CONCEPTIONS AND CONSEQUENCES OF WHAT WE CALL ARGUMENTATION, JUSTIFICATION, AND PROOF

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Argumentation, justification, and proof are conceptualized in many ways in extant mathematics education literature. At times, the descriptions of these objects and processes are compatible or complementary; at other times, they are inconsistent and even contradictory. The inconsistencies in definitions and use of the terms argumentation, justification, and proof highlight the need for scholarly conversations addressing these (and other related) constructs. Collaboration is needed to move toward, not one-size-fits-all definitions, but rather a framework that highlights connections among them and exploits ways in which they may be used in tandem to address overarching research questions. Working group leaders aim to facilitate discussions and collaborations among researchers and to advance our collective understanding of argumentation, justification and proof, particularly the relationships among these important mathematical constructs. Working group sessions will provide opportunities to engage with a panel of researchers and other participants who approach these aspects of reasoning from different perspectives, as well as to: hear findings from a recent analysis of these constructs in research; reflect on one's own work and position it with respect to the field; and contribute to moving the field forward in this area.

Keywords: Reasoning and Proof; Advanced Mathematical Thinking

Brief History of the Working Group

This is a new working group intended to advance the field's collective understanding of the interrelated objects and processes of argumentation, justification, and proof. We reviewed the prior 20 years of PME-NA proceedings (1995 – 2014) and prior 10 years of PME proceedings (2005 – 2014) to determine whether any previous working groups have focused on these topics. We found one related working group and two related discussion groups; however no group focused on the connections among these three constructs.

A working group on "Learning and Teaching with Proof" was facilitated by Stylianou and Blanton (2004) at PME-NA 26. That working group focused specifically on the development of proof across K-16, whereas the proposed working group focuses on the field's understanding and study of not only proof, but also argumentation and justification, as well as the interrelationships among them. A discussion group on argumentation in mathematics education convened at PME 30 that focused partly on defining and discussing the role of argumentation in mathematics education and research (Schwarz & Boero, 2006). While informative to the current efforts, the focus of that discussion group was distinct from the present working group. Most recently, a discussion group at PME 33 focused on the value of "generic proofs," a particular type of proof presentation (Leron & Zaslavsky, 2009), a topic which is much more specific than this proposed working group. Thus, we classify the present working group as new, while recognizing the contributions of earlier efforts in prior PME-NA and PME meetings. Additionally, none of the leaders of the proposed working group have participated in the aforementioned working groups, providing a further distinction between prior work and the present group. The leaders of this working group are researchers working in

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different areas of argumentation, justification and proof. Recent collaborations and conversations have led us to consider a need for the field at-large to converse about these interrelated objects and processes, which subsequently led to this newly proposed working group.

Focal Issues

There is a large and growing body of research in mathematics education focused on argumentation, justification, and proof. The research on proof, for example, includes studies on: the role of proof in the discipline; proof in school mathematics and at the undergraduate level; what counts as a proof; proof schemes and categories; teachers' conceptions of proof; students' abilities to write valid proofs; and what teaching proof looks like in classrooms at various levels (e.g., Boero, 2007; Harel & Sowder, 2007; Reid & Knipping, 2010; Stylianou, Blanton, & Knuth, 2009). At the same time, researchers and policy documents have issued calls to engage school children with disciplinary practices such as constructing viable arguments, justifying conclusions, critiquing the reasoning of others, and constructing proof for mathematical assertions (National Council of Teachers of Mathematics [NCTM], 2000; National Governors Association Center for Best Practices [NGA] & Council of Chief State School Officers [CCSSO], 2010).

