MODERN CONSTRAINTS ON F-TERM SUSY HYBRID INFLATION MODELS

by

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To my parents for all their dedication and nurturing, to my brother Ben for his friendship, to Calleigh for her love and care, and to friends for their lifelong companionship.
Nature is satisfied with little; and if she is, I am also.

*Baruch Spinoza*
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We study modifications of supersymmetric hybrid inflation, which continues to be one of the most popular inflationary models. The seminal formulation considered the $V_{G+\Delta V}$ potential, in which one can show that $\Delta_R \sim \left(\frac{M}{m_P}\right)^2$, which indicates that the breaking scale $M \sim 10^{16}$ GeV. This is a non-trivial fact, and provides a clue the group may be a Grand Unified Theory (GUT). Inspired by this, we consider inflating while constraining the breaking scale $M$ at the MSSM gauge coupling unification scale, $2.86 \times 10^{16}$ GeV. We show that one can inflate successfully; in particular, we use non-minimal Kähler to achieve, for the recent Planck bounds $0.945 < n_s < 0.975$, $r \simeq 3 \times 10^{-4}$ in the case where $V$ is bounded from below, and $r \simeq 1 \times 10^{-2}$ where this condition is relaxed. Unfortunately, GUTs tend to predict topological defects. To ameliorate this problem, we consider the addition of a Planck-suppressed term which gives rise to shifted inflation, where one inflates in a similar way except that $|\phi| \neq 0$. We show that one can inflate successfully; one achieves similar results as in the standard case, including the large $r$ solutions particular to non-minimal Kähler contributions. We achieve $r \simeq 0.02$, which is similar to the non-minimal standard case. Finally, we consider a generalization of the model to include Planck-suppressed R-symmetry violation, parametrized by $\alpha$. One can generate masses more naturally in MSSM by treating R-symmetry as approximate, and we discover that, keeping to the standard inflationary track $|\phi| = 0$, the effect is to raise $r$ in the preferred $n_s$ range by about four orders of magnitude as compared with the standard case, for $\alpha \simeq 10^{-9}$. By considering $\alpha \simeq 10^{-7}$, one can achieve $r \simeq 10^{-4}$. This is fairly remarkable in that it is done with only minimal Kähler.
Chapter 1

INTRODUCTION

Within the past one hundred years, our knowledge of the Universe has increased many fold. In 1929, Edwin Hubble argued that his published data “indicate a linear relationship between distances and velocities [of galaxies]” [28]. In other words, galaxies are moving away from us (and from all points in space) at a rate proportional to their distances from us. Since that time, our species has developed a standard model of cosmology, which explains much of what we observe in the Universe today. Yet despite these successes, most of the energy density of the Universe is of unknown origin; approximately 26.7% consists of some unknown matter; about 68.3% consists of “dark energy”, which has the effect of increasing the rate at which the Universe is expanding; and only about 4.9% consists of known matter [2]. This is humbling, and reminds us of how much we have yet to discover.

In this dissertation, we discuss one particular type of inflationary model. Cosmic inflation seeks to explain a number of seemingly inconsistent observations about the Universe. According to the inflationary hypothesis, the Universe experienced an enormous expansion a fraction of a second after the Big Bang–more precisely, this occurred at about $10^{-32}$ s.

In this introduction, we write the necessary equations from general relativity, and proceed to discuss the essential cosmological equations and concepts. We then define and discuss inflation in Chapter 2. The type of inflationary model we discuss in this dissertation is introduced in Chapter 3. Each research topic is presented separately; in Chapter 4 we consider SUSY hybrid inflation while constraining the breaking scale $M$, in Chapter 5 we present SUSY shifted inflation, and in Chapter 6 we discuss the
topic of R-symmetry violation within SUSY hybrid inflation. For excellent reviews and books on cosmology, see [11, 42, 44, 32, 26, 62].

1.1 General Relativity

1.1.1 Tensors

We will briefly discuss tensors and tensor calculus. A square differential element of distance, \( ds^2 \), can be written in terms of the metric \( g_{\mu\nu} \):

\[
ds^2 = g_{\mu\nu}dx^\mu dx^\nu
\]

The metric is a tensor, by which it is meant that it transforms properly under a change of coordinates. A tensor with one index (for example, \( A_\mu \)) is a \textit{rank-one} tensor; with two indices (for example, \( g_{\mu\nu} \)) one has a \textit{rank-two} tensor, and so on. A rank-one tensor obeys the coordinate transformation rule

\[
A'_\nu = \frac{\partial x_\nu}{\partial x'_\mu} A_\mu.
\]

Similarly, a rank-two tensor obeys

\[
A'_{\mu\nu} = \frac{\partial x_\mu}{\partial x'_\alpha} \frac{\partial x_\nu}{\partial x'_\beta} A_{\alpha\beta}
\]

A tensor with a lower index (like \( A_\mu \)) is called a \textit{covariant} tensor. A tensor with an upper index (like \( A^\mu \)) is a \textit{contravariant} tensor. Note the convention that a lower index in a denominator sums as if it were an upper index in the numerator. We can change an upper index into a lower index by summing over the metric:

\[
A_\mu = g_{\mu\nu} A^\nu.
\]

It can be proved that \( g_{\mu\nu} \) and \( g^{\mu\nu} \) are covariant and contravariant tensors, respectively (see Appendix A). One should also keep in mind that \( d^4 x \) is not invariant; it can be shown that the volume element is

\[
d^4 x' \sqrt{g'} = d^4 x \sqrt{g}.
\]
1.1.1.1 Curvature, covariant derivatives, and the Riemann tensor

In “flat” space, a vector remains the same if it is moved from point A to point B; contrarily, a vector moved around a sphere will look different when it arrives at its original location. This difference can be written in general as

\[ \delta A^\nu = -\Gamma^\nu_{\alpha\beta} A^\alpha dx^\beta. \]

By varying the constant \( g_{\mu\nu} A^\mu A^\nu \) and setting it to 0, we can find \( \Gamma^\nu_{\alpha\beta} \) in terms of the metric and its derivatives. By interchanging the indices (which we can do twice) and subtracting the last two equations from the first, we have

\[ g_{\alpha\beta} \Gamma^\beta_{\mu\nu} = \frac{1}{2} \left( -\frac{\partial g_{\mu\nu}}{\partial x^\alpha} + \frac{\partial g_{\mu\alpha}}{\partial x^\nu} + \frac{\partial g_{\alpha\nu}}{\partial x^\mu} \right), \]

which we can write as

\[ \Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^\sigma_\alpha ( -g_{\mu\nu,\alpha} + g_{\mu\alpha,\nu} + g_{\alpha\nu,\mu} ). \]

Here I am introducing the useful notation \( g_{\mu\nu,\alpha} \equiv \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \). We have also used the fact that \( \Gamma^\nu_{\alpha\beta} \) is symmetrical in its lower indices. Since we know that moving a vector from one point to another will alter the vector, we define the covariant derivative:

\[ \nabla_\sigma A^\mu \equiv \frac{\partial A^\mu}{\partial x^\sigma} + \Gamma^\mu_{\sigma\alpha} A^\alpha \equiv A^\mu;_\sigma. \]

Finally, we define the Riemann tensor \( R^\mu_{\nu\alpha\beta} \):

\[ (\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha) A^\mu = R^\mu_{\nu\alpha\beta} A^\nu. \]

With a good amount of algebra, we can solve for \( R^\mu_{\nu\alpha\beta} \):

\[ R^\mu_{\nu\alpha\beta} = \Gamma^\mu_{\nu\beta,\alpha} - \Gamma^\mu_{\nu\alpha,\beta} + \Gamma^\mu_{\sigma\alpha} \Gamma^\sigma_{\nu\beta} - \Gamma^\mu_{\sigma\beta} \Gamma^\sigma_{\nu\alpha}. \]

We can also define the Ricci tensor \( R_{\mu\nu} = R^\alpha_{\nu\alpha\beta} \) and the Ricci scalar \( R = g^{\mu\nu} R_{\mu\nu} \).
1.1.1.2 The Einstein Field Equations

The Einstein Field Equations (EFEs) are most compactly written in terms of the Einstein tensor

\[ G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}, \]

which has the interesting property that \( G_{\mu\nu}^{\mu\nu} = 0 \). Similarly, the energy-momentum tensor obeys \( T_{\mu\nu}^{\mu\nu} = 0 \). Therefore, we can write

\[ G_{\mu\nu} = \kappa T_{\mu\nu}. \]

Finally, we write down the EFEs as

\[ G_{\mu\nu} = 8\pi G T_{\mu\nu} \equiv T_{\mu\nu}, \quad (1.1) \]

where \( c = 1 = 8\pi G = m_P^{-2} \), where \( G \) is Newton’s constant. Throughout we will use \( m_P \), the reduced Planck mass, which is \( m_P = 2.436 \times 10^{18} \text{GeV} \). Today, it is known that the Universe is accelerating; thus the EFEs today are \( R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \equiv T_{\mu\nu} \). The cosmological constant is \( \Lambda \), and this gives rise to the phenomenon of “dark energy”. However, since we are interested in the history of Universe, we use 1.1.

1.2 Standard Big Bang Cosmology

Now we begin to discuss modern cosmology—we start with the metric, then derive the essential equations (the Friedman equation, the acceleration equation, and the continuity equation), after which essential cosmological concepts and parameters are described.

1.2.1 The Cosmological Principle and the Robertson-Walker metric

We are lucky enough to live in a Universe which in fact appears to us both homogeneous and isotropic on large scales, which allows us to much more easily solve the EFEs 1.1. By homogeneous, it is meant that matter and radiation are evenly distributed over space. By isotropic, it is meant that matter and radiation look the same regardless of where one looks in the sky. It is possible to imagine an isotropic distribution of galaxies that is not homogeneous; for example, galaxies could be distributed in
concentric rings around Earth. This would seem to imply that the Earth is in a rather special place. The Copernican principle rejects this idea and instead states that the Earth does not occupy a special place in the Universe. When combined with isotropy, the Copernican principle implies both homogeneity and isotropy. The idea that the Universe on large scales is homogeneous and isotropic is known as the cosmological principle. We start with the cosmological principle, from which we will write down the metric. Applying the cosmological principle, one obtains the Robertson-Walker metric

\[ ds^2 = -dt^2 + a^2(t) \left( \frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin \theta d\phi^2) \right). \]

(1.2)

The parameter \( a(t) \) is the scale factor, and \( k \) is a dimensionless parameter which describes the spatial curvature.

1.2.2 The Friedmann, acceleration, and continuity equations

The components of the RW metric are

\[ g_{00} = -1, \quad g_{11} = \frac{a^2(t)}{1 - kr^2}, \quad g_{22} = a^2(t)r^2, \quad g_{33} = a^2(t)r^2 \sin \theta, \]

where all non-diagonal terms are zero.

The task now is to compute the \( \Gamma \)s and then the Einstein tensor \( G_{\mu\nu} \). The last step is to write down the energy-momentum tensor \( T_{\mu\nu} \), applying the relevant symmetries. One can show that \( T_{\mu\nu} \) takes the form

\[ T_{00} = \rho, \quad T_{ii} = g_{ii}P, \]

where \( \rho \) is the energy density and \( P \) is the pressure. The Ricci tensor can be shown to be

\[ R_{00} = -3 \frac{\ddot{a}}{a}, \quad R_{11} = \frac{2\dot{a}^2 + a\ddot{a} + 2K}{1 - Kr^2}, \quad R_{22} = \frac{2\dot{a}^2 + a\ddot{a} + 2K}{r^2}, \quad R_{33} = \frac{2\dot{a}^2 + a\ddot{a} + 2K}{r^2 \sin^2 \theta}, \]

from which it can be shown that the Ricci scalar is

\[ R = 6 \left( \frac{\dot{a}^2}{a^2} + \frac{\dddot{a}}{a} + \frac{K}{a^2} \right). \]
From the 00 term and any of the $ii$ terms, respectively, we derive the Friedmann and acceleration equations; finally, matter conservation ($T^\mu_{\nu\mu} = 0$) yields the continuity equation. Before we write these down, let us define the useful Hubble parameter, which is

$$H \equiv \frac{\dot{a}}{a}. \quad (1.3)$$

The Friedmann, acceleration, and continuity equations are written below. (It should be noted that only two of these equations are independent, as 1.6 can be derived from 1.4 and 1.5.)

$$H^2 = \frac{\rho}{3} - \frac{K}{a^2}, \quad (1.4)$$
$$\frac{\dot{a}}{a} = \frac{1}{6} (\rho + 3P), \quad (1.5)$$
$$0 = \dot{\rho} + 3H(\rho + P). \quad (1.6)$$

### 1.2.3 Essential cosmological concepts

#### 1.2.3.1 Equation of state and density parameters

We can now begin to solve the equations 1.4-1.6. First we introduce the useful concept of the equation of state, $w$:

$$w \equiv \frac{P}{\rho}, \quad (1.7)$$

where $P$ is the pressure and $\rho$ is the energy density. Further, it is useful to define the energy density parameter $\Omega$:

$$\Omega \equiv \frac{\rho}{\rho_c}, \quad (1.8)$$

where the critical density $\rho_c$ is $\rho$ such that $K = 0$:

$$\rho_c \equiv 3H^2. \quad (1.9)$$

Thus we can easily parametrize the geometry: when $\Omega = 1$, $K = 0$ and we have “flat” space. The continuity equation 1.6 can now be written as

$$\frac{\dot{\rho}}{\rho} = -3\frac{\dot{a}}{a}(1 + w). \quad (1.10)$$
\[ \begin{array}{|c|c|c|c|}
\hline
w & H(a) & \rho & a(t) \\
\hline
w = 0 \text{ (matter)} & a^{-3/2} & a^{-3} & t^{2/3} \\
\hline
w = \frac{1}{3} \text{ (radiation)} & a^{-2} & a^{-4} & t^{1/2} \\
\hline
\end{array} \]

**Table 1.1:** A useful table specifying \( H, \rho, \) and \( a. \)

If \( w \neq -1, \) one can solve this to obtain

\[ \rho \propto a^{-3(1+w)} \Rightarrow H \propto a^{-\frac{3}{2}(1+w)}. \]

Further, if we begin with \( H \propto a^{-\frac{3}{2}(1+w)} \) and solve for \( t, \) we derive

\[ t \propto a^{\frac{3(1+w)}{4}} \Rightarrow a(t) \propto t^{\frac{2}{3(1+w)}}, \quad (1.11) \]

More concretely, non-relativistic matter (called “matter”) is given by \( w = 0, \) whereas relativistic matter (“radiation”) is given by \( w = 1/3. \) Substituting in these values, we write Table 1.1 for reference.

### 1.2.3.2 The particle horizon and comoving distances

The concept of distance in cosmology is somewhat intricate. Consider the RW metric 1.2

\[ ds^2 = -dt^2 + a^2(t)d\chi^2, \quad (1.12) \]

where we have written the spatial terms as \( d\chi^2. \) The second term, \( a^2(t)d\chi^2, \) can increase or decrease over time, and represents physical distances in the Universe. However, \( d\chi^2 \) is constant in time, even in an expanding Universe; hence, it cannot be a physical distance. Instead, we regard \( d\chi \) as the *comoving distance*. Its relationship to physical distance can be derived by considering the distance a photon traverses in a given time. One has \( ds^2 = 0 \Rightarrow dt = a(t)d\chi \Rightarrow d\chi = \frac{dt}{a(t)}. \) By integrating, we find the comoving distance traversed by a photon in a time \( t - t_0. \) We refer to this as the *comoving particle horizon*, given by

\[ \tau = \int_{t_0}^{t} \frac{dt}{a(t)}. \quad (1.13) \]
Similarly, the physical distance a photon travels in a time $t - t_0$ is the particle horizon. The particle horizon is a fundamental concept, for it defines the farthest observable distance for an observer $O$. In other words, if a radio signal is emitted at time $t_0$, it will be observable to observer $O$ at $t$. The particle horizon is equal to the comoving horizon, $\tau$, times the scale factor:

$$d_p(t) = a(t) \int_{t_0}^{t} \frac{dt'}{a(t')}.$$  \hfill (1.14)

It is useful to rewrite the comoving particle horizon $\tau$ via

$$\frac{dt}{a(t)} = \frac{da}{a H a^2} = \frac{d \ln a}{a H},$$

and thus $\tau$ is

$$\tau = \int \frac{1}{a H} d \ln a.$$

The parameter $\frac{1}{a H}$ is referred to as the comoving hubble radius:

$$\tau_{hr} = \frac{1}{a H}.$$  \hfill (1.15)

### 1.2.3.3 A few words about the thermal history of the Universe

We first note that as the Universe ages, it expands and its temperature scales as $a^{-1}$. The Planck scale, which we take to be $\rho^{1/4} \sim 10^{19}$ GeV, is the scale at which quantum corrections to general relativity should render the current theory invalid. During the period where $10^{19} \gtrsim \rho^{1/4} \gtrsim 10$ eV, the Universe was a plasma of relativistic particles. Current understanding indicates that a GUT phase transition occurred around $10^{16}$ GeV. The electroweak (spontaneous symmetry breaking) phase transition took place at around $300 GeV$. Nucleosynthesis, the production of atomic nuclei, occurs around 0.1 MeV. The decoupling of photons, which gave rise to the Cosmic Microwave Background Radiation (CMBR), occurred at 0.1 eV. This corresponds to about 380,000 years after the big bang. The temperature of the background today is $\approx 2.7$ K, which corresponds to about $2 \times 10^{-4}$ eV.
1.3 Problems in Standard Big Bang Cosmology

Despite its successes, the SBB model nonetheless inadequately explains several observed features of the Universe; these are: homogeneity and isotropy, flatness, and a lack of topological defects. These problems are sufficiently addressed by successful inflationary models.

1.3.1 The horizon problem

The comoving Hubble radius $\tau_{hr}$ is integral to understanding the early Universe. For instance, if $\tau_{hr}$ has been increasing since the beginning of the Universe, then at early times the comoving particle horizon must have been quite small. This, in turn, constrains our ability to explain the homogeneity and isotropy of the early Universe. In particular, the (Figure 1.1) Cosmic Microwave Background Radiation (CMBR), which was emitted about 380,000 years after the Big Bang, we know is homogeneous and isotropic to one part in $10^{-5}$. Yet, at this time, the comoving particle horizon may have been far smaller than the observable Universe—in other words, the CMBR on one side of the Universe could have not communicated with the CMBR on the other side.

What we will now show is that this is in fact the case.

