SOME INVERSE PROBLEMS FOR HYPERBOLIC
PARTIAL DIFFERENTIAL EQUATIONS

by

Zachary J. Bailey

A dissertation submitted to the Faculty of the University of Delaware in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics

Spring 2018

© 2018 Zachary J. Bailey
All Rights Reserved
SOME INVERSE PROBLEMS FOR HYPERBOLIC
PARTIAL DIFFERENTIAL EQUATIONS

by

Zachary J. Bailey

Approved: _______________________________________________________
Louis Rossi, Ph.D.
Chair of the Department of Mathematical Sciences

Approved: _______________________________________________________
George Watson, Ph.D.
Dean of the College of Arts and Sciences

Approved: _______________________________________________________
Ann L. Ardis, Ph.D.
Senior Vice Provost for Graduate and Professional Education
I certify that I have read this dissertation and that in my opinion it meets the academic and professional standard required by the University as a dissertation for the degree of Doctor of Philosophy.

Signed: ____________________________________________
Rakesh, Ph.D.
Professor in charge of dissertation

I certify that I have read this dissertation and that in my opinion it meets the academic and professional standard required by the University as a dissertation for the degree of Doctor of Philosophy.

Signed: ____________________________________________
Fioralba Cakoni, Ph.D.
Member of dissertation committee

I certify that I have read this dissertation and that in my opinion it meets the academic and professional standard required by the University as a dissertation for the degree of Doctor of Philosophy.

Signed: ____________________________________________
David L. Colton, Ph.D.
Member of dissertation committee

I certify that I have read this dissertation and that in my opinion it meets the academic and professional standard required by the University as a dissertation for the degree of Doctor of Philosophy.

Signed: ____________________________________________
Yuk J. Leung, Ph.D.
Member of dissertation committee
ACKNOWLEDGEMENTS

First and foremost I would like to thank my thesis advisor, Dr. Rakesh. His guidance, patience, and wisdom helped to mold me into the mathematician I am today. My knowledge and understanding of mathematics has grown considerably over the last five years and I owe most of that growth to Rakesh.

I would also like to thank all of my fellow graduate students for their support, senses of humor, and understanding. Additionally I want to thank all the great professors I had and worked with at the University of Delaware. A big thanks is due to Professors Cakoni, Colton, Leung for taking the time to serve on my dissertation committee.
TABLE OF CONTENTS

LIST OF FIGURES .................................................. viii
ABSTRACT ......................................................... ix

Chapter

1 INTRODUCTION AND RESULTS ................................. 1

1.1 Background ......................................................... 1
1.2 1-D Hyperbolic System ......................................... 1
1.3 Backscattering in 3-D ........................................... 5
1.4 A Useful Proposition ........................................... 9

2 REFLECTION DATA IN A 1-D COMPLEX-VALUED
HYPERBOLIC SYSTEM ................................................. 10

2.1 Introduction ....................................................... 10
2.1.1 The Progressing Wave Expansion ......................... 11

2.2 Proof of Theorem 2.2 ............................................ 14
2.3 Proof of Theorem 2.1 ............................................ 16
2.3.1 Preliminaries ................................................ 16
2.3.2 Stability .................................................... 18
2.3.3 Continuity .................................................. 19

3 TRANSMISSION DATA IN A 1-D COMPLEX-VALUED
HYPERBOLIC SYSTEM ............................................... 22

3.1 Introduction ..................................................... 22
3.2 The Relationship Between $A, B, L$, and $R$ ................. 27
3.3 Proof of Theorem 3.2 .......................................... 30
3.4 Proof of Theorem 3.1 ........................................ 32
  3.4.1 Local Inversion ........................................ 32
  3.4.2 Global Inversion ........................................ 38
3.5 Proof of Proposition 3.3 ................................. 41

4 OFFSET SOURCE-RECEIVER BACKSCATTERING ............ 44
  4.1 Introduction .............................................. 44
  4.2 Proof of Theorem 4.2 .................................... 46
    4.2.1 Identity for the Difference of Two Solutions .......... 46
    4.2.2 Weighted Ellipsoidal Mean Value ...................... 49
    4.2.3 Stability Estimate ................................... 52
  4.3 Proof of Lemma 4.1 ...................................... 57
    4.3.1 Parametrizing $\partial E(a,\tau)$ ....................... 57
    4.3.2 Bounding the Terms $D$ and $J$ and Derivatives on $\partial E(a,\tau)$ .. 58

5 SPHERICAL WAVE BACKSCATTERING ......................... 62
  5.1 Introduction and Results ................................ 62
  5.2 Proof of Theorem 5.1 .................................... 65
    5.2.1 The Progressing Wave Expansion ...................... 65
    5.2.2 The Transport Equation .............................. 67
  5.3 Proof of Theorem 5.2 .................................... 68
    5.3.1 The Goursat Problem ................................ 68
    5.3.2 An Identity for the Difference of Two Solutions .... 71
    5.3.3 Parametrizing $\partial H(a,\tau)$ for $-1 < \tau < 1$ .... 74
    5.3.4 Estimating $p$ by $M(p)$ ........................... 77
    5.3.5 The Volterra Estimate .............................. 78
    5.3.6 Stability Estimate ................................. 79

6 UNIQUENESS FROM BACKSCATTERING DATA FOR SMALL
   COEFFICIENTS ............................................. 86
  6.1 Introduction .............................................. 86
  6.2 Proof of Part (a) of Theorem 6.1 ........................ 88
6.3 Part (b) of Theorem 6.1 ........................................ 90
6.4 Part (b) of Theorem 6.2 ........................................ 93

BIBLIOGRAPHY ...................................................... 95
## LIST OF FIGURES

2.1 The triangle $OAB$ ................................................. 15
2.2 The trapezoid $OPQB$ ............................................. 17
3.1 Characteristics for $L$ .................................................. 24
3.2 Energy Estimate on $OABC$ .......................................... 30
3.3 Region in Lemma 3.2 .................................................. 32
3.4 Energy Estimates on $D_0C_0BA$ and $PQRS$ .................. 41
4.1 The upper half of $E(a, \tau)$ ....................................... 50
5.1 Exterior of the light cone, $-|x| < t < |x| < 1$ ....................... 69
5.2 Downward facing light cone, $-1 \leq t \leq -|x|$ ................. 70
5.3 Relation between $a \in S$ and $x \in \partial H(a, \tau)$ ............... 75
We consider four inverse problems for hyperbolic PDEs with two of them associated with one space dimension and two of them associated with three space dimensions.

The first two problems are inverse problems associated to one space dimensional hyperbolic systems of PDEs with complex coefficients where the goal is the recovery of a single complex coefficient from either the reflection data or the transmission data. We show that the map sending the coefficient to the reflection/transmission data is injective and stable and we also characterize the range of this map for the transmission data case.

The other two problems are associated with a single hyperbolic PDE with a zero order coefficient and the goal is the recovery of this coefficient from two different types of “backscattering data” - backscattering data coming from a fixed offset distribution of sources and receivers on the boundary or backscattering data coming from a single incoming spherical wave. For these problems we prove a stability result provided the difference of the two coefficients is horizontally or angularly controlled respectively.

Our work adapts the techniques used by Eemeli Blästen, Rakesh and Gunther Uhlmann to solve problems similar to theirs.
Chapter 1
INTRODUCTION AND RESULTS

1.1 Background

In this thesis, we study problems originating in two different areas. Motivated by applications to optics, we study two one dimensional inverse problems for a hyperbolic system with complex coefficients. Motivated by applications in seismology we study two types of formally determined inverse problems for the three dimensional wave equation.

Throughout the thesis $B$ will denote the open unit ball in $\mathbb{R}^3$, $S$ will denote the unit sphere which is also the boundary of $B$, $\mathbb{R}^3_+$ will denote the half-space $x_3 \geq 0$ in $\mathbb{R}^3$, $e_1 = (1,0,0)$, $e_3 = (0,0,1)$, and $\Box$ will denote the wave operator $\partial_{tt} - \Delta_x$ on $\mathbb{R}^n \times \mathbb{R}$.

1.2 1-D Hyperbolic System

We study two inverse problems for a 1-D complex hyperbolic system of PDEs. In both problems we consider an acoustic medium occupying the interval $[0, \infty)$ and a complex valued function $q(x)$ on $[0, \infty)$ representing some acoustic property of the medium. We assume that the medium is uniform for $x > X$ for some $X > 0$ and that $q(x) = 0$ for $x > X$. The medium is excited by a point source at $x = 0$ and we measure
the medium response either at \( x = 0 \) or at \( x = X \) and the goal is the recovery of \( q(\cdot) \).

For both problems \( L(x, t) \) and \( R(x, t) \) will represent the left and right moving waves in the acoustic medium.

For the first problem, the interaction of the waves with the medium is modeled by the following initial boundary-value problem (IBVP) where the boundary condition represents the point source exciting the medium.

\[
\begin{align*}
L_x - L_t &= qR, \quad x \geq 0, t \in \mathbb{R} \\
R_x + R_t &= \overline{q}L, \quad x \geq 0, t \in \mathbb{R} \\
(L + R)(0, t) &= \delta(t), \quad t \in \mathbb{R} \\
L = R &= 0, \quad t < 0.
\end{align*}
\] (1.1)

When \( q \) is complex-valued, the PDE models a one dimensional diffraction grating, see [20], [17], [30], and when \( q \) is real-valued the PDE is related to Webster’s horn equation - see [23] for details.

When \( q \) is real valued, Symes in [31] and [32] (also see [3], [4], [6], [28] for related material) studied the inverse problem of recovering \( q \) given the reflection data. In these papers, Symes proved that the map

\[ q(x) \mapsto L(0, t)|_{t \in [0, 2X]} \]

is injective, locally Lipschitz continuous with a locally Lipschitz continuous inverse. In addition, he characterized the range of this map. Symes tackled these problems using a downward continuation method which utilizes the fact that the one dimensional hyperbolic system is well posed even when the roles of time and space variables are reversed.

Rakesh in [23] studied (1.1)-(1.4), for complex-valued \( q \in L^2[0, X] \), the problem of recovering \( q \) from the reflection data. Rakesh proved that the map

\[ q(x) \mapsto L(0, t)|_{t \in [0, 2X]} \]
is injective, locally Lipschitz continuous with a locally Lipschitz continuous inverse. In addition, he characterized the range of this map.

In chapter 3, we study the inverse problem associated with the transmission data where the source is at $x = 0$ but the receiver is at $x = X$. The goal is the inversion of the map

\[ q \rightarrow R(X, \cdot). \]

**When $q$ is real**, Carroll and Santosa showed in [7] that if the transmission data, $R(X,t)$ is known for all real $t$ then one can recover reflection data $L(0,t)$ for all $t$, hence one can recover $q$ by using results known for reflection data. For a piecewise constant medium, Claerbout in [8] had already provided a complete solution of the problem, including showing that one needed $R(X,t)$ only for $X \leq t \leq 3X$ to recover $q$. Then Rakesh and Sacks adapted Claerbout’s ideas to the continuous case in [25] and [22]. They proved uniqueness, constructed an inverse and indirectly characterized the range of this map.

In chapter 3 we study the **complex $q$ case** for the transmission data problem. By adapting the method in [25] and [22] to the complex case, we show that the nonlinear map

\[ \mathcal{F} : L^2[0,X] \rightarrow L^2[X,3X] \]

\[ q(\cdot) \mapsto R(X, \cdot) \]

is injective and one may characterize the range of this map and construct its inverse.

For the second problem, the interaction of the waves with the medium is modeled by a small modification of (1.1)- (1.4), the $\bar{q}$ in (1.2) is replaced by $-\bar{q}$ and this has a significant impact on the behavior of the solutions of the system. Specifically suppose
\(L(x, t), R(x, t)\) are solutions of the IBVP

\[
\begin{align*}
L_x - L_t &= qR, \\ x &\geq 0, \ t \in \mathbb{R} \\
R_x + R_t &= -\bar{q}L, \\ x &\geq 0, \ t \in \mathbb{R} \\
L &= R = 0, \\ x &\geq 0, \ t < 0 \\
(L + R)(0, t) &= \delta(t), \quad t \in \mathbb{R}.
\end{align*}
\]  

Our goal is to study inversion from reflection data for this problem, that is study the inversion of the map

\(q \to L(0, \cdot).\)

Belishev in [1] studied this problem using his Boundary Control method. He proved the map was injective, characterized its range and constructed its inverse. We struggled to understand some of his arguments and we wanted to obtain the results based on, at least in our opinion, the simpler downward continuation idea of Symes. In chapter 2, we show that when the domain of this map is restricted to bounded functions the map is injective, continuous and stable and we give a procedure for constructing its inverse. We were unable to obtain optimal results which would include a characterization of the range of this map for the appropriate domain.

Inversion from both reflection data and transmission data for the system (1.1)-(1.4) has been thoroughly studied. Gel’fand, Levitan, Krein, Symes (see [31]), proved that one can uniquely recover a real-valued \(q\) from reflection data. Rakesh in [23] proved that one can uniquely recover a complex-valued \(q\) from reflection data. Rakesh and Sacks in [25] and [22] studied inversion from transmission data for real-valued \(q\). They proved that one can uniquely recover a real-valued \(q\) from transmission data. In chapter 3, we prove that one can uniquely recover a complex-valued \(q\) from transmission data.
Inversion from either reflection data or transmission data has not been as thoroughly studied for the system (1.5)-(1.8). This is due to the fact that the system (1.1)-(1.4) directly corresponds to a second order hyperbolic PDE, but (1.5)-(1.8) does not. Belishev in [1] proved injectivity for the reflection data inverse problem in (1.5)-(1.8), and in chapter 2 we prove the same result using a different method. We think that the methods used in chapter 3 would also apply to the transmission data inverse problem for (1.5)-(1.8), but we are not aware of any published results on the subject.

1.3 Backscattering in 3-D

Our problem in $\mathbb{R}^3$ is loosely motivated by applications to geophysics. Our goal is to study formally determined\(^1\) inverse problems in $\mathbb{R}^3$ for hyperbolic PDEs and backscattering problems of various types are perhaps the simplest of these, even though all these inverse backscattering problems remain unsolved. We give partial results for two types of backscattering problems.

The well-known backscattering problem consists of the following. Suppose $q$ is a smooth real valued function on $\mathbb{R}^3$ with support in $B$ (q represents some acoustic property of the medium) and, for each unit vector $\omega \in \mathbb{R}^3$, let $U(x, t, \omega)$ be the response of the medium to a plane wave coming in from the direction $\omega$, that is $U(x, t, \omega)$ is the solution of the IVP

$$\begin{align*}
(\Box - q)U(x, t, \omega) &= 0, & (x, t) &\in \mathbb{R}^3 \times \mathbb{R} \\
U(x, t) &= \delta(t - x \cdot \omega), & x &\in \mathbb{R}^3, \ t < -1.
\end{align*}$$

If one defines

$$u(x, t, \omega) = U(x, t, \omega) - \delta(t - x \cdot \omega)$$

\(^1\) An inverse problem is said to be formally determined if the parameter count of the data equals the parameter count of the unknown function. Some inverse problems are over-determined and our focus is on formally determined inverse problems.
then a good candidate for the far-field data in the direction of the unit vector $\theta$ and delay $s \in \mathbb{R}$ is

$$\alpha(\theta, \omega, s) := \lim_{r \to \infty} ru(r \theta, r - s, \omega),$$

where the limit is known to exist as a distribution in $s$ (see [26]). So the backscattering data is defined to be

$$\beta(\omega, s) := \alpha(-\omega, \omega, s), \quad s \in \mathbb{R}, \quad \omega \in \mathbb{R}^3, \quad |\omega| = 1.$$

The inverse backscattering problem is the inversion of the map

$$q \to \beta(\omega, s),$$

which remains one of the major open problems in the field of Inverse Problems - even the injectivity of this map has not been established.

In [26], Rakesh and Uhlmann took a small step towards proving the injectivity of this map. They showed that if $q_1 \neq q_2$ and $q_1 - q_2$ is angularly controlled then the backscattering data for $q_1$ and $q_2$ would also be different (as functions of $\omega$ and $s$). Note that a function $p(x)$ on $\mathbb{R}^3$ with support in $B$ is said to be angularly controlled if there is constant $C$ independent of $\rho$ such that

$$\int_{|x| = \rho} |(x_i \partial_j - x_j \partial_i) p(x)|^2 dS_x \leq C \int_{|x| = \rho} |p(x)|^2 dS_x, \quad \forall \rho \in (0, 1), \quad \forall i < j. \quad (1.9)$$

Please see [26], [27] for a characterization of functions with angular control in terms of their spherical harmonic expansions, and also for examples.

In [27], Rakesh and Uhlmann studied an inverse problem which could also be considered a type of inverse backscattering problem. Suppose $q(x)$ is a smooth function in $\mathbb{R}^3$ with support in $B$. For each $a \in S = \partial B$, let $U^a(x, t)$ be the medium response to a source located at $a$, that is $U^a(x, t)$ is the solution of the IVP

$$U_{tt}^a - \Delta U^a - q(x)U^a = \delta(x - a, t), \quad x \in \mathbb{R}^3, t \in \mathbb{R} \quad (1.10)$$

$$U^a(x, t) = 0, \quad t < 0. \quad (1.11)$$
So if the medium response is measured at the source location then the data could be considered backscattering data. Rakesh and Uhlmann considered the inverse backscattering problem of inverting the map

\[ q \rightarrow U^a(a, t)|_{a \in S, t > 0}. \]

They showed that if \( q_1 \neq q_2 \) and \( q_1 - q_2 \) is angularly controlled then the backscattering data for \( q_1 \) and \( q_2 \) must be different as a function on \( S \times (0, \infty) \). They also proved this result when \( q_1 \geq q_2 \) (without requiring \( q_1 - q_2 \) to be angularly controlled). Further, Blasten in [2] showed that this map is logarithmically stable, again when \( q_1, q_2 \) are restricted so that \( q_1 - q_2 \) is angularly controlled.

However, the injectivity of the map for general \( q \) is an open question. We take a small step forward for the general \( q \) case. In chapter 6 we prove that the map

\[ q \rightarrow U^a(a, t)|_{a \in S, t > 0}. \]

is injective for small \( q \). This requires an appeal to an identity from [21] and [12] about recovering a function supported in a ball from its spherical mean values over all spheres centered on the unit sphere.

Motivated by the above results and problems, in this thesis we study two other types of inverse back-scattering problems.

**Offset source-receiver pairs**

Below \( A \) will represent the hyperplane \( x_3 = 0 \) in \( \mathbb{R}^3 \). Consider a compactly supported smooth function \( q(x) \) on \( \mathbb{R}^3_+ \) representing some acoustic property of the medium occupying \( \mathbb{R}^3_+ \). For a fixed \( h > 0 \), and for any \( a \in A \), the medium is probed by a source located at \( a - he_1 \) and the medium response is measured at \( a + he_1 \), so the \( a \) centered source-receiver pair is offset by a fixed vector \( 2he_1 \), as its center \( a \) is allowed to vary on
the hyperplane $A$. The goal is the recovery of $q$ from this offset backscattering data. More specifically, for each $a = (a_1, a_2, 0) \in A$, let $U^a(x, t)$ be the solution of the IBVP

$$
\begin{align*}
\partial_t^2 U^a - \Delta U^a - q_j(x) U^a &= 0, \quad x \in \mathbb{R}_+^3, \ t \in \mathbb{R} \\
\partial_{x_3} U^a(x_1, x_2, 0, t) &= \delta(x_1 - a_1 - h, x_2 - a_2, t), \quad x \in \partial \mathbb{R}_+^3, \ t \in \mathbb{R} \\
U^a(x, t) &= 0, \quad t < 0.
\end{align*}
$$

The goal is the inversion of the offset “backscattering” map

$$
q \to U^a(a + he_1, t)|_{a \in A, t \in \mathbb{R}}.
$$

In chapter 4 we prove results similar to the results in [27] and [2]. We prove that this map is injective and logarithmically stable provided $q_1 - q_2$ is horizontally controlled in the sense that there is a constant $C$ independent of $x_3$ such that

$$
\int_{\mathbb{R}^2} |\partial_i(q_1 - q_2)(x_1, x_2, x_3)|^2 \, dx_1 \, dx_2 \leq C \int_{\mathbb{R}^2} |(q_1 - q_2)(x_1, x_2, x_3)|^2 \, dx_1 \, dx_2, \quad \forall x_3 > 0, \ i = 1, 2.
$$

We have also made some progress for the general $q$ problem. We have uniqueness results when $q_1 \geq q_2$ or when $h = 0$ and $q_1, q_2$ are small - where an identity from [5] plays an important role.

**Incoming spherical wave**

Again the medium is represented by a smooth function $q(x)$ on $\mathbb{R}^3$ with support in $B$. The medium is probed by an incoming spherical wave and the medium response is measured on $\partial B$ - hence this may also be considered as a type of back-scattering data. More specifically, let $U(x, t)$ be the solution of the IVP

$$
\begin{align*}
U_{tt} - \Delta U - q(x) U &= 0, \quad x \in \mathbb{R}^3, \ t \in \mathbb{R} \\
U(x, t) &= \frac{2\delta(t + |x|)}{|x|}, \quad t < -1
\end{align*}
$$

and the goal is the inversion of the map

$$
q \to U_{\partial B \times (-1, 1)}.
$$
As before, in chapter 5, we prove the injectivity and the logarithmic stability of this map when $q_1 - q_2$ is angularly controlled.

1.4 A Useful Proposition

Since it is easier to manipulate integrals on $\mathbb{R}^3$ than surface integrals, we will occasionally convert surface integrals to integrals on $\mathbb{R}^3$ with the help of the following well known proposition, see [15], [19].

**Proposition 1.1.** Suppose $\varphi(x)$ is a smooth function on $\mathbb{R}^n$ and $\nabla \varphi(x) \neq 0$ whenever $\varphi(x) = 0$. Then for any $f \in C^\infty_c(\mathbb{R}^n)$, we have

\[
\int_{\mathbb{R}^n} f(x)|\nabla \varphi(x)| \delta(\varphi(x)) \, dx = \int_{\varphi(x)=0} f(x) \, dS_x.
\]
Chapter 2

REFLECTION DATA IN A 1-D COMPLEX-VALUED HYPERBOLIC SYSTEM

2.1 Introduction

Consider an acoustic medium represented by the one-dimensional interval $0 \leq x \leq X$. We probe the medium by an acoustic wave generated by a source at the left end, $(x = 0)$, and we wish to recover the acoustic property of the medium from the medium response also at the left end over a finite time interval.

Let $q(x)$ be a complex-valued function representing the acoustic property we seek and let $L(x, t)$ and $R(x, t)$ represent the left and right-moving waves at a displacement $x$ and time $t$. The experiment is modeled as the IBVP

$$L_x - L_t = qR, \quad x \geq 0, \ t \in \mathbb{R} \quad (2.1)$$
$$R_x + R_t = -\overline{q}L, \quad x \geq 0, \ t \in \mathbb{R} \quad (2.2)$$
$$L = R = 0, \quad x \geq 0, \ t < 0 \quad (2.3)$$
$$(L + R)(0, t) = \delta(t), \quad t \in \mathbb{R}. \quad (2.4)$$

Our goal is to recover the property $q(x)$ given the data $\frac{1}{2}(L - R)(0, t)$. 

2.1.1 The Progressing Wave Expansion

We guess that the solution of (2.1)-(2.4) will have the form

\[ L(x, t) = l(x, t)H(t - x) \quad (2.5) \]
\[ R(x, t) = r(x, t)H(t - x) + \delta(t - x), \quad (2.6) \]

for some functions \( l(x, t) \) and \( r(x, t) \). Then

\[ L_x(x, t) = l_x(x, t)H(t - x) - l(x, t)\delta(t - x) \]
\[ L_t(x, t) = l_t(x, t)H(t - x) + l(x, t)\delta(t - x) \]
\[ R_x(x, t) = r_x(x, t)H(t - x) - r(x, t)\delta(t - x) - \delta'(t - x) \]
\[ R_t(x, t) = r_t(x, t)H(t - x) + r(x, t)\delta(t - x) + \delta'(t - x). \]

So (2.1) is equivalent to

\[ (l_x - l_t)H(t - x) - 2l\delta(t - x) = qrH(t - x) + q\delta(t - x), \]

that is

\[ (l_x - l_t)H(t - x) = qrH(t - x) + (q + 2l)\delta(t - x). \]

This forces \( 2l = -q \) on \( t = x \) and \( l_x - l_t = qr \) on \( x \leq t \). Also (2.2) is equivalent to

\[ (r_x + r_t)H(t - x) = -qrH(t - x), \]

which forces \( r_x + r_t = -qr \) on \( x \leq t \). Finally, (2.4) becomes

\[ (l + r)(0, t)H(t) + \delta(t) = \delta(t), \]

forcing

\[ (l + r)(0, t) = 0, \quad t \geq 0. \]
Thus (2.5)-(2.6) is a solution of (2.1)-(2.4) if \( l(x, t), r(x, t) \) solves the characteristic BVP

\[
\begin{align*}
 l_x - l_t &= qr, \quad 0 \leq x \leq t \quad (2.7) \\
r_x + r_t &= -\overline{q}l, \quad 0 \leq x \leq t \quad (2.8) \\
(l + r)(0, t) &= 0, \quad t \in \mathbb{R} \quad (2.9) \\
l(x, x) &= -\frac{q(x)}{2}, \quad x \geq 0. \quad (2.10)
\end{align*}
\]

Since
\[
\frac{1}{2} (L - R)(0, t) = -\frac{1}{2} \delta(t) + \frac{1}{2} (l - r)(0, t),
\]
knowing \((L - R)(0, \cdot)\) is equivalent to knowing \((l - r)(0, \cdot)\), and since \((l + r)(0, \cdot) = 0\), knowing \((l - r)(0, \cdot)\) is equivalent to knowing \(l(0, \cdot)\). This suggests the following.

The existence and uniqueness of solutions \( l, r \in L^2 \) with \( L^2 \) traces on horizontal, vertical, and slope \( \pm 1 \) lines, given \( q \in L^\infty([0, X]; \mathbb{C}) \), can be shown using standard energy estimates for first-order hyperbolic systems, see [9], [11], [16], [18]. Define the forward map:

\[
\mathcal{F} : L^\infty[0, X] \rightarrow L^2[0, 2X]
\]

\[
q(\cdot) \mapsto l(0, \cdot).
\]

Our goal is to study the inversion of \( \mathcal{F} \). We prove that \( \mathcal{F} \) is continuous and \( \mathcal{F}^{-1} \) is continuous (i.e. stability) in the \( L^2 \) norm which also gives us injectivity.