Yet, as the field moves forward to strive for maximizing students' learning opportunities for engaging in these disciplinary practices, mathematics educators need to refine their notions of these terms in scholarly activities and in policy documents (Cai & Cirillo, 2014). How, when, and why decisions related to word choices are made (e.g., 'argument' versus 'proof') in curriculum materials, policy documents, and research is an open question. In fact, some researchers have hinted that these choices are not always purposeful. For example, Lynn Steen, a member of the 1989 NCTM Standards Committee, claimed that uncertainty about the role of proof in school mathematics caused NCTM in its first *Standards* (1989) document to resort to, what he called, "euphemisms" such as "'justify,' 'validate,' 'test conjectures,' [and] 'follow logical arguments'" (Steen, 1999, p. 274). Rarely, he stated, did the document use the term 'proof.' Although Steen's comments were published over 15 years ago, we would argue that his proposition, that the role of proof (as well as argumentation and justification) in school mathematics is uncertain, continues to be true today.

Building on NCTM's (2000) document, which recommended that students' experiences with reasoning and proof include making and testing conjectures, judging the validity of arguments, and constructing proofs, the *Common Core State Standards for Mathematics* (*CCSSM*; NGA & CCSSO, 2010) continued this emphasis. Although proof is no longer included as an explicit standard, the authors added attention to argumentation through the third Standard for Mathematical Practice. Proof and justification (or proving, justifying, etc.) are also included, with proof appearing most often in the high school standards. As the field continues to grapple with the meanings and interconnections of argumentation, justification, and proof, the usage of these varying terms could point to potential challenges of implementing and studying this aspect of CCSSM.

One challenge of reading extant research or developing a research agenda related to these disciplinary practices is that the classifications offered differ according to the perspective of the researcher, the focus of the research, and the particular data being analyzed (Reid & Knipping, 2010). Only recently have we begun to see mathematics educators offering explicit definitions of these constructs in their work. This is ironic given the importance of definitions in the field of mathematics itself. A review of the literature reveals much more specific attention to proof than to justification and argumentation in mathematics. Table 1 provides some definitions of the three constructs.

When considering the definitions provided in Table 1, one might notice various things. For example, two of the authors describe proof as an argument. This is interesting given that Cabassut and colleagues (2012) claimed that opposing views exist in the field: On the one hand, it has become customary in mathematics education to use the term 'argumentation' for reasoning which is not yet a

Table 1. Definitions of Proof/Proving, Argumentation, and Justification.

Proof/Proving	Argumentation	Justification/Justify
"the process employed by an individual to remove or create doubts about the truth of an observation" (Harel & Sowder, 1998, p. 241)	"mathematical explanation intended to convince oneself or others about the truth of a mathematical idea" (Mueller, Yankelewitz, & Maher, 2012, p. 376)	"to provide sufficient reason for" (National Research Council, 2001, p. 130)
"arguments consisting of logically rigorous deductions of conclusions from hypotheses" (NCTM, 2000, p. 55)	"discursive exchange among participants for the purpose of convincing others through the use of certain modes of thought" (Wood, 1999, p. 172)	"an argument that demonstrates (or refutes) the truth of a claim that uses accepted statements and mathematical forms of reasoning" (Staples, Bartlo, & Thanheiser, 2012, p. 448)
"a mathematical argument, a connected sequence of assertions for or against a mathematical claim, with the following characteristics: (1) It uses statements accepted by the classroom community (*set of accepted statements*) that are true and available without further justification; (2) It employs forms of reasoning (*modes of argumentation*) that are valid and known to, or within the conceptual reach of, the classroom community; and (3) It is communicated with forms of expression (*modes of argument representation*) that are appropriate and known to, or within the conceptual reach of, the classroom community." (Stylianides, 2007, p. 291)	"the process of making an argument, that is, drawing conclusions based on a chain of reasoning" (Umland & Sriraman, 2014, p. 44)	

proof, on the other hand, other researchers, such as Balacheff (1988) and Duval (2007), believe that argumentation and proof are fundamentally different.