Starting from 1.11, and substituting this into $\frac{1}{aH}$ (calling $n \equiv \frac{2}{3(1+w)}$), the comoving particle horizon is

$$\tau_{hr} \propto \frac{t^{1-n}}{n}.$$ 

The slope, then, is

$$\frac{d}{dt}\tau_{hr} \propto \frac{1}{n}(1 - n)t^{-n},$$

which is positive only if $n < 1$, corresponding to $w > -\frac{1}{3}$. Thus, during matter and radiation domination the particle horizon becomes smaller as one traces the evolution of the Universe backwards in time. Solving for $\tau$ from the beginning of the Universe, one gets

$$\tau = \int_0^t \frac{dt}{a(t)} \propto \int_0^t t^{\frac{2}{3(1+w)}} dt \propto a^{\frac{1}{2(1+3w)}},$$

9
using the fact that $t \propto a^{\frac{2(1+w)}{3}}$ (start from $H \propto a^{-\frac{3}{2}(1+w)}$, and solve for $t$). Let us quantify the problem. The particle horizon is, at $t = t_1$:

$$d_{p,t_1}(t_1) = a(t_1) \int_0^{t_1} \frac{dt}{a(t)}.$$

We will use the notation that today $t = t_0$; hence, today this horizon has been stretched to

$$d_{p,t_0}(t_1) = a(t_0) \int_0^{t_1} \frac{dt}{a(t)},$$

whereas the present horizon is

$$d_{p,t_0}(t_0) = a(t_0) \int_0^{t_0} \frac{dt}{a(t)} \equiv \int_0^{t_0} \frac{dt}{a(t)}.$$

We have used the convention that $a(t_0) = 1$. For matter and radiation domination $a \propto t^n$ (where $0 < n < 1$), we obtain

$$d_{p,t_0}(t_0) = \int_0^{t_0} t^{-n} dt \propto \frac{t_0^{-n+1}}{1-n} \propto \frac{1}{a(t_0)} \propto \frac{1}{a(t_0)H(t_0)}.$$

Similarly,

$$d_{p,t_0}(t_1) \propto \frac{1}{a(t_1)H(t_1)}.$$
and we can define the fraction \( f \equiv \frac{d_{p,t_0}(t_0)}{d_{p,t_0}(t_1)} \), which is the ratio of the size of the current horizon to the current size of the horizon at some earlier time \( t_1 \). It is

\[
f = \frac{a(t_1)H(t_1)}{a(t_0)H(t_0)}.
\]

If we take \( t_1 \) to be the time of recombination, the Universe is matter dominated and \( a \propto t^{2/3}, H \propto t^{-1}, \) and \( aH \propto a^{-1/2} \). Finally, we invoke the redshift \( 1 + z = \frac{a(t_0)}{a(t_1)} \), and substitute in our expression for \( aH \):

\[
f \propto \frac{a(t_1)^{-1/2}}{a(t_0)^{-1/2}} \propto (z + 1)^{1/2}.
\]

At recombination, \( z \simeq 1100 \Rightarrow f \simeq 1000^{1/2} \approx 35 \); this means that a causally connected patch of the CMBR corresponds to an angle of \( 1/35 \approx 2^\circ \). This is rather problematic, because the CMBR is isotropic to about one part in \( 10^{-5} \)!

### 1.3.2 The Flatness Problem

The second problem is how one explains the observation that the Universe appears very “flat”—i.e., \( \Omega \simeq 1 \). We can quantify this by rewriting the Friedmann equation 1.4 as

\[
1 = \frac{\rho}{3H^2} - \frac{K}{a^2H^2} = \frac{\rho}{\rho_c} - \frac{K}{a^2H^2},
\]

using 1.9. Then, using 1.8,

\[
|1 - \Omega| \propto \frac{1}{(aH)^2},
\]

which must increase over time, since the \( \tau_{hr} \) increases. Thus,

\[
\frac{d}{dt} |1 - \Omega| > 0
\]

for \( w > -\frac{1}{3} \). If \( |1 - \Omega| \simeq 0 \) today, it must have been exceptionally small in the early Universe; in fact, at the GUT scale, \( |1 - \Omega| \simeq \mathcal{O}(10^{-55}) \).

### 1.3.3 Topological Defects and Galactic Structure

Further, one has the problem of excessive topological defects in many Grand Unified Theories (GUTs). In particular, magnetic monopoles can be produced, which
is highly problematic. This can be remedied by inflation if such defects are produced beforehand. This is discussed further in Chapter 5. The existence of local anisotropies in the Universe can also be explained via inflation.
Chapter 2

INFLATION

From Section 1.3, we can see how to ameliorate the problems in SBB cosmology. Both the horizon and flatness problems arise from the fact that the comoving hubble radius \( \tau_{hr} \) increases over time; thus, one can simply solve both problems by hypothesizing that there was a period of time in the early Universe when \( \tau_{hr} \) decreased. Subsequently, \( \tau_{hr} \) increased as it would under SBB cosmology, matching its value today. At the time of decoupling, therefore, various patches of the surface of last scattering were in fact in causal contact from the beginning of the Universe. For excellent reviews and discussions on inflation, see [11, 42, 44, 32, 26]

2.1 Definition

From the above discussion we can see how to define inflation; we require the comoving hubble radius to decrease:

\[
\frac{d}{dt} \left( \frac{1}{aH} \right) < 0. \tag{2.1}
\]

An equivalent definition is

\[
\ddot{a} > 0. \tag{2.2}
\]

From the acceleration equation 1.5, we can see that a third equivalent definition of inflation is

\[
-\frac{1}{6} (\rho + 3P) = \frac{\ddot{a}}{a} > 0 \Rightarrow \frac{P}{\rho} \equiv w < -\frac{1}{3}. \tag{2.3}
\]

There are therefore three equivalent conditions under which inflation takes place. These are grouped below.
\[ \frac{d}{dt} \left( \frac{1}{aH} \right) < 0 \]
\[ \ddot{a} > 0 \]
\[ w < -\frac{1}{3} \]

2.2 Dynamics of inflation

2.2.1 Equation of motion

The simplest and easiest way to create inflation is to hypothesize the existence of some scalar field $\phi$, which is constant in space but not in time. The lagrangian density therefore is

\[ \mathcal{L} = -\frac{1}{2} g_{\mu \nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) = \frac{L}{\sqrt{-g}}. \]

To find the equation of motion, we can use the Euler-Lagrange equation

\[ 0 = \frac{\partial L}{\partial \phi} - \frac{\partial}{\partial \alpha} \left( \frac{\partial L}{\partial (\partial_\alpha \phi)} \right), \]

which yields

\[ 0 = \dddot{\phi} + 3H \dot{\phi} + \frac{\partial V}{\partial \phi}. \]

(2.4)

2.2.2 Slow-roll inflation

The slow-roll approximation is frequently used in constructing inflationary models. It allows one to more simply solve 2.4. By rewriting the acceleration equation 1.5, we can understand how to define slow-roll. We start by defining the parameter

\[ \tilde{\epsilon} \equiv -\frac{\dot{H}}{H^2}, \]

from

\[ \frac{\ddot{a}}{a} = -\frac{1}{6} (\rho + 3P) = \dot{H} + H^2 \]

\[ = H^2 \left( \frac{\dot{H}}{H^2} + 1 \right) \]

\[ \equiv H^2 (1 - \tilde{\epsilon}). \]
Note that if $\bar{\epsilon} < 1$, then $\ddot{a} > 0$. Further, we will show inflation proceeds when the potential energy $V$ dominates over the kinetic energy $\frac{1}{2} \dot{\phi}^2$. First, let us write the energy-momentum tensor in terms of the action for a scalar field; by varying the action $S_\phi = \int dx^4 \sqrt{-g} \mathcal{L}$, one can compute the EFEs 1.1. By comparing to 1.1, one can see that the energy momentum tensor is

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_\phi}{\delta g^{\mu\nu}},$$

which, with some work, gives

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi + g_{\mu\nu} \mathcal{L}.$$

Knowing that $T_{00} = \rho$ and $T_{ii} = g_{ii} P$, we compute the energy density and pressure:

$$T_{00} = \rho = \frac{1}{2} \dot{\phi}^2 + V(\phi) \quad (2.5)$$

$$P = \frac{1}{2} \dot{\phi}^2 - V(\phi). \quad (2.6)$$

We should note that 2.4 is analogous to mechanical roll down a potential $V$ with a friction term $3H \dot{\phi}$. We have already seen that inflation occurs when $w < -\frac{1}{3}$, but for the field to roll slowly we require that $|\dot{\phi}| \ll \frac{\dot{\phi}}{3H}$; by taking $w = -1$, we find

$$-1 = w = \frac{P}{\rho} = \frac{\frac{1}{2} \dot{\phi}^2 - V(\phi)}{\frac{1}{2} \dot{\phi}^2 + V(\phi)} \Rightarrow \frac{1}{2} \dot{\phi}^2 \ll V(\phi). \quad (2.7)$$

We can rewrite the Friedman equation 1.4 as $H^2 = \dot{\phi}^2 \simeq \frac{V}{3}$; in other words, the energy density is dominated by $V(\phi)$ relative to the kinetic energy. The equation of motion 2.4 is

$$0 = 3H \dot{\phi} + \frac{\partial V}{\partial \phi} \Rightarrow \dot{\phi}^2 = \frac{1}{3} \frac{V^2}{\phi} \Rightarrow \frac{\dot{\phi}^2}{V} = \frac{1}{6} \left( \frac{V_{,\phi}}{V} \right)^2 \ll 1,$$

where $V_{,\phi} \equiv \frac{\partial V}{\partial \phi}$. We now define the slow-roll parameter $\epsilon \equiv \frac{M_p^2}{2} \left( \frac{V_{,\phi}}{V} \right)^2$. The second slow-roll parameter we derive by differentiating $-\dot{\phi} = \sqrt{\frac{1}{3} \frac{V_{,\phi}}{V}}$, which gives

$$-\frac{\ddot{\phi}}{H \dot{\phi}} \equiv \eta = \frac{\eta}{M_p^2} = \frac{\epsilon}{M_p^2},$$

15
where $\eta \equiv M_p^2 \frac{V_{,\phi \phi}}{V}$ is the second slow-roll parameter. Since $\epsilon$ is much smaller than 1, we therefore need $|\eta| \ll 1$. Note that we have added $M_p$ as needed to ensure that both slow-roll parameters are dimensionless. In summary, inflation will proceed when both slow-roll parameters $\ll 1$. These are sufficient but not necessary conditions for inflation, because inflation can occur when these conditions are violated. Inflation therefore occurs when both $\epsilon \ll 1$ and $|\eta| \ll 1$, where the slow-roll parameters during inflation are

$$\epsilon = \frac{m_p^2}{2} \left( \frac{V_{,\phi}}{V} \right)^2 \quad (2.8)$$

$$\eta = m_p^2 \frac{V_{,\phi \phi}}{V} \quad (2.9)$$

One can derive higher-order slow-roll parameters, and we would like to include here for completeness one higher-order term:

$$\xi^2 = \frac{m_p^4}{2} \frac{V'V'''}{V^2}. \quad (2.10)$$

One could ask what the form of the scale factor takes during inflation. From 1.10, we can see that inflation implies $H \simeq constant$; hence, $\dot{a} = aH \Rightarrow \ln a = Ht + C \Rightarrow a \propto e^{HT}$, which is termed de-Sitter. We can write a table analogous to 1.1; during inflation, $H$, $\rho$, and $a(t)$ is given in Table 2.1.

<table>
<thead>
<tr>
<th>$w$</th>
<th>$H(a)$</th>
<th>$\rho$</th>
<th>$a(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w = -1$ (inflation)</td>
<td>constant</td>
<td>constant</td>
<td>$e^{HT}$</td>
</tr>
</tbody>
</table>

**Table 2.1:** $H$, $\rho$, and $a$ during inflation.

Finally, we must quantify the amount of inflation. We will parametrize this as the number of efoldings

$$N \equiv \ln \frac{a_{end}}{a_{beg}},$$
which we can write as

\[ N = \int_{t_{\text{beg}}}^{t_{\text{end}}} \frac{da}{a} = \int_{t_{\text{beg}}}^{t_{\text{end}}} H dt = \int_{t_{\text{beg}}}^{t_{\text{end}}} \frac{H d\phi}{d\phi/dt}. \]

We use \( 3H \dot{\phi} \simeq -\frac{\partial V}{\partial \phi} \) and \( V \simeq 3H^2 \) to obtain an approximate but nonetheless useful formula:

\[ N \simeq \frac{1}{m_p^2} \int_{\phi_{\text{beg}}}^{\phi_{\text{end}}} \frac{V d\phi}{V_{,\phi}} = \frac{1}{m_p} \int_{\phi_{\text{beg}}}^{\phi_{\text{end}}} \frac{d\phi}{\sqrt{2\epsilon}}. \]  (2.11)

We will primarily use this expression for the number of e-foldings. One generally requires about 50 – 60 e-foldings for successful inflation.

### 2.3 \( \Delta_s^2, r \) and \( n_s \)

In this section we discuss the three primary observational parameters to which we fit our subsequent models. First, we write down the primordial scalar amplitude \( \Delta_R \). For a very short mathematical introduction to the derivation of \( \Delta_R^2 \), see Appendix A.1.2. Starting from A.1.2, we write the primordial scalar amplitude in dimensionless form:

\[ \Delta_R^2 = \frac{H^2}{8\pi^2\epsilon}. \]

Substituting equalities for \( H^2 \) and \( \epsilon \), we will obtain the following form:

\[ \Delta_R^2 = \frac{1}{12\pi^2 m_p^6 V_{,\phi}^2}. \]  (2.12)

The spectrum \( \Delta_R^2 \) is known by observation to deviate from scale invariance. We quantify this via the scalar spectral index \( n_s \) (which we will call the spectral index):

\[ n_s - 1 = \frac{d \ln \Delta_R^2}{d \ln k}, \]

which we opportunistically write as \( \frac{d \ln \Delta_R^2}{d N} = \frac{d N}{d \ln k} \). Further, using \( \frac{d}{dN} = m_p^2 V_{,\phi} \frac{d}{d\phi} \),

\[
\begin{align*}
\frac{d \ln \Delta_R^2}{d N} &= \frac{d}{dN} \left( \ln \frac{H^2}{8\pi^2\epsilon} \right) \\
&= m_p^2 V_{,\phi} \frac{d}{d\phi} \left( \ln \frac{V^3}{12\pi^2 m_p^6 V_{,\phi}} \right) \\
&= 3m_p^2 V_{,\phi}^2 V^2 - 2m_p^2 V_{,\phi} V \\
&= 6\epsilon - 2\eta.
\end{align*}
\]
Starting from $k = aH$, we write $\ln k = \ln a + \ln H = N + \ln H$; further,

$$\frac{dN}{d\ln k} = \left(\frac{d\ln k}{dN}\right)^{-1} = \left(1 + \frac{d\ln H}{dN}\right)^{-1} = (1 - \epsilon)^{-1} \approx 1 + \epsilon.$$ 

Finally, we write the spectral index, to first approximation, as

$$n_s = 1 + 6\epsilon - 2\eta. \quad (2.13)$$

The “spectral tilt” refers to $1 - n_s$. Further, a “red-tilted” spectrum indicates $n_s < 1$; conversely, a “blue-tilted” spectrum indicates $n_s > 1$.

The scalar spectral index can “run”, meaning that it is not scale independent. This is parametrized by the spectral running $\alpha_s$, given by

$$\alpha_s = 16\epsilon\eta - 24\epsilon^2 - 2\xi^2 \quad (2.14)$$

Inflationary models must keep $|\alpha_s| \lesssim 0.01$, and Planck indicates that negative values are favored.

Next, we can derive the tensor-to-scalar ratio $r$, which is defined to be $r \equiv \frac{\Delta_t^2}{\Delta_s^2}$, where the subscript $t$ denotes the tensor perturbations. One can show that $\Delta_t^2 = \frac{2H^2}{\pi^2 m_p^2}$, and therefore, to first approximation,

$$r = 16\epsilon. \quad (2.15)$$

The (scalar) spectral index 2.13 and the tensor-to-scalar ratio 2.15 are the primary observational parameters to which we will fit our data in the proceeding chapters.

2.4 Observation

In this section we briefly summarize the relevant results from the WMAP 7-year and 9-year data, and Planck mission’s recent data release. This dissertation discusses results published at different times; Chapter 4 contains results published after the release of the most recent results (Planck 2013); Chapter 5 contains results published only after the WMAP 7-year data release; and Chapter 6 contains results published
only after the WMAP 9-year data release. For convenience, the various results are summarized below in Table 2.2, where we consider the observational parameters most pertinent to our results.

**Note:** The recent measurement by BICEP2 [1] has the potential to alter the inflationary landscape greatly. During March 2014, the BICEP2 team released data which indicated that \( r \approx 0.2 \). There has been, however, substantial debate regarding the veracity of the results. It has been argued [23, 50] that the BICEP2 signal may not originate wholly from inflation. We do not discuss these results further in this dissertation, as all the research presented here was completed before the BICEP2 data release.

<table>
<thead>
<tr>
<th>Observational Parameter</th>
<th>WMAP 7-year Range</th>
<th>WMAP 9-year Range</th>
<th>Planck Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n_s )</td>
<td>[0.92, 1.016]</td>
<td>[0.92, 1.02]</td>
<td>[0.945, 0.975]</td>
</tr>
<tr>
<td></td>
<td>( \approx 0.968 \pm 4\sigma )</td>
<td>( \approx 0.971 \pm 4\sigma )</td>
<td>( \approx 0.9603 \pm 4\sigma )</td>
</tr>
<tr>
<td>( \Delta^2_R )</td>
<td>[2.21, 2.65]</td>
<td>[2.271, 2.583]</td>
<td>[2.072, 2.393]</td>
</tr>
<tr>
<td></td>
<td>( = 2.43 \times 10^{-9} \pm 2\sigma )</td>
<td>( = 2.427 \times 10^{-9} \pm 2\sigma )</td>
<td>( = 2.215 \times 10^{-9} \pm 2\sigma )</td>
</tr>
<tr>
<td>( r )</td>
<td>( &lt; 0.13 )</td>
<td>( &lt; 0.13 )</td>
<td>( &lt; 0.11 )</td>
</tr>
</tbody>
</table>

**Table 2.2:** Here we summarize the essential results \( (r, \Delta^2_R, n_s) \) of the WMAP seven-year [33] and nine-year [27] releases, and the Planck 2013 [55] release. Each of these results is quoted variously in this dissertation.

### 2.5 A few important approximations

Here, we quickly discuss two important approximations. The Lyth bound reveals a bound in the change in the inflaton field during the course of inflation. Additionally, we can relate \( r \) to the energy scale of inflation.
2.5.1 The Lyth bound

We can relate the change in the inflaton field to $r$ by beginning with 2.11 and taking the approximation that $\epsilon$ is approximately constant during inflation:

\[
N_e = \frac{1}{m_P} \int_{\phi_{\text{beg}}}^{\phi_{\text{end}}} \frac{d\phi}{\sqrt{2\epsilon}} \approx \frac{1}{\sqrt{2\epsilon}} \frac{\Delta \phi}{m_P}
\]

\[
\Rightarrow \frac{\Delta \phi}{m_P} \approx \sqrt{\frac{r}{8N_e}}.
\]

Taking $N_e \approx 50$, this yields

\[
\frac{\Delta \phi}{m_P} \approx O(1) \times \sqrt{\frac{r}{0.01}},
\]

which is known as the “Lyth bound”. Generally speaking, we see that models of inflation where $r \gtrsim 0.01$ require trans-Planckian inflaton values.

2.5.2 $r$ and the energy scale of inflation

By starting with 2.15 and applying 2.12, we can write the energy scale of inflation $V^{1/4}$ in terms of $r$; this is

\[
V^{1/4} \approx (7r \Delta^2 R m_P^4)^{1/4}
\]

\[
\approx 3r^{1/4} \times 10^{16}
\]

in units of GeV. Thus, $r$ values of order $O(1) - O(-1)$ indicate that the energy scale of inflation is around GUT scale.

2.6 Models of inflation

There is a wide variety of inflationary models. Here, we present a brief discussion of the most basic models, and then focus more closely on $\phi^2$ inflation.

- **Large field inflation** In these models, the inflaton begins far away from its minimum and slowly approaches its minimum. One simple example is $\phi^2$ inflation, discussed in more detail below. The inflaton is generally around or larger than (trans-Planckian) the Planck scale near the start of observable inflation. The tensor-to-scalar ratio $r$ is generally $r \gtrsim 0.1$. (See 2.16.)

- **Small field inflation, also called “new inflation”** In these models, the inflaton is relatively more near its minimum, as in the example $V = V_0 - g\phi^n$. These models produce very small $r$, at least if one avoids fine-tuning.
• Hybrid inflation This is essentially a hybrid between a $\phi^2$ model and the higgs potential $(\phi^2 - \mu)^2$. The potential is (in the simple formulation) written as a function of the inflaton $\phi$ and an additional “waterfall” field $\chi$. The premise is that $\phi$ rolls down a local minimum ($\frac{\partial V}{\partial \chi} = 0$ and $\frac{\partial^2 V}{\partial \chi^2} > 0$), falling down a valley at a critical point, and ending up at $\chi = 0$.