\textbf{Remark.} We do not attempt to characterize the range of \( \mathcal{F} \) here. This was done in [1] using the \textit{boundary control method}. We use an approach similar to [6], [31], [32], [23] but have yet to determine the solvability for \( q \in L^2 \). If the RHS of (2.2) is \( +\overline{q}l \) instead of \(-\overline{q}l\), then we can extend the domain \( \mathcal{F} \) to all of \( L^2[0, X] \) and characterize the range as in [23].
Theorem 2.1. \( \mathcal{F} \) is injective, locally Lipschitz continuous, and has a locally Lipschitz continuous inverse in the \( L^2 \) norm on \( L^\infty[0,X] \). In other words, if \( q_i \in L^\infty[0,X], \ i = 1, 2, \) and \( L_i, R_i \), are solutions to (2.1)-(2.4) then there are constants \( C_1, C_2 > 0 \) so that

\[
\| q_1 - q_2 \|_{L^2[0,X]}^2 \leq C_1 \| (l_1 - l_2)(0,\cdot) \|_{L^2[0,2X]}^2,
\]

and

\[
\| (l_1 - l_2)(0,\cdot) \|_{L^2[0,2X]}^2 \leq C_2 \| q_1 - q_2 \|^2,
\]

where \( C_1 \) depends only on \( \| l_2(0,\cdot) \|_{L^2[0,2X]} \) and \( X \), and \( C_2 \) depends only on \( X \), \( \| q_i \|_\infty \).

We also have a bound on \( q \) in terms of \( \mathcal{F}(q) \):

Theorem 2.2. For any \( q \in L^\infty[0,X] \), if \( l, r \) satisfy (2.7)-(2.10), then

\[
\| q \|_{L^2[0,X]}^2 \leq 4 \| l(0,\cdot) \|_{L^2[0,2X]}^2 \leq (1 + 4X \| q \|_\infty) \| q \|_{L^2[0,X]}^2.
\]

We will make frequent use of the following lemma:

Lemma 2.1. Suppose \( D \) is an open subset of \( \mathbb{R}^2 \) and \( u, v \) is a solution of the system

\[
\begin{align*}
 u_x - u_t &= f, \quad (x,t) \in D \\
 v_x + v_t &= g, \quad (x,t) \in D.
\end{align*}
\]

Then for any \( (x,t) \in D \),

\[
(|u|^2 + |v|^2)_x - (|u|^2 - |v|^2)_t = 2 \text{Re}(\bar{u}f + v\bar{g}),
\]

and

\[
(|u|^2 - |v|^2)_x - (|u|^2 + |v|^2)_t = 2 \text{Re}(\bar{u}f - v\bar{g}).
\]

13
Proof. To show (2.11), compute
\[
(|u|^2 + |v|^2)_x - (|u|^2 - |v|^2)_t \\
= (u\overline{u}_x + u_x\overline{u}) + (v\overline{v}_x + v_x\overline{v}) - (u\overline{u}_t + u_t\overline{u}) + (v\overline{v}_t + v_t\overline{v}) \\
= 2 \text{Re} (\overline{u}(u_x - u_t) + v(\overline{v}_x + \overline{v}_t)) \\
= 2 \text{Re}(\overline{u}f + v\overline{g})
\]
The proof of (2.12) is similar. \qed

In particular, if \( u = l, \ v = r \) are the solutions of (2.7)-(2.10), then \( f = qr, \ g = -\overline{q}l \), which gives
\[
(|l|^2 + |r|^2)_x - (|l|^2 - |r|^2)_t = 0,
\]
and
\[
(|l|^2 - |r|^2)_x - (|l|^2 + |r|^2)_t = 4 \text{Re}(\overline{l}qr).
\]
The proofs in this chapter rely on using Stokes’ theorem and integrating these identities over various sections of the triangle (see figure 2.1)

\[
OAB = \{(x,t) \in \mathbb{R}^2 : 0 \leq x \leq X, \ x \leq t \leq 2X - x \}.
\]

2.2 Proof of Theorem 2.2

For the lower bound, integrate (2.13) over OAB to get
\[
0 = \iint_{OAB} (|l|^2 + |r|^2)_x - (|l|^2 - |r|^2)_t \, dx \, dt \\
= \int_{\partial OAB} (|l|^2 - |r|^2) \, dx + (|l|^2 + |r|^2) \, dt \\
= 2 \int_{OA} |l|^2 \, dx - 2 \int_{AB} |r|^2 \, dx + \int_{BO} |l|^2 + |r|^2 \, dt \\
= 2 \int_{OA} |l|^2 \, dx + 2 \int_{BA} |r|^2 \, dx - \int_{OB} |l|^2 + |r|^2 \, dt.
\]
Define
\[ h(x) := r(x, 2X - x). \]

From (2.9)-(2.10), \( l = -q/2 \) on \( OA \) and \( r = -l \) on \( OB \). Thus
\[
\frac{1}{2} \|q\|_{L^2[0,X]}^2 + 2\|h\|_{L^2[0,X]}^2 = 2\|l(0, \cdot)\|_{L^2[0,2X]}^2,
\]
which yields
\[
\|q\|_{L^2[0,X]}^2 \leq 4\|l(0, t)\|_{L^2[0,2X]}^2.
\]

For the upper bound fix \( \tau \) and, integrate (2.14) on \( OAQR \), noting that \( |l|^2 - |r|^2 = 0 \)

on \( OR \), to get
\[
\iint_{OAQR} 4 \text{Re}(q^* r) \, dx \, dt = \int_{\partial OAQR} (|l|^2 + |r|^2) \, dx + (|l|^2 - |r|^2) \, dt
\]
\[
= 2 \int_{OA} |l|^2 \, dx - 2 \int_{QA} |r|^2 \, dx - \int_{RQ} |l|^2 + |r|^2 \, dx.
\]

Define the energy at time \( \tau \),
\[
E(\tau) := 2 \int_{QA} |r|^2 \, dx + \int_{RQ} |l|^2 + |r|^2 \, dx, \quad 0 \leq \tau \leq 2X.
\]
with the understanding that \(2 \int_{QA} |r|^2 \, dx = 0\) for \(0 \leq t \leq X\). Then using (2.10) we have

\[
E(\tau) = \frac{1}{2} \|q\|^2 - \iint_{OAC} 4 \text{Re}(q^* r) \, dx \, dt \leq \frac{1}{2} \|q\|^2 + \iint_{OAC} 2|q|(|l|^2 + |r|^2) \, dx \, dt.
\]

If \(\|q\|_\infty = M\), then

\[
E(\tau) \leq \frac{1}{2} \|q\|^2 + 2M \int_0^\tau E(s) \, ds,
\]

so by Gronwall’s inequality,

\[
E(t) \leq e^{4XM} \frac{1}{2} \|q\|^2.
\]

At \(t = 2X\), this is

\[
2\|h\|_{L^2[0,2X]}^2 \leq e^{4XM} \frac{1}{2} \|q\|^2,
\]

and combining this with (2.15) we get

\[
\|l(0, \cdot)\|_{L^2[0,2X]}^2 \leq \frac{1}{4} + e^{4XM} \frac{1}{4} \|q\|_{L^2[0,2X]}^2.
\]

(2.16)

2.3 Proof of Theorem 2.1

2.3.1 Preliminaries

We will use the following lemma repeatedly throughout this section:

Lemma 2.2. Let \(l, r\) be the solution to (2.7)-(2.10) associated to a \(q \in L^\infty[0, X]\). Then

\[
\int_x^{2X-x} |l(x,t)|^2 + |r(x,t)|^2 \, dt \leq \int_0^{2X} |l(0,t)|^2 \, dt
\]

Proof. Integrating (2.13) on the trapezoid \(OPQB\) (see figure 2.2) we get

\[
0 = \iint_{OPQB} (|l|^2 + |r|^2)_x - (|l|^2 - |r|^2)_t \, dx \, dt
\]

\[
= \int_{\partial OPQB} (|l|^2 - |r|^2) \, dx + (|l|^2 + |r|^2) \, dt
\]

\[
= 2 \int_{OP} |l|^2 \, dx + \int_{PQ} |l|^2 + |r|^2 \, dt + 2 \int_{BQ} |r|^2 \, dx - \int_{OB} |l|^2 + |r|^2 \, dt,
\]
Figure 2.2: The trapezoid $OPQB$

Thus

$$\int_{PQ} |l|^2 + |r|^2 \, dt \leq \int_{PQ} |l|^2 + |r|^2 \, dt + 2 \int_{OP} |l|^2 \, dx + 2 \int_{BQ} |r|^2 \, dx$$

$$= \int_{OB} |l|^2 + |r|^2 \, dt.$$

On the triangle $OAB$ (see figure 2.1), Let $l_i, r_i, i = 1, 2$ be the solutions of

$(l_i)_x - (l_i)_t = q_i r_i, \quad 0 \leq x \leq X, \quad x \leq t \leq 2X - x \quad (2.17)$

$(r_i)_x + (r_i)_t = -q_i l_i, \quad 0 \leq x \leq X, \quad x \leq t \leq 2X - x \quad (2.18)$

$(l_i + r_i)(0, t) = 0, \quad 0 \leq t \leq 2X \quad (2.19)$

$l_i(x, x) = -\frac{q_i(x)}{2}, \quad 0 \leq x \leq X. \quad (2.20)$
If $p = q_1 - q_2$, $l = l_1 - l_2$, $r = r_1 - r_2$, then $l, r$, solves

\begin{align}
  l_x - l_t &= pr_2 + q_1 r, \quad 0 \leq x \leq X, \quad x \leq t \leq 2X - x \tag{2.21} \\
  r_x + r_t &= -pl_2 - q_1 l, \quad 0 \leq x \leq X, \quad x \leq t \leq 2X - x \tag{2.22} \\
  (l + r)(0, t) &= 0, \quad 0 \leq t \leq 2X \tag{2.23} \\
  l(x, x) &= -p(x) \tag{2.24} \\
\end{align}

Here we used the fact that $q_2 r_2 - q_1 r_1 = pr_2 + q_1 r$ and $-q_2 l_2 + q_1 l_1 = -pl_2 - q_1 l$.

We will use energy estimates where we define the "energy" $J(x)$ for the difference of two pairs of solutions $(l_1, r_2), (l_2, r_2)$ as

\[ J(x) := \int_x^{2X-x} |l(x, t)|^2 + |r(x, t)|^2 \, dt, \quad 0 \leq x \leq X \tag{2.25} \]

and define the energy for the pair $(l_2, r_2)$ as

\[ J_2(x) := \int_x^{2X-x} |l_2(x, t)|^2 + |r_2(x, t)|^2 \, dt, \quad 0 \leq x \leq X. \]

Note that lemma 2.2 says

\[ J_2(x) \leq J_2(0). \tag{2.26} \]

### 2.3.2 Stability

We first prove the continuity of $F^{-1}$. We also show that $J(x) \leq CJ(0)$, which will be used to prove continuity of $F$ in the following subsection. We work again on the trapezoid $OPQB$ in figure 2.2. Applying lemma 2.1 to (2.21)-(2.22), we get

\[
\int\int_{OPQB} 2 \text{Re}(p \bar{r}_2 - pr_2) \, dx \, dt \\
= \int_{\partial OPQB} (|l|^2 - |r|^2) \, dx + (|l|^2 + |r|^2) \, dt \\
= \int_{PQ} |l|^2 + |r|^2 \, dt + 2 \int_{BQ} |r|^2 \, dx + 2 \int_{OP} |l|^2 \, dx - \int_{OB} |l|^2 + |r|^2 \, dt.
\]
Using (2.10) and (2.25), we can write this as

\[ J(x) + \frac{1}{2} \| p \|_{L^2[0,x]}^2 + 2 \int_{BQ} |r|^2 \, dx = J(0) + \int_{OPQB} 2 \text{Re}(p \bar{r}_2 - p r_l) \, dx \, dt. \]

Thus for any \( \epsilon > 0 \), we have

\[ J(x) + \frac{1}{2} \| p \|_{L^2[0,x]}^2 \leq J(0) + \int_{0}^{x} \frac{\epsilon}{\epsilon} |p|^2 (|l_2|^2 + |r_2|^2) + \frac{1}{\epsilon} (|l|^2 + |r|^2) \, dx \, dt \]

\[ = J(0) + \epsilon J_2(x) \cdot \int_{0}^{x} |p(y)|^2 \, dy + \frac{1}{\epsilon} \int_{0}^{x} J(y) \, dy, \quad 0 \leq x \leq X. \]

Using (2.26), this becomes

\[ J(x) + \left( \frac{1}{2} - \epsilon J_2(0) \right) \| p \|_{L^2[0,x]}^2 \leq J(0) + \int_{0}^{x} J(y) \, dy, \quad 0 \leq x \leq X. \]

Let \( \epsilon \) be small enough so that \( C_\epsilon = \frac{1}{2} - \epsilon J_2(0) > \frac{1}{4} \). Then

\[ J(x) + \frac{1}{4} \| p \|_{L^2[0,x]}^2 \leq J(0) + \frac{1}{\epsilon} \int_{0}^{x} J(y) \, dy \leq J(0) + \frac{1}{\epsilon} \int_{0}^{x} \left( J(y) + \frac{1}{4} |p(y)|^2 \right) \, dy. \]

Thus by Gronwall’s inequality,

\[ J(x) + \frac{1}{4} \| p \|_{L^2[0,x]}^2 \leq e^{x/\epsilon} J(0), \quad 0 \leq x \leq X. \]  

(2.27)

Hence, using (2.23)

\[ \| p \|_{L^2[0,X]}^2 \leq C \cdot J(0) = C(\| l(0, \cdot) \|_{L^2[0,2X]}^2 + \| r(0, \cdot) \|_{L^2[0,2X]}^2) = 2C \| l(0, \cdot) \|_{L^2[0,2X]}^2, \]

where \( C \) depends on \( X \) and \( \| l_2(0, \cdot) \| \|^2 \). If we fix \( l_2 \) we get local Lipschitz continuity for \( \mathcal{F}^{-1} \).

### 2.3.3 Continuity

Let us fix a positive finite \( M \) a priori so that \( \| q_i \|_\infty \leq M, \ i = 1, 2 \). Then from (2.16), we have

\[ J_2(0) \leq \frac{1 + e^{4XM}}{2} \| q_2 \|^2 \leq \frac{M^2}{2} (1 + e^{4XM}) = C_M. \]  

(2.28)
Write
\[ l f - r \bar{g} = l p r_2 + l q_1 r + r p l_2 + r q_1 \bar{l} = p(l r_2 + r \bar{r}_2) + q_1 l r, \]
then apply lemma 2.1 to (2.21)-(2.22) and integrating over \( OAQR \) (see figure 2.1), we get
\[
2 \iint_{O\bar{A}QR} \text{Re}(p(l r_2 + r \bar{r}_2) + q_1 l r) \\
= \iint_{O\bar{A}QR} (|l|^2 - |r|^2)x - (|l|^2 + |r|^2)t \, dx \, dt \\
= \int_{O\bar{A}QR} (|l|^2 + |r|^2) \, dx + (|l|^2 - |r|^2) \, dt \\
= 2 \int_{O\bar{A}} |l|^2 \, dx - 2 \int_{Q\bar{A}} |r|^2 \, dx - \int_{R\bar{Q}} |l|^2 + |r|^2 \, dx - \int_{O\bar{R}} |l|^2 - |r|^2 \, dt. 
\tag{2.29}
\]
Redefine the horizontal energy \( E \) at time \( 0 \leq \tau \leq 2X \), as (see figure 2.1)
\[ E(\tau) := \int_{R\bar{Q}} |l|^2 + |r|^2 \, dx + 2 \int_{Q\bar{A}} |r|^2 \, dx, \]
then from (2.9) and (2.29) we have
\[ E(\tau) = 2 \int_{O\bar{A}} |l|^2 \, dx - 2 \iint_{O\bar{A}QR} \text{Re}(p(l r_2 + r \bar{r}_2 + 2q_1 l r)) \, dx \, dt. \]
So from (2.10) and (2.26)
\[
E(\tau) \leq \frac{1}{2} \|p\|_{L^2[0,X]}^2 + 2 \iint_{O\bar{A}QR} |p l r_2| + |p r l_2| + 2|q_1 l r| \, dx \, dt \\
\leq \frac{1}{2} \|p\|_{L^2[0,X]}^2 + \int_{O\bar{A}QR} |p|^2(|l|^2 + |r|^2) + (1 + 2|q_1|)(|l|^2 + |r|^2) \, dx \, dt \\
\leq \frac{1}{2} \|p\|_{L^2[0,X]}^2 + \int_0^X |p(y)|^2 J_2(y) \, dy + (1 + 2M) \int_0^{\tau} E(s) \, ds \\
\leq \frac{1 + X J_2(0)}{2} \|p\|_{L^2[0,X]}^2 + (1 + 2M) \int_0^{\tau} E(s) \, ds, \quad 0 \leq \tau \leq 2X. 
\]
Then by Gronwall’s inequality and (2.28), we have
\[
E(\tau) \leq e^{(1+2M)2X} \left( \frac{1 + X J_2(0)}{2} \|p\|_{L^2[0,X]}^2 \right) \leq e^{(1+2M)2X} \left( \frac{1 + XC_M}{2} \right) \|p\|^2 = C \|p\|_{L^2[0,X]}^2.
\]
Thus taking $\tau = 2X$ (see figure 2.1), we have

$$2 \int_{BA} |r|^2 \, dx = E(2X) \leq C\|p\|^2. \quad (2.30)$$

Now, applying lemma 2.1 to (2.21)-(2.22) and integrating over $OAB$, we get

$$\iint_{OAB} 2 \, \text{Re}(p\tilde{r}_2 - pr\tilde{l}_2) \, dx \, dt = \int_{\partial OAB} (|l|^2 - |r|^2) \, dx + (|l|^2 + |r|^2) \, dt$$

$$= 2 \int_{OA} |l|^2 \, dx + 2 \int_{BA} |r|^2 \, dx - J(0).$$

Hence, using (2.10), (2.26), and 2.30 for any $\delta > 0$, we have

$$J(0) = \frac{1}{2}\|p\|_{L^2[0,X]}^2 + 2 \int_{BA} |r|^2 \, dx - \iint_{OAB} 2 \, \text{Re}(p\tilde{r}_2 - pr\tilde{l}_2) \, dx \, dt$$

$$\leq \left(1/2 + C\right)\|p\|^2 + \int_{OAB} \delta |p|^2 (|l|^2 + |r|^2) + \frac{1}{\delta} (|l|^2 + |r|^2) \, dx \, dt$$

$$\leq \left(1/2 + C + \delta J_2(0)\right)\|p\|^2 + \frac{1}{\delta} \int_0^X J(y) \, dy.$$

From (2.27), we have (recall our choice of $\epsilon$),

$$J(0) \leq (1/2 + C + \delta J_2(0))\|p\|^2 + \frac{Xe^{X/\epsilon}}{\delta} J(0)$$

Set $\delta = 2X e^{X/\epsilon}$, so that

$$J(0) \leq (1 + 2C + 4X e^{X/\epsilon} J_2(0))\|p\|^2 \leq C'\|p\|^2,$$

where

$$C' = 1 + 2C + 4X e^{X/\epsilon} C_M.$$
Chapter 3

TRANSMISSION DATA IN A 1-D COMPLEX-VALUED HYPERBOLIC SYSTEM

3.1 Introduction

Consider, for some $X > 0$, a one-dimensional acoustic medium occupying $[0, \infty)$ that is homogeneous for $x > X$. We probe the medium by an acoustic wave generated by a source at the left end, $x = 0$, and we wish to recover the acoustic property of the medium from a measurement of the medium response at $x = X$ over a finite time interval. In chapter 2 we studied the recovery problem for medium response data at the same end as the source, called reflection data. In this chapter we study the recovery problem for data for the medium response at $x = X$ away from the source, called the transmission data.

Let $L(x,t)$ and $R(x,t)$ represent the left and right-moving waves at a displacement $x$ and time $t$. Let $q(x)$ be a complex-valued function supported in $[0, X]$ representing the acoustic property. We model the acoustic interaction as the IVP

\[
L_x - L_t = qR, \quad x \geq 0, \quad t \in \mathbb{R} \tag{3.1}
\]
\[
R_x + R_t = \bar{q}L, \quad x \geq 0, \quad t \in \mathbb{R} \tag{3.2}
\]
\[
L = R = 0, \quad x \geq 0, \quad t < 0 \tag{3.3}
\]
\[
(L + R)(0,t) = \delta(t), \quad t \in \mathbb{R}. \tag{3.4}
\]

The inverse problem in question is: Given $R(X,t)$ for $X \leq t \leq 3X$, determine $q(x)$ for $0 \leq x \leq X$. Note the difference between (3.2) and (2.2). This is a significant
We guess that the solution of (3.1)-(3.4) will have the form

\begin{align*}
L(x, t) &= l(x, t) H(t - x), \\
R(x, t) &= r(x, t) H(t - x) + \delta(t - x),
\end{align*}

for some \(l, r\). Also (3.3) are automatically satisfied and (3.4) forces \(l(0, t) + r(0, t) = 0\) for \(t \geq 0\). Then

\begin{align*}
L_x(x, t) &= l_x(x, t) H(t - x) - l(x, t) \delta(t - x) \\
L_t(x, t) &= l_t(x, t) H(t - x) + l(x, t) \delta(t - x) \\
R_x(x, t) &= r_x(x, t) H(t - x) - r(x, t) \delta(t - x) - \delta'(t - x) \\
R_t(x, t) &= r_t(x, t) H(t - x) + r(x, t) \delta(t - x) + \delta'(t - x).
\end{align*}

So

\begin{align*}
L_x - L_t &= (l_x - l_t) H(t - x) - 2l \delta(t - x), \\
R_x + R_t &= (r_x + r_t) H(t - x).
\end{align*}

Hence our guess will satisfy (3.1) and (3.2) if

\begin{align*}
(l_x - l_t) H(t - x) - 2l \delta(t - x) &= qr H(t - x) + q \delta(t - x) \\
(r_x + r_t) H(t - x) &= \bar{q} l H(t - x),
\end{align*}

forcing \(l = -q/2\) on \(t = x\). Hence our guess is a solution of (3.1) and (3.4) if \(l, r\) solves

\begin{align*}
l_x - l_t &= qr, \quad 0 \leq x \leq t \tag{3.7} \\
r_x + r_t &= \bar{q} l, \quad 0 \leq x \leq t \tag{3.8} \\
(l + r)(0, t) &= 0, \quad 0 \leq t \tag{3.9} \\
l(x, x) &= -\frac{q(x)}{2}, \quad 0 \leq x. \tag{3.10}
\end{align*}
The inverse problem is then, given $r(X,t)$ for $X \leq t \leq 3X$, determine $q(x)$ for $0 \leq x \leq X$.

Note that the existence and uniqueness of solutions $l, r \in L^2$ with $L^2$ traces on horizontal, vertical, and slope $\pm 1$ lines, given $q \in L^2([0,X]; \mathbb{C})$, can be shown using standard energy estimates for first-order hyperbolic systems, see [9], [11], [16], [18].

Since $q(x) = 0$ outside of $[0,X]$, we can state a simple yet important property of the solution $L(x,t)$ on the vertical line $x = X$.

**Lemma 3.1.** If $q \in L^2(\mathbb{R})$ is supported in $[0,X]$ and if $L, R$ solves (3.1)-(3.4), then the trace $L(X,t)$ vanishes almost everywhere for $t \in \mathbb{R}$.

**Proof.** For any $T > 0$, differentiate $L$ on the line $t = X + T - x$ (see figure 3.1),

$$\frac{d}{dx} L(x, X + T - x) = L_x(x, X + T - x) - L_t(x, X + t - x).$$
Since $q = 0$ for $x \geq X$, this is zero by (3.1). Thus $L$ is constant a.e. on the line $t = X + T - x$. When $X + T < x$, $t < 0$ so from (3.3) we get that $L = 0$ a.e. on $t = X + T - x$ for all $x \geq X$. Since $T$ was arbitrary, we get that at $x = X$,

$$L(X, X + T - X) = L(X, T) = 0, \quad \forall T \in \mathbb{R}.$$ 

Let $A, B$ be the solution for the following sideways IVP,

$$A_x - A_t = qB, \quad x \geq 0, \ t \in \mathbb{R} \quad (3.11)$$

$$B_x + B_t = qA, \quad x \geq 0, \ t \in \mathbb{R} \quad (3.12)$$

$$(A + B)(0, t) = 0, \quad t \in \mathbb{R} \quad (3.13)$$

$$A(0, t) = \delta(t), \quad t \in \mathbb{R}. \quad (3.14)$$

We will study the inverse problem in question by studying a closely related inverse problem for $A$ and $B$.

We now continue with a progressing wave expansion for $A$ and $B$. We guess

$$A(x, t) = (H(t + x) - H(t - x))a(x, t) + \delta(t + x) \quad (3.15)$$

$$B(x, t) = (H(t + x) - H(t - x))b(x, t) - \delta(t - x), \quad (3.16)$$

hence (3.13) and (3.14) are satisfied. Further

$$A_x = (\delta(t + x) + \delta(t - x))a + (H(t + x) - H(t - x))a_x + \delta'(t + x)$$

$$A_t = (\delta(t + x) - \delta(t - x))a + (H(t + x) - H(t - x))a_t + \delta'(t + x)$$

$$B_x = (\delta(t + x) + \delta(t - x))b + (H(t + x) - H(t - x))b_x + \delta'(t - x)$$

$$B_t = (\delta(t + x) - \delta(t - x))b + (H(t + x) - H(t - x))b_t - \delta'(t - x),$$
so

\[ A_x - A_t = (H(t + x) - H(t - x))(a_x - a_t) + 2\delta(t - x)a \]
\[ B_x + B_t = (H(t + x) - H(t - x))(b_x + b_t) + 2\delta(t + x)b. \]

Therefore (3.11)-(3.12) becomes

\[ (H(t + x) - H(t - x))(a_x - a_t) + 2\delta(t - x)a = (H(t + x) - H(t - x))qb - q\delta(t - x) \]
\[ (H(t + x) - H(t - x))(b_x + b_t) + 2\delta(t + x)b = (H(t + x) - H(t - x))\bar{q}a + \bar{q}\delta(t + x). \]

Thus our guess for \( A \) and \( B \) is a solution of (3.11)-(3.14) if \( a, b \) solve the Goursat problem,

\[ a_x - a_t = qb, \quad 0 \leq |t| \leq x \] \hspace{1cm} (3.17)
\[ b_x + b_t = \bar{q}a, \quad 0 \leq |t| \leq x \] \hspace{1cm} (3.18)
\[ (a + b)(0, t) = 0, \quad t \in \mathbb{R} \] \hspace{1cm} (3.19)
\[ a(x, x) = -\frac{q(x)}{2}, \quad b(x, -x) = \frac{\bar{q}(x)}{2}, \quad 0 \leq x \leq X. \] \hspace{1cm} (3.20)

Notice (3.20) forces \( q(0) = -\bar{q}(0) \) but this does not matter if \( q \in L^2[0, X] \).

