

*Exploring Side-Side-Angle Triangle Congruence Criterion*

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Abstract

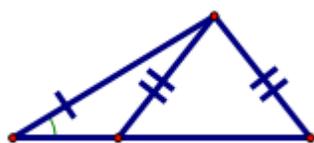
We describe an exploratory task intended to support students' conceptual understandings of triangle congruence with particular emphasis on the Side-Side-Angle (SSA) case. We reveal how SSA, often dismissed, is actually a challenging and interesting case for exploration.

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## Exploring Side-Side-Angle Triangle Congruence Criterion

Through our past experiences as students, teachers, and observers of teachers, we have noticed that the teaching of triangle congruence tends to be done by decree. More specifically, it is fairly typical for teachers to *tell* their students which triangle congruence criteria are valid and then have students use those postulates in proofs. We were also told (or we told our students) that some triangle congruence criteria are invalid. For example, when discussing Side-Side-Angle (SSA), which is *not* considered a valid congruence criterion, we tended to show a counterexample of two non-congruent triangles with the same SSA configurations (**fig. 1**). With the exception of Hypotenuse-Leg, it is *atypical* to explore situations where SSA actually does work to prove triangles congruent. Rather, when SSA is introduced, one can usually expect jokes from the students (or the teachers) about how SSA (spelled backwards) is a “bad” word, both inside and outside of mathematics.



**Fig. 1** An ambiguous case of SSA

Here, we describe a task intended to support students’ conceptual understandings of triangle congruence. Through the development of the activity and our students’ explorations, we, ourselves, realized that we needed to know more about SSA. As illustrated later, we learned that SSA guarantees a congruent copy of a triangle more often than it does not. In the remainder of this article, we describe the task and then reveal how SSA, often dismissed, is actually a challenging and interesting case for exploration.

### THE HIDDEN TRIANGLE EXPLORATION

The Hidden Triangle Exploration (HTE) is a rich, student-centered task of triangle congruence

criteria. In this task, students work in groups with geometry tools (compasses, rulers, and protractors) and a set of paper triangles concealed in an envelope. Each student takes a turn trying to get their group members to draw an exact (congruent) copy of a “hidden” triangle. The student giving the clues (the director) is free to measure and select which particular pieces of information to share with their peers (the drawers). For example, consider the following exchange as Jamie gives clues to her group members, Kenji and Joe:

Jamie: Okay, so first you want to draw a line segment that is six centimeters.

Kenji: The base is six?

Jamie: Yeah, the bottom segment is six centimeters. Maybe it will help if you label that segment AB.

Joe: Okay, what else?

Jamie: Then at point A, draw a forty degree angle.

Kenji: How do we know how long to draw it?

Jamie: Well that’s the next part I was going to give you. The side that forms the angle should be five-point-five [see **fig. 2**].

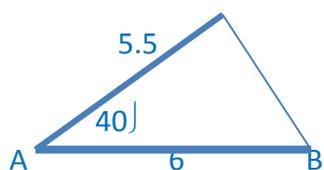
Kenji: Okay, what else?

Joe: What do you mean what else? Don’t you have a triangle?

Kenji: How would I have a triangle?

Joe: Well now you can just draw in the last side and we’re done.

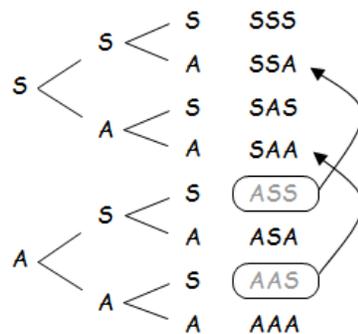
Jamie: Okay let’s see if your triangles are the same as mine.