2.6.1 A simple example: $\phi^2$ inflation

Let us take the simplest case $V = \lambda \phi^2$, where $\lambda$ has mass dimension 2. Before inflation, the Universe has a large energy $V^{1/4}$, and thus the inflaton $\phi$ has a large value. The inflaton then falls toward its minimum at $\phi = 0$, causing the observable Universe to inflate. The slow-roll parameters 2.8 are

$$\epsilon = \eta = 2 \left( \frac{m_p}{\phi} \right)^2 \quad (2.19)$$

We can compute $n_s$ via 2.13:

$$n_s = 1 - 8 \left( \frac{m_p}{\phi} \right)^2.$$ 

We now know that $n_s$ is close to 0.96; imposing this constraint, we see that

$$\frac{\phi}{m_p} = 10\sqrt{2}.$$ 

We can also use the fact that $\epsilon = \eta$ to write $n_s = 1 - \frac{r}{4}$. Therefore, $r = 0.16$ for $n_s = 0.96$. We can easily find the number of e-foldings

$$N_e = \frac{\phi^2 - \phi^2_e}{4m_p^2}.$$ 

Inflation will end when $\epsilon = 1 = \eta$, which yields $\phi = \sqrt{2} m_p$; computing the number of e-foldings, we have

$$N_e = \frac{198}{4} \approx 50.$$ 

Lastly, we should compute $\lambda$ by application of Eq 2.12. We can write

$$48\pi^2 \Delta^2_R = \frac{\lambda}{m_p^2} \left( \frac{\phi}{m_p} \right)^4 \Rightarrow \frac{\lambda}{m_p^2} \approx 2.6 \times 10^{-11}.$$ 

Thus we can compute the energy scale of inflation in this case directly:

$$V^{1/4} = \lambda^{1/4} \phi^{1/2} \approx 2 \times 10^{16}.$$
in GeV. It is interesting to use the approximation \(2.17\), which yields, for \(r = 0.16\), 
\[ V^{1/4} \approx 1.9 \times 10^{16} \text{ GeV}. \]

2.7 Supersymmetry

In the subsequent chapters, we work within the context of supersymmetry (SUSY). A thorough exposition of the subject is far beyond the scope of this dissertation, but it will suffice to present the motivations and essential premise of SUSY. For excellent reviews on SUSY, see \([68, 45, 48]\).

2.7.1 What is the appeal of Supersymmetry?

Although thus far evidence for SUSY is scant, there are a number of good motivations for it. When one looks at the running of the electromagnetic, weak, and strong coupling constants in the standard model, one finds that they almost converge at a high energy scale. SUSY greatly improves this unification, which occurs around the GUT scale, or around \(10^{16} \) GeV. Thus, GUTs are naturally discussed within the context of SUSY. Second, SUSY provides a solution to the hierarchy problem in the standard model. That is, the radiative corrections to the mass of the Higgs are so large that tremendous fine-tuning is required to keep the Higgs vev at 246 GeV. SUSY solves this problem via cancellations of the divergences, which occurs without fine tuning since the masses of the sparticles (supersymmetric particles) are degenerate with those of the particles. Third, SUSY provides dark matter candidates—i.e., stable particles of sufficient mass—the lightest supersymmetric particle (LSP).

2.7.2 Essentials of global SUSY

SUSY extends spacetime \((x^\mu)\) to superspace \((X = (x^\mu, \theta^\alpha, \bar{\theta}^{\dot{\alpha}}))\). In doing so, one promotes fields to superfields, and thus SUSY Lagrangians are invariant under SUSY transformations. There are three types of fields of interests: left-handed chiral superfields, right-handed chiral superfields, and a vector superfield.

A crucial step is to realize that \(\mathcal{L}\) is invariant under SUSY transformations if one writes \(\mathcal{L} = \mathcal{L}_F + \mathcal{L}_D\), where \(\mathcal{L}_F\) represents one term from the left-handed chiral
superfield and its corresponding term from the right-handed chiral superfield, and $\mathcal{L}_D$ represents a term from the vector superfield.

It turns out that the F-term is $\mathcal{L}_F = \sum_i |\partial_{z_i} W|^2$, where $W$ is the superpotential. The D-term in the abelian case is $\mathcal{L}_D = |D|^2$, where $D = g \sum_i q_i |\phi_i|^2$. The $q_i$ are the charges under $U(1)_R$. In this dissertation we take the D-terms to be zero, which can be done by the arbitrary constraint $|\phi| = 0 = |\bar{\phi}|$. For our purposes, we need to remember that the F-term SUSY potential is written as

$$V_F = \sum_i \left| \frac{\partial W}{\partial z_i} \right|^2.$$  \hfill (2.20)

We should be cognizant of the fact that when $V = 0$, SUSY is broken.

One can also impose a useful symmetry on SUSY called R-symmetry. This is a continuous $U(1)$ symmetry, often referred to as $U(1)_R$. In SUSY it becomes quite difficult to ensure baryon and lepton number conservation, but one can solve this problem by imposing $U(1)_R$ and breaking it to a discrete $Z_2$ symmetry dubbed R-parity. For an excellent discussion and review, see [6].

### 2.7.3 Supergravity

When one makes SUSY a local symmetry, one is lead to supergravity (SUGRA). SUGRA effects become important especially near the Planck scale. The $F$-term SUGRA scalar potential is a function of both the superpotential $W$ and the Kähler potential $K$ of a given theory, and may be written down using the formula

$$V_F = e^{K/m_P^2} \left( K_{ij}^{-1} D_{z_i} W D_{z_j} W^* - 3m_P^{-2} |W|^2 \right),$$ \hfill (2.21)

where $z_i \in \{ s, \phi, \bar{\phi}, \cdots \}$, and we have used the shorthand

$$K_{ij} = \frac{\partial^2 K}{\partial z_i \partial z_j^*},$$

$$D_{z_i} W = \frac{\partial W}{\partial z_i} + m_P^{-2} \frac{\partial K}{\partial z_i} W,$$

$$D_{z_i} W^* = (D_{z_i} W)^*.$$
The Kähler potential is a SUSY generalization of kinetic terms; it arises from nonrenormalizeable terms and thus doesn’t appear in global SUSY (since gravity is nonrenormalizeable the Kähler potential should be included). One can see that in the limit $m_P \to \infty$, the SUGRA F-term potential reduces to $K_{ij}^{-1} D_{z_i} W D_{z_j} W^*$. Minimal Kähler refers to $\sum_i |\Phi_i|^2$, whereas a more general so-called non-minimal Kähler includes higher-order Planck-suppressed terms (see 3.2.1). This will be discussed in more detail in Section 3.2.1.
3.1 Introduction

Modern cosmology has seen rapid developments due to experiments such as COBE and WMAP. Augmenting their unprecedented successes will be Planck, which may for the first time yield direct evidence of inflation. At the same time, there are enormous strides being made in particle physics, with the LHC having made perhaps the greatest single discovery in the field in decades, while the testing of supersymmetry (SUSY) is highly anticipated. For the first time particle and cosmological models can be tested with precision, and the deep connections between these two fields motivate us to consider the effects that particle physics considerations have on inflationary models. This has led to models such as, among others, SUSY hybrid inflation.

The standard version of SUSY hybrid inflation [21, 17, 49, 46, 37, 51, 4] remains one of the most successful and well-motivated inflationary models. It is the most general non-trivial model one can write with a gauge singlet field $S$ and supermultiplets $\Phi$ and $\bar{\Phi}$ that respects $U(1)_R$, such that the latter two fields belong to non-trivial representations of the gauge group $G$. It has a connection to particle physics in that within it grand unified theories (GUTs) are naturally incorporated. In this model the gauge group $G$ is broken to a subgroup $H$ at the end of inflation, and the energy scale at which this occurs is related to local temperature anisotropies in the cosmic microwave background radiation [21]; this scale happens to be close to the GUT scale, indicating that $G$ may be related to a GUT. In addition, supergravity (SUGRA) corrections remain under control [43] because this model can yield solutions within the WMAP nine-year $2\sigma$ bounds without trans-Planckian inflaton field values.
3.2 Superpotential

We desire the most general superpotential that satisfies the following properties:

- is gauge invariant
- satisfies R-symmetry
- contains a scalar field which can help drive inflation
- is renormalizable (contains only terms that have mass dimension [3])

We will call the scalar field $S$. The most general non-trivial renormalizable superpotential one can write involving a singlet superfield $S$ and two conjugate supermultiplets $\Phi$ and $\overline{\Phi}$ that preserves a gauge group $G$ and $U(1)_R$ R-symmetry is [21, 17]

$$W = \kappa S(\overline{\Phi} \Phi - M^2), \quad (3.1)$$

where $M$ is the energy scale at which $G$ breaks and $\kappa$ is a dimensionless coupling which we take to be positive without loss of generality since we can absorb the phase of $\kappa$ into that of $S$. The global SUSY F-terms are given by

$$V_F \equiv \sum_i \left| \frac{\partial W}{\partial z_i} \right|^2. \quad (3.2)$$

Here, $z_i \in \{s, \phi, \overline{\phi}\}$, where $s$, $\phi$, and $\overline{\phi}$ are the scalar components of the superfields $S$, $\Phi$, and $\overline{\Phi}$, respectively. We choose to set the D-terms to zero by imposing $|\overline{\Phi}| = |\Phi|$, for convenience.

Using Equation (3.2), the tree-level global SUSY potential in the D-flat direction is

$$V_F = \kappa^2 (M^2 - |\phi|^2)^2 + 2\kappa^2 |s|^2 |\phi|^2. \quad (3.3)$$

A plot of this potential in field space is shown in Figure 3.1. Inflation proceeds along the local minimum $|\phi| = 0$ (the inflationary track), beginning at large $|s|$ (top of Figure 3.1). An instability occurs at the waterfall point $|\tilde{s}_{c}|^2 = M^2$, which is the value of $|s|$ such that $0 = \left. \frac{\partial^2 V}{\partial |\phi|^2} \right|_{|\phi|=0}$ (the subscript “c” denotes “critical”, and the symbol $|\tilde{s}_{c}|$ will be used later to denote the dimensionful inflaton field at the critical point; we maintain...
Figure 3.1: The tree-level, global scalar potential $V$ in standard hybrid inflation. The variables $s$ and $\phi$ are the scalar components of the superfields $S$ and $\Phi$. The same notation here for consistency.) At this point the field falls naturally into one of two global minima at $|\phi|^2 = M^2$. This coincides with the breaking of the gauge group $G$. At large $|s|$, the scalar potential is approximately quadratic in $|\phi|$, whereas at $|s| = 0$ Equation (3.3) becomes a Higgs potential.

Along the inflationary track the potential is flat ($V = \kappa^2 M^4$), and thus one cannot end inflation. One-loop radiative corrections (RC), which should be added for consistency in any case–since SUSY is broken during inflation–can alleviate this problem. SUSY is restored after inflation, when the field evolves to one of its global minima (where $V = 0$). The radiative corrections [16] can be calculated from the equation

$$\Delta V = \frac{1}{64\pi^2} \sum_{i,j} (-1)^{2j} (1 + 2j) \mathcal{M}^4_i \left\{ \ln \left( \frac{\mathcal{M}^4_i}{Q^2} \right) - 3/2 \right\},$$

(3.4)

where for scalars we have $j = 0$, and for fermions we have $j = 1/2$. See Appendix A.2.1 for the calculation. Finally, we derive the radiative corrections $\Delta V = \frac{N\kappa^4 m^4}{8\pi^2} F(x)$,
where

\[ F(x) = \frac{1}{4} \left( (x^4 + 1) \ln \left( \frac{x^4 - 1}{x^4} \right) + 2x^2 \ln \frac{x^2 + 1}{x^2 - 1} + 2 \ln \left( \frac{\kappa^2 m^2 x^2}{Q^2} \right) - 3 \right). \] (3.5)

Here, \( x = |s|/|\tilde{s}_c| \) is a convenient reparametrization of the inflaton field, \( N \) is the dimensionality of the representation of the fields \( \Phi \) and \( \overline{\Phi} \), and \( Q \) is the renormalization scale. The total potential, considering only the global and radiative correction terms, is

\[ V_{G+\Delta V} = \kappa^2 (M^2 - |\phi|^2) + 2\kappa^2 |s|^2 |\phi|^2 + \frac{N\kappa^4 m^4}{8\pi^2} F(x). \] (3.6)

The seminal formulation \([21]\) considered the \( V_{G+\Delta V} \) potential. In this scenario, one can show that \( \Delta R \sim \left( \frac{M}{m_P} \right)^2 \), which indicates that the breaking scale \( M \sim 10^{16} \) GeV. This is a non-trivial fact, and provides a clue that \( G \) may be a GUT. This approximation should hold to at least an order of magnitude with the addition of terms considered in this dissertation.

### 3.2.1 Non-minimal Kähler and SUGRA corrections

We will have occasion to use both minimal (canonical) and non-minimal forms of the Kähler potential. In general, \( K \) should be expanded in inverse powers of the cutoff scale \( M_* \); we refer to this as “non-minimal Kähler”, and it is

\[
K = |S|^2 + |\Phi|^2 + |\overline{\Phi}|^2 + \frac{\kappa_S}{4} \frac{|S|^4}{M_*^4} + \frac{\kappa_{\Phi}}{4} \frac{|\Phi|^4}{M_*^4} + \frac{\kappa_{\overline{\Phi}}}{4} \frac{|\overline{\Phi}|^4}{M_*^4} \\
+ \kappa_{SS} \frac{|S|^2 |\Phi|^2}{M_*^2} + \kappa_{SS} \frac{|S|^2 |\overline{\Phi}|^2}{M_*^2} + \kappa_{\Phi \overline{\Phi}} \frac{|\Phi|^2 |\overline{\Phi}|^2}{M_*^2} + \frac{\kappa_{SS}}{6} \frac{|S|^6}{M_*^4} + \cdots. \] (3.7)

By “minimal Kähler”, we mean \( K = |S|^2 + |\Phi|^2 + |\overline{\Phi}|^2 \). From 2.21, we can see that \( K_{ij} = \frac{\partial^2 K}{\partial z_i \partial z_j} = 1 \) in this case. Further, in this work we will consider the first-order SUGRA correction term, which is \( V_{SUGRA} = \kappa^2 M^4 \frac{|S|^4}{m_P^4} \). This is computed from 2.21.

### 3.2.2 Soft terms

Since \( V \neq 0 \) during inflation, SUSY is broken during inflation. Therefore, in principle there should be “soft” breaking terms in the potential 3.3. However, one
could ask whether the Universe can be inflated properly without such terms. In the next section, we will consider this question.

We will consider gravity-mediated breaking only, where a hidden sector which breaks SUSY communicates gravitationally with the observable sector. Via \([53]\), the potential is

\[
V = |W_a|^2 + m_{3/2}^2 |y_a|^2 + m_{3/2} \{ y_a W_a + (A - 3)W + h.c. \},
\]

where \(y_a\) is an observable sector field, \(W_a \equiv \frac{\partial W}{\partial y_a}\), and \(m_{3/2}\) is the gravitino mass. The first term yields the global terms, the second is the soft mass squared term, and the third is proportional to each \(S\)-term in \(W\). The \(y_a\) are \(S, \Phi, \bar{\Phi}\). We will take the superpotential as

\[
W = \kappa S(\Phi \Phi - M^2) + \frac{\alpha}{m_p} S^4,
\]

in reference to \([61]\); to obtain the potential for \([3.3]\), we will set \(\alpha = 0\). The third term \(m_{3/2} \{ y_a W_a + (A - 3)W + h.c. \}\) is

\[
V = m_{3/2} \left\{ - (A - 2) M^2 \kappa s + (A + 1) \frac{\alpha}{m_p} S^4 + h.c. \right\}.
\]

Taking \(\alpha\) and \(s\) as complex, we can write \(\alpha = |\alpha|e^{i\theta_\alpha}\) and \(s = |s|e^{i\theta_s}\); adding the hermitian conjugate terms, one has

\[
V_{soft} = am_{3/2} M^2 \kappa |s| + m_{3/2}^2 |s|^2 + \frac{2bm_{3/2} |\alpha||s|^4}{m_p},
\]

where \(a = 2(2 \cos \theta_s - |A| \cos (\theta_A + \theta_s))\) and \(b = |A| \cos (\theta_A + \theta_\alpha + 4\theta_s) + \cos (\theta_\alpha + 4\theta_s)\).

### 3.3 Summary of results

#### 3.3.1 \(V_{G+\Delta V+soft+SUGRA}\) Results

By starting with \([3.1]\), and including the radiative corrections, soft terms, and the first-order SUGRA term, one obtains

\[
V_{G+\Delta V+soft+SUGRA} = \kappa^2 (M^2 - |\phi|^2)^2 + 2\kappa^2 |s|^2 |\phi|^2 + \frac{N\kappa^4 m^4}{8\pi^2} F(x) \tag{3.8}
\]

\[
+ \kappa^2 M^4 \frac{|S|^4}{m_p^4} + am_{3/2} M^2 \kappa |s|, \tag{3.9}
\]

where \(\alpha = 0\). In Figure 3.2 we present the results of \([61]\), in which one takes \(V_{G+\Delta V+soft+SUGRA}\) and \(m_{3/2} = 10^3\) GeV. (The effect of the mass squared term is
Figure 3.2: The spectral index $n_s$ versus $\kappa$ and the tensor-to-scalar ratio $r$ versus $n_s$, from 3.2.

relatively suppressed compared to the linear term.) When only (minimal) SUGRA and radiative corrections are added to the global potential, a numerical lower bound on the scalar spectral index, $n_s \approx 0.985$, develops. This corresponds to the $V_{G+\Delta V+\text{soft+SUGRA}}$ potential with $a = 0$, and is depicted in both 6.1(a) and 6.1(b). The 2σ bounds on $n_s$, $n_s = 0.971 \pm 0.010$ [27], indicates that this lower bound is somewhat disfavored \footnote{In [63] it is also shown that $a = -1$ reduces $n_s$ significantly.}. The linear term, incidentally, is also important in explaining the MSSM $\mu$-problem [20, 31, 56]. However, the aforementioned model produces very small $r$. Specifically, we know from [61] that one can obtain $r \lesssim 10^{-5}$ in the red-tilted region in this case, although $r$ is seven orders of magnitude smaller than the upper limit at the central $n_s$ value. By contrast, the WMAP nine-year upper limit is $r < 0.13$. Planck will be able to detect $r$ of the order $\sim 10^{-2}$ [55]; if gravity waves exist at this order of magnitude, Planck will greatly reduce the number of viable inflationary models, or at the least, will rule out a considerable amount of parameter space in models in which large gravity waves can be produced.

Much of the work that has been done in these models pertains to increasing $r$, thereby allowing modern experiments such as Planck to potentially rule out such
models. In subsequent work [58], it was shown that non-minimal Kähler can increase $r$ by three orders of magnitude. This is shown in Figure 3.3.

![Figure 3.3](image)

Figure 3.3: The tensor-to-scalar ratio $r$ versus the spectral index $n_s$, from [58]. Here, non-minimal Kähler is considered, which raises $r$ by three orders of magnitude. However, in the most favored region $n_s \approx 0.96$, $r$ is raised by about 10 orders of magnitude.

### 3.4 Room for improvement

In this dissertation, we examine ways in which SUSY hybrid inflation can be improved. First, in 4 we ask if a theoretically interesting constraint on $M$ precludes one from achieving inflation. We consider a solution to one of the main drawbacks of the model, topological defects, which is discussed in 5. Lastly, we examine a well-motivated generalization of the model in 6, in which the R-symmetry is somewhat relaxed. A description of and motivation for each case is below.
3.4.1 Shifted inflation

One potential drawback of this model is that, since symmetry breaking occurs at the end of inflation, topological defects, such as magnetic monopoles and domain walls, can be produced in quantities sufficient to contradict experiment. One way of ameliorating the problem of topological defects is to work with a gauge group that does not produce topological defects during breaking, such as ‘flipped SU(5)’ group (SU(5) × U(1)_X). Cosmological results do not differ tremendously [59]. If one adopts, for example, flipped SU(5) (N = 10) as opposed to U(1) (N = 1). However, it is interesting to ask whether one can construct a model in which one has more flexibility with regard to the gauge group. For instance, if the breaking occurs before inflation the topological defects will be inflated away [29].