\( A \) and \( B \) exhibit a symmetry which is characterized in the following proposition.

**Proposition 3.1.** If \( q \in L^2[0, X] \), \( A, B \) solves (3.11)-(3.14) and \( a, b \) solves (3.17)-(3.20), then

\[ A(x, t) = -\overline{B}(x, -t), \]

and

\[ a(x, t) = -\overline{b}(x, -t). \]

**Proof.** Let

\[ g(x, t) = -\overline{a}(x, -t) \]
\[ f(x, t) = -\overline{b}(x, -t). \]
Then

\[ f_x - f_t = -(b_x + b_t)(x, -t) = -q\bar{a}(x, -t) = qg \]
\[ g_x + g_t = -(a_x - a_t)(x, -t) = -\bar{q}b(x, -t) = \bar{q}f, \]

and

\[ f(x, x) = -\bar{b}(x, -x) = -\frac{q(x)}{2} \]
\[ g(x, -x) = -\bar{a}(x, x) = \frac{q(x)}{2}. \]

From the well posedness of (3.17)-(3.20), we get that

\[ a(x, t) = f(x, t) = -\bar{b}(x, -t) \]
\[ b(x, t) = g(x, t) = -\bar{a}(x, -t). \]

From (3.15)-(3.16), we get

\[ A(x, t) = -\bar{B}(x, -t). \]

\[ \square \]

3.2 The Relationship Between \( A, B, L, \) and \( R \)

For \( x \geq 0, s, t \in \mathbb{R}, \) define

\[ f(x, t, s) = \left[R(x, t - s)A(x, s) - L(x, t - s)B(x, s)\right] \]
\[ g(x, t, s) = \left[R(x, t - s)A(x, s) + L(x, t - s)B(x, s)\right], \]

then

\[ \frac{\partial f}{\partial x} - \frac{\partial g}{\partial s} = \left[R_x A + R A_x - L_x B - LB_x + R_t A - RA_t + L_t B - LB_t\right] \]
\[ = \left[(R_x + R_t)A + R(A_x - A_t) - (L_x - L_t)B - L(B_x + B_t)\right] \]
\[ = \left[\bar{q}LA + qRB - qRB - \bar{q}LA\right] \]
\[ = 0. \] (3.22)
Integrating this identity on the rectangular region $0 \leq x \leq X$ and $-T \leq s \leq T$, we have

\[ 0 = \int_{\partial([0,X] \times [-T,T])} g \, dx + f \, ds \]
\[ = \int_0^X g(x,t,-T) \, dx - \int_0^X g(x,t,T) \, dx + \int_{-T}^{T} f(X,t,s) \, ds - \int_{-T}^{T} f(0,t,s) \, ds. \]

Since, for $0 \leq x \leq X$ and large $T$, $A(x,\pm T)$ and $B(x,\pm T)$ are 0 and hence $f(x,t,\pm T)$, $g(x,t,\pm T)$ are 0, letting $T \to \infty$ we have

\[ \int_{-\infty}^{\infty} f(X,t,s) \, ds = \int_{-\infty}^{\infty} f(0,t,s) \, ds. \]

We may write this as

\[ R(X,t) *_t A(X,t) - L(X,t) *_t B(X,t) = R(0,t) *_t A(0,t) - L(0,t) *_t B(0,t). \]

From lemma 3.1 we have $L(X,t) = 0$ for all $t \in \mathbb{R}$. Since $L(0,t) + R(0,t) = \delta(t)$,

\[ R(X,t) *_t A(X,t) = A(0,t) - L(0,t) *_t (A + B)(0,t) = A(0,t) = \delta(t). \quad (3.23) \]

From (3.15) and (3.6), we define

\[ j(t) := \begin{cases} a(X,t-X), & : 0 \leq t \leq 2X \\ 0, & : \text{otherwise} \end{cases} \]
\[ e(t) := \begin{cases} r(X,t+X), & : 0 \leq t < \infty \\ 0, & : t < 0 \end{cases} \]

Then for $q \in L^2[0,X]$ we have

\[ A(X,t) = j(t+X) + \delta(t+X) \quad (3.24) \]
\[ R(X,t) = e(t-X) + \delta(t-X), \quad (3.25) \]

where $j$ is supported in $[0,2X]$ and $e$ is supported in $[0,\infty)$, $j \in L^2(\mathbb{R})$, $e \in L^2(\mathbb{R})$.  

28
Substituting (3.24) and (3.25) in (3.23), we get

\[ \begin{align*}
\delta(t) &= (\delta(t - X) + e(t - X)) \ast (\delta(t + X) + j(t + X)) \\
&= (\delta(t) + e(t)) \ast (\delta(t) + j(t)).
\end{align*} \tag{3.26} \]

Using the support of \( e(t) \) and \( j(t) \), (3.26) may be written as the volterra equation

**Proposition 3.2.**

\[ j(t) + e(t) + \int_0^t j(s)e(t - s) \, ds = 0, \quad 0 \leq t \leq 2X. \tag{3.27} \]

So if we know \( e \) on \([0, 2X]\) we can solve for \( j \) on \([0, 2X]\). This says that if we know \( R(X,t) \) for \( X \leq t \leq 3X \), we can recover \( A(X,t) \) for \( |t| \leq X \). Thus the inverse problem reduces to:

**Given** \( A(X,t) \) **for** \(-X \leq t \leq X\), recover \( q(x) \) **for** \( 0 \leq x \leq X\).

Note that from (3.21), knowing \( A(X,\cdot) \) gives us \( B(X,\cdot) \) as well. We now state the main results of this chapter.

**Theorem 3.1.** The forward map

\[ \mathcal{F} : L^2[0, X] \to L^2[-X, X] \]

\[ q \mapsto a(X,\cdot) \]

is a continuous injection with continuous inverse whose range is the open subset

\[ G = \{ j \in L^2[0, 2X] : 1 + \hat{j}(\zeta) \neq 0, \quad \forall \zeta \in \{ \zeta \in \mathbb{C} : \text{Im} \zeta < 0 \}, \] \]

where \( \hat{j} \in L^2(-\infty, \infty) \) is the Fourier Transform of \( j \).
We have an identity that relates $\|q\|_{L^2[0,X]}$ to $\mathcal{F}(q)$.

**Theorem 3.2.** Given $q \in L^2[0,X]$ for which $a$ and $b$ satisfy (3.17)-(3.20), we have that
\[
\|q\|_{L^2[0,X]}^2 = \frac{1}{\pi} \left\| \hat{\frac{j}{1+j}} \right\|^2_{(-\infty,\infty)},
\]
where $j(t+X) = a(X,t)$ in $|t| < x$.

### 3.3 Proof of Theorem 3.2

Let $l, r$ solve (3.7)-(3.10). Integrating
\[
0 = (|l|^2 - |r|^2)_x - (|l|^2 + |r|^2)_t
\]
over the trapezoidal region $OABC$ in the $x,t$ plane with vertices $O(0,0)$, $A(X,X)$, $B(X,T)$, and $C(0,T)$ (with $T > X$), and using (3.9) and (3.10) and lemma 3.1,

**Figure 3.2:** Energy Estimate on $OABC$
\[ 0 = 2 \int_{OA} |l|^2 \, dx + \int_{AB} |l|^2 - |r|^2 \, dt - \int_{CB} |l|^2 + |r|^2 \, dx - \int_{CO} |l|^2 - |r|^2 \, dt \]
\[ = \frac{1}{2} \int_{0}^{X} |q(x)|^2 \, dx - \int_{AB} |r(X,t)|^2 \, dt - E(T), \]
where
\[ E(T) = \int_{0}^{X} |l(x,T)|^2 + |r(x,T)|^2 \, dx \]
is the energy at time \( T \) in the interval \( 0 \leq x \leq X \). Hence
\[ \frac{1}{2} \int_{0}^{X} |q(x)|^2 \, dx = E(T) + \int_{X}^{T} |r(X,t)|^2 \, dt. \tag{3.28} \]

Since \( q \in L^2(0,X) \), (3.28) implies that \( \int_{X}^{T} |R(X,t)|^2 \, dt \) (which is \( \int_{AB} |R(X,t)|^2 \) )
is bounded above by a finite quantity independent of \( T \), implying \( R(X,t) \in L^2(X,\infty) \).

Letting \( T \to \infty \) we get
\[ \frac{1}{2} \int_{0}^{X} |q(x)|^2 \, dx = \int_{X}^{\infty} |r(X,t)|^2 \, dt = \int_{0}^{\infty} |e(t)|^2 \, dt, \tag{3.29} \]
from (3.25) and the following proposition, proved in section 3.5.

**Proposition 3.3.** \( \lim_{T \to \infty} E(T) = 0. \)

Thus we can estimate \( \|q\|_{L^2[0,X]} \) in terms of the transmission data \( e(t) \) for \( t \in (0,\infty) \). However, we can do better. We can estimate \( \|q\| \) by \( e(\cdot) \) on \( [0,2X] \),
because \( e(\cdot) \) on \( [0,\infty) \) can be determined by \( e(\cdot) \) on \( [0,2X] \) as seen below.

Using (3.27) and the Fourier transform, we obtain
\[ (1 + \hat{e})(1 + \hat{j}) = 1, \tag{3.30} \]

31
giving
\[ \hat{e} = -\frac{\hat{j}}{1 + j}. \]  
(3.31)

Note that this means that knowing \( e(t) \) on \([0, 2X]\), we can recover \( e(t) \) on \([0, \infty)\) because we can determine \( j \) on \([0, 2X]\) by solving the Volterra equation (3.27). More importantly, (3.29) implies
\[ \int_0^X |q(x)|^2 \, dx = 2 \int_0^\infty |e(t)|^2 \, dt = \frac{1}{\pi} \int_{-\infty}^{\infty} |\hat{\epsilon}(\omega)|^2 \, d\omega = \frac{1}{\pi} \int_{-\infty}^{\infty} \left| \frac{\hat{j}(\omega)}{1 + \hat{j}(\omega)} \right|^2 \, d\omega. \]  
(3.32)

3.4 Proof of Theorem 3.1

3.4.1 Local Inversion

We want to locally reconstruct \( q \) given the right-hand data of the above problem, \( A(X, t) \). For this it suffices to know \( a(X, t) \), where \( a, b \) solve (3.17)-(3.20) in the region \( \mathcal{R} \).

Figure 3.3: Region in Lemma 3.2
For some $M > 0$ and $0 \leq \alpha < \beta \leq X$, define the complete metric space

$$K = \{ q \in L^2[\alpha, \beta] : \| q \|_{L^2[\alpha, \beta]} \leq M \}.$$ 

In this subsection we will denote $\| q \|_{L^2[\alpha, \beta]}$ as just $\| q \|$.

**Proposition 3.4.** Given $f, g \in L^2[\beta, 2X - \beta]$ and a large enough positive number $M$, there is a unique $q \in K$ and $L^2$ functions $a(x, t)$ and $b(x, t)$ which solve the boundary value problem

\begin{align*}
    a_x - a_t &= q(x)b, \quad |t| \leq x, \quad \alpha \leq x \leq \beta \tag{3.33} \\
    b_x + b_t &= \overline{q}(x)a, \quad |t| \leq x, \quad \alpha \leq x \leq \beta \tag{3.34} \\
    a(\beta, t) &= f(t), \quad b(\beta, t) = g(t), \quad -\beta \leq t \leq \beta, \tag{3.35}
\end{align*}

for which

$$a(x, x) = -\frac{q(x)}{2}, \quad \alpha \leq x \leq \beta, \tag{3.36}$$

provided $\beta - \alpha$ is small enough, specifically if

$$\beta - \alpha \leq \rho < \min \left\{ \frac{1}{4M^2}, \frac{1}{2304(\| f \|^2_{L^2[-\beta, \beta]} + \| g \|^2_{L^2[-\beta, \beta]})}, X \right\},$$

where $M^2 \geq 6(\| f \|^2 + \| g \|^2)$.

We will prove proposition 3.4 by showing that the map

$$\mathcal{C} : K \to K$$

$$q(x) \mapsto -2a(x, x),$$

is a contraction. To show this we repeatedly employ the following sideways estimate.

**Lemma 3.2.** Suppose $a, b$ are square integrable on $\{(x, t) : \alpha \leq x \leq \beta, \ |t| \leq x \}$ with $L^2$ traces on horizontal, vertical, and slope $\pm1$ lines. If (distributionally), $a$ and $b$ satisfy

\begin{align*}
    a_x - a_t &= F, \quad |t| \leq x, \quad \alpha \leq x \leq \beta \\
    b_x + b_t &= G, \quad |t| \leq x, \quad \alpha \leq x \leq \beta,
\end{align*}
then
\[ J(x) = \int_{PQ} |a|^2 + |b|^2 \, dt - 2 \int_{SPQR} \text{Re}(\pi F + b\bar{G}) \, dx \, dt, \]
where we define (see figure 3.4)
\[ J(x) := 2 \int_{RQ} |a|^2 \, dx + 2 \int_{SP} |b|^2 \, dx + \int_{SR} |a|^2 + |b|^2 \, dt. \]

Proof. Integrate the identity
\[ (|a|^2 + |b|^2)_x - (|a|^2 - |b|^2)_t = 2 \text{Re}(\pi F + b\bar{G}) \]
over the region \( \{(x, t) : \alpha \leq x \leq \beta, |t| \leq x\} \).

If \( q \in L^2[\alpha, \beta] \) then the system (3.33)-(3.35) has a solution \( a, b \) with \( L^2 \) traces on horizontal, vertical, and \( \pm 1 \) lines. We use lemma 3.2 on \( a, b \) with \( F = qb \) and \( G + \bar{q}a \). Then the \( J \) corresponding to these \( a, b \), satisfies
\[ J(x) = \int_{PQ} |a|^2 + |b|^2 \, dt - 2 \int_{SPQR} 2q\bar{a}b \, dx \, dt \]
\[ J(x) \leq J(\beta) + \int_{SPQR} 2|q(y)||(|a|^2 + |b|^2) \, dx \, dt \]
\[ \leq J(\beta) + 2 \int X |q(y)|J(y) \, dy, \]
Hence Gronwall’s inequality implies
\[ J(x) \leq J(\beta)e^{\int_x^\beta 2|q(y)| \, dy} \leq J(\beta)e^{2\sqrt{\beta-\alpha}\|q\|}, \quad x \in [\alpha, \beta]. \]
Thus, using the upper bound on \( \beta - \alpha \), we have
\[ \|C(q)\|^2 = 4 \int_{\alpha}^{\beta} |a(x, x)|^2 \, dx \leq 2J(\alpha) \leq 2E(\beta)e^{2\sqrt{\beta-\alpha}\|q\|} \leq 2(\|f\|^2 + \|g\|^2)e^{2\sqrt{\beta-\alpha}M} \leq 2e(\|f\|^2 + \|g\|^2) \leq 6(\|f\|^2 + \|g\|^2) \leq M^2, \]
implying that \( a(x, x) \in K \).

To show \( C \) is a contraction we will use lemma 3.2 on a difference of two solutions to (3.33)-(3.35). Suppose \( q_j \in K \) and \( a_j, b_j, j = 1, 2 \) are the corresponding solutions of (3.33) - (3.35). Let

\[
p = q_2 - q_1, \quad a = a_2 - a_1, \quad b = b_2 - b_1.
\]

then

\[
a_x - a_t = pb_2 + q_1 b, \quad |t| \leq x, \quad \alpha \leq x \leq \beta \tag{3.37}
\]
\[
b_x + b_t = \bar{p}a_2 + \bar{q} a, \quad |t| \leq x, \quad \alpha \leq x \leq \beta. \tag{3.38}
\]

Note that the \( f \) and \( g \) in (3.35) are the same for \( j = 1, 2 \). Thus

\[
a(\beta, \cdot) = a_2(\beta, \cdot) - a_1(\beta, \cdot) = f(\cdot) - f(\cdot) = 0
\]
\[
b(\beta, \cdot) = b_2(\beta, \cdot) - b_1(\beta, \cdot) = g(\cdot) - g(\cdot) = 0,
\]

which implies \( J(\beta) = 0 \).

Define \( J^*(x) := \sup_{x \leq y \leq \beta} J(y) \). Then noting

\[
C(q_2)(x, x) - C(q_1)(x, x) = -2(a_2(x, x) - a_1(x, x)) = -2a(x, x),
\]

we have

\[
\|C(q_2) - C(q_1)\|^2 = \int_{\alpha}^{\beta} |C(q_2)(y) - C(q_1)(y)|^2 \, dy = 4 \int_{NQ} |a|^2 \, dy \leq 2J(\alpha). \tag{3.39}
\]

So \( C \) will be a contraction for small \( \beta - \alpha \) if we can estimate \( J(\alpha) \) in terms of \( \beta - \alpha \) and \( \|p\| \).
To apply lemma 3.2 to (3.37)-(3.38), note that
\[
\pi F + bG = p\pi b_2 + q_1\pi b + p\pi_2 b + q_1\pi b = p(\pi b_2 + \pi_2 b) + 2q_1\pi b,
\]
and
\[
\int_{PQ} |a|^2 + |b|^2 \, dt = J(\beta).
\]
So lemma 3.2 implies
\[
J(x) \leq J(\beta) + 2\int_{SPQR} 2|q_1(y)| |a| |b| + |p(y)||(a| |b_2| + |a_2| |b|) \, dx \, dt
\]
\[
\leq J(\beta) + 2\int_x^\beta |q_1(y)| dy \int_y^x (|a|^2 + |b|^2)(y, t) \, dt
\]
\[
+ 2\int_{SPQR} |p(y)||(a| |b_2| + |a_2| |b|) \, dx \, dt
\]
\[
= 2\int_{MPQN} |p(y)||(a| |b_2| + |a_2| |b|) + 2\int_x^\beta |q_1(y)| J(y) \, dy.
\]
Gronwall’s inequality, noting that \( q_1 \in K \) and the bound on \( \beta - \alpha \), implies that
\[
J(x) \leq e^{2\int_\alpha^\beta |q_1(y)| \, dy} 2\int_{MPQN} |p(y)||(a| |b_2| + |a_2| |b|)
\]
\[
\leq 2e^{2M\sqrt{\beta - \alpha}} \left( \int_{MPQN} |p(y)||(a| |b_2| + |a_2| |b|) \right)
\]
\[
\leq 6\int_{MPQN} |p(y)||(a| |b_2| + |a_2| |b|).
\]
(3.40)

Now
\[
\left( \int_{MPQN} |p| |a| |b_2| \right)^2 \leq \int_{MPQN} |pb_2|^2 \int_{MPQN} |a|^2
\]
\[
\leq \|pb_2\|_{L^2(MPQN)}^2 \int_{\alpha}^{\beta} J(y) \, dy
\]
\[
\leq \|pb_2\|_{L^2(MPQN)}^2 (\beta - \alpha) J^*(\alpha).
\]
and,
\[
\left( \int_{MPQN} |p| |a_2| |b| \right)^2 \leq \|pa_2\|_{L^2(MPQN)}^2 (\beta - \alpha) J^*(\alpha).
\]
Using these two relations in (3.40), we get
\[
J(x) \leq 6\sqrt{\beta - \alpha} \sqrt{J^*(\alpha)} (\|pa_2\|_{L^2(MPQN)} + \|pb_2\|_{L^2(MPQN)}), \quad \alpha \leq x \leq \beta.
\]
Thus
\[
J^*(\alpha) \leq 6\sqrt{\beta - \alpha} \sqrt{J^*(\alpha)(\|pa_2\|_{L^2(MPQN)} + \|pb_2\|_{L^2(MPQN)})},
\] (3.41)
implying
\[
\sqrt{J^*(\alpha)} \leq 6\sqrt{\beta - \alpha}(\|pa_2\|_{L^2(MPQN)} + \|pb_2\|_{L^2(MPQN)}).
\] (3.42)
Now it remains to estimate \(\|pa_2\|_{L^2(MPQN)}\) and \(\|pb_2\|_{L^2(MPQN)}\) in terms of \(\|p\|\). We have
\[
\|pa_2\|_{L^2(MPQN)}^2 = \int_{\alpha}^{\beta} |p(x)|^2 \, dx \int_{\alpha}^{\beta} |a_2(x, t)|^2 \, dt \leq \int_{\alpha}^{\beta} |p(x)|^2 J_2(x) \, dx,
\] (3.43)
where \(J_2\) is \(J\) with \(a, b\) replaced by \(a_2, b_2\). If we repeat the calculation used in the derivation of (3.40) with these functions instead, 3.2 would yield, noting that \(F = q_2b_2\) and \(G = \overline{q}_2a_2\),
\[
J_2(x) = \int_{PQ} |a_2|^2 + |b_2|^2 - 4 \int_{SPQR} \text{Re}(\overline{a}qb) \, dx \, dt
\leq J_2(\beta) + 4 \int_{SPQR} \overline{q}_2 |a_2| |b_2| \, dx \, dt
\leq J_2(\beta) + 2 \int_{SPQR} \overline{q}_2^2 (|a_2|^2 + |b_2|^2) \, dx \, dt
= J_2(\beta) + 2 \int_{x}^{\beta} |q_2(y)| J_2(y) \, dy
\]
Applying Gronwall’s inequality, we get
\[
J_2(x) \leq e^{2 \int_{x}^{\beta} |q_2(y)| \, dy} J_2(\beta) \leq e^{2M\sqrt{\beta - \alpha}} J_2(\beta).
\]
Since \(2M\sqrt{\beta - \alpha} \leq 1\), we have
\[
J_2(x) \leq 3J_2(\beta).
\]
Using this in (3.43), we obtain
\[
\|pa_2\|_{L^2(MPQN)}^2, \|pb_2\|_{L^2(MPQN)}^2 \leq 3J_2(\beta) \int_{\alpha}^{\beta} |p(x)|^2 \, dx = 3\|p\|^2_{L^2[0,\beta]}J_2(\beta).
\]
Using this in (3.42) we obtain
\[
\sqrt{J^*(\alpha)} \leq 24\sqrt{\beta - \alpha} \sqrt{J_2(\beta)} \|p\|.
\]
Using this in (3.39) we obtain (taking a square root),
\[ \|C(q_2) - C(q_1)\| \leq 48\sqrt{\beta - \alpha}\sqrt{J_2(\beta)}\|p\| = 48\sqrt{\beta - \alpha}\sqrt{J_2(\beta)}\|q_2 - q_1\|. \]

Noting that
\[ J_2(\beta) = \int_{-\beta}^{\beta} |f(t)|^2 + |g(t)|^2 \, dt = \|f\|^2 + \|g\|^2, \]
\( C \) will be a contraction if \( \beta - \alpha \) is small enough so that
\[ 48\sqrt{\beta - \alpha}\sqrt{J_2(\beta)} < 1. \]

Along with the earlier condition that \( 2M\sqrt{\beta - \alpha} \leq 1 \), we see that \( C \) is a contraction if
\[ \beta - \alpha \leq \min \left\{ \frac{1}{4M^2}, \frac{1}{2304(\|f\|^2 + \|g\|^2)}, X \right\}. \]

3.4.2 Global Inversion

We now prove that \( F \) is injective and that the range of \( F \) is
\[ G = \{ j \in L^2[0,2X] : 1 + \hat{j}(\zeta) \neq 0, \forall \zeta \in \{ z \in \mathbb{C} : \implies z < 0 \} \}. \]

Suppose \( j \in G \). We show there is a unique \( q \) and functions \( a, b \) so that
\[ a_x - a_t = q(x)b, \quad |t| \leq x, \quad 0 \leq x \leq X \quad (3.44) \]
\[ b_x + b_t = \overline{q}(x)a, \quad |t| \leq x, \quad 0 \leq x \leq X \quad (3.45) \]
\[ a(X,t) = j(t+X), \quad b(X,t) = -\overline{j}(t+X), \quad |t| \leq X; \quad (3.46) \]
for which
\[ a(x,x) = -\frac{q(x)}{2}, \quad \text{and} \quad b(x,-x) = \frac{\overline{q}(x)}{2}, \quad 0 \leq x \leq X, \quad (3.47) \]
proving the claim about the range of \( F \) and the injectivity of \( F \). The existence of a unique \( a, b, q \), satisfying the above BVP will be done by repeated application of proposition 3.4 in steps of size \( \rho_0 > 0 \) over the interval \([0, X]\), where \( \rho_0 \) will be determined below.
Let
\[ \mu_0^2 = \frac{4}{\pi} \int_{-\infty}^{\infty} \left| \frac{j(\omega)}{1 + j(\omega)} \right|^2 d\omega. \]

Based on theorem 3.2, we expect that \( \|q\|_{L^2[0,X]} \leq \frac{1}{2} \mu_0 \) for the unique solution mentioned in the previous paragraph. Define
\[ J(x) := \int_{-x}^{x} |a(x,t)|^2 + |b(x,t)|^2 dt. \]

Using lemma 3.2 on (3.44) - (3.46) with \( \beta = X \), we get that for \( \alpha \leq x \leq X \),
\[ J(x) \leq \int_{-x}^{x} |a(X,t)|^2 + |b(X,t)|^2 dt + 4 \int_{SPQR} |q\bar{a}b| \]
\[ \leq 2\|j\|^2_{L^2[0,2X]} + 2 \int_{x}^{X} |q(y)|J(y) \, dy. \]

Hence by Gronwall’s inequality,
\[ J(x) \leq 2\|j\|^2_{L^2[0,2X]} e^{\int_{x}^{X} 2|\bar{q}(y)| \, dy} \]
\[ \leq 2\|j\|^2_{L^2[0,2X]} e^{2\sqrt{X}\|q\|_{L^2[0,X]}} \]
\[ \leq 2\|j\|^2_{L^2[0,2X]} e^{\mu_0 \sqrt{X}}. \]

Note that the last line is obtained from \( \|q\|_{L^2[0,X]} \leq \frac{1}{2} \mu_0 \) because of theorem 3.2.