**Fig. 2** Jamie’s triangle

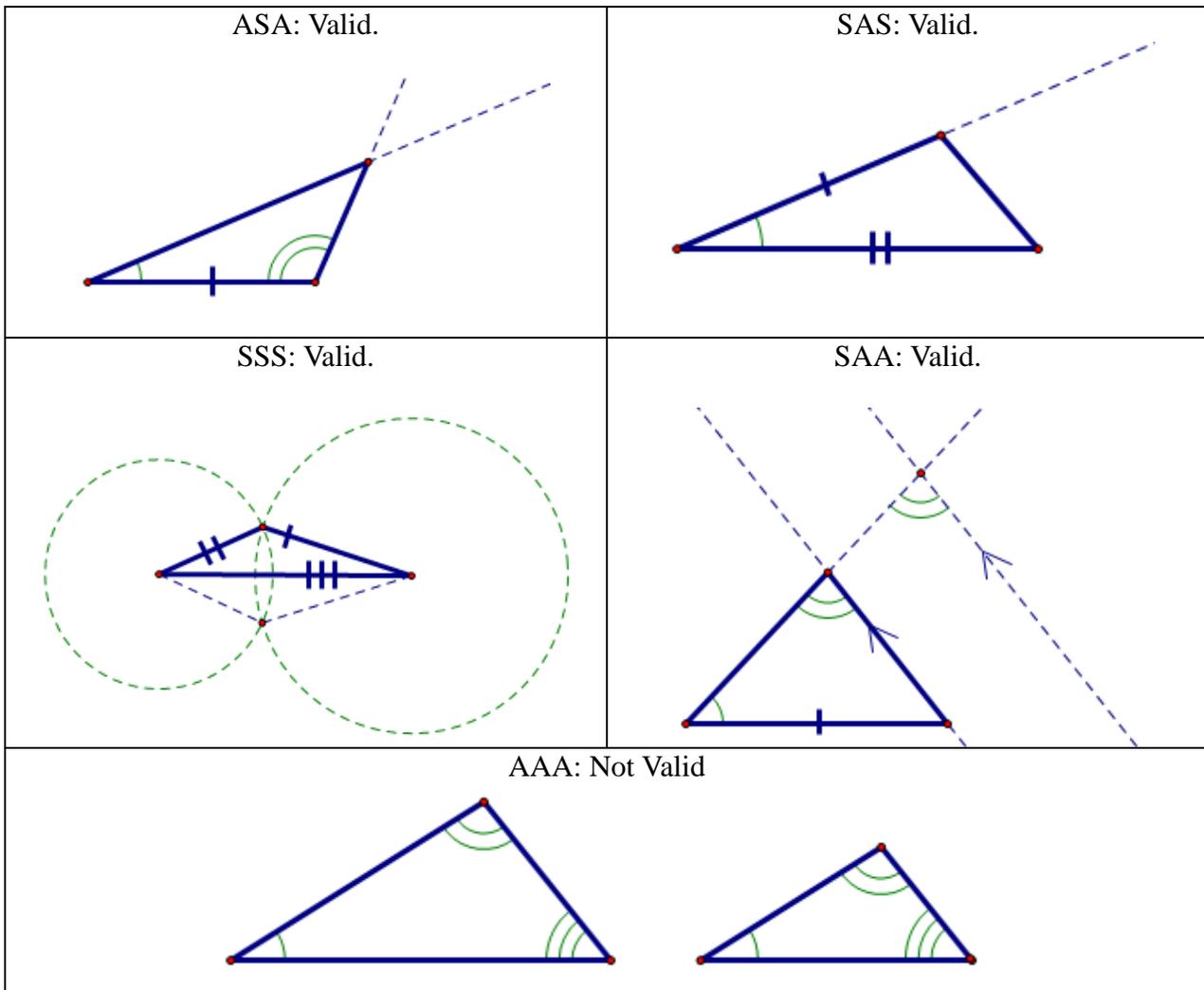
In the example above, Jamie provided her peers with the measurements of a side ( $\overline{AB}$ , 6cm), an angle ( $40^\circ$ ), and a second side (5.5cm), making use of SAS to help Kenji and Joe copy the hidden triangle. Some important goals of this activity include discovering that only three (not six) pieces of information are needed to copy a triangle and determining which combinations of three pieces of information will assist in making exact copies of hidden triangles and which will not.