This can be done by inflating along a shifted track such that the symmetry breaking precedes observable inflation [15, 29, 21]. However, one can solve this problem in the standard case with an appropriate gauge group choice in which no topological defects are produced, such as G = SU(5) × U(1)_X, referred to as flipped SU(5) [18, 25, 7, 19, 3, 66, 52]. (The extra U(1)_X contains a part of U(1)_em, ensuring that there are no stable monopoles. See [19] for details.) We will see that a sufficiently red-tilted spectrum can be produced, as well as considerable gravity waves (r). This is discussed in 5, which has largely been taken from the author’s published paper [15].

3.4.2 R-symmetry violation

Another way in which these models can be improved is via considering more carefully particle physics considerations. For instance, when doing inflation in the context of flipped SU(5) [36, 60], the R-charges are determined in part by the inflationary sector. By imposing a U(1)_R symmetry (“R-symmetry”), as is standard practice, one ensures that proton decay occurs only via the six-dimensional operator (the $5_h \overline{5}_h$ term is disallowed). The importance of this is that we preclude rapid proton decay. However, one also prohibits terms such as the quartic couplings $10, 10, \overline{10}_H \overline{10}_H$, which give rise to right-handed neutrino masses. (Note that the decay of Majorana
right-handed neutrinos can explain the observed baryon asymmetry via leptogenesis [24]. (For non-thermal leptogenesis in inflation, see [38].) If we reassign $R$-charges such that $10^i 10_j \overline{10}_H \overline{10}_H$ is allowed, we end up prohibiting the Yukawa term which gives rise to down-type quark masses ($y_{ij}^{(d)} 10^i 10_j 5_h$). This outcome is not necessarily catastrophic, since the relevant quark masses may be generated radiatively. On the other hand, one may invoke a “double seesaw” mechanism [35] to account for the lack of large right-handed neutrino masses. A simpler solution to this problem is to allow higher-order (Planck scale suppressed) $R$-symmetry violating terms in the superpotential, while enforcing $R$-symmetry for renormalizable terms. With this motivation in mind, we wish to explore the effects of these additional terms on the inflationary dynamics. Introducing these terms raises the question of whether such a modified model can support successful inflation in the context of global SUSY alone: we find that this is not the case, leading us to incorporate SUGRA when constructing our model.

The material in Chapter 6 is taken largely from the author’s published paper [14]. There, we consider the inflationary implications of such a scenario, within the context of standard SUSY hybrid inflation, which we briefly review in the Section 2. In Section 6.1 we discuss our model including the leading nonrenormalizable terms, and in Section 6.2 we detail our results. There, we compare our results with WMAP nine-year data, and discuss our model in light of the Planck mission.

### 3.4.3 Constraining $M$

The scenario described in this chapter, while perfectly consistent with the current observations, requires some modification if one desires to incorporate values of $M$ that are comparable or equal to $M_{\text{gut}}$. This is indispensable in cases where $G_{\text{gut}}$ includes non abelian factors besides $G_{\text{sm}}$, which are expected to disturb the successful gauge coupling unification within MSSM. Thus, in this dissertation we attempt to inflate successfully while constraining $M$ at GUT scale. We note that the observationally favored values (close to 0.96) for $n_s$ with $M$ equal to the SUSY GUT scale can be readily achieved within SUSY hybrid inflation by invoking a specific type of
non-minimal Kähler, first proposed in [58]. In particular, a convenient choice of the next-to-minimal and the next-to-next-to-minimal term of the adopted Kähler generates [58, 4] a positive mass (quadratic) term for the inflaton and a sizeable negative quartic term which allows one to build a model of the hilltop type [12] in most of the allowed parameter space. Our objectives can also be achieved in smaller regions of the allowed parameter space even with monotonic inflationary potential and therefore complications related to the initial conditions of the model can be safely eluded. Acceptable $n_s$ values within this scenario are accompanied with an enhancement of the running of $n_s$, $\alpha_s$, and the scalar-to-tensor ratio, $r$; these can be said to be maximum within the model if we take into account that $M$’s larger than $M_{\text{gut}}$ are arguably less plausible. Note, in passing, that the reduction of $n_s$ by generating a negative mass (quadratic) term for the inflaton, as done in [8], is not suitable for our purposes since $M$ remains well below $M_{\text{gut}}$.

This scenario is discussed in Chapter 4. The material in Chapter 4 is taken largely from the author’s published paper [13]. There, we briefly recall in Section 4.2 the observational and theoretical constraints imposed on our model. In Section 4.3 we exhibit our updated results, and our conclusions are summarized in 4.4.
4.1 Introduction

Here we consider whether one can successfully inflate via Eq 3.1 while constraining the breaking scale $M$ at the value such that the MSSM gauge couplings unify at the GUT scale (see Section 3.4.3). In order to ensure the boundedness of the potential from below, we consider non-minimal Kähler Eq 3.7 up to five orders in the inflaton field—this potential we refer to as case I. However, we nonetheless present results of the simpler case where one has only two non-minimal Kähler terms, which we refer to as case II. We will use the symbol $V_{HI0}$ to refer to $\kappa^2 M^4$.

We end up with the following inflationary potential, from

$$V_{HI} \simeq V_{HI0} \left( 1 + c_{HI} + \sum_{\nu=1}^{5} (-1)^\nu c_{2\nu K} \left( \frac{\sigma}{\sqrt{2m_p}} \right)^{2\nu} \right) \quad (4.1)$$

where $\sigma = \sqrt{2}|S|$ is the canonically normalized inflaton field. The contribution of RCs (see 3.5) read

$$c_{HI} = \frac{\kappa^2 N}{8\pi^2} F(x) \quad (4.2)$$

We consider $N = 2$ and $N = 10$ in this paper.

The remaining coefficients, $c_{2\nu K}$, in 4.1 can be expressed as functions of the $\kappa$’s in 4.5 [5]. From them only the first two play a crucial role during the inflationary dynamics; they are
\[ c_{2K} = \kappa_{4s}, \quad (4.3) \]
\[ c_{4K} = \frac{1}{2} - \frac{7\kappa_{4s}^2}{4} + \kappa_{4s}^3 - \frac{3\kappa_{6s}}{2} \quad (4.4) \]

The residual higher order terms in the expansion of 4.1 prevent a possible runaway behavior of the resulting \( V_{HI} \) – see point 8 of 4.2. The terms necessary to ensure boundedness are

\[ c_{6K} = -\frac{2}{3} + \frac{3\kappa_{4s}}{2} - \frac{7\kappa_{4s}^2}{4} + \kappa_{4s}^3 + \frac{10\kappa_{6s}}{3} \]
\[ -3\kappa_{4s}\kappa_{6s} + 2\kappa_{8s}, \]
\[ c_{8K} = \frac{3}{8} - \frac{5\kappa_{10s}}{2} - \frac{13\kappa_{4s}}{24} + \frac{41\kappa_{10s}^2}{32} \]
\[ -\frac{7\kappa_{4s}^3}{4} + \kappa_{4s}^4 - \frac{13\kappa_{6s}}{4} + \frac{143\kappa_{4s}\kappa_{6s}}{24} \]
\[ -\frac{9\kappa_{4s}\kappa_{6s}}{2} + \frac{9\kappa_{6s}^2}{4} - \frac{39\kappa_{8s}}{8} + \frac{4\kappa_{4s}\kappa_{8s}}{5}, \quad (4.5) \]
\[ c_{10K} = -\frac{2}{15} + \frac{32\kappa_{10s}}{5} + 3\kappa_{12s} + \frac{\kappa_{4s}}{24} - 5\kappa_{10s}\kappa_{4s} \]
\[ -\frac{13\kappa_{4s}^2}{24} + \frac{41\kappa_{6s}^2}{32} - \frac{7\kappa_{4s}^4}{4} + \kappa_{4s}^5 \]
\[ +\frac{5\kappa_{6s}}{3} - \frac{29\kappa_{4s}\kappa_{6s}}{6} + \frac{103\kappa_{4s}^2\kappa_{6s}}{12} \]
\[ -6\kappa_{4s}\kappa_{6s} - 5\kappa_{6s}^2 + \frac{27\kappa_{4s}\kappa_{6s}^2}{4} + 5\kappa_{8s} \]
\[ -\frac{67\kappa_{4s}\kappa_{8s}}{8} + 6\kappa_{4s}^2\kappa_{8s} - 6\kappa_{6s}\kappa_{8s}. \]

Let us note, lastly, that the soft SUSY breaking terms (3.2.2) of the order of \((1-10)\) TeV do not have an appreciable effect due to large value of \(M\) that we employ.

### 4.2 Constraining the Model Parameters

Our inflationary model can be qualified by imposing a number of observational (1-3) and theoretical (4-8) requirements, specified below. Primarily, we mandate that the essential inflationary constraints be satisfied. The observational constraints are applications of those discussed in Chapter 2, which we would like to reiterate. The theoretical constraints, which we enumerate, are specific to this model.
1. The number of e-foldings must be at least enough to resolve the horizon and flatness problems of standard Big Bang cosmology.

2. The scalar amplitude $\Delta^2_R$ (see 2.12) must be consistent with the data [2], i.e.

$$\Delta^2_R = \frac{1}{12\pi^2 m_p^3} \frac{V_{HI}(\sigma_*)}{V_{HI,\sigma}(\sigma_*)^2} \simeq 2.2 \times 10^{-9}. \quad (4.6)$$

3. The scalar spectral index $n_s$, its running, $a_s = \frac{dn_s}{d\ln k}$, and the scalar-to-tensor ratio, $r$, given by

$$n_s = 1 - 6\epsilon_\ast + 2\eta_\ast, \quad (4.7)$$

$$\alpha_s = 16\epsilon_\ast\eta_\ast - 24\epsilon_\ast^2 - 2\xi_\ast^2, \quad (4.8)$$

where $\xi^2$ is given in 2.10 and all the variables with the subscript $\ast$ are evaluated at $\sigma = \sigma_\ast$; the observational constraints from Planck are:

$$n_s = 0.9603 \pm 0.014 \Rightarrow 0.945 \lesssim n_s \lesssim 0.975, \quad (4.9)$$

$$\alpha_s = -0.0134 \pm 0.018 \quad \mathrm{and}$$

$$r < 0.11, \quad (4.10)$$

at 95% confidence level (c.l.).

We further impose the following upper bound:

$$|\alpha_s| \ll 0.01, \quad (4.12)$$

since, within the cosmological models with running $\alpha_s$, $|\alpha_s|$’s of order 0.01 are encountered [2].

We further constrain the potential as per the following considerations:

4. The $G_{\text{gut}}$ breaking scale has to be determined by the unification of the MSSM gauge coupling constants, i.e.,

$$gM \simeq 2 \cdot 10^{16}\text{GeV}, \quad (4.13)$$

with $g \simeq 0.7$ being the value of the unified gauge coupling constant. Here $gM$ is the mass at the SUSY vacuum.
5. It is reasonable to ask $V_{HI}$ to be bounded from below as $\sigma \to \infty$. Given our ignorance, however, for the pre-inflationary (i.e. for $\sigma > \sigma_*$) cosmological evolution we do not impose this requirement as an absolute constraint.

6. Depending on the values of the coefficients in 4.1, $V_{HI}$ is a either monotonic function of $\sigma$ or develops a local minimum and maximum. The latter case may jeopardize the implementation of SUSY hybrid inflation if $\sigma$ gets trapped near the minimum of $V_{HI}$. It is, therefore, crucial to indicate the regions where $V_{HI}$ is a monotonically increasing function of $\sigma$.

7. Inflation proceeds such that $\sigma$ rolls from $\sigma_{\text{max}}$, which is the point where the maximum of $V_{HI}$ lies, down to smaller values. Therefore a mild tuning of the initial conditions is required [8] in order to obtain acceptable $n_s$ values, since for lower $n_s$ values we must set $\sigma_*$ closer to $\sigma_{\text{max}}$. We quantify the amount of tuning in the initial conditions via the quantity [8]:

$$\Delta m_* = (\sigma_{\text{max}} - \sigma_*) / \sigma_{\text{max}}. \quad (4.14)$$

Large $\Delta m_*$ values correspond to a more natural scenario.

4.3 Results

Our inflationary model depends on the following parameters:

$$\kappa, \kappa_{4s}, \kappa_{6s}, \kappa_{8s}, \kappa_{10s}, \kappa_{12s}, N, \sigma_* \quad (4.15)$$

with $M$ fixed from 4.13. In our computation, we use as input parameters $\kappa_{8s}, \kappa_{10s}$ and $\kappa_{12s}$. We restrict $\kappa$ and $\sigma_*$ such that Eqs. 2.11 and (4.6) are fulfilled. The restrictions on $n_s$ from 4.9 can be met by adjusting $\kappa_{4s}$ and $\kappa_{6s}$, whereas the last three parameters of $K$ control mainly the boundedness and the monotonicity of $V_{HI}$; we thus take them into account only if we impose restriction 6 of 4.2. In these cases we set $\kappa_{10s} = -1$ and $\kappa_{12s} = 0.5$ throughout and we verify that these values do not play a crucial role in the inflationary dynamics. We briefly comment on the impact of the variation of $\kappa_{8s}$ and $N$ on our results. Using Eq. (4.8) we can extract $\alpha_s$ and $r$.

Following the strategy of [58] we choose the sign of $c_{2s} = \kappa_{4s}$ to be negative – cf. [8]. As a consequence, fulfilling of 4.7 requires a negative $c_{2s} k$ or positive $\kappa_{6s}$. The structure of $V_{HI}$ is displayed in 4.2 where we show the variation of $V_{HI}$ as a function of $\sigma$ for $\kappa = 0.018$ and $\kappa_{4s} = -0.0443, \kappa_{6s} = 0.736, \kappa_{8s} = -1.5$ (gray line).
Figure 4.1: Allowed (lightly gray shaded) region, as determined by the restrictions 1-6 of 4.2, in the $\kappa - (-\kappa_{4s})$ (a), $\kappa - \kappa_{6s}$ (b), $\kappa - |\alpha_s|$ (c) and $\kappa - r$ (d) plane for $N = 10$, $\kappa_{8s} = -1.5$, $\kappa_{10s} = -1$ and $\kappa_{12s} = 0.5$. In the hatched regions $V_{HI}$ remains monotonic. The conventions adopted for the various lines are also shown. The thin lines (case II) are obtained by setting $c_{6K} = c_{8K} = c_{10K} = 0$ in 4.1.
Figure 4.2: Variation of $V_{HI}$ as a function of $\sigma$ for $n_s = 0.96$, setting $N = 10$, $\kappa = 0.018$, $\kappa_{10s} = -1$, $\kappa_{12s} = 0.5$ and $\kappa_{4s} = -0.0443, \kappa_{6s} = 0.736, \kappa_{8s} = -1.5$ $[\kappa_{4s} = -0.0415, \kappa_{6s} = 0.656, \kappa_{8s} = -0.5 ]$ (gray [light gray] line). The values of $\sigma^*, \sigma_0, \sigma_{\text{max}}$ and $\sigma_{\text{min}}$ are also depicted.

or $\kappa_{4s} = -0.0415, \kappa_{6s} = 0.656, \kappa_{8s} = -0.5$ (light gray line). These parameters yield $n_s = 0.96$, $r \simeq 0.00019$ and $\alpha_s \simeq 0.0054$ [$\alpha_s \simeq 0.0037$] (gray [light gray] line). The values of $\sigma_0/\kappa \simeq 19.03$ [$\sigma_0/\kappa \simeq 18.4$] (gray [light gray] line) and $\alpha_s/\kappa \simeq 1.42$ are also depicted. In the first case (gray line) $V_{HI}$ remains monotonic due to the larger $|\kappa_{8s}|$ value employed. Contrarily, $V_{HI}$ develops the minimum-maximum structure, in the second case (light gray line) with the maximum located at $\sigma_{\text{max}}/\kappa = 26.6 \{27.2\}$ and the minimum at $\sigma_{\text{min}}/\kappa = 53.8 \{63.5\}$ – the values obtained via B.3 and B.4 are indicated in curly brackets. We find that $\Delta_m \simeq 0.31$.

Applying the constraints of Sec. 4.2, we can identify the allowed regions in the $\kappa - (-\kappa_{4s})$, $\kappa - \kappa_{6s}$, $\kappa - |\alpha_s|$ and $\kappa - r$ planes – see Figure 4.1. The conventions adopted for the various lines are also shown. In particular, the thick and thin gray dashed [dot-dashed] lines correspond to $n_s = 0.975$ [$n_s = 0.946$], whereas the thick and thin gray solid lines are obtained by fixing $n_s = 0.96$ – see Eq. (4.9). The thick lines are obtained setting $\kappa_{8s} = -1.5$ which – together with the universally selected $\kappa_{10s}$ and $\kappa_{12s}$ above – ensures the fulfilment of restriction 5 of 4.2; the faint lines correspond to the choice $c_{6K} = c_{8K} = c_{10K} = 0$, which does not ensure the boundedness of $V_{HI}$. From the panels (a), (b) and (c) we see that the thin lines almost coincide with the thick
ones for $\kappa \leq 0.01$, and then deviate and smoothly approach some plateau. The regions allowed by imposing the constraints 1-5 of 4.2 are denoted by light gray shading. In the hatched subregions, requirement 6 is also met. On the other hand, the regions surrounded by the thin lines are actually the allowed ones, when only the restrictions 1-4 of 4.2 are satisfied. The various allowed regions are cut at low $\kappa$ values since the required $\kappa_{6s}$ reaches rather high values (of order 10), which starts looking unnatural. At the other end, 4.9 and $\sigma_\ast \simeq m_P$ bounds the allowed areas in the case of bounded or unbounded $V_{HI}$ respectively. For both cases we remark that $|\kappa_{4s}|$ increases with $\kappa$, whereas $\kappa_{6s}$ drops with $\kappa$, which is clear from Fig. 2. For fixed $\kappa$, increasing $|\kappa_{4s}|$ means decreasing $\kappa_{6s}$. Moreover, $|\kappa_{4s}|$ is restricted to somewhat small values in order to avoid the well-known [39, 46] $\eta$ problem. On the other hand, no tuning for $\kappa_{6s}$ is needed since it is of order unity for most $\kappa$ values.

From 4.1 we observe that for increasing $\kappa$ beyond 0.01, $|\alpha_\ast|$ corresponding to the bold lines precipitously drops at $\kappa \simeq 0.02$, changes sign and rapidly saturates the bound of 4.12 along the thick black solid line. In other words, for every $\kappa$ in the vicinity of $\kappa \simeq 0.2$ we have two acceptable $\kappa_{6s}$ values, as shown in 4.1 with two different $\alpha_\ast$ values of either sign. Furthermore, from 4.1 we remark that $r$ is largely independent of the $n_s$ value, and so the various types of lines coincide for both bounded and unbounded $V_{HI}$. We also see that $r$ increases almost linearly with $\kappa$ and reaches its maximal value which turns out to be: (i) $r \simeq 2.9 \cdot 10^{-5}$ as $\alpha_\ast$ approaches the bound of 4.12, for bounded $V$; (ii) $r \simeq 0.01$ as the inequality $\sigma_\ast \leq m_P$ is saturated for $n_s \simeq 0.975$ and unbounded $V_{HI}$. Therefore, lifting restriction 5 of 4.2 allows larger $r$. However, non vanishing $c_{\nu K}$’s perhaps corresponds to a more natural scenario.