Since \( \int_{-x}^{x} |a(x,t)|^2 + |b(x,t)|^2 dt = \int_{SR} |a|^2 + |b|^2 \leq J(x) \), we get
\[ \int_{-x}^{x} |a(x,t)|^2 + |b(x,t)|^2 dt \leq 2e^{\mu_0 \sqrt{X}} \|j\|^2_{L^2[0,2X]} \quad \text{(3.48)} \]

This suggests how we choose the \( M_0 \) and \( \rho_0 \) that will work for all \( \alpha \in [0,X] \).

Define \( M_0, \rho_0 \) by
\[ M_0^2 = 12 e^{\mu_0 \sqrt{X}} \|j\|^2 \]
\[ \rho_0 = \min \left\{ \frac{1}{4M_0^2}, X, \frac{e^{-\mu_0 \sqrt{X}}}{4608 \|j\|^2_{L^2[0,2X]}} \right\}. \]
Now apply proposition 3.4 with \( \alpha = X - \rho_0, \beta = X, f(\cdot) = j(X + \cdot), g(\cdot) = -\overline{j}(X + \cdot) \).

We use \( M = M_0 \) and since

\[
\rho_0 \leq \frac{1}{4608\|j\|_{L^2[0,2X]}^2},
\]

we may use \( \rho = \rho_0 \). So there is a unique \( q \in L^2[X - \rho_0, X] \), and \( a, b \), which satisfy (3.44)-(3.46) over the interval \([X - \rho_0, X]\) (and not \([0, X]\)). So we have managed to recover \( q \) over the interval \([X - \rho_0, X]\).

Proposition 3.4 guarantees that \( \|q\|_{L^2[X - \rho_0, X]} \leq M_0 \) but from the proof of theorem 3.2, we can say instead that

\[
\|q\|_{L^2[X - \rho_0, X]} \leq \frac{1}{2}\mu_0.
\]

Now we show the induction step. Suppose for some \( x_0 < X \) we have found a unique \( q \in L^2[x_0, X] \) with \( \|q\|_{[x_0, X]} \leq \mu_0 \) and \( a, b \), which satisfy (3.44)-(3.46) over the region \(-x \leq t \leq x, x_0 \leq x \leq X\). Then as in (3.48), we have

\[
\int_{-\beta}^{\beta} |a(x_0, t)|^2 + |b(x_0, t)|^2 \, dt \leq e^{\|q\|_{L^2[X - \rho_0, X]} \sqrt{X}} \|j\|^2 \leq e^{\mu_0 \sqrt{X}} \|j\|^2_{L^2[X - \beta, X + \beta]}.
\]

Apply Proposition 3.4 to the case where \( \alpha = x_0 - \rho_0, \beta = x_0, j(X + \cdot) = a(x_0, \cdot), -\overline{j}(X + \cdot) = b(x_0, \cdot) \). Since

\[
M_0^2 = 12e^{\mu_0 \sqrt{X}} \|j\|^2_{L^2[X - \beta, X + \beta]} \geq 6(\|a(x_0, \cdot)\|^2_{L^2[-\beta, \beta]} + \|b(x_0, \cdot)\|^2_{L^2[-\beta, \beta]}),
\]

we may take \( M = M_0 \). Further, since

\[
\frac{1}{2304(\|a(x_0, \cdot)\|^2_{L^2[-\beta, \beta]} + \|b(x_0, \cdot)\|^2_{L^2[-\beta, \beta]})} \geq \frac{e^{-\mu_0 \sqrt{X}}}{4608\|j\|^2_{L^2[X - \beta, X + \beta]}} \geq \rho_0,
\]

we may use \( \rho = \rho_0 \). So there is a unique \( q \) in \( L^2[x_0 - \rho_0, x_0] \) and \( a, b \), which satisfy (3.33)-(3.35). So we have found an extension for \( a, b, q \) so that (3.44)-(3.46) is valid over \([x_0 - \rho_0, X]\) instead of just \([x_0, X]\). Further, as before we note that \( \|q\|_{[x_0 - \rho_0, X]} \leq \mu_0 \).
So in each induction step, we recover $q$ over an interval which is $\rho_0$ units longer than the previous interval and hence we will cover $[0, X]$ in a finite number of steps. Note that the size of $\rho_0$ is completely determined by our data $j(\cdot)$ via Theorem 2.

### 3.5 Proof of Proposition 3.3

From lemma 3.2, we know for pairs of solutions $l, r$ to (3.7)-(3.10)

$$4 \text{Re}(\overline{l}rq) = (|l|^2 + |r|^2)_x - (|l|^2 - |r|^2)_t.$$

For $T, p > 0$ and $T - p > 2X$, define the regions in figure 3.4. Integrating over $CDBA$

![Figure 3.4: Energy Estimates on $D_0C_0BA$ and $PQSR$](image)

and noting that $l = 0$ on $BA$, we have

$$\iint_{CDBA} 2 \text{Re}(\overline{l}rq) = \int_{\partial CDBA} (|l|^2 - |r|^2) \, dx + (|l|^2 + |r|^2) \, dt$$

$$= - \int_{DB} 2|r|^2 \, dx + \int_{BA} (|l|^2 + |r|^2) \, dt - 2 \int_{CA} |l|^2 \, dx - \int_{DC} |l|^2 + |r|^2 \, dt. \quad (3.49)$$

41
So if we define

\[ H(x) := 2 \int_{DB} |r|^2 \, dx + 2 \int_{CA} |l|^2 \, dx + \int_{DC} |l|^2 + |r|^2 \, dt \]

then (3.49) implies

\[ H(x) = \int_{BA} |l|^2 + |r|^2 \, dt - \iint_{CDBA} 4 \text{Re}(lrq). \]

Hence

\[ H(x) \leq H(X) + 2 \iint_{CDBA} |q|(|l|^2 + |r|^2) \]

\[ \leq 2 \int_{x}^{X} |q(y)|H(y) \, dy. \]

Thus by Gronwall's inequality,

\[ H(x) \leq e^{2\int_{0}^{x} |q(y)|} H(X) \leq e^{2\sqrt{X\|q\|_{L^2(0,x)}}} H(X), \quad 0 \leq x \leq X. \quad (3.50) \]

Integrating over \([0, X]\), we get

\[ \iint_{C_{00}BA} |l|^2 + |r|^2 \leq \int_{0}^{X} H(x) \, dx \leq C_{x,q} H(X). \]

Recall

\[ E(t) = \int_{0}^{X} |l(x,t)|^2 + |r(x,t)|^2 \, dx. \]

Then integrating the identity

\[ 0 = (|l|^2 - |r|^2)_x - (|l|^2 + |r|^2)_t. \]

over \(PQRS\), we get

\[ 0 = \int_{\partial PQRS} (|l|^2 + |r|^2) \, dx + (|l|^2 - |r|^2) \, dt \]

\[ = \int_{PQ} |l|^2 + |r|^2 \, dx + \int_{QS} |l|^2 - |r|^2 \, dt - \int_{RS} |l|^2 + |r|^2 \, dx - \int_{PR} |l|^2 - |r|^2 \, dt \]

Noting (3.9) and the fact that \(l(X,t) = 0\), we have

\[ 0 = E(T_1) - E(T_2) - \int_{T_1}^{T_2} |r(X,t)|^2 \, dt, \]
thus
\[ E(T_2) + \int_{T_1}^{T_2} |r(X,t)|^2 \, dt = E(T_1), \]
which implies \( E(T_2) \leq E(T_1) \), so \( E(t) \) is a decreasing non-negative function of \( t \). For fixed \( p > 0 \), we have
\[
0 \leq pE(T) \leq \int_{T-p}^{T} E(t) \, dt \leq \int_{C_0D_0BA} |l|^2 + |r|^2 \\
\leq C_{X,q} H(X) \\
= C_{X,q} \int_{T-X-p}^{T+X+p} |r(X,t)|^2 \, dt \quad \text{(because \( l = 0 \) on \( x = X \))}
\]
Letting \( T \to \infty \), we get the integral on the RHS goes to 0 because \( r(X,t) \in L^2[X, \infty) \) from (3.29). Thus \( E(T) \to 0 \) as \( T \to \infty \).
4.1 Introduction

Define

- \( \mathbb{R}^3_+ := \{ x \in \mathbb{R}^3 : x_3 \geq 0 \} \),
- \( A := \partial \mathbb{R}^3_+ = \{ x \in \mathbb{R}^3 : x_3 = 0 \} \).

Our goal is to recover the acoustic properties of a medium, represented by the half-space \( \mathbb{R}^3_+ \), from the data generated by fixed offset source/receiver pairs on the boundary, \( A \).

Let \( \partial_i = \partial_{x_i}, x = (x_1, x_2, x_3), a = (a_1, a_2, 0) \in A \), and \( h > 0 \). Let \( q \in C^\infty_c(\mathbb{R}^3) \) represent the acoustic property of the medium. For any \( a \in A \), the source/receiver pairs are at \( a - he_1 \) and \( a + he_1 \) respectively. If \( U^a(x, t) \) is the medium response, then \( U^a(x, t) \) is the solution of the IBVP

\[
(\Box - q)U^a(x, t) = 0, \quad x \in \mathbb{R}^3_+, \ t \in \mathbb{R} \\
\partial_3U^a(x, t) = \delta(x_1 - a_1 + h, x_2 - a_2, t), \quad x \in A, \ t \in \mathbb{R} \\
U^a(x, t) = 0, \quad x \in \mathbb{R}^3_+, \ t < 0.
\]

Goal: Suppose \( q(x) = 0 \) for \( 0 \leq x_3 \leq \epsilon_0 \) and \( |x| \) large enough. Given \( U^a(a + he_1, t) \) for all \( a \in A \) and \( t \in \mathbb{R} \), determine \( q \).
Extend $q$ as an even function of $x_3$ and consider $V^a(x,t)$, the solution of the IVP

$$(\Box - q)V^a(x,t) = \delta(x - a + he_1, t), \quad x \in \mathbb{R}^3, \ t \in \mathbb{R} \quad (4.1)$$
$$V^a(x,t) = 0, \quad x \in \mathbb{R}^3, \ t < 0 \quad (4.2)$$

As shown in [24], we have $U^a(x,t) = -2V^a(x,t)$ on $\mathbb{R}_+^3$, hence the problem is equivalent to determining a $q(x)$ which is even in $x_3$, supported away from $\partial\mathbb{R}_+^3$, from a knowledge of $V^a(a + he_1,t)$ for all $a \in A$ and all $t$.

In order to avoid dealing with the distributional terms, from Romanov in [29] we can use the progressing wave expansion to represent $V^a$ as a smooth term plus a distributional term.

**Theorem 4.1** (Romanov). Let $q \in C_c^\infty(\mathbb{R}^3)$, $a \in A$, and $0 < h < 1$. Then the IVP (4.1)-(4.2) has a unique distributional solution $V^a$ which can be written as

$$V^a(x,t) = \frac{1}{4\pi} \frac{\delta(t - |x - a + he_1|)}{|x - a + he_1|} + v^a(x,t)H(t - |x - a + he_1|) \quad (4.3)$$

where $v^a(x,t)$ is the smooth solution of the Goursat problem

$$\Box v^a - qv^a = 0, \quad x \in \mathbb{R}^3, \ t \geq |x - a + he_1|$$
$$v^a(x,|x - a + he_1|) = \frac{1}{8\pi} \int_0^1 q(a - he_1 + s(x - a + he_1)) \, ds, \quad x \in \mathbb{R}^3.$$

In the following theorem we show that the forward map

$$\mathcal{F} : q(x) \mapsto V^a(a + he_1,t)$$

is injective and stable under perturbations, given that the $q$ belong to a certain family of functions.
In the literature objects like \(V^a(a + he_1, t)\) are often referred to as the “backscattering data” (see [26], [2]). In order to discuss the stability of \(F\) in a meaningful way, we define a “backscattering data norm” for a function \(W^a(x, 2\tau)\) as:

\[
\|W^a(x, 2\tau)\|_{BS}^2 = \sup_{0 < \sigma < \sqrt{1 - h^2}} \int_{\mathbb{R}^2} |\partial_\sigma(W^a)(a + he_1, 2\tau)|^2 \, da_1 \, da_2,
\]

where

\[
\sigma = \sqrt{\tau^2 - h^2}.
\]  

(4.4)

We will use this for \(W^a = V^a_1 - V^a_2\) where \(V^a_i\) are the solutions to (4.1)-(4.2) corresponding to \(q_i, i = 1, 2\). The reader should note that, even though the solutions \(V^a_i\) are distributions, this integral makes sense because the difference is a function because of theorem 4.1. We show stability in this backscattering norm.

**Definition 4.1.** We say \(p \in C^\infty_c(\mathbb{R}^3)\) is **horizontally controlled** if there exists a constant \(C_0\) independent of \(x_3\) so that

\[
\int_{\mathbb{R}^2} |\nabla_{1,2}p(x_1, x_2, x_3)|^2 \, dx_1 \, dx_2 \leq C_0 \int_{\mathbb{R}^2} |p(x_1, x_2, x_3)|^2 \, dx_1 \, dx_2, \quad 0 \leq x_3 \leq 1,
\]

(4.5)

where \(\nabla_{1,2} = e_1 \partial_1 + e_2 \partial_2\).

**Theorem 4.2.** Let \(U^a_j\) be solutions of (4.1)-(4.2) corresponding to potentials \(q_j \in C^\infty_c(\mathbb{R}^3), j = 1, 2\). Given \(M > 0\), if \(\|q_j\|_{C^7(\mathbb{R}^3)} \leq M\) and \(q_1 - q_2\) are horizontal controlled, then we have the following stability estimate

\[
\|q_1 - q_2\|^2_{L^2(\mathbb{R}^2 \times [0, \sqrt{1 - h e_1^2}])} \leq C_{C_0, M, \text{supp} (q_1 - q_2)} \|V^a_1 - V^a_2\|^2_{BS}.
\]

(4.6)

### 4.2 Proof of Theorem 4.2

#### 4.2.1 Identity for the Difference of Two Solutions

In this subsection we construct an identity for the difference of two solutions \(V^a_i\). This identity is an integral equation that represents \((V^a_1 - V^a_2)(a + he_1, t)\) as integrals
of $q_1 - q_2$ and is crucial in proving theorem 4.2.

Let $\tau > 0$, and suppose $q_j(x) \in C_c^\infty(\mathbb{R}^3)$ and $V^a_j$, $j = 1, 2$, the corresponding solutions of (4.1)-(4.2). Define $p = q_1 - q_2$ and $W^a = V^a_1 - V^a_2$, then

$$\Box W^a - q_1 W^a = pV_2^a, \quad x \in \mathbb{R}^3, \ t \in \mathbb{R} \quad (4.7)$$

$$W^a(x,t) = 0, \quad x \in \mathbb{R}^3, \ t < 0. \quad (4.8)$$

Define

$$V^*(x,t) := V^{a+2h}_1(x, 2\tau - t);$$

then $V^*$ is the solution of the backward IVP

$$\Box V^* - q_1 V^* = \delta(x - a - he_1, 2\tau - t), \quad x \in \mathbb{R}^3, \ t \in \mathbb{R} \quad (4.9)$$

$$V^*(x,t) = 0, \quad x \in \mathbb{R}^3, \ t > 2\tau. \quad (4.10)$$

Since the intersection of the supports of $V^*$ and $W^a$ is compact, we can integrate by parts to get

$$W^a(a + he_1, 2\tau) = \iint_{\mathbb{R}^3 \times \mathbb{R}} W^a(x,t) \delta(x - a - he_1, 2\tau - t) \ dx \ dt$$

$$= \iint_{\mathbb{R}^3 \times \mathbb{R}} W(\Box V^* - q_1 V^*) \ dx \ dt$$

$$= \iint_{\mathbb{R}^3 \times \mathbb{R}} (\Box W - q_1 W^a)V^* \ dx \ dt$$

$$= \iint_{\mathbb{R}^3 \times \mathbb{R}} p(x)V_2^a(x,t)V^{a+2he_1}_1(x,2\tau - t) \ dx \ dt. \quad (4.11)$$

Using the result in [29], we can write

$$V_2^a(x,t) = \frac{1}{4\pi} \frac{\delta(t - |x - a + he_1|)}{|x - a + he_1|} + v_2^a(x,t)$$

$$V^{a+2he_1}_1(x,2\tau - t) = \frac{1}{4\pi} \frac{\delta(2\tau - t - |x - a + he_1|)}{|x - a - he_1|} + v^{a+2he_1}_1(x,2\tau - t),$$

47
where \( v_2^a(x, t) \) is supported in \( t \geq |x - a + he_1| \), \( v_1^{a+2he_1}(x, t) \) is supported and smooth in \( t \geq |x - a - he_1| \). Thus (4.11) may be written as

\[
W^a(a + he_1, 2\tau) = \frac{1}{16\pi^2} \int_{\mathbb{R}^3 \times \mathbb{R}} p(x) \frac{\delta(t - |x - a + he_1|)}{|x - a + he_1|} \frac{\delta(2\tau - t - |x - a + he_1|)}{|x - a + he_1|} \, dx \, dt
\]

\[
+ \frac{1}{4\pi} \int_{\mathbb{R}^3 \times \mathbb{R}} p(x) v_2^a(x, t) \frac{\delta(2\tau - t - |x - a + he_1|)}{|x - a + he_1|} \, dx \, dt
\]

\[
+ \frac{1}{4\pi} \int_{\mathbb{R}^3 \times \mathbb{R}} p(x) v_1^{a+2he_1}(x, 2\tau - t) \frac{\delta(t - |x - a + he_1|)}{|x - a + he_1|} \, dx \, dt
\]

\[
+ \int_{\mathbb{R}^3} p(x) v_1^{a+2he_1}(x, 2\tau - t) v_2^a(x, t) \, dx \, dt.
\]

Simplifying,

\[
W^a(a + he_1, 2\tau) = \frac{1}{16\pi^2} \int_{\mathbb{R}^3} p(x) \frac{\delta(2\tau - |x - a + he_1| - |x - a + he_1|)}{|x - a + he_1|} \, dx
\]

\[
+ \frac{1}{4\pi} \int_{\mathbb{R}^3} p(x) v_2^a(x, 2\tau - |x - a + he_1|) \, dx \, dt
\]

\[
+ \frac{1}{4\pi} \int_{\mathbb{R}^3} p(x) v_1^{a+2he_1}(x, 2\tau - |x - a + he_1|) \, dx \, dt
\]

\[
+ \int_{\mathbb{R}^3} p(x) v_1^{a+2he_1}(x, 2\tau - t) v_2^a(x, t) \, dx \, dt.
\]  

(4.12)

Since \( v_2^a(x, t) \) is supported in \( t \geq |x - a + he_1| \) and \( v_1^{a+2he_1}(x, t) \) is supported in \( t \geq |x - a - he_1| \), the middle two integrals are really integrals over the ellipsoid

\[
E(a, \tau) := \{ x \in \mathbb{R}^3 : |x - a + he_1| + |x - a - he_1| \leq 2\tau \}.
\]

The last integral is over the region \( 2\tau - t \geq |x - a - he_1| \), \( t \geq |x - a + he_1| \), i.e. over

\[
\{(x, t) : x \in E(a, \tau), \ |x - a + he_1| \leq t \leq 2\tau - |x - a - he_1| \}.
\]

So we can rewrite (4.12) as

\[
W^a(a + he_1, 2\tau) = \frac{1}{16\pi^2} \int_{\mathbb{R}^3} p(x) \frac{\delta(2\tau - |x - a + he_1| - |x - a + he_1|)}{|x - a + he_1|} \, dx
\]

\[
+ \int_{E(a, \tau)} p(x) k(x, a, he_1, \tau) \, dx.
\]

(4.13)
where

\[
k(x, a, he_1, \tau) = \frac{1}{4\pi} \left( v_2^o(x, 2\tau - |x - a - he_1|) + v_1^{a+2he_1}(x, 2\tau - |x - a + he_1|) \right)
+ \int_{|x-a+he_1|}^{2\tau-|x-a-he_1|} v_1^{a+2he_1}(x, 2\tau - t)v_2^o(x, t) \, dt.
\]

We note that because \( p(x) \) is supported away from \( x_3 = 0 \), we never have \(|x - a \pm he_1| = 0\) in or close to the support of \( p \); hence we may consider \( k \) as a smooth function of \( x, a, he_1, \) and \( \tau \). Using method similar to the proofs of theorems 1.2, and 1.3 in [2] we have the bounds, recalling that \( \sigma \) is defined by (4.4),

\[
\sup_{0 < \sigma \leq \sqrt{1-h^2}} \sup_{a \in A} \int_{\partial E(a, \tau)} |k(x, a, he_1, \tau)|^2 \, dS_x \leq K
\]

\[
\sup_{0 < \sigma \leq \sqrt{1-h^2}} \sup_{a \in A} \int_{E(a, \tau)} |\partial_\alpha k(x, a, he_1, \tau)|^2 \, dS_x \leq K \tag{4.14}
\]

for some \( K = K(M) > 0 \).

### 4.2.2 Weighted Ellipsoidal Mean Value

We would like to get rid of the distributional term in (4.13). Noting that on \( \partial E(a, \tau) \),

\[
|\nabla_x(2\tau - |x - a + he_1| - |x - a - he_1|)| = \frac{2|\tau(x - a + he_1) - |x - a + he_1|he_1|}{|x - a + he_1| |x - a - he_1|}, \tag{4.15}
\]

we can write the first integral in (4.13) as

\[
\frac{1}{16\pi^2} \int_{\mathbb{R}^3} p(x) \frac{\delta(2\tau - |x - a + he_1| - |x - a - he_1|)}{|x - a + he_1| |x - a - he_1|} \, dx
= \frac{1}{32\pi^2} \int_{\partial E(a, \tau)} \frac{p(x)}{|\tau(x - a + he_1) - |x - a + he_1|he_1|} \, dS_x
\]

For fixed \( 0 < h < 1 \), define

\[
D(x) := |\tau(x - a + he_1) - |x - a + he_1|he_1|,
\]
and define the operator $M$ to be

$$(M\rho)(a,\tau) = \frac{1}{32\pi^2} \int_{\partial E(a,\tau)} \frac{\rho(x)}{||\tau(x-a+he_1) - |x-a+he_1|e_1||} \, dS_x$$

$$= \frac{1}{32\pi^2} \int_{\partial E(a,\tau)} \frac{\rho(x)}{D(x)} \, dS_x, \quad a \in A, \quad \tau \geq 0.$$  \hspace{1cm} (4.16)

**Figure 4.1:** The upper half of $E(a,\tau)$

We have the following lemma, whose proof is in section 4.3.

**Lemma 4.1.** Points $x = (x_1, x_2, x_3) \in \partial E(a,\tau)$ can be parametrized by $\theta \in [0, 2\pi)$, representing the rotation of $x - a$ about $e_3$, and $x_3 \in [-\sigma, \sigma]$, so that

$$x - a = \left( \frac{\sqrt{\sigma^2 + h^2}}{\sigma} \sqrt{\sigma^2 - x_3^2} \cos \theta, \sqrt{\sigma^2 - x_3^2} \sin \theta, \ x_3 \right).$$

The surface element $dS_x$ on $\partial E(a,\tau)$ is

$$dS_x = J(\theta, x_3) \, d\theta \, dx_3,$$

where

$$J(\theta, x_3)^2 = (\sigma^2 - x_3^2) \left( \cos^2 \theta + \left( 1 + \frac{h^2}{\sigma^2} \right) \sin^2 \theta \right) + (\sigma^2 + h^2)x_3^2(\sigma^2) \left( \frac{\sin^2 \theta}{\sigma^2} + \cos^2 \theta \right)^2.$$  

Moreover, for $x \in \partial E(a,\tau)$, and $h < \tau \leq 1$, we have the following estimates on $D(x)$, $J(x,\theta)$, and their derivatives in $\sigma$:

(a) $|D(x)| = O(\sigma^4)$.  

50
(b) \( \frac{1}{|D(x)|} = O(\sigma^{-4}), \sigma \to 0. \)

(c) \( |\partial_\sigma D(x)| = O(\sigma^{-7}) + O(\sigma^{-4}) \frac{1}{\sqrt{\sigma^2 - x_3^2}}, \sigma \to 0 \)

(d) \( \sigma \leq J(\theta, x_3) \leq \tau. \)

(e) \( |\partial_\sigma J(\theta, x_3)| = O(\sigma^{-4}), \sigma \to 0 \)

We prove lemma 4.1 in section 4.3. We will frequently refer back to this lemma in the following estimates.