Reid and Knipping argued that while ignoring multiple uses of 'proof' can lead to communicative challenges, the answer is not to insist on one "correct" usage (Reid & Knipping, 2010; Herbst & Balacheff, 2009). In particular, Herbst and Balacheff (2009) stated:

> If the field is in a deadlock as regards to what we mean by "proof," we contend this is so partly because of the insistence on a comprehensive notion of proof that can serve as referent for every use of the word…. We have argued that to make it operational for understanding and appraising the mathematics of classrooms we need at least three meanings of the word. (p. 62)

Awareness of the different uses of proof is an important step in deciphering and making progress in mathematics education research (Reid & Knipping, 2010). In fact, Balacheff (2002) claimed that "research speaks in a very confusing way about the topic" (p. 39).

Conflicting Constructs, Methodologies, and Findings from the Research Literature

In a forthcoming review of the literature, Stylianides, Stylianides, and Weber (forthcoming) asserted that the field has made substantial progress in tackling these issues. Researchers have developed powerful conceptual constructs and methodologies, and our collective understanding of how students perceive proof and how justification, argumentation, and proof are taught in K-12 and university classrooms has very much improved. Nonetheless, there are several places where different researchers have generated conclusions that conflict with one another. Here, we present a few such contradictions.

First, in the debate on how proof should be introduced to students, Marty (1991) proposed teaching students the rules of logic and standard proof techniques while deliberately keeping the content shallow, suggesting that the notion of proof as a justification to prove mathematically meaningful statements should come later. Alibert and Thomas (1991) suggested the opposite; students should initially be engaged in argumentation about the veracity of interesting mathematical statements. In socially negotiating what counts as an acceptable argument, the instructor can lead students to produce the standard norms of proof. Both researchers presented research results that supported their point of view.

Second, as mentioned above, there is a debate as to whether argumentation and proof are activities that are deeply intertwined or fundamentally separate. For instance, numerous authors have proposed that students will be more successful at proof writing if they base their proofs on informal arguments, often involving the inspection of examples and graphical reasoning (e.g., Garuti, Boero, & Lemut, 1998; Raman, 2003; Sandefur, Mason, Stylianides & Watson, 2013; Weber & Alcock, 2004). However, Duval (2007) contended that argumentation and proof occur in different semiotic registers. Due to the different goals and demands of each activity, they must occur separately.

Third, several researchers have presented classroom studies that demonstrate that even young children are capable of writing proofs (e.g., Maher & Martino, 1996). Yet there are numerous studies that suggest that high school geometry students and even advanced mathematics majors struggle to write relatively simple proofs (e.g., Moore, 1994; Senk, 1989). Is it possible that young children have greater competency at proving than most geometry students or mathematics majors? Or are their proofs being evaluated according to different standards?

Fourth, a number of researchers have defined proof to be a convincing justification (cf., Balacheff, 2008). Yet there are other mathematics educators who emphasize that some proofs do not always provide full justification to mathematicians, and mathematicians sometimes are convinced by justifications that they would not call proofs (e.g., de Villiers, 1990; Tall, 1989; Rodd, 2002; Weber, Inglis, & Mejía-Ramos, 2014).

What we contend is that these, and other, inconsistent claims are based on researchers holding different conceptions of argumentation, justification, and proof. What we hope to accomplish in this working group is to develop a better understanding of these differences, an appreciation for what different perspectives of proof can accomplish, and a framework that describes how different perspectives can complement each other, rather than oppose each other and lead to inconsistent findings.

Frameworks for conducting research: The case of proof

In the forthcoming review, Stylianides et al. (forthcoming) described three broad research traditions with respect to proof. One theoretical frame is to view *proving as problem solving*. In these studies, little emphasis is given to the issues of what constitutes a proof, why students engage in this activity, and how they interpret the proofs that they produce. Issues of argumentation and justification are typically ignored. Instead, these studies focus on what competencies are needed to

successfully write a proof and design instruction that helps students develop these competencies (e.g., Selden & Selden, 2013; Weber, 2001).

Another school of thought views *proving as convincing* (e.g., Harel & Sowder, 1998). Researchers in this perspective seek to determine what types of justifications students find convincing and attempt to develop instruction that leads students to transform their standards of conviction to those held by mathematicians (e.g., Harel, 2002; Recio & Godino, 2001). Careful attention is paid between arguments that are complete convincing justifications and those that merely increase one's belief in the likelihood of a statement.