As with any good task, the HTE can be modified so that it is more or less open-ended. In the version of the HTE described here, the director decides which triangle (e.g., acute, obtuse) and which combinations of sides and angles to use to conjecture about congruence criteria (e.g., SAS). In our experiences, students quickly discover that they can reproduce hidden triangles using ASA and SAS. After students discover that the minimum number of parts needed is three, they can work through each case systematically (see **fig 3**). Students usually find that AAA produces similar but not congruent triangles. It typically takes more effort and insight to reproduce triangles by SSS and AAS (see **fig. 4** for a summary). When it came to SSA, however, it was sometimes the case that the triangle the students produced was truly the only possible triangle that could be drawn. Because the textbooks that we have taught from either oversimplified or ignored the case of SSA, we were not initially well-prepared to deal with this situation. Therefore, in the remainder of this article, we explore the reasons that SSA is so often successful.



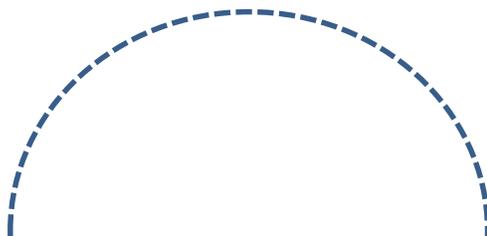
**Fig. 3** A tree diagram of triangle congruence criteria for exploration

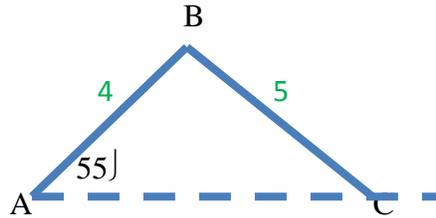
**Fig. 4** Congruence cases explored through the HTE



### SSA IN THE HTE

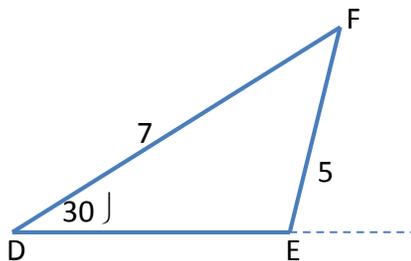
When only one triangle can be formed, SSA determines a unique triangle that is congruent to the original triangle. If two triangles can be formed, we have the ambiguous case of SSA. First, we examine a case of uniqueness. Suppose that during the activity a director gives the drawers a  $55^\circ$  angle ( $\angle A$ ) where one of the angle's rays is a 4 cm side and the next side length is 5 cm (see **fig. 5**). All the possible points that are five centimeters away from B are represented by the circle centered at B. There is only one point, C, where the circle intersects with the ray coming from A (now  $\overline{AC}$ ). By connecting A, B, and C, the drawers produced the only unique triangle from the given SSA information.



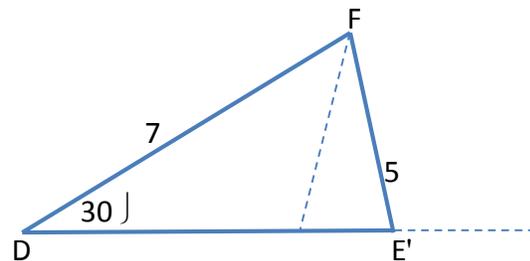


**Fig. 5** A unique case of SSA

Now let us examine an ambiguous case. Suppose a student director, Grady, was looking at triangle DEF as shown in **figure 6**. Grady is exploring the SSA case so he directs his partners, Maurice and Emma, to draw triangle DEF with  $\angle D$  measuring  $30^\circ$  and side lengths 7 and 5. From our experiences with the HTE, Maurice and Emma would typically draw the acute rather than the obtuse triangle (see **fig. 7**). When comparing their triangles, the group finds that they have two different triangles. This is good! Now they must ask themselves why this occurred. But, perhaps Grady's original triangle was acute triangle DE'F. Then Maurice and Emma might copy Grady's triangle without the group ever discovering a second possible option (DEF). As a result, they inductively reason that SSA works in every case. The latter scenario has played out in our classrooms when we facilitated the HTE. To better prepare for this situation, we needed to do some additional research to understand how often and under what conditions triangles can be reproduced by SSA.