Since \( p = 0 \) near \( x_3 = 0 \) and the integrand is even in \( x_3 \), we get

\[
(Mp)(a, \tau) = \frac{1}{16\pi^2} \int_0^\sigma \int_0^{2\pi} p(x) \frac{J(\theta, x_3)}{D(x)} d\theta dx_3
\]

We can compute

\[
\frac{\partial x}{\partial \sigma} = \left( \frac{(\sigma^4 + h^2 x_3^2)}{\sigma^2 \sqrt{\sigma^2 + h^2} \sqrt{\sigma^2 - x_3^2}}, \frac{\sigma}{\sqrt{\sigma^2 - x_3^2}} \sin \theta, 0 \right).
\]

which gives, recalling \( \tau^2 = \sigma^2 - h^2 \),

\[
\left| \frac{\partial x}{\partial \sigma} \right| \leq \frac{\sqrt{(\sigma^4 + h^2 x_3^2)^2 + \sigma^6 \tau^2}}{\sigma^2 \tau \sqrt{\sigma^2 - x_3^2}} \approx \frac{1}{\sigma^2 \sqrt{\sigma^2 - x_3^2}}.
\] (4.17)

Define the upper half of \( \partial E(a, \tau) \) by

\[
\partial^+ E(a, \tau) := \partial E(a, \tau) \cap \mathbb{R}_+^3.
\] (4.18)

Then, since at \( x = a + \sigma e_3 \), \( J = \tau^2 \sigma^2 \) and \( D = \tau \sigma \), so

\[
\partial_\sigma (Mp)(a, \tau) = \frac{\tau^2 \sigma + \sigma^2 \tau}{16\pi} p(a + \sigma e_3)
\]

\[
+ \frac{1}{16\pi^2} \int_0^\sigma \int_0^{2\pi} (\nabla p \cdot x_\sigma) \frac{J}{D} + p \frac{DJ_\sigma - D_\sigma J}{D^2} d\theta dx_3
\]

\[
= \frac{\tau^2 \sigma + \sigma^2 \tau}{16\pi} p(a + \sigma e_3) + \frac{1}{16\pi^2} \int_{\partial^+ E(a, \tau)} \frac{x_\sigma \cdot \nabla_{1,2} p(x)}{D(x)} dS_x
\]

\[
+ \frac{1}{16\pi^2} \int_{\partial^+ E(a, \tau)} p(x) \left( \frac{DJ_\sigma - D_\sigma J}{JD^2} \right)(x) dS_x.
\] (4.19)
Then,
\[
\frac{(\tau + \sigma)^2 \sigma^2 \tau^2}{256\pi^2} |p(a + \sigma e_3)|^2 \leq 3 |\partial_\sigma (Mp)(a, \tau)|^2 + \frac{3}{256\pi^4} \left( \int_{\partial^+ E(a,\tau)} \frac{x_\sigma \cdot \nabla_{1,2} p(x)}{D} \ dS_x \right)^2
\]
\[+ \frac{3}{256\pi^4} \left( \int_{\partial^+ E(a,\tau)} p(x) \frac{D J_\sigma - D J }{JD^2} \ dS_x \right)^2.
\]

Using Cauchy-Schwarz and (4.17),
\[
(\tau + \sigma)^2 \sigma^2 \tau^2 |p(a + \sigma e_3)|^2
\leq 768 \pi^2 |\partial_\sigma (Mp)(a, \tau)|^2 + \frac{3}{\sigma^2 \pi^2} \int_{\partial^+ E(a,\tau)} \frac{|\nabla_{1,2} p(x)|^2}{\sqrt{\sigma^2 - x_3^2}} \ dS_x \cdot \int_{\partial^+ E(a,\tau)} \frac{1}{D^2} \ dS_x
\]
\[+ \frac{3}{\pi^2} \int_{\partial^+ E(a,\tau)} |p(x)|^2 \ dS_x \int_{\partial^+ E(a,\tau)} \left( \frac{D J_\sigma - D J }{JD^2} \right)^2 \ dS_x.
\]

Then appealing to lemma 4.1, we have
\[
\int_{\partial^+ E(a,\tau)} \left( \frac{D J_\sigma - D J }{JD^2} \right)^2 \ dS_x \leq |\partial^+ E(a, \tau)| O(\sigma^{-32}), \quad \sigma \to 0,
\]
\[\int_{\partial^+ E(a,\tau)} \frac{1}{D^2} \ dS_x \leq |\partial^+ E(a, \tau)| O(\sigma^{-8}), \quad \sigma \to 0.
\]

Thus
\[
|p(a + \sigma e_3)|^2 \leq |\partial_\sigma (Mp)(a, \tau)|^2 + C_\sigma \int_{\partial^+ E(a,\tau)} \frac{|\nabla_{1,2} p(x)|^2 + |p(x)|^2}{\sqrt{\sigma - x_3}} \ dS_x, \quad (4.20)
\]

where \( C_\sigma \) is \( O(\sigma^{-34}) \) as \( \sigma \to 0 \).

### 4.2.3 Stability Estimate

Recall \( W^a(a + he_1, 2\tau) \) is the difference of the two solutions \( V_2 \) and \( V_1 \) to (4.1)-(4.2). From (4.13), we get
\[
-(Mp)(a, \tau) = -W^a(a + he_1, 2\tau) + \int_{E(a,\tau)} p(x) k(x, a, he_1, \tau) \ dx. \quad (4.21)
\]
So
\[-\partial_\sigma (M p)(a, \tau) = - \partial_\sigma W^a(a + he_1, 2\tau) + \int_{E(a, \tau)} p(x) \partial_\sigma k(x, a, he_1, \tau) \, dx \]
\[+ \frac{\sigma}{\tau} \int_{\partial E(a, \tau)} p(x) k(x, a, he_1, \tau) \frac{|x - a - he_1|}{D^2} \, dS_x.\]

Recall (4.14), then from (4.32), the fact that $p$ is even in $x_3$, and Cauchy-Schwarz

\[|\partial_\sigma (M p)(a, \tau)| \leq |\partial_\sigma W^a(a + he_1, 2\tau)| + \int_{E^+(a, \tau)} |p(x)| \, dx + C_\sigma \int_{\partial^+ E(a, \tau)} |p(x)| \, |k| \, dS_x\]
\[|\partial_\sigma (M p)(a, \tau)|^2 \leq |\partial_\sigma W^a(a + he_1, 2\tau)|^2 + \int_{E^+(a, \tau)} |p(x)|^2 \, dx + C_\sigma^2 \int_{\partial^+ E(a, \tau)} |p(x)|^2 \, dS_x.\]

where $C_\sigma = O(\sigma^{-7})$ as $\sigma \to 0$ via lemma 4.1. Then from (4.20), we get that

\[|p(a + \sigma e_3)|^2 \leq |\partial_\sigma W^a(a + he_1, 2\tau)|^2 + \int_{E^+(a, \tau)} |p(x)|^2 \, dx\]
\[+ C_\sigma \int_{\partial^+ E(a, \tau)} |\nabla_{1,2} p(x)|^2 + |p(x)|^2 \frac{dS_x}{\sqrt{\sigma - x_3}}.\]

(4.22)

where the (new) $C_\sigma = O(\sigma^{-34})$ as $\sigma \to 0$. Let $a = (a', 0)$ for $a' \in \mathbb{R}^2$. Henceforth, we will refer to integrals over $a$ as

\[\int_A f(a) \, dS_a = \int_{\mathbb{R}^2} f(a', 0) \, da'.\]

Integrate (4.22) over $a \in A$ to get

\[\int_{\mathbb{R}^2} |p(a + \sigma e_3)|^2 \, da' \leq \int_{\mathbb{R}^2} |\partial_\sigma W^a(a + he_1, 2\tau)|^2 \, da' + \int_{\mathbb{R}^2} \int_{E^+(a, \tau)} |p(x)|^2 \, dx \, da'\]
\[+ C_\sigma \int_{\mathbb{R}^2} \int_{\partial^+ E(a, \tau)} |\nabla_{1,2} p(x)|^2 + |p(x)|^2 \frac{dS_x}{\sqrt{\sigma - x_3}} \, dS_x \, da'.\]

(4.23)

The next few steps will help us change the order of integration and reduce the integrals over $E$ and $\partial E$ to integrals over $\mathbb{R}^3$.

From lemma 4.1 we can describe $\partial E(a, \tau)$ as the points $x \in \mathbb{R}^3$ so that

\[\frac{\sigma}{\tau} (x_1 - a_1)^2 + (x_2 - a_2)^2 = \sigma^2 - x_3^2 \quad (x_1, x_2) \in \mathbb{R}^2, \ |x_3| \leq \sigma.\]
So we have \( \partial E(a, \tau) = \{x \in \mathbb{R}^3 : \varphi(x, a) = 0\} \) where

\[
\varphi(x, a) = \frac{\sigma}{\tau} (x_1 - a_1)^2 + (x_2 - a_2)^2 + x_3^2 - \sigma^2.
\]

Now

\[
|\nabla_{a'} \varphi(x, a)| = |(2\sigma/\tau(x_1 - a_1), 2(x_2 - a_2), 0)| = 2\sqrt{\frac{\sigma^2}{\tau^2}(x_1 - a_1)^2 + (x_2 - a_2)^2}.
\]

Since \( 0 \leq \sigma/\tau \leq 1 \), we have

\[
2\sqrt{\frac{\sigma^2}{\tau^2}(x_1 - a_1)^2 + \frac{\sigma}{\tau}(x_2 - a_2)^2} \leq |\nabla_{a'} \varphi(x, a)| \leq 2\sqrt{\frac{\sigma}{\tau}(x_1 - a_1)^2 + (x_2 - a_2)^2}.
\]

Thus, on \( \varphi(x, a) = 0 \), we have

\[
2\frac{\sigma}{\tau}\sqrt{\sigma^2 - x_3^2} \leq |\nabla_{a'} \varphi(x, a)| \leq 2\sqrt{\sigma^2 - x_3^2}.
\quad (4.24)
\]

Noting that \( \partial E(a, \tau) \) is empty when \( \sigma < |x_3| \) and that \( 0 < \sigma/\tau < 1 \), we have

\[
\int_{\mathbb{R}^2} \delta(\varphi(x, a)) \, da' = \int_{\varphi(x,a)=0} \frac{1}{|\nabla_{a'} \varphi(x, a)|} \, dS_{a'}
\]

\[
\leq \frac{\tau}{2\sigma\sqrt{\sigma^2 - x_3^2}} \int_{\varphi(x,a)=0} \, dS_{a'}
\]

\[
= \frac{\tau}{2\sigma\sqrt{\sigma^2 - x_3^2}} \cdot 4\sqrt{\sigma^2 - x_3^2} \cdot C(h/\tau)
\]

\[
= \left\{ \begin{array}{ll}
\frac{2\pi}{\sigma} C(h/\tau) & : |x_3| < \sigma \\
0 & : |x_3| > \sigma
\end{array} \right.,
\]

where \( C(h/\tau) \) is the circumference of the ellipse with eccentricity \( h/\tau \) and semi-major axis of length \( \tau \) (see §19.1 in [10]). Thus

\[
\int_{\mathbb{R}^2} \delta(\varphi(x, a)) \, da' \leq \frac{\pi \tau}{\sigma} H(\sigma - |x_3|).
\quad (4.25)
\]

Also

\[
\nabla_x \varphi(x, a) = 2(\sigma/\tau(x_1 - a_1), x_2 - a_2, x_3),
\]

so

\[
|\nabla_x \varphi(x, a)| = 2\sqrt{\frac{\sigma^2}{\tau^2}(x_1 - a_1)^2 + (x_2 - a_2)^2 + x_3^2}.
\]

54
Hence for any integrable, even in $x_3$, non-negative $f$ on $\mathbb{R}^3$ with $f(x_1, x_2, 0) = 0$, noting that $0 < \sigma/\tau < 1$, we have

\[
\int_{\mathbb{R}^2} \int_{\partial^+ E(a, \tau)} f(x) \, dS_x \, da' = \int_{\mathbb{R}^2} \int_{\partial^+ E(a, \tau)} f(x) H(\sigma - |x_3|) \, dS_x \, da'
= \int_{\mathbb{R}^2 \times (0, \sigma)} f(x) \int_{\mathbb{R}^2} \sqrt{\frac{\sigma^2}{\tau^2} (x_1 - a_1)^2 + (x_2 - a_2)^2 + x_3^2 \cdot \delta(\varphi(x, a))} \, da' \, dx
\leq \int_{\mathbb{R}^2 \times (0, \sigma)} f(x) \int_{\mathbb{R}^2} \varphi(x, a) + \sigma^2 \cdot \delta(\varphi(x, a)) \, da' \, dx
= \sigma \int_{\mathbb{R}^2 \times (0, \sigma)} f(x) \int_{\mathbb{R}^2} \delta(\varphi(x, a)) \, da' \, dx
\leq \frac{\pi \tau}{2} \int_{\mathbb{R}^2 \times (0, \sigma)} f(x) \, dx,
\]

(4.26)

where $\mathbb{R}^2 \times (0, \sigma) = \{x \in \mathbb{R}^3 : 0 < x_3 < \sigma\}$. The integral of $H(-\varphi(x, a))$ in $a$ is just the area of the ellipse

\[
\frac{\sigma}{\tau(\sigma^2 - x_3^2)} (x_1 - a_1)^2 + \frac{1}{\sigma^2 - x_3^2} (x_2 - a_2)^2 = 1,
\]

i.e.

\[
\int_{\mathbb{R}^2} H(-\varphi(x, a)) \, da' = \pi \sqrt{\frac{\tau}{\sigma}} (\sigma^2 - x_3^2).
\]

(4.27)

Then for a non-negative integrable $f$,

\[
\int_{\mathbb{R}^2} \int_{E^+(a, \tau)} f(x) \, dx \, da' = \int_{\mathbb{R}^2} f(x) H(\sigma - x_3) \int_{\mathbb{R}^2} H(-\varphi(x, a)) \, da' \, dx
= \pi \sqrt{\frac{\tau}{\sigma}} \int_{\mathbb{R}^2 \times (0, \sigma)} f(x) (\sigma^2 - x_3^2) \, dx
\leq \pi \sqrt{\tau \sigma^3/2} \int_{\mathbb{R}^2 \times (0, \sigma)} f(x) \, dx.
\]

(4.28)

So using (4.26), (4.28), and letting $x' = (x_1, x_2)$, (4.23) implies

\[
\int_{\mathbb{R}^2} |p(x', \sigma)|^2 \, dx' \ll \|W^a\|_{B^S} + C_{\sigma, 1} \int_{0}^{\sigma} \int_{\mathbb{R}^2} |p(x', x_3)|^2 + \frac{\|\nabla_{1,2} p(x', x_3)\|^2 + |p(x', x_3)|^2}{\sqrt{\sigma - x_3}} \, dx' \, dx_3
\]

where $x' = (x_1, x_2) \in \mathbb{R}^2$ and $C_{\sigma, 1} = O(\sigma^{-34})$ as $\sigma \to 0$. But $p$ is horizontally controlled, so from (4.5), we can say

\[
\int_{\mathbb{R}^2} |p(x', \sigma)|^2 \, dx' \ll \|W^a\|_{B^S} + C_{\sigma, 1} \int_{0}^{\sigma} \int_{\mathbb{R}^2} |p(x', x_3)|^2 + \frac{C_0 |p(x', x_3)|^2}{\sqrt{\sigma - x_3}} \, dx' \, dx_3.
\]

(4.29)
Define
\[ P(x_3) := \int_{\mathbb{R}^2} |p(x', x_3)|^2 \, dx'; \]
then noting that \( 0 < \sigma \leq 1 - h^2 \leq 1 \), (4.29) becomes
\[
P(\sigma) \lesssim \|W^a\|^2_{BS} + C_{\sigma, 2} \int_0^{\sigma} P(x_3) + \frac{P(x_3)}{\sqrt{\sigma - x_3}} \, dx_3 \\
= \|W^a\|^2_{BS} + C_{\sigma, 2} \int_0^{\sigma} \left(1 + \frac{1}{\sqrt{\sigma - x_3}}\right) P(x_3) \, dx_3 \\
\lesssim \|W^a\|^2_{BS} + C_{\sigma, 3} \int_0^{\sigma} \frac{P(x_3)}{\sqrt{\sigma - x_3}} \, dx_3,
\]
(4.30)
where \( C_{\sigma, 3} = O(\sigma^{-34}) \) as \( \sigma \to 0 \). Let \( \epsilon > 0 \) and consider \( \sigma \) so \( \epsilon < \sigma \). Applying (4.30) twice, there is a \( C_\epsilon > 0 \) such that
\[
P(\sigma) \leq \|W^a\|^2_{BS} + C_\epsilon \int_0^{\sigma} \frac{\|W^a\|^2_{BS} + \int_{x_3}^{\sigma} \frac{P(s)}{\sqrt{\sigma - x_3}} \, ds}{\sqrt{\sigma - x_3}} \, dx_3 \\
= \|W^a\|^2_{BS} + C_\epsilon \int_0^{\sigma} \int_0^{x_3} \frac{P(s)}{\sqrt{\sigma - x_3} \sqrt{\sigma - s}} \, ds \, dx_3 \\
= \|W^a\|^2_{BS} + C_\epsilon \int_0^{\sigma} P(s) \int_s^{\sigma} \frac{1}{\sqrt{(\sigma - x_3)(x_3 - s)}} \, dx_3 \, ds \\
= \|W^a\|^2_{BS} + \pi C_\epsilon \int_0^{\sigma} P(s) \, ds.
\]
Thus by Gronwall’s inequality,
\[
P(\sigma) \leq e^{\pi C_\epsilon} \|W^a\|^2_{BS}, \quad \epsilon < \sigma < \sqrt{1 - he_1^2}.
\]
Since this holds for all \( \sigma \in (\epsilon, 1] \), we have that
\[
\int_{\epsilon}^{\sqrt{1 - he_1^2}} P(\sigma) \, d\sigma = \|p\|^2_{L^2(\mathbb{R}^2 \times [\epsilon, \sqrt{1 - he_1^2}])} \leq (1 - \epsilon) e^{\pi C_\epsilon} \|W^a\|^2_{BS}.
\]
Because \( p \) is supported away from \( x_3 = 0 \), we have that \( P(\sigma) = 0 \) for \( \sigma \) small enough. Hence, we can find \( C > 0 \) independent of \( \sigma \) that depends only on \( C_0, M \), and \( \text{supp} \, p \) so that
\[
\|p\|^2_{L^2(\mathbb{R}^2 \times 0, \sqrt{1 - he_1^2})} \leq C \|W^a\|^2_{BS}.
\]
4.3 Proof of Lemma 4.1

4.3.1 Parametrizing \( \partial E(a, \tau) \)

Recall for \( \tau > h \),
\[
\partial E(a, \tau) = \{ x \in \mathbb{R}^3 : |x - a + he_1| + |x - a - he_1| = 2\tau \}.
\]

For \( x \in \partial E(a, \tau) \) and \( \sigma^2 = \tau^2 - h^2 \), we have
\[
|x - a + he_1| = 2\tau - |x - a - he_1| \\
|x-a+he_1|^2 - |x-a-he_1|^2 = 4\tau^2 - 4\tau|x-a-he_1| \\
4(x-a) \cdot he_1 = 4\tau^2 - 4\tau|x-a-he_1| \\
\tau^2|x-a-he_1|^2 = (\tau^2 - (x-a) \cdot he_1)^2 \\
\tau^2|x-a|^2 - 2\tau^2(x-a) \cdot he_1 + \tau^2h^2 = \tau^4 - 2\tau^2(x-a) \cdot he_1 + ((x-a) \cdot he_1)^2 \\
\tau^2|x-a|^2 - h^2(x_1-a_1)^2 = \tau^2(\tau^2 - h^2) \\
\sigma^2(x_1-a_1)^2 + \tau^2(x_2-a_2)^2 + \tau^2x_3^2 = \tau^2\sigma^2.
\]

Thus the ellipsoid \( \partial E(a, \tau) \) can be parametrized by
\[
\frac{(x_1-a_1)^2}{\tau^2} + \frac{(x_2-a_2)^2}{\sigma^2} + \frac{x_3^2}{\sigma^2} = 1. \tag{4.31}
\]

Reparametrize \( \partial E(a, \tau) \) by \( \sigma, x_3 \) and \( \theta \), the angle of rotation of \( x \) in the plane.

Then
\[
x - a = \left( \frac{\tau}{\sigma} \sqrt{\sigma^2 - x_3^2} \cos \theta, \sqrt{\sigma^2 - x_3^2} \sin \theta, x_3 \right) \\
= \left( \frac{\sqrt{\sigma^2 + h^2}}{\sigma} \sqrt{\sigma^2 - x_3^2} \cos \theta, \sqrt{\sigma^2 - x_3^2} \sin \theta, x_3 \right).
\]
To compute $dS_x$ in terms of $\theta$ and $x_3$, note

$$x_\theta = \left( -\frac{\tau}{\sigma} \sqrt{\sigma^2 - x_3^2} \sin \theta, \sqrt{\sigma^2 - x_3^2} \cos \theta, 0 \right)$$

$$x_{x_3} = \left( -\frac{\tau x_3}{\sigma \sqrt{\sigma^2 - x_3^2}} \cos \theta, -\frac{x_3}{\sqrt{\sigma^2 - x_3^2}} \sin \theta, 1 \right),$$

and

$$x_\theta \times x_{x_3} \cdot e_1 = \sqrt{\sigma^2 - x_3^2} \cos \theta,$$

$$x_\theta \times x_{x_3} \cdot e_2 = \frac{\tau}{\sigma} \sqrt{\sigma^2 - x_3^2} \sin \theta,$$

$$x_\theta \times x_{x_3} \cdot e_3 = \frac{\tau x_3}{\sigma}.$$

Thus

$$J(\theta, x_3) := |x_\theta \times x_{x_3}| = \frac{\tau}{\sigma} \sqrt{\frac{\sigma^2}{\tau^2} (\sigma^2 - x_3^2) \cos^2 \theta + (\sigma^2 - x_3^2) \sin^2 \theta + x_3^2},$$

so

$$dS_x = \frac{\tau}{\sigma} \sqrt{\frac{\sigma^2}{\tau^2} (\sigma^2 - x_3^2) \cos^2 \theta + (\sigma^2 - x_3^2) \sin^2 \theta + x_3^2} \, d\theta \, dx_3,$$

where $|x_3| \leq \sigma$ and $0 \leq \theta \leq 2\pi$.

4.3.2 Bounding the Terms $D$ and $J$ and Derivatives on $\partial E(a, \tau)$

Recall

$$D = |\tau(x - a + he_1)| - |x - a + he_1| |he_1| = |x - a + he_1| \left| \frac{\tau}{|x - a + he_1|} - he_1 \right|.$$

Note that $\tau > h$, because if $\tau \leq h$, then $W^a(a + he_1, 2\tau) = 0$. We show that $|x - a + he_1|$ and $\left| \frac{\tau}{|x - a + he_1|} - he_1 \right|$ are finite and bounded away from 0 for $x \in \partial E(a, \tau)$.

Given any two unit vectors $k$ and $l$, and $\tau > 0$, we have

$$\tau - h \leq |\tau k + hl| \leq \tau + h,$$
thus
\[ \tau - h \leq \left| \frac{x - a + he_1}{|x - a + he_1|} - he_1 \right| \leq \tau + h. \]

By the symmetry of \( E(a, \tau) \), the largest and smallest values of \( |x - a + he_1| \) on \( \partial E(a, \tau) \) occur on the intersection of \( A \) with \( \partial E(a, \tau) \) at \( x - a = \pm \tau e_1 \). So
\[ \tau - h = | - \tau e_1 + he_1 | \leq |x - a + he_1| \leq | \tau e_1 + he_1 | = \tau + h. \]

Thus, for \( \tau > h \),
\[ \frac{\sigma^4}{(\tau + h)^2} = (\tau - h)^2 \leq D \leq (\tau + h)^2 = \frac{\sigma^4}{(\tau - h)^2}. \] (4.32)

To compute \( D_\sigma \), let \( \omega = x - a + he_1 \). Then
\[ D^2 = |\tau \omega + |\omega| he_1|^2 = \tau^2 |\omega|^2 + 2\tau |\omega| \cdot he_1 + |\omega|^2 h^2 = |\omega|^2 \left[ \tau^2 + 2\tau \left( \frac{\omega}{|\omega|} \cdot he_1 \right) + h^2 \right] \]
and differentiating in \( \sigma \) gives, noting that \( \frac{\partial \tau}{\partial \sigma} = \frac{\tau}{\sigma} \),
\[ DD_\sigma = \sigma |\omega|^2 + \tau^2 |\omega| \partial_\sigma |\omega| + \frac{\sigma}{\tau} |\omega| (\omega \cdot he_1) + \tau \partial_\sigma |\omega| (\omega \cdot he_1) + \tau |\omega| \partial_\sigma (\omega \cdot he_1) + |\omega| \partial_\sigma |\omega| h^2 \]
\[ = \sigma |\omega|^2 + \frac{\sigma}{\tau} |\omega| (\omega \cdot he_1) + (|\omega|^2 + \tau (\omega \cdot he_1 + \tau^2 |\omega|) \partial_\sigma |\omega| + \tau |\omega| \partial_\sigma (\omega \cdot he_1) \]
(4.33)

To solve for \( D_\sigma \) we compute \( \partial_\sigma |\omega| \) and \( \partial_\sigma (\omega \cdot he_1) \). From (4.31), we have
\[ |\omega|^2 = (x_1 - a_1 + h)^2 + (x_2 - a_2)^2 + x_3^2 \]
\[ = (x_1 - a_1 + h)^2 - \frac{\sigma^2 (x_1 - a_1)^2}{\tau^2} + \sigma^2 \]
\[ = \sigma^2 + \frac{(\tau^2 - \sigma^2)(x_1 - a_1)^2 + 2\tau^2 h_1 (x_1 - a_1) + h^2 \tau^2}{\tau^2} \]
\[ = \sigma^2 + \frac{h^2}{\tau^2} (x_1 - a_1)^2 + 2h_1 (x_1 - a_1) + h^2 \]
\[ = \sigma^2 + \frac{h^2 (\sigma^2 - x_3^2)}{\sigma^2} \cos^2 \theta + \frac{2h_1 \tau \sqrt{\sigma^2 - x_3^2}}{\sigma} \cos \theta + h^2 \]

and
\[ \omega \cdot he_1 = \frac{h \tau \sqrt{\sigma^2 - x_3^2}}{\sigma} \cos \theta \]
So
\[ |\omega| \partial_\sigma |\omega| = \sigma + \frac{h^2 x^2_3}{\sigma^3} \cos^2 \theta + \left( \frac{h \sqrt{\sigma^2 - x^2_3}}{\tau} + \frac{h \tau}{\sqrt{\sigma^2 - x^2_3} - \frac{h \tau \sqrt{\sigma^2 - x^2_3}}{\sigma^2}} \right) \cos \theta, \]
\[ \partial_\sigma (\omega \cdot he_1) = \left( \frac{h \sqrt{\sigma^2 - x^2_3}}{\tau} + \frac{h \tau}{\sqrt{\sigma^2 - x^2_3} - \frac{h \tau \sqrt{\sigma^2 - x^2_3}}{\sigma^2}} \right) \cos \theta. \]

Then (4.33) implies
\[ \begin{align*}
DD_\sigma &= \sigma |\omega|^2 + \frac{\sigma}{\tau} |\omega| (\omega \cdot he_1) \\
&+ \left( h^2 + \tau \left( \frac{\omega}{|\omega|} \cdot he_1 \right) + \tau^2 \right) \left( \sigma + \frac{h^2 x^2_3}{\sigma^3} \cos^2 \theta \right) \\
&+ \left( h^2 + \tau \left( \frac{\omega}{|\omega|} \cdot he_1 \right) + \tau^2 \right) \left[ \frac{h \sqrt{\sigma^2 - x^2_3}}{\tau} + \frac{h \tau}{\sqrt{\sigma^2 - x^2_3} - \frac{h \tau \sqrt{\sigma^2 - x^2_3}}{\sigma^2}} \right] \cos \theta \\
&+ \tau |\omega| \left( \frac{h \sqrt{\sigma^2 - x^2_3}}{\tau} + \frac{h \tau}{\sqrt{\sigma^2 - x^2_3} - \frac{h \tau \sqrt{\sigma^2 - x^2_3}}{\sigma^2}} \right) \cos \theta.
\end{align*} \]

Note that
\[ \left| h^2 + \tau \left( \frac{\omega}{|\omega|} \cdot he_1 \right) + \tau^2 \right| \leq (\tau + h)^2. \]

From (4.32), \( 1/D = O(\sigma^{-4}) \) as \( \sigma \to 0 \). Hence, as \( \sigma \to 0 \),
\[ |D_\sigma| \leq O(\sigma^{-7}) + \tau h \frac{(\tau + h)^2 + \tau |x - a + he_1|}{D \sqrt{\sigma^2 - x^2_3}} = O(\sigma^{-7}) + O(\sigma^{-4}) \frac{1}{\sqrt{\sigma^2 - x^2_3}} \quad (4.34) \]

To bound \( J \) and \( \partial_\sigma J \), recall
\[ J(\theta, x_3) = \frac{\tau}{\sigma} \sqrt{\frac{\sigma^2}{\tau^2} (\sigma^2 - x^2_3) \cos^2 \theta + (\sigma^2 - x^2_3) \sin^2 \theta + x^2_3}. \]

Using calculus, we can determine that
\[ \min_{\theta, x_3} J = \sigma \leq J \leq \tau = \max_{\theta, x_3} J. \]
Additionally,

\[ \partial_\sigma J = \sigma \cos^2 \theta + \left( \sigma - \frac{(\sigma^2 - \tau^2)x_3^2}{\sigma^3} \right) \sin^2 \theta + \frac{(\sigma^2 - \tau^2)x_3^2}{\sigma^3} \]

\[ = \frac{\sigma^4 \cos^2 \theta + (\sigma^4 - (\sigma^2 - \tau^2)x_3^2) \sin^2 \theta + (\sigma^2 - \tau^2)x_3^2}{\sigma^3 J} \]

\[ = \frac{\sigma^4 + (\sigma^2 - \tau^2)x_3^2 \sin^2 \theta}{\sigma^3 J}. \]

Thus using the fact that \( \sigma \leq J \leq \tau \), we have

\[ |\partial_\sigma J| \leq \frac{\sigma^4 + h^2 x_3^2}{\sigma^4} \leq 1 + \frac{h^2}{\sigma^4} < \infty, \quad h < \tau < 1. \]

So

\[ |\partial_\sigma J| = O(\sigma^{-4}), \quad \sigma \to 0. \quad (4.35) \]
Chapter 5

SPHERICAL WAVE BACKSCATTERING

5.1 Introduction and Results

Our goal is to determine the acoustic properties of a medium in $\mathbb{R}^3$ which is homogeneous outside the unit ball. We excite the medium by an incoming spherical wave and measure the medium response at every point on the unit sphere over a finite time interval.