We observe that the optimistic restriction 5 in 4.2 can be met in very limited slices of the allowed (lightly gray shaded) areas, only when the boundedness of $V_{HI}$ has been ensured. In these regions $\sigma_\ast$ also turns out to be rather large ($10M$), and we therefore observe a mild dependence of our results on $c_{6K}$ (or $\kappa_{8s}$). This point is further clarified in 4.1 where we list the model parameters and predictions for $n_s \simeq 0.96$, $N = 10, \kappa = 0.005, 0.01, 0.02$ and various $\kappa_{8s}$ values. We remark that for $\kappa = 0.005$
the results are practically unchanged for varying $\kappa_{8s}$. The dependence on $\kappa_{8s}$ starts to become relevant for $\kappa \approx 0.01$ and crucially affects the results for $\kappa = 0.02$; here, for $\kappa_{8s} = -2$ the solution obtained belongs to the branch with $\alpha_s < 0$ and not in the branch with $\alpha_s > 0$, as is the case with $\kappa = 0.005$ and 0.01. Listed is also the quantity $\Delta_{m*}$ which takes rather natural values for the selected $\kappa$ – the entries without a value assigned indicate that $V_{HI}$ is a monotonic function of $\sigma$.

As shown in 4.1, $|\kappa_{4s}|$ ranges between about 0.015 and 0.05 for the case with bounded $V_{HI}$ or 0.042 for unbounded $V_{HI}$. For each of these $\kappa_{4s}$ values and every $\kappa$ in the allowed range found in 4.1, we vary $\kappa_{6s}$ in order to obtain $n_s$ in the observationally favored region of 4.9 and we extract the resulting $r$. Our results are presented in 4.3, where we display the allowed region in the $n_s - r$ plane for bounded (upper plot) or unbounded (lower plot) $V_{HI}$. Along the dashed lines of both plots $\kappa_{6s}$ ranges between 9 and 26 whereas along the solid line of the upper [lower] plot $\kappa_{6s}$ varies between 0.69 and 0.75 [0.39 and 1.15]. From the upper plot we see that the maximal for $r$ is about $2.9 \cdot 10^{-4}$ and turns out to be nearly independent of $n_s$. Interestingly, this value is included in the region with monotonic $V_{HI}$ depicted by the hatched region. From the lower plot we see that there is a mild dependence of the largest $r$ from $n_s$; thus, the maximal $r = 0.006$ is achieved for $n_s = 0.975$. No region with monotonic $V_{HI}$ is located

<table>
<thead>
<tr>
<th>$-\kappa_{8s}$</th>
<th>$\kappa$ (10^{-2})</th>
<th>$\sigma_s/M$</th>
<th>$\kappa_{4s}$ (10^{-2})</th>
<th>$\kappa_{6s}$</th>
<th>$\Delta_{m*}$ (%)</th>
<th>$\alpha_s$ (10^{-3})</th>
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Table 4.1: Model parameters and predictions for $N = 10$ and $n_s \simeq 0.96$. We take $\kappa_{10s} = -1, \kappa_{12s} = 0.5$ and various $\kappa_{8s}$'s.
Figure 4.3: Variation of $r$ as a function of $n_s$ for $N = 10$. We set $\kappa_{10s} = -1$, $\kappa_{12s} = 0.5$, $\kappa_{4s} = -0.05$ (solid line) and $\kappa_{4s} = -0.015$ (dashed line) for Figure 4.3 (case I). For Figure 4.3 (case II) we set $c_{6K} = c_{8K} = c_{10K} = 0$ and $\kappa_{4s} = -0.042$ (solid line) or $\kappa_{4s} = -0.015$ (dashed line). The shaded region between the two curves is approximately the allowed region. In the hatched region $V_{HI}$ remains monotonic.

in this case, however.

4.3.1 Bounds on the parameters

Summarizing our findings from Figs. 4.1 and 4.3 for $n_s$ in the range given by 4.9 and imposing the restrictions 1-6 of 4.2, the various quantities are bounded as follows:

\[
\{4.9 \cdot 10^{-2}\} \quad 1.5 \lesssim \frac{k}{10^{-2}} \lesssim 2.3, \quad (4.16)
\]
\[
\{1.4\} \quad 4 \lesssim \frac{-k_{4s}}{10^{-2}} \lesssim 7.95, \quad (4.17)
\]
\[
0.68 \lesssim \kappa_{6s} \lesssim 0.77 \quad \{10\}, \quad (4.18)
\]
\[
\{5.7 \cdot 10^{-2}\} \quad 0.4 \lesssim \frac{|\alpha_s|}{10^{-2}} \lesssim 1, \quad (4.19)
\]
\[
\{1.7 \cdot 10^{-3}\} \quad 1.3 \lesssim \frac{r}{10^{-4}} \lesssim 2.9. \quad (4.20)
\]

Note that the limiting values obtained without imposing the monotonicity of $V_{HI}$ – requirement 5 in 4.2 – are indicated in curly brackets. In the corresponding
region, $\Delta m_s$ ranges between 16 and 32%. As can be deduced from the data of 4.1, $\Delta m_s$ increases with $\kappa$'s. Small $\Delta m_s$ values indicate a second mild tuning (besides the one needed to avoid the $\eta$ problem), which is however a common feature in the models of hilltop inflation. If we ignore requirement 5 of 4.2, then confining $n_s$ in the range of 4.9 we obtain the following ranges:

\begin{align}
4.9 \cdot 10^{-3} & \lesssim \frac{\kappa}{10^{-1}} \lesssim 1, \\
1.4 & \lesssim \frac{-\kappa_{4s}}{10^{-2}} \lesssim 4.7, \\
0.4 & \lesssim \kappa_{6s} \lesssim 10, \\
5.7 \cdot 10^{-1} & \lesssim \frac{|\alpha_s|}{10^{-3}} \lesssim 6, \\
1.4 \cdot 10^{-5} & \lesssim \frac{r}{10^{-2}} \lesssim 1.
\end{align}

(4.21) \ (4.22) \ (4.23) \ (4.24) \ (4.25)

Obviously, no solutions with monotonic $V_{HI}$ are achieved in this case whereas $\Delta m_s$ varies between 16 and 29%. The maximal $r$ is reached for the maximal $n_s$ in 4.9 and as $\sigma_s \sim m_P$.

**Table 4.2:** Model parameters and predictions for $N = 2$ and $n_s \simeq 0.96$. We set $\kappa_{8s} = -1.5$, $\kappa_{10s} = -1$ and $\kappa_{12s} = 0.5$.

<table>
<thead>
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<th>$\kappa$</th>
<th>$\kappa_{8s}$</th>
<th>$\kappa_{10s}$</th>
<th>$\kappa_{12s}$</th>
<th>$\Delta m_s$</th>
<th>$\alpha_s$</th>
<th>$r$</th>
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<td>$(10^{-2})$</td>
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<td>5.7</td>
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<tr>
<td>2</td>
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<td>5.1</td>
<td>0.816</td>
<td>35</td>
<td>6.3</td>
<td>23</td>
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</table>

So far we focused on flipped $SU(5)$, employing $N = 10$ in our investigation. However, our results are not drastically affected even in the case of $G_{10}$ for most values of $\kappa$, as can be inferred by comparing the results (for $\kappa_{8s} = -1.5$) listed in Tables 4.1 and 4.2 where we use $N = 10$ and $N = 2$ respectively. This signals the fact that the SUGRA corrections to $V_{HI}$ originating from the last term in the sum of 4.1 dominate over the radiative corrections which are represented by $c_{HI}$. The discrepancy between
the two results ranges from 6 to 20%, increasing with $\kappa$, and it is essentially invisible in the plots of 4.1. On the other hand, we observe that in the $N = 10$ case the enhanced $c_{HI}$ creates a relatively wider space with monotonic $V_{HI}$; this space is certainly smaller for $N = 2$, as shown from our outputs for $\kappa = 0.02$.

### 4.4 Summary

Inspired by the recently released results by the PLANCK collaboration on the inflationary observables, we have reviewed and updated the nonminimal version of SUSY hybrid inflation arising from F-terms. In our formulation, inflation is based on a unique renormalizable superpotential, employs a quasi-canonical Kähler and is followed by the spontaneous breaking at $M_{\text{gut}}$ of a GUT symmetry which is taken to be $G_{lr}$ or $G_{5x}$. As suggested first in [58], $n_s$ values close to 0.96 in conjunction with the fulfilment of 4.13 can be accommodated by considering an expansion of the Kähler – see 4.5 – up to twelfth order in powers of the various fields with suitable choice of signs for the coefficients $\kappa_{4s}$ and $\kappa_{6s}$.

Fixing $n_s$ at its central value, we obtain $\{7.8 \cdot 10^{-2}\} 1.57 \lesssim \kappa/10^{-2} \lesssim 2.2$ with $\{2\} 4.2 \lesssim -\kappa_{4s}/10^{-2} \lesssim 7.2$ and $0.72 \lesssim \kappa_{6s} \lesssim 0.79 \{10\}$, while $|\alpha_s|$ and $r$ assume the values ($\{0.1\} 0.45 - 1 \cdot 10^{-2}$ and ($\{3.5 \cdot 10^{-3}\} 1.4 - 1.9 \cdot 10^{-4}$ respectively – recall that the limiting values in the curly brackets are achieved without imposing the monotonicity of $V_{HI}$. With a non-monotonic $V_{HI}$, $\Delta_{ms}$ ranges between 16 and 30%. It is gratifying that there is a sizable portion of the allowed parameter space where $V_{HI}$ remains a monotonically increasing function of $\sigma$; thus, unnatural restrictions on the initial conditions for inflation due to the appearance of a maximum and a minimum of $V_{HI}$ can be avoided. On the other hand, if we do not insist on the boundedness of $V_{HI}$, $\kappa$ reaches 0.1 with $\kappa_{4s} = -0.046$ and $\kappa_{6s} = 0.4$ with the resulting $\alpha_s$ and $r$ both $\approx 0.006$. Finally SUSY hybrid inflation can be followed by a successful scenario of non-thermal leptogenesis [38, 34] for both $G_{\text{gut}}$'s considered here.
Chapter 5
SHIFTED INFLATION

5.1 Introduction

In order to achieve ‘shifted’ inflation, we need an alternative track to \(|\phi| = 0\). We can find such a track by including in \(W_{st}\) an additional non-renormalizable term. Inspired by [29], we consider \(\beta S (\bar{\Phi} \Phi)^2 / M_*^2\), and thus define the shifted superpotential \(W_{sh}\) as

\[
W_{sh} = \kappa S (\bar{\Phi} \Phi - M^2) - \beta S (\bar{\Phi} \Phi)^2 / M_*^2,
\]

(5.1)

\[
= \kappa S \left[ \bar{\Phi} \Phi - M^2 \right] - \xi (\bar{\Phi} \Phi)^2 / M_*^2,
\]

(5.2)

where \(\beta\) is a positive dimensionless coupling, \(M_*\) is the cutoff scale, and we have defined \(\xi \equiv \beta M^2 / \kappa M_*^2\) for convenience. The (global SUSY) scalar potential in this model appears as

\[
V_{global, sh} = \kappa^2 \left( \left( |\phi|^2 - M^2 \right) - \xi \frac{|\phi|^4}{M^2} \right)^2 + \sigma^2 |\phi|^2 \left( 1 - 2 \xi \frac{|\phi|^2}{M^2} \right)^2,
\]

(5.3)

where we \(\sigma\) is the canonically normalized inflaton field \(\sigma \equiv \sqrt{2} |s|\). As can be seen in Fig. 5.1, this potential contains three tracks: the standard inflationary track \(|\phi| = 0\), and two additional tracks at constant values

\[
|\phi| = \pm \frac{M}{\sqrt{2} \xi}.
\]

(5.4)

The shape of the potential along these tracks is very similar to that of the standard track, and so we may expect inflation to occur in a qualitatively similar way. However, if we choose one of the shifted tracks from Eq. (5.4), then \(|\phi| \neq 0\) and thus
the gauge symmetry $G$ is already broken. If the breaking of $G$ produces topological defects, then such production will occur before inflation. We can therefore inflate away these defects, giving us more freedom in regards to our choice of gauge group. Thus we attempt to find inflationary solutions along a shifted track where $|\phi| \neq 0$. We wish to mention that there exist a number of interesting intermediate scenarios where inflation takes place partially along each of the standard and shifted tracks; see Ref. [29]. Those will not be of interest to us here–instead, we will consider the case where inflation takes place only along Eq. (5.4). We may only have values

$$\frac{1}{8} \leq \xi < \frac{1}{4},$$

in order to ensure that $V_{\text{min}} = 0$ and that the shifted track remains lower than the standard track [29]. Additionally, we will assume that $G$ is broken down to the Standard Model in a single stage, so that multiple periods of inflation are not needed to handle topological defects produced by subsequent stages of symmetry breaking.

We once again include the radiative corrections, via 3.4. Further we have defined
the useful parametrizations
\[
m^2 \equiv M^2 \left( \frac{1}{4\xi} - 1 \right),
\]
\[
x \equiv \frac{\sigma}{m},
\]
which we will use throughout our analysis. The inflaton rolls down its valley until it reaches the instability point, and thus undergoes a ‘waterfall’ transition to the vev (see 3). The field then falls toward one of the \( V = 0 \) vacua at \( \sigma = 0 \). In the shifted case, however, there exists two additional vacua at the \( |\phi| \) values
\[
|\phi|^2_{\pm} \xrightarrow{\sigma=0} \frac{M^2}{2\xi} \left[ 1 \pm \sqrt{1 - 4\xi} \right].
\]
(5.6)
As described in Ref. \([29]\), \( |\phi|_- \) appears earlier than \( |\phi|_+ \) as the inflaton rolls, and the system evolves into \( |\phi|_- \) before \( |\phi|_+ \) exists as a minimum. Hence the appropriate choice of global vev is \( \langle |\phi| \rangle = |\phi|_- \).

In Chapter 3, we discussed the contribution of soft SUSY breaking terms to SUSY hybrid inflation. We may write the effective linear and mass-squared soft terms in the form
\[
\Delta V_{\text{soft}} = \frac{1}{\sqrt{2}} am_{3/2} k m^3 x + \frac{1}{2} M_{\sigma}^2 m^2 x^2 + M_{\phi}^2 \left( \frac{M^2}{\xi} \right),
\]
(5.7)
\[
a = 2 |A - 2| \cos[\arg S + \arg (A - 2)],
\]
(5.8)
with \( m_{3/2} \sim 1 \) TeV. A useful fact to note is that \( a \) does not vary much over the course of inflation if \( \arg S \) is initially very small \([69]\); for simplicity, we thus assume that this initial condition has been made and treat \( a \) as a parameter of order one. In \([60]\), it was shown that taking the soft masses \( M_{\sigma} \) and \( M_{\phi} \) at intermediate scales can reduce \( n_s \) for \( M_S^2 < 0 \). If one chooses \( M_S^2 > 0 \), large-\( r \) solutions can be produced \([67]\). In both cases, the results were produced in the standard hybrid scenario. On the other hand, it has been shown that these intermediate scales are not crucial \([61, 58]\) in either case. It is arguably more natural to take the soft masses around the TeV scale. We decide here to \( M_{\sigma}, M_{\phi} \sim m_{3/2} \).
We will consider minimal and non-minimal Kähler potentials. For reasons that will become clear later, we will choose $M_* = m_P$ in the non-minimal Kähler case.¹

We write the full inflationary potential as

$$V = V_{\text{global}}(\phi = \frac{M}{\sqrt{2}\xi}) + \Delta V_{\text{SUGRA}} + \Delta V_{\text{1-loop}} + \Delta V_{\text{soft}},$$

$$= \kappa^2 m^4 \left[ A + \frac{1}{2} B \left( \frac{m}{m_P} \right)^2 x^2 + \frac{1}{4} C \left( \frac{m}{m_P} \right)^4 x^4 + \frac{\kappa^2}{4\pi^2} F(x) \right]$$

$$+ \frac{1}{\sqrt{2}} am_{3/2} \kappa m^3 x + \frac{1}{2} M^2 m^2 x^2 + M^2 \phi \left( \frac{M^2}{\xi} \right). \quad (5.9)$$

The effective coefficients $A, B, C$ are functions of the couplings $\kappa_i$ in the Kähler potential, and of the quantity $\phi_P \equiv |\phi|/m_P = (M/m_P)/\sqrt{2}\xi$. See Appendix B.3 for the explicit expressions. We decide to take the couplings $\kappa_i$ in the natural range $-1 \lesssim \kappa_i \lesssim 1$. This produces the following functions, representing the extreme values of the $\kappa_i$

$$A_{\text{max}} = 1 + 4\phi_P^2 + 13\phi_P^4,$$  \quad (5.10)

$$A_{\text{min}} = 1 - 2\phi_P^4,$$  \quad (5.11)

$$B_{\text{max}} = 1 + 16\phi_P^2,$$  \quad (5.12)

$$B_{\text{min}} = -1 - 4\phi_P^2,$$  \quad (5.13)

$$C_{\text{max}} = \frac{19}{4},$$  \quad (5.14)

$$C_{\text{min}} = -\frac{113}{64}. \quad (5.15)$$

The behavior of these functions is depicted in Fig. 5.2. Note that the vacuum potential is now $V_0 = \kappa^2 m^4 A$, so we require that $A \gtrsim 0$. In order that perturbativity be preserved, we also enforce $|B|, |C| < A$.

¹ In the cases where we use the minimal Kähler potential, $M_*$ will enter only via the definition of $\xi$. 

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Figure 5.2: The depicted curves indicate the upper and lower possible values of $A$ (red), $B$ (blue), and $C$ (green). The solid lines signify the upper bounding function for each parameter according to Eqs. (5.10)–(5.15), while the dashed curves represent the lower bounding function. These parameters may be further constrained by other considerations, as explained in the text.

We have written down the potential $V(x)$, and now we write explicitly the slow-roll parameters 2.8 in terms of our parametrization:

$$
\epsilon = \frac{1}{2} \left( \frac{m_p}{m} \right)^2 \left( \frac{V'}{V} \right)^2, \quad (5.16)
$$

$$
\eta = \left( \frac{m_p}{m} \right)^2 \left( \frac{V''}{V} \right), \quad (5.17)
$$

where primes denote differentiation with respect to $x$. The number of e-foldings (Equation 2.11) is given by

$$
N_0 = \left( \frac{m}{m_p} \right)^2 \int_{x_e}^{x_0} \left( \frac{V}{V'} \right) dx, \quad (5.18)
$$

where a subscript ‘0’ denotes a value taken when the pivot scale $k_0 = 0.002 \text{ Mpc}^{-1}$ crosses the horizon. To leading order in the slow roll parameters, the scalar spectral index, tensor-to-scalar ratio, and primordial scalar amplitude appear as, in our
parametrization,

\[ n_s \simeq 1 - 6\epsilon + 2\eta, \quad (5.19) \]

\[ r \simeq 16\epsilon, \quad (5.20) \]

\[ \Delta^2_R \simeq \frac{m^2}{12\pi^2 m_P^6} \left( \frac{V^3}{(V')^2} \right), \quad (5.21) \]

These quantities will be compared with the experimental measurements from WMAP7 [33].

From Eq. (5.9), one can see clearly the similarities between this shifted model and the standard hybrid scenario. The most crucial difference is the use of the mass parameter \( m \) in the shifted model, analogous to the breaking scale \( M \) in the standard case. From Eq. (5.5), we see that have \( m \leq M \); this has the effect of reducing the vacuum potential \( V_0 \) in the shifted case as compared to the standard. On the other hand, we can raise \( V_0 \) is we allow \( \mathcal{A} \) to be significantly greater than unity. If we take \( M \lesssim 0.1 m_P \), we have \( \phi_P \lesssim 0.2 \), and we see from Fig. 5.2 that the system is confined in a region where \( \mathcal{A} \sim 1 \).

As in the previous chapter, we consider the radiative and soft corrections. We should note that the factor \( N \) corresponding to the size of the gauge representation of \( \phi \) and \( \phi \) in the standard case is missing in the shifted case; this is because the gauge symmetry \( G \) is broken. Also it should be noted that the soft mass-squared term for \( \phi, \phi \) is nonzero in shifted inflation. However, since we take soft masses of order \( \sim 1 \) TeV, we do not expect this to have a substantial effect on the results.