**Fig. 6** Acute  $\sphericalangle (30^\circ, 7, 5)$

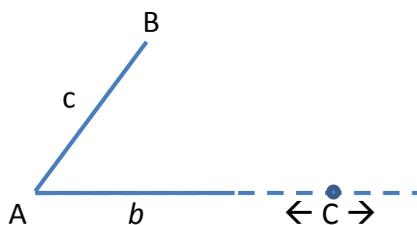


**Fig. 7** Obtuse  $\sphericalangle (30^\circ, 7, 5)$

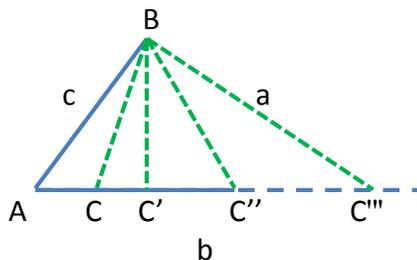
**STUDYING SSA MORE CLOSELY**

Recall that students choose from triangles provided by their teacher, and, once they select a

triangle, they choose any combination of sides and angles that satisfy the case they are exploring at the time (here, two sides and a non-included angle). Here we consider all options of the SSA case that could be explored through the HTE. Because every triangle has at least two acute angles, we begin with an acute angle. Acute  $\angle A$  is drawn such that it is determined by two rays,  $b$  and  $c$  where  $c$  is a known length and  $b$  is unknown (see **fig. 8**). Because we are working on SSA, we also know the length of side  $a$  ( $\overline{BC}$ ). However, we do not know where to place  $a$  without knowing where point C is (as determined by  $b$ , which is unknown) or the measure of angle B. Thus, we can imagine  $a$  as a hinged side hanging from point B (see **fig. 9**). Given the three possible scenarios for the relationship between  $a$  and  $c$  ( $a < c$ ,  $a = c$ ,  $a > c$ ), any type of triangle that exists could be drawn (**fig. 9**). For example, an obtuse scalene triangle (ABC), a right scalene triangle (ABC'), an acute isosceles triangle (ABC''), and so forth could be formed depending on the measure of angle A and the lengths of  $a$  and  $c$ . (For a full list of all seven possible triangle types, see Column I of **Table 1**.)



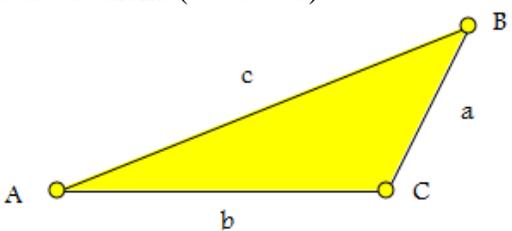
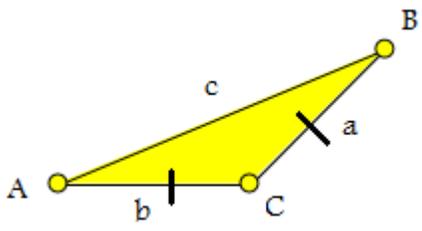
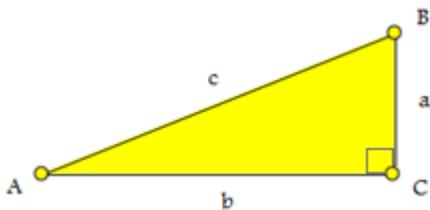
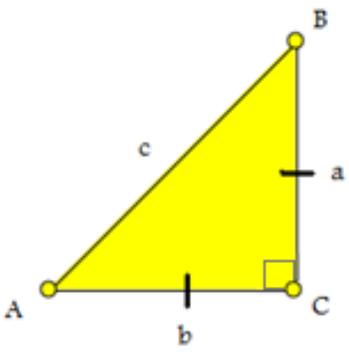
**Figure 8**