Let $B$ denote the open unit ball in $\mathbb{R}^3$ and $S = \partial B$. Suppose $q \in C^\infty_c(\mathbb{R}^3)$ with $\text{supp } q \subseteq B$ and $U(x,t)$ is the solution of the IVP

$$\Box U - q(x)U = 0, \quad (x,t) \in \mathbb{R}^3 \times \mathbb{R} \quad (5.1)$$

$$U(x,t) = \frac{2\delta(t + |x|)}{|x|}, \quad x \in \mathbb{R}^3, \ t < -1. \quad (5.2)$$

We have two goals in this chapter:

1. Show that (5.1)-(5.2) has a unique distributional solution $U(x,t)$.

2. Solve the inverse problem: given $U(a,t)$ for all $a \in S$, $t \in (-1,1)$, recover $q(x)$ for $x \in B$ and show the recovery is unique and stable under perturbations.
In the literature objects similar to $U(a,t)$ for $(a,t) \in S \times (-1,1)$ are referred to as “backscattering data”, see [26], [2]. In order to discuss these results in a meaningful way, we define a “backscattering data norm” for the difference of two solutions $U_1$ and $U_2$ of (5.1)-(5.2):

**Definition 5.1.** For solutions $U_i$, $i = 1, 2$, of (5.1)-(5.2), we define the **backscattering data norm** to be

$$
\|U_1 - U_2\|_{BS}^2 = \sup_{-1<\tau<1} \int_S \left( \partial_\tau(U_1 - U_2)(a, \tau) \right)^2 dS_a.
$$

The reader should note that, even though the solutions $U_i$ are distributions, this integral makes sense because the difference $U_1 - U_2$ is a function, which we prove below.

The inverse problem under consideration is the recovery of $q$ in $\mathcal{B}$ given $U$ on $S \times (-1,1)$. Consider the forward map:

$$
\mathcal{F} : q(x) \mapsto U(a, \tau)|_{a \in S, -1<\tau<1}.
$$

We show that $\mathcal{F}$ is injective and stable under perturbations, provided $q$ belongs to a family of smooth functions whose pairwise differences are *angularly controlled*.

**Definition 5.2 (Angular Control).** Given $p \in C_0^\infty(\mathbb{R}^3)$, we say $p$ is **angularly controlled** if there is a constant $C_0$ independent of $\rho$ so that for any $0 < \rho < 1$,

$$
\sum_{i<j} \int_{|x|=\rho} |\Omega_{ij}p(x)|^2 dS_x \leq C_0 \int_{|x|=\rho} |p(x)|^2 dS_x,
$$

(5.3)

where $\Omega_{ij} = x_i \partial_j - x_j \partial_i$ are the angular derivatives.

**Theorem 5.1 (Existence and uniqueness for the forward problem).** If $q \in C_0^\infty(\mathbb{R}^3)$ and $\text{supp } q \subseteq B$, then there is a unique distribution $U(x,t)$ on $\mathbb{R}^3 \times \mathbb{R}$ that satisfies (5.1)-(5.2). Further, for any positive integer $m$,

$$
U(x,t) = \frac{2\delta(t + |x|)}{|x|} - V(x,t) + \sum_{k=0}^{m} H(t + |x|) a_k(x) \frac{(t + |x|)^k}{k!} + r(x,t), \quad (x,t) \in \mathbb{R}^3 \times \mathbb{R}
$$
where $V$ is the unique distributional solution of the IVP

\[ \Box V - qV = 8\pi\delta(x,t), \quad (x,t) \in \mathbb{R}^3 \times \mathbb{R} \tag{5.4} \]
\[ V = 0, \quad t < 0, \tag{5.5} \]

and the $a_k$ are given by the recurrence relation

\[ a_0(x) = \int_1^\infty q(sx) \, ds, \tag{5.6} \]
\[ a_{k+1}(x) = -\frac{1}{2} \int_1^\infty s(q + \Delta)a_k(sx) \, ds, \quad k = 0, 1, 2, \ldots \tag{5.7} \]

and $r(x,t) \in C^{m-1}(\mathbb{R}^3 \times \mathbb{R})$ is the unique solution of the IVP

\[ \Box r - qr = (q + \Delta)a_m(x) \varphi^m, \quad x \in \mathbb{R}^3, \ t \in \mathbb{R} \tag{5.8} \]
\[ r(x,t) = 0, \quad t < -1. \tag{5.9} \]

Note $\text{supp} \ r \subseteq \{(x,t) \in \mathbb{R}^3 \times \mathbb{R} : t + |x| \geq 0\}$.

**Theorem 5.2** (Stability of Inverse Problem). Let $0 < b < 1$ and $U_i$, $i = 1, 2$, the solutions of (5.1)-(5.2) corresponding to $q_i \in C^\infty_c(\mathbb{R}^3)$ with $\text{supp} \ q_i \subseteq \{x : |x| < 1 - b\}$.

If $\|q_i\|_{C^7(\mathbb{R}^3)} \leq M$ and $q_1 - q_2$ is angularly controlled then

\[ \|q_1 - q_2\|_{L^2(|x|=\rho)} \leq e^{C/\rho^4} \|U_1 - U_2\|_{BS}, \quad 0 < \rho < 1 \tag{5.10} \]

and if $\|U_1 - U_2\|_{BS} < e^{-1}$ then

\[ \|q_1 - q_2\|_{L^2(B)} \leq C' \left( \log \frac{1}{\|U_1 - U_2\|_{BS}} \right)^{-1/4} \tag{5.11} \]

where $C, C' > 0$ depend only on $b$, $M$, and $C_0$.

Note that when $\|U_1 - U_2\|_{BS} \geq e^{-1}$ then trivially $\|q_1 - q_2\|_{L^2(B)} \ll \|U_1 - U_2\|_{BS}$.
5.2 Proof of Theorem 5.1

Our objective is to show that the IVP (5.1)-(5.2) has a unique solution. To do this, we construct $U$ as the sum of explicitly given distributions and functions, plus a remainder term $r$ that is smooth enough to apply the standard well-posedness theory for IVP for hyperbolic PDE.

5.2.1 The Progressing Wave Expansion

For $k \geq 0$ and $s \in \mathbb{R}$, define

$$s_+^k = \begin{cases} s^k, & s > 0 \\ 0, & s \leq 0 \end{cases}$$

and define

$$\varphi^k(x,t) = \begin{cases} \frac{(t + |x|)^k}{k!}, & k \geq 1 \\ H(t + |x|), & k = 0 \\ \delta^{(-k-1)}(t + |x|), & k \leq -1 \end{cases}$$

Let $V(x,t)$ be the unique distributional solution of (5.4)-(5.5) whose existence is guaranteed by [29]. We seek $U(x,t)$ in the form

$$U(x,t) = \frac{2\delta(t + |x|)}{|x|} - V(x,t) + \sum_{k=0}^{m} a_k(x)\varphi^k(x,t) + r(x,t)$$

for some smooth functions $a_k$. 

65
From (5.12) we know that

\[ \partial_j \varphi^k(x, t) = \frac{(t + |x|)^{k-1}}{(k-1)!} \cdot \frac{x_j}{|x|} = \varphi^{k-1}(x, t) \frac{x_j}{|x|}, \]

\[ \partial_{jj} \varphi^k(x, t) = \varphi^{k-2}(x, t) \frac{x_j^2}{|x|^2} + \varphi^{k-1}(x, t) \frac{|x|^2 - x_j^2}{|x|^3}, \]

\[ \Delta_x \varphi^k(x, t) = \varphi^{k-2}(x, t) + \frac{2 \varphi^{k-1}(x, t)}{|x|}, \]

\[ \partial_t \varphi^k(x, t) = \varphi^{k-1}(x, t), \]

\[ \partial_{tt} \varphi^k(x, t) = \varphi^{k-2}(x, t), \]


Thus

\[ (\Box - q) \varphi^k(x, t) = - \varphi^{k-1}(x, t) - q(x) \varphi^k(x, t), \]

\[ - \Delta a_k(x) \varphi^k(x, t) - 2 \varphi^{k-1}(x, t) \]

\[ = -2 \left( \frac{a_k(x) + x \cdot \nabla a_k(x)}{|x|} \right) \varphi^{k-1}(x, t) - (q(x)a_k(x) + \Delta a_k(x)) \varphi^k(x, t). \] (5.13)

From [14] we know that

\[ (\Box - q) \left( \frac{2\delta(t + |x|)}{|x|} \right) = 8\pi \delta(x, t) - q(x) \frac{2\delta(t + |x|)}{|x|}. \] (5.14)

So from (5.12), (5.13) and (5.14) we get

\[ (\Box - q) \left( U(x, t) - \frac{2\delta(t + |x|)}{|x|} + V(x, t) - \sum_{k=0}^{m} a_k(x) \varphi^k(x, t) \right) \]

\[ = (\Box - q) U(x, t) + q(x) \varphi^{k-1}(x, t) + (q(x)a_k(x) + \Delta a_k(x)) \varphi^k(x, t) \]

\[ = (\Box - q) U(x, t) + (x \cdot \nabla a_0(x) + a_0(x) + q(x) + \frac{2\delta(t + |x|)}{|x|} + (q + \Delta) a_m(x) \varphi^m(x, t) \]

\[ + \frac{1}{|x|} \sum_{k=0}^{m-1} [\frac{1}{|x|} (q + \Delta) a_k + 2x \cdot \nabla a_{k+1}(x) + 2a_{k+1}(x)] \varphi^k(x, t) \]

\[ = (\Box - q) r(x, t). \] (5.15)
Noting (5.15), we seek $a_0(x)$ that satisfies the transport equation

$$x \cdot \nabla a_0(x) + a_0(x) = -q(x)$$  \hspace{1cm} (5.16)$$

and $a_k(x)$ so that

$$|x|(q + \Delta)a_k + 2x \cdot \nabla a_{k+1}(x) + 2a_{k+1}(x) = 0.$$  \hspace{1cm} (5.17)$$

If we do this and we choose $a_k(x) = 0$ for $|x| \geq 1$, then to construct $U$ as a solution of (5.1)-(5.2) we would want

$$r(x, t) := U(x, t) - 2 \frac{\delta(t + |x|)}{|x|} + V(x, t) - \sum_{k=0}^{m} a_k(x) \varphi^k(x, t)$$

to be a solution of the IVP

$$\Box r - qr = (q + \Delta)a_m(x)\varphi^m, \quad x \in \mathbb{R}^3, \quad t \in \mathbb{R}$$

$$r(x, t) = 0, \quad t < -1.$$  

Then we would have (5.1). Note that this guarantees since $q \in C^\infty$, the smoothness of $r$ only depends on $\varphi^m$ which is $C^{m-1}$ and $a_m$, which we will fix to be as smooth as $q$. Such an $r$ exists and is unique by the standard well-posedness theory. Finally, since $q$ is supported in $|x| < 1$, implies $a_k$ is supported in $|x| < 1$, and for such $x$, $\varphi^k(x, t) = 0$ for $t < -1$. Combined with the fact that $V(x, t)$ is supported in $|x| \leq t$, we get (5.2).

### 5.2.2 The Transport Equation

We now construct $a_0, a_k$, by letting $x = r\theta$ where $|\theta| = 1$. Then (5.16) becomes

$$r\theta \cdot \nabla a_0(r\theta) + a_0(r\theta) = -q(r\theta)$$

$$r \left( \frac{d}{dr} a_0 \right)(r\theta) + a_0(r\theta) = -q(r\theta)$$

$$\frac{d}{dr} (ra_0(r\theta)) = -q(r\theta).$$
Thus, noting $r = |x|$ and $a_0(x) = 0$ for $|x| > 1$,

$$-ra_0(-r\theta) = -\int_{-\infty}^{-r} q(\sigma\theta) \, d\sigma$$

$$a_0(-x) = \int_{-\infty}^{-|x|} \frac{q(\sigma\theta)}{|x|} \, d\sigma,$$

and using a change of variables, $\sigma = s|x|$, we get

$$a_0(x) = \int_{-\infty}^{-1} q(-sx) \, ds = -\int_{1}^{\infty} q(sx) \, ds = \int_{1}^{\infty} q(sx) \, ds.$$

To get a recursive relation for $a_k$ we observe similarly that (5.17) becomes

$$\frac{d}{dr}(ra_{k+1}(s\theta)) = -\frac{r}{2}(q + \Delta)a_k(s\theta),$$

Since $a_k(x) = 0$ for $|x| > 1$, we get

$$-ra_{k+1}(-r\theta) = -\frac{1}{2} \int_{-\infty}^{-r} \sigma(q + \Delta)a_k(\sigma\theta) \, d\sigma,$$

i.e. as before

$$a_{k+1}(x) = \frac{1}{2} \int_{-\infty}^{-1} s(q + \Delta)a_k(-sx) \, ds = -\frac{1}{2} \int_{1}^{\infty} s(q + \Delta)a_k(sx) \, ds.$$

Thus (5.1) holds. Uniqueness follows from the fact that $V$ and $r$ are unique and all the other terms are known.

5.3 Proof of Theorem 5.2

5.3.1 The Goursat Problem

In this subsection we write the solution $U$ of (5.1)-(5.2) in terms of its singularity and a smooth term. The smooth term will be shown to solve a Goursat problem and this information will be crucial in obtaining an identity useful in proving theorem 5.2. This can be neatly summed up in the following Lemma:
Figure 5.1: Exterior of the light cone, $-|x| < t < |x| < 1$

Lemma 5.1. If $q \in C_c^\infty(\mathbb{R}^3)$ and $U$ is the solution of (5.1)-(5.2), then

$$U(x, t) = \frac{2\delta(t + |x|)}{|x|} + H(t + |x|)u(x, t), \quad -|x| < t < |x|$$

where $u(x, t)$ is the solution of the exterior Goursat problem

$$(\Box - q)u = 0, \quad -|x| < t < |x| \quad (5.18)$$
$$u(x, -|x|) = \int_{1}^{\infty} q(sx) \, ds, \quad x \in \mathbb{R}^3. \quad (5.19)$$

Proof. If

$$u(x, t) = \sum_{k=0}^{m} a_k(x) \varphi^k(x, t) + r(x, t), \quad -|x| \leq t < |x|$$

then, by theorem (5.1),

$$(\Box - q)u = q(x) \frac{2\delta(t + |x|)}{|x|}, \quad -|x| \leq t < |x|.$$
Thus (5.18) is satisfied. To verify (5.19), note that $a_k(x)\varphi^k(x,t) = 0$ on $t = -|x|$ for $k \geq 1$, implying

$$u(x,-|x|) = a_0(x) + r(x,-|x|) = \int_{1}^{\infty} q(s x) \, ds + r(x,-|x|),$$

so showing $r(x,-|x|) = 0$ would be enough. We know

$$(\Box - q)r(x,t) = (\Delta + q) a_m(x) \varphi^m(x,t), \quad (x,t) \in \mathbb{R}^3 \times \mathbb{R}$$

$$r(x,t) = 0, \quad t < -1.$$ 

Since $\text{supp}(\varphi^m) \subseteq \{t + |x| \geq 0\}$, we know $r(x,t)$ is the solution of the IVP

$$(\Box - q)r(x,t) = 0, \quad t \leq -|x|$$

$$r(x,t) = 0, \quad t < -1$$

So from the standard theory (see [9], [11], [13]), $r = 0$ on $t < -|x|$ and since $r$ is continuous on $\mathbb{R}^3 \times \mathbb{R}$, we have $r = 0$ on $t = -|x|$. 

\[ \Box \]
5.3.2 An Identity for the Difference of Two Solutions

Suppose $0 < b < 1$, $q_i$ are smooth functions on $\mathbb{R}^3$ with support in $\{ x \in \mathbb{R}^3 : |x| \leq 1 - b \}$, and $U_i$ the solution of (5.1)-(5.2) corresponding to $q = q_i$, $i = 1, 2$.

Let $W = U_1 - U_2$ and $p = q_1 - q_2$, then $W(x, t)$, supported in $|x| + t \geq 0$, is the solution of the IVP

$$(\Box - q_1)W = pU_2, \quad (x, t) \in \mathbb{R}^3 \times \mathbb{R}$$

$$W = 0, \quad t < -1.$$  

Given $a \in S$ and $\tau \in (-1, 1)$, let $G_{a,\tau}(x, t)$ be the solution of the backward IVP

$$(\Box - q_1)G_{a,\tau}(x, t) = \delta(x - a, \tau - t), \quad (x, t) \in \mathbb{R}^3 \times \mathbb{R}$$

$$G_{a,\tau} = 0, \quad t > \tau.$$  

The existence of a $G$ supported in $|x - a| \leq \tau - t$ was shown in [27]. So we have

$$W(a, \tau) = \int\int_{\mathbb{R}^3 \times \mathbb{R}} W(x, t)\delta(x - a, \tau - t) \, dx \, dt$$

$$= \int\int_{\mathbb{R}^3 \times \mathbb{R}} W(x, t)(\Box - q_1)G_{a,\tau}(x, t) \, dx \, dt$$

$$= \int\int_{\mathbb{R}^3 \times \mathbb{R}} G_{a,\tau}(x, t)p(x)U_2(x, t) \, dx \, dt.$$  

(5.20)

Note that the boundary terms in the integral vanish because the intersection of the supports of $W(x, t)$ and $G(x, t)$ is compact since $\tau \in (-1, 1)$. We know that (see [29], [27])

$$G_{a,\tau}(x, t) = \frac{1}{4\pi} \frac{\delta(\tau - t - |x - a|)}{|x - a|} + g_{a,\tau}(x, t)$$

where $g_{a,\tau}(x, t)$ is supported in $|x - a| + t \leq \tau$ and smooth in that region. From lemma 5.1, we have

$$U_2(x, t) = \frac{2\delta(t + |x|)}{|x|} + u_2(x, t)$$

71
where \( u_2(x, t) \) is supported in \( t + |x| \geq 0 \) and smooth in \(-|x| < t < |x|\). Hence, from (5.20) for \( \tau \in (-1, 1) \), \( a \in S \), we have

\[
W(a, \tau) = \int \int_{\mathbb{R}^3 \times \mathbb{R}} p(x) \left( \frac{1}{4\pi} \frac{\delta(\tau - t - |x - a|)}{|x - a|} + g_{a, \tau}(x, t) \right) \left( \frac{2\delta(t + |x|)}{|x|} + u_2(x, t) \right) \, dx \, dt
\]

\[
= \int_{\mathbb{R}^3} p(x) \left[ \int_{\mathbb{R}} \frac{1}{2\pi} \frac{\delta(\tau - t - |x - a|)\delta(t + |x|)}{|x - a| |x|} + 2g_{a, \tau}(x, t) \frac{\delta(t + |x|)}{|x|} \right] \, dx + \int_{\mathbb{R}^3} p(x) k(x, a, \tau) \, dx, \quad (5.21)
\]

where

\[
k(x, a, \tau) = \frac{2g_{a, \tau}(x, -|x|)}{|x|} + \frac{u_2(x, \tau - |x - a|)}{4\pi|x - a|} + \int_{\mathbb{R}} g_{a, \tau}(x, t) u_2(x, t) \, dt. \quad (5.22)
\]

For \( a \in S, \tau \in (-1, 1) \), define the region

\[
H(a, \tau) = \{ x \in \mathbb{R}^3 : |x - a| - |x| \leq \tau \}.
\]

Since \( u_2(x, t) \) is supported in \( t + |x| \geq 0 \) and \( g_{a, \tau}(x, t) \) is supported in \( |x - a| + t \leq \tau \), we deduce that \( k(x, a, \tau) \) is supported in \( H(a, \tau) \). Since \( k \) is multiplied by \( p \) in the second integral of (5.21), we only need to consider \( k \) in the support of \( p \), i.e. \( \{ |x| \leq 1 - b \} \).

Thus we are only considering where \( |x| \leq 1 - b \) and \( |x - a| - |x| \leq \tau \). Because of this, singularity in the second term in the RHS of (5.22) is not an issue. The only issue is in the first term where \( |x| \) is near 0. To remedy this we restrict \( \tau \) to \((-1, 1 - 4\epsilon]\), where \( \epsilon > 0 \). Then for \( |x| < \epsilon \),

\[
\tau \leq 1 - 4\epsilon < 1 - 2\epsilon = \min(|x - a| - |x|) \leq |x - a| - |x|. \quad (5.23)
\]

So \( \text{supp} \, k \cap \text{supp} \, p \subseteq \{|x| \geq \epsilon \} \). This tells us we can regard \( k(x, a, \tau) \) as a smooth function supported in

\[
H(a, \tau) \cap \{ x \in \mathbb{R}^3 : \epsilon < |x| < 1 - b \}.
\]

Because of this, the derivative \( \partial_{\tau} k(x, a, \tau) \) is also smooth and supported in this region. The problem is that any bound on these functions would depend on \( \epsilon, u_2, \) and \( g_{a, \tau} \).
We can bound the $\frac{1}{|x|}$ term above by $\frac{1}{\epsilon}$ and, using method similar to the proofs of theorems 1.2, and 1.3 in [2], we can obtain the following bound on $k$ and $\partial_{\tau}k$:

$$\sup_{-1<\tau\leq 1-4\epsilon} \sup_{|a|=1} \left| k(x,a,\tau) \right|^2 dS_x \leq \frac{K}{\epsilon}$$

$$\sup_{-1<\tau\leq 1-4\epsilon} \sup_{|a|=1} \left| \partial_{\tau}(\tau k(x,a,\tau)) \right|^2 dS_x \leq \frac{K}{\epsilon} \quad (5.24)$$

for some $K = K(M, b, B) > 0$.