### 5.2 Results

#### 5.2.1 Minimal Kähler

In the case of minimal Kähler, we have

\[ \mathcal{A} \to 1 + 2 \left[ \phi_P^2 + \phi_P^4 \right], \quad (5.22) \]

\[ B \to 2\phi_P^2, \quad (5.23) \]

\[ C \to \frac{1}{2}. \quad (5.24) \]
As mentioned earlier, the cutoff $M_*$ only appears in the definition of $\xi$ in the minimal Kähler case.

Fixing $M_* \simeq m_P$ implies that $1 \lesssim \beta \lesssim 100$. For simplicity, we will fix $\beta \simeq 1$, which leads to cutoff values below the Planck scale.

With Ref. [61] as our inspiration, we choose $a = -1$ for the coefficient of the linear soft term, and hold the soft masses at the TeV scale. In this case, we expect to find regions of parameter space where the spectral index $n_s$ can be smaller than 1 in accordance with the latest WMAP data. For comparison, choosing $a = 0, +1$ leads to $n_s \gtrsim 0.985$, lying just outside the WMAP $1\sigma$ bound.

The results of the calculations for the minimal Kähler case are displayed in Panels (a)–(c) of Fig. 5.3. In these plots, we fix $\xi$ between its maximum and minimum values, and the number of e-foldings is $50 \lesssim N_0 \lesssim 60$. The behavior exhibited here is indeed very similar to the analogous results from the standard hybrid case.

As in the standard hybrid case, the primary effect of the negative linear soft term is to reduce $n_s$ to within the pertinent bounds. However, $r$ is very small, similar to the standard case. In the next section, we consider non-minimal Kähler, which, as we will show, drastically increases $r$.

### 5.2.2 Non-minimal Kähler

Previous research has shown that non-minimal Kähler increases $r$ by multiple orders of magnitude. Specifically, the addition of two non-minimal Kähler terms yields large $r$ solutions [65, 58]. These solutions are produced more copiously with a negative second derivative, indicating that a (+quadratic − quartic) structure in the potential. This is also preferable for the production of $n_s < 1$; see [12, 54].

We decide to include more than two non-minimal Kähler terms in this dissertation in order to more fully explore large-$r$ solutions. The non-minimal Kähler terms are grouped for convenience in the parameters $A, B$ and $C$.

We further impose the constraint $V' \gtrsim 0$ in this model, in order to prevent local minima from forming along the inflationary track; these minima would prevent the
Figure 5.3: Results of our numerical calculations for the shifted hybrid inflation model with a minimal Kähler potential. We have taken $\xi$ and $N_0$ near their maximum and minimum values. The light (dark) lines indicate 50 (60) e-foldings.
inflaton field from falling smoothly down the track and toward the waterfall.

Before describing the results in detail, it is worthwhile to examine $r$ at its largest values via analytical approximation. Large-$r$ solutions will be found only close to the Planck scale, since the energy scale of inflation will be large in this case. It is convenient to define $f \equiv \sigma/m_p$, which will increase toward unity as $r$ increases. Then, the polynomial terms in the potential will dominate—in other words, the radiative corrections will be suppressed—unless $\kappa$ is very large. In this case we may approximate the potential as

$$V \approx V_0 \left[ 1 + \frac{1}{2} \tilde{B} f^2 + \frac{1}{4} \tilde{C} f^4 \right],$$

where $\tilde{B} \equiv B/A$, and $\tilde{C}$ is defined similarly. Approximating $V(x_0) \approx V_0 = \kappa^2 m^4 A$ in Eqs. (5.16) and (5.17), and using these expressions in Eq. (5.20), we obtain

$$r \approx 8f^2 \left( \tilde{B} + \tilde{C} f^2 \right)^2,$$

which can be rewritten in the alternate forms

$$r \approx 8f^2 \left( -2\tilde{C} f^2 + \eta \right)^2,$$

$$r \approx \frac{8}{9} f^2 \left( 2\tilde{B} + \eta \right)^2.$$

These expressions are useful in predicting the dependence of $r$ in the large-$r$ region.

We give a simple example here: since we have taken $V' \gtrsim 0$ during inflation, it is implied that $\tilde{C} \sim C$ should be negative and approaching zero as $r$ approaches its largest values [67].

We have obtained an approximate expression for $r$ where the radiative corrections may be neglected. Large $\kappa$ values, however, invalidate this approximation. In Ref. [59] this is described in detail; here, the radiative corrections are integral in bounding $r$. This occurs indirectly through $N_0$; as $\kappa$ increases radiative correction terms become comparable to the polynomial terms in the integrand of Eq. (5.18). This places an upper bound on $r$ in these models. The upper limit is $r$ roughly the lower value which the Planck satellite should be able to measure, adding to the excitement of these models.
Figure 5.4: Results of our numerical calculations for the shifted hybrid inflation model with a non-minimal Kähler potential. The color- and symbol-coding indicates regions of varying interest.
<table>
<thead>
<tr>
<th>Fundamental parameter</th>
<th>Range</th>
<th>Scale type</th>
<th>Derived quantity</th>
<th>Constraining range</th>
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<td>( \kappa )</td>
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<td>( M/m_p )</td>
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<td>( B )</td>
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<td>(\log)</td>
<td>(r)</td>
<td>(&lt; 1)</td>
</tr>
<tr>
<td>( C )</td>
<td>([\max(C_{\min}, -A), \min(C_{\max}, A)])</td>
<td>(\text{linear})</td>
<td>(N_0)</td>
<td>([50, 60])</td>
</tr>
<tr>
<td>( a )</td>
<td>([-1, 0, 1])</td>
<td>(\text{linear})</td>
<td>(\text{na})</td>
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<tr>
<td>( x_0 )</td>
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<td>(\text{na})</td>
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</tbody>
</table>

**Table 5.1:** Ranges specified for the fundamental parameters in Eq. (5.9), and constraints on the derived quantities. The parameter \( a \) takes discrete values, and \( x_0 \) fall anywhere between the waterfall and the Planck scale. Here we reference the WMAP 7-year analysis [33]. It should be noted that the constraint placed manually on \( r \) is designed only to eliminate spurious results, and is not related to the bound given by WMAP7.

The numerical calculations we done via a parameter study, where random values of \( A, B, C \) were generated within the ranges specified by Eqs. (5.10)–(5.15); additionally, \(|B|, |C| < A\). From experience we know that successful inflation occurs for positive values of \( B \), and thus randomly generate \( \log B \) to better examine variation over many orders of magnitude. The parameter \( A \) and \( C \) were randomly generated on a linear scale. See Table 6.1 for the ranges of the parameters. Further, recall that \( M_* = m_P \) here. Since we wish to explore the largest values of \( r \), we choose the cutoff at the Planck scale.

The results of our numerical calculations for the case of shifted inflation with a non-minimal Kähler potential are displayed in Fig. 5.4. These plots exhibit many similarities to the standard hybrid inflation results, but also some (largely quantitative) differences. Panel (a) of Fig. 5.4 shows the region in the \((n_s, r)\) plane where successful inflation may occur in this model. We can see that, within the allowed range, these observables are essentially uncorrelated. This panel also serves to define our color- and symbol-coding. We have highlighted two regions of interest (as well as the overlap between these regions), namely points with large \( r \)-values (\(\gtrsim 10^{-4}\)) and points within
a 1σ range of the \( n_s \) central value as given by WMAP7.

As can be seen in Fig. 5.4, the largest values of the tensor-to-scalar ratio we obtain is \( r \sim 0.02 \). We note that the largest value of \( r \) obtainable in this model is somewhat smaller than it was in the standard hybrid case, as given in Refs. [65, 59]. Recall that the largest \( r \)-values occur for inflaton value close to the Planck scale \((f \sim 1)\). In our parametrization, this is equivalent to \( \sigma \simeq m_P \); however, in the standard hybrid scenario it is more convenient to use \(|s| = \sigma / \sqrt{2}\). Thus \( f \sim 1 \) is equivalent to \(|s| \simeq m_P\).

One can see from Panel (b) the upper bound on \( r \) due to the radiative correction terms, as discuss previously. However, one cannot see the lowering of \( r \) at large values of \( \kappa \). This is because one finds less solutions at sufficiently large \( \kappa \). This is indicated from Table 6.1, where one can see that we have allowed for \( \kappa \) to be somewhat larger than 1; nonetheless, the numerical results yield only \( \kappa \lesssim 0.03 \). This can be understood by considering the relation

\[
\kappa = \frac{\beta}{\xi} \left( \frac{M}{m_P} \right)^2.
\]  

(5.26)

At first, it would seem that \( \kappa \) increases with decreasing \( \xi \). However, if we fix the value of \( M \), decreasing \( \xi \) will raise \( m \) up to a maximum of \( m = M \) for \( \xi = 1/8 \). For successful inflation, the vacuum potential \( V_0 \simeq \kappa^2 m^4 \) is roughly constant at a value near the GUT scale; thus, as \( m \) increases \( \kappa \) must decrease. Hence, the largest values of \( \kappa \) should correspond to the largest values of \( \xi \). This result can be shown more rigorously by eliminating \( M \) in favor of \( V_0 \) in Eq. (5.26), and noting that \( V_0 \) can no longer be treated as constant if \( \xi \) becomes too close to \( 1/4 \). Then, recalling \( \beta \lesssim 1 \) and \( M/m_P \lesssim 0.1 \), we note an upper limit \( \kappa \lesssim 0.04 \). We can see that the numerical results are consistent with our analysis from Fig. 5.4.

Roughly, \( m \) varies with \( \kappa \) via a power law. Thus, an upper bound on \( \kappa \) also leads to a lower bound on \( m \). From \( V_0 \sim \text{constant} \), we derive an lower bound \( m/m_P \gtrsim 5 \times 10^{-3} \). Panel (c) of Fig. 5.4 depicts this bound. Panel (d) depicts the \( m \) versus the quadratic coefficient \( B \). It is interesting to note that the limits on \( \kappa \) and \( m \) are largely responsible for the differences in the results of the present shifted hybrid model as compared to those of the standard hybrid scenario. We know that the limit as
$B \to 0$ corresponds to the global SUSY version of the version. In the standard hybrid model this behavior can be seen clearly from Ref. [59]. There, the mass parameter $M$ tends toward the global SUSY prediction of $\sim$ GUT scale in this limit. In our model, we expect that $m$ should also tend toward a constant value around the same scale. However, the low point density prevents us from making any such declarations.

In Panel (e), we display $r$ versus $B$. The relationship is closely linear on a log scale. Recall that in the standard hybrid case, a linear relationship forms between $r$ and the quadratic coefficient as well. This was the case, however, only at the largest values of $r$.

Other differences from the standard hybrid scenario may be attributed to small differences in the definitions of such parameters as $x$ and the coefficients of various terms in the potential. These we have already noted. There is also a difference in the way the quartic coefficient was generated: in the standard case, the fundamental couplings $\kappa_S$ and $\kappa_{SS}$ were generated on a log scale and the quartic coefficient was then derived from them—contrarily, in the shifted model we have directly generated $C$ on a linear scale. In addition, the allowed range in our model is somewhat more limited by the constraint $|C| \lesssim A \sim 1$. The quartic coefficient versus $r$ is displayed in Panel (f) of Fig. 5.4.

Because of the appreciable similarity between the inflationary potentials of the standard and shifted hybrid scenarios, many of the issues exhibited before will persist here and may be handled in a similar way as in the standard case. However, there is one aspect in which the shifted model has an advantage over the standard hybrid model—as we have discussed, the primary motivation for employing shifted hybrid inflation is to inflate away problematic topological defects. This is because the production of topological defects occurs at the end of inflation in the standard SUSY hybrid scenario, when the gauge group $G$ is broken at the waterfall transition. Depending on the gauge group, these defects may be cosmic strings (e.g. from $G = U(1)$), whose density must be sufficiently suppressed [10] to agree with observations, or monopoles (from models such as $G = SU(4) \times SU(2) \times SU(2)$ and $G = SU(5)$) which must almost completely inflated
away. By choosing the shifted track for the entire duration of observable inflation, we have ensured that the density of these objects will be sufficiently suppressed due to the fact that they are produced before inflation. In this dissertation, we have shown that this issue may be resolved without a substantial change to the predictions of observable parameters.

5.3 Summary

We have thoroughly discussed shifted supersymmetric hybrid inflation, and in particular we have included contributions from SUGRA, soft SUSY-breaking, and the radiative corrections. If the linear soft term is negative, a red-tilted spectrum with $n_s$ spanning the WMAP7 2σ range can be obtained; this is with only minimal Kähler. If non-minimal Kähler is considered, we obtain drastically larger $r$ while maintaining a red-tilted spectrum. Specifically, the tensor-to-scalar ratio $r$ in our model is boosted up to $r \simeq 0.02$, which is potentially observable by the Planck Satellite. In contrast to the standard hybrid scenario, the shifted model ensures that any topological defects produced in the breaking of the gauge symmetry are inflated away. Future work can extend this discussion to other inflationary scenarios, such as smooth hybrid inflation [40, 41, 30, 63, 69] and warm inflation [9] models.
Chapter 6

THE EFFECTS OF R-SYMMETRY VIOLATION

6.1 Introduction

We now wish to determine the effects of allowing $R$-symmetry violation beyond the renormalizable level. First, let us list which additional terms one can consider in this type of model. The three lowest-order nonrenormalizable $R$-violating terms one can write with the aforementioned superfields (respecting gauge symmetry) are

$$\frac{\alpha}{m_P}S^4, \quad \frac{\beta}{m_P}S^2(\Phi\Phi), \quad \frac{\gamma}{m_P}(\Phi\Phi)^2,$$

where $\alpha$, $\beta$, and $\gamma$ are dimensionless, and are sufficiently small such that each term is a perturbation about the standard case. Further, $m_P = M_P/\sqrt{8\pi} \approx 2.4 \times 10^{18}$ GeV is the reduced Planck mass. Along the inflationary track, only the first term will lift the potential. Therefore, in this dissertation we consider solely the inflationary ramifications of the $S^4$ term, so that our superpotential is

$$W = \kappa S(\Phi\Phi - M^2) + \frac{\alpha}{m_P}S^4. \quad (6.1)$$

We will use $m = \frac{M}{m_P}$ as the breaking scale. Other inflationary tracks may be available via the inclusion the $\beta$ and $\gamma$ terms. Inclusion of these terms may lead to a form of shifted inflation; however, we do not discuss this here.

It is important to ask the following question: Can we drive inflation with this $S^4$ term, without radiative corrections nor any additional terms? Let us compute the global, tree-level scalar potential along the inflationary track. We do this via Equation (3.2), which yields the dimensionless potential ($\mathcal{V} \equiv V/m_P^4$)

$$\mathcal{V}_F \big|_{|\phi|=0} = \kappa^2 m^4 - 8\cos(\theta_\alpha + 3\theta_s)|\alpha|x^3\kappa s_c^3 m^2 + 16x^6|\alpha|^2 s_c^6, \quad (6.2)$$

60
where $\theta_\alpha$ and $\theta_s$ are the phases of $\alpha$ and $s$, respectively, and where we have defined the following dimensionless parameters:

$$x \equiv \frac{|s|}{|\tilde{s}_c|}, \quad s_c \equiv \frac{|\tilde{s}_c|}{m_p}, \quad m \equiv \frac{M}{m_p}.$$  

The symbol $|\tilde{s}_c|$, as before, denotes the inflaton field at the waterfall point and its dimensionless value $s_c$ is given by

$$-\kappa m^2 + s_c^2 (\kappa + 4 |\alpha| s_c) = 0. \quad (6.3)$$

It can be shown that, using just Equation (6.2), one cannot obtain a red-tilted spectrum while simultaneously satisfying the slow-roll conditions (to be defined somewhat later).

An analytical calculation reveals that, by imposing the condition $n_s < 1$ in the slow-roll approximation, the inflaton field at the start of inflation is necessarily trans-Planckian (see Appendix B.4.2). We therefore cannot, in this scenario, achieve a suitable spectral tilt without additional terms. In other words, one cannot have successful inflation using only the global SUSY terms.

In order that our model yield more experimentally favorable results, we include soft [63] and SUGRA corrections [43, 63] to the global plus RC potential. The soft terms are derived in a gravity-mediated SUSY-breaking scenario [53]; including the soft mass squared terms, they are

$$a m_{3/2} \kappa m^2 s_c x, \quad m_{3/2}^2 s_c^2 x^2, \quad b m_{3/2} |\alpha| s_c^4 x^4,$$

where the last term is a direct consequence of our $S^4$ term in $W$, and $m_{3/2}$ is the gravitino mass ($\sim$ TeV) divided by the Planck scale. We write the effective coefficients of the soft terms as

$$a = 2 \left[2 \cos(\theta_s) - |A| \cos(\theta_A + \theta_s)\right],$$

$$b = 2 \left[|A| \cos(\theta_A + \theta_\alpha + 4 \theta_s) + \cos(\theta_\alpha + 4 \theta_s)\right], \quad (6.4)$$

where each $\theta_i$, $i \in \{A, \alpha, s\}$, is the phase of a complex parameter, and $A$ is the trilinear coupling. Note that we cannot take $\alpha$ real without loss of generality, since we have
already absorbed the phase of $\kappa$ into that of $s$; therefore, we consider the most general case where $\alpha$, $s$, and $A$ are complex.

While $\theta_A$ and $\theta_\alpha$ are components of couplings, $\theta_s$ is a dynamical field. For the sake of simplicity, we minimize the potential with respect to $\theta_s$ so as to define the inflaton field purely as $|s|$. As a result (see Appendix B.4.1), we choose the following values of the phases

$$\theta_s = l\pi, \quad \theta_A = n\pi, \quad \theta_\alpha = p\pi,$$

such that $l$, $n$, and $p$ are all odd integers. With these choices, the effective coefficients are

$$a = -2(2 + |A|), \quad b = 2(|A| - 1),$$

and in conjunction with these phase choices we additionally impose the condition that $|A| < 1$, or equivalently $b < 0$ (see Appendix B.4.1). Henceforth, we drop the bars on $A$ and $\alpha$ with the understanding that they represent the moduli of the corresponding complex quantities.

The SUGRA scalar potential is given by 2.21, and we include SUGRA correction terms up to sixth order in the inflaton field $|s|$, consistent with our inclusion of the $\alpha^2|s|^6$ global SUSY term; they are:

$$\frac{1}{2} \kappa^2 m^4 s_c^4 x^4, \quad \frac{2}{3} m^4 \kappa^2 s_c^6 x^6, \quad -12 \kappa m^2 \alpha s_c^5 x^5.$$

Hence, with the addition of the soft SUSY-breaking, SUGRA, and 1-loop radiative correction terms to Equation (6.2), the full scalar potential, scaled by $1/m_P^4$, becomes:

$$V = \kappa^2 m^4 - 8\alpha \kappa s_c^3 m^2 x^3 + 16 \alpha^2 s_c^6 x^6 + \frac{m^4 \kappa^4 N}{8\pi^2} F(x)$$

$$+ a m_{3/2} \kappa s_c x + b m_{3/2} \alpha s_c^4 x^4 + m_{3/2}^2 s_c^2 x^2$$

$$+ \frac{1}{2} m^4 \kappa^2 s_c^4 x^4 + \frac{2}{3} m^4 \kappa^2 s_c^6 x^6 - 12 \kappa m^2 \alpha s_c^5 x^5. \tag{6.5}$$

In solving the essential cosmological equations, we employ the slow-roll approximation throughout, in which inflation occurs while the slow-roll parameters are less than unity. In our notation these are written as:

$$\epsilon = \frac{1}{4s_c^2 \left( \frac{\dot{V}}{V} \right)^2}, \quad \eta = \frac{1}{2s_c^2} \frac{\dot{V}}{V}, \quad \xi^2 = \frac{1}{4s_c^2} \frac{\dddot{V} \dot{V}}{V^2}.$$
Here, the prime (′) denotes a derivative with respect to x. Inflation ends either when the slow-roll parameters become unity, or when the inflaton field reaches the waterfall point at x = 1. Observable inflation starts at x₀, defined at the pivot scale k₀ = 0.002 Mpc⁻¹, and ends at xₑ. With this, the number of e-foldings becomes, to leading order,

\[ N_0 \approx 2s_c^2 \int_{x_e}^{x_0} \frac{V}{V'} dx, \]  

(6.6)

while the usual definitions hold for

\[ r \approx 16\epsilon, \quad n_s \approx 1 - 6\epsilon + 2\eta, \quad \frac{dn_s}{d\ln k} \approx 16\epsilon \eta - 24\epsilon^2 - 2\xi^2. \]  

(6.7)

The amplitude of the curvature perturbation is given, to leading order, by

\[ \Delta^2_R \approx \frac{s_c^2 V^3}{6\pi^2 V'^2}. \]  

(6.8)

For higher order expressions see [64]. Note that Equations (6.7) and (6.8) are evaluated at the pivot scale.