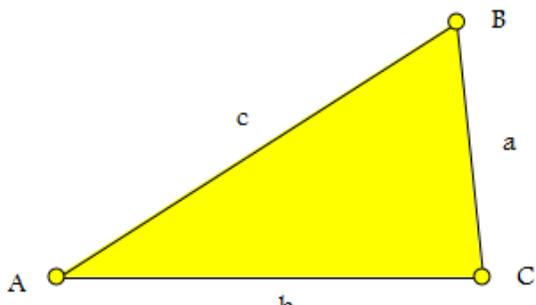
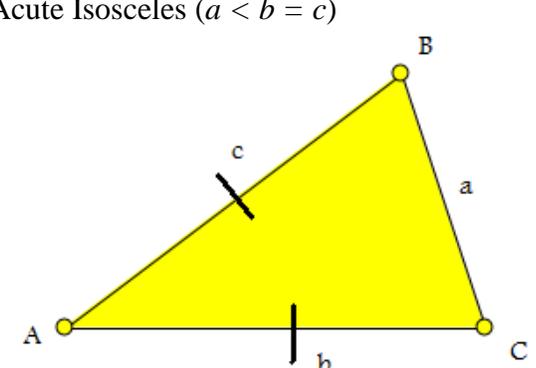
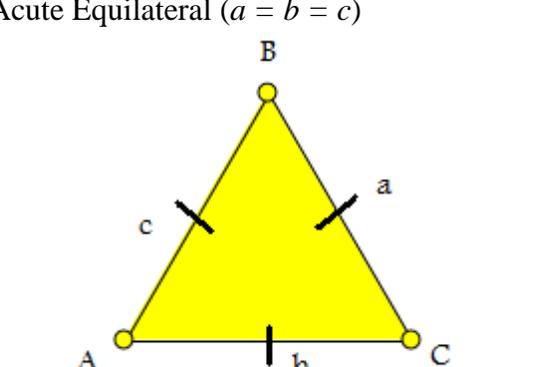


**Figure 9**

**Table 1** A case by case view of SSA for every type of triangle

I	II	III	IV
Triangle type given to be reproduced	Pieces of information given	Number of triangles that	ASS Case

		can be drawn	
Obtuse Scalene ( $a < b < c$ ) 	$\angle A c a$ $\angle A b a$ $\angle B a b$ $\angle B c b$ $\angle C b c$ $\angle C a c$	2 2 1 2 1 1	ASs II ASs II AsS ASs II AsS AsS
Obtuse Isosceles ( $a = b < c$ ) 	$\angle A c a$ $\angle A b a$ $\angle C b c$	(or $\angle B c b$ ) (or $\angle B a b$ ) (or $\angle C a c$ )	2 1 1 ASs II AsS AsS
Right Scalene ( $a < b < c$ ) 	$\angle A c a$ $\angle A b a$ $\angle B a b$ $\angle B c b$ $\angle C b c$ $\angle C a c$	1 2 1 1 1 1	ASs I ASs II AsS ASs I AsS: HL AsS: HL
Right Isosceles ( $a = b < c$ ) 	$\angle A c a$ $\angle A b a$ $\angle C b c$	(or $\angle B c b$ ) (or $\angle B a b$ ) (or $\angle C a c$ )	1 1 1 ASs I AsS AsS: HL
Acute Scalene ( $a < b < c$ )	$\angle A c a$ $\angle A b a$ $\angle B a b$ $\angle B c b$ $\angle C b c$	2 2 1 2 1	ASs II ASs II AsS ASs II AsS

	$\angle C a c$		1	AsS
Acute Isosceles ( $a < b = c$ ) 	$\angle A c a$ $\angle B a b$ $\angle B c b$	(or $\angle A b a$ ) (or $\angle C a c$ ) (or $\angle B c b$ )	2 1 1	<b>ASs II</b> AsS AsS
Acute Equilateral ( $a = b = c$ ) 	$\angle A c a$	(or $\angle A b a$ ) $\angle B a b$ $\angle B c b$ $\angle C b c$ $\angle C a c$ )	1	AsS