A third framework treats *proof as socially embedded activity*. In this perspective, researchers focus on what it is students, from their perspective, are *actually* doing when they engage in proving (e.g., Herbst & Brach, 2006) and how students' and teachers' activities are shaped by social and institutional constraints (e.g., Herbst & Chazan, 2003). Researchers often focus on the cognitive, social, and pedagogical goals that proof can and should play in the classroom beyond being a skill to acquire or a means of conviction (Staples, Bartlo, & Thanheiser, 2012).

In the review, Stylianides and colleagues found that each perspective (a) used different conceptions of argumentation, justification, and/or proof, (b) addressed different questions, (b) developed different theoretical constructs to understand and investigate these questions, and (d) measured the success of instructional interventions in different ways. They also found that considerable progress has been made within each of these perspectives that has enhanced the field's understanding of proof and related constructs. As one can see, many varying conceptions of argumentation, justification, and proof exist in our field. It would be an unrealistic and inappropriate goal of the working group to try to reach a consensus on what argumentation, justification, and proof are. Indeed, in mathematical practice, mathematicians adopt different standards and perspectives on these objects and corresponding processes depending upon their aims and context (Weber, 2014). Rather what we seek to explore is the different ways that researchers conceptualize these constructs (including, but not necessarily limited to, the perspectives above), the consequences of such conceptualizations, and how they might work in tandem to address overarching research questions.

Balacheff (2002) put forth a set of recommendations to help address what he called a research "deadlock" (p. 1). Heeding his recommendations, we will consider the following during the working group: (a) looking for a common lexicon to improve available definitions; (b) engaging with different research programs and their possible contrasts and relationships; (c) considering the theoretical commonalities and divergences, and possibly turning them into research questions; (d) discussing different methodologies, their benefits, and possibly limitations; and (e) acknowledging accepted results or turning objections and differences into research problems.

Plan for the Working Group

The overarching goal of the working group is to facilitate discussion and collaboration among researchers in the field doing work in this area – at various stages of their careers – and to advance our collective understanding of argumentation, justification and proof, and relationships among these important mathematical constructs. Aligned with these goals to advance our collective understanding, we anticipate organizing our time together in the following manner.

Session 1: *Where are we now? Where is the field? Where are you?*

In Session 1, we begin with introductions and discerning interests in the working group, and then provide an overview of the goals for group. We then engage participants in two activities in order to establish some common ground for our collective work and to provide a reflective opportunity for participants to position their own thinking and work with respect to the field. The key activities in Session 1 are as follows:

- We will facilitate Introductions, Define the Problem, and outline Working Group Goals.
- Keith Weber, a co-author of the proof chapter in the forthcoming NCTM handbook, will offer an historical view of the use of these constructs in mathematics and educational research. The talk will highlight convergence in the literature as well as contradictions in definitions and research results. The purpose of the talk will not be to call for convergence, but rather to highlight different traditions and where points of disagreement may lie, and to propose ways in which different traditions may inform each other to advance the field collectively.
- All working group participants produce a diagram or concept map with the terms *argumentation, justification,* and *proof* to elicit personal conceptions and uses of these terms and how they are interrelated. Small groups will then compare and discuss these representations.

Session 2: *How have individual researchers used these constructs to support their work? What choices have they made?*

Session 2 features an interactive panel discussion by three invited math educators who focus on one or more notions of argumentation, justification and proof in their research. The panel of experienced researchers will share their expertise through a facilitated format, beginning with set questions and then evolving into a question-and-answer period and group discussion. Potential guiding questions include the following:

- *What process(es) and object(s) (justifying-justification, argumentation-arguments, and proving-proof) are central to your work? Why did you choose these? How have you conceptualized or defined them for your work?*
- *How do you see these conceptualizations mattering for the work you (we) do with teachers, with students, and/or as researchers as members of the mathematics and/or mathematics education community?*
- *From your perspective, where is the field now with its understanding of these processes and objects, how to foster them in classrooms, and how to support teacher learning of pedagogies to organize student participation in these critical processes?*

