THE PURPOSE OF THIS STUDY?
This is a Senior Thesis project conducted by Alex Frketic. The purpose of this study is to better understand students’ obstacles in developing fluency with logarithmic expressions. We estimate that approximately 400 MATH 115 students will participate in the pre/Post-test components of the study and 20 participants in the interview component. You are being invited to participate because you are enrolled in MATH 115, Pre-Calculus, at the University of Delaware. The only reason why you would be excluded from volunteering in this study is if you drop MATH 115.

WHAT WILL YOU BE ASKED TO DO?
This study consists of two components: a pre/Post-test component and an interview component. This consent form describes the interview component.

Approximately 20 MATH 115 students who have completed the Pre-test will be invited to participate in three one-on-one interviews. The 20 students will be selected based on their performance on the Pre-test. The interviews will be audio-recorded and transcribed.

The first interview will take place in early September; it will take about 30 minutes to complete and will probe your responses to the Pre-test. The second interview will take about one hour and will be scheduled mid to late October. The goal of this interview is to assess your understanding of the logarithm concepts that were addressed in the first few logarithm lessons in class. You will be asked to work with logarithmic
expressions and explain your thinking. You will also be asked to respond to some Likert items assessing your comfort with logarithms. The third interview will take about one hour and will be scheduled in November, after the logarithm lessons are completed. The goal of the third interview is to assess your understanding of logarithms. You will be asked to work on some logarithm problems similar to those in the second interview and to explain your thinking.

**WHAT ARE THE POSSIBLE RISKS AND DISCOMFORTS?**
This research involves no risks to participants.

**WHAT ARE THE POTENTIAL BENEFITS?**
You will not directly benefit from taking part in this research. However, the knowledge gained from this study may contribute to our understanding of students’ perceptions of logarithms and what obstacles and misconceptions they have when learning them.

**NEW INFORMATION THAT COULD AFFECT YOUR PARTICIPATION:**
None

**HOW WILL CONFIDENTIALITY BE MAINTAINED? WHO MAY KNOW THAT YOU PARTICIPATED IN THIS RESEARCH?**
All names will be replaced by code numbers. An electronic file linking names and code numbers will be encrypted and stored on a secure, university-maintained server. Paper copies of the consent forms and data will be stored in a locked file cabinet in an office on campus. Electronic copies of all data will be stored in password-protected files on a secure university-maintained server. Only the project staff will have access to the secured data. All data will be kept indefinitely. The research team will make every effort to keep all research records that identify you confidential. The findings of this research may be presented or published. If this happens, no information that reveals your identity will be shared.

Digital audio files from the interviews will be stored online in a password-protected folder on a secure university-maintained server. At no time will a student’s identity be revealed during these recordings. The transcripts will only include the participant’s code number (no names).

The confidentiality of your records will be protected to the extent permitted by law. Your research records may be viewed by the University of Delaware Institutional Review Board, which is a committee formally designated to approve, monitor, and review biomedical and behavioral research involving humans. Records relating to this
research will be kept for at least three years after the research study has been completed.

WILL THERE BE ANY COSTS TO YOU FOR PARTICIPATING IN THIS RESEARCH?
There are no costs to you for participating in this research study.

WILL YOU RECEIVE ANY COMPENSATION FOR PARTICIPATION?
If you participate in this component of the research, you will receive compensation for your time and effort. Students will earn 5 course points for participating in each of the first and second interviews (5 points per interview, for a total of 10 points). Students who complete all three interviews will receive a $30 Main Street gift card at the conclusion of the third interview. Students who do not consent to participate will be able to earn these 10 points by completing other assignments from their MATH 115 instructor at the end of the semester. The written assignments will be of comparable demand on your time and effort.

DO YOU HAVE TO TAKE PART IN THIS STUDY?
Taking part in this research study is entirely voluntary. You do not have to participate in this research. If you choose to take part, you have the right to stop at any time. If you decide not to participate or if you decide to stop taking part in the research at a later date, there will be no penalty or loss of benefits to which you are otherwise entitled. Your decision to stop participation, or not to participate, will not influence current or future relationships with the University of Delaware. As a student, if you decide not to take part in this research, your choice will have no effect on your academic status or your grade in the class.

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Your signature on this form means that: 1) you are at least 18 years old; 2) you have read and understand the information given in this form; 3) you have asked any questions you have about the research and the questions have been answered to your
satisfaction; and 4) you accept the terms in the form and volunteer to participate in the study. You will be given a copy of this form to keep.

<table>
<thead>
<tr>
<th>Printed Name of Participant</th>
<th>Signature of Participant</th>
<th>Date</th>
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<tr>
<td>Person Obtaining Consent (PRINTED NAME)</td>
<td>Person Obtaining Consent (SIGNATURE)</td>
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Appendix H

A SIDE-BY-SIDE COMPARISON OF THE MEAN SCORES