Now that we have determined that there are seven possible triangles for the students to work with, we need to consider the different combinations of sides and angles that could be selected by the students. In the case of an obtuse scalene triangle, there are six different possible combinations of one angle and one side that could be selected (see Column II of **Table 1**). However, if the triangle is an obtuse isosceles triangle, there are only half as many unique cases to consider (Column II, **Table 1**). To summarize the cases, there are always six unique cases for scalene triangles, three for isosceles triangles, and one for equilateral triangles. This gives us a total of 28 unique cases of SSA that could be explored during the HTE. Next, we need to determine the number of cases out of 28 that are

ambiguous. To do so, we consider a proof that was inspired by the proof of Hirschhorn (1990).

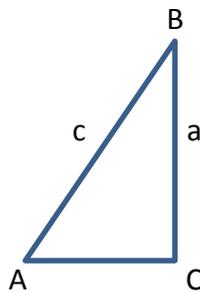
Hirshhorn wrote a proof of Yeshurun and Kay's (1983) SsA Theorem which stated:

*If the first and second sides and the angle opposite the first side of one triangle are congruent to the first and second sides and the angle opposite the first side of another triangle, respectively, such that the length of the first side is not less than that of the second, then the triangles are congruent.* (Yeshurun & Kay 1983, p. 364)

In other words, two triangles are congruent by SsA when the side opposite the given angle is not less than the second given side (e.g., in **fig. 9**, when  $a \geq c$ ).

In the proof presented here, we consider three cases of SSA. And, although our proof uses the Law of Sines, it is a geometric proof in the sense that it explains when and how two triangles can (and, unlike Hirshhorn's proof, when they cannot) be determined congruent by SSA. In order to understand the proof, it is useful to establish the notation used for the altitude of the triangle. Consider the triangle ABC where  $\angle C$  is a right angle (**fig. 10**). Then, using basic trigonometry  $\sin A = \frac{a}{c}$ . Thus  $a = c \sin A$ .

Now consider a triangle where  $\angle C$  is acute or obtuse. The altitude of that triangle would still be represented by  $c \sin A$ , as long as A is an acute angle.

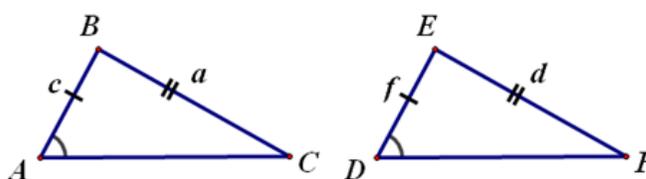


**Figure 10** Triangle ABC with altitude  $c \sin A$

The proof (see **fig 11**) explores three cases of SSA (written ASS for ease of following the argument. In Case One, ASs (I) represents the situation where  $\triangle ABC \cong \triangle DEF$  given that  $a = c \sin A$  (i.e.,  $\angle C$  is a right angle). In Case Two, ASs (II) represents the situation where we can produce two *different* triangles given that  $c > a > c \sin A$ . Finally, in Case Three, AsS, we proved that when  $a \geq c$  (i.e., we have an acute triangle with  $a > c$ , an isosceles triangle with  $a = c$ , or an obtuse triangle where  $\angle A$  is an obtuse angle), then  $\triangle ABC \cong \triangle DEF$ . To summarize, we proved: the case where the given  $\angle A$  is acute but one of the other angles of the triangle is a right angle (ASs I); the possibility of an ambiguous case (ASs II) where the first given side after the angle is larger than the second given side; and the SsA (or AsS) Theorem stated earlier where the first side after the angle is less than or equal to the second given side. Hypotenuse-Leg is a special case of Case Three. These are the only three possible situations that can occur when considering the SSA case, using actual triangles.

In **Table 1**, we determined how many possible triangles can be drawn given particular combinations of an angle and two sides (Column II of **Table 1**) within particular triangles (Column I of **Table 1**). Counting up the instances, only 9 of 28 possible configurations (32%) yield an ambiguous case (marked in red). When we take all possible ways that students could give directions for the HTE, including repetition, they are even less likely to come across an ambiguous case (only 26% of the time). (By repetition, we mean that in an equilateral triangle, for example, any combination of an angle followed by two sides would be the same case.) Thus, the HTE task provides an opportunity to point out that inductive reasoning, while useful, can sometimes be problematic. This discovery can motivate the study of formal deductive reasoning (i.e., proof).