Then we can write (5.21) as

$$W(a,\tau) = \frac{1}{2\pi} \iint_{\mathbb{R}^3} p(x) \frac{\delta(\tau - |x - a| + |x|)}{|x| |x-a|} \, dx + \iint_{H(a,\tau)} p(x)k(x,a,\tau) \, dx \quad (5.25)$$

where the second integrand is smooth. The first integral becomes a surface integral on $\partial H(a,\tau)$ and to compute it, let $\psi(x) = |x-a| - |x| - \tau$; then

$$\nabla \psi(x) = \frac{x-a}{|x-a|} - \frac{x}{|x|} = \frac{|x|(x-a) - |x-a|x}{|x| |x-a|}$$

and on $\partial H(a,\tau)$, $\varphi(x) = 0$, so we can substitute $|x|$ for $|x-a| - \tau$ and get

$$|\nabla \psi(x)| = \frac{|x|(x-a) - |x-a|x}{|x| |x-a|} = \frac{|\tau x + a|x|}{|x| |x-a|}.$$ 

Using proposition 1.1,

$$\int_{\mathbb{R}^3} f(x)\delta(\psi(x)) \, dx = \int_{\psi(x)=0} \frac{f(x)}{|\nabla \psi(x)|} \, dS_x,$$

we rewrite the first term in (5.25) as

$$\frac{1}{2\pi} \iint_{\mathbb{R}^3} p(x) \frac{\delta(\tau - |x - a| + |x|)}{|x| |x-a|} \, dx = \frac{1}{2\pi} \int_{\partial H(a,\tau)} \frac{p(x)}{|\tau x + a|x|} \, dS_x.$$ 

Define the operator $M$ as

$$(Mp)(a,\tau) := \frac{1}{2\pi} \int_{\partial H(a,\tau)} \frac{p(x)}{|\tau x + a|x|} \, dS_x, \quad a \in S, \quad \tau \in (-1, 1-4\epsilon);$$

then we can rewrite (5.25) as

$$W(a,\tau) = (Mp)(a,\tau) + \iint_{H(a,\tau)} p(x)k(x,a,\tau) \, dx, \quad a \in S, \quad \tau \in (-1, 1-4\epsilon). \quad (5.26)$$
5.3.3 Parametrizing $\partial H(a, \tau)$ for $-1 < \tau < 1$

**Proposition 5.1.** $\partial H(a, \tau)$ may be parametrized by $\rho = |x|$, $\theta$ the rotation of $x$ about $a$, and $\tau$. The surface element $dS$ for $\partial H$ is then

$$dS = \sqrt{\rho(\rho + \tau)(1 - \tau^2)} \, d\rho \, d\theta, \quad \frac{1 - \tau}{2} \leq \rho \leq 1, \, 0 \leq \theta \leq 2\pi.$$

Moreover, for $x \in \partial H(a, \tau_0)$, we have

$$|\tau x + a| = \sqrt{\rho(\rho + \tau)(1 - \tau^2)},$$

and we can estimate $1/ \sin \varphi$ by

$$\frac{1}{\sin \varphi} \leq \frac{2\rho}{(1 + \tau)\sqrt{1 - \tau}\sqrt{2\rho - (1 - \tau)}}.$$

**Proof.** To any $x \in \mathbb{R}^3$, we associate its spherical coordinates $(\rho, \theta, \varphi)$ where $\rho = |x|$, $\varphi$ is the angle between $x$ and $a$ and $\theta$ is the rotation about $a$:

$$x = \rho \omega, \quad \omega = (\cos \theta \sin \varphi, \sin \theta \sin \varphi, \cos \varphi).$$

For any $x \in \partial H(a, \tau)$ we have $|x - a| = \rho + \tau$. Squaring both sides and noting that $|a| = 1$, we have

$$\rho^2 - 2\rho \cos \varphi + 1 = \rho^2 + 2\rho \tau + \tau^2,$$

which simplifies to

$$\cos \varphi = \frac{1 - 2\tau \rho - \tau^2}{2\rho} \quad \text{(5.27)}$$

with

$$\sin \varphi = \frac{\sqrt{4\rho^2 - (1 - 2\rho \tau - \tau^2)^2}}{2\rho} = \frac{\sqrt{(2\rho - 1 + 2\rho \tau + \tau^2)(2\rho + 1 - 2\rho \tau - \tau^2)}}{2\rho} = \frac{\sqrt{2\rho(1 + \tau - (1 - \tau^2))(1 - \tau^2) + 2\rho(1 - \tau)}}{2\rho} = \frac{\sqrt{1 - \tau^2}(2\rho - (1 - \tau))(2\rho + 1 + \tau)}{2\rho}.$$
Noting that $1 + \tau \leq 2\rho + 1 + \tau$ we get

$$\frac{1}{\sin \varphi} \leq \frac{2\rho}{|1 + \tau|\sqrt{1 - \tau^2} - 2\rho - (1 - \tau)} = \frac{2\rho}{(1 + \tau)\sqrt{1 - \tau^2} - 2\rho - (1 - \tau)}.$$  

For (5.27), we can compute

$$\frac{\partial \phi}{\partial \rho} = \frac{\cos \varphi + \tau}{\rho \sin \varphi},$$

so $\partial H(a, \tau)$ may be parametrized by $\rho$, $\theta$, and $\tau$. and

$$\frac{\partial x}{\partial \rho} = \omega + \rho \frac{\partial \omega}{\partial \rho}, \quad \text{and} \quad \frac{\partial x}{\partial \theta} = \rho \frac{\partial \omega}{\partial \theta}.$$  

Combining these, we get

$$\left| \frac{\partial x}{\partial \rho} \times \frac{\partial x}{\partial \theta} \right|^2 = \rho^2 \left| \left( \omega + \rho \frac{\partial \omega}{\partial \rho} \right) \times \frac{\partial \omega}{\partial \theta} \right|^2$$  

$$= \rho^2 (|\omega + \rho \omega_{\rho}|^2 |\omega_{\theta}|^2 - (\omega \cdot \omega_{\theta} + \rho \omega_{\rho} \cdot \omega_{\theta})^2).$$
To simplify this, compute

$$\omega_\rho = \frac{\partial \varphi}{\partial \rho} (\cos \theta \cos \varphi, \sin \theta \cos \varphi, -\sin \varphi),$$

$$\omega_\theta = (-\sin \theta \sin \varphi, \cos \theta \sin \varphi, 0),$$

$$|\omega + \rho \omega_\rho|^2 = \left( \sin \varphi + \rho \frac{\partial \varphi}{\partial \rho} \cos \varphi \right)^2 + \left( \cos \varphi - \rho \frac{\partial \varphi}{\partial \rho} \sin \varphi \right)^2$$

$$= 1 + \rho^2 \left( \frac{\partial \varphi}{\partial \rho} \right)^2$$

$$= 1 + \frac{(\cos \varphi + \tau)^2}{\sin^2 \varphi} = \frac{1 + \tau^2 + 2 \tau \cos \varphi}{\sin^2 \varphi},$$

$$|\omega_\theta|^2 = \sin^2 \varphi,$$

$$(\omega + \rho \omega_\rho) \cdot \omega_\theta = 0.$$  

Then,

$$\left| \frac{\partial x}{\partial \rho} \times \frac{\partial x}{\partial \theta} \right|^2 = \rho^2 (1 + \tau^2 + 2 \tau \cos \varphi)$$

$$= \rho^2 \left( \frac{\rho + \rho \tau^2 + \tau - 2 \tau^2 \rho - \tau^3}{\rho} \right)$$

$$= \rho (\rho - \tau^2 \rho + \tau - \tau^3)$$

$$= \rho (\rho + \tau) (1 - \tau^2).$$

So we can write the surface element $dS$ for $\partial H(a, \tau)$ as

$$dS = \sqrt{\rho (\rho + \tau) (1 - \tau^2)} \, d\rho \, d\theta. \quad (5.28)$$

Moreover, for $x \in \partial H(a, \tau),$

$$|\tau x + a\rho|^2 = \tau^2 \rho^2 + \rho^2 + 2 \tau \rho a \cdot x = \rho^2 (\tau^2 + 1) + 2 \tau \rho^2 \cos \varphi$$

$$= \rho^2 (\tau^2 + 1) + 2 \tau \rho^2 \left( \frac{1 - 2 \tau \rho - \tau^2}{2 \rho} \right)$$

$$= \rho^2 (\tau^2 + 1) + \tau \rho - 2 \tau^2 \rho^2 - \tau^3 \rho$$

$$= \rho^2 + \tau \rho - \tau^2 \rho^2 - \tau^3 \rho$$

$$= \rho (\rho + \tau) (1 - \tau^2).$$
5.3.4 Estimating $p$ by $M(p)$

We can compute $\partial_\tau M(p)$ using the following proposition,

**Proposition 5.2.** If $p(x)$ is a smooth function on $\mathbb{R}^3$ with support in $B$, then for all $(a, \tau) \in S \times (-1, 1)$, we have

$$\partial_\tau (Mp)(a, \tau) = \frac{1}{2} \left( \frac{1 - \tau}{2} a \right) + \frac{1}{2\pi} \int_{\partial H(a, \tau)} \frac{|x - a|}{|\tau x + a||x||} \cdot (\alpha \cdot \nabla) p(x) \, dS_x \quad (5.29)$$

where $\alpha = \frac{\partial x}{\partial \tau} / \left| \frac{\partial x}{\partial \tau} \right| = \frac{\partial x}{\partial \tau} / |x - a|$, and $0 \leq \varphi \leq \pi$ is the angle between $x$ and $a$. Moreover, we have $x \perp \alpha$ for all $x \in \partial H(a, \tau)$.

**Proof.** To any $x \in \mathbb{R}^3$, we associate its spherical coordinates $(\rho, \theta, \varphi)$ where $\rho = |x|$, $\varphi$ is the angle between $x$ and $a$ and $\theta$ is the rotation about $a$. Then from proposition 5.1, we can write

$$(Mp)(a, \tau) = \frac{1}{2\pi} \int_{\partial H(a, \tau)} \frac{p(x)}{|\tau x + a||x||} \, dS_x = \frac{1}{2\pi} \int_{\frac{1-\tau}{2}}^{1} \int_{0}^{2\pi} p(x) \sqrt{\rho(\rho + \tau)(1 - \tau^2)} \, d\theta \, d\rho \quad (5.30)$$

Thus, from proposition 5.1 we get

$$(Mp)(a, \tau) = \frac{1}{2\pi} \int_{\frac{1-\tau}{2}}^{1} \int_{0}^{2\pi} p(x) \, d\theta \, d\rho, \quad a \in S, \tau \in (-1, 1).$$

Taking a derivative in $\tau$, we get

$$\frac{\partial}{\partial \tau} (Mp)(a, \tau) = \frac{1}{4\pi} \int_{0}^{2\pi} p \left( \left( \frac{1 - \tau}{2} \right) a \right) \, d\theta + \frac{1}{2\pi} \int_{\frac{1-\tau}{2}}^{1} \int_{0}^{2\pi} \frac{\partial}{\partial \tau} p(x) \, d\theta \, d\rho$$

$$= \frac{1}{2} p \left( \frac{1 - \tau}{2} a \right) + \frac{1}{2\pi} \int_{\frac{1-\tau}{2}}^{1} \int_{0}^{2\pi} \left( \nabla p \cdot \frac{\partial x}{\partial \tau} \right) \, d\theta \, d\rho$$

$$= \frac{1}{2} p \left( \frac{1 - \tau}{2} a \right) + \frac{1}{2\pi} \int_{\partial H(a, \tau)} \frac{\nabla p \cdot \frac{\partial x}{\partial \tau}}{|\tau x + a||x||} \, dS_x. \quad (5.30)$$

In spherical coordinates,

$$x = (\rho \cos \theta \sin \varphi, \rho \sin \theta \sin \varphi, \rho \cos \varphi),$$

77
which gives
\[ \frac{\partial x}{\partial \tau} = (\rho \cos \theta \cos \varphi \frac{\partial \varphi}{\partial \tau}, \rho \sin \theta \cos \varphi \frac{\partial \varphi}{\partial \tau}, -\rho \sin \varphi \frac{\partial \varphi}{\partial \tau}), \]
and
\[ \left| \frac{\partial x}{\partial \tau} \right| = \rho \left| \frac{\partial \varphi}{\partial \tau} \right|. \]

We can compute \( \partial \varphi / \partial \tau \) by differentiating (5.27), to get
\[ -\sin \varphi \frac{\partial \varphi}{\partial \tau} = \frac{-2\rho - 2\tau}{2\rho}, \]
thus
\[ \left| \frac{\partial x}{\partial \tau} \right| = \frac{\rho + \tau}{\sin \varphi} = \frac{|x - a|}{\sin \varphi}. \]

Letting \( \alpha = \frac{\partial x / \partial \tau}{\left| \partial x / \partial \tau \right|} \), (5.30) gives
\[
\frac{\partial}{\partial \tau} (Mp)(a, \tau) = \frac{1}{2} p \left( 1 - \frac{\tau}{2} a \right) + \frac{1}{2\pi} \int_{\partial H(a, \tau)} \left( \alpha \cdot \nabla \right) p(x) \frac{|x - a|}{|\tau x + a|} \sin \varphi \, dS_x.
\]

To show that \( x \perp x_\tau \), we note that only \( \varphi \) depends on \( \tau \), thus \( |x| = \rho \) implies
\[ 2x \cdot x_\tau = \frac{\partial}{\partial \tau} (\rho^2) = 0. \]

\( \square \)

### 5.3.5 The Volterra Estimate

Let \( e_i, i = 1, 2, 3 \) denote the standard unit vectors in \( \mathbb{R}^3 \). For any \( x \in \mathbb{R}^3 \) and \( i, j = 1, 2, 3 \) define the vectors \( T_{i,j} = x_i e_j - x_j e_i \) and define \( \Omega_{ij} = T_{ij} \cdot \nabla \). For any \( v, w \in \mathbb{R}^3 \) we have
\[ |v|^2 w = \sum_{i<j} (w \cdot T_{ij}) T_{ij} + (v \cdot w)v. \]

Applying this to \( v = x \) and \( w = \alpha \), noting that \( x \perp \alpha \), we get
\[ |x|^2 (\alpha \cdot \nabla p)(x) = \sum_{i<j} (\alpha \cdot T_{ij})(T_{ij} \cdot \nabla p)(x) = \sum_{i<j} (\alpha \cdot T_{ij})(\Omega_{ij} p)(x). \]
Since $|T_{ij}| \leq 2|x|$, we have

$$|\alpha \cdot T_{ij}| \leq 2|x|,$$

so

$$|(\alpha \cdot \nabla)p(x)| \leq \frac{2}{|x|} \sum_{i<j} |\Omega_{ij}p(x)|. \tag{5.31}$$

From proposition 5.1 and (5.31), proposition 5.2 implies

$$\frac{1}{2} \left| p \left( \frac{1 - \tau}{2} a \right) \right| \leq |\partial_\tau (Mp)(a, \tau)| + \frac{1}{\pi} \sum_{i<j} \int_{\frac{1-\tau}{2}}^1 \int_{\frac{1-\tau}{2}}^{2\pi} \frac{|\rho + \tau|}{\rho} \cdot |\Omega_{ij}p(x)| \sin \varphi \, d\theta \, d\rho. \tag{5.32}$$

From proposition 5.1 the fact that $\frac{1}{2} \leq \rho + \tau \leq 1$, (5.32) becomes

$$\frac{1}{2} \left| p \left( \frac{1 - \tau}{2} a \right) \right| \leq |\partial_\tau (Mp)(a, \tau)| + \frac{2}{\pi} \sum_{i<j} \int_{\frac{1-\tau}{2}}^1 \int_{\frac{1-\tau}{2}}^{2\pi} \frac{(\rho + \tau)|\Omega_{ij}p(x)|}{(1 + \tau)\sqrt{1 - \tau} \sqrt{2\rho - (1 - \tau)}} \, d\theta \, d\rho$$

$$\left| p \left( \frac{1 - \tau}{2} a \right) \right| \leq 2 |\partial_\tau (Mp)(a, \tau)| + \frac{4}{\pi \sqrt{1 - \tau}} \sum_{i<j} \int_{\frac{1-\tau}{2}}^1 \int_{\frac{1-\tau}{2}}^{2\pi} \frac{|\Omega_{ij}p(x)|}{\sqrt{2\rho - (1 - \tau)}} \, d\theta \, d\rho.$$

Applying Cauchy-Schwarz, for all $(a, \tau) \in S \times (-1, 1)$, we have

$$\left| p \left( \frac{1 - \tau}{2} a \right) \right|^2 \leq |\partial_\tau (Mp)(a, \tau)|^2$$

$$\left| p \left( \frac{1 - \tau}{2} a \right) \right|^2 \leq |\partial_\tau (Mp)(a, \tau)|^2 + \frac{\sqrt{1 + \tau}}{1 - \tau} \sum_{i<j} \int_{\frac{1-\tau}{2}}^1 \int_{\frac{1-\tau}{2}}^{2\pi} \frac{|\Omega_{ij}p(x)|^2}{\sqrt{2\rho - (1 - \tau)}} \, d\theta \, d\rho$$

$$\left| p \left( \frac{1 - \tau}{2} a \right) \right|^2 \leq |\partial_\tau (Mp)(a, \tau)|^2 + \frac{\sqrt{1 + \tau}}{1 - \tau} \sum_{i<j} \int_{\partial H(a, \tau)} |\partial_\tau (x + a|x||\sqrt{2|x| - (1 - \tau)} \, dS_x. \tag{5.33}$$

### 5.3.6 Stability Estimate

From (5.26), we get

$$(Mp)(a, \tau) = W(a, \tau) - \int_{H(a, \tau)} p(x)k(x, a, \tau) \, dx$$

$$= W(a, \tau) - \int_{\mathbb{R}^3} p(x)k(x, a, \tau)H(\tau + |x| - |x - a|) \, dx$$
\[ \partial_x (Mp)(a, \tau) \]
\[ = \partial_x W(a, \tau) - \int_{H(a, \tau)} p(x) \partial_x k(x, a, \tau) \, dx - \int_{\mathbb{R}^3} p(x) k(x, a, \tau) \delta(\tau + |x| - |x - a|) \, dx \]
\[ = \partial_x W(a, \tau) - \int_{H(a, \tau)} p(x) \partial_x k(x, a, \tau) \, dx - \int_{\partial H(a, \tau)} p(x) k(x, a, \tau) \frac{|x| |x - a|}{|\tau x + a|} \, dS_x. \]

Using the AM-GM and Cauchy Schwarz inequalities and (5.24), we get
\[ |\partial_x (Mp)(a, \tau)|^2 \lessgtr |\partial_x W(a, \tau)|^2 + \left( \int_{H(a, \tau)} p(x) \partial_x k(x, a, \tau) \, dx \right)^2 \]
\[ + \left( \int_{\partial H(a, \tau)} p(x) k(x, a, \tau) \frac{|x| |x - a|}{|\tau x + a|} \, dS_x \right)^2 \]
\[ \lessgtr |\partial_x W(a, \tau)|^2 + \frac{K^2}{\epsilon^2} \int_{H(a, \tau)} |p(x)|^2 \, dx + \frac{K^2}{\epsilon^2} \left( \int_{\partial H(a, \tau)} p(x) \frac{|x| |x - a|}{|\tau x + a|} \, dS_x \right)^2 \]
\[ \lessgtr |\partial_x W(a, \tau)|^2 + \frac{K^2}{\epsilon^2} \int_{H(a, \tau)} |p(x)|^2 \, dx \]
\[ + \frac{K^2}{\epsilon^2} \int_{\partial H(a, \tau)} \frac{|p(x)|^2}{|\tau x + a|} \, dS_x \int_{\partial H(a, \tau)} \frac{|x|^2 |x - a|^2}{|\tau x + a|} \, dS_x \]
\[ \lessgtr |\partial_x W(a, \tau)|^2 + \frac{K^2}{\epsilon^2} \int_{H(a, \tau)} |p(x)|^2 \, dx + \frac{K^2}{\epsilon^2} \int_{\partial H(a, \tau)} \frac{|p(x)|^2}{|\tau x + a|} \, dS_x. \tag{5.34} \]

Note that we bounded the integral \( \int_{\partial H(a, \tau)} \frac{|x|^2 |x - a|^2}{|\tau x + a|} \, dS_x \) by using proposition 5.1 and computing
\[ \int_{\partial H(a, \tau)} \frac{|x|^2 |x - a|^2}{|\tau x + a|} \, dS_x = \int_0^{2\pi} \int_{\frac{\pi}{2}}^0 \rho^2 (\rho + \tau)^2 \, d\rho \, d\theta \leq 4\pi (1 + \tau). \]

Combining (5.34) with (5.33) gives
\[ |p \left( \frac{1}{2}(1 - \tau) a \right)|^2 \lessgtr |\partial_x W(a, \tau)|^2 + \int_{H(a, \tau)} |p(x)|^2 \, dx + \int_{\partial H(a, \tau)} \frac{|p(x)|^2}{|\tau x + a|} \, dS_x \]
\[ + \sum_{i < j} \frac{\sqrt{1 + \tau}}{1 - \tau} \int_{\partial H(a, \tau)} \frac{|\Omega_{ij} p(x)|^2}{|\tau x + a| \sqrt{2|x| - (1 - \tau)}} \, dS_x. \]
Thus integrating over $a \in S$, we get

\[
\begin{align*}
&\int_S |p\left(\frac{1}{2}(1-\tau)a\right)|^2 \, dS_a \\
&\int_S |\partial_{\tau}W(a,\tau)|^2 \, dS_a + \int_S \int_{H(a,\tau)} |p(x)|^2 \, dx \, dS_a + \int_S \int_{\partial H(a,\tau)} \frac{|p(x)|^2}{|\tau x + a||x|} \, dS_x \, dS_a \\
&+ \frac{\sqrt{1+\tau}}{1-\tau} \sum_{i<j} \int_S \int_{\partial H(a,\tau)} \frac{\Omega_{ij} p(x)^2}{|\tau x + a||x| \sqrt{2|x|} - (1-\tau)} \, dS_x \, dS_a.
\end{align*}
\]

(5.35)

where the constant is $O(1/\epsilon^2)$ as $\epsilon \to 0$.

Next we get two estimates for integrals of $\delta$ and $H$ in order to convert the $dx \, dS_a$ and $dS_x \, dS_a$ integrals into just $dx$ integrals. Since $a \in S$, write it in spherical coordinates ($\varphi, \theta$) where $\varphi$ is the angle between $a$ and $e_3$ and $\theta$ is the rotation about $e_3$,

\[
a = (\cos \varphi \sin \theta, \sin \varphi \sin \theta, \cos \varphi).
\]

For fixed $x$, we write

\[
|x - a|^2 = |x|^2 + 1 - 2|x| \cos \varphi
\]

Then

\[
\int_S \delta(|x - a|^2 - (\tau + |x|)^2) \, dS_a = 2\pi \int_0^\pi \delta(1 - 2|x| \cos \varphi - \tau^2 - 2\tau |x|) \sin \varphi \, d\varphi
\]

\[
= 2\pi \int_{-1}^1 \delta(1 - 2|x| s - \tau^2 - 2\tau |x|) \, ds
\]

\[
= \frac{\pi}{|x|} \int_{-1}^1 \delta \left( s - \frac{1 - \tau^2 - 2\tau |x|}{2|x|} \right) \, ds
\]

\[
= \begin{cases} 
\frac{\pi}{|x|} : -1 < \frac{1 - \tau^2 - 2\tau |x|}{2|x|} < 1 \\
0 : Otherwise
\end{cases}
\]

81
Since $\tau \in (-1, 1)$, we can write the inequality in the last line as

\[
-1 < \frac{1 - \tau^2 - 2\tau|x|}{2|x|} < 1 \iff -2|x| < 1 - \tau^2 - 2\tau|x| < 2|x|
\]

\[
\iff (2\tau - 2)|x| < 1 - \tau^2 < (2\tau + 2)|x|
\]

\[
\iff 0 < \frac{(1 - \tau)(1 + \tau)}{2} < (1 + \tau)|x|
\]

\[
\iff \frac{1 - \tau}{2} < |x|
\]

So

\[
\int_S \delta(|x - a|^2 - (\tau + |x|)^2) \, dS_a = \frac{\pi}{|x|} H(\tau + 2|x| - 1). \quad (5.36)
\]

Similarly, for $-1 < \tau < 1$, we have

\[
\int_S H(|x - a|^2 - (\tau + |x|)^2) \, dS_a = 2\pi \int_{-1}^{1} H \left( s - \frac{1 - \tau^2 - 2\tau|x|}{2|x|} \right) \, ds
\]

\[
\leq \begin{cases} 4\pi & : \frac{1 - \tau^2 - 2\tau|x|}{2|x|} < 1 \\ 0 & : \frac{1 - \tau^2 - 2\tau|x|}{2|x|} \geq 1 \end{cases}
\]

\[
= 4\pi H(\tau + 2|x| - 1).
\]

Let $f$ be a smooth, non-negative integrable function supported in $B$, then for $-1 < \tau < -1$, and the fact that

\[
|\nabla_x(|x - a|^2 - (\tau + |x|)^2)| = \left| 2(x - a) - 2(\tau + |x|) \frac{x}{|x|} \right| = 2 \left| a + \tau \frac{x}{|x|} \right| \leq 2(|\tau| + 1),
\]
we have,

\[
\int_S \int_{\partial H(a,\tau)} f(x) \, dS_x \, dS_a = 2 \int_S \int_{\mathbb{R}^3} f(x) \left| a + \tau \frac{x}{|x|} \right| \delta(|x-a|^2 - (\tau + |x|)^2) \, dx \, dS_a \\
= 2 \int_{\mathbb{R}^3} f(x) \int_S \left| a + \tau \frac{x}{|x|} \right| \delta(|x-a|^2 - (\tau + |x|)^2) \, dS_a \, dx \\
\leq 2(|\tau| + 1) \int_{\mathbb{R}^3} f(x) \int_S \delta(|x-a|^2 - (\tau + |x|)^2) \, dS_a \, dx \\
= 2(|\tau| + 1) \pi \int_{|x| \geq \frac{1+\tau}{\tau}} \frac{f(x)}{|x|} \, dx \\
\leq 4\pi \int_{|x| \geq \frac{1+\tau}{\tau}} \frac{f(x)}{|x|} \, dx,
\]

and

\[
\int_S \int_{H(a,\tau)} f(x) \, dx \, dS_a = \int_S \int_{\mathbb{R}^3} f(x)H(|x-a|^2 - (\tau + |x|)^2) \, dx \, dS_a \\
\leq 4\pi \int_{|x| \geq \frac{1+\tau}{\tau}} f(x) \, dx.
\]

Using this, (5.35) becomes

\[
\int_S |p \left( \frac{1}{2} (1 - \tau)a \right)|^2 \, dS_a \lesssim \|W\|^2_{BS} + \int_{|x| \geq \frac{1+\tau}{\tau}} |p(x)|^2 \, dx + \int_{|x| \geq \frac{1+\tau}{\tau}} \frac{|p(x)|^2}{|x|} \, dx \\
+ \frac{1}{1-\tau} \sum_{i<j} \int_{|x| \geq \frac{1+\tau}{\tau}} \frac{|\Omega_{ij} p(x)|^2}{|x|^{2|1/2-1/2|-(1-\tau)}} \, dx
\]

where the constant is \(O(1/\epsilon^2)\) as \(\epsilon \to 0\), which may be rewritten as

\[
\int_S |p \left( \frac{1}{2} (1 - \tau)a \right)|^2 \, dS_a \lesssim \|W\|^2_{BS} + \int_{|x| \geq \frac{1+\tau}{\tau}} \frac{|p(x)|^2}{|x|} \, dx \\
+ \frac{1}{1-\tau} \sum_{i<j} \int_{|x| \geq \frac{1+\tau}{\tau}} \frac{|\Omega_{ij} p(x)|^2}{|x|^{2|1/2-1/2|-(1-\tau)}} \, dx. \quad (5.37)
\]

Define

\[
P(\rho) := \int_{|x| = \rho} |p(x)|^2 \, dS_x,
\]

83
then the angular control condition (5.3) is equivalent to the statement that

\[ \sum_{i<j} \int_{|x|=\rho} |\Omega_{ij}(x)|^2 \, dS_x \leq C_0 P(\rho), \quad \forall \rho \in (0, 1). \]

So from (5.3) and (5.37) we can say

\[
\frac{1}{(1-\tau)^2} P \left( \frac{1-\tau}{2} \right) \leq \frac{1}{\epsilon^2} \|W\|_{BS}^2 + \frac{1}{\epsilon^2} \int_{\frac{1}{2-s}}^{1} \frac{P(\rho)}{\rho} \, d\rho + \frac{C_0}{1-\tau} \int_{\frac{1}{2-s}}^{1} \frac{P(\rho)}{\rho^{\sqrt{2\rho} - (1-\tau)}} \, d\rho,
\]

i.e.

\[
P \left( \frac{1-\tau}{2} \right) \leq \frac{(1-\tau)^2}{\epsilon^2} \|W\|_{BS}^2 + \frac{(1-\tau)^2}{\epsilon^2} \int_{\frac{1}{2-s}}^{1} \frac{P(\rho)}{\rho} \, d\rho + (1-\tau) \int_{\frac{1}{2-s}}^{1} \frac{P(\rho)}{\rho^{\sqrt{2\rho} - (1-\tau)}} \, d\rho
\]

for all \( \tau \in (-1, 1 - 4\epsilon]\) where the constant in the inequality is independent of \( \epsilon \) and \( \tau \).