In our numerical calculations we take \( m_{3/2} = 1 \text{ TeV}/m_P, Q = 10^{15} \text{ GeV}/m_P, \) and since we are implicitly embedding our model in flipped SU(5), we take \( N = 10 \) [60]. In addition, we impose the ranges in Table 6.1.

### 6.2 Results

#### 6.2.1 Overview

Previous studies have shown that small gravity waves are generated using minimal Kähler and a TeV-scale positive soft SUSY-breaking mass squared term (i.e., \( m_{3/2}^2 x^2 \), with \( m_{3/2} \sim 10^{-16} \)) [61, 15, 60]. Specifically, when the lowest-order SUGRA correction term and a negative linear soft term (\( a = -1 \)) are added to the global SUSY plus TeV-scale positive soft mass squared plus RC potential, one finds that \( r \sim 10^{-12.5} \) around the WMAP nine-year central value \( n_s = 0.971 \) [61]. Alternatively, using non-minimal Kähler in shifted inflation with positive TeV-scale soft mass squared and \( a = 1, 0, \) or \( -1 \) [15] (See also [59, 57, 69] for further references.), or non-minimal Kähler with with same \( a \) values, and large, positive soft mass squared terms (\( \sim 10^{-5} \)) [65], one
Table 6.1: These are the ranges specified for the fundamental parameters in Equation (6.5), and constraints placed on derived quantities, that we have used in our numerical calculations. Note that $x$ can take on any value between the waterfall point and the Planck scale. The experimental bounds on $r$, $\Delta^2_R$, and $n_s$ are from the WMAP nine-year analysis + eCMB + BAO + $H_0$ [27]. The numerical constraints on the quantities $r$ and $\Delta^2_R$ are the experimental bounds; however, the numerical constraints on $n_s$ differ slightly from the experimental bounds. This has been done for ease of plotting.

can generate $r \sim 10^{-2}$ with red spectral tilt. In this paper we find that the solutions follow curves of a similar shape to those presented in [61], as can be seen in Figure 6.1. By employing minimal Kähler, positive TeV-scale soft mass squared terms, and a negative linear and a negative $\alpha$-dependent quartic soft term, we obtain in this paper $r \sim 10^{-8.5}$ around $n_s \simeq 0.965$ for $\alpha = 10^{-9}$; in [61], one obtains $r$ values four orders of magnitude lower than this, at $n_s \simeq 0.965$. As we will describe, this model yields even larger gravity waves ($\sim 10^{-4}$, see Figure 6.4(b)) with red spectral tilt, and we expect that with non-minimal Kähler this model can yield solutions similar to [65]. While the full set of results is outside the reach of current experiments such as Planck, a model in which solutions are tending toward larger $r$ solutions is nonetheless preferred.

6.2.2 The Effect of the Parameters on the Model

Our potential is dependent upon the inflaton field $x$ and the parameters $A$, $\alpha$, $\kappa$, and $m$. Our new parameter $\alpha$, which parametrizes the amount of $R$-symmetry violation beyond the renormalizable level, yields qualitatively and quantitatively distinct results from the standard case.

<table>
<thead>
<tr>
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<th>Range</th>
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<tr>
<td>$s_c$</td>
<td>$[0, 1]$</td>
<td>linear</td>
<td>$N_0$</td>
<td>[50, 60]</td>
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</table>
Figure 6.1: The tensor-to-scalar ratio, \( r \), versus the scalar spectral index, \( n_s \), is depicted. Three curves for large values of \( \alpha \) are shown in 6.1(a), and a larger range of \( \alpha \) is taken in 6.1(b). Here, the number of e-foldings and \( A \) have been fixed at 50 and \( 10^{-1} \), respectively. With \( \alpha = 0 \), we produce solutions closely matching the \( a = -1 \) case in [61]. Note that this curve (in 6.1(b)) does not produce false vacua.

The negative \( \alpha \)-dependent terms in (6.5) create false vacua in some regions of parameter space, i.e. the general behavior of the potential changes from that of Figure 6.2(a) to that of 6.2(b). (Note that the inflaton field rolls from right to left in these figures.) We cannot produce a successful inflationary scenario from Figure 6.2(b) as the system will become trapped in the false vacuum. Rejecting solutions for which this occurs produces gaps in the parameter space such as those seen in Figures 6.1 and 6.6. Note that these vacua did not appear in [61].

Figure 6.1 depicts the effects of \( \alpha \) on \( r \). (See Figure 6.6 for further results. Note the similarity of these curves to those in [61].) The potential in Equation (6.5) differs from that in [61] by two higher-order SUGRA correction terms, an \( \alpha \)-dependent quartic soft term, and two global \( \alpha \)-dependent terms (\( a \) is also different). The effect of \( \alpha \) is to raise \( r \), particularly in red-tilted regions. This is primarily a result of the global term proportional to \( \alpha \). The \( \alpha = 10^{-9} \) curve is raised by three to six orders of magnitude for \( 0.92 < n_s < 0.98 \), as compared to the \( \alpha = 0 \) case in [61].

Our model greatly benefits from the fact that we can, as noted, generate larger gravity waves than the standard (\( \alpha = 0 \)) case. However, \( \alpha \) cannot be raised arbitrarily,
Figure 6.2: The qualitative change in behavior caused by the negative $\alpha$ terms. Figure (a) depicts a potential that is well-behaved, i.e., the field will roll toward the global minimum, while Figure (b) depicts a potential with a false vacuum. Mathematical solutions which produce false vacua are not acceptable inflationary scenarios.

since we require $R$-symmetry violation to be small. We find no need to impose an upper bound, though, because our parameter study yields a numerical upper bound $\alpha \sim 10^{-7}$. This can be understood mathematically by noting that $|V'|$ increases faster than $|V|$ as $\alpha \rightarrow 10^{-7}$ (from smaller $\alpha$). The e-foldings constraint (6.6) becomes impossible to satisfy at large $\alpha$, because its integrand is suppressed by a large $V'$, and the limits of the integral can only be marginally altered.

We find that $x_0$ can vary over at least two orders of magnitude until $\alpha \sim 10^{-8}$; then, $x_0$ is compressed to $\sim 10$. Likewise, the end of inflation is pushed toward waterfall, $x_e = 1$, as $\alpha \rightarrow 10^{-7}$. Thus the distance in $x$ over which inflation occurs approaches an approximately constant value. If we take solutions to $\Delta^2_{R} = 2.427 \times 10^{-9}$ (Figure 6.3(a)) and then impose the constraint that the number of e-foldings be between 50 and 60 (Figure 6.3(b)), we observe that, for many orders of magnitude in $\alpha$, requiring sufficient inflation decreases by at least an order of magnitude the number of solutions relative to those obtained merely from the curvature perturbation constraint. Achieving a sufficient amount of inflation severely limits the number of viable solutions generally, but is most limiting at large $\alpha$. The curvature perturbation constraint also
Numerically, we obtained our results using two independent methods: a continuation method and a parameter study. The results of the former are seen in Figures 6.1 and 6.6. The latter results are presented in Figures 6.4 and 6.5. Figure 6.4(a) shows the presence of qualitatively new “horizontal solutions” in the $r - n_s$ plane. If we drop all the $\alpha$-dependent terms in the scalar potential (6.5) except the global term linear in $\alpha$ (which is $-8\alpha \kappa s_\epsilon^3 m^2 x^3$), the number of horizontal solutions in this region increases. On the other hand, dropping this term and keeping the others does not produce a viable inflationary scenario. Since the global $\alpha$ term is the most dominant $\alpha$-dependent term, it is primarily responsible for producing these horizontal solutions.

We find that limits on $r$ naturally arise in our model (see Figure 6.5(a)). Solutions producing $r \sim 10^{-4}$ can be produced throughout the range of $\alpha$ that we have taken, although all of these except the solutions near the upper limit of $\alpha$ correspond to...
Figure 6.4: (a) The tensor-to-scalar ratio, $r$, versus the scalar spectral index, $n_s$, for $\alpha = 10^{-9}$, $A = 10^{-4}$, and $N_0 \in [50, 60]$. One can see the “horizontal solutions”, which yield $r$ values somewhat above $10^{-8}$. (b) The tensor-to-scalar ratio, $r$, versus the scalar spectral index, $n_s$ for solutions corresponding to the ranges in Table 6.1. Solutions are color-coded as follows: blue - $(10^{-14} < \alpha \leq 10^{-12})$, green - $(10^{-12} < \alpha \leq 10^{-11})$, yellow - $(10^{-11} < \alpha \leq 10^{-10})$, red - $(10^{-10} < \alpha \leq 10^{-9})$, cyan - $(10^{-9} < \alpha \leq 10^{-8})$, magenta - $(10^{-8} < \alpha \leq 10^{-7})$. A blue-tilted spectrum. Figure 6.4(b) depicts this behavior; note that the only large-$r$ solutions corresponding to red spectral tilt are colored magenta and cyan (meaning that $10^{-9} < r \leq 10^{-7}$). Figure 6.5(a) also indicates that the smallest-$r$ solutions that can be produced are increasingly larger as $\alpha$ increases, so that at the upper limit of $\alpha$ only $r \sim 10^{-4}$ can be produced. This effect can be understood by using the following approximation of the energy scale of inflation: $\mathcal{V}_0^{1/4} \sim \kappa^{3/2} m^2 / \sqrt{\lambda^2 x_0^2} \alpha$, where $\lambda \equiv \left[ (3456 \Delta_{R|x_0}^2 \pi^2) \right]^{1/4}$. As mentioned, the integrand in Equation (6.6) is suppressed by the fact that $|\mathcal{V}|$ increases faster than $|\mathcal{V}|$ as $\alpha$ increases. To compensate for this effect the $\kappa^2 m^4$ term in $\mathcal{V}$ increases, raising the numerator in $\mathcal{V}_0^{1/4}$. This prohibits small-$r$ solutions in large-$\alpha$ regions (recall that $r \propto \mathcal{V}_0^{1/4}$).

The upper limit on $r$ of $\sim 10^{-4}$, approximately constant over many orders of magnitude of $\alpha$, is a consequence of the Lyth Bound [47] $r \lesssim \mathcal{O}(10^{-2}) \times m^2(x_0 - x_e)^2$. 

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Figure 6.5: (a) The tensor-to-scalar ratio, $r$, versus $\alpha$ using the constraints in Table 6.1. (b) The scalar spectral index, $n_s$, versus $\alpha$ using the constraints in Table 6.1. Note that in the red-tilted region where $\alpha \to 0$, solutions are sparse.

The largest $m$ values obtainable ($m \sim 10^{-1}$) correspond to very small values of $x_0 - x_e$, and therefore limit $r$ below $10^{-4}$. Solutions with $m \sim 10^{-2}$ correspond to $x_0 \approx 10$ and $x_e \approx 1$; thus, $m^2(x_0 - x_e)^2 \sim 10^{-2}$ for the largest $r$ values obtainable in our model.

The parameter $A$, arising from the gravity-mediated soft-SUSY breaking terms, does not have any discernible effect on our results in the red-tilted region. This is expected from the fact that the soft terms are suppressed by the gravitino mass ($\sim$ TeV).

The plots in Figure 6.6 depict the effects of increasing $\alpha$. Figures 6.6(a), 6.6(c), and 6.6(d) are in direct reference to [61]. We can see from Figure 6.6(c) that large $\alpha$ boosts $\kappa$, especially in the red-tilted region. Similarly, Figure 6.6(a) shows that, for the same breaking scale $M$, $\kappa$ is boosted as $\alpha$ increases. Note that $m$ is also boosted by $\alpha$ (Figure 6.6(b)). We have discussed that larger gravity waves are produced via increasing $\alpha$. This effect is evident from Figures 6.6(b) and 6.6(c), which show that $m$ and $\kappa$ values are raised as $\alpha$ increases, particularly in red-tilted regions. Figure
Figure 6.6: Here we plot our numerical results in which the number of e-foldings and $A$ have been kept fixed at 50 and $10^{-4}$, respectively. The absolute value of the running of $n_s$, $\log_{10} |\frac{dn_s}{d\ln k}|$, is plotted in 6.6(d).

6.6(d) depicts the absolute value of the running of $n_s$, $\log_{10} |\frac{dn_s}{d\ln k}|$. In our model, $\frac{dn_s}{d\ln k}$ is negative, and $|\frac{dn_s}{d\ln k}| \sim \mathcal{O}(10^{-3}) - \mathcal{O}(10^{-4})$.

6.3 Summary

In this dissertation we have considered the effects of the $R$-symmetry violating $\frac{\alpha}{m_P} S^4$ term in the standard hybrid inflationary scenario. By allowing for $R$-symmetry violation beyond the renormalizable level, we can give masses to right-handed neutrinos and down-type quarks in a simple way within flipped $SU(5)$, providing a particularly well-motivated inflationary scenario. We compare our results to the WMAP nine-year parameters $n_s$ and $r$, finding that we can easily generate red spectral tilt values within
the observational bounds. This type of model can generate larger gravity waves than the standard \((\alpha = 0)\) case; raising \(\alpha\) from \(10^{-14}\) to \(10^{-9}\) corresponds to an increase in \(r\) from \(10^{-12}\) to \(10^{-8}\) at the WMAP nine-year central \(n_s\) value [27]. This result has been achieved using a minimal Kähler potential, positive TeV-scale soft mass squared terms, a negative linear and a negative quartic soft term, and SUGRA correction terms up to sixth order in \(x\). Additional interesting results were observed in the parameter study, which yielded a region of qualitatively new “horizontal solutions” in the red-tilted region, above the main \(r - n_s\) curve in Figure 6.4(a). The parameter study yields \(r\) values as large as \(\sim 10^{-4}\) in the red-tilted region. Finally, we observe that \(|\alpha|\) develops a numerical upper bound of \(\sim 10^{-7}\), while \(|\frac{dn_s}{dk}\| \lesssim \mathcal{O}(10^{-3}) - \mathcal{O}(10^{-4})\).
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Appendix A
GENERAL RELATIVITY, COSMOLOGY, AND SUPERSYMMETRY

A.1 General Relativity

A.1.1 Proving that the metric is a tensor

We start with
\[
\frac{\partial x'^\alpha}{\partial x^\beta} \frac{\partial x^\beta}{\partial x'_{\mu}} = \frac{\partial x'^\alpha}{\partial x^\beta} g^{\beta\mu} \frac{\partial x_{\mu}}{\partial x'_{\sigma}}.
\]

and we note that
\[
\frac{\partial x_{\mu}}{\partial x'_{\sigma}} = \frac{\partial x'^\sigma}{\partial x^\mu} \frac{\partial x^\mu}{\partial x'_{\sigma}}.
\]

By a simple substitution for \(dx_{\mu}\) we have
\[
\frac{\partial x'^\alpha}{\partial x^\beta} \frac{\partial x^\beta}{\partial x'_{\sigma}} = \frac{\partial x'^\alpha}{\partial x^\beta} g^{\beta\mu} \frac{\partial x^\mu}{\partial x'_{\sigma}}.
\]

The LHS is
\[
\frac{\partial x'^\alpha}{\partial x^\beta} \frac{\partial x^\beta}{\partial x'_{\sigma}} = \partial x'^\alpha,
\]
giving us
\[
\frac{\partial x'^\alpha}{\partial x^\beta} \frac{\partial x^\beta}{\partial x'_{\sigma}} = \frac{\partial x'^\alpha}{\partial x^\beta} g^{\beta\mu} \frac{\partial x^\mu}{\partial x'_{\sigma}}.
\]

Finally, we can rewrite this as
\[
\partial x'^\alpha = \partial x'_{\sigma} g^{\sigma\alpha} \Rightarrow g'^{\sigma\alpha} = \frac{\partial x'^\alpha}{\partial x^\beta} \frac{\partial x^\mu}{\partial x'_{\sigma}} g^{\beta\mu}.
\]

Similarly,
\[
g'_{\mu\nu} = \frac{\partial x'^\alpha}{\partial x^\mu} \frac{\partial x^\beta}{\partial x'_{\sigma}} g^{\alpha\beta}.
\]

Thus, the metric transforms as a tensor.
An important consideration for the following discussion is the evolution of scales— in particular, the evolution of the comoving wavenumber \( k \). We will be concerned with whether or not \( k^{-1} \) is larger or smaller than the comoving Hubble radius \( \frac{1}{aH} \). The convention is so refer to the “horizon” as \( aH \), so that \( k \) “exits the horizon” when \( k < aH \), and is inside the horizon when \( k > aH \).