Three researchers, whose work is prominent in this area, will serve on the panel:

- Kristen Bieda, Michigan State University, has investigated the interaction between opportunities in curriculum to engage in justification and proof and how teachers enact those opportunities with students in middle school classrooms. Bieda is particularly interested in teachers' goals for justification in their classroom and how they modify and deploy curricular tasks to achieve those goals.
- Anna Conner, University of Georgia, studies the role of the teacher in collective argumentation, specifically how teachers learn to support their students in making mathematical arguments. Within this, Conner also studies teachers' beliefs about proof and how this may influence the argumentation in their classes.
- Pablo Mejía-Ramos, Rutgers University, focuses on the reasoning processes involved in the three main argumentative activities related to the notion of proof in university mathematics: constructing a new proof, reading a given proof, and presenting a known proof. Two of the main goals of Mejía-Ramos's research are (1) to better understand the ways in which different types of students and mathematicians engage in these argumentative activities, and (2) to identify effective strategies for performing such activities.

Samuel Otten, University of Missouri, will moderate the panel discussion.

Session 3: ***Where are you now? What are next steps?***

We anticipate pursuing three general goals in Session 3 and recognize that the particulars of this session will be in response to the first two days. The three goals are:

- Gaining clarity in situating oneself and one's own work within the broader field. Toward this end, we revisit the set of individual diagrams generated in Session 1. Participants will have opportunities to revise their diagrams. We anticipate discussing two or three representations that emerge regarding the relationships among argumentation, justification and proof as "prototypical" views. We will discuss their similarities and differences, consequences for the field, and whether they are complementary or contradictory.
- Generating key questions or suggestions for the field to better leverage the work it is doing. First in small group discussion, and then with the full group, we will develop a handful of key questions that the collective identifies as important to furthering individuals' and the field's work with respect to argumentation, justification and proof. Subgroups may organize around these questions.
- Organizing work on a white paper and identifying next steps. We will share an outline for a white paper, and elicit input for the focus of Year 2 work at the next PME-NA conference. Given the group's interest and the key questions they develop, subgroups can contribute specific sections to the white paper. Subgroups may also organize around a focal process. Finally, a subgroup may plan to take up particular key questions or activities during the year.

Anticipated Follow-up Activities

We anticipate two follow-up activities and products for the first iteration of this working group. First, we have inquired with editors of some of the leading mathematics education journals in the field regarding the potential for publishing a summary of a panel discussion (similar to Sfard, Nesher, Streefland, Cobb, & Mason, 1998), or, alternatively, a research commentary. The commentary might provide recommendations for how mathematics education researchers report what they mean by argumentation, justification and/or proof in their work. Such a commentary may also document key questions for the field as well as share core ideas from discussions and differences in perspectives from organizers, panel members, as well as working group participants.

We also anticipate the development of a white paper from participants of the working group. The paper will have multiple sections documenting ideas from our generative work together, as well as solidifying and extending some of those ideas. A white paper format affords flexibility in how the document is organized and how authorship is assigned (e.g., authorship of sub-sections within the larger white paper). While not suggesting a definitive structure *a priori*, one potential format would include perspectives from developing sub-groups within the working group based on participants' diagrams and particular conceptions of argumentation, justification and proof. This activity will allow for further engagement within and beyond this meeting of the working group. The white paper would be developed over the months following PME-NA 37 with the goal of sharing the document widely (e.g., through our institutions' open-access digital repositories). Thus, such products fulfill our purpose of creating a working group at PME-NA: to engage colleagues in discussion of the different understandings and operations of argumentation, justification and proof. While consensus on these points is not anticipated, and not necessarily a goal, efforts towards consensus regarding how mathematics educators convey their conceptions of these objects and processes is.

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