Let \( s = \frac{1-\tau}{4} \) and recall (5.23), which implies

\[ \epsilon \leq \frac{1-\tau}{4} = s \leq \frac{1}{2}. \]

Then there is a constant so that for \( s \in [\epsilon, 1/2] \),

\[
P(2s) \leq \frac{s^2}{\epsilon^2} \|W\|_{BS}^2 + \frac{s^2}{\epsilon^2} \int_{2s}^{1} \frac{P(\rho)}{\rho} \, d\rho + s \int_{2s}^{1} \frac{P(\rho)}{\rho^{\sqrt{\rho} - 2s}} \, d\rho
\]

\[
\leq \frac{s^2}{\epsilon^2} \|W\|_{BS}^2 + \frac{s}{\epsilon^2} \int_{2s}^{1} P(\rho) \, d\rho + \int_{2s}^{1} \frac{P(\rho)}{\sqrt{\rho} - 2s} \, d\rho
\]

This implies for some \( C_1 > 0 \),

\[
P(2s) \leq \frac{4C_1 s^2}{\epsilon^2} \|W\|_{BS}^2 + \frac{2C_1 s}{\epsilon^2} \int_{2s}^{1} \frac{P(\rho)}{\sqrt{\rho} - 2s} \, d\rho. \tag{5.38}
\]

Apply this inequality again in the right hand side of (5.38) to get

\[
P(2s) \leq \frac{4C_1 s^2}{\epsilon^2} \|W\|_{BS}^2 + \frac{2C_1 s}{\epsilon^2} \int_{2s}^{1} \frac{P(\rho)}{\sqrt{\rho} - 2s} \, d\rho
\]

\[
= \left( \frac{4C_1 s^2}{\epsilon^2} + \frac{2I(s)C_1^2 s}{\epsilon^4} \right) \|W\|_{BS}^2 + \frac{2C_1^2 s}{\epsilon^4} \int_{\rho}^{1} \frac{\rho P(\rho)}{\sqrt{\rho - 2s \sqrt{\rho - \rho}}} \, d\rho \, d\rho. \tag{5.39}
\]
where
\[ I(s) = \int_{2s}^{1} \frac{\rho^2}{\sqrt{\rho - 2s}} \, d\rho. \]

Note that if \( f \in L^\infty \), then
\[
\int_A \int_x^1 \frac{xf(y)}{\sqrt{x-A\sqrt{y-x}}} \, dy \, dx \leq \int_A f(y) \int_A^y \frac{1}{\sqrt{x-A\sqrt{y-x}}} \, dx \, dy
= \pi \int_A f(y) \, dy.
\]

So (5.39), \( 0 < \rho < 1 \), and the fact that \( I(s) \leq 1 \) gives
\[
P(2s) \leq \left( \frac{4C_1 s^2}{\epsilon^2} + \frac{2C_2^2 s}{\epsilon^4} \right) \|W\|_{BS}^2 + \frac{2C_2^2 s \pi}{\epsilon^4} \int_{2s}^{1} P(\rho) \, d\rho.
\]

Because \( \epsilon \leq s \leq 1/2 \) we get
\[
P(2s) \leq \frac{C_1 + C_2^2}{\epsilon^4} \|W\|_{BS}^2 + \frac{C_2^2 \pi}{\epsilon^4} \int_{2s}^{1} P(\rho) \, d\rho. \tag{5.40}
\]

Thus by Gronwall’s inequality,
\[
P(2s) \leq \exp \left( (1 - 2s)C_1^2 \pi / \epsilon^4 \right) \frac{C_1 + C_2^2}{\epsilon^4} \|W\|_{BS}^2.
\]

Since \( \epsilon \leq s \leq 1/2 \) and \( 0 < X < e^X \), we can find a \( C_2 > 0 \) so that
\[
\frac{1}{2} \sqrt{P(2s)} = \|p(x)\|_{|x|=2s} \leq \frac{1}{2} \|W\|_{BS} e^{C_2 / \epsilon^4}.
\]

Since \( \epsilon \leq s \), the right hand side of the above equation is minimized by setting \( s = \epsilon \).

Hence for a fixed, arbitrary \( \epsilon > 0 \),
\[
\frac{1}{2} \sqrt{P(2\epsilon)} = \|p(x)\|_{|x|=2\epsilon} \leq \|W\|_{BS} e^{C_2 / \epsilon^4}. \tag{5.41}
\]

Since \( \epsilon \) was arbitrary, we have (5.10). From Lemma 5.3 in [2], we get (5.11).
Chapter 6
UNIQUENESS FROM BACKSCATTERING DATA FOR SMALL COEFFICIENTS

6.1 Introduction

Below \( A = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_3 = 0\} \), \( x = (x_1, x_2, x_3) \in \mathbb{R}^3 \), \( q \in C_c^\infty(\mathbb{R}^3) \) with \( q(x) \) even in \( x_3 \). For each \( a \in A \) and \( h \in (0, 1) \), let \( U^a(x, t) \) be the solution of the IVP

\[
(\Box - q)U^a(x, t) = \delta(x - a + he_1, t), \quad (x, t) \in \mathbb{R}^3 \times \mathbb{R} \tag{6.1}
\]

\[
U^a(x, t) = 0, \quad x \in \mathbb{R}^3, t < 0. \tag{6.2}
\]

We can decompose \( U^a \) (see theorem 4.1) as

\[
U^a(x, t) = \frac{1}{4\pi} \frac{\delta(t - |x - a + he_1|)}{|x - a + he_1|} + u^a(x, t)H(t - |x - a + he_1|),
\]

where \( u^a(x, t) \) is smooth in the conical region \( t - |x - a + he_1| \geq 0 \) and is the solution of the Goursat problem

\[
(\Box - q)u^a(x, t) = 0, \quad |x - a + he_1| < t \tag{6.3}
\]

\[
u^a(x, t) = \int_0^1 q(a - he_1 + s(x - a + he_1)) \, ds, \quad |x - a + he_1| = t \tag{6.4}
\]

\[
u^a(x, t) = 0, \quad t < 0. \tag{6.5}
\]

Our goal is to study the recovery of \( q(x) \) over \( \{x \in \mathbb{R}^3 : -\sqrt{1-h^2} < x_3 < \sqrt{1-h^2}\} \) from the data \( U^a(a + he_1, t) \) for all \( a \in A \) and \( h < t < 1 \) with a fixed \( h \).
In chapter 4 we proved uniqueness and stability under the assumption that the difference of two $q_i$ exhibited 'planar control'. In this chapter, we will answer the question without this assumption but with the assumption that either the norms $\|q_i\|_{C^7(\mathbb{R}^3)}$ are small enough or that $q_1 \geq q_2$.

**Theorem 6.1.** Suppose $q_i \in C^\infty_c(\mathbb{R}^3)$, and let $u_i^a$, $i = 1, 2$ be solutions of (6.3)-(6.5).

(a) If $q_1 \geq q_2$ and $u_1^a(a + he_1, 2\tau) = u_2^a(a + he_1, 2\tau)$ for a fixed $a \in A$, $0 < h < 1$ and for all $\tau \in (h, \infty)$, then $q_1(x) = q_2(x)$ on $\mathbb{R}^3$.

(b) Let $h = 0$. There is an $M > 0$ such that if $\|q_i\|_{C^7(\mathbb{R}^3)} \leq M$ for $i = 1, 2$ and $u_1^a(a, \tau) = u_2^a(a, \tau)$ for all $a \in A$ and all $\tau \in (0, 1)$ then $q_1(x) = q_2(x)$ on $\{x \in \mathbb{R}^3 : -1 < x_3 < 1\}$.

We can answer this question for the backscattering problem on the sphere as well. Let $B$ be the open unit ball in $\mathbb{R}^3$ and $S = \partial B$. If $q \in C^\infty_c(\mathbb{R}^3)$ (not necessarily even in $x_3$) and $U^a(x, t)$ is the solution of the IVP

\[
(\Box - q)U^a(x, t) = \delta(x - a, t), \quad (x, t) \in \mathbb{R}^3 \times \mathbb{R} \quad (6.6)
\]

\[
U^a(x, t) = 0, \quad t < 0. \quad (6.7)
\]

for $a \in S$ instead of the plane $A$, then we have the following theorem:

**Theorem 6.2.** Suppose $q_i \in C^\infty_c(\mathbb{R}^3)$ with support in $B$ and $u_i^a$, $i = 1, 2$ are solutions of (6.6)-(6.7). Then there is an $M > 0$ such that if $\|q_i\|_{C^7(\mathbb{R}^3)} \leq M$ for $i = 1, 2$ and $U_1^a(a, \tau) = U_2^a(a, \tau)$ for all $a \in S$ and all $\tau \in (0, 1)$ then $q_1 = q_2$ on $B$.

Note that we can prove part (a) of theorem 6.1 for the problem in theorem 6.2. The proof is almost exactly the same so we only prove it once.
6.2 Proof of Part (a) of Theorem 6.1

Let \( \varphi(x, \tau) = 2\tau - |x - a + he_1| - |x - a - he_1| \) and define

\[
E(a, \tau) := \{ x : \varphi(x, \tau) \geq 0 \}, \quad \partial E(a, \tau) := \{ x : \varphi(x, \tau) = 0 \}.
\]

Let \( p = q_1 - q_2 \) and \( W^a = U^a_1 - U^a_2 \). From section 4.2.3 we have

\[
(Mp)(a, \tau) = W^a(a + he_1, 2\tau) - \int_{E(a, \tau)} p(x)k(x, a, he_1, \tau) \, dx, \tag{6.8}
\]

where

\[
(Mp)(a, \tau) = \frac{1}{32\pi^2} \int_{\partial E(a, \tau)} \frac{p(x)}{|x - a + he_1||x - a - he_1|} \, dS_x,
\]

and

\[
k(x, a, he_1, \tau) = \frac{1}{4\pi} \left( \frac{u^2_1(x, 2\tau - |x - a - he_1|)}{|x - a - he_1|} + \frac{u^{a+2h}_1(x, 2\tau - |x - a + he_1|)}{|x - a + he_1|} \right)
+ \int_{|x - a + he_1|}^{2\tau - |x - a - he_1|} u^{a+2h}_1(x, 2\tau - t)u^a_2(x, t) \, dt.
\]

From (4.15) we have

\[
|\nabla_x \varphi(x, \tau)| = \frac{2}{|x - a + he_1||x - a - he_1|} D(x). \tag{6.9}
\]

So, letting \( k(x, a, he_1, \tau) = k(x, \tau) \),

\[
\int_{E(a, \tau)} p(x)k(x, \tau) \, dx = \int_{\varphi(x, \tau) \geq 0} p(x)k(x, \tau) \, dx
= \int_{\mathbb{R}^3} p(x)k(x, \tau)H(\varphi(x, \tau)) \, dx
= \int_{\mathbb{R}^3} p(x)k(x, \tau) \int_{\mathbb{R}} H(2\tau - 2\tau')\delta(\varphi(x, \tau')) \, d\tau' \, dx
= \int_{\mathbb{R}} H(2\tau - 2\tau') \int_{\mathbb{R}^3} p(x)k(x, \tau)\delta(\varphi(x, \tau')) \, dx \, d\tau'
= \int_0^\tau \int_{\varphi(x, \tau') = 0} p(x)k(x, \tau)|\nabla_x \varphi(x, \tau')|^{-1} \, dS_x \, d\tau'.
\]
\( \partial E(a, \tau) \) is empty for \( \tau < h \), so the above equation may be rewritten as

\[
\int_{E(a, \tau)} p(x)k(x, \tau) \, dx = \int_{h}^{\tau} \int_{\varphi(x, \tau')=0} p(x)k(x, \tau)|\nabla_x \varphi(x, \tau')|^{-1} \, dS_x \, d\tau'.
\]

Introducing

\[
\sigma := \sqrt{\tau^2 - h^2}, \quad \sigma' = \sqrt{\tau'^2 - h^2}.
\]

we have

\[
\int_{E(a, \tau)} p(x)k(x, \tau) \, dx = \int_{0}^{\sigma} \int_{\partial E(a, \tau')} p(x)k(x, \tau)|\nabla_x \varphi(x, \tau')|^{-1} \, dS_x \, d\sigma'.
\]

Noting (6.9),

\[
|\nabla_x \varphi(x, \tau')|^{-1} = \frac{|x - a + he_1| |x - a - he_1|}{2D(x)},
\]

since \( W^a(a + h, 2\tau) = 0 \) (by assumption) we can write (6.8) as

\[
(Mp)(a, \tau) = -\int_{0}^{\sigma} \int_{\partial E(a, \tau')} p(x)k(x, \tau)|\nabla_x \varphi(x, \tau')|^{-1} \, dS_x \, d\sigma'
\]

\[
= -\int_{0}^{\sigma} \int_{\partial E(a, \tau')} p(x)k(x, \tau)\frac{|x - a + he_1| |x - a - he_1|}{2D(x)} \, dS_x \, d\sigma'
\]

\[
\leq \int_{0}^{\sigma} \int_{\partial E(a, \tau')} p(x)|k(x, \tau)|\frac{|x - a + he_1| |x - a - he_1|}{2D(x)} \, dS_x \, d\sigma'.
\]

Since \( q_1 \geq q_2 \), we have \( p \geq 0 \). Since \( k \) is smooth in the support of \( p \), and \( 0 \leq \frac{\sigma}{\tau} \leq 1 \) we can find \( C_0 \) independent of \( \sigma \) so that

\[
(Mp)(a, \tau) \leq C_0 \int_{0}^{\sigma} \frac{1}{32\pi^2} \int_{\partial E(a, \tau')} p(x) \frac{D(x)}{|D(x)|} \, dS_x \, d\sigma',
\]

i.e.

\[
(Mp)(a, \tau) \leq C_0 \int_{0}^{\sigma} (Mp)(a, \tau') \, d\sigma'.
\]

Since \( (Mp)(a, \tau) \geq 0 \), Gronwall’s inequality implies

\[
(Mp)(a, \tau) = 0, \quad 0 < \sigma
\]

Thus \( p = 0 \) on \( \partial E(a, \tau) \) for all \( \tau > h \), so \( p = 0 \) on \( \mathbb{R}^3 \).
6.3 Part (b) of Theorem 6.1

For \( h = 0 \) and \( a \in A \), recall (4.21)

\[
(Mp)(a, \tau) = -8\pi \int_{|x-a| \leq \tau} p(x)k(x, a, \tau) \, dx,
\]

(6.10)

where

\[
(Mp)(a, \tau) := \frac{1}{4\pi \tau^2} \int_{|x-a|=\tau} p(x) \, dS_x.
\]

From [5], we know that

\[
2\partial_\tau (\tau (Mp)(a, \tau)) = w(a, \tau), \quad a \in A, \ \tau \in (0, 1),
\]

where \( w(x, t) \) is the solution of

\[
\square w = 0, \quad x \in \mathbb{R}^3, \ x_3 > 0, \ t > 0 \quad (6.11)
\]

\[
w(x, 0) = p(x), \quad w_t(x, 0) = 0, \quad x \in \mathbb{R}^3, \ x_3 > 0 \quad (6.12)
\]

\[
\partial_{x_3} w(x, t) = 0, \quad x \in \mathbb{R}^3, \ x_3 = 0, \ t > 0. \quad (6.13)
\]

From equations (12),(13) in [5] we get that (since \( M(a, \tau) = 0 \) for \( \tau > 1 \)),

\[
\int_{x_3 > 0} \frac{|p(x)|^2}{x_3} \, dx = \int_0^1 \int_{\mathbb{R}^2} \frac{w^2(a, \tau)}{\tau} \, da' \, d\tau.
\]

Since \( p \) is supported away from \( x_3 = 0 \), we have for some small \( \epsilon > 0 \), that

\[
\int_{x_3 > \epsilon} \frac{|p(x)|^2}{x_3} \, dx = \int_{x_3 > 0} \frac{|p(x)|^2}{x_3} \, dx
\]

\[
\leq 4 \int_0^1 \int_{\mathbb{R}^2} \frac{[(Mp)(a, \tau)]^2 + \tau^2[\partial_\tau (Mp)(a, \tau)]^2}{\tau} \, da' \, d\tau. \quad (6.14)
\]

Define

\[
K := \sup_{\text{supp} \ p} \{ |k| + |k_\tau| \},
\]

then from (6.10), we have

\[
|(Mp)(a, \tau)|^2 \leq 64\pi^2 \frac{4\pi \tau^3}{3} K^2 \int_{|x-a| \leq \tau} |p(x)|^2 \, dx,
\]

(6.15)
\[ \partial_{\tau} (Mp)(a, \tau) = -8 \pi \int_{|x-a| \leq \tau} p(x)k(x,a,\tau) \, dx - 8 \pi \int_{|x-a|=\tau} p(x)k(x,a,\tau) \, dS_x, \]

so that
\[
|\partial_{\tau} (Mp)(a, \tau)|^2 \leq 128 \pi^2 K^2 \left[ \frac{4 \pi \tau^3}{3} \int_{|x-a| \leq \tau} |p(x)|^2 \, dx + 4 \pi \tau^2 \int_{|x-a|=\tau} |p(x)|^2 \, dS_x \right].
\]

From (6.15)-(6.16) we have
\[
((Mp)(a, \tau))^2 + \tau^2 |\partial_{\tau} (Mp)(a, \tau)|^2 \leq 512 \pi^3 (\tau^3 + \tau^5) K^2 \int_{|x-a| \leq \tau} |p(x)|^2 \, dx
\]
\[
+ 512 \pi^3 \tau^2 K^2 \int_{|x-a|=\tau} |p(x)|^2 \, dS_x.
\]

Thus (6.14) becomes
\[
\int_{x_3 > \epsilon} \frac{|p(x)|^2}{x_3} \, dx
\]
\[
\leq 2048 \pi^3 K^2 \int_0^1 \int_{\mathbb{R}^2} \left( \tau^2 + \tau^4 \right) \left[ \int_{|x-a| \leq \tau} |p(x)|^2 \, dx + \tau^2 \int_{|x-a|=\tau} |p(x)|^2 \, dS_x \right] \, \, da' \, d\tau
\]
\[
\leq 2048 \pi^3 K^2 \int_0^1 \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} |p(x)|^2 \left[ \delta(\tau^2 - |x-a|^2) + H(\tau^2 - |x-a|^2) \right] \, dx \, da' \, d\tau.
\]

From (4.25) and (4.27) one can compute
\[
\int_{\mathbb{R}^2} \delta(\tau^2 - |x-a|^2) \, da' = \pi H(\tau - |x_3|),
\]
and
\[
\int_{\mathbb{R}^2} H(\tau^2 - |x-a|^2) \, da' = \pi (\tau^2 - x_3^2) H(\tau - |x_3|).
\]

Since \( p \) is even in \( x_3 \) and supported away from \( x_3 = 0 \),
\[
\int_{x_3 > 0} \frac{|p(x)|^2}{x_3} \, dx \leq 4096 \pi^4 K^2 \int_0^1 \int_{|x_3| \leq \tau} |p(x)|^2 \, dx \, d\tau \leq 4096 \pi^4 K^2 \int_{|x_3| \leq 1} |p(x)|^2 \, dx
\]
\[
\leq 4096 \pi^4 K^2 \int_{|x_3| \leq 1} \frac{|p(x)|^2}{x_3} \, dx
\]
\[
= 9192 \pi^4 K^2 \int_{0 < x_3} \frac{|p(x)|^2}{x_3} \, dx.
\]
So if $K$ satisfies
\[ K < \frac{1}{64\pi^2\sqrt{2}}, \]
then $p = 0$.

We have from the definition of $k$ and $K$, that
\[ K \leq \frac{\|u\|_\infty}{2\pi\epsilon} + 2\|u\|_\infty^2 + \frac{\|u_t\|_\infty}{\pi\epsilon} + 2\|u_t\|_\infty^2 \leq \frac{\|u\|_*}{\pi\epsilon} + 2\|u\|_*^2, \quad (6.17) \]
where
\[ \|u\|_* = \sup_{i=1,2, x \in \mathbb{R}^3, 0 < t < 1} (|u_i| + |\partial_t u_i|). \]

We have the following lemma from theorem 1.3 in [2]:

**Lemma 6.1.** Given $a \in A$, $q \in C^\infty_c(\mathbb{R}^3)$ with $\|q\|_{C^7(\mathbb{R}^3)} \leq M$ supported away from $x_3 = 0$, and $u^a(x,t)$ the solution of the Goursat problem
\[ \Box u^a - q u^a = 0, \quad |x - a| \leq t \quad (6.18) \]
\[ u^a(x,t) = \int_0^1 q(a + s(x - a)) \, ds, \quad |x - a| = t \quad (6.19) \]
\[ u^a(x,t) = 0, \quad t < 0. \quad (6.20) \]

There is a $C > 0$ so that
\[ \sup_{\mathbb{R}^3 \times (0,1)} |u(x,t) + u_t(x,t)| \leq CM. \]

Combine this with (6.17) to get
\[ K \leq \frac{CM}{\pi\epsilon} + 2C^2M^2. \]

We can thus find $M$ small enough, dependent on the size of $\epsilon$ and $C$, to get the desired bound on $K$. 

6.4 Part (b) of Theorem 6.2

From Proposition 2 in [27], if $U_1^a = U_2^a$ on $S \times [0,1]$, then

$$(Mp)(a, \tau) = -8\pi \int_{|x-a| \leq \tau} p(x)k(x,a,\tau) \, dx,$$  \hfill (6.21)

where

$$(Mp)(a, \tau) := \frac{1}{4\pi \tau^2} \int_{|x-a| = \tau} p(x) \, dS_x,$$

and for $|x-a| \leq \tau$,

$$k(x,a,\tau) = \frac{(u_1^a + u_2^a)(x, 2\tau - |x-a|)}{4\pi |x-a|} + \int_{|x-a|}^{2\tau-|x-a|} u_1^a(x, 2\tau - t)u_2^a(x, t) \, dt \hfill (6.22)$$

is smooth in the support of $p$. Thus if $w(x,t)$ solves the IVP

$$\Box w(x,t) = 0, \quad x \in \mathbb{R}^3, \ t \in \mathbb{R}$$

$$w(x,0) = 0, \quad x \in \mathbb{R}^3$$

$$w_t(x,0) = p(x), \quad x \in \mathbb{R}^3,$$

then via Kirchoff’s formula,

$$w(a, \tau) = \tau (Mp)(a, \tau), \quad a \in S, \ \tau \in (0,1).$$

Thus from (6.21)

$$w(a, \tau) = -8\pi \tau \int_{|x-a| \leq \tau} p(x)k(x,a,\tau) \, dx, \quad a \in S, \ \tau \in (0,1).$$

Thus

$$w_\tau(a, \tau) = -8\pi \left[ \int_{|x-a| \leq \tau} p(x)(k(x,a,\tau) + \tau k_x(x,a,\tau)) \, dx + \tau \int_{|x-a| = \tau} pk \, dS_x \right] \hfill (6.23)$$

Since $p$ is supported in $B$, $w(a, \tau)$ vanishes for $\tau > 1$. Noting this and from Equation (8) in [12], we have

$$\int_{\mathbb{R}^3} |p(x)|^2 \, dx = \int_{|a|=1} \int_0^1 2\tau w_\tau(a, \tau)^2 \, d\tau \, dS_a. \hfill (6.24)$$
So from (6.23), letting 
\[ K := \sup_{\text{supp } p} \{|k| + |k_\tau|\}, \]
we have
\[
\int_{\mathbb{R}^3} |p(x)|^2 \, dx \leq 128\pi^2 \int_{|a|=1} \int_0^1 \left[ \frac{4\pi \tau^3}{3} \int_{|x-a| \leq \tau} p^2(k + \tau k_\tau)^2 + 4\pi \tau^3 \int_{|x-a| = \tau} p^2 k^2 \right] \, d\tau \, dS_a 
\]
\[
\leq 1024\pi^3 K^2 \int_{|a|=1} \int_0^1 \left[ \int_{|x-a| \leq \tau} p^2 + \int_{|x-a| = \tau} p^2 \right] \, d\tau \, dS_a 
\]
\[
\leq 2048\pi^3 K^2 \int_{|a|=1} \int_{|x-a| \leq 1} |p(x)|^2 \, dx \, dS_a 
\]
\[
\leq 9192\pi^4 K^2 \int_{\mathbb{R}^3} |p(x)|^2 \, dx. 
\]

The last line comes from the fact that, for \(0 < \tau < 1\),
\[
\int_{|a|=1} \int_{|x-a| \leq \tau} |f(x)| \, dx = 4\pi \int_{|x| \geq 1-\tau} |f(x)| \, dx. 
\]

If we can find \(K\) so that
\[ K < \frac{1}{64\pi^2\sqrt{2}}, \]
then \(p = 0\) on \(B\). Since \(S\) is compact and \(p\) is supported away from \(S\), we have that \(\text{supp } p\) has a positive minimum distance, \(\epsilon\), from \(S\). Thus (6.22) implies for \(0 < \tau < 1\),
\[
K \leq \frac{\|u\|_\infty}{2\pi\epsilon} + 2\|u\|_\infty^2 + \frac{\|u_t\|_\infty}{\pi\epsilon} + 2\|u_t\|_\infty^2 \leq \frac{\|u\|_*}{\pi\epsilon} + 2\|u\|_*^2, \quad (6.25) 
\]
where
\[ \|u\|_* = \sup_{i=1,2} (|u_i| + |\partial_t u_i|). \]

For \(a \in S\) instead of \(A\), lemma 6.1 holds from theorem 1.3 in [2]. We can thus find \(M\) small enough, dependent on the size of \(\epsilon\) and \(C\), to get the desired bound on \(K\).


[12] D. Finch and Rakesh. Trace identities for solutions of the wave equation with
initial data supported in a ball. Mathematical Methods in the Applied Sciences,


[14] F.G. Friedlander and M. Joshi. Introduction to the theory of distributions. Cam-


1982.


for the design of nonuniform fiber bragg gratings. IEEE Journal of Quantum

[21] D. Finch, S. Patch and Rakesh. Determining a function from its mean values over
a family of spheres. SIAM Journal on Mathematical Analysis, 35(5):1213–1240,
2004.

[22] Rakesh. Characterization of transmission data for webster’s horn equation. Inverse

[23] Rakesh. A one dimensional inverse problem for a hyperbolic system with complex


[25] Rakesh and P. Sacks. Impedance inversion from transmission data for the wave

[26] Rakesh and G. Uhlmann. Uniqueness for the inverse back-scattering problem for