In the comoving gauge (see [11]) perturbations are characterized purely by

\[
\delta g_{ij} = a^2(1 - 2\zeta)\delta_{ij} + a^2h_{ij},
\]

where \( h_{ij} \) is a transverse and traceless tensor. The parameter \( \zeta \) is a scalar and is referred to as the *comoving curvature perturbation*; it has the advantageous property that for adiabatic matter fluctuations it is time-independent on superhorizon \((k \ll aH)\) scales:

\[
\lim_{k \ll aH} \dot{\zeta} = 0.
\]

By varying the action, and parametrizing it via \( v \equiv z\zeta \) and \( z^2 = 2a^2\epsilon \), one can derive the Mukhanov-Sasaki equation:

\[
v''_k + \omega^2 v_k = 0,
\]

where \( \omega^2 \equiv k^2 - \frac{z''}{z} \). Since \( a \propto e^{Ht} \) in de Sitter space, we can use the relationship \( \tau = \int \frac{dt}{a} \) to get \( a = -\frac{1}{H\tau} \). Finally, \( \tau^{-1} \propto \sqrt{|\frac{z''}{z}|} \propto aH \). Horizon scales correspond to \( k = aH = \sqrt{|\frac{z''}{z}|} \), and therefore superhorizon scales correspond to \( k^2 \ll |\frac{z''}{z}| \). We can write \( \omega^2 \) as \( \omega^2 \equiv k^2 - \frac{2}{\tau^2} \), which indicates that on superhorizon scales one has \( \omega^2 \approx -\frac{2}{\tau^2} \); the equation one has to solve therefore is

\[
v''_k - \frac{2}{\tau^2}v_k = 0.
\]

We thus have \( \frac{v''}{v_k} \propto \tau^{-2} \propto \frac{z''}{z} \). Recalling that \( \zeta = v_kz^{-1} \), we can see that \( \zeta \) is constant on superhorizon scales. By solving \( A.1 \), one obtains

\[
v_k = \frac{e^{-ik\tau}}{\sqrt{2k}} \left(1 - \frac{i}{k\tau}\right),
\]
which, on superhorizon scales, yields \( v_k = \frac{1}{\sqrt{2}} \frac{1}{k^{3/2}} \). By doing a mode expansion with \( v_k^* = a_k^- v_k + a_k^+ v_k^* \), one calculates the zero-point mode fluctuations to be \( < 0 | v_k v_k^* | 0 > = |v_k|^2 \delta (\vec{k} + \vec{k}^*) \). On superhorizon scales we multiply \( v_k \) by its complex conjugate to obtain

\[
|v_k|^2 = \frac{(aH)^2}{2k^3} \equiv P_v.
\]

Correspondingly,

\[
P_\zeta = P_v |_{\vec{k} = aH}.
\]

Finally, we derive the scalar perturbations used here, and stated in 2.3

\[
\Delta_s^2 = \frac{k^3}{2\pi^2} P_\zeta = \frac{H^2}{8\pi^2 \epsilon} \frac{V^3}{12\pi^2 m_{pl}^6 V^2 \phi}.
\]

### A.2 SUSY hybrid inflation

#### A.2.1 Deriving the radiative corrections

The terms \( M_i^2 \) are the eigenvalues of the mass matrix

\[
M_{ij} = \frac{\partial^2 V}{\partial q_i \partial q_j},
\]

for the scalars. After computing these terms, one applies \( \tilde{\phi} = 0 = \phi = \tilde{\phi}^* = \phi^* \), and then finds that the only nonzero terms are \( \phi, \tilde{\phi}, \phi^* \), and \( \tilde{\phi}^* \). One therefore has

\[
M^2 = \begin{pmatrix}
0 & A & B & 0 \\
A & 0 & 0 & B \\
B & 0 & 0 & A \\
0 & B & A & 0
\end{pmatrix},
\]

where \( A = \kappa^2 S^* S = \kappa^2 |S|^2 \) and \( B = -\kappa^2 m^2 \). The eigenvalues are

\[
\pm (A \pm B).
\]

This yields four eigenvalues, which are, in our parametrization,

\[
M_i^2 = \begin{cases}
\kappa^2 m^2(x^2 - 1) \\
\kappa^2 m^2(x^2 + 1) \\
-\kappa^2 m^2(x^2 - 1) \\
-\kappa^2 m^2(x^2 + 1).
\end{cases}
\]
Thus, from (3.4) one has

$$\Delta V_s = (128\pi^2)^{-1} F,$$

where

$$F = 2\kappa^4 m^4 \left( x^2 - 1 \right)^2 \left( \ln \left( \frac{\kappa^2 m^2 (x^2 - 1)}{Q^2} \right)^2 - 3 \right)$$

$$+ 2\kappa^4 m^4 \left( x^2 + 1 \right)^2 \left( \ln \left( \frac{\kappa^2 m^2 (x^2 + 1)}{Q^2} \right)^2 - 3 \right).$$

This can be put into the form

$$\Delta V_s = \frac{\kappa^4 m^4}{32\pi^2} \left\{ 2 \left( x^4 + 1 \right) \ln \frac{\kappa^2 m^2 x^2}{Q^2} - 3 \left( x^4 + 1 \right) + \left( x^4 + 1 \right) \ln \frac{x^4 - 1}{x^4} + 2x^2 \ln \frac{x^2 + 1}{x^2 - 1} \right\}.$$

For the fermions, the mass matrix is

$$M_{ij} = \frac{\partial^2 W}{\partial Q_i \partial Q_j},$$

where $Q_i$ and $Q_j$ are the superfields. This yields this simple mass matrix

$$M^* M = \begin{pmatrix} \kappa^2 |s|^2 & 0 \\ 0 & \kappa^2 |s|^2 \end{pmatrix},$$

which yields $M_i = \pm \kappa |s|$. This gives

$$\Delta V_f = -(32\pi^2)^{-1} \left[ 2\kappa^2 m^4 x^4 \left( \ln \left( \frac{\kappa^2 x^2 m^2}{Q^2} \right) \right) - 3 \right],$$

which produces 3.4.
Appendix B

SHIFTED, COMCONSTRAINED M AND R-SYMMETRY VIOLATION

B.1 Numerical calculations

Here we give a short introduction on how to numerically solve inflationary models, focusing on the models presented in this dissertation. The first goal is to deduce the scalar potential $V$. Chapter 3 discussed how one computes of $V$, beginning with the global terms 3.3. One can then choose to consider the effects of soft breaking terms (see 3.2.2), and non-minimal Kähler and SUGRA correction terms (3.7 and 2.21). One then has a potential $V$ as a function of the free parameters: $V = V(\kappa, M, \kappa_S, S, \ldots)$. The task is then to simultaneously solve the observational constraints

$$\Delta^2_R = \frac{1}{12\pi^2 m_{pl}^6 V_\phi^2} V^3$$

$$N = \frac{1}{m_p^2} \int_{\phi_{end}}^{\phi_{beg}} \frac{V d\phi}{V_\phi}$$

One fixes $\Delta^2_R$ from experiment (see 2.2), and requires at least 50 e-foldings. However, one also needs to ensure that $\epsilon \ll 1$ and $|\eta| \ll 1$ during inflation; thus for solutions where $|\eta|$ is approaching one, one needs to also simultaneously solve $|\eta| = 1$. It is common in SUSY hybrid inflation models for $|\eta| > \epsilon$. One can also fix $n_s$, however this choice can leave out interesting solutions since $r$ can vary considerably over the valid $n_s$ range (see Table 2.2).

Let us concentrate on Figure 6.1. The author uses Mathematica, and we give here a brief description of the code used. First, one must define the relevant functions; these include $V$, $\epsilon$ and $\eta$, $\Delta^2_R$, $n_s$, $r$, and $N_e$. One defines a function in Mathematica by simply writing, for example,

$V[x, \kappa, m] := m^4 \kappa^2 - 8 \kappa \alpha x^3 m^5 + \ldots$
where we have defined a potential $V = V(x, \kappa, m)$. The author then uses the (built-in) FindRoot function in conjunction with a Do loop. First one needs a cell in which one defines the relevant functions $[\text{CapitalDelta}]R\text{squared}[xs, [\text{Kappa}], m]$ and $\text{Ne}[x, [\text{Kappa}], m, m]$, and all the constant parameters. For instance, in the below code the parameter PR appears. Since it is not a variable for which Mathematica is solving, it must be defined previously. All of these definitions can be contained in one cell, and it must be evaluated before the below code can be run. Code one would write to solve for B.1 and B.2 would take the form

```mathematica
xsg = 1.05550;
mg = 0.001387;
Do[
  SOL = FindRoot[{PR == \[CapitalDelta]Rsquared[xs, \[Kappa], m],
    50 == NIntegrate[Ne[x, \[Kappa], m, m], {x, xe, xs}],
    {xs, xsg/n, xsg*n}, {m, mg/n, mg*n}];

  , {\[Kappa], 10^-3.260728, 10^-2, 0.001}];
```

The `FindRoot` function takes the provided guesses $xsg = 1.05550$ and $mg = 0.001387$ and searches for solutions to the equations $PR == [\text{CapitalDelta}]R\text{squared}[xs, [\text{Kappa}], m]$ and $50 == N\text{Integrate}[\text{Ne}[x, [\text{Kappa}], m, m], \{x, xe, xs\}]$ with those guesses. The function $N\text{Integrate}[\text{integrand}[x], \{x, x1, x2\}]$ is a built-in Mathematica function (no definitions or prerequisites are needed to use it) that integrates $\text{integrand}[x]$ from $x1$ to $x2$. The Do loop runs over $[\text{Kappa}]$ from $10^{-3.260728}$ to $10^{-2}$, by the iteration $0.001$.

### B.2 Constraining M

\[
\sigma_{\text{max}} \simeq \frac{m_P \sqrt{\pi |c_{2\kappa}| + \sqrt{\pi^2 c_{2\kappa}^2 + N \kappa^2 |c_{2\kappa} k|}}}{\sqrt{2\pi |c_{2\kappa} k|}}, \tag{B.3}
\]
and a local minimum at the inflaton-field value:

\[ \sigma_{\text{min}} \simeq m_p \sqrt{3|c_{2\kappa}x| + \sqrt{9c_{2\kappa}x^2 + 32|c_{2\kappa}k_{c_{2\kappa}}x|}} \]  \hspace{1cm} (B.4)

In deriving B.3 we keep terms up to the fourth power of \( \sigma \), whereas for B.4 we focus on the last three terms of the expansion in the right-hand side of 4.1. For this reason, the latter result is independent of \( c_{HI} \) and \( c_{2\kappa} \).

### B.3 Functional Forms of \( A, B, C \)

Before writing down the expressions for \( A, B, C \), it is convenient to define some auxiliary functions of the couplings in the Kähler potential:

\[
\begin{align*}
c_0 &= 1 - \frac{1}{2} (\kappa_{S\Phi} + \kappa_{S\Phi\Phi}), \\
c_1 &= 1 + \frac{1}{8} \left[ 4\kappa_{S\Phi}^2 - \kappa_{S\Phi\Phi} - 4\kappa_{S\Phi\Phi\Phi} + 8\kappa_{S\Phi}(-1 + \kappa_{S\Phi}) \\
&\quad + 4(-2 + \kappa_{S\Phi})\kappa_{S\Phi\Phi} - \kappa_{S\Phi\Phi\Phi} + \kappa_{\Phi} + 4\kappa_{\Phi\Phi} + \kappa_{\Phi} \right], \\
c_2 &= 1 + \frac{1}{2} \left[ -\kappa_{SS\Phi} - \kappa_{SS\Phi\Phi} + (-2 + \kappa_{S\Phi})\kappa_{S\Phi} + (-2 + \kappa_{S\Phi})\kappa_{S\Phi} \right] \\
&\quad + 2\kappa_{S}(-1 + \kappa_{S\Phi} + \kappa_{S\Phi}).
\end{align*}
\]

Notice that each of these reduces to 1 in the case of minimal Kähler. We may also write the function

\[ \gamma_S = 1 - \frac{7}{2} \kappa_S + 2\kappa_S^2 - 3\kappa_{SS}, \]

which is the same parameter that appears in the quartic coefficient of the standard hybrid inflation model, and also reduces to 1 for minimal Kähler. In the case of non-minimal Kähler with couplings \(-1 \lesssim \kappa_i \lesssim 1\), we obtain the ranges

\[
\begin{align*}
0 &\lesssim c_0 \lesssim 2, \\
-1 &\lesssim c_1 \lesssim \frac{13}{2}, \\
-2 &\lesssim c_2 \lesssim 8, \\
-\frac{113}{32} &\lesssim \gamma_S \lesssim \frac{19}{2}.
\end{align*}
\]  \hspace{1cm} (B.5)
This range for the $\kappa_i$ couplings is somewhat more restrictive than needed; for perturbativity, we must only have $|\kappa_i| \lesssim O(1)$, and there is some ambiguity involved with how the combinatoric factors are written down in the Kähler potential. Looking slightly ahead, it will be most convenient if we are able to treat $c_0, c_1, c_2$ and $\gamma_S$ as independently varying parameters, so that the quantities $A, B, C$ may be varied independently. From the definitions of these $c_i$’s above, one can readily verify that this is not quite the case within the specified ranges; there exist some interdependencies via the underlying couplings, which lead to some regions of the $(c_0, c_1, c_2)$ being impossible to access. However, relaxing to somewhat larger values of the $|\kappa_i|$’s (but still order 1) has the effect of $c_0, c_1, c_2$ becoming essentially independent within their ranges in Eqs. (B.5). Based on these arguments, and for concreteness and simplicity, we will take these ranges for $c_0, c_1, c_2$, and assume that they may vary independently within these ranges.

Now, we define $A, B, C$ as the coefficients of the terms constant, quadratic, and quartic (respectively) in $|s|/m_P$ in the normalized potential $V/\kappa^2 m^4$. These quantities appear in the form

$$A = 1 + 2c_0 \phi_P^2 + 2c_1 \phi_P^4,$$

$$B = -\kappa_S + 2c_2 \phi_P^2,$$

$$C = \frac{\gamma_S}{2}.$$

Again, we take these parameters as independent of one another (although this assumption only approximately holds, to within factors of order unity), which makes the numerical calculations more tractable. Using these expressions with the ranges in Eqs. (B.5), the extremal functions in Eqs. (5.10)–(5.15) can easily be verified.

### B.4 R-symmetry violation

#### B.4.1 Minimization of the Potential With Respect to $\theta_S$

The phase-dependent terms in our potential are, including just the global SUSY and soft SUSY-breaking terms:

$$V(\theta) \equiv -c_1 \cos(\delta) + c_2 (2 \cos(\theta_S) - |A| \cos(\psi)) + c_3 (|A| \cos(\delta + \psi) + \cos(\theta_S + \delta)),$$
where the phases and coefficients (the latter are dimensionful) are

$$
\delta \equiv \theta_\alpha + 3\theta_s, \quad \psi \equiv \theta_s + \theta_A
$$

$$
c_1 \equiv \frac{8|\alpha||S|^3 M \kappa}{m_P}, \quad c_2 \equiv 2m_{3/2}M^2 \kappa |s|, \quad c_3 \equiv \frac{2m_{3/2}|\alpha||S|^4}{m_P}.
$$

We seek to minimize this function with respect to $\theta_s$, thus we impose the conditions

$$
d V_\theta \frac{d\theta_s}{d\theta} = 3c_1 \sin(\delta) + c_2 [-2 \sin(\theta_s) + |A| \sin(\psi)] - 4c_3 [|A| \sin(\delta + \psi) + \sin(\theta_s + \delta)] = 0,
$$

$$
d^2 V_\theta \frac{d\theta_s^2}{d\theta^2} = 9c_1 \cos(\delta) + c_2 [-2 \cos(\theta_s) + |A| \cos(\psi)] - 16c_3 [|A| \cos(\delta + \psi) + \cos(\theta_s + \delta)] > 0.
$$

The first condition is met trivially by setting

$$
\sin(\theta_s) = 0, \quad \sin(\psi) = 0, \quad \sin(\delta + \psi) = 0, \quad \sin(\theta_s + \delta) = 0,
$$

which implies that each argument is an integer multiple of $\pi$. Finally, the second condition can be satisfied with

$$
\cos(\theta_s) < 0, \quad \cos(\psi) > 0, \quad \cos(\delta) > 0, \quad \cos(\theta_s + \delta) < 0, \quad \cos(\delta + \psi) > 0.
$$

The last inequality is a consequence of the previous four, but we can nonetheless ensure that the second condition is satisfied by declaring $|A| < 1$, which, from Equation (6.4), is equivalent to $b < 0$. With these phases the linear coefficient is $a = -2(2 + |A|) < 0$. Finally, the only phase-dependent SUGRA correction term we are including is $V_{\theta,SUGRA} = -\frac{128 M^2 |M|^5 |\alpha|}{m_P^3} \cos(\delta)$, which will always be negative with our chosen phases.

**B.4.2 Blue-Tilt in Standard Global Plus $S^4$ Case**

It is interesting to investigate whether inflation can be driven exclusively using the new $S^4$ contribution without extending to SUGRA. In global SUSY, the scalar potential acquires two new terms (see Equation (6.2)) which impart a nonzero slope along the inflationary valley. For $|\alpha|$ small enough to constitute a perturbation, the $|s|^6$ term is strongly subdominant to the $|s|^3$ term, and thus the latter is chiefly responsible for meaningful alterations to the inflationary dynamics.
Some care must be taken in choosing the phases of the complex quantities in this version of the model. The new term in the superpotential induces an additional SUSY vacuum appearing along the $|\phi| = 0$ direction. This vacuum is gauge symmetric and must be avoided if the symmetry breaking structure of the model is to be preserved. If the phase factor $\cos(\theta_\alpha + 3\theta_S)$ is positive, a stable inflationary valley along $|\phi| = 0$ leads to this vacuum. Therefore we require this phase factor to be negative. To achieve this, one (but not both) of $\theta_\alpha, \theta_S$ must be an odd integer multiple of $\pi$. For simplicity, we choose $\theta_S = 0, \theta_\alpha = n\pi$ ($n$ odd). Additionally, these phases result in a slight ($\sim$ cubic) uplifting of the potential, consistent with a perturbation to the original hybrid model.

With these phases chosen, we examine the constraint imposed on the parameters of the model by the slow roll conditions $\epsilon_0 < 1, \eta_0 < 1$. We find that the constraint from $\eta_0$ is convincingly more stringent than that from $\epsilon_0$, and so we will concentrate on the former. In terms of dimensionful quantities, we may write analytically

$$\eta_0 = 24m_P|\alpha||s_0|\left[\frac{kM^2 + 10|\alpha||s_0|^3/m_P}{(kM^2 + 4|\alpha||s_0|^3/m_P)^2}\right],$$

$$\simeq \frac{24m_P|\alpha||s_0|}{kM^2 + 4|\alpha||s_0|^3/m_P}.$$  \hspace{1cm} (B.6)

The denominator of this expression is strikingly close to Equation (6.3) specifying the waterfall point $\tilde{s}_c$, after adjusting for phase differences (which amounts to a sign flip for the $\alpha$-dependent term). Approximating $|s_0| \approx \tilde{s}_c$, we have

$$\eta_0 \simeq \frac{24m_P|\alpha||s_0|}{k\tilde{s}_c^2}.$$  \hspace{1cm} (B.6)

Taking $\eta_0 < 1$ and rearranging, we may cast the slow roll condition in the form

$$\frac{|\alpha|}{k} \simeq \frac{1}{24} \left(\frac{|s_0|}{m_P}\right) \left(\frac{\tilde{s}_c}{|s_0|}\right)^2.$$  \hspace{1cm} (B.7)

Next, we investigate the condition for obtaining a red-tilted spectrum in this model. Since $\eta_0 > 0$ in the present case, $n_s < 1$ requires $6\epsilon_0 > 2\eta_0$ according to Equation (6.7), or

$$\frac{\epsilon_0}{\eta_0} > \frac{1}{3}.$$
Using the same approximation stated above, we may write

\[ \epsilon_0 \simeq 144 \left( \frac{\alpha}{\kappa} \right)^2 \left( \frac{|s_0|}{s_c} \right)^4. \]

Combining this with Equation (B.7) yields a red tilt condition

\[ \left| \frac{\alpha}{\kappa} \right| \left( \frac{|s_0|}{s_c} \right)^2 \left( \frac{|s_0|}{m_P} \right) > \frac{1}{18} \sim 10^{-1}, \]

\[ \left| \frac{\alpha}{\kappa} \right| \gtrsim 10^{-1} \left( \frac{m_P}{s_c} \right) \left( \frac{s_c}{|s_0|} \right)^3. \]  

(B.8)

Taking $1/24 \sim 10^{-1}$ in Equation (B.7), the slow roll and red tilt conditions may be consolidated into one expression,

\[ 10^{-1} \left( \frac{m_P}{s_c} \right) \left( \frac{s_c}{|s_0|} \right)^3 \lesssim \left| \frac{\alpha}{\kappa} \right| \lesssim 10^{-1} \left( \frac{|s_0|}{m_P} \right) \left( \frac{s_c}{|s_0|} \right)^2. \]  

(B.9)

The middle portion $|\alpha|/\kappa$ may safely be omitted, after which several common factors may be canceled from the inequality. Simplifying Equation (B.9) readily leads to

\[ |s_0| \gtrsim m_P. \]  

(B.10)

In other words, the global-SUSY version of the model under the slow roll approximation may only lead to $n_s < 1$ for trans-Planckian values of the inflaton field. Since the SUGRA sector is suppressed by powers of $|s_0|/m_P$, it is then inconsistent to treat the model only using global SUSY.

In addition to these arguments, the assumptions placed on the phases for this version of the model are also problematic. Above, we commented that the phase factor $\cos(\theta_\alpha + 3\theta_S)$ must be negative in order to preserve gauge symmetry breaking. Referring to Appendix B.4.1, however, the potential is minimized with respect to $\theta_S$ for $\cos(\theta_\alpha + 3\theta_S) > 0$. Therefore an unattractive choice must be made: allow the phase $\theta_S$ to vary dynamically (potentially ruining inflation in the process), or sacrifice gauge symmetry breaking (a defining characteristic of hybrid inflation). This issue compounds the difficulties in obtaining a red-tilted spectrum via slow roll. Thus we conclude that successful inflation may not be implemented in our $S^4$ model using only global SUSY.
Appendix C

PUBLICATION RIGHTS

This dissertation contains material from three published papers. Two of those papers have been published in an APS journals; the third was published by Elsevier. The papers are, in order

- Red spectral tilt and observable gravity waves in shifted hybrid inflation
- R-symmetry breaking in supersymmetry hybrid inflation
- Upper Bound on the Tensor-to-Scalar Ratio in GUT-Scale Supersymmetric Hybrid Inflation

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