**Figure 11** A Geometric Proof of SSA Using the Law of Sines



Given:  $\angle A \cong \angle D, a = d, c = f$ .

By the Law of Sines,  $\frac{\sin A}{a} = \frac{\sin C}{c}$ .

**Case One: ASs (I)** We want to show that ASs (I) is a valid congruence theorem given the additional constraint that  $a = c \sin A$ .

By substitution of  $a = c \sin A$  into the equation for the Law of Sines,  $\frac{\sin A}{c \sin A} = \frac{\sin C}{c}$ .

As a result,  $\sin C = 1$ . In a triangle, this means that  $m\angle C = 90^\circ$ .

By the Law of Sines,  $\frac{\sin D}{d} = \frac{\sin F}{f}$ .

From substitution according to the given statements,  $d = f \sin D$ . Using this equation as well as the Law of Sines, we can conclude similarly that  $m\angle F = 90^\circ$ .

Since  $m\angle C = m\angle F = 90^\circ$ , we can conclude that  $\angle C \cong \angle F$ . By the AAS congruence theorem,  $\triangle ABC \cong \triangle DEF$ .

Now,

Because  $\frac{\sin A}{a} = \frac{\sin C}{c}$ , we know  $\sin C = \frac{c \sin A}{a}$ .

Because  $\angle A \cong \angle D$ ,  $\sin A = \sin D$ .

Since  $a = d$  and  $c = f$ , by substitution we know  $\sin C = \frac{f \sin D}{d}$ . But  $\sin F$  is also  $\frac{f \sin D}{d}$  by the Law of Sines, so  $\sin C = \sin F$ .

**Case Two: ASs (II)** We want to show that ASs (II) is the ambiguous case when  $c > a > c \sin A$

Let  $\sin C = \gamma$ .

Because  $a > c \sin A$ , we know that  $\gamma \neq 1$ . There are two angles  $\alpha$  and  $\beta$  with a positive measure less than  $180^\circ$  that satisfy  $\sin \alpha = \sin \beta = \gamma$ . One angle is acute, and the other is obtuse.

We can make two triangles in this case, one where  $m\angle C = \alpha$ , and one where  $m\angle C = \beta$ .

**Case Three: AsS (III)** We want to show that AsS (III) is a valid congruence theorem given the additional constraint of  $a \geq c$  (where  $c \geq c \sin A$ ).

Since  $a \geq c$ ,  $m\angle A \geq m\angle C$ .

Since  $\angle C$  is not the largest angle in the triangle,  $\angle C$  is acute.

In a similar manner, because  $d \geq f$ ,  $\angle F$  is acute.

Since  $\angle C$  and  $\angle F$  are both acute angles and  $\sin C = \sin F$ ,  $\angle C \cong \angle F$ .

Thus,  $\triangle ABC \cong \triangle DEF$ .

## CONCLUSION

There are multiple motivations for having students *explore* triangle congruence criteria. For

example, NCTM (2000) encourages teachers to allow students to create their own conjectures, solve problems, and reason through ideas via exploration. In terms of content, triangle congruence is an important geometry topic in the Common Core State Standards (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010) where the ambiguous case, which illustrates that “Side-Side-Angle is not a congruence criterion” (p. 74), is explicitly mentioned. The HTE is a rich task that supports students’ understandings of triangle congruence criteria and provides students with opportunities to engage in the Standards for Mathematical Practice such as using appropriate tools, attending to precision, and constructing viable arguments and critiquing the reasoning of others. Although in our own education as students and then teachers, SSA was given little attention, we believe that rather than being set aside, SSA can be a rich opportunity for exploration. At a minimum, we found it helpful as teachers to enter discussions of triangle congruence criteria with a solid understanding of the SSA case